Relativistic Quantum Mechanics and Field Theory of Arbitrary Spin

Volodimir Simulik

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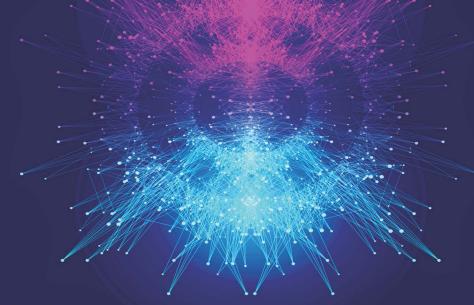
RELATIVISTIC QUANTUM MECHANICS AND FIELD THEORY OF ARBITRARY SPIN

CLASSICAL AND QUANTUM MECHANICS

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Preface

This book is about relativistic quantum mechanics and field theory of arbitrary spin. An original author's approach to the problem is considered.

The first goal is the derivation of the arbitrary spin relativistic wave equation without redundant components. The special procedure of synthesis of higher spin relativistic wave equations is suggested. The generalization for an arbitrary spin is fulfilled. This approach is based on three level scheme: (i) relativistic canonical quantum mechanics, (ii) the Foldy–Wouthuysen type canonical field model, (iii) locally covariant field theory.

Relativistic wave equations for particle-antiparticle doublets are found step by step. The start is given from the relativistic canonical quantum mechanics, after that the transition to the Foldy–Wouthuysen type canonical field model is fulfilled. Finally, on the basis of inverse Foldy–Wouthuysen type transformations locally covariant Dirac-like equations are derived. The examples of different spin particle-antiparticle doublets and useful multiplets are considered.

The second goal is the proof of the Fermi–Bose duality property of a few main equations of field theory, which before were known to have only single Fermi (or single Bose) property. The proof of the existence of bosonic symmetries, solutions, conservation laws of the Dirac equation and spinor field is the first step of this program. The fermionic properties of the Maxwell equations are considered as well. The Fermi–Bose duality of few new equations for higher spin also is considered.

The third goal is at least brief consideration of other corresponded original ideas and approaches. They are the application of classical electrodynamics to inneratomic problems, formulation of quantum electrodynamics in the terms of field strengths, symmetries of relativistic hydrogen atom, model of atom without quantum theory, etc.

Professionals know that it takes a lot of space to present a new approach.

On the other hand modern journals with impact factor prefer to publish brief or middle size articles. Therefore, an idea to explain it in book seems to be not only useful but a necessary as well. Indeed, four first Chapters here have the features of an introduction. Only after that the main ideas are presented and explained.

Furthermore, the proof of our main assertions appeals to new mathematical objects. They are the gamma matrix representation of 28-dimensional SO(8) algebra, which contains the standard and additional spin operators, and the 64-dimensional representations of the Clifford algebras $C\ell^{\mathbb{R}}(4,2)$ and $C\ell^{\mathbb{R}}(0,6)$ in the terms of Dirac γ matrices. These SO(8) and Clifford algebras are determined as the algebras over the field of real numbers. The new possibilities in comparison with standard 16-dimensional Clifford–Dirac algebra are demonstrated. Therefore, these algebras are useful not only in this book frameworks, but can contribute into other branches of contemporary theoretical physics. These new objects are introduced here in the Chapter 1.

The diversity of the approaches to the problem of the Dirac equation derivation (Chapter 2) demonstrates that the variety of approaches to the field theory model in general also is possible. In the Chapter 4 the helpfulness of the Foldy–Wouthuysen representation is considered. The main results are presented in the Chapters 5–10.

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Symbols

$C\ell^{\mathbb{R}}(4,2)$ and $C\ell^{\mathbb{R}}(0,6)$	–	two realizations of the 64-dimensional Clifford algebra over the field of real numbers
$C\ell^{\mathbb{C}}(1,3)$	–	Clifford algebra over the field of complex numbers, which gamma matrix representation determines the Clifford–Dirac algebra
SO(1,5)	–	15-dimensional special orthogonal group
SO(8)	–	28-dimensional special orthogonal group
SU(2)	–	special unitary group
M(1,3)	–	Minkowski space-time

R^3 — space of coordinates

$R^3_{\vec{k}}$ — momentum space

$H^{3,4}$ — quantum-mechanical Hilbert space of 4-component functions

$S^{3,4} \subset H^{3,4} \subset S^{3,4*}$ — quantum-mechanical rigged Hilbert space of 4-component functions

$S^{3,4}$ — Schwartz test function space of 4-component functions

$S^{3,4*}$ — Schwartz generalized function space of 4-component functions

$H^{3,N}$ — quantum-mechanical Hilbert space of N-component functions

$S^{3,N} \subset H^{3,N} \subset S^{3,N*}$ — quantum-mechanical rigged Hilbert space of N-component functions

$S^{3,N}$ — Schwartz test function space of N-component functions

$S^{3,N*}$ — Schwartz generalized function space of N-component functions

$L_+^{\uparrow} = SO(1,3)$ — proper ortochronous Lorentz group

$\mathcal{L} = SL(2,C)$ — universal covering of the proper ortochronous Lorentz group

$P_+^{\uparrow} = T(4) \times) L_+^{\uparrow} \supset L_+^{\uparrow}$ — proper ortochronous Poincaré group

$\mathcal{P}$ — universal covering of the proper ortochronous Poincaré group

$D(\frac{1}{2}, \frac{1}{2})$ — vector representation of the Lie algebra of Lorentz group

$D(1,0) \oplus (0,0)$ — tensor-scalar representation of the Lie algebra of Lorentz group

$D(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ — spinor representation of the Lie algebra of Lorentz group

FW	–	Foldy–Wouthuysen
RCQM	–	relativistic canonical quantum mechanics
SUSY	–	supersymmetry
IFR	–	inertial frame of reference

Chapter 1

On the "Old" and "New" Gamma Matrix Representations of the Clifford Algebra

The gamma matrix representation of 28-dimensional SO(8) algebra, which contains the standard and additional spin operators, is under consideration. The 64-dimensional representations of the Clifford algebras $C\ell^{\mathbb{R}}(4,2)$ and $C\ell^{\mathbb{R}}(0,6)$ in the terms of Dirac γ matrices are considered as well. The SO(8) and Clifford algebras are determined as the algebras over the field of real numbers. The relationships between the suggested representations of the SO(8) and Clifford algebras are investigated. The role of matrix representations of such algebras in the quantum field theory is considered. The role of gamma matrix representation of SO(8) algebra in the proof of Fermi–Bose duality property of the Dirac and higher spin Dirac-like equations is discussed briefly. Recent failed interpretation is overcome.

1.1.   Introduction to the Chapter 1

The role of the standard gamma matrix representations of the 16-dimensional Clifford algebra $C\ell^{\mathbb{C}}(1,3)$ (the Clifford–Dirac algebra) in the quantum field theory and its applications are well known. Recently we have found the 64-dimensional representations of the Clifford algebras $C\ell^{\mathbb{R}}(4,2)$ and $C\ell^{\mathbb{R}}(0,6)$ in the terms of Dirac γ matrices. The 28-dimensional Lie algebra SO(8) in the

terms of Dirac γ matrices, which is linked with $C\ell^{\mathbb{R}}(0,6)$, has been introduced as well. These SO(8) and Clifford algebras are determined as the algebras over the field of real numbers. It is naturally to suppose that gamma matrix representations of these algebras are useful in theoretical physics as well. Below the description of gamma matrix representations of algebras SO(8), $C\ell^{\mathbb{R}}(4,2)$ and $C\ell^{\mathbb{R}}(0,6)$ is given.

Consider the goal and subject of this chapter in some details.

The matrix representations of the SO(1,n), or SO(n), algebras (see, e.g., [1]) have the important applications in modern theoretical physics. First, these mathematical objects are useful in the relativistic quantum mechanics and field theory [1, 2]. Especially, the representations of SO(1,3) and SO(4) algebras have been applied.

The matrix representations of the Clifford algebra (on the Clifford algebras see, e.g., [3, 4]) have the important applications in modern theoretical physics as well. These mathematical objects simplify the calculations in the relativistic quantum mechanics and field theory. The most useful is the gamma matrix representation of the 16-dimensional Clifford algebra $C\ell^{\mathbb{C}}(1,3)$ over the field of complex numbers. In physics this object is called the Clifford–Dirac algebra.

At first we investigated the different special representations [5–11] of the 16-dimensional Clifford algebra $C\ell^{\mathbb{C}}(1,3)$ in the case of the Maxwell equations in the terms of electromagnetic field strengths. Recently, see [12–18], we were able to introduce the more powerful and useful as the Clifford–Dirac algebra mathematical objects. Over the field of real numbers the 64-dimensional gamma matrix representation of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ and 28-dimensional gamma matrix representation of SO(8) algebra have been introduced.

The gamma matrix representation of 64-dimensional $C\ell^{\mathbb{R}}(0,6)$ algebra was put into consideration in [12–18]. The algebra was considered over the field of real numbers and for the 4×4 γ matrices. Mathematically correct interpretation of $C\ell^{\mathbb{R}}(0,6)$ algebra was given in [19–24], and especially in [23, 24].

The gamma matrix representation of SO(8) algebra containing two independent sets of SU(2) generators was put into consideration in [15–18]. This algebra was considered over the field of real numbers, in terms of anti-Hermitian operators and for the 4×4 γ matrices. The interesting problem known from [16–18] is the relationship between the 64-dimensional $C\ell^{\mathbb{R}}(0,6)$ and 28-dimensional SO(8) algebras. Today's point of view on this problem is considered briefly in [23, 24] and in details below.

1.2.   Notations, Assumptions and Definitions

One of the principal objects of the relativistic quantum mechanics and field theory is the Dirac equation, see, e.g. [2]. This equation describes the particle-antiparticle doublet with spins s=(1/2,1/2) or, in other words, spin 1/2 fermion-antifermion doublet. The presentation of Dirac equation operator $D \equiv i\gamma^\mu \partial_\mu - m$, as well as a list of other operators of quantum spinor field, in the terms of γ matrices gives a possibility to use the anti-commutation relations between the Clifford algebra operators directly for finding the symmetries, solutions, conservation laws, fulfilling the canonical quantization and calculating the interaction processes in the quantum field models. The important fact is that application of the Clifford algebra essentially simplifies the calculations.

We use the definition of Clifford algebra given in [3, 4].

Thus, for the case of free non-interacting spinor field the Dirac equation has the form

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \tag{1.1}$$

where

$$x \in M(1,3), \quad \partial_\mu \equiv \partial/\partial x^\mu, \quad \mu = \overline{0,3}, \quad j = 1,2,3, \tag{1.2}$$

$M(1,3) = \{x \equiv (x^\mu) = (x^0 = t, \overrightarrow{x} \equiv (x^j))\}$ is the Minkowski space-time and 4-component function $\psi(x)$ belongs to rigged Hilbert space

$$S^{3,4} \subset H^{3,4} \subset S^{3,4*}. \tag{1.3}$$

Note that due to a special role of the time variable $x^0 = t \in (x^\mu)$ (in obvious analogy with nonrelativistic theory), in general consideration one can use the quantum-mechanical rigged Hilbert space (1.3). Here the Schwartz test function space $S^{3,4}$ is dense in the Schwartz generalized function space $S^{3,4*}$ and $H^{3,4}$ is the quantum-mechanical Hilbert space of 4-component functions over $R^3 \subset M(1,3)$. For the manifestly covariant field theory the rigged Hilbert space is taken as $S^{4,4} \subset H^{4,4} \subset S^{4,4*}$.

In order to finish with notations, assumptions and definitions let us note that here the system of units $\varepsilon = \mu = \hbar = c = 1$ is chosen, the metric tensor in Minkowski space-time $M(1,3)$ is given by

$$g^{\mu\nu} = g_{\mu\nu} = g_\nu^\mu, \ (g_\nu^\mu) = \mathrm{diag}\,(1,-1,-1,-1); \quad x_\mu = g_{\mu\nu}x^\mu, \tag{1.4}$$

and summation over the twice repeated indices is implied.

The Dirac γ matrices are taken in the standard Dirac-Pauli representation

$$\gamma^0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \quad \gamma^\ell = \begin{vmatrix} 0 & \sigma^\ell \\ -\sigma^\ell & 0 \end{vmatrix}, \quad \ell = 1,2,3, \tag{1.5}$$

where the Pauli matrices are given by

$$\sigma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma^2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \sigma^1\sigma^2 = i\sigma^3, \quad 123! - \text{circle.} \tag{1.6}$$

1.3. Standard Matrix Representations of the $C\ell^{\mathbb{C}}(1,3)$ Clifford and the SO(1,5) Lie Algebras

Four operators (1.5) satisfy the anti-commutation relations of the Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu, \gamma^{\dagger\ell} = -\gamma^\ell, \gamma^{\dagger 0} = \gamma^0, \tag{1.7}$$

and realize the 16-dimensional (2^4=16) matrix representation of the Clifford algebra $C\ell^{\mathbb{C}}(1,3)$ over the field of complex numbers.

For our purposes we introduce the additional matrix

$$\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3 = -i\begin{vmatrix} 0 & I \\ I & 0 \end{vmatrix}, \quad I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \tag{1.8}$$

Such matrix satisfies the anti-commutation relations of the Clifford algebra as well

$$\gamma^{\bar\mu}\gamma^{\bar\nu} + \gamma^{\bar\nu}\gamma^{\bar\mu} = 2g^{\bar\mu\bar\nu}, \quad \bar\mu,\bar\nu = \overline{0,4}, (g^{\bar\mu\bar\nu}) = (+----), \tag{1.9}$$

where metric tensor is given by 5×5 matrix. Contrary to four generators (1.5) of this Clifford algebra the operator $\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$ is dependent. Therefore, γ^4 does not contribute to the definition of this algebra as 16-dimensional (2^4=16) $C\ell^{\mathbb{C}}(1,3)$.

Here and in our publications (see, e.g. the articles [5–24]) we use the anti-Hermitian $\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$ matrix instead of the Hermitian γ^5 matrix of other authors [2]. Our γ^4 is equal to $i\gamma^5_{\text{standard}}$. Notation γ^5 is used in [12–24] and below for a completely different matrix $\gamma^5 \equiv \gamma^1\gamma^3\hat{C}$.

The 16 elements of the γ matrix representation of the $C\ell^{\mathbb{C}}(1,3)$ algebra are given by the Table 1.1, where $I_4 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4$ is 4×4 unit matrix, $s^{05},...,s^{45}$

Table 1. 1. The 16 elements of the γ matrix representation of the $C\ell^{\mathbb{C}}(1,3)$ algebra

$\mathbb{I}_4$					
	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$	$\gamma^0\gamma^4$	$2s^{05} \equiv \gamma^0$
		$\gamma^1\gamma^2$	$\gamma^1\gamma^3$	$\gamma^1\gamma^4$	$2s^{15} \equiv \gamma^1$
			$\gamma^2\gamma^3$	$\gamma^2\gamma^4$	$2s^{25} \equiv \gamma^2$
				$\gamma^3\gamma^4$	$2s^{35} \equiv \gamma^3$
					$2s^{45} \equiv \gamma^4$

are five elements of SO(1,5) Lie algebra determined at once below and in Table 1.2.

In the papers [25, 26] 16 elements of the Clifford-Dirac algebra are linked to the 15 elements of the SO(3,3) Lie algebra. In our papers [12–18] we were able to give the link between the γ matrix representation of the $C\ell^{\mathbb{C}}(1,3)$ algebra (Table 1.1) and the 15 elements of the SO(1,5) Lie algebra:

$$s^{\tilde{\mu}\tilde{\nu}} = \left\{ s^{\bar{\mu}\bar{\nu}} \equiv \frac{1}{4}\left[\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}\right], \ s^{\bar{\mu}5} = -s^{5\bar{\mu}} = \frac{1}{2}\gamma^{\bar{\mu}} \right\}, \quad \tilde{\mu}, \tilde{\nu} = \overline{0,5}, \ \bar{\mu}, \bar{\nu} = \overline{0,4}. \quad (1.10)$$

The generators (1.10) satisfy the commutation relations of the SO(1,5) Lie algebra:

$$\left[s^{\tilde{\mu}\tilde{\nu}}, s^{\tilde{\rho}\tilde{\sigma}}\right] = -g^{\tilde{\mu}\tilde{\rho}}s^{\tilde{\nu}\tilde{\sigma}} - g^{\tilde{\rho}\tilde{\nu}}s^{\tilde{\sigma}\tilde{\mu}} - g^{\tilde{\nu}\tilde{\sigma}}s^{\tilde{\mu}\tilde{\rho}} - g^{\tilde{\sigma}\tilde{\mu}}s^{\tilde{\rho}\tilde{\nu}}; \quad (g^{\tilde{\mu}\tilde{\nu}}) = \mathrm{diag}(+1,-1,-1,-1,-1,-1).$$
$$(1.11)$$

The 15 elements of the γ matrix representation of the SO(1,5) algebra are given by the Table 1.2.

Thus, it is proved that one and the same 15 not unit elements determine both the γ matrix representation of the $C\ell^{\mathbb{C}}(1,3)$ Clifford algebra and the γ matrix representation of the SO(1,5) Lie algebra (similarly for SO(6) algebra). Note that set of elements of the SO(1,5) representation (1.10) contains the spin operators of the spinor field and Dirac theory. Note that here in (1.10) the anti-Hermitian realization of the SO(1,5) operators is chosen, for the reasons see, e.g., [12–18],

Table 1. 2. The 15 elements of the γ matrix representation of the SO(1,5) algebra

$s^{01} \equiv \frac{1}{2}\gamma^0\gamma^1$	$s^{02} \equiv \frac{1}{2}\gamma^0\gamma^2$	$s^{03} \equiv \frac{1}{2}\gamma^0\gamma^3$	$s^{04} \equiv \frac{1}{2}\gamma^0\gamma^4$	$s^{05} \equiv \frac{1}{2}\gamma^0$
	$s^{12} \equiv \frac{1}{2}\gamma^1\gamma^2$	$s^{13} \equiv \frac{1}{2}\gamma^1\gamma^3$	$s^{14} \equiv \frac{1}{2}\gamma^1\gamma^4$	$s^{15} \equiv \frac{1}{2}\gamma^1$
		$s^{23} \equiv \frac{1}{2}\gamma^2\gamma^3$	$s^{24} \equiv \frac{1}{2}\gamma^2\gamma^4$	$s^{25} \equiv \frac{1}{2}\gamma^2$
			$s^{34} \equiv \frac{1}{2}\gamma^3\gamma^4$	$s^{35} \equiv \frac{1}{2}\gamma^3$
				$s^{45} \equiv \frac{1}{2}\gamma^4$

for the possibility and mathematical correction to choose the anti-Hermitian operators see, e.g., [27, 28].

Briefly these reasons are as follows. Some commutation and anti-commutation relations, as well as some transformations between different representations, are valid only in the terms of anti-Hermitian operators.

Comparison of the Tables 1.1 and 1.2 shows evidently that $C\ell^{\mathbb{C}}(1,3)$ and SO(1,5) are completely different algebras. Moreover, the algebra SO(1,5) can not be the subalgebra of the $C\ell^{\mathbb{C}}(1,3)$ algebra. Indeed, the first one is Clifford and the second is Lie algebra, which have different structure. In particular, Lie algebras do not contain the unit element as $\mathtt{I}_4 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4$ from $C\ell^{\mathbb{C}}(1,3)$.

1.4. Matrix Representations of the $C\ell^{\mathbb{R}}(0,6)$, $C\ell^{\mathbb{R}}(4,2)$ Clifford and the SO(8) Lie Algebras

In addition to the Dirac equation the γ matrix representations of the Clifford algebra are used widely for the multicomponent Dirac-like equations of arbitrary spin such as the Bhabha [29, 30], Bargman–Wigner [31], Rarita–Schwinger (for the field with the spin s=3/2 [32]), Iwanenko-Landau-Dirac–Kähler (see, e.g., [33]) equations.

Note that today the problem of arbitrary spin description is presented by a set of different approaches [34–67] where the different wave equations for higher and arbitrary spin are suggested (on our results see [19–22, 68–71]). Note that these equations can describe both fermions and bosons. Moreover, new equations for an arbitrary spin [19–22, 68–71], which are derived from the

relativistic canonical quantum mechanics, are based on the γ matrix representations of the Clifford algebra as well. Here at first the proper arbitrary spin wave equations are listed [34–46], further, the equations for the particles of spin 3/2 [47–54, 56–65, 67] and spin 2 [55, 66] are presented.

Thus, the application of the Clifford algebra in the quantum theory is more wide then the 4-component Dirac equation and corresponded spinor field. It is evident that the program of finding of more wide matrix representations then the 16-dimensional $C\ell^{\mathbb{C}}(1,3)$ and 15-dimensional SO(1,5) algebras representations is the interesting task. The corresponding matrix representations can be useful both in the known quantum field theory models and in the development of new approaches.

This problem has been considered in [12–22], where the new 64-dimensional matrix representations of the $C\ell^{\mathbb{R}}(0,6)$ and $C\ell^{\mathbb{R}}(4,2)$ Clifford algebras have been found as well as the 28-dimensional representation of the SO(8) Lie algebra, all over the field of real numbers. Contrary to the well-known matrix representation of the SO(1,5) algebra (15 generators of which accurate to the 1/2 coefficient coincide with 15 not unit elements of the standard γ matrix representation of the $C\ell^{\mathbb{C}}(1,3)$ Clifford algebra and, furthermore, determine the 16-dimensional representation of this Clifford algebra as well) 64-dimensional matrix representations of the $C\ell^{\mathbb{R}}(0,6)$ and $C\ell^{\mathbb{R}}(4,2)$ Clifford algebras essentially do not coincide with 28-dimensional γ matrix representation of the SO(8) Lie algebra. The 28 generators of the SO(8) algebra γ matrix representation do not realize the Clifford algebra even after multiplication by factor 2 and adding the unit element. In order to find some γ matrix representation of the Clifford algebra on the basis of 28 SO(8) generators one must essentially expands the list of these operators. Thus, one comes to the 64-dimensional matrix representations of the $C\ell^{\mathbb{R}}(0,6)$ and $C\ell^{\mathbb{R}}(4,2)$ Clifford algebras over the field of real numbers, which have been found earlier in [12–22].

By taking into account these new γ matrix representations of the Clifford and SO(8) algebras the Bose symmetries, solutions and conservation laws for the standard Dirac equation and spinor field with nonzero mass have been found [12–18]. Therefore, the Fermi-Bose duality of these equation has been proved. These interesting results are presented below in the Chapter 10. Furthermore, in [19–22, 70, 71] the Fermi-Bose duality of the multicomponent Dirac-like equations for higher spin has been proved as well (see the Chapter 7).

Let us consider briefly the new matrix representations of the Clifford and SO(8) algebras and the history how these algebras have been put into consider-

ation. We have started from the case of *massless Dirac equation* and the *spinor field of zero mass*. Indeed, at first the Bose symmetries of the massless Dirac equation have been found. We have used essentially so-called *Pauli-Gürsey-Ibragimov* symmetry [72–74] of the Dirac equation with zero mass, i.e., the fact that this equation is invariant with respect to the transformations generating by the eight operators

$$\{\gamma^2\hat{C}, i\gamma^2\hat{C}, \gamma^2\gamma^4\hat{C}, i\gamma^2\gamma^4\hat{C}, \gamma^4, i\gamma^4, i, \mathrm{I}\}, \tag{1.12}$$

where imaginary unit $i = \sqrt{-1}$ is considered as the operator, $\hat{C}$ is the operator of complex conjugation, $\hat{C}\psi = \psi^*$ (the operator of involution in $\mathrm{H}^{3,4}$), i.e. these operators are here the nontrivial orts of the algebra. Of course, the *Pauli-Gürsey-Ibragimov* algebra is defined over the field of real numbers.

At first we have proved that six of the operators (1.12)

$$s_{\mathrm{PGI}}^{01} = \frac{i}{2}\gamma^2\hat{C}, \; s_{\mathrm{PGI}}^{02} = -\frac{1}{2}\gamma^2\hat{C}, \; s_{\mathrm{PGI}}^{03} = -\frac{i}{2}\gamma^4, \; s_{\mathrm{PGI}}^{23} = \frac{i}{2}\gamma^2\gamma^4\hat{C}, \tag{1.13}$$

$$s_{\mathrm{PGI}}^{31} = -\frac{1}{2}\gamma^2\gamma^4\hat{C}, \; s_{\mathrm{PGI}}^{12} = -\frac{i}{2},$$

realizes the additional $\mathrm{D}(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$ spinor representation [5–11] of the Lie algebra of *universal covering* $\mathcal{L} = \mathrm{SL}(2,\mathrm{C})$ of the proper ortochronous Lorentz group $\mathrm{L}_+^\uparrow = \mathrm{SO}(1,3) = \{\Lambda = (\Lambda_\nu^\mu)\}$. Moreover, six corresponding operators $s^{\mu\nu} = \{s^{01}, s^{02} s^{03}, s^{23}, s^{31}, s^{12}\}$ from the set (1.10) commute with the operators from (1.13).

Further, the application of the simplest linear combinations of the generators $s^{\mu\nu}$ (1.10) and (1.13) gives the possibility [5–11] to find Bose representations of the Lorentz group $\mathcal{L}$ and the Poincaré group $\mathcal{P} \supset \mathcal{L} = \mathrm{SL}(2,\mathrm{C})$ (here $\mathcal{P}$ is the universal covering of the proper ortochronous Poincaré group $\mathrm{P}_+^\uparrow = \mathrm{T}(4)\times)\mathrm{L}_+^\uparrow \supset \mathrm{L}_+^\uparrow$), with respect to which the massless Dirac equation is invariant. Thus, we have found the Bose $\mathrm{D}(1,0)\oplus(0,0)$ and $\mathrm{D}(\frac{1}{2},\frac{1}{2})$ representations of the Lie algebra of the Lorentz group $\mathcal{L}$ together with the tensor-scalar of the spin s=(1,0) and vector representations of the Lie algebra of the Poincaré group $\mathcal{P}$, with respect to which the Dirac equation with m=0 is invariant.

After that the goal of further investigations has been formulated as follows. *To find the similar symmetries of the Dirac equation with nonzero mass.* Our first idea in this direction was to find the complete set of combinations of the Pauli-Gürsey-Ibragimov operators (1.12) and the elements of the matrix representation of the $C\ell^\mathbb{C}(1,3)$ algebra demonstrated in the Table 1.1. This program

has been fulfilled in [12–22] and resulted in the 64 elements of the matrix representation of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$. After that on the basis of these 64 elements the 28 generators of algebra SO(8) have been found [15–18].

Below the results are presented in modern mathematical notations of Clifford algebras and in valid interpretation.

Consider the fact that seven γ matrices

$$\gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3, \gamma^5 = \gamma^1\gamma^3\hat{C}, \gamma^6 = i\gamma^1\gamma^3\hat{C}, \gamma^7 = i\gamma^0, \tag{1.14}$$

where $\gamma^0 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \gamma^k = \begin{vmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{vmatrix}$, σ^k are the standard Pauli matrices (1.6) and operator $\hat{C}$ is defined after the formula (1.12), satisfy the anticommutation relations

$$\gamma^A\gamma^B + \gamma^B\gamma^A = -2\delta^{AB}, \quad A, B = \overline{1,7}, \tag{1.15}$$

of the Clifford algebra generators over the field of real numbers. Due to the evident fact that only six operators of (1.14) are linearly independent, $\gamma^4 = -i\gamma^7\gamma^1\gamma^2\gamma^3$, it is the representation of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ of the dimension $2^6 = 64$.

The first 16 operators are given in the Table 1.1, the next 16 are found from them with the help of the multiplication by imaginary unit $i = \sqrt{-1}$. Last 32 are found from first 32 with the help of multiplication by operator $\hat{C}$ of complex conjugation. Thus, if to introduce the notation "stand CD" ("stand" and CD are taken from standard Clifford–Dirac) for the set of 16 matrices from the Table 1.1, then the set of 64 elements of $C\ell^{\mathbb{R}}(0,6)$ algebra representation will be given by

$$\left\{(\text{stand CD}) \cup i \cdot (\text{stand CD}) \cup \hat{C} \cdot (\text{stand CD}) \cup i\hat{C} \cdot (\text{stand CD})\right\}. \tag{1.16}$$

As the consequences of the equalities $\gamma^4 \equiv \prod_{\mu=0}^{3} \gamma^\mu \rightarrow \prod_{\bar{\mu}=0}^{4} \gamma^{\bar{\mu}} = -\mathrm{I}$, known from the standard Clifford–Dirac algebra $C\ell^{\mathbb{C}}(1,3)$, and the anticommutation relations (1.15), in $C\ell^{\mathbb{R}}(0,6)$ algebra for the matrices γ^A (1.14) the following extended equalities are valid: $\gamma^7 \equiv -\prod_{\underline{A}=1}^{6} \gamma^{\underline{A}} \rightarrow \prod_{A=1}^{7} \gamma^A = \mathrm{I}, \quad \gamma^5\gamma^6 = i$.

Another realization $C\ell^{\mathbb{R}}(4,2)$ can be formed directly by the four γ^μ generators of $C\ell^{\mathbb{C}}(1,3)$ taking together with the simplest independent generators from (1.12) of the Pauli–Gürsey–Ibragimov algebra [72–74].

The main structure elements of such joint set are given by $(\gamma^0, \gamma^1, \gamma^2, \gamma^3, i, \hat{C}I_4)$, where γ^μ are 4×4 Dirac matrices in standard representation. It is easy to see that simplest set of the Clifford–Dirac algebra generators can be constructed from these elements in the form $(i\gamma^0, i\gamma^1, \gamma^2, i\gamma^3, \hat{C}I_4, i\hat{C}I_4)$. Therefore, the 4×4 matrix generators of the corresponding Clifford–Dirac algebra over the field of real numbers can be found by the simple redefinition

$$\tilde{\gamma}^1 \equiv i\gamma^1, \quad \tilde{\gamma}^2 \equiv i\gamma^3, \quad \tilde{\gamma}^3 \equiv \hat{C}I_4, \quad \tilde{\gamma}^4 \equiv i\hat{C}I_4, \quad \tilde{\gamma}^5 \equiv i\gamma^0, \quad \tilde{\gamma}^6 \equiv -\gamma^2, \quad (1.17)$$

of the matrices $(i\gamma^0, i\gamma^1, \gamma^2, i\gamma^3, \hat{C}I_4, i\hat{C}I_4)$, where γ^μ are given in (1.5), (1.6).

Matrices (1.17) together with the matrix $\tilde{\gamma}^7 \equiv \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \tilde{\gamma}^4 \tilde{\gamma}^5 \tilde{\gamma}^6 = \gamma^4$ satisfy the anti-commutation relations of the Clifford–Dirac algebra generators in the form

$$\tilde{\gamma}^A \tilde{\gamma}^B + \tilde{\gamma}^B \tilde{\gamma}^A = 2g^{AB}; \quad A, B = \overline{1,7}, \ (g^{AB}) = (+ + + + - - -). \quad (1.18)$$

As well as in (1.14) among the generators of (1.18) only the 4+2=6 matrices (1.17) are independent and form the basis of the algebra. Therefore, the found above algebra over the field of real numbers is defined as $C\ell^{\mathbb{R}}(4,2)$ and the dimension of this algebra is $2^6 = 64$.

Operators (1.14) generate also the 28 matrices

$$s^{\widetilde{AB}} = \{s^{AB} = \frac{1}{4}[\gamma^A, \gamma^B], s^{A8} = -s^{8A} = \frac{1}{2}\gamma^A\}, \quad \widetilde{A}, \widetilde{B} = \overline{1,8}, \quad (1.19)$$

which satisfy the commutation relations of the Lie algebra SO(8)

$$[s^{\widetilde{AB}}, s^{\widetilde{CD}}] = \delta^{\widetilde{AC}} s^{\widetilde{BD}} + \delta^{\widetilde{CB}} s^{\widetilde{DA}} + \delta^{\widetilde{BD}} s^{\widetilde{AC}} + \delta^{\widetilde{DA}} s^{\widetilde{CB}}. \quad (1.20)$$

It is evident that here we have the algebra over the field of real numbers as well. Furthermore, it is evident that 28 elements (1.19) of SO(8) do not form any Clifford algebra and do not form any subalgebra of the Clifford algebra. It is independent from the Clifford algebra mathematical object. Note that here (as in (1.10)) the anti-Hermitian realization of the SO(8) operators is chosen, for the reasons see, e.g., [12–18] and [27, 28].

The explicit form of the 28 elements of the γ matrix representation of the SO(8) algebra is given in the Table 1.3.

The start of these investigations has been given in [12–15]. The exact interpretation of considered algebraic objects is given for the first time in [24].

Table 1. 3. The 28 elements of the γ matrix representation of the SO(8) algebra

$s^{12} \equiv \frac{1}{2}\gamma^1\gamma^2$	$s^{13} \equiv \frac{1}{2}\gamma^1\gamma^3$	$\frac{1}{2}\gamma^1\gamma^4$	$\frac{1}{2}\gamma^1\gamma^5$	$\frac{1}{2}\gamma^1\gamma^6$	$\frac{1}{2}\gamma^1\gamma^7$	$s^{18} \equiv \frac{1}{2}\gamma^1$
	$s^{23} \equiv \frac{1}{2}\gamma^2\gamma^3$	$\frac{1}{2}\gamma^2\gamma^4$	$\frac{1}{2}\gamma^2\gamma^5$	$\frac{1}{2}\gamma^2\gamma^6$	$\frac{1}{2}\gamma^2\gamma^7$	$s^{28} \equiv \frac{1}{2}\gamma^2$
		$\frac{1}{2}\gamma^3\gamma^4$	$\frac{1}{2}\gamma^3\gamma^5$	$\frac{1}{2}\gamma^3\gamma^6$	$\frac{1}{2}\gamma^3\gamma^7$	$s^{38} \equiv \frac{1}{2}\gamma^3$
			$\frac{1}{2}\gamma^4\gamma^5$	$\frac{1}{2}\gamma^4\gamma^6$	$\frac{1}{2}\gamma^4\gamma^7$	$s^{48} \equiv \frac{1}{2}\gamma^4$
				$\frac{1}{2}\gamma^5\gamma^6$	$\frac{1}{2}\gamma^5\gamma^7$	$s^{58} \equiv \frac{1}{2}\gamma^5$
					$\frac{1}{2}\gamma^6\gamma^7$	$s^{68} \equiv \frac{1}{2}\gamma^6$
						$s^{78} \equiv \frac{1}{2}\gamma^7$

1.5. Relationship between the γ Matrix Representations of the SO(8) and $C\ell^{\mathbb{R}}(0,6)$ Algebras

In our earlier publications on this subject [15–18] (and in some publications of other authors) the interpretation of the relationship between the γ matrix representation of SO(8) algebra and the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ was not correct. Nevertheless, in the last time publications [19–22] the classification of the found 64-dimensional γ matrix representation of the Clifford algebra has been given. After that the situation became clear. The publications [23, 24] are free already from the interpretation difficulties.

Consider briefly the assertions, which mast be used in correct interpretations of found mathematical objects.

(i) The Clifford algebra (and Clifford–Dirac matrix representation as well) can not be 29-dimensional because the dimension of $C\ell^{\mathbb{R}}(m,n)$ algebra is 2^{m+n}. Therefore, the chain of possible dimensions is given by 16, 32, 64, 128, ... (ii) For the same reasons the 29-dimensional algebra can not be any subalgebra of the Clifford algebra. (iii) Further, such representations can not be "exclusive" because all gamma matrix representations of that kind were considered already in [12–18] and [19–22]. (iiii) Furthermore, 29 dimensions are related somewhere with SO(8) algebra. Unfortunately, the algebra SO(8) belongs to Lie algebras and Lie algebras does not contain unit element. Therefore, the dimen-

sion of SO(8) algebra is 28, not 28+1. (V) It is evident that any Lie algebra can not coincide (or be the subalgebra) of any Clifford algebra. Clifford and Lie algebras are different algebras with different structure. For example, contrary to Lie algebras, Clifford algebras contain the unit element. (VI) The chain of possible dimensions for SO(m.n), or SO(n), algebras is given by 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ... (For the dimension of SO(n) algebra the formula $n(n-1)/2$ is valid.) Therefore, the most close to each other are 16-dimensional Clifford $C\ell^{\mathbb{C}}(1,3)$ and 15-dimensional SO(1,5) algebras over the field of complex numbers. Below in Table 1.1 and Table 1.2 the explicit forms of corresponding gamma matrix representations are compared. There is no doubt in difference between the elements of these representations. The comparison of introduced in [12–18] gamma matrix representations of $C\ell^{\mathbb{R}}(0,6)$ and SO(8) algebras is given in [23, 24] and here. In this case the difference between the elements of corresponding representations is evident. The comparison of the sets from (1.16) and Table 1.3 can be fulfilled very easy. It is evident that these sets are completely different

Note that the analysis of physical phenomena is impossible without mathematical correctness of consideration. Note further that the γ matrix representations of the algebras SO(8) and $C\ell^{\mathbb{R}}(0,6)$ are the basis of the proof of the nonzero mass Dirac equation Fermi–Bose duality [12–18]. Thus, these objects are useful for contemporary classical and quantum field theory.

1.6.　　On the Possible Relation to SUSY

Note that the results of [12–18] are not related to SUSY (supersymmetry).

(i) SUSY partner for considered in [12–18] spin 1/2 particle-antiparticle doublet s=(1/2,1/2) is the spin 0 particle-antiparticle doublet s=(0,0). In some variants of SUSY the role of such super-partner can play the spin 1 particle-antiparticle doublet s=(1,1). In [12–18] another object, namely spin s=(1,0) particle-antiparticle multiplet, has been considered. Thus, results of [12–18] do not relate to SUSY.

(ii) SUSY is based on fermion-boson super-multiplets. Such objects are not considered in [12–18]]. Thus, results of [12–18] do not relate to SUSY.

(iii) SUSY is based on super-space, which in addition to Minkowski space-time contains non-commutative spinor coordinates. Such objects are not considered in [12–18] as well. Thus, results of [12–18] do not relate to SUSY as well.

Moreover, super-partners with essentially large mass in comparison with ordinary elementary particles are absent in [12–18].

It means that above mentioned concepts, definitions and notations determined SUSY as the concrete fixed model of physical reality. It is evident that in [12–18] we deal with another theoretical model. In [12–18] following [75–77] such phenomena were named the Fermi–Bose duality. If in future we will be able to construct on the basis of [12–18] the new principal model of physical reality it will be not contemporary SUSY.

Another question arises "Why somebody needs to follow SUSY today?" Note that today hypothesis on SUSY is not so popular as 20 years ago! The subject of [12–18] publications was the Fermi–Bose duality in the sense of [75–77], without appealing to SUSY.

1.7. Generalization on Arbitrary Dimensions

Matrix representations of the Clifford and SO(8) algebras considered above for the case of 4×4 gamma matrices can be generalized for an arbitrary dimensions N=2s+1. The beginning was given in [20–22]. Such generalization is necessary for the relativistic canonical quantum mechanics and field theory of arbitrary spin [19–22]. The notation N=2s+1 was given for the particle singlet of arbitrary spin and the notation 2N was given for the particle-antiparticle doublet. Note that Pauli matrices exist only when the values of N are even.

Note further that in higher dimensions (as it follows, e.g., from [33] and [78]) the more wide algebras as $C\ell^{\mathbb{R}}(0,6)$ or SO(8) can be found.

1.8. Brief Consideration of Application

The first application of the matrix representations of the algebras $C\ell^{\mathbb{R}}(0,6)$ and SO(8) is the symmetry analysis (the search of groups and algebras with respect to which the equation is invariant) of the Dirac equation with nonzero mass. It is easy to understand that the Foldy–Wouthuysen (FW) representation [79, 80] is preferable for such analysis. Indeed, in this representation one must calculate the commutation relations of possible pure matrix symmetry operators from (1.16) and Table 1.3 only with two elements of the FW equation operator: γ^0 and i. After the determining of the symmetries of the FW equation one can find the symmetries of the Dirac equation on the basis of the inverse FW transfor-

mation. Note that after such transformation only the small part of symmetry operators will be pure matrix, the main part of operators will contain the non-local operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ and the functions of it. The 31-dimensional Lie algebra SO(6)$\oplus i\gamma^0$SO(6)$\oplus i\gamma^0$ has been found, which is formed by the elements from $C\ell^{\mathbb{R}}(0,6)$ and is the maximal pure matrix algebra of invariance of the Dirac equation in the FW representation. For the Dirac equation only the part of this algebra is pure matrix, other elements contain the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$, see, e.g., [12–18].

Furthermore, in the FW representation [79, 80] two subsets (s^{23}, s^{31}, s^{12}) and (s^{45}, s^{64}, s^{56}) of operators $s^{\widetilde{AB}}$ from the Table 1.3 (i) determine two different sets of SU(2) spin 1/2 generators, (ii) commute between each other and (iii) commute with the operator of the Dirac equation in the FW representation. Therefore, we can use here the methods developed in [5–11] for the case $m = 0$ (the brief consideration is given in section 1.4 above). The start of such investigations for $m \neq 0$ has been presented in [12–18], where on this bases the Bose symmetries, Bose solutions and Bose conservation laws for the Dirac equation with nonzero mass have been found. Among the Bose-symmetries the important Lorentz and Poincaré algebras of invariance of the Dirac equation with nonzero mass were found (see also the Chapter 10 here).

Thus, the algebras $C\ell^{\mathbb{R}}(0,6)$ and SO(8) are the basis of the proof [12–18] of the nonzero mass Dirac equation Fermi–Bose duality.

1.9. Other Useful Gamma Matrix Representations of the SO(8) Algebra Over the Field of Real Numbers

It follows from the consideration above that the algebra SO(8) over the field of real numbers, which was introduced in [12–18] and, finally, in [24], is the main algebraic object for our further consideration.

In **the fundamental FW representation**, the 28 orts of the real algebra SO(8) are given by the formulas (1.19), where the 6 generators of the corresponding Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ representation, together with the matrix operator γ^7, are given in (1.14).

In the standard **Pauli–Dirac representation**, the so-called covariant (or local) representation, the corresponding 28 orts of the SO(8) algebra are the consequences of the inverse for the FW [79] transformation and are given by the

corresponding generators

$$\widetilde{s}^{\widetilde{A}\widetilde{B}} = \{\widetilde{s}^{AB} = \frac{1}{4}[\widetilde{\gamma}^A, \widetilde{\gamma}^B], \widetilde{s}^{A8} = -\widetilde{s}^{8A} = \frac{1}{2}\widetilde{\gamma}^A\}, \tag{1.21}$$

where $\widetilde{A}, \widetilde{B} = \overline{1,8}$, $A, B = \overline{1,7}$. In expressions (1.21) 6 generators $\widetilde{\gamma}^A = V^+\gamma^A V^-$ of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ representation and operator $\widetilde{\gamma}^7$, together with operators $\widetilde{\gamma}^0 = V^+\gamma^0 V^-$ and $\widetilde{C} = V^+\hat{C}V^-$, are non-local and have the form:

$$\overrightarrow{\widetilde{\gamma}} = \overrightarrow{\gamma}\frac{-\overrightarrow{\gamma}\cdot\nabla + m}{\widehat{\omega}} + \overrightarrow{p}\frac{-\overrightarrow{\gamma}\cdot\nabla + \widehat{\omega} + m}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{1.22}$$

$$\widetilde{\gamma}^4 = \gamma^4\frac{-\overrightarrow{\gamma}\cdot\nabla + m}{\widehat{\omega}}, \tag{1.23}$$

$$\widetilde{\gamma}^5 = \widetilde{\gamma}^1\widetilde{\gamma}^3\widetilde{C}, \quad \widetilde{\gamma}^6 = i\widetilde{\gamma}^1\widetilde{\gamma}^3\widetilde{C}, \quad \widetilde{\gamma}^7 = i\widetilde{\gamma}^0, \tag{1.24}$$

$$\widetilde{\gamma}^0 = \gamma^0\frac{-\overrightarrow{\gamma}\cdot\nabla + m}{\widehat{\omega}}, \quad \widetilde{C} = (\mathrm{I} + 2\frac{i\gamma^1\partial_1 + i\gamma^2\partial_2}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}})\hat{C}, \tag{1.25}$$

where $\widehat{\omega} \equiv \sqrt{-\triangle + m^2}$.

In the **bosonic representation** of the FW model, where the proof of the bosonic properties of the FW and Dirac equation is most convenient, the corresponding 28 orts of the SO(8) algebra are given by

$$\breve{s}^{\widetilde{A}\widetilde{B}} = \{\breve{s}^{AB} = \frac{1}{4}[\breve{\gamma}^A, \breve{\gamma}^B], \breve{s}^{A8} = -\breve{s}^{8A} = \frac{1}{2}\breve{\gamma}^A\}, \tag{1.26}$$

where $\widetilde{A}, \widetilde{B} = \overline{1,8}$, $A, B = \overline{1,7}$. In formulas (1.26) the 6 generators of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ representation, together with operators $\breve{\gamma}^0\breve{\gamma}^7$, operators i and $\breve{C}$, have the form

$$\breve{\gamma}^0 = \begin{vmatrix} \sigma^3 & 0 \\ 0 & \sigma^1 \end{vmatrix}, \quad \breve{\gamma}^1 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & i & i \\ -1 & i & 0 & 0 \\ 1 & i & 0 & 0 \end{vmatrix}, \tag{1.27}$$

$$\breve{\gamma}^2 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -i & i \\ 0 & 0 & -1 & -1 \\ -i & 1 & 0 & 0 \\ i & 1 & 0 & 0 \end{vmatrix}, \quad \breve{\gamma}^3 = -\begin{vmatrix} \sigma^2 & 0 \\ 0 & i\sigma^2 \end{vmatrix}\hat{C}, \tag{1.28}$$

$$
\breve{\gamma}^4 = \begin{vmatrix} i\sigma^2 & 0 \\ 0 & -\sigma^2 \end{vmatrix} \hat{C}, \quad \breve{\gamma}^5 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & i & -i \\ 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \end{vmatrix}, \tag{1.29}
$$

$$
\breve{\gamma}^6 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & -i & -i \\ 0 & 0 & 1 & -1 \\ -i & -1 & 0 & 0 \\ -i & 1 & 0 & 0 \end{vmatrix}, \quad \breve{\gamma}^7 = \gamma^7 = i\gamma^0, \tag{1.30}
$$

$$
\breve{i} = \begin{vmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^1 \end{vmatrix}, \quad \breve{C} = \begin{vmatrix} \sigma^3 & 0 \\ 0 & I_2 \end{vmatrix} \hat{C}. \tag{1.31}
$$

Transition from the fundamental representation of the SO(8) algebra to the bosonic representation is fulfilled by the transformation $\breve{\gamma}^A = W\gamma^A W^{-1}$ with the help of the operator W:

$$
W = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2}C & 0 \\ 0 & -C & 0 & 1 \\ 0 & -C & 0 & -1 \end{vmatrix}, \quad W^{-1} = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -C & -C \\ 0 & i\sqrt{2}C & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix}, \tag{1.32}
$$

$$
WW^{-1} = W^{-1}W = I_4.
$$

where C is the simple operator of complex conjugation.

In **the bosonic representation for the massless Dirac equation and slightly generalized Maxwell equations with extended symmetry properties** [8–11] the corresponding 28 orts of the SO(8) algebra are given by

$$
\acute{s}^{\widetilde{AB}} = \{ \acute{s}^{AB} = \frac{1}{4}[\acute{\gamma}^A, \acute{\gamma}^B], \acute{s}^{A8} = -\acute{s}^{8A} = \frac{1}{2}\acute{\gamma}^A \}, \tag{1.33}
$$

where $\widetilde{A}, \widetilde{B} = \overline{1,8}$, $A, B = \overline{1,7}$. In formulas (1.26) the 6 generators of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ representation, together with operators $\acute{\gamma}^0\acute{\gamma}^7$, operators $\acute{i}$ and $\acute{C}$, have the form

$$
\acute{\gamma}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \hat{C}, \quad \acute{\gamma}^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \hat{C}, \quad \acute{\gamma}^2 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \hat{C},
$$

$$\acute{\gamma}^3 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \hat{C}, \ \acute{\gamma}^4 = i, \ \acute{\gamma}^5 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \hat{C}, \ \acute{\gamma}^6 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{vmatrix} \hat{C},$$

$$(1.34)$$

$$\acute{\gamma}^7 = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix} \hat{C}, \quad \acute{i} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{vmatrix}, \quad \acute{C} = \begin{vmatrix} C & 0 & 0 & 0 \\ 0 & -C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix}.$$

In **the relativistic canonical quantum mechanics** [19–22] (axiomatic foundations are given briefly in [20]) the 6 generators $\bar{\gamma}^A$, together with $\bar{\gamma}^7$, are given by $\bar{\gamma}^A = v\gamma^A v$ (operator v is known as $v = \begin{vmatrix} I_2 & 0 \\ 0 & CI_2 \end{vmatrix}$) and have an explicit form:

$$\bar{\gamma}^1 = \gamma^1 \hat{C}, \ \bar{\gamma}^2 = \gamma^0 \gamma^2 \hat{C}, \ \bar{\gamma}^3 = \gamma^3 \hat{C}, \ \bar{\gamma}^4 = \gamma^0 \gamma^4 \hat{C} \bar{\gamma}^5 = \gamma^1 \gamma^3 \hat{C}, \ \bar{\gamma}^6 = -i\gamma^2 \gamma^4 \hat{C}, \ \bar{\gamma}^7 = i,$$

$$(1.35)$$

where contrary to simple complex conjugation C operator $\hat{C}$ is 4×4 matrix $I_4 C$.

1.10. Matrix Representation of the $C\ell^{\mathbb{R}}(3,1)$ Algebra in the Terms of Real Gamma Matrices and Corresponded $C\ell^{\mathbb{R}}(0,6)$

Matrix representations of the Clifford algebras in the terms of real gamma matrices are known from [81, 82]. Although the final physical results do not depend on the applied representation of the Dirac matrices, e.g., due to the invariance of traces of products of Dirac matrices, the appropriate choice of the representation used may facilitate the analysis.

The generators of $C\ell^{\mathbb{R}}(3,1)$ algebra in the terms of real gamma matrices

$$\gamma_r^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad \gamma_r^1 = \begin{vmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \tag{1.36}$$

$$\gamma_r^2 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \gamma_r^3 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

were applied in [83] for Courant–Snyder theoretical model of two-dimensional coupled linear optics. The author of [84] used another unitary equivalent representation

$$\tilde{\gamma}_r^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \tilde{\gamma}_r^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \tag{1.37}$$

$$\tilde{\gamma}_r^3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad \tilde{\gamma}_r^4 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix},$$

The generators (1.36) (as well as (1.37)) can be rewritten in the terms of Pauli matrices (1.6) as 2×2 block matrices. The corresponded for (1.36) explicit forms are given by

$$\gamma_r^0 = \begin{vmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{vmatrix}, \gamma_r^1 = \begin{vmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{vmatrix}, \gamma_r^2 = \begin{vmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{vmatrix}, \gamma_r^3 = \begin{vmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{vmatrix}.$$
$$\tag{1.38}$$

The real matrices (1.36) (or (1.38)) satisfy the anti-commutation relations (1.7) of the Clifford algebra generators, in which the metric tensor $g^{\mu\nu} = \text{diag}(- + + +)$.

The complete set of 16 elements of the Clifford algebra $C\ell^{\mathbb{R}}(3,1)$ gamma matrix representation is given by the Table 1.1. Here all 16 elements are real matrices. The complete set of 64 elements of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ gamma matrix representation is given by (1.16).

The link with SU(2) spin of fermionic-antifermionic particle doublet, with SO(5,1) and SO(8) Lie algebras gamma matrix representations, is not so direct and evident as in consideration above. Nevertheless, the SO(8) Lie algebra gamma matrix representation and complete above given consideration can be repeated on the basis of (1.36) or (1.37).

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Chapter 2

Twenty Six Variants of the Dirac Equation Derivation

The 26 different ways of the Dirac equation derivation are presented. The various physical principles and mathematical formalisms are used. Three original approaches of the authors to the problem are given. They are (i) the generalization of H. Sallhofer derivation, (ii) the obtaining of the massless Dirac equation from the Maxwell equations in maximally symmetrical form, (iii) the derivation of the Dirac equation with nonzero mass from the relativistic quantum mechanics of the fermion-antifermion spin s=1/2 doublet. Today we are able to demonstrate new features of our derivations given in original papers. In some sense the role of the Dirac equation today is demonstrated.

2.1. Introduction to the Chapter 2

The Dirac equation is one of the fundamental equations of modern theoretical physics. It is in service more than 90 years (1928–2019). The application today is much wider than the areas of quantum mechanics, quantum field theory, atomic and nuclear physics, solid state physics. Let us recall only that the first analysis of this equation enabled Dirac to give the theoretical prediction of the positron, which was discovered experimentally by Anderson in 1932. The recent well-known application of the massless Dirac equation to the graffen ribbons is an example of the contemporary possibilities of this equation.

The successful derivation of some equation of mathematical physics is the

first step to successful application. In such process the essence of the corresponding model of nature, the mathematical principles and the physical foundations are visualized. Here we deal with the different approaches to the problem of the Dirac equation derivation

Here the original investigation of the problem of the Dirac equation derivation is presented. The different approaches, which are based on the various mathematical and physical principles, are considered (26 methods). The importance of place of the Dirac equation in modern theoretical physics is discussed.

In our recent publications [2–8] we were able to extend the domain of the Dirac equation application. We proved [2–8] that this equation has not only fermionic but also the bosonic features, can describe not only fermionic, but also the bosonic states.

Note that in modern quantum field theory the Dirac equation is applied not only to the spin 1/2 spinor field description but in its different generalizations is the base for the construction of the models of higher and arbitrary spin elementary particles as well.

Among the results of this monograph the original method of the derivation of the standard Dirac and the Dirac-like equation for arbitrary spin is suggested. The Dirac equation is derived here from the 4-component formalism of relativistic canonical quantum mechanics (RCQM). In order to determine the place of our derivation among the other known methods we consider below the different ways of the Dirac equation derivation.

Thus, it is evident that the methods of the derivation of the Dirac equation cause interest of the researchers. Indeed, the new ways of the Dirac equation derivation automatically visualize the ground principles, which are in the foundations of the modern description of the elementary particles. Hence, the active consideration of the different ways of the Dirac equation derivation is the subject of many contemporary publications. Usually the start from the different basic principles and assumptions is considered.

An introduction to the Dirac equation consideration is based on different variants of its derivation. Below a brief review of the 26 different ways of the Dirac equation derivation is given.

Here the system of units $\hbar = c = 1$ is chosen, the metric tensor in Minkowski space-time $M(1,3)$ is given by $g^{\mu\nu} = g_{\mu\nu} = g_\nu^\mu, (g_\nu^\mu) = \text{diag}(1,-1,-1,-1)$, $x_\mu = g_{\mu\nu}x^\mu$, and summation over the twice repeated indices is implied. Note that in quantum-mechanical theories, where the time variable plays the role of a parameter, the rigged Hilbert space $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$

is used. Nevertheless, in the formalism of the manifestly covariant field theory the rigged Hilbert space is taken as $S^{4,4} \subset H^{4,4} \subset S^{4,4*}$.

2.2.   Derivation of the Dirac Equation from Start

2.2.1.   Dirac's Derivation

At first, one should note **the elegant derivation given by Paul Dirac** in his book [9] (of course, this consideration is based on origins of [1]). Until today it is very interesting for the readers to feel Dirac's way of thinking and to follow his logical steps. One can follow his start from the principles of linearity, manifestly Lorentz invariance, relationship to the Klein–Gordon equation and finish in determination of α and β matrices explicit forms.

Nevertheless, the Dirac's consideration of the Schrödinger–Foldy equation (our suggestion to call this equation as the Schrödinger–Foldy equation is considered after few steps below) $\left\{ p_0 - \left(m^2 c^2 + p_1^2 + p_2^2 + p_3^2 \right)^{\frac{1}{2}} \right\} \psi = 0$, which was essentially used in his derivation [9], was quite hasty. Especially his assertion that Schrödinger–Foldy equation is unsatisfactory from the point of view of the relativistic theory. Dirac's doubts were overcome later in [10–12] after the Foldy–Wouthuysen (FW) consideration, in which the corresponding canonical representation of the Dirac equation has been suggested and interpreted.

Today the significance of the Schrödinger–Foldy equation is confirmed by few hundreds publications about FW (see, e.g., [13–29] and the references therein) and the spinless Salpeter [30–47] equations (together with its generalizations), which have wide-range application in contemporary theoretical physics. Our contribution in consideration of the FW and Schrödinger–Foldy equations is presented in [48–55].

Suggestion to call the main equation of the contemporary N-component relativistic canonical quantum mechanics as the Schrödinger–Foldy equation was given in our publications. Our motivation was as follows. The start of this model has been given in [10]. In the papers [11, 12] the two-component version of equation $i \partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$ was considered and called as the Schrödinger equation. The one-component version of this equation was suggested in [30] and is called in the literature as the spinless Salpeter equation. In the papers considering the Lévy flights the spinless Salpeter equation is called as the Lévy-Schrödinger equation. Thus, the one-component case is identified well. But what about many-component case?

Taking into account the L. Foldy's contribution [10–12] in the construction of 2- and 4-component relativistic canonical quantum mechanics (RCQM) and his proof of the principle of correspondence between the relativistic and non-relativistic quantum mechanics, we propose *to call the N-component equations of this type as the Schrödinger-Foldy equations*.

2.2.2. Van der Waerden–Sakurai Derivation

In this version of the Dirac equation derivation the spin of the electron is incorporated into the nonrelativistic theory, see, e.g., [56]. Above considered Dirac's derivation was supported by the application of the relation $(\vec{\sigma} \cdot \widehat{\vec{p}})(\vec{\sigma} \cdot \widehat{\vec{p}}) = \widehat{\vec{p}}^2$ and the factorization procedure in the form $(\widehat{p}^0 + \vec{\sigma} \cdot \widehat{\vec{p}})(\widehat{p}^0 - \vec{\sigma} \cdot \widehat{\vec{p}}) = m^2$ is used. (Here $\widehat{\vec{p}} \equiv (\widehat{p}^j) = -i\nabla$ is the quantum-mechanical momentum operator, $\widehat{p}^0 \equiv i\partial_0$ is the operator of energy and $\vec{\sigma}$ are the Pauli matrices; the standard representation is given in (1.6).) Then the procedure of transition from 2-component to 4-component equation is fulfilled and explained.

2.2.3. Briefly on "Factorization" of the Klein–Gordon Operator

In the well-known *N.N. Bogoliubov and D.V. Shirkov book* [57] one can find a review of the Dirac theory and two different ways of the Dirac equation derivation. First, it is the presentation of the Klein–Gordon equation in the form of a first-order differential system of equations, i.e., so-called "factorization" of the Klein–Gordon operator in contemporary form. Following Dirac [1, 9] and [56] author's demonstrated that

$$\Box - m^2 = (i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m), \tag{2.1}$$

where the matrix operators γ^ν (1.5) obey the conditions $\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu = 2g^{\mu\nu}$ and $g^{\mu\nu}$ is the corresponding metric tensor. Assertion (2.1) is the main step of the given in [57] Dirac equation derivation.

2.2.4. Principle of Least Action for the Spinor Field

Application of the least action principle to the spinor field presents the next independent variant of the Dirac equation derivation. Again one can refer to [57], where the Lagrange approach is considered and the Dirac equation is derived from the **variational Euler–Lagrange least action principle**.

The Lagrange function, for which the Euler–Lagrange equations coincide with the free Dirac equation (1.1), is given by

$$\mathcal{L}_\Psi(x) = \frac{i}{2}\left(\overline{\psi}(x)\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - \frac{\partial\overline{\psi}}{\partial x^\mu}\gamma^\mu\psi(x)\right) - m\overline{\psi}(x)\psi(x), \tag{2.2}$$

where $\overline{\psi}(x)$ is the independent variational variable (in the set of solutions of the Dirac equation (1.1) $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$).

The Dirac equation in external electromagnetic field of potentials $A^\mu(x)$ follows from the principle of least action, in which the Lagrange function have the form $\mathcal{L} = \mathcal{L}_\Psi + \mathcal{L}_{\text{Int}}$, where the interaction Lagrangian is given by

$$\mathcal{L}_{\text{Int}}(x) = e\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x). \tag{2.3}$$

2.2.5. Ryder's Derivation

In the *L. Ryder book* [58] (second edition) the Dirac equation is derived **from the manifestly covariant transformational properties of the 4-component spinor**.

2.2.6. Start from the Initial Geometric Properties of the Space-Time and Electron

The derivation of the Dirac equation from the initial geometric properties of the space-time and electron together with wide-range discussion of the geometric principles of the electron theory is the main content of the *J. Keller book* [59]. The ideas of *V. Fock and D. Iwanenko* [60, 61] on the geometrical sense of Dirac γ-matrices are the basis of the approach.

2.2.7. Origin in Bargman–Wigner Classification

One should point out the derivation of the Dirac equation based on the Bargman–Wigner classification of the irreducible unitary representations of the Poincaré group, see, e.g., [62, 63]. It is an illustrative demonstration of the possibilities of the group-theoretical approach to the elementary particle physics. Indeed, within the group-theoretic approach to the elementary particles physics, the Dirac equation follows from the Bargman–Wigner classification [62, 63] of the elementary particles of arbitrary mass and spin on the basis of irreducible

unitary representations of the Poincaré group. The Dirac Hamiltonian corresponds to the irreducible representation of this group, which is characterized by the eigen values $m > 0$ and $s = 1/2$ of the corresponded Casimir operators.

2.2.8.   The Partial Case of the Bargman–Wigner Equation

Dirac equation is the partial case of the Bargman–Wigner equation [63] for the arbitrary spin fields. Indeed, after the substitution s=1/2 case into 2s·4-component Bargman–Wigner equation [63] one comes to 4-component Dirac equation. Nevertheless, such derivation is week. Note that Bargman–Wigner equation has been introduced as the 2s·4-component generalization of the Dirac equation.

2.2.9.   Inverse Foldy–Wouthuysen Transformation

In *L. Foldy and S. Wouthuysen* paper [10], and in *L. Foldy* articles [11, 12], one can easy find the inverse problem, in which the Dirac equation can be derived from the canonical FW equation.

Note that the FW transformation was introduced in the Dirac theory in order to transform the Dirac equation into the special canonical form, in which the quantum-mechanical interpretation is more evident in comparison with standard manifestly covariant form. Indeed, in the FW representation (contrary to the Dirac representation) the operators of position and velocity are defined well and the spin commutes with the Hamiltonian itself (not only together with orbital angular momentum). The orbital angular momentum commutes here with the Hamiltonian as well. The FW representation is considered here in the Chapters 4, 6.

Nevertheless, the operators of energy and spin in the FW transformation still can not be interpreted in the framework of quantum mechanics. The operator of energy leads to the solutions with positive and negative energies and spin projection for the particle and antiparticle is similar. The well-defined energy and spin operators are obtained after the extended FW transformation, which translates the Dirac model into the well-defined quantum-mechanical model without negative energies and with different projections of spin for the particle and antiparticle. Such extended transformation has been suggested in our articles [48–55] and the corresponding representation of the Dirac theory has been called as the RCQM. This model and its relation with the Dirac theory is considered below in the Chapter 5.

Not a matter of fact the FW representation is well-known, see, e.g., [13–29] and the references therein. Therefore, the derivation of the Dirac equation from the FW equation must be considered in the list of Dirac equation derivations. Nevertheless, it is only the transition from one representation of the spinor field to another, from the close to quantum mechanics nonlocal canonical representation to the local field manifestly covariant picture.

2.2.10. Feynman's Derivation

The intriguing derivation was suggested by Feynman (for the SO(1,1) case with one spatial dimension and one time dimension) and was given as a problem in his book with Hibbs [64]. This approach was continued in [65], where the path integral constitutes a generalization of Feynman's checkerboard model to 3+1 space-time dimensions.

2.3.   Unexpected Derivations

2.3.1.   Sallhofer's Derivation from the Maxwell Electrodynamics in Medium

The first one who should be mentioned is the austrian scientist H. Sallhofer. The following assertion has been proved [66, 67]. *Matrix multiplication of the Maxwell equations in medium (without currents and charges) on the Pauli vector* $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ *gives the massless stationary Dirac equation.* Not stopping at this, but taking his result as a basis, he put into consideration for the electric and magnetic permeabilities of the inner-atomic medium the representation as follows (here $\hbar \neq c \neq 1$)

$$\varepsilon(\vec{x}) = 1 - \frac{\Phi(\vec{x}) - mc^2}{\hbar \widetilde{\omega}}, \quad \mu(\vec{x}) = 1 - \frac{\Phi(\vec{x}) + mc^2}{\hbar \widetilde{\omega}}, \quad (2.4)$$

where $\Phi(\vec{x}) = -Ze^2/|\vec{x}|$ is the external Coulomb field, $\hbar$ is the Planck constant and c is the velocity of light. Further, the Sommerfeld–Dirac formula for the hydrogen spectrum fine structure has been derived [68, 69] from the Maxwell equations in medium (2.4) (for the first time after Dirac from a fundamentally different equation and even without appealing to quantum mechanics). Thus, papers [66–69] contain considerably more information than simply the derivation of the Dirac equation from the Maxwell equations.

2.3.2. In Addition to Ryder's Derivation

Some remarks about Ryder's derivation of the Dirac equation were considered in [70], where the formalism of [58] was repeated together with taking into account the physical meaning of the negative energies and the relative intrinsic parity of the elementary particles. The authors believe that the derivation from [58] is improved.

2.3.3. Derivation from the Langevin Equation Associated with a Two-Valued Process

The object of contribution [71] was to show that the relativistic equation for spin-1/2 particles can be obtained by an enlargement of the theory of stochastic processes one step beyond the theory of complex measures. The quaternion measurable processes were introduced and the Dirac equation was derived from the Langevin equation associated with a two-valued process. A direct derivation of the Dirac equation via quaternion measures has been suggested.

2.3.4. Derivation from the Conservation Law of Spin 1/2 Current

The author of [72] was able to derive the Dirac equation from the conservation law of spin 1/2 current. The requirement that this current is conserved leads to a unique determination of the Lorentz invariant equation satisfied by the relativistic spin 1/2 field. If so, then what is distinguished the conservation law of spin-current in comparison with other constants of motion? Therefore, it is logical to expect the successful derivations of the Dirac equation from other conservation laws of the spinor field.

Let us briefly comment that the complete list of fundamental conservation laws for the spinor field is the Noether consequence of the Dirac equation and its symmetries. The 10 main conserved quantities are the consequences of the Poincaré symmetry. Moreover, different additional and so-called hidden symmetries are known, see, e.g., [2–8]. The additional and hidden conservation laws are known for the spinor field as well [6–8], where not only the Fermi but also the Bose constants of motion are found. Note especially the list of additional conservation laws for the spinor field, which was found in [73].

Moreover, as soon as the conserved currents follow directly from some equation of motion, therefore, the validity of the inverse problem is really expected. And not only spin 1/2 conserved current can be useful.

2.3.5. Derivation from the Master Equation of Poisson Process

The Dirac equation was derived [74] from the master equation of Poisson process by analytic continuation. The extension to the case where a particle moves in an external field was given. It was shown that the generalized master equation is closely related to the three-dimensional Dirac equation in an external field.

2.3.6. Derivation from the Relativistic Newton's Second Law

In [75], a method of deriving the Dirac equation from the relativistic Newton's second law was suggested. However, for these purposes the author put into consideration the list of his own definitions, concepts and, even, special models. Indeed, this derivation is possible in a new formalism, which relates the special form of relativistic mechanics to the quantum mechanics. The author suggested a concept of a velocity field. At first, the relativistic Newton's second law was rewritten as a field equation in terms of the velocity field, which directly reveals a new relationship linked to the quantum mechanics. After that it was shown that the Dirac equation can be derived from the field equation in a rigorous and consistent manner. The impression is that the author seeks for coordination of his approach with the well-tested model of Dirac.

2.3.7. A Geometrical Derivation of the Dirac Equation

A geometrical derivation of the Dirac equation, by considering a spin 1/2 particle traveling with the speed of light in a cubic space-time lattice, was made in [76]. The mass of the particle acts to flip the multi-component wave function at the lattice sites. Starting with a difference equation for the case of one spatial and one time dimensions, the authors generalize the approach to higher dimensions. Interactions with external electromagnetic and gravitational fields are also considered. Nevertheless, the idea of such derivation is based on the Dirac's observation that the instantaneous velocity operators of the spin 1/2 particle (hereafter called by the generic name "the electron") have eigenvalues $\pm c$. Note that today this fact is considered as a difficulty in the Dirac equation quantum-mechanical interpretation and was demonstrated, explained and overcome in [10].

It will be useful to consider the link of this approach with the geometric consideration of V. Fock, D. Iwanenko, J. Keller [59–61] briefly presented here in subsection 2.2.6. Unfortunately, in [76] such link is absent. The appealing

to such assumption as a $\pm c$ velocity of the massive fermion is a shortcoming of the formalism of [76].

2.3.8.   Start from the Geodesic Equation

Using the mathematical tool of Hamilton's bi-quaternions, the authors of [77] propose a derivation of the Dirac equation from the geodesic equation. Such derivation is given in the program of application of the theory of scale relativity to the purposes of microphysics at recovering quantum mechanics as a new non-classical mechanics on a non-derivable space-time.

2.3.9.   Derivation from a Generally Covariant Field Equation for Gravitation and Electromagnetism

M. Evans was successful to express his equation of general relativity (generally covariant field equation for gravitation and electromagnetism [78]) in spinor form, thus producing the Dirac equation in general relativity [79]. The Dirac equation in special relativity is recovered in the limit of Euclidean or flat space-time. Thus, the Dirac equation was derived from a generally covariant field equation for gravitation and electromagnetism.

2.3.10.   Derivation from the Unique Conditions for Wave Function

Author of [80] first determines that each eigenfunction of a bound particle is a specific superposition of plane wave states that fulfills the averaged energy relation. After that the Schrodinger and Dirac equations were derived as the unique conditions the wave function must satisfy at each point in order to fulfill the corresponding energy equation. The Dirac equation involving electromagnetic potentials has been derived.

2.3.11.   Derivation by Factorizing the Algebraic Relation Satisfied by the Classical Hamiltonian

Author of this approach [81] started from classical-quantum correspondence, that associates a linear differential operator with a classical Hamiltonian, and that leads to regard energy and momentum as operators of this kind. The Dirac equation is derived by factorizing the algebraic relation satisfied by the classical Hamiltonian. The Whitham consideration [82] of the theory of classical waves

was used, where for any linear wave equation one may define different "'wave modes'", each of which is characterized by a *dispersion relation*, i.e., an explicit dependence of the frequency ω as a function of the spatial wave (co-)vector **k**, $\omega = W(\mathbf{k}; X)$ (in the general case of heterogeneous propagation, when the dispersion depends indeed on the space-time position X). It turns out [82] that for a given wave mode the wave vector **k** propagates along the bicharacteristics of a certain linear partial differential equation of the first order. When the latter equation is put into characteristic form, one obtains a *Hamiltonian system*, in which the Hamiltonian is none other than the dispersion relation W defining the given wave mode of the wave equation considered. Therefore, with certain precautions, which are made necessary by the existence of several wave modes, one may *recover the wave equation* from the dispersion relation alone. It is the basis of the derivation of the Dirac equation in [81] for the free particle, for the case of external electromagnetic field and for the case of static gravitational field. Note that the main point in such derivation is near standard factorization of the Klein–Gordon operator considered above in subsection 2.2.3.

2.3.12. Origin in Conformal Differential Geometry

In [83] the Dirac equation is derived by conformal differential geometry. The Hamilton–Jacobi equation for the particle is found to be linearized, exactly and in closed form, by an ansatz solution that can be straightforwardly interpreted as the quantum wave function of the 4-spinor solution of Dirac's equation. All quantum features arise from the subtle interplay between the conformal curvature acting on the particle as a potential and the particle motion which affects the geometric prepotential associated to the conformal curvature itself. Finally, the Dirac equation is found in the form $\widehat{D}_+\widehat{D}_-\psi = \widehat{D}_-\widehat{D}_+\psi = 0$, $\widehat{D}_\mp \equiv \gamma^\mu(p_\mu - eA_\mu) \mp m$.

2.3.13. Recent Derivation from Principles of Information Processing

Recently, in [84] the derivation of the Dirac equation from principles of information processing has been presented. It has been shown, without using the relativity principle, how the Dirac equation in three space-dimensions emerges from the large-scale dynamics of the minimal nontrivial quantum cellular automaton satisfying unitarity, locality, homogeneity, and discrete isotropy. The

Dirac equation is recovered for small wave-vector and inertial mass, whereas Lorentz covariance is distorted in the ultra-relativistic limit. The automaton can thus be regarded as a theory unifying scales from Planck to Fermi.

2.4. Our Approaches to the Problem of the Dirac Equation Derivation

We have contributed in the methods of Dirac equation derivation thrice: (i) the Sallhofer's derivation from the Maxwell electrodynamics in medium was generalized to the case of non-stationary equation and the complete set of corresponding transformations was found, (ii) the massless Dirac equation was derived from the maximally symmetrical form of the Maxwell equations, (iii) the Dirac equation with nonzero mass was derived from the RCQM of spin 1/2 particle-antiparticle doublet. This determines our interest to the problem.

2.4.1. Generalization of Sallhofer's Derivation

As it is already presented in subsection 2.3.1 above in the papers [67–69] the link between the Dirac equation in external electromagnetic field and the Maxwell equations in medium has been given. In the paper [85] Sallhofer's idea was generalized to the case of non-stationary Dirac equation and is presented in the framework of another formalism. Moreover, in [85] the complete set of linking transformation was found. The main assertion of [85] is as follows.

Maxwell's equations of source-free electrodynamics

$$\frac{\varepsilon}{c}\partial_0 \overrightarrow{E} - \operatorname{curl}\overrightarrow{H} = 0, \frac{\mu}{c}\partial_0 \overrightarrow{H} + \operatorname{curl}\overrightarrow{E} = 0, \quad \operatorname{div}\overrightarrow{E} = 0, \operatorname{div}\overrightarrow{H} = 0, \quad (2.5)$$

(c is the light velocity, ε and μ are the electric and magnetic permeabilities of the medium) are linked to the Dirac-like equation

$$\left[\overrightarrow{\alpha}\cdot\nabla - \left|\begin{matrix} \varepsilon I_2 & 0 \\ 0 & \mu I_2 \end{matrix}\right| \frac{1}{c}\frac{\partial}{\partial t}\right]\psi^{\mathrm{el}} = 0, \quad (2.6)$$

where $\overrightarrow{\alpha} = \left|\begin{matrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{matrix}\right|$, $\overrightarrow{\sigma} = \{\sigma^1,\sigma^2,\sigma^3\}$ are the Pauli matrices (1.6), ψ^{el} is

one of eight columns known from [85]

$$\psi^{\text{el}} = \begin{vmatrix} iE^3 \\ i(E^1 + iE^2) \\ H^3 \\ H^1 + iH^2 \end{vmatrix}, \tag{2.7}$$

I_2 is 2×2 unit matrix (In (2.5), (2.7) and below $(\overrightarrow{E}, \overrightarrow{H})$ are the electromagnetic field strengths).

By applying the matrix-differential operator of (2.6) to the column (2.7) we obtain a system of four equations in which the imaginary and real parts can easily be separated. Then, by natural requiring that the imaginary and real parts be equal to zero independently, we come to Maxwell's equations (2.5).

It is evident that column ψ^{el} (2.7) contains only six real functions $(\overrightarrow{E}, \overrightarrow{H})$, while four-component Dirac spinor ψ^D contains eight similar real functions, which in general are not related to $(\overrightarrow{E}, \overrightarrow{H})$. Therefore, the components of the column ψ^{el} (2.7) are the subset of the components of Dirac spinor ψ^D. Hence, there is no one to one correspondence between components of ψ^{el} (2.7) and ψ^D.

Therefore, the inverse assertion that the massless Dirac equation follows from the Maxwell equations (2.5) is not so evident. In order to fulfill such inverse derivation it is necessary to put $\varepsilon = \mu = 1$ and to complete the 6 real components of (2.7) to the 8 real components of the standard Dirac spinor.

The complete set of columns, which can be chosen as ψ^{el} in (2.7), is eight. In the notations

$$E^{\pm} = E^1 \pm iE^2, \quad H^{\pm} = H^1 \pm iH^2, \tag{2.8}$$

they are given by:

$$\psi^1 = \begin{vmatrix} iE^3 \\ iE^+ \\ H^3 \\ H^+ \end{vmatrix}, \psi^2 = \begin{vmatrix} -E^3 \\ -E^+ \\ iH^3 \\ iH^+ \end{vmatrix}, \psi^3 = \begin{vmatrix} H^3 \\ H^+ \\ iE^3 \\ iE^+ \end{vmatrix}, \psi^4 = \begin{vmatrix} iH^3 \\ iH^+ \\ -E^3 \\ -E^+ \end{vmatrix}, \tag{2.9}$$

$$\psi^5 = \begin{vmatrix} -iH^- \\ iH^3 \\ E^- \\ -E^3 \end{vmatrix}, \psi^6 = \begin{vmatrix} H^- \\ -H^3 \\ iE^- \\ -iE^3 \end{vmatrix}, \psi^7 = \begin{vmatrix} E^- \\ -E^3 \\ -iH^- \\ iH^3 \end{vmatrix}, \psi^8 = \begin{vmatrix} iE^- \\ -iE^3 \\ H^- \\ -H^3 \end{vmatrix}.$$

(The set (2.9) includes column (2.7) as ψ^1).

The validity and suitability of the set (2.9) can *be proven* by direct substitution of every column of this set for ψ^{el} in (2.6). In order to prove that it is really a complete set let us recall the eight Pauli–Gürsey–Ibragimov operators [86–88] $\{\gamma^2\hat{C}, i\gamma^2\hat{C}, \gamma^2\gamma^4\hat{C}, i\gamma^2\gamma^4\hat{C}, \gamma^4, i\gamma^4, i, I\}$ ($\hat{C}$ is the 4×4 matrix operator of complex conjugation, $\hat{C}\psi = \psi^*$, the operator of involution in $H^{3,4}$). These generators form a complete set of pure matrix operators, which leave the massless Dirac equation being invariant. The verification that eight columns from (2.9) are obtained by action of every Pauli–Gürsey–Ibragimov operator on the function ψ^{el} (2.7) finishes the proof. As soon as we have in $\{\gamma^2\hat{C}, i\gamma^2\hat{C}, \gamma^2\gamma^4\hat{C}, i\gamma^2\gamma^4\hat{C}, \gamma^4, i\gamma^4, i, I\}$ the complete set of operators, therefore, the (2.9) is the complete set too.

Note that the four columns ψ^{3-6} can be chosen in (2.9) only together with simultaneous interchange $\varepsilon \leftrightarrow \mu$ in (2.6).

Note, further, that in [89–91] different useful matrix representations of the Lie algebras of the Lorentz and Poincaré groups have been found on the basis of Pauli–Gürsey–Ibragimov operators [86–88]. At first, it is the additional $D(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lie algebra of *universal covering* $\mathcal{L} =$ SL(2,C) of the proper ortochronous Lorentz group $L_+^\uparrow = SO(1,3) = \{\Lambda = (\Lambda_v^\mu)\}$. Further, the application of the simplest linear combinations of both standard and additional $D(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ generators gives the possibility to find Bose representations of the Lorentz group $\mathcal{L}$ and the Poincaré group $\mathcal{P} \supset \mathcal{L} =$ SL(2,C) (here $\mathcal{P}$ is the universal covering of the proper ortochronous Poincaré group $P_+^\uparrow = T(4)\times)L_+^\uparrow \supset L_+^\uparrow$), with respect to which the massless Dirac equation is invariant. Thus, we have found [89–91] the Bose $D(1,0)\oplus(0,0)$ and $D(\frac{1}{2}, \frac{1}{2})$ representations of the Lie algebra of the Lorentz group $\mathcal{L}$ together with the tensor-scalar of the spin s=(1,0) and vector representations of the Lie algebra of the Poincaré group $\mathcal{P}$, with respect to which the Dirac equation with m=0 is invariant.

Assuming harmonic time dependence in the form

$$\Psi = \psi e^{-i\widetilde{\omega}t} \rightarrow \frac{\partial}{\partial(ct)}\Psi = -i\frac{\widetilde{\omega}}{c}\Psi, \qquad (2.10)$$

from (2.6) the electromagnetic amplitude equation

$$\left[\overrightarrow{\alpha} \cdot \nabla + i\frac{\widetilde{\omega}}{c} \left| \begin{array}{cc} \varepsilon I_2 & 0 \\ 0 & \mu I_2 \end{array} \right| \right] \psi^{el} = 0 \qquad (2.11)$$

follows. A comparison with the Dirac amplitude equation

$$
\left[\vec{\alpha} \cdot \nabla + i\frac{\widetilde{\omega}}{c} \left|
\begin{array}{cc}
\left(1 - \frac{\Phi(\vec{x}) - mc^2}{\hbar\omega}\right) I_2 & 0 \\
0 & \left(1 - \frac{\Phi(\vec{x}) + mc^2}{\hbar\omega}\right) I_2
\end{array}
\right| \right] \psi^D = 0 \quad (2.12)
$$

in external Coulomb field $\Phi(\vec{x}) = -Ze^2/r$ immediately shows the complete formal agreement of electrodynamics with the Dirac theory. A consequence of such a comparison are the *Sallhofer's* formulas (2.4), on the basis of which the description of hydrogen atom relativistic spectrum was presented in [68, 69] by means of (2.4) and (2.5), i.e., in terms of Maxwell's electrodynamics.

Similarly to the consideration above of column ψ^{el} (2.7) and four-component Dirac spinor ψ^D, here again there is no one to one correspondence between 6 real components of ψ^{el} (2.11) and 8 real components of ψ^D (2.12).

Not a meter of fact that above some formal agreement of electrodynamics with the Dirac theory is presented the fine structure of hydrogen atom spectrum on this basis has been found both with application of (2.5) [68, 69] and with application of 8-component form of Maxwell equations [90, 91] (especially in [92]). Such Maxwell system is considered in subsection 2.4.2 below. The components of these slightly generalized Maxwell equations possess one to one correspondence with the Dirac theory.

It is better to consider the above given presentation of [85] together with our paper [93]. In [93] different relations between the free Maxwell equations (2.5), when $\varepsilon = \mu = 1$, and the massless Dirac equations were investigated: The relationships between the Lagrangians, symmetries, solutions of free Maxwell equations and Lagrangians, symmetries, solutions of the massless Dirac equations were found. Furthermore, the relationships between the conservation laws for the electromagnetic and massless spinor field were found in [93] as well.

Note that recently the authors of [94] also marked the difference between the approaches in [66, 67] and [85, 93]. Moreover, author of [66, 67] named his model "The Maxwell-Dirac isomorphism". Note further that "isomorphism" in [66–69], as it is evident from consideration above, can be considered rather as the physical, not as the mathematical assertion. In the papers [95–98] we named it (together with the case of free Maxwell equations in vacuum in [93]) as the "relationship" between the Dirac and the Maxwell equations. J. Keller in his review [99] named it the "mapping" of the Maxwell formalism into the Dirac formalism.

Thus, it is better to speak carefully that derivation of the Dirac equation in

[66–69, 85] and here above is only partially based on mathematics and includes so called physical intuition. Nevertheless, let us note that many important steps in physics were made only on the basis of physical intuition without any appealing to mathematical derivations.

Finally, the results considered above are very interesting and can be useful for further investigations.

2.4.2. Origin in Maximally Symmetrical Form of the Maxwell Equations

More then ten years ago we already presented our own independent derivation of the Dirac equation [90, 91, 97, 98]. The Dirac equation was derived from slightly generalized Maxwell equations with gradient-like current and charge densities. Such Maxwell system includes magnetic gradient-like current and charge densities (in another interpretation system contains additional scalar field). This form of the Maxwell equations, which is directly linked to the Dirac equation, is the maximally symmetrical variant of Maxwell system. Such Maxwell equations are invariant with respect to a 256-dimensional algebra (the well-known algebra of conformal group has only 15 generators). Of course, we derived only massless Dirac equation.

Note that J.C. Maxwell derived a system of equations for describing electromagnetic phenomena on the basis of a generalized rewriting of all known electrodynamics laws of Faraday, Ampere, Weber, etc., as well as from the principle of symmetry. Looking for the equations for inneratomic problems in the framework of classical electrodynamics we recall the Maxwell's idea about symmetry principle. By analogy with Maxwell's suggestions in the articles [90, 91, 97, 98], perhaps the most symmetrical form of the Maxwell equations with 256 dimensional invariance algebra is introduced. It is precisely this system of Maxwell equations that is directly related to the massless Dirac equation and can describe the spectrum of hydrogen. The massless Dirac equation follows from such non-ordinary Maxwell equations.

Consider this way of the Dirac equation derivation in some details.

The above mentioned equations for the system of electromagnetic and scalar fields $(\vec{E}, \vec{H}, E^0, H^0)$ have the form:

$$\partial_0 \vec{E} = \text{curl}\,\vec{H} - \text{grad}E^0, \quad \partial_0 \vec{H} = -\text{curl}\,\vec{E} - \text{grad}H^0,$$
$$\text{div}\,\vec{E} = -\partial_0 E^0, \quad \text{div}\,\vec{H} = -\partial_0 H^0. \tag{2.13}$$

Here and below the system of units $\hbar = c = 1$ is used. The equations (2.13) are nothing more than the weakly generalized Maxwell equations ($\varepsilon = \mu = 1$) with gradient-like electric and magnetic sources $j_\mu^e = -\partial_\mu E^0$, $j_\mu^{\text{mag}} = -\partial_\mu H^0$, i.e.,

$$\vec{j}_e = -\text{grad}E^0, \quad \vec{j}_{\text{mag}} = -\text{grad}H^0, \quad \rho_e = -\partial_0 E^0, \quad \rho_{\text{mag}} = -\partial_0 H^0. \quad (2.14)$$

In terms of complex 4-component object

$$\mathcal{E} \equiv \begin{vmatrix} \vec{\mathcal{E}} \\ \mathcal{E}^0 \end{vmatrix} = \text{column}\left|E^1 - iH^1, E^2 - iH^2, E^3 - iH^3, E^0 - iH^0\right|, \quad (2.15)$$

and, further, in terms of following complex tensor

$$\mathcal{E} = (\mathcal{E}^{\mu\nu}) \equiv \begin{vmatrix} 0 & \mathcal{E}^1 & \mathcal{E}^2 & \mathcal{E}^3 \\ -\mathcal{E}^1 & 0 & i\mathcal{E}^3 & -i\mathcal{E}^2 \\ -\mathcal{E}^2 & -i\mathcal{E}^3 & 0 & i\mathcal{E}^1 \\ -\mathcal{E}^3 & i\mathcal{E}^2 & -i\mathcal{E}^1 & 0 \end{vmatrix} \quad (2.16)$$

equations (2.13) can be rewritten in the manifestly covariant forms

$$\partial_\mu \mathcal{E}_\nu - \partial_\nu \mathcal{E}_\mu + i\varepsilon_{\mu\nu\rho\sigma}\partial^\rho \mathcal{E}^\sigma = 0, \quad \partial_\mu \mathcal{E}^\mu = 0, \quad (2.17)$$

(vector form) and tensor-scalar form:

$$\partial_\nu \mathcal{E}^{\mu\nu} = \partial^\mu \mathcal{E}^0. \quad (2.18)$$

It is useful also to consider the following form of equations (2.13), (2.17), (2.18):

$$\left(i\partial_0 - \vec{S} \cdot \vec{p}\right)\vec{\mathcal{E}} - i\text{grad}\mathcal{E}^0 = 0, \quad \partial_\mu \mathcal{E}^\mu = 0, \quad (2.19)$$

where $\vec{S} \equiv (S^j)$ are the generators of irreducible representation $D(1)$ of the group $SU(2)$:

$$S^1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{vmatrix}, S^2 = \begin{vmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{vmatrix}, S^3 = \begin{vmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}; \vec{S}^2 = 1(1+1)I. \quad (2.20)$$

The general solution of equations (2.13), (2.17)–(2.19) was found in [95], their symmetry properties were considered in [89], the application (after formulation in specific medium (2.4)) to the hydrogen spectrum description was given

in [92, 96]. The solution was found in the space $(S(R^4) \otimes C^4)^*$ of Schwartz's generalized functions directly by application of Fourier method. In terms of helicity amplitudes $c^{\bar{\alpha}}(\vec{k})$ this solution has the form

$$\mathcal{E}(x) = \int d^3k \sqrt{\frac{2\tilde{\omega}}{(2\pi)^3}} \left\{ \begin{array}{l} \left[c^1 e_1 + c^3 (e_3 + e_4)\right] e^{-ikx} + \\ \left[c^{*2} e_1 + c^{*4} (e_3 + e_4)\right] e^{ikx} \end{array} \right\}, \quad \tilde{\omega} \equiv \sqrt{\vec{k}^2}, \tag{2.21}$$

where $kx \equiv \tilde{\omega}t - \vec{k}\vec{x}$, $\bar{\alpha} = 1, 2, 3, 4$, and 4-component basis vectors e_α are taken in the form

$$e_1 = \left| \begin{array}{c} \vec{e_1} \\ 0 \end{array} \right|, \quad e_2 = \left| \begin{array}{c} \vec{e_2} \\ 0 \end{array} \right|, \quad e_3 = \left| \begin{array}{c} \vec{e_3} \\ 0 \end{array} \right|, \quad e_4 = \left| \begin{array}{c} 0 \\ 1 \end{array} \right|. \tag{2.22}$$

Here the 3-component basis vectors which, without any loss of generality, can be taken as

$$\vec{e_1} = \frac{1}{\tilde{\omega}\sqrt{2(k^1 k^1 + k^2 k^2)}} \left| \begin{array}{c} \tilde{\omega}k^2 - ik^1 k^3 \\ -\tilde{\omega}k^1 - ik^2 k^3 \\ i(k^1 k^1 + k^2 k^2) \end{array} \right|, \quad \vec{e_2} = \vec{e_1}^*, \quad \vec{e_3} = \frac{\vec{k}}{\tilde{\omega}}, \tag{2.23}$$

are the eigenvectors for the quantum-mechanical helicity operator for the spin $s = 1$.

Note that if the quantities E^0, H^0 in equations (2.13) are some given functions, for which the representation

$$E^0 - iH^0 = \int d^3k \sqrt{\frac{2\tilde{\omega}}{(2\pi)^3}} \left(c^3 e^{-ikx} + c^4 e^{ikx} \right) \tag{2.24}$$

is valid, then (2.13) are the Maxwell equations with the given sources, $j_\mu^e = -\partial_\mu E^0$, $j_\mu^{mag} = -\partial_\mu H^0$ (namely these 4 currents we call the gradient-like sources). In this case the general solution of the Maxwell equations (2.13), (2.17)–(2.19) with the given sources, as follows from (2.21), has the form

$$\vec{E}(x) = \int d^3k \sqrt{\frac{\tilde{\omega}}{2(2\pi)^3}} \left(c^1 \vec{e}_1 + c^2 \vec{e}_2 + \alpha \vec{e}_3 \right) e^{-ikx} + c.c,$$
$$\vec{H}(x) = i \int d^3k \sqrt{\frac{\tilde{\omega}}{2(2\pi)^3}} \left(c^1 \vec{e}_1 - c^2 \vec{e}_2 + \beta \vec{e}_3 \right) e^{-ikx} + c.c, \tag{2.25}$$

where the amplitudes of longitudinal waves $\vec{e}_3 \exp(-ikx)$ are $\alpha = c^3 + c^4$, $\beta = c^3 - c^4$ and c^3, c^4 are determined by the functions E^0, H^0 according to the formula (2.24).

The following assertion is valid.

Equations (2.13), (2.17)–(2.19) are directly related to the free massless Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) = 0. \tag{2.26}$$

The proof is as follows.

Note at first that there is no reason to appeal here to the stationary case as it was done in [66–69], where the case with nonzero interaction and mass $m \neq 0$ was considered. Indeed. the substitution of

$$\psi = \begin{vmatrix} E^3 + iH^0 \\ E^1 + iE^2 \\ iH^3 + E^0 \\ -H^2 + iH^1 \end{vmatrix} = U\mathcal{E}, \quad U = \begin{vmatrix} 0 & 0 & C_+ & C_- \\ C_+ & iC_+ & 0 & 0 \\ 0 & 0 & C_- & C_+ \\ C_- & iC_- & 0 & 0 \end{vmatrix}; C_\mp \equiv \frac{1}{2}(C\mp 1), \tag{2.27}$$

(C is the operator of complex conjugation, $C\mathcal{E}^1 = \mathcal{E}^{1*}$; complex field strength $\mathcal{E}$ is known from (2.15)) into Dirac equation (2.26) with γ matrices in standard Pauli–Dirac representation (1.5) guarantees its transformation into the generalized Maxwell equations (2.13), (2.17)–(2.19).

Moreover, the substitution of column (2.15)

$$\mathcal{E} = \begin{vmatrix} E^1 - iH^1 \\ E^2 - iH^2 \\ E^3 - iH^3 \\ E^0 - iH^0 \end{vmatrix} = U^{-1}\psi, \quad U^{-1} = U^\dagger = \begin{vmatrix} 0 & C_+ & 0 & C_- \\ 0 & iC_- & 0 & iC_+ \\ C_+ & 0 & C_- & 0 \\ C_- & 0 & C_+ & 0 \end{vmatrix}, \tag{2.28}$$

into the equation

$$\widetilde{\gamma}^\mu \partial_\mu \mathcal{E}(x) = 0 \tag{2.29}$$

with $\widetilde{\gamma}^\mu = U^{-1}\gamma^\mu U$ matrices

$$\widetilde{\gamma}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} C, \quad \widetilde{\gamma}^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} C, \tag{2.30}$$

$$\widetilde{\gamma}^2 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} C, \quad \widetilde{\gamma}^3 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} C,$$

guarantees its transformation into the generalized Maxwell equations (2.13), (2.17)–(2.19) as well.

Proved assertions demonstrate both the derivation of the slightly generalized Maxwell equations from the massless Dirac equation and the derivation of the massless Dirac equation from such kind of Maxwell system.

Note specially that ordinary Maxwell equation follow from (2.13), (2.17)–(2.19), (2.29) in the case $E^0 = H^0 = 0$.

The transitions (2.27), (2.28) are based on unitary transformation. The unitary properties of the operator U can be verified easily by taking into account that the equations

$$(AC)^\dagger = CA^\dagger, \quad aC = Ca^*, \quad (aC)^* = Ca \tag{2.31}$$

hold for an arbitrary matrix A and a complex number a. We note that in the real algebra (i.e., the algebra over the field of real numbers) and in the Hilbert space of quantum mechanical amplitudes the operator U in (2.27), (2.28) has all properties of unitarity: $UU^{-1} = U^{-1}U = 1$, $U^{-1} = U^\dagger$, plus linearity.

Due to the unitarity of the operator U the $\widetilde{\gamma}^\mu$ matrices (2.30) still obey the anticommutation relations of Clifford–Dirac algebra

$$\widetilde{\gamma}^\mu\widetilde{\gamma}^\nu + \widetilde{\gamma}^\nu\widetilde{\gamma}^\mu = 2g^{\mu\nu} \tag{2.32}$$

and have the same Hermitian properties as the Pauli - Dirac γ^μ matrices (1.5):

$$\widetilde{\gamma}^{0\dagger} = \widetilde{\gamma}^0, \quad \widetilde{\gamma}^{k\dagger} = -\widetilde{\gamma}^k. \tag{2.33}$$

The complete set of 8 transformations like (2.27), which relates generalized Maxwell equations (2.13) and massless Dirac equation (2.26), was found in [89, 93]. The explicit form is as follows

$$\psi^{\mathrm{I}} = \begin{vmatrix} E^3 + iH^0 \\ E^1 + iE^2 \\ iH^3 + E^0 \\ -H^2 + iH^1 \end{vmatrix}, \psi^{\mathrm{II}} = \begin{vmatrix} iE^3 - H^0 \\ iE^1 - E^2 \\ -H^3 + iE^0 \\ -H^1 - iH^2 \end{vmatrix}, \psi^{\mathrm{III}} = \begin{vmatrix} iE^1 + E^2 \\ -iE^3 - H^0 \\ -H^1 + iH^2 \\ H^3 + iE^0 \end{vmatrix},$$

$$\psi^{\mathrm{IV}} = \begin{vmatrix} -E^1 + iE^2 \\ E^3 - iH^0 \\ -iH^1 - H^2 \\ iH^3 - E^0 \end{vmatrix}, \psi^{\mathrm{V}} = \begin{vmatrix} -H^3 + iE^0 \\ -H^1 - iH^2 \\ iE^3 - H^0 \\ iE^1 - E^2 \end{vmatrix}, \psi^{\mathrm{VI}} = \begin{vmatrix} -H^1 + iH^2 \\ H^3 + iE^0 \\ iE^1 + E^2 \\ -iE^3 - H^0 \end{vmatrix}, \tag{2.34}$$

$$\psi^{VII} = \begin{vmatrix} iH^3 + E^0 \\ -H^2 + iH^1 \\ E^3 + iH^0 \\ E^1 + iE^2 \end{vmatrix}, \psi^{VIII} = \begin{vmatrix} -iH^1 - H^2 \\ iH^3 - E^0 \\ -E^1 + iE^2 \\ E^3 - iH^0 \end{vmatrix}.$$

Relationship between the generalized Maxwell equations (2.13), (2.17)–(2.19) and massless Dirac equation (2.26) is considered here in the way similar to the [93] and [85, 89]. The *proof* that the number of such transformation is eight is similar to the proof given above for the set (2.9). This number is eight, because the number of pure matrix Pauli–Gürsey–Ibragimov operators [86–88] $\{\gamma^2\hat{C}, i\gamma^2\hat{C}, \gamma^2\gamma^4\hat{C}, i\gamma^2\gamma^4\hat{C}, \gamma^4, i\gamma^4, i, I\}$, which leave the massless Dirac equation being invariant, is eight.

Note that equations (2.13), (2.17)–(2.19), (2.29) are the maximally symmetrical form of the Maxwell system. Indeed, here both Dirac-like and Maxwell-like symmetries are natural and are ready for application. The Bose symmetries of the massless Dirac equation and Fermi symmetries of the the slightly generalized Maxwell equations together with other corresponded results were considered in [89–91].

Further, contrary to the formalism of the item 2.4.1 above, here there is one-to-one correspondence between the solutions of the massless Dirac equation (2.26) and the slightly generalized Maxwell equations (2.13), (2.17)–(2.19). Therefore, the presented here derivation of the massless Dirac equation is direct and simple.

2.4.3. Derivation of the Dirac Equation from the Relativistic Canonical Quantum Mechanics

In our recent papers [48–55, 100], the Dirac equation with nonzero mass has been derived from the quantum-mechanical relativistic equation for the fermion-antifermion particle doublet of spins $s = 1/2$. We proposed to call such 4-component relativistic equation, which is the starting point of this derivation, as the Schrödinger–Foldy equation (the one-component case is often called as the spinless Salpeter equation [30–46] or as the equation for Lévy flight [47]). Corresponded model of the physical reality, which is based on such equation of motion, was called by us as RCQM of particle-antiparticle doublet. The arbitrary spins and arbitrary dimensions have been considered.

For the partial case of spin 1/2 fermion-antifermion doublet the 4-

component Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \end{vmatrix}, \tag{2.35}$$

and is considered in the rigged Hilbert space $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$, where $S^{3,4}$ is the 4-component Schwartz test function space over the space $R^3 \subset M(1,3)$ and $H^{3,4}$ is the Hilbert space of the 4-component square-integrable functions over the $x \in R^3 \subset M(1,3)$

$$H^{3,4} = L_2(R^3) \otimes C^{\otimes 4} = \{ f = (f^4) : R^3 \to C^{\otimes 4}; \int d^3 x |f(t, \overrightarrow{x})|^2 < \infty \}, \tag{2.36}$$

where $d^3 x$ is the Lebesgue measure in the space $R^3 \subset M(1,3)$ of the eigenvalues of the position operator $\overrightarrow{x}$ of the Cartesian coordinate of the particle in an arbitrary-fixed inertial frame of references, $M(1,3)$ is the Minkowski space. Further, $S^{3,4*}$ is the space of the 4-component Schwartz generalized functions. The space $S^{3,4*}$ is conjugated to that of the Schwartz test functions $S^{3,4}$ by the corresponding topology (see, e.g., [101]).

The following assertion is valid.

The link between the quantum-mechanical equation (2.35) and the Dirac equation

$$i\partial_0 \psi(x) = (\overrightarrow{\alpha} \cdot \overrightarrow{p} + \beta m)\psi(x), \quad \overrightarrow{\alpha} \equiv \gamma^0 \overrightarrow{\gamma}, \beta \equiv \gamma^0, \tag{2.37}$$

is given by the operator

$$V = \frac{i\gamma^\ell \partial_\ell + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix}, \quad V^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix} \frac{-i\gamma^\ell \partial_\ell + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}, \tag{2.38}$$

$$VV^{-1} = V^{-1}V = I, \quad \widehat{\omega} = \sqrt{-\Delta + m^2},$$

where C is the operator of complex conjugation, $C\psi^1 = \psi^{1*}$, (the operator of involution in $H^{3,1}$). The following relationships are valid

$$V(\partial_0 + i\widehat{\omega})V^{-1} = \partial_0 + i(\overrightarrow{\alpha} \cdot \overrightarrow{p} + \beta m), \tag{2.39}$$

for the anti-Hermitian operators of corresponding equations, and

$$\psi^{\text{Dirac}}(x) = V f^{\text{Sr}-\text{Foldy}}(x), \tag{2.40}$$

for the corresponding general solutions. The vice-verse relations are valid as well, but for the derivation of the Dirac equations directly the formulas (2.39), (2.40) are used.

The proof of this assertion is fulfilled by direct calculations of (2.39) and (2.40). In order to verify (2.40) the explicit forms of corresponding general solutions are used. These solutions are given below.

Note that due to the presence of the operator C our extended FW operator (2.38) is well-defined only in application to anti-Hermitian operators. For the possibility and mathematical correctness in using the anti-Hermitian operators in quantum theory see, e.g., [102, 103].

In (2.40) the general solution $f^{\text{Sr}-\text{Foldy}}(x)$ of the Schrödinger–Foldy equation of motion (2.35) has the form

$$f^{\text{Sr}-\text{Foldy}}(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3 k e^{-ikx} \tag{2.41}$$

$$[a^1(\overrightarrow{k})\mathrm{d}_1 + a^2(\overrightarrow{k})\mathrm{d}_2 + a^3(\overrightarrow{k})\mathrm{d}_3 + a^4(\overrightarrow{k})\mathrm{d}_4],$$

where the orts $\{\mathrm{d}_\alpha\}$ of the Cartesian basis are given by

$$\mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, \mathrm{d}_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}, \tag{2.42}$$

and the following notations

$$kx \equiv \widetilde{\omega}t - \overrightarrow{k}\,\overrightarrow{x}, \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \tag{2.43}$$

are used.

The interpretation of the amplitudes in the general solution (2.41) follows from equations

$$\widehat{\overrightarrow{p}}e^{-ikx}\mathrm{d}_{\tilde{\alpha}} = \overrightarrow{k}e^{-ikx}\mathrm{d}_{\tilde{\alpha}}, \quad \tilde{\alpha} = 1, 2, 3, 4, \tag{2.44}$$

$$s^3 d_1 = \frac{1}{2} d_1, \, s^3 d_2 = -\frac{1}{2} d_2, \, s^3 d_3 = -\frac{1}{2} d_3, \, s^3 d_4 = \frac{1}{2} d_4, \qquad (2.45)$$

$$g d_1 = -e d_1, \, g d_2 = -e d_2, \, g d_3 = e d_3, \, g d_4 = e d_4, \qquad (2.46)$$

on eigen vectors and eigenvalues of the operators of stationary complete set $(\widehat{\overrightarrow{p}}, s_z \equiv s^3, g)$. Thus, the functions $a^1(\overrightarrow{k}), a^2(\overrightarrow{k})$ are the momentum-spin amplitudes of the particle (e.g., electron) with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(-e)$ and spin projections $(\frac{1}{2}, -\frac{1}{2})$, respectively. Further, the functions $a^3(\overrightarrow{k}), a^4(\overrightarrow{k})$ in (2.41) are the momentum-spin amplitudes of the antiparticle (e.g., positron) with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(+e)$ and spin projections $(-\frac{1}{2}, \frac{1}{2})$, respectively.

Taking into account the Pauli principle and the fact that experimentally positron is observed as the mirror reflection of an electron, the operators of the charge sign and the spin of the s=(1/2,1/2) particle-antiparticle doublet are taken in the form

$$g \equiv -\gamma^0 = \begin{vmatrix} -I_2 & 0 \\ 0 & I_2 \end{vmatrix}, \qquad \overrightarrow{s} = \frac{1}{2} \begin{vmatrix} \overrightarrow{\sigma} & 0 \\ 0 & -C\overrightarrow{\sigma}C \end{vmatrix}, \qquad (2.47)$$

where $\overrightarrow{\sigma}$ are the standard Pauli matrices (1.6), C is the operator of complex conjugation, I_2 is 2×2 unit matrix.

In the choice of the spin (2.47) the principle of correspondence and heredity with the FW representation is used, where the particle-antiparticle doublet spin operator is given by

$$\overrightarrow{s}_{FW} = \frac{1}{2} \begin{vmatrix} \overrightarrow{\sigma} & 0 \\ 0 & \overrightarrow{\sigma} \end{vmatrix}. \qquad (2.48)$$

The link between the spins (2.47) and (2.48) $v\overrightarrow{s}_{FW}v = \overrightarrow{s}$, $v\overrightarrow{s}v = \overrightarrow{s}_{FW}$ is given by the transformation operator v

$$v = v^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix}, \qquad vv^{-1} = v^{-1}v = I, \qquad (2.49)$$

which extends the FW transformation in (2.38). Note that transformation v is valid only for the case of anti-Hermitian form of spins (2.47), (2.48).

In (2.40) the general solution $\psi^{\text{Dirac}}(x)$ of the Dirac equation (2.37) is given by

$$\psi(x) = V^+\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^{\mathrm{r}}(\vec{k}) v_{\mathrm{r}}^-(\vec{k}) + e^{ikx} a^{*\check{\mathrm{r}}}(\vec{k}) v_{\check{\mathrm{r}}}^+(\vec{k}) \right],$$

(2.50)

$$\mathrm{r} = (1,2), \check{\mathrm{r}} = (3,4).$$

The amplitudes $a^\alpha(\vec{k})$ in (2.50) are the same as in (2.41) in the corresponding relativistic canonical quantum mechanics. The basis vectors are changed and now have the form

$$v_1^-(\vec{k}) = N \begin{vmatrix} \widetilde{\omega}+m \\ 0 \\ k^3 \\ k^1+ik^2 \end{vmatrix}, \quad v_2^-(\vec{k}) = N \begin{vmatrix} 0 \\ \widetilde{\omega}+m \\ k^1-ik^2 \\ -k^3 \end{vmatrix}, \quad (2.51)$$

$$v_3^+(\vec{k}) = N \begin{vmatrix} k^3 \\ k^1+ik^2 \\ \widetilde{\omega}+m \\ 0 \end{vmatrix}, \quad v_4^+(\vec{k}) = N \begin{vmatrix} k^1-ik^2 \\ -k^3 \\ 0 \\ \widetilde{\omega}+m \end{vmatrix},$$

where

$$N \equiv \frac{1}{\sqrt{2\widetilde{\omega}(\widetilde{\omega}+m)}}, \quad \widetilde{\omega} \equiv \sqrt{\vec{k}^2+m^2}. \quad (2.52)$$

Thus, the orts $v_\alpha^\pm(\vec{k})$ are the standard 4-component Dirac spinors. Their conditions of orthonormalization and completeness are well known, see, e.g., [57].

Hence, the Dirac equation (2.37) is derived from the quantum-mechanical equation (2.35).

2.5.   Conclusion and Perspectives

Original investigations of such important problem of modern theoretical physics as the derivation of the Dirac equation are presented. A variety of approaches and appeals to independent mathematical formalisms are demonstrated. In particular, the derivation of the Dirac equation from the principles of information processing, or the obtaining of this equation on the basis of conformal differential geometry (in general 26 different methods), are considered.

At first consider briefly the derivation of the Dirac equation given in subsection 2.4.3.

Note that the RCQM, which is based on Schrödinger–Foldy equation (2.35) and its general solution (2.41) is free from difficulties of the Dirac model and, moreover, is free from the difficulties of the FW model (contradictions of the canonical FW representation with main principles of quantum mechanics are considered briefly in subsection 2.2.9). Here in (2.41) the negative energies are absent and the adventages of the spin operator (2.47) in comparison with the FW spin (2.48) are evident as well. As a consequences on the basis of (2.35) the self-consistent relativistic quantum-mechanical model can be built, which is similar to the von Neumann's non-relativistic consideration [104].

Furthermore, the natural generalization of the FW transformation is considered. The extended FW operator, which relates the Dirac and pure quantum-mechanical models, is presented in (2.38).

The Schrödinger–Foldy equation (2.35) is the main equation of the RCQM of spin s=(1/2,1/2) particle-antiparticle doublet. Hence, the Dirac equation has been derived from the more fundamental model of the same physical reality, which is formulated as the RCQM of the spin s=(1/2,1/2) particle-antiparticle doublet.

In general, many faces of the Dirac equation derivation prove the interest to the problem.

The performed analysis gives an opportunity to compare different methods of the Dirac equation derivation. Probably the simplest is considered in [57] approach (subsection 2.2.3), which is based on the start from the Klein–Gordon equation. Nevertheless, such approach becomes quite difficult, if to begin from the Klein-Gordon equation derivation. It is more interesting to compare the fundamentals of different conclusions, their physical and philosophical significance. In this case it should be mentioned the subsection 2.4.3, where the Dirac equation is derived from a relativistic canonical mechanics of the fermion-antifermion doublet, which is more fundamental model of physical reality than the Dirac model of the spinor field at the classical level. However, the most fundamental is the derivation of the Dirac equation from the principle of the Euler–Lagrange least action in the application to the classical spinor field (subsection 2.2.4).

A review of the 26 different derivations of the Dirac equation demonstrates (in particular) that presented in [48–55] and here method of the Dirac equation (and the Dirac-like equation for arbitrary spin) derivation is original and new.

In general, the presented review demonstrates evidently that the Dirac equation is related to the fundamental laws and principles of nature.

It follows from the above given consideration that new methods of the Dirac equation derivation are not stopped and *to be continued*.

The main conclusion from the above considered variety of the approaches to the Dirac equation derivation is the evident signal that the similar variety in the approaches to the modern quantum field theory and standard model is expected as well. For example, new approach to quantum electrodynamics is the content of the book [105] and papers [106, 107]. One of the goals of the consideration in this book below is to present few new approaches of such kind.

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Chapter 3

On the Representations of the Poincaré Group for the Local and Canonical Fields

This chapter contains minimum information on the representations of the Poincaré group for the local and canonical fields, which is necessary for understanding of our further consideration. Our appealing to the prime anti-Hermitian generators of the Lie algebra of the Poincaré group is explained.

We use the notation $\mathcal{L} = \mathrm{SL}(2,\mathrm{C})$ for the *universal covering* of the proper ortochronous Lorentz group $\mathrm{L}_+^\uparrow = \mathrm{SO}(1,3) = \{\Lambda = (\Lambda_\nu^\mu)\}$. And the notation $\mathcal{P} \supset \mathcal{L}$ for the universal covering of the proper ortochronous Poincaré group $\mathrm{P}_+^\uparrow = \mathrm{T}(4) \times) \mathrm{L}_+^\uparrow \supset \mathrm{L}_+^\uparrow)$ is used. For the real parameters of the $\mathcal{P}$ group the 4-vector of translations $a = (a^\mu) \in \mathrm{M}(1,3)$ and the tensor $\omega \equiv (\omega^{\mu\nu} = -\omega^{\nu\mu})$ are chosen. Here $\omega^{\mu\nu}$ is the angle of rotation in the Cartesian plane $\mu\nu \subset \mathrm{M}(1,3)$. For the notation of element of the group $\mathcal{P} \supset \mathcal{L}$ the same symbols (a, ω) are used.

The necessary notations for the Minkowski space-time $\mathrm{M}(1,3)$ and metric tensor $g^{\mu\nu}$ together with brief consideration of the rigged Hilbert space are given in the Chapter 1. The Lorentz square of the 4-vector in $\mathrm{M}(1,3)$ is given as

$$x^2 = x^\mu x^\nu g_{\mu\nu} = x^\mu x_\mu = x_0^2 - \vec{x}^2; \quad \vec{x} = (x^j) \in \mathrm{R}^3 \subset \mathrm{M}(1,3). \tag{3.1}$$

The arbitrary $\mathcal{P}$-transformation of the field $\varphi = (\varphi^\mathrm{A}(x))_{n=1}^\mathrm{A}$ (as an arbitrary-fixed A-component $\mathcal{P}$-covariant), which is generated by the $\mathcal{P}$-transformation

in M(1,3)

$$x \to \acute{x} = \Lambda x + a \Leftrightarrow x^\mu \to \acute{x}^\mu = \Lambda_\nu^\mu x^\nu + a^\mu, \quad \Lambda = \Lambda(\omega) \equiv (\Lambda_\nu^\mu) \in \mathcal{L}, \quad (3.2)$$

infinitesimally (inf) has the form

$$\varphi(x) \to \acute{\varphi}(x) \equiv (R(a.\omega))\varphi(x) \stackrel{\text{inf}}{=} (1 - ia^\mu p_\mu^{\text{stand}} - \frac{i}{2}\omega^{\mu\nu} j_{\mu\nu}^{\text{stand}})\varphi(x), \quad (a,\omega) \in \mathcal{P}. \tag{3.3}$$

Generators of the infinitesimal transformations (3.3) satisfy the commutation relations of the Lie algebra of the Poincaré group $\mathcal{P}$ in the covariant form

$$[p_\mu^{\text{stand}}, p_\nu^{\text{stand}}] = 0, \quad [p_\mu^{\text{stand}}, j_{\rho\sigma}^{\text{stand}}] = i(g_{\mu\rho} p_\sigma^{\text{stand}} - g_{\mu\sigma} p_\rho^{\text{stand}}), \tag{3.4}$$

$$[j_{\mu\nu}^{\text{stand}}, j_{\rho\sigma}^{\text{stand}}] = -i(g_{\mu\rho} j_{\nu\sigma}^{\text{stand}} + g_{\rho\nu} j_{\sigma\mu}^{\text{stand}} + g_{\nu\sigma} j_{\mu\rho}^{\text{stand}} + g_{\sigma\mu} j_{\rho\nu}^{\text{stand}}),$$

and exponential series

$$(a,\omega) \to R(a.\omega) = \exp(-ia^\mu p_\mu^{\text{stand}} - \frac{i}{2}\omega^{\mu\nu} j_{\mu\nu}^{\text{stand}}) \tag{3.5}$$

determines the finite $\mathcal{P}$-representations $R(a.\omega)$ in the space $\varphi(x)$.

$\mathcal{P}$-representation, which is determined by the formulas (3.3), or (3.5), is called a *local* if $\mathcal{P}$-generators $(p_\mu^{\text{stand}}, j_{\mu\nu}^{\text{stand}})$ have the form of the following Lie operators

$$p_\rho^{\text{stand}} \equiv i\partial_\rho, \quad j_{\rho\sigma}^{\text{stand}} \equiv m_{\rho\sigma}^{\text{stand}} + s_{\rho\sigma}^{\text{stand}}, \tag{3.6}$$

where

$$\partial_\rho \equiv \partial/\partial x^\rho, \quad m_{\rho\sigma}^{\text{stand}} \equiv x_\rho p_\sigma^{\text{stand}} - x_\sigma p_\rho^{\text{stand}}, \tag{3.7}$$

which, certainly, satisfy the commutation relations (3.4). Here $s_{\rho\sigma}^{\text{stand}}$ are the generators of pure matrix A-dimensional representation of the group $\mathcal{L}$, i.e., the operators of the pure matrix $\mathcal{L}$-transformations $\omega \to R(\omega)$ (we call the set of 6 independent matrix generators $s_{\rho\sigma}^{\text{stand}}$ as the Lorentz spin).

Note that operators $m_{\rho\sigma}^{\text{stand}}$, or $s_{\rho\sigma}^{\text{stand}}$, satisfy last formula in the commutation relations (3.4) itself.

The characteristic feature of the locality of the $\mathcal{P}$-transformation of the field $\varphi(x)$ is its explicit form

$$\varphi(x) \to \acute{\varphi}(x) = R_1(\omega)\varphi(\Lambda^{-1}(x-a)), \tag{3.8}$$

which is the consequence of the fact that generators (3.6) belong to the class of the Lie operators. Note that in the class of the Lie operators the matrix part $R_1(\omega)$ (the transformation of form) and the differential part $R_2(a, \omega)$ $((R_2(a, \omega)\varphi)(x) \equiv \varphi(\Lambda^{-1}(x-a)))$ of the $\mathcal{P}$-transformation (3.8) are separate and independent.

Below we often use the $\mathcal{P}$-representations for the Foldy–Wouthuysen (FW) fields [1, 2]. Such representations of the Poincaré group are determined by the sets of the generators, the part of which with necessity is given by the non-local, non-Lie operators, which does not belong to the class of the Lie operators. The first example is given by the operator of energy $\widehat{p}_0 = \gamma^0 \widehat{\omega} \equiv \gamma^0 \sqrt{-\Delta + m^2}$ in the FW representation. Contrary to the operators (3.6) some generators of the sets $(p_0, p_j, j_{mn}, j_{0k})^{\text{non}-\text{local, non}-\text{Lie}}$ can not be presented as a sum of differential and matrix parts. Nevertheless, such sets $(p_0, p_j, j_{mn}, j_{0k})^{\text{non}-\text{local, non}-\text{Lie}}$ of $\mathcal{P}$-generators satisfy the commutation relations of the Lie algebra of $\mathcal{P}$-group as well.

Note that for our purposes (in order to demonstrate not only Fermi but the Bose features of the Dirac equation and the spinor field as well) we use the $\mathcal{P}$-generators in anti-Hermitian form

$$p_\rho = \partial_\rho \equiv \partial/\partial x^\rho, \quad j_{\rho\sigma} = m_{\rho\sigma} + s_{\rho\sigma}, \tag{3.9}$$

where

$$m_{\rho\sigma} \equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho, \tag{3.10}$$

and spin operators $s_{\rho\sigma}$ are ant-Hermitian as well. These generators are directly related to the real parameters of the $\mathcal{P}$ group (the 4-vector of translations $a = (a^\mu) \in M(1,3)$ and the tensor $\omega \equiv (\omega^{\mu\nu} = -\omega^{\nu\mu})$). The authors of [3, 4] recommend such operators for physicists. In our publications we call such operators as primary or prime generators. The necessity of the forms (3.9), (3.10) for our derivations and proofs is demonstrated below in next chapters. The link with the ordinary $\mathcal{P}$-generators is given by

$$p_\rho^{\text{stand}} = i p_\rho \equiv i \partial_\rho, \quad j_{\rho\sigma}^{\text{stand}} = i j_{\rho\sigma} \equiv m_{\rho\sigma}^{\text{stand}} + s_{\rho\sigma}^{\text{stand}}, \tag{3.11}$$

and vice versa

$$p_\rho = -i p_\rho^{\text{stand}}, \quad j_{\rho\sigma} = -i j_{\rho\sigma}^{\text{stand}}. \tag{3.12}$$

The prime operators (3.9) satisfy the commutation relations of the Lie algebra of the group $\mathcal{P}$ in the form given by the formulas (4.2) in the Chapter 4.

It is useful to compare the SU(2) algebra commutation relations for the standard and for the prime anti-Hermitian generators. They are given by

$$\left[s_{\text{stand}}^{j}, s_{\text{stand}}^{\ell}\right] = i\varepsilon^{j\ell n} s_{\text{stand}}^{n}, \quad \left[s^{j}, s^{\ell}\right] = \varepsilon^{j\ell n} s^{n}, \quad \varepsilon^{123} = +1, \tag{3.13}$$

where $\varepsilon^{j\ell n}$ is the Levi-Civita tensor.

Below we essentially use the induced representations of the $\mathcal{P}$ group (see, e.g., [5], for corresponding definitions an considerations).

The proof of bosonic properties of the Dirac equation and operator link between the relativistic canonical quantum mechanics (RCQM) and FW representation are possible only in terms of anti-Hermitian prime operators. Moreover, in our program of synthesis of covariant particle equations some commutation and anti-commutation relations between the necessary operators are valid if the anti-Hermitian forms of corresponding operators are chosen.

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Chapter 4

Dirac Equation in the Canonical Foldy–Wouthuysen Representation

In this chapter the necessity of the Foldy–Wouthuysen (FW) representation [1–3] for the quantum-mechanical interpretation of the Dirac equation is demonstrated. The review and systematization of arguments on impossibility of the Dirac spinor $\psi(x)$ interpretation in the role of quantum-mechanical wave function of spin s=1/2 particle-antiparticle fermionic doublet is given. The sence of the FW interpretation is considered.

4.1. Introduction to the Chapter 4

The start of the relativistic quantum mechanics was given by Paul Dirac with his well known equation for electron. More precisely, in Dirac's 4-component model, the spin s=(1/2,1/2) particle-antiparticle doublet of two fermions was considered (in particular, the electron-positron doublet). Nevertheless, the quantum-mechanical interpretation of the Dirac equation, which should be similar to the physical interpretation of the non-relativistic Schrödinger equation, is not evident and is hidden deeply in the Dirac model. In order to visualize the quantum mechanical interpretation of the Dirac equation, transformation to the canonical (quantum-mechanical type) representation was suggested [1, 2]. In this FW representation of the Dirac equation, the quantum-mechanical in-

terpretation is much more clear. Nevertheless, the direct and evident quantum-mechanical interpretation of the spin s=(1/2,1/2) particle-antiparticle doublet can be fulfilled only within the framework of the relativistic canonical quantum mechanics (RCQM), see the consideration in next chapter. On the other hand in quantum-mechanical interpretation the FW representation has the evident advantages in comparison with the Dirac model. The goal of this chapter is to demonstrate the meaning of the FW representation.

4.2. Covariant and Schrödinger Forms of the Dirac Equation

Here we consider the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0; \quad \partial_{\mu} \equiv \partial/\partial x^{\mu}, \ \mu = \overline{0,3} = 0, 1, 2, 3, \qquad (4.1)$$

$$\psi \in S(M(1,3)) \times C^4 \equiv S^{4,4},$$

as an equation in the *test function space* $S^{4,4}$ of the 4-component functions over the Minkowski space M(1,3). We note that the complete set $\{\psi\} \equiv \Psi$ of solutions of the equation (1) contains the generalized solutions belonging to the space $S^{4,4*} \supset S^{4,4}$ of the Schwartz's *generalized* functions, $\{\psi^{\text{gen}}\} \subset S^{4,4*}$. The mathematically well-defined consideration of this fact demands (see, e.g., the book [4] on the axiomatic approach to the field theory) the functional representation of the elements of $\psi^{\text{gen}} \subset S^{4,4*}$, which makes the consideration very complicated. Hence, we recall that the *test function space* $S^{4,4}$ is *dense* in $S^{4,4*}$. It means that any element $\psi^{\text{gen}} \subset S^{4,4*}$ can be approximated (with arbitrary degree of accuracy) by an element $\psi \in S^{4,4}$ from the corresponding Cauchy sequence in the space $S^{4,4}$. Therefore, here for the equation (1) we have restricted ourselves to the supposition $\psi \in S^{4,4}$. Such supposition is physically verified and essentially simplifies consideration without any loss of generality and mathematical correctness.

Consider briefly the conventional relativistic invariance of equation (1). In the modern consideration, it is the invariance with respect to the universal covering $\mathcal{P} \supset \mathcal{L}=$SL(2,C) of the proper ortochronous inhomogeneous Poincaré group $P_{+}^{\uparrow} \supset L_{+}^{\uparrow} = $ SO(1,3). For our purposes it is convenient to rewrite in the space $S^{4,4}$ the standard Fermi (spinor) representation of the group $\mathcal{P}$ in terms of the *primary $\mathcal{P}$-generators* $(p \equiv (p_{\mu}), j \equiv (j_{\mu\nu}))$. By the definition, these generators are associated with the *real parameters* $(a, \omega) \equiv (a^{\mu}, \omega^{\mu\nu} = -\omega^{\nu\mu})$ with

well-known physical meaning. The generators $(p_\mu, j_{\mu\nu})$ are the orts of the *real P-algebra*, i.e., the algebra over the real numbers (we use the similar symbols for groups and their algebras). The necessity of using the primary P-generators and the real P-algebra will be evident below in our search for the additional (hidden) symmetries of the equation (4.1).

In arbitrary representations, the primary P-generators $(p_\mu, j_{\mu\nu})$, as the orts of the real Lie P-algebra, satisfy the commutation relations

$$[p_\mu, p_\nu] = 0, \quad [p_\mu, j_{\rho\sigma}] = g_{\mu\rho}p_\sigma - g_{\mu\sigma}p_\rho, \tag{4.2}$$

$$[j_{\mu\nu}, j_{\rho\sigma}] = -g_{\mu\rho}j_{\nu\sigma} - g_{\rho\nu}j_{\sigma\mu} - g_{\nu\sigma}j_{\mu\rho} - g_{\sigma\mu}j_{\rho\nu},$$

and generate a P-representation, which is defined by an exponential series

$$(a, \omega) \in P \rightarrow \widehat{F}(a, \omega) = \exp(a^\mu p_\mu + \frac{1}{2}\omega^{\mu\nu}j_{\mu\nu}) \stackrel{\text{inf}}{=} 1 + a^\mu p_\mu + \frac{1}{2}\omega^{\mu\nu}j_{\mu\nu}, \tag{4.3}$$

where symbol $\stackrel{\text{inf}}{=}$ defines "infinitesimally, i.e., in the neighborhood of the unit element of the group P".

The primary generators $(p_\mu, j_{\mu\nu})$ for the ordinary *local* spinor (Fermi) P-representation in $S^{4,4}$ are the following *Lie operators*

$$p_\rho = \partial_\rho \equiv \partial/\partial x^\rho, \quad j_{\rho\sigma} = m_{\rho\sigma} + s_{\rho\sigma} \quad (m_{\rho\sigma} \equiv x_\rho\partial_\sigma - x_\sigma\partial_\rho, \quad s_{\rho\sigma} \equiv \frac{1}{4}[\gamma_\rho, \gamma_\sigma]). \tag{4.4}$$

The operators $p_\mu, j_{\mu\nu}$ from (4.4) commute with the Diracian $i\gamma^\mu\partial_\mu - m$. Therefore, formulae (4.3) with generators (4.4) define the *local* spinor (Fermi) P^F-representation of the group P in the form

$$\psi(x) \rightarrow \psi'(x) = [\widehat{F}(a, \omega) \equiv \widehat{F}_1(\omega)\widehat{F}_2(a, \omega)]\psi(x) \stackrel{\text{inf}}{=} (1 + a^\mu p_\mu + \frac{1}{2}\omega^{\mu\nu}j_{\mu\nu})\psi(x), \tag{4.5}$$

which is the *P-group of invariance* of equation (4.1). In formula (4.5) the following notations are used:

$$\psi, \psi' \in S^{4,4}, \quad \widehat{F}_1(\omega) \equiv \exp(\frac{1}{2}\omega^{\mu\nu}s_{\mu\nu}) \stackrel{\text{inf}}{=} (1 + \frac{1}{2}\omega^{\mu\nu}s_{\mu\nu}) \sim (\frac{1}{2}, 0) \otimes (0, \frac{1}{2}), \tag{4.6}$$

$$\widehat{F}_2(a, \omega)\psi(x) \equiv \exp(a^\mu p_\mu + \frac{1}{2}\omega^{\mu\nu}m_{\mu\nu})\psi(x) = \psi(\Lambda^{-1}(x - a)) \tag{4.7}$$

$$\stackrel{\text{inf}}{=} (1 + a^\mu p_\mu + \frac{1}{2}\omega^{\mu\nu}m_{\mu\nu})\psi(x), \quad \Lambda^{-1} \in P_+^\uparrow.$$

Note that the primary $\mathcal{P}$-generators from (4.2)–(4,4) are not our own suggestion. For the possibility and mathematical correction of choosing the anti-Hermitian operators like (4.2)–(4.4) in physics see, e.g., [5, 6].

We pay attention to the following detail of the mathematical correctness of consideration. The space $S^{4,4}$ is the common domain of definitions and values both for the generators $(p_\mu, j_{\mu\nu})$ (4.4) and for all functions from them, which we use here (in particular, for the exponential series (4.3) convergent in the space $S^{4,4}$). Further, we mark that usually the fermionic $\mathcal{P}$-transformations of the field ψ are written in the form

$$\psi(x) \to \psi'(x') = \widehat{F}_1(\omega)\psi(x), \quad x' = \Lambda x + a, \quad \Lambda \in P_+^\uparrow. \qquad (4.8)$$

However, this form (contrary to formula (4.5)) does not demonstrate manifestly the mathematical definition of the group of invariance of equation (4.1), which is given by the definition: $\mathcal{P}^F$ is a group of invariance of equation (4.1), if $\mathcal{P}^F \Psi = \Psi$; or for arbitrary solution of equation (4.1): if from $\psi(x) \in \Psi$ results $\widehat{F}(a, \omega)\psi(x) \in \Psi \subset S^{4,4}$, where $\widehat{F}(a, \omega)$ is given by (4.5).

Let us note further that in conventional field-theoretical approach, instead of our primary Lie generators (4.4), the following operators are used

$$p_\rho^{\text{stand}} = i p_\rho \equiv i \partial_\rho, \quad j_{\rho\sigma}^{\text{stand}} = i j_{\rho\sigma} \equiv m_{\rho\sigma}^{\text{stand}} + s_{\rho\sigma}^{\text{stand}} \qquad (4.9)$$

$$(m_{\rho\sigma}^{\text{stand}} \equiv i x_\rho \partial_\sigma - i x_\sigma \partial_\rho, \quad s_{\rho\sigma}^{\text{stand}} \equiv \frac{i}{4}[\gamma_\rho, \gamma_\sigma]).$$

It is evident that these generators are associated with the pure imaginary parameters $(-ia^\mu, -i\omega^{\mu\nu})$ and with corresponding $\mathcal{P}$-algebra over complex numbers. Below, as well as in other our publications on the symmetries, primary generators (4.4) and customary operators (4.9) should not be confused.

We recall that both pure matrix operators $s_{\rho\sigma}$ and pure differential operators $m_{\rho\sigma}$ from (4.4) satisfy the same commutation relations as $[j_{\mu\nu}, j_{\rho\sigma}]$ in (4.2). However, the operators $s_{\rho\sigma}$ and $m_{\rho\sigma}$ (contrary to their sum $j_{\rho\sigma}$) *are not* the symmetry operators (operators of invariance) of equation (4.1) (operator $\widehat{q}$ is called *a symmetry operator* or the operator of invariance of equation (4.1), if equality $\widehat{q}\Psi = \Psi$ is valid, where $\Psi \equiv \{\psi\} \subset S^{4,4}$ is the complete set of solutions of equation (4.1), see e.g., the corresponding definition in [7]). Therefore, both pure matrix $\mathcal{L}$-representation $\widehat{F}_1(\omega)$ (4.6) and infinite-dimensional $\mathcal{L}$-representation

$$\widehat{F}_2(\omega)\psi(x) \equiv \exp(\frac{1}{2}\omega^{\mu\nu}m_{\mu\nu})\psi(x) = \psi(\Lambda^{-1}(x)) \overset{\text{inf}}{=} (1 + \frac{1}{2}\omega^{\mu\nu}m_{\mu\nu})\psi(x) \quad (4.10)$$

in $S^{4,4}$ are not the groups of invariance of equation (4.1). As a consequence of these facts (see arbitrary conventional consideration in relevant papers, handbooks and monographs), both matrix $\mathcal{L}$-spin $s_{\rho\sigma}$ and orbital angular momentum $m_{\rho\sigma}$ (differential $\mathcal{L}$ angular momentum) do not generate the conserved in time integral constants of motion, i.e., being taken separately both spin and orbital angular momenta of the field ψ are not conserved. Therefore, both pure matrix $\widehat{F}_1(\omega)$ (4.6) and infinite-dimensional $\widehat{F}_2(\omega)$ representations in $S^{4,4}$ of the Lorentz group $\mathcal{L}$ are not the groups of invariance of the Dirac equation (4.1).

Besides the local $\mathcal{P}^F$-representation, the so-called *induced* (see, e.g., [2, 8]) $\mathcal{P}^F$-representation for the field ψ is useful and meaningful. Mathematically this representation can be related to the *conception of special role of the time variable $t \in (-\infty, \infty) \subset M(1,3)$*. Indeed, in the general consideration of the Dirac equation in the axiomatic approach, it follows from equation (4.1) that the Dirac field ψ satisfies identically the Klein – Gordon equation

$$(\partial^\mu \partial_\mu + m^2 \equiv \partial_0^2 - \triangle + m^2)\psi(x) = 0, \quad \psi \in S^{4,4*}, \tag{4.11}$$

which is the equation of hyperbolic type. As a consequence of this fact, the generalized solutions of the Dirac equation (4.1) are the *ordinary functions* of the time variable $x^0 = t \in (-\infty, \infty) \subset M(1,3)$ (they are the generalized functions of the variables $\overrightarrow{x} \equiv (x^\ell) \in R^3 \subset M(1,3)$ only). Therefore, due to a special role of the time variable $x^0 = t \in (x^\mu)$ (in obvious analogy with nonrelativistic theory), one can use in general consideration *the quantum-mechanical rigged Hilbert space* $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$, $S^{3,4} \equiv S(R^3) \times C^4$, where the space $H^{3,4}$ is defined by (2.36). The space $H^{3,4}$ is the quantum-mechanical Hilbert space of the 4-component functions over $R^3 \subset M(1,3)$ (depending parametrically on $x^0 = t$), which are the square modulus integrable over *the Lebesgue measure* d^3x in the space $R^3 \subset M(1,3)$. Just the space R^3 is interpreted as the coordinate spectrum of the quantum-mechanical particles described by the field ψ.

In this concept, the Dirac equation in the Schrödinger form

$$i\partial_0 \psi = H\psi \leftrightarrow (\partial_0 - \widetilde{p}_0)\psi = 0; \quad H \equiv \gamma^0(-i\gamma^k \partial_k + m), \quad \widetilde{p}_0 = -iH, \tag{4.12}$$

(which is completely equivalent to the equation (4.1)) in the integral form is given by

$$\psi(t) = u(t_0, t)\psi(t_0); \quad u(t_0, t) = \exp[-iH(t - t_0)]; \quad \psi(t_0), \psi(t) \subset S^{3,4*}. \tag{4.13}$$

In formula (4.13), the unitary operator $u(t_0, t)$ (with arbitrary-fixed parameters $t_0, t \in (-\infty, \infty) \subset M(1,3)$) is the operator of automorphism in the rigged Hilbert space (1.3) (below we set $t_0 = 0$).

Recall that the test function space $S^{3,4}$ has a few wonderful features. It consists of the functions being infinitely smooth (infinitely differentiable with respect to x^ℓ) rapidly decreasing at the infinity $|\overrightarrow{x}| \to \infty$ in arbitrary direction in R^3 together with its derivatives of arbitrary orders. Further, it contains the space of *the finite functions* with the same properties. Moreover, the space $S^{3,4}$ is *kernel* in the triple $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$ (1.3). The last one means that this space is dense both in the $H^{3,4}$ and $S^{3,4*}$ spaces. Therefore, below we restrict our consideration to the suggestion $\psi \in S^{3,4}$ in equation (4.12). Such restriction is both mathematically correct and technically appropriate (it does not require the using of the functional form of the elements $\psi \in S^{3,4*}$). It is also physically motivated. Indeed, an arbitrary measurement of a construction from $\psi^{\text{gen}} \subset S^{3,4*}$ by an equipment of an arbitrary degree of accuracy can be successfully approximated (with forward fixing arbitrarily precise degree of accuracy) by the corresponding constructions from the prelimit functions $\psi \in S^{3,4} \subset H^{3,4} \subset S^{3,4*}$. Thus, the suggestion $\psi \in S^{3,4} \subset H^{3,4} \subset S^{3,4*}$ for the solutions of (4.12) is well-defined too.

For definiteness below we use the Pauli–Dirac representation of the Clifford–Dirac γ-matrices. The explicit forms are given in (1.5), (1.6). In this representation, the SU(2)-spin primary generators are given by the matrices $s_{\ell n} \equiv \frac{1}{4}[\gamma_\ell, \gamma_n]$. Two of them are quazidiagonal and the s_z-matrix is diagonal:

$$\overrightarrow{s} = (s_{23}, s_{31}, s_{12}) \equiv (s^\ell) = \frac{1}{2} \begin{vmatrix} \overrightarrow{\sigma} & 0 \\ 0 & \overrightarrow{\sigma} \end{vmatrix} \to s_z \equiv s^3 = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \quad (4.14)$$

Therefore, the operator s_z contains the projections of the spin $\overrightarrow{s}$ on the axis z of the quantum-mechanical spin-$\frac{1}{2}$ doublet of particles.

The primary $\mathcal{P}$-generators of the induced $\mathcal{P}$-representation in the space $S^{3,4}$ have the form of the following anti-Hermitian operators

$$\widetilde{p}_0 = -iH \equiv -\gamma_0(\gamma^\ell \partial_l + im), \ \widetilde{p}_k = p_k = \partial_k, \ \widetilde{j}_{kl} = j_{kl} = m_{kl} + s_{kl}, \quad (4.15)$$

$$\widetilde{j}_{ok} = t\partial_k - \frac{1}{2}\{x_k, \widetilde{p}_0\},$$

where $\{A, B\} \equiv AB + BA$ and $t \in (-\infty, \infty)$ is the arbitrary-fixed parameter (the primary generators $\widetilde{q} = (\widetilde{p}_\mu, \widetilde{j}_{\mu\nu})$ (4.15) coincide with the corresponding opera-

tors $-i\widetilde{q}^{\text{stand}}$, where $\widetilde{q}^{\text{stand}}$ are given by the formulae (126)–(129) in [2]). Using the Heisenberg commutation relations in the form $[x^\ell, \partial_j] = \delta_{\ell j}$ and the SU(2)-relations for s^ℓ (4.14), it is easy to show that the operators (4.15) satisfy the commutation relations of $\mathcal{P}$-algebra in the manifestly covariant form (4.2). Furthermore, generators (4.15) commute with the operator $(\partial_0 - \widetilde{p}_0)$ of the equation (4.12). Therefore, as a consequence of anti-Hermiticity of arbitrary operator from (4.15), the induced $\mathcal{P}$-representation

$$(a, \omega) \in \mathcal{P} \rightarrow \widetilde{U}(a, \omega) = \exp(a^\mu \widetilde{p}_\mu + \frac{1}{2}\omega^{\mu\nu}\widetilde{j}_{\mu\nu}) \qquad (4.16)$$

in the space $S^{3,4}$ is unitary and is the group of invariance of equation (4.14). Hence, in the definition

$$w^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\widetilde{p}_\rho \widetilde{j}_{\nu\sigma} \rightarrow w^0 = s^\ell \partial_\ell \equiv \overrightarrow{s}^{\text{stand}} \cdot \overrightarrow{p}^{\text{stand}}, \qquad (4.17)$$

the main Casimir operators for the generators (4.15) are given by

$$\widetilde{p}^\mu \widetilde{p}_\mu \equiv \widetilde{p}_0^2 - \partial_\ell^2 \equiv m^2 I_4, \quad I_4 \equiv \text{diag}(1,1,1,1), \qquad (4.18)$$

$$\mathcal{W} \equiv w^\mu w_\mu = m^2 \overrightarrow{s}^2 = m^2 \frac{1}{2}(\frac{1}{2}+1)I_4. \qquad (4.19)$$

According to the Bargman–Wigner classification, just this fact means that in equation (4.12) (for which the induced $\mathcal{P}^F$-representation (4.16) is the group of invariance) the field ψ is the Fermi field (the field of quantummechanical spin-$\frac{1}{2}$ doublet of particles with the mass m).

The relativistic invariance of the spinor field theory with respect to the representation (4.16) has some special features. It should be stressed once more that in the induced $\mathcal{P}^F$-representation (4.16) the time $t = x^0$ plays a special role in comparison to the role of space variables x^ℓ. Moreover, we use in the definition of Hilbert space $H^{3,4}$ such $\mathcal{P}$-non-covariant objects as the Lebesgue measures d^3x (or d^3k in the momentum representation of the rigged Hilbert space $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$ for the field states $\psi \in S^{3,4}$). Nevertheless, the theory of the spinor field ψ based on the induced $\mathcal{P}^F$-representation (4.16) is obviously relativistic invariant. The proof is given in the text after formulae (4.15) and by the Bargman–Wigner analysis of the Casimir operators (4.18), (4.19).

Now we are in position to give some comparison of the local and induced $\mathcal{P}^F$-representations (4.5) and (4,16). It is easy to see that in the set of solutions

$\Psi = \{\psi\}$ of equations (4.12)=(4.1) the local $\mathcal{P}$-representation (4.5) and the induced $\mathcal{P}$-representation (4.16) coincide. Moreover, the main Casimir operators for the Lie $\mathcal{P}^F$-generators (4.4) have the form

$$p^{\text{loc } 2} \equiv p^\mu p_\mu \equiv \partial^\mu \partial_\mu, \quad W^{\text{loc}} \equiv m^2 \overrightarrow{s}^2 = \frac{3}{4} \partial^\mu \partial_\mu. \tag{4.20}$$

Therefore, the eigenvalues of the main Casimir's for the $\mathcal{P}^F$-representations (4.5) and (4.16) coincide. Note, by the way, that as it is easy to show formulae (4.20) have the same form for arbitrary Lie operators (4.4) (i.e., for arbitrary local $\mathcal{P}$-representations), for which the Lorentz spins $s_{\mu\nu}$ generate the $\mathcal{L}$-representations $(s,0) \otimes (0,s)$.

Mark that the local generators (4.4) in $S^{4,4}$ are the functions of 14 independent operators $x^\rho, \partial_\rho, s_{\rho\sigma}$ (the Lorentz spin operators $s_{\rho\sigma}$ are the independent orts of the CD-algebra). The conception of Hermiticity or anti-Hermiticity in $S^{4,4}$ is not inherent both for these 14 operators and $\mathcal{P}^F$-generators (4.4). Therefore, the concept of unitarity is not inherent for the local $\mathcal{P}$-representation (4.5) as well. It means that the $\mathcal{P}$-representation (4.5) itself (and, similarly, its generators (4.4)) does not contain the information "what quantum-mechanical particles are described by the filed ψ from equation (4.1)".

Contrary to these facts, the generators (4.15) of induced $\mathcal{P}^F$-representation in the space $S^{3,4}$ are the anti-Hermitian functions of (particularly) other 11 independent operators $\overrightarrow{x} = (x^\ell), \partial_\ell, s_{k\ell}, \gamma^0, m$ or are the functions of the standard Hermitian in $S^{3,4}$ operators: $\overrightarrow{x} = (x^\ell), \overrightarrow{p}^{\text{stand}} = (-i\partial_\ell), \overrightarrow{s}^{\text{stand}} = (is_{\ell n}), \gamma^0, m$. Hence, the induced $\mathcal{P}^F$-representation (4.16) is unitary in the quantum-mechanical rigged Hilbert space $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$. However, the restriction of the local $\mathcal{P}^F$-representation (4.5) on the set $\Psi \subset S^{3,4}$ of solutions of equation (4.1)=(4.12) coincides with the induced $\mathcal{P}^F$-representation (4.16).

4.3. Link between the Canonical Foldy–Wouthuysen and the Covariant Pauli–Dirac Representations of the Dirac Theory

Both covariant and Schrödinger forms of the Dirac equation considerede above have the difficulties in quantum-mechanical interpretation. These shortcomings

are related to the Pauli–Dirac representation of the spinor field ψ. The step forward in the quantum-mechanical interpretation of the Dirac theory is given by the transition to the FW representation. In this representation, the equation for the spinor field (the FW equation) has a form

$$(i\partial_0 - \gamma^0 \widehat{\omega})\phi(x) = 0, \quad x \in M(1,3), \ \phi \in S^{3,4} \subset H^{3,4} \subset S^{3,4*}; \qquad (4.21)$$

and is linked with the Dirac equation

$$(i\partial_0 - H)\psi(x) = 0, \quad H \equiv \vec{\alpha} \cdot \widehat{\vec{p}} + \beta m, \ \vec{\alpha} = \gamma^0 \vec{\gamma}, \ \beta = \gamma^0, \qquad (4.22)$$

by the FW transformation $V^{\pm}$:

$$V^{\pm} \equiv \frac{\pm \vec{\gamma} \cdot \widehat{\vec{p}} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}, \qquad (4.23)$$

where $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$, $\widehat{\vec{p}} = -i\nabla$, $\nabla \equiv (\partial_\ell)$, $\partial_\ell \equiv \partial/\partial x^\ell$, $I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$.

Relations between the states vectors in the covariant Pauli–Dirac and canonical FW representations (relations between the solutions of the Dirac and FW equations) are given by

$$\phi(x) = V^+ \psi(x), \quad \psi(x) = V^- \phi(x). \qquad (4.24)$$

Relations between the operators in the covariant Pauli–Dirac and canonical FW representations have the form

$$q^{FW} = V^+ q^{PD} V^-, \quad q^{PD} = V^- q^{FW} V^+, \qquad (4.25)$$

where q^{PD} and q^{FW} are the arbitrary operators in the covariant Pauli–Dirac and canonical FW representations, respectively. For example, the relationship between the FW and Dirac Hamiltonians (and vice versa) is given by

$$V^+ \gamma^0 \widehat{\omega} V^- = \vec{\alpha} \cdot \widehat{\vec{p}} + \beta m, \quad V^- (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m) V^+ = \gamma^0 \widehat{\omega}. \qquad (4.26)$$

The relationship between the FW spin operator $\vec{s}$ (4.14) and Dirac spin has the form

$$\vec{s}_{PD} = V^+ \vec{s} V^- = \vec{s} - \frac{\vec{\gamma} \times \nabla}{2\widehat{\omega}} + \frac{\nabla \times (\vec{s} \times \nabla)}{\widehat{\omega}(\widehat{\omega} + m)}. \qquad (4.27)$$

It is easy to verify that well-defined Dirac spin (4.27) commutes with the operator $i\partial_0 - (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m)$ of the Dirac equation (4.22) and satisfies the SU(2) commutation relations.

The solution of the FW equation (4.21) is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \{e^{-ikx}[a^1(\vec{k})d_1 + a^2(\vec{k})d_2] + e^{ikx}[a^{*3}(\vec{k})d_3 + a^{*4}(\vec{k})d_4]\},$$

(4.28)

where the $a^{\tilde{\alpha}}(\vec{k})$ are the quantum-mechanical momentum-spin amplitudes and the orts $\{d_{\tilde{\alpha}}\}$ of the 4-dimensional Cartesian basis have the form

$$d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, d_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, d_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \qquad (4.29)$$

The solution of the Dirac equation (4.22) is given by

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k [a^r(\vec{k})v_r^-(\vec{k})e^{-ikx} + b^{*r}(\vec{k})v_r^+(\vec{k})e^{ikx}], \quad r = 1,2,$$

(4.30)

$$b^1(\vec{k}) \equiv a^3(\vec{k}), \quad b^2(\vec{k}) \equiv a^4(\vec{k}).$$

Here, similarly to (4.28), $kx \equiv \widetilde{\omega}t - \vec{k}\vec{x}$, $\widetilde{\omega} \equiv \sqrt{\vec{k}^2 + m^2}$, and the 4-component Dirac spinors are given by

$$v_r^-(\vec{k}) = N \begin{vmatrix} (\widetilde{\omega}+m)d_r \\ (\vec{\sigma} \cdot \vec{k})d_r \end{vmatrix}, \quad v_r^+(\vec{k}) = N \begin{vmatrix} (\vec{\sigma} \cdot \vec{k})d_r \\ (\widetilde{\omega}+m)d_r \end{vmatrix}; \qquad (4.31)$$

$$N \equiv \frac{1}{\sqrt{2\widetilde{\omega}(\widetilde{\omega}+m)}}, \quad d_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad d_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}.$$

4.4. Difficulties in the Quantum-Mechanical Interpretation of the Dirac Theory

Recall that for most tasks of atomic and nuclear physics, the following assertion is adopted.

Ansatz. Energetic spectrum and other quantum-mechanical properties of the electron-positron doublet of mass m and charges (e^-, e^+) (or other fermionic charges), which interacts with the external electromagnetic field of potentials $A = (A^\mu)$, satisfactorily is described satisfactorily by the Dirac equation

$$[\gamma^\mu (i\partial_\mu - eA_\mu(x) - m)] \psi(x) = 0. \tag{4.32}$$

This Ansatz is also the basis of the Hartree-Fock-Dirac multi-particle method. Furthermore, it is known that, for asymptotically large space-time distances (where experimental measurements of characteristics of initial and final states are carried out), the quantum-mechanical properties of a spin $s = 1/2$ doublet must be described by a corresponding free equation (which does not contain an external field $A = (A^\mu)$. If the above-mentioned Anzac is flawless, then the equation (4.22), as a partial case of the equation (4.32), should perfectly describe all the quantum-mechanical properties of a free doublet. However, as shown in the original paper [1], and in many publications until today [2, 9–14], the quantum-mechanical description of a free spin $s = 1/2$ doublet by equation (4.22) has fundamental difficulties. What is the influence of these difficulties at the accuracy of the methods of atomic calculations (such as Hartree-Fock-Dirac)? The question is open.

Below the brief review of the difficulties of the quantum-mechanical Dirac equation (4.22) is presented.

The velocity operator in a quantum-mechanical theory, based on the Dirac equation (4.22), is not the operator of the velocity of the spin $s = 1/2$ particle-antiparticle doublet.

Indeed, the operator

$$\overrightarrow{\dot{x}} = \frac{i}{\hbar} [H, \overrightarrow{x}] = c\gamma^0 \overrightarrow{\gamma} \tag{4.33}$$

(here and in (4.34) below $\hbar \neq c \neq 1$) in accordance with the principles of heredity and correspondence with non-relativistic quantum mechanics, should be called the operator of the velocity of a doublet of spin $s = 1/2$ particles. In (4.33) H is the Hamiltonian of the Dirac equation (4.22), $\hbar$ is the Planck constant and c is the velocity of light in the vacuum. In his monograph [15] Dirac was unsuccessful in attempts to give the interpretation of such "velocity". The statements of [1] are evident. Indeed, since $(\gamma^0 \gamma^\ell)^2 = 1$, then the absolute value of the projection of this "velocity" on a given direction is always equal to c, which is nonsense from the point of view of modern physics. Moreover, due

to the fact that $[\gamma^0\gamma^1, \gamma^0\gamma^2] \neq 0$, if a projection of velocity is determined in one direction, it is impossible to determine two other projections at the same time. It is clear that all this contradicts with the experiments on measuring the electron or positron velocity.

The coordinate operator in quantum-mechanical theory, based on the Dirac equation (4.22), is not the operator of the coordinate of the spin $s = 1/2$ doublet (in particular, in the case of the interaction of an electron with a nucleus field $A_0 = Ze/|\overrightarrow{x}|$, the variable $\overrightarrow{x}$ is not the coordinate of the electron motion relative to the point nucleus). This statement is a direct consequence of the previous statement about velocity. The uncertainty (blurriness) of the particle coordinates makes necessary an appealing to the so-called Schrdinger's "trembling" of the electron (zitterbewegung), which in turn generates a set of additional theoretical problems and is artificial in areal of experimental facts.

The spin operator (4.14) in a quantum-mechanical theory, based on the Dirac equation (4.22), is not the spin of $s = 1/2$ particle-antiparticle doublet.

Direct calculation of the commutators of the projections $\overrightarrow{s} = (s_{23}, s_{31}, s_{12}) \equiv (s^\ell)$ of spin (4.14) with the Dirac Hamiltonian $H \equiv \overrightarrow{\alpha} \cdot \widehat{\overrightarrow{p}} + \beta m$ gives

$$\left[H, \overrightarrow{s}\right] = ic\gamma^0(\overrightarrow{\gamma} \times \widehat{\overrightarrow{p}}); \quad \widehat{\overrightarrow{p}} \equiv -i\nabla. \tag{4.34}$$

It means that the spin of the Dirac particle-antiparticle doublet does not commute with the Hamiltonian. Hence, the spin of the Dirac particle-antiparticle doublet (none of its components separately) is not a constant of motion. Therefore, since the Dirac equation (4.22) is considered to be a quantum mechanical equation for a free particle-antiparticle doublet, then non-conservation of the spin (4.14) means that, according to the free Dirac equation, spin-flip processes (spin turning) are possible even for a free particle-antiparticle doublet, which contradicts with the experiment.

Of course, the total angular momentum (orbital angular nomentum plus spin angular momentum)

$$\overrightarrow{j} = \overrightarrow{x} \times \widehat{\overrightarrow{p}} + \overrightarrow{s} \tag{4.35}$$

commutes with the Hamiltonian of the Dirac equation

$$\left[H, \overrightarrow{j}\right] = 0, \tag{4.36}$$

which is considered until today as an achievement of the Dirac theory. Nevertheless, taking into account (4.34) and the comments given, it is evident that

this fact improves the situation only with respect to the total angular momentum (4.35) and, in any case, not with respect to the spin angular momentum (4.14).

Taking into account (4.35) and (4.36) Dirac in [15] has come to a conclusion "This result one can interpret by saying *the electron has a spin angular momentum* $\vec{s}$, which must be added to the orbital angular momentum $\vec{x} \times \widehat{\vec{p}}$, before one gets the constant of the motion." Taking into account the consideration [1], this conclusion of Dirac is incorrect at least methodically. The fact is that in this very representation (even without transition to the FW representation) there are such "true coordinate" and "true spin" that, expressed in such a valid coordinate, the orbital angular momentum itself, taken separately, commutes with the Dirac Hamiltonian H (4.22), as well as taken separately "true spin" $\vec{s}_{\text{PD}}$ (4.27). Corresponding angular momentum is given by

$$\vec{x}_{\text{PD}} \times \widehat{\vec{p}} \tag{4.37}$$

and commutes with the Hamiltonian H (4.22) of the Dirac equation. Such "true coordinate" has the form

$$\vec{x}_{\text{PD}} = V^{+}\vec{x}V^{-} = \vec{x} + \frac{i\vec{\gamma}}{2\widehat{\omega}} - \frac{\vec{s} \times \widehat{\vec{p}}}{\widehat{\omega}(\widehat{\omega}+m)} - \frac{i\widehat{\vec{p}}(\vec{\gamma} \cdot \widehat{\vec{p}})}{2\widehat{\omega}^{2}(\widehat{\omega}+m)}. \tag{4.38}$$

In [1] such "true spin" and "true coordinate" were called mean spin and mean coordinate, respectively. Thus, both orbital angular momentum and spin angular momentum are integrals of motion. So today, the reminded arguments of Dirac, based on assertions (4.35) and (4.36), should be considered unconvincing and unnecessary.

Unfortunately, Dirac's argument is still used in modern textbooks on quantum mechanics in order to introduce the concept of spin. Here and below, on the bases of the results [1] and [16, 17], we emphasize the presence of a mathematically correct and physically substantiated definition of spin in the theory of a spinor field.

Dirac spinors are not the eigen vectors of the spin projection $s_z = s^3$ **of the spin (4.14).**

It is well-known that the spinors $v_{\text{r}}^{\mp}(\vec{k})$ (4.31) are called the spin states of the doublet. Nevertheless, they are not the eigen vectors of the operator $s_z = s^3$ from $\vec{s}$ (4.14). They are the eigen vectors of the operator $s_z^{\text{PD}} \equiv s_{\text{PD}}^3 \subset \vec{s}_{\text{PD}}$ from the mean spin operator for the spinor field ψ, which is given in (4.27) (and in the table 1 in [1]) and which is obtained from the $\vec{s}$ (4.14) by the inverse

FW-transformation [1]. Indeed, it is easy to see that the following equations are valid:

$$s_z v_1^{\mp}(\overrightarrow{k}) \neq \frac{1}{2} v_1^{\mp}(\overrightarrow{k}), \quad s_z v_2^{\mp}(\overrightarrow{k}) \neq -\frac{1}{2} v_2^{\mp}(\overrightarrow{k}); \tag{4.39}$$

$$s_z^{\text{PD}} v_1^{\mp}(\overrightarrow{k}) = \frac{1}{2} v_1^{\mp}(\overrightarrow{k}), \quad s_z^{\text{PD}} v_2^{\mp}(\overrightarrow{k}) = -\frac{1}{2} v_2^{\mp}(\overrightarrow{k}), \tag{4.40}$$

(the assertions (4.39), (4.40) are our small addition [16] to the consideration [1]). Moreover, operator $\overrightarrow{s}$ (4.14) does not commute with both the prime Dirac operator from equation (1) and the operator H from (4.22), which is already considered above. Therefore, for the spinor field ψ in the PD-representation the operator $\overrightarrow{s}$ (4.14) can not be interpreted as the quantum-mechanical spin operator for the spin $\frac{1}{2}$-doublet of particles and does not generate the spin conservation law (even in the absence of interaction), which contradicts the experiment. Furthermore, as it is already considered above, the operator $\overrightarrow{x} = (x^\ell) \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ in the PD-representation for the field ψ also cannot be interpreted as the quantum-mechanical operator of 3-coordinate for the spin $\frac{1}{2}$-doublet of particles. Nevertheless, operators $\overrightarrow{s}$ (4.14) and $\overrightarrow{x}$ are the important structure operators (which possess the physically meaningful quantum-mechanical spectra) for the construction of generators (4.4), (4.12), (4.15) and corresponding local and induced $\mathcal{P}^{\mathrm{F}}$-representations.

Shortcomings of the Dirac Hamiltonian

The Hamiltonian $H \equiv \overrightarrow{\alpha} \cdot \overrightarrow{p} + \beta m$ (4.22) of the free Dirac equation $(i\partial_0 - H)\psi(x) = 0$ contains non-diagonal operators that mix the components of the wave functions of the electron and positron. Therefore, for example, the function

$$\psi_{\overrightarrow{k}}^-(x) = a^{\mathrm{r}}(\overrightarrow{k}) v_{\mathrm{r}}^-(\overrightarrow{k}) e^{-ikx} = e^{-i\omega t} \psi_{\overrightarrow{k}}^-(\overrightarrow{x}), \tag{4.41}$$

which is eigen function of the operator H (4.22) with eigen value $E^+ = \widetilde{\omega} = \sqrt{\overrightarrow{k}^2 + m^2}$, is interpreted as a "quantum-mechanical" wave function of (free) electron. However, it is 4-component, not 2-component, as it is required by the quantum mechanics of an electron (i.e., single spin $s = 1/2$ particle from an electron-positron doublet). Further, the spectrum of the Hamiltonian H (4.22) is sign-undetermined, so for an antiparticle in a "quantum-mechanical state"

$$\psi_{\overrightarrow{k}}^+(x) = b^{*\mathrm{r}}(\overrightarrow{k}) v_{\mathrm{r}}^+(\overrightarrow{k}) e^{ikx} = e^{i\omega t} \psi_{\overrightarrow{k}}^+(\overrightarrow{x}), \tag{4.42}$$

which is eigen state for the operator H (4.22), a negative (complete relativistic) energy $E^- = -\sqrt{\vec{k}^2 + m^2}$ is corresponded that is the physical nonsense.

Remark. In order to overcome the last difficulty, Dirac proposed the concept of "vacuum", containing completely covered states with negative energy, and with the status of the unobserved substance. And positron introduced by him, he interpreted in a known way as a "hole" in such vacuum. Despite the useful historical and heuristic significance of these considerations of Dirac, the consistent physical and mathematical logic of quantum theory does not require the manipulation with such an unobserved substance.

One of the fundamental reasons of the shortcomings listed here is the attempt to use in the theory only manifestly relativistic invariant concepts in the local representation of a spinor field, such as a local representation of the group $\mathcal{P}$, a relativistic invariant Lebesgue-Stieltjes measure, etc. Therefore, the way out of such situation can be found in the rejection of manifestly local relativistic covariance (or local invariance) without any loss of the ordinary relativistic invariance of the theory of the spinor field in the modern definition of the invariance of the field equation in relation to one or another transformation.

All above-mentioned conceptual difficulties in quantum-mechanical interpretation of the free Dirac equation are also actual for the Dirac equation (4.32), which includes the interaction with external potential field $A = (A^\mu)$. It gives rise to known difficulties in the problem of obtaining a quasi-relativistic quantum-mechanical approximation based on the equation (4.32).

It is the search for a mathematically correct transformation for the diagonalisation of the Hamiltonian of the free Dirac equation in order to provide a consistent interpretation of the theory of the spinor field, which led to the FW transformation, canonical FW representation and the FW relativistic equation of motion. For the case of the free equation (4.22) there is an exact FW transformation, and for the equation (4.32) containing the interaction with the external electromagnetic field, the transformation exists only as an approximation in the form of the series in degrees $1/m$ [1] or as the series in degrees of charge e [12].

4.5. Advantages of the Foldy–Wouthuysen Representation in the Quantum-Mechanical Interpretation of the Dirac Theory

As shown in [1] the above considered shortcomings follow from the non-diagonality of the Hamiltonian $H = i\widetilde{p}_0$ in (4.12) and (4.22). Therefore, in the PD-representation of the field ψ, the particle and antiparticle states are mixed. The progressive way of moving forward was suggested in [1]. The FW-representation of the spinor field theory is free from the above shortcomings, e.g., the coordinate $\overrightarrow{x} = (x^\ell)$ of the FW-spinor ϕ and spin $\overrightarrow{s}$ (4.14) in this representation have direct physical meaning of corresponding quantum-mechanical observables of spin-$\frac{1}{2}$ doublet of particles. It is important that just the components of this spin $\overrightarrow{s}$ are the elements of the Clifford–Dirac algebra in the Pauli–Dirac representation. The above arguments force us [18, 19] to start from the FW representation of the spinor field (not from the Pauli–Dirac representation) and to consider the Clifford–Dirac algebra just inside the FW representation of the spinor field.

Thus, the way to overcoming of the above considered difficulties of the quantum-mechanical interpretation of the Dirac equation is in transition to the FW representation [1].

In the FW representation the spin operator $\overrightarrow{s}$ (4.14) commutes with the FW Hamiltonian

$$\left[\overrightarrow{s}, \gamma^0\widehat{\omega}\right] = 0. \tag{4.43}$$

Moreover, mean spin operator $\overrightarrow{s}_{\text{PD}}$ (4.27), which is found from the operator $\overrightarrow{s}$ (4.14) on the basis of FW transformation $\overrightarrow{s}_{\text{PD}} = V^+\overrightarrow{s}V^-$, commutes with the Dirac Hamiltonian

$$\left[\overrightarrow{s}_{\text{PD}}, \overrightarrow{\alpha} \cdot \widehat{\overrightarrow{p}} + \beta m\right] = 0. \tag{4.44}$$

As a consequence of (4.43) the situation with the equations (4.39) on the eigen vectors and eigen values of the spin projection operator $s_z = s^3$ from $\overrightarrow{s}$ (4.14) in the FW representation is improved as well. Indeed, the requirements of the foundations of quantum mechanics in the FW representation are fulfilled. It is easy to show that basis vectors $\{d_\alpha\}$ (4.29) of the 4-dimensional Cartesian basis are the eigen vectors of the spin projection operator $s_z = s^3$

$$s^3 d_1 = +\frac{1}{2}d_1, \quad s^3 d_2 = -\frac{1}{2}d_2, \quad s^3 d_3 = +\frac{1}{2}d_3, \quad s^3 d_4 = -\frac{1}{2}d_4. \tag{4.45}$$

Furthermore, as a consequence of (4.44) the situation with the equations on the eigen vectors and eigen values (4.39) is improved in the case of the mean spin projection operator s^3_{PD}, see (4.40). Therefore, the Dirac spinors $v^{\mp}_r(\overrightarrow{k})$ (4.31) are the eigen vectors of the mean spin projection operator s^3_{PD}, which is found from the operator s^3 in (4.14) on the basis of FW transformation $\overrightarrow{s}_{PD} = V^+ \overrightarrow{s} V^-$. Operator $V^{\pm}$ is given in (4.23).

The velocity operator in the FW representation is well defined

$$\dot{\overrightarrow{x}} = i\left[\gamma^0\widehat{\omega}, \overrightarrow{x}\right] = \gamma^0 \frac{\widehat{\overrightarrow{p}}}{\widehat{\omega}}. \tag{4.46}$$

Similarly, in the FW representation all other deficiencies of the quantum-mechanical interpretation of the Dirac equation are corrected. Moreover, if to start from the FW representation and to fulfill the inverse transformation to the Dirac model, then as a consequence the well-defined quantum-mechanical theory is obtained for the Dirac equation and the spinor field as well. Therefore, in our quest of the hidden Bose properties of the Dirac equation [18, 19] and [20, 21] we have started from the FW representation. For the same reason in [18–21] the gamma matrix representation of the standard Clifford algebra, which elements are linked $s_{\ell n} \equiv \frac{1}{4}[\gamma_\ell, \gamma_n]$ with the spin $s = 1/2$ (4.14), and Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ over the field of real numbers [22] we introduced into the FW representation.

Note that contrary to many authors, which investigated the FW transformation from Dirac to canonical representations (the generalization of the FW transformation for the fields of arbitrary spin, different variants of the FW transformation in the case of external interaction) [23–34], we have considered [18–21] the inverse FW transformation from the FW canonical representation to the Dirac model of the $s = 1/2$ spinor field.

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Chapter 5

Relativistic Canonical Quantum Mechanics of Arbitrary Spin

The axiomatic level description of the relativistic canonical quantum mechanics (RCQM) of the particle singlet and particle-antiparticle doublet of an arbitrary mass and spin has been given. The level of von Neumann's consideration of non-relativistic case is maintained. The introduced equations are without redundant components. The partial examples for spins s = 0, 1/2, 1, 3/2, 2 are presented.

5.1. Introduction to the Chapter 5

In previous chapter the advantages of the Foldy–Wouthuysen (FW) representation in quantum-mechanical description of the spin s=1/2 particle-antiparticle doublet in comparison with the Dirac model [1, 2] is demonstrated. Nevertheless, the transition to the FW representation is only the first step for quantum-mechanical interpretation of this model. Indeed, the positive and negative parts of solution, which lead to both positive and negative energies, here still exist [3, 4]. Moreover, in the FW representation the spin of the antiparticle is not the mirror reflection of the particle spin. These difficulties of the Dirac model are not overcome by the transition to the FW representation. Some other difficulties exist as well.

The complete quantum-mechanical picture of the spin s=1/2 particle-antiparticle doublet is possible in the framework of another model. We call

this model as the RCQM [5]. The foundations of this model of physical reality are presented below. We are able to fulfill this task for the particle singlet and particle-antiparticle doublet of an arbitrary mass and spin. The axiomatic level of von Neumann's [6] consideration of non-relativistic case is upheld.

The relativistic quantum mechanics under consideration is called *canonical* due to three main reasons. (i) The model under consideration has direct link with non-relativistic quantum mechanics based on non-relativistic Schrödinger equation. The principles of heredity and correspondence with other models of physical reality leads directly to non-relativistic Schrödinger quantum mechanics. (ii) The FW model is already called by many authors as the canonical representation of the Dirac equation or a canonical field model, see, e.g., the paper [4]. And the difference between the field model given by FW and the RCQM is minimal – in corresponding equations it is only the presence and absence of beta matrix. (iii) The list of relativistic quantum-mechanical models is long, see, e.g., the review [7]. The Dirac model and the FW model are called by the "'old'" physicists as the relativistic quantum mechanics as well (one of our tasks in this book is to show in visual and demonstrative way that these models have only weak quantum-mechanical interpretation). Further, the fractional relativistic quantum mechanics and the proper-time relativistic quantum mechanics can be listed (recall matrix formulation by W. Heisenberg, Feynman's sum over path's quantum theory, many-worlds interpretation by H. Everett), etc.

Therefore, in order to avoid a confusion the model under consideration must have its proper name. Due to the reasons (i)–(iii) the best name for it is RCQM.

Note that here only the first-order particle and the field equations (together with their canonical nonlocal pseudo-differencial representations) are considered. The second order equations (like the Klein–Gordon–Fock equation) are not the subject of this investigation.

The goals of this chapter are as follows: to formulate the axiomatic foundations of the RCQM of an arbitrary spin and mass, to demonstrate and explain this model on the examples of the spin s=0, s=1/2, s=1, s=3/2, s=2 singlets, spin s=(0,0), s=(1/2,1/2), s=(1,1), s=(3/2,3/2), s=(2,2) particle-antiparticle doublets, spin s=(1,0) multiplet and spin s=(1,0,1,0), s=(2,0,2,0), s=(2,1,2,1) particle-antiparticle multiplets.

This chapter plays a special role in our consideration. It is the starting platform of our program of **synthesis of covariant particle equations of arbitrary mass and spin**. On the basis of RCQM we are able to construct the canonical FW model of arbitrary spin and, finally, to derive the manifestly covariant

equations of arbitrary mass and spin.

Such procedure of synthesis of covariant particle equations of arbitrary mass and spin consists of three levels, which are linked as follows.

COVARIANT LOCAL FIELD THEORY OF ARBITRARY SPIN

↑

CANONICAL FIELD THEORY OF ARBITRARY SPIN

↑

RELATIVISTIC CANONICAL QUANTUM MECHANICS

OF ARBITRARY SPIN

This program has the beginning here, a continuation in the Chapter 6 (the FW type canonical field equations of arbitrary mass and spin) and is finished in the Chapter 7 as the covariant local field theory of arbitrary mass and spin.

5.2. Dirac's Comment

Note that the square-root operator equation (2.35), which is the main equation of RCQM, has been rejected by Dirac. In his consideration in [2] (chapter 11, section 67) of the main steps of [1] Dirac discussed this equation. His comment was as follows.

Let us consider first the case of the motion of an electron in the absence of an electromagnetic field, so that the problem is simply that of the free particle, as dealt with in §30, with the possible addition of internal degrees of freedom. The relativistic Hamiltonian provided by classical mechanics for this system is given by equation 23 of §30, and leads to the wave equation

$$\left\{ p_0 - \left(m^2 c^2 + p_1^2 + p_2^2 + p_3^2 \right)^{\frac{1}{2}} \right\} \psi = 0, \qquad (5)$$

where the p's are interpreted as operators in accordance with (4). Equation (5), although it takes into account the relation between energy and momentum required by relativity, is yet unsatisfactory from the point of view of relativistic theory, because it is very unsymmetrical between p_0 and the other p's, so much so that one cannot generalize it in a relativistic way to the case when there is a field present. We must therefore look for a new wave equation.

Today another reason of Dirac's rejection of the square-root operator equation is evident as well. The operations with the Dirac's Hamiltonian are too much easier than the operations with the pseudo-differential Hamiltonian of the equation

$$i\partial_0 f(x) = \sqrt{-\Delta + m^2} f(x) \tag{5.1}$$

for the N-component wave function

$$f \equiv \text{column}(f^1, f^2, ..., f^N), \quad N = 2s + 1, \quad f \in H^{3,N}, \tag{5.2}$$

in the case of particle singlet and for the M-component wave function (M = 2N = 2(2s + 1)) in the case of particle-antiparticle doublet. Here $H^{3,N}$ is the Hilbert space of N-component functions. Note that namely (5.1) is the equation of motion in the RCQM.

Nevertheless, today, contrary to the year 1928, the definition of the pseudo-differential (non-local) operator

$$\widehat{\omega} \equiv \sqrt{\widehat{\vec{p}}^2 + m^2} = \sqrt{-\Delta + m^2} \geq m > 0, \tag{5.3}$$

$$\widehat{\vec{p}} \equiv (\widehat{p}^j) = -i\nabla, \quad \nabla \equiv (\partial_\ell),$$

is well known. The action of the operator (5.3) in the coordinate representation (see, e.g., [8]) is given by

$$\widehat{\omega} f(t, \vec{x}) = \int d^3 y K(\vec{x} - \vec{y}) f(t, \vec{y}), \tag{5.4}$$

where the function $K(\vec{x} - \vec{y})$ has the form $K(\vec{x} - \vec{y}) = -\frac{2m^2 K_2(m|\vec{x} - \vec{y}|)}{(2\pi)^2 |\vec{x} - \vec{y}|^2}$ and $K_v(z)$ is the modified Bessel function (Macdonald function), $|\vec{a}|$ designates the norm of the vector $\vec{a}$.

Further, the following integral form

$$(\widehat{\omega} f)(t, \vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3 k e^{i\vec{k}\vec{x}} \widetilde{\omega} \widetilde{f}(t, \vec{k}), \tag{5.5}$$

$$\widetilde{\omega} \equiv \sqrt{\vec{k}^2 + m^2}, \quad \widetilde{f} \in \widetilde{H}^{3,N},$$

of the operator $\widehat{\omega}$ is used often, see, e.g., [4, 9], where f and $\widetilde{f}$ are linked by the 3-dimensional Fourier transformations

$$f(t, \vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3 k e^{i\vec{k}\vec{x}} \widetilde{f}(t, \vec{k}) \Leftrightarrow \tag{5.6}$$

$$\widetilde{f}(t, \overrightarrow{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{-i\overrightarrow{k}\,\overrightarrow{x}} \widetilde{f}(t, \overrightarrow{x}),$$

(in (5.6) $\overrightarrow{k}$ belongs to the spectrum $\mathrm{R}^3_{\overrightarrow{k}}$ of the operator $\widehat{\overrightarrow{p}}$, and the parameter $t \in (-\infty, \infty) \subset \mathrm{M}(1,3)$).

Note that the space of states $\mathrm{H}^{3,\mathrm{N}}$ is invariant with respect to the Fourier transformation (5.6). Therefore, both $\overrightarrow{x}$-realization $\mathrm{H}^{3,\mathrm{N}}$ and $\overrightarrow{k}$-realization $\widetilde{\mathrm{H}}^{3,\mathrm{N}}$ of the space of states are suitable for the purposes of our consideration. In the $\overrightarrow{k}$-realization the Schrödinger–Foldy equation has the algebraic-differential form

$$i\partial_t \widetilde{f}(t, \overrightarrow{k}) = \sqrt{\overrightarrow{k}^2 + m^2}\,\widetilde{f}(t, \overrightarrow{k}); \quad \overrightarrow{k} \in \mathrm{R}^3_{\overrightarrow{k}}, \quad \widetilde{f} \in \widetilde{\mathrm{H}}^{3,\mathrm{N}}. \tag{5.7}$$

Below in the places, where misunderstanding is impossible, the symbol "tilde" is omitted.

Thus, today on the basis of above given definitions the difficulties, which stopped Dirac in 1928, can be overcome.

5.3.   Foldy's Contribution

The name of the person, whose contribution in the theoretical model based on the equation (5.1) was decisive, is Leslie Lawrance Foldy (1919–2001).

His interesting biography is presented in [10]. *Les was born in Sabinov, Czechoslovakia, on 26 October 1919, into a family with Hungarian roots. His parents named him Laszlo Földi. In the turbulent times following World War I, he immigrated with his parents to the US in 1921. His father changed the family's last name and Les's first name; Les later added his middle name, unaware of its more common spellings.*

The first step of Foldy's contribution is visualization of the quantum mechanical interpretation of the Dirac equation on the basis of transformation to the canonical (quantum-mechanical) representation [3]. This transformation was suggested together with the Netherlander Siegfried (Sieg) Wouthuysen (pronounced Vout-high-sen). In this FW representation of the Dirac equation the quantum-mechanical interpretation is much more clear. Nevertheless, the direct and evident quantum-mechanical interpretation of the spin s=(1/2,1/2) particle-antiparticle doublet can be fulfilled only within the framework of the RCQM: the start was given in [4].

In our investigations we always marked the role of L. Foldy. Taking into account the L. Foldy's contribution in the construction of RCQM and his proof of the principle of correspondence between RCQM and non-relativistic quantum mechanics, we propose [11, 12] *to call the N-component equation (3) as the Schrödinger–Foldy equation*. Note here that equation (5.1), which is a direct sum of one component spinless Salpeter equations [13], has been introduced in the formula (21) of [4]. Furthermore, note here that the corresponding non-local Poincaré group representation generators, which determine the Poincaré symmetry of the equation (5.1), are known from the formulas (B-25)–(B-28) of the L. Foldy's paper [4].

5.4. On the Relativistic Canonical Quantum Mechanics Status Quo

Contrary to the times of papers [1, 3, 4, 13], the RCQM today is sufficiently tested and generally accepted theory. The spinless Salpeter equation has been introduced in [13]. The allusion on the RCQM and the first steps are given in [4], where the Salpeter equation for the 2s+1-component wave function was considered and the cases of s=1/2, s=1 were presented as an examples. In [14] L. Foldy continued his investigations [4] by the consideration of the relativistic particle systems with interaction. The interaction was introduced by the specific group-theoretical method.

After that in the RCQM were developed both the construction of mathematical foundations and the solution of concrete quantum-mechanical problems for different potentials. Some mathematical foundations and spectral theory of pseudo-differential operator $\sqrt{\widehat{\vec{p}}^{\,2} + m^2} - Ze^2/r$ were given in [15–18].

The application of the RCQM to the quark-antiquark bound state problem can be found in [19, 20]. The numerical solutions of the RCQM equation for arbitrary confining potentials were presented in [20]. In [21] the spinless Salpeter equation for the N particle system of spinless bosons in gravitational interaction was applied. In [22] a lower bound on the maximum mass of a boson star on the basis of the Hamiltonian $\sqrt{\widehat{\vec{p}}^{\,2} + m^2} - \alpha/r$ has been calculated.

In [23] results calculated by the author with the spinless Salpeter equation are compared with those obtained from Schrodinger's equation for heavy-quark systems, heavy-light systems, and light-quark systems. In each case the Salpeter

energies agree with experiment substantially better than the Schrodinger energies. The paper [24] deal with an investigation of the exact numerical solutions. The spinless Salpeter equation with the Coulomb potential is solved exactly in momentum space and is shown to agree very well with a coordinate-space calculation.

In [25, 26] the problem of spectrum of energy eigenvalues calculations on the basis of the spinless Salpeter equation is considered. The spinless relativistic Coulomb problem is studied. It was shown how to calculate, by some special choices of basis vectors in the Hilbert space of solutions, for the rather large class of power-law potentials, at least upper bounds on these energy eigenvalues. The authors of [25, 26] proved that for the lowest-lying levels, this may be done even analytically.

In the paper [27] the spinless Salpeter equation was rewritten into integral and integro-differential equations. Some analytical results concerning the spinless Salpeter equation and the action of the square-root operator have been presented. Further, in [28] F. Brau constructed an analytical solution of the one-dimensional spinless Salpeter equation with a Coulomb potential supplemented by a hard core interaction, which keeps the particle in the x positive region. In the context of RCQM based on the spinless Salpeter equation it was shown [29] how to construct a large class of upper limits on the critical value, $g_c^{(\ell)}$, of the coupling constant, g, of the central potential, $V(r) = -gv(r)$. In [30] a lower bounds on the ground state energy, in one and three dimensions, for the spinless Salpeter equation applicable to potentials, for which the attractive parts are in $L^p(R^n)$ for some p ¿ n (n = 1 or 3), are found. An extension to confining potentials, which are not in $L^p(R^n)$, is also presented.

In the paper [31], the authors used the theory of fractional powers of linear operators to construct a general (analytic) representation theory for the square-root energy operator $\gamma^0 \sqrt{\widehat{\vec{p}}^2 + m^2} + V$ of FW canonical field theory, which is valid for all values of the spin. The example of the spin 1/2 case, considering a few simple yet solvable and physically interesting cases, is presented in details in order to understand how to interpret the operator. Note that corresponding results for the RCQM can be found from the FW canonical field theory results [31] with the help of our transformation, see, e.g., operator (33) in [11] or added to the FW transformation part of (2.38).

Using the momentum space representation, the authors of [32] presented an analytical treatment of the one-dimensional spinless Salpeter equation with a

Coulomb interaction. The exact bound-state energy equation was determined. The results obtained were shown to agree very well with exact numerical calculations existing in the literature. In [33] an exact analytical treatment of the spinless Salpeter equation with a one dimensional Coulomb interaction in the context of quantum mechanics with modified Heisenberg algebra implying the existence of a minimal length was presented. The problem was tackled in the momentum space representation. The bound-state energy equation and the corresponding wave functions were exactly obtained. The probability current for a quantum spinless relativistic particle was introduced [8] based on the Hamiltonian dynamics approach using the spinless Salpeter equation. The correctness of the presented formalism was illustrated by examples of exact solutions to the spinless Salpeter equation including the new ones. Thus, in [8] the partial wave packet solutions of this equation have been presented: the solutions for free massless and massive particle on a line, for massless particle in a linear potential, plane wave solution for a free particle (this formula is given here below for N-component case as the solution of the equation (5.1)), the solution for free massless particle in three dimensions.

Further, in the paper [34] other time dependent wave packet solutions of the free spinless Salpeter equation are given. Taking into account the relation of such wave packets to the Lévy process the spinless Salpeter equation (in one dimensional space-time) is called in [34] as the Lévy-Schrödinger equation. The several examples of the characteristic behavior of such wave packets have been shown, in particular of the multimodality arising in their evolutions: a feature at variance with the typical diffusive unimodality of both the corresponding Lévy process densities and usual Schrödinger wave functions.

A generic upper bound is obtained [35] for the spinless Salpeter equation with two different masses. Analytical results are presented for systems relevant for hadronic physics: Coulomb and linear potentials when a mass is vanishing. A detailed study for the classical and the quantum motion of a relativistic massless particle in an inverse square potential has been presented recently in [36]. The quantum approach to the problem was based on the exact solution of the corresponding spinless Salpeter equation for bound states. Finally, in [36] the connection between the classical and the quantum descriptions via the comparison of the associated probability densities for momentum has been made.

The goal of the recent paper [37] is a comprehensive analysis of the intimate relationship between jump-type stochastic processes (e.g., Lévy flights) and nonlocal (due to integro-differential operators involved) quantum dynam-

ics. In [37] a special attention is paid to the spinless Salpeter (here, $m \geq 0$) equation and the evolution of various wave packets, in particular to their radial expansion in 3D. Foldy's synthesis of *covariant particle equations* is extended to encompass free Maxwell theory, which however is devoid of any *particle* content. Links with the photon wave mechanics are explored. The authors of [37] takes into account our results [11] presented also in more earlier preprint, see the last reference in [37].

In the papers [11, 12], where we started our investigations in RCQM, this relativistic model for the test case of the spin s=(1/2,1/2) particle-antiparticle doublet is formulated. In [11], this model is considered as the system of the axioms on the level of the von Neumann monograph [38], where the mathematically well-defined consideration of the nonrelativistic quantum mechanics was given. Furthermore, in [11, 12] the operator link between the spin s=(1/2,1/2) particle-antiparticle doublet RCQM and the Dirac theory is given and Foldy's synthesis of *covariant particle equations* is extended to the start from the RCQM of the spin s=(1/2,1/2) particle-antiparticle doublet. In [5, 38–40] the same procedure is fulfilled for the spin s=(1,1), s=(1,0,1,0), s=(3/2,3/2), s=(2,2), s=(2,0,2,0) and spin s=(2,1,2,1) RCQM. The corresponding equations, which follow from the RCQM for the covariant local field theory, are introduced.

Therefore, here and in [5, 38–40] we are not going to formulate a new relativistic quantum mechanics! The foundations of RCQM based on the spinless Salpeter equation are already formulated in [7, 8, 11–37].

5.5. Axiomatic Foundations of the Relativistic Canonical Quantum Mechanics of an Arbitrary Spin

The RCQM of the particle singlet and the particle-antiparticle doublet of an arbitrary spin was suggested in sections 2 and 18 of [38]. This model can be formulated at the level of von Neumann's consideration [6]. The difference is only in relativistic invariance and in the consideration of multi-component and multidimensional objects.

We start in section 7 of [38] from the example of spin s=1/2 particle-antiparticle doublet. The RCQM of the arbitrary spin particle-antiparticle doublet (or particle singlet) can be formulated similarly as the corresponding gen-

eralization of this partial case.

Below the brief presentation of the *list of the axioms* is given. Note that some particular content of these axioms is already given in section 2 of [38], where the beginning of this investigation was laid.

5.5.1. On the Space of States

The space of states of isolated arbitrary spin particle singlet in an arbitrarily-fixed inertial frame of reference (IFR) in its $\overrightarrow{x}$-realization is the Hilbert space

$$\mathrm{H}^{3,\mathrm{N}} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes \mathrm{N}} = \{ f = (f^{\mathrm{N}}) : \mathrm{R}^3 \to \mathrm{C}^{\otimes \mathrm{N}}; \tag{5.8}$$

$$\int d^3x |f(t,\overrightarrow{x})|^2 < \infty \}, \quad \mathrm{N} = 2s+1,$$

of complex-valued N-component square-integrable functions of $x \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (similarly, in momentum, $\overrightarrow{p}$-realization). In (5.8) d^3x is the Lebesgue measure in the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ of the eigenvalues of the position operator $\overrightarrow{x}$ of the Cartesian coordinate of the particle in an arbitrary-fixed IFR. Further, $\overrightarrow{x}$ and $\widehat{\overrightarrow{p}}$ are the operators of canonically conjugated dynamical variables of the spin s=(1/2,1/2) particle-antiparticle doublet, and the vectors f, $\tilde{f}$ in $\overrightarrow{x}$- and $\overrightarrow{p}$-realizations are linked by the 3-dimensional Fourier transformation (the variable t is the parameter of time-evolution).

The mathematical correctness of the consideration demands the application of the rigged Hilbert space

$$\mathrm{S}^{3,\mathrm{N}} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^{\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}, \tag{5.9}$$

where the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$ is the core (i.e., it is dense both in $\mathrm{H}^{3,\mathrm{N}}$ and in the space $\mathrm{S}^{3,\mathrm{N}*}$ of the N-component Schwartz generalized functions). The space $\mathrm{S}^{3,\mathrm{N}*}$ is conjugated to that of the Schwartz test functions $\mathrm{S}^{3,\mathrm{N}}$ by the corresponding topology (see, e.g., [41]).

Strictly speaking, the mathematical correctness of consideration demands to make the calculations in the space $\mathrm{S}^{3,\mathrm{N}*}$ of generalized functions, i.e., with the application of cumbersome functional analysis (see, e.g., [42]). Nevertheless, one can take into account the properties of the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$ in the triple (5.9). The space $\mathrm{S}^{3,\mathrm{N}}$ is dense both in quantum-mechanical space $\mathrm{H}^{3,\mathrm{N}}$ and in the space of generalized functions $\mathrm{S}^{3,\mathrm{N}*}$. Therefore, any physical state $f \in \mathrm{H}^{3,\mathrm{N}}$ can be approximated with an arbitrary precision by the corresponding elements of the Cauchy sequence in $\mathrm{S}^{3,\mathrm{N}}$, which converges to the given

$f \in \mathrm{H}^{3,\mathrm{N}}$. Further, taking into account the requirement to measure the arbitrary value of the quantum-mechanical model with non-absolute precision, it means that all concrete calculations can be fulfilled within the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$. Thus, such consideration allows us to perform, without any loss of generality, all necessary calculations in the space $\mathrm{S}^{3,\mathrm{N}}$ at the level of correct differential and integral calculus.

Furthermore, the mathematical correctness of the consideration demands to determine the domain of definitions and the range of values for any used operator and for the functions of operators. Note that if the kernel space $\mathrm{S}^{3,\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}}$ is taken as the common domain of definitions of the generating operators $\overrightarrow{x} = (x^j)$, $\overrightarrow{\widehat{p}} = (\widehat{p}^j)$, $\overrightarrow{s} \equiv (s^j) = (s_{23}, s_{31}, s_{12})$ of coordinate, momentum and spin, respectively, then this space appears to be also the range of their values. Moreover, the space $\mathrm{S}^{3,\mathrm{N}}$ appears to be the common domain of definitions and values for the set of all below mentioned functions from the 9 operators $\overrightarrow{x} = (x^j)$, $\overrightarrow{\widehat{p}} = (\widehat{p}^j)$, $\overrightarrow{s} \equiv (s^j)$ (for example, for the generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ of the irreducible unitary representations of the Poincaré group $\mathcal{P}$ and for different sets of commutation relations). Therefore, in order to guarantee the realization of the principle of correspondence between the results of cognition and the instruments of cognition in the given model, it is sufficient to take the algebra $\mathrm{A_S}$ of the all sets of observables of the given model in the form of converged in $\mathrm{S}^{3,\mathrm{N}}$ Hermitian power series of the 9 generating operators $\overrightarrow{x} = (x^j)$, $\overrightarrow{\widehat{p}} = (\widehat{p}^j)$, $\overrightarrow{s} \equiv (s^j)$.

Note that the Schrödinger–Foldy equation (5.1) has generalized solutions, which do not belong to the space $\mathrm{H}^{3,\mathrm{N}}$ (5.8). Therefore, the application of the rigged Hilbert space $\mathrm{S}^{3,\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}$ (5.9) is necessary.

Some other details of motivations of the choice of the spaces (5.8), (5.9) (and all necessary notations) are given in [12], where the corresponding 4-component spaces are considered.

Note finally that in the case of arbitrary spin particle-antiparticle doublet the dimension of spaces (5.8), (5.9) is M=2N=2(2s+1).

5.5.2. On the Time Evolution of the State Vectors

The time dependence of the state vectors $f \in \mathrm{H}^{3,\mathrm{N}}$ (time t is the parameter of evolution) is given either in the integral form by the unitary operator

$$u(t_0, t) = \exp[-i\widehat{\omega}(t - t_0)]; \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \qquad (5.10)$$

(below $t_0 = t$ is put), or in the differential form by the Schrödinger–Foldy equation of motion (5.1) with the wave function (5.2). In terms of operator (5.3)–(5.5) this equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0. \tag{5.11}$$

Note that here the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is the relativistic analog of the energy operator (Hamiltonian) of nonrelativistic quantum mechanics. The Minkowski space-time M(1,3) is pseudo Euclidean with metric $g = \mathrm{diag}(+1, -1, -1, -1)$.

The step from the particle singlet of arbitrary spin to the corresponding particle-antiparticle doublet is evident.

Thus, for the arbitrary spin particle-antiparticle doublet the system of two N-component equations $(i\partial_0 - \widehat{\omega})f(x) = 0$ and $(i\partial_0 - \widehat{\omega})f(x) = 0$ is used. Therefore, the corresponding Schrödinger–Foldy equation is given by (5.11), where the 2N-component wave function is the direct sum of the particle and antiparticle wave functions, respectively. Due to the historical tradition of the physicists the antiparticle wave function is put in the down part of the 2N-column.

The general solution of the Schrödinger–Foldy equation of motion (5.11) (in the case of particle-antiparticle arbitrary spin doublet) has the form

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{-ikx} a^{2N}\left(\overrightarrow{k}\right) d_{2N}, \tag{5.12}$$

$$kx \equiv \widetilde{\omega}t - \overrightarrow{k}\,\overrightarrow{x}, \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2},$$

where the orts of the N-dimensional Cartesian basis are given by

$$d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{vmatrix}, ..., d_N = \begin{vmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{vmatrix}. \tag{5.13}$$

The action of the pseudo-differential (non-local) operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is explained in (5.4), (5.5).

5.5.3.   On the Fundamental Dynamical Variables

The dynamical variable $\vec{x} \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (as well as the variable $\vec{k} \in \mathrm{R}^3_{\vec{k}}$) represents the external degrees of freedom of the arbitrary spin particle-antiparticle doublet. The spin $\vec{s}$ of the particle-antiparticle doublet is the first in the list of the carriers of the internal degrees of freedom. Taking into account the Pauli principle and the fact that experimentally an antiparticle is observed as the mirror reflection of a particle, the operators of the charge sign and the spin of the arbitrary particle-antiparticle doublet are taken in the form

$$g \equiv -\Gamma^0_{2N} \equiv -\sigma^3_{2N} = \left| \begin{matrix} -\mathrm{I}_N & 0 \\ 0 & \mathrm{I}_N \end{matrix} \right|, \quad \vec{s}_{2N} = \left| \begin{matrix} \vec{s}_N & 0 \\ 0 & -\hat{C}\vec{s}_N\hat{C} \end{matrix} \right|, \quad N = 2s+1, \quad (5.14)$$

where Γ^0_{2N} is the $2N \times 2N$ Dirac Γ^0 matrix, σ^3_{2N} is the $2N \times 2N$ Pauli σ^3 matrix, $\hat{C}$ is the operator of complex conjugation in the form of $N \times N$ diagonal matrix, the operator of involution in $\mathrm{H}^{3,2N}$, and I_N is $N \times N$ unit matrix.

Thus, the spin is given by the generators of SU(2) algebra!

The spin matrices $\vec{s}_{2N}$ from (5.14) satisfy the commutation relations

$$\left[s^j_{2N}, s^\ell_{2N} \right] = i\varepsilon^{j\ell n} s^n_{2N}, \quad \varepsilon^{123} = +1, \quad (5.15)$$

of the algebra of SU(2) group, where $\varepsilon^{j\ell n}$ is the Levi-Civita tensor and $s^j = \varepsilon^{j\ell n}s_{\ell n}$ are the Hermitian $2N \times 2N$ matrices (5.14) – the generators of a 2N-dimensional reducible representation of the spin group SU(2) (universal covering of the SO(3)$\subset$SO(1,3) group). The spin matrices $\vec{s}_N$ of the singlet satisfy the SU(2) commutation relations as well.

The Casimir operator for the RCQM representation of SU(2) spin given in (5.14) has the form

$$\vec{s}^{\,2}_{2N} = 2s(s+1)\mathrm{I}_{2N}, \quad (5.16)$$

where I_{2N} is $2N \times 2N$ unit matrix.

Above in the text of this axiom the case of arbitrary spin *particle-antiparticle doublet* has been considered. For the case of arbitrary spin *particle singlet* the operator of the charge sign is absent. The spin operator $\vec{s}_N$ is given by the Hermitian $N \times N$ matrices – the generators of a N-dimensional irreducible representation of the spin group SU(2). Therefore, the SU(2) commutation relations in such notations have the explicit form

$$\left[s^j_N, s^\ell_N \right] = i\varepsilon^{j\ell n} s^n_N, \quad \varepsilon^{123} = +1, \quad (5.17)$$

and the corresponding Casimir operator is given by

$$\overrightarrow{s}_N^2 = s(s+1)I_N. \tag{5.18}$$

The validity of these assertions is proved by numerous partial examples presented below.

5.5.4. On the External and Internal Degrees of Freedom

The coordinate $\overrightarrow{x}$ (as an operator in $H^{3,2N}$) is an analog of the discrete index of generalized coordinates $q \equiv (q_1, q_2, ...)$ in non-relativistic quantum mechanics of the finite number degrees of freedom. In other words the coordinate $\overrightarrow{x} \in R^3 \subset M(1,3)$ is the continuous carrier of the external degrees of freedom of a multiplet (the similar consideration was given in [43]). The coordinate operator together with the operator $\widehat{\overrightarrow{p}}$ determines the operator $m_{ln} = x_l\widehat{p}_n - x_n\widehat{p}_l$ of an orbital angular momentum, which also is connected with the external degrees of freedom.

However, the RCQM doublet has the additional characteristics such as the spin operator $\overrightarrow{s}$ from (5.14), which is the carrier of the internal degrees of freedom of this multiplet. The set of generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ (considered below in (5.20), (5.21)) of the main dynamical variables (see also formulas (73) in [38]) of the doublet are the functions of the following basic set of 9 functionally independent operators

$$\overrightarrow{x} = (x^j), \quad \widehat{\overrightarrow{p}} = (\widehat{p}^j), \quad \overrightarrow{s}_{2N} \equiv \left(s_{2N}^j\right) = (s_{23}, s_{31}, s_{12}). \tag{5.19}$$

Note that spin $\overrightarrow{s}_{2N}$ from (5.14) commutes both with $(\overrightarrow{x}, \widehat{\overrightarrow{p}})$ and with the operator $i\partial_t - \sqrt{-\Delta + m^2}$ of the Schrödinger–Foldy equation (5.1). Thus, for the free doublet the external and internal degrees of freedom are independent. Therefore, 9 operators (5.19) in $H^{3,2N}$, which have the univocal physical sense, are the *generating* operators not only for the 10 $\mathcal{P}$ generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ (5.20), (5.21), but also for other operators of any experimentally observable quantities of the doublet.

5.5.5. On the Algebra of Observables

Using the operators of canonically conjugated coordinate $\overrightarrow{x}$ and momentum $\overrightarrow{p}$ (where $\left[x^j, \widehat{p}^\ell\right] = i\delta^{j\ell}, \quad \left[x^j, x^\ell\right] = \left[\widehat{p}^j, \widehat{p}^\ell\right] = 0,$) in $H^{3,2N}$, being completed by

the operators $\vec{s}_{2N}$ and g from (5.14), we construct the algebra of observables as the Hermitian functions of 10 ($\vec{x}$, $\vec{p}$, $\vec{s}_{2N}$, $-\Gamma^0_{2N}$) generating elements of the algebra.

5.5.6.   On the Relativistic Invariance of the Theory

The relativistic invariance of the model under consideration (the relativistic invariance of the Schrödinger–Foldy equation (5.11)) requires, as a first step, consideration of its invariance with respect to the proper ortochronous Lorentz $L^\uparrow_+ =$ SO(1,3)=$\{\Lambda = (\Lambda^\mu_\nu)\}$ and Poincaré $P^\uparrow_+ = T(4) \times) L^\uparrow_+ \supset L^\uparrow_+$ groups. This invariance in an arbitrary relativistic model is the implementation of the Einstein's relativity principle in the special relativity form. Note that the mathematical correctness requires the invariance mentioned above to be considered as the invariance with respect to the *universal coverings* $L = $ SL(2,C) and $P \supset L$ of the groups $L^\uparrow_+$ and $P^\uparrow_+$, respectively.

The main information on the group P, which is necessary for our consideration, is collected in the Chapter 3. For the group P we choose real parameters: the 4-vector of translations $a = (a^\mu) \in$ M(1,3) and the tensor $\omega \equiv (\omega^{\mu\nu} = -\omega^{\nu\mu})$ with well-known physical meaning. For the standard P generators (p^{stand}_μ, $j^{\text{stand}}_{\mu\nu}$) (3.6) we use commutation relations in the manifestly covariant form (3.4).

The following assertion should be noted. Not a matter of fact that *non-covariant* objects such as the Lebesgue measure d^3x and *non-covariant (non-Lie)* generators of algebras are explored, the model of RCQM of arbitrary spin is a relativistic invariant in the following sense. The Schrödinger–Foldy equation (5.11) and the set of its solution $\{f\}$ (5.12) are invariant with respect to the irreducible unitary representation of the group P, the N $\times$ N matrix-differential generators of which for an arbitrary spin are given by the following non-local operators

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \tag{5.20}$$

$$\widehat{j}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n},$$

$$\widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \check{s}_\ell\right), \tag{5.21}$$

where the orbital parts of the generators are not changed under the transition from one spin value to another. Under such transitions only the spin parts of the expressions (5.20), (5.21), which are determined by the explicit form of spin

$\overrightarrow{s} = (s^{\ell n})$ from (5.14), are changed. Indeed, the direct calculations visualize that generators (5.20), (5.21) commute with the operator of equation (5.11) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$. In formulas (5.20), (5.21), the SU(2)-spin generators $s^{\ell n}$ have particular specific forms for each representation of the SU(2) group (see the list of examples below).

Note that the generators (5.20), (5.21) are known from the formulae (B-25) (B-28) of the paper [4].

Thus, the irreducible unitary representation of the Poincaré group $\mathcal{P}$ in the space (5.9), with respect to which the Schrödinger-Foldy equation (5.11) and the set of its solution $\{f\}$ (5.12) are invariant, is given by a series converges in this space

$$(a, \omega) \rightarrow U(a, \omega) = \exp(-ia^0 \widehat{p}_0 - i\overrightarrow{a}\,\widehat{\overrightarrow{p}} - \frac{i}{2}\omega^{\mu\nu}\widehat{j}_{\mu\nu}) \qquad (5.22)$$

where the generators $(\widehat{p}^\mu, \widehat{j}^{\mu\nu})$ are given in (5.20), (5.21) with the arbitrary values of the SU(2) spins $\overrightarrow{s} = (s^{\ell n})$ (5.14), (5.15).

The validity of this assertion is verified by the following three steps. (i) The calculation that the $\mathcal{P}$-generators (5.20), (5.21) commute with the operator $i\partial_0 - \widehat{\omega}$ of the Schrödinger–Foldy equation (5.11). (ii) The verification that the $\mathcal{P}$-generators (5.20), (5.21) satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$. (iii) The proof that generators (5.20), (5.21) realize the spin s(s+1) irreducible representation of this group (or spin 2s(s+1) reducible representation in the case of doublet). Therefore, the Bargman–Wigner classification on the basis of the corresponding Casimir operators calculation should be given. These three steps can be made by direct and non-cumbersome calculations.

The expression (5.22) is well known, but rather formal. In fact the transition from a Lie algebra to a finite group transformations in the case of non-Lie generators is a rather non-trivial action. The mathematical justification of (5.22) can be fulfilled in the framework of Schwartz test function space and will be given in next special publication.

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_N, \qquad (5.23)$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = s(s+1)m^2 I_N, \qquad (5.24)$$

where w_μ is the Pauli–Lubanski pseudovector, I_N is the $N \times N$ unit matrix and s $=1/2, 1, 3/2, 2, ...$

Note that together with the generators (5.20), (5.21) another set of 10 operators commutes with the operator of equation (5.11), satisfies the commutation relations (3.4) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (5.20) together with the orbital parts of the generators $\widetilde{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (5.21). Thus, this second set of Poincaré generators is given by

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \tag{5.25}$$

$$\widehat{m}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell, \quad \widehat{m}_{0\ell} = -\widehat{m}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\}.$$

It is evident that in the case s=0 only generators (5.25) form the Poincaré symmetry.

Note that the modern definition of $\mathcal{P}$ invariance (or $\mathcal{P}$ symmetry) of the equation of motion (5.11) in $H^{3,N}$ is given by the following assertion, see, e.g., [44]. *The set* $F \equiv \{f\}$ *of all possible solutions of the equation (5.11) is invariant with respect to the $\mathcal{P}^f$-representation of the group $\mathcal{P}$, if for arbitrary solution f and arbitrarily-fixed parameters (a, ω) of $\mathcal{P}$-transformation the assertion*

$$(a, \omega) \to U(a, \omega)\{f\} = \{f\} \equiv F \tag{5.26}$$

is valid.

Note further that in general the above given definition is valid for arbitrary equation of mathematical physics and arbitrary transformation, which is analyzed for the symmetry property.

In spite of the fact that in RCQM many manifestly non-covariant objects are used, *the model under consideration is relativistic invariant in the sense of the definition given above*.

5.5.7. On the Dynamic and Kinematic Aspects of the Relativistic Invariance

Consider briefly some detalizations of the relativistic invariance of the Schrödinger–Foldy equation (5.11). Note that for the free particle-antiparticle doublet of arbitrary spin the equation (5.11) has one and the same explicit form

in arbitrary-fixed IFR (its set of solutions is one and the same in every IFR). Therefore, the algebra of observables and the conservation laws (as the functionals of the free particle-antiparticle doublet states) have one and the same form too. This assertion explains the dynamical sense of the $\mathcal{P}$ invariance (the invariance with respect to the dynamical symmetry group $\mathcal{P}$).

Another, kinematic, aspect of the $\mathcal{P}$ invariance of the RQCM model has the following physical sense. Note at first that any solution of the Schrödinger–Foldy equation (5.11) is determined by the concrete given set of the amplitudes $\{A\}$. It means that if f with the fixed set of amplitudes $\{A\}$ is the state of the doublet in some arbitrary IFR, then for the observer in the (a, ω)-transformed IFR′ this state f' is determined by the amplitudes $\{A'\}$. The last ones are received from the given $\{A\}$ by the unitary $\mathcal{P}^A$-transformation (5.22).

5.5.8.   On the Clifford–Dirac Algebra

This axiom is additional and is not necessary. Nevertheless, such axiom is very useful for the dimensions, where the Γ matrices exist.

Application of the Clifford–Dirac algebra is the useful method of calculations in RCQM. Three different definitions of the Clifford algebra and their equivalence are considered in [45]. In different approaches to the relativistic quantum mechanics the matrix representation of the Clifford algebra in terms of the Dirac gamma matrices is used. This representation is called the Clifford–Dirac algebra. We use also the gamma matrix representation of 64-dimensional extended Clifford algebra over the field of real numbers. This our generalization of Clifford–Dirac algebra is considered in the Chapter 1.

For our purposes the anti-commutation relations of the Clifford–Dirac algebra generators are taken in the general form

$$\Gamma_{2N}^{\bar{\mu}}\Gamma_{2N}^{\bar{\nu}} + \Gamma_{2N}^{\bar{\nu}}\Gamma_{2N}^{\bar{\mu}} = 2g^{\bar{\mu}\bar{\nu}}; \qquad (5.27)$$

$$\bar{\mu}, \bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+ - - -),$$

where $\Gamma_{2N}^{\bar{\mu}}$ are the 2N × 2N Dirac $\Gamma^{\bar{\mu}}$ matrices (2N × 2N generalization of the Dirac 4 × 4 γ matrices), $\Gamma_{2N}^4 \equiv \Gamma_{2N}^0\Gamma_{2N}^1\Gamma_{2N}^2\Gamma_{2N}^3$. Here and in our publications (see, e.g., the articles [46–53]) we use the $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$ matrix instead of the γ^5 matrix of other authors. Our γ^4 is equal to $i\gamma^5_{standard}$. Notation γ^5 is used in [46–53] for a completely different matrix $\gamma^5 \equiv \gamma^1\gamma^3 C$. As well as the element $\Gamma_{2N}^4 \equiv \Gamma_{2N}^0\Gamma_{2N}^1\Gamma_{2N}^2\Gamma_{2N}^3$ of (5.27) is dependent the algebra basis is formed by

4=1+3 independent elements. Therefore, such Clifford algebra over the field of complex numbers is denoted $C\ell^{\mathbb{C}}(1,3)$ and the dimension of the algebra is $2^4 = 16$.

Note that relations (5.27) are valid only for the dimensions, where the Γ_{2N} are defined (where the matrix representation of the Clifford algebra exists). Therefore, the relations (5.27) can be useful not for all multiplets of RCQM. Nevertheless, existing relations (5.27) are very useful in order to operate with spins, with standard FW transformation and in order to formulate the FW transformation generalizations for the dimensions 2(2s+1)>4.

It has been explained in [46–53] that the Clifford–Dirac algebra should be introduced into consideration in the FW representation [3] of the spinor field. The reasons are as follows. Part of the Clifford–Dirac algebra operators are directly related to the spin 1/2 doublet operators $(s^1 \equiv \frac{1}{2}\gamma^2\gamma^3, s^2 \equiv \frac{1}{2}\gamma^3\gamma^1, s^3 \equiv \frac{1}{2}\gamma^1\gamma^2)$ (in the anti-Hermitian form). In the FW representation for the spinor field [3] these spin operators commute with the Hamiltonian and with the operator of the FW equation of motion $i\partial_0 - \gamma^0\hat{\omega}$. In the Pauli–Dirac representation these operators do not commute with the Dirac equation operator. Only the sums of the orbital operators and such spin operators commute with the Diracian. So *if we want to relate the orts γ^μ of the Clifford–Dirac algebra with the actual spin we must introduce this algebra into the FW representation.*

Of course, the Clifford–Dirac algebra can be introduced into the RCQM as well. This opportunity is equal to the above considered. We choose the first one and introduce the Clifford–Dirac algebra into the FW representation because it is closer to the Dirac model. Indeed, the FW and Dirac representations are linked by the unitary FW operator.

Here in general N-dimensional formalism of RCQM the situation is similar. The anti-commutation relations of the Clifford–Dirac algebra generators (5.27) and corresponding Γ_{2N} matrices must be introduced in the FW representation. Therefore, in order to apply (5.27) in RCQM one must transform the Γ_{2N} matrices from the FW representation into the RCQM representation. Corresponding operator transformation is given by

$$v_{2N} = \begin{vmatrix} I_N & 0 \\ 0 & \hat{C}I_N \end{vmatrix}, \quad v_{2N}^{-1} = v_{2N}^\dagger = v_{2N}, \tag{5.28}$$

$$v_{2N}v_{2N} = I_{2N}, \quad N = 2s+1,$$

where $\hat{C}I_N$ is the $N \times N$ operator of complex conjugation. Indeed, the operator (5.28) translates any operator from canonical field FW representation into the

RCQM representation and vice versa:

$$v_{2N}\hat{q}_{cf}^{anti-Herm}v_{2N} = \hat{q}_{qm}^{anti-Herm}, \tag{5.29}$$

$$v_{2N}\hat{q}_{qm}^{anti-Herm}v_{2N} = \hat{q}_{cf}^{anti-Herm}.$$

Here $\hat{q}_{qm}^{anti-Herm}$ is an arbitrary operator from the RCQM of the 2N-component particle-antiparticle doublet in the anti-Hermitian form, e.g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion (5.11), the operator of spin $\vec{s}_{2N}$ (5.14) taken in anti-Hermitian form, etc., $\hat{q}_{cf}^{anti-Herm}$ is an arbitrary operator from the canonical field theory of the 2N-component particle-antiparticle doublet in the anti-Hermitian form. Thus, the only warning is that operators here must be taken in anti-Hermitian form, see Chapter 3 for the details and see [54, 55] for the mathematical correctness of anti-Hermitian operators application (note that contrary to [56, 57], where non-Hermitian operators have been considered, we appeal in [5, 11, 12, 38–40] and here only to anti-Hermitian operators and corresponding Foldy–Wouthuysen transformation).

Further, the operator (5.28) translates

$$\phi = v_{2N}f, \quad f = v_{2N}\phi, \tag{5.30}$$

the solution (5.12) of the Schrödinger–Foldy equation (5.11) into the solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx}a^N(\vec{k})d_N + e^{ikx}a^{*\check{N}}(\vec{k})d_{\check{N}} \right], \tag{5.31}$$

$N = 1, 2, ..., N, \quad \check{N} = N+1, N+2, ..., 2N$, of the FW equation

$$(i\partial_0 - \Gamma_{2N}^0\widehat{\omega})\phi(x) = 0, \quad \Gamma_{2N}^0 \equiv \sigma_{2N}^3 = \begin{vmatrix} I_N & 0 \\ 0 & -I_N \end{vmatrix}, \tag{5.32}$$

$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad N = 2s+1$, and vice verso. Thus, the transformation (5.28), (5.29) translates the matrices Γ_{2N}^0 and

$$\Gamma_{2N}^j = \begin{vmatrix} 0 & \Sigma_N^j \\ -\Sigma_N^j & 0 \end{vmatrix}, \quad \Gamma_{2N}^4 = \Gamma_{2N}^0\Gamma_{2N}^1\Gamma_{2N}^2\Gamma_{2N}^3, \quad j = 1, 2, 3, \tag{5.33}$$

into the RCQM representation

$$\bar{\Gamma}_{2N}^{\bar{\mu}} = v_{2N}\Gamma_{2N}^{\bar{\mu}}v_{2N}, \tag{5.34}$$

where matrices $\bar{\Gamma}^{\bar{\mu}}_{2N}$ satisfy the anti-commutation relations

$$\bar{\Gamma}^{\bar{\mu}}_{2N}\bar{\Gamma}^{\bar{\nu}}_{2N} + \bar{\Gamma}^{\bar{\nu}}_{2N}\bar{\Gamma}^{\bar{\mu}}_{2N} = 2g^{\bar{\mu}\bar{\nu}}; \quad \bar{\mu},\bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+----), \quad (5.35)$$

of the Clifford–Dirac algebra generators as well. In (5.33) Σ^j_N are the $N \times N$ Pauli matrices. *Note that Pauli matrices exist only if* the values of N are even. The explicit forms of the RCQM representation of the $\bar{\Gamma}^{\bar{\mu}}_{2N}$ matrices are given by

$$\bar{\Gamma}^0_{2N} = \Gamma^0_{2N}, \quad \bar{\Gamma}^1_{2N} = \Gamma^1_{2N}\widehat{C}, \quad \bar{\Gamma}^2_{2N} = \Gamma^0_{2N}\Gamma^2_{2N}\widehat{C}, \quad (5.36)$$

$$\bar{\Gamma}^3_{2N} = \Gamma^3_{2N}\widehat{C}, \quad \bar{\Gamma}^4_{2N} = \Gamma^0_{2N}\Gamma^4_{2N}\widehat{C},$$

where $\widehat{C}$ is the $2N \times 2N$ operator of complex conjugation and matrices $\Gamma^{\bar{\mu}}_{2N}$ are given in (5.32), (5.33).

Note that in the terms of $\bar{\Gamma}^{\bar{\mu}}_{2N}$ matrices (5.36) the RCQM spin operator (5.14) has the form

$$\overrightarrow{s} = \frac{i}{2}(\bar{\Gamma}^2_{2N}\bar{\Gamma}^3_{2N}, \bar{\Gamma}^3_{2N}\bar{\Gamma}^1_{2N}, \bar{\Gamma}^1_{2N}\bar{\Gamma}^2_{2N}). \quad (5.37)$$

Note further that formula (5.37) is valid for the multiplets of arbitrary dimension but only for the spin s=1/2, whereas the formula (5.14) is valid for arbitrary spin. Note also that in some dimensions the gamma matrices can not be defined. Furthermore, the complete analogy between the (5.37) and the particle-antiparticle spin s=1/2 doublet of arbitrary dimension in the FW representation exists

$$\overrightarrow{s}_{FW} = \frac{i}{2}(\Gamma^2_{2N}\Gamma^3_{2N}, \Gamma^3_{2N}\Gamma^1_{2N}, \Gamma^1_{2N}\Gamma^2_{2N}). \quad (5.38)$$

It is very useful to consider a wider then $C\ell^{\mathbb{C}}(1,3)$ Clifford–Dirac algebra. In [46–53] such additional matrix representations of the Clifford algebras have been introduced for the purposes of finding links between the fermionic and bosonic states of the spinor field. New extended representation can be formed by the generators of $C\ell^{\mathbb{C}}(1,3)$ together with the generators of the Pauli–Gürsey–Ibragimov algebra [58–60]. Here this new mathematical object (new gamma matrix representation of the Clifford algebra) is considered in the Chapter 1.

The main structure elements of such set are given by $(\gamma^0, \gamma^1, \gamma^2, \gamma^3, i, \widehat{C}I_4)$, where γ^μ are 4×4 Dirac matrices in standard representation. It is easy to see that simplest set of the Clifford–Dirac algebra generators can be constructed from these elements in the form $(i\gamma^0, i\gamma^1, \gamma^2, i\gamma^3, \widehat{C}I_4, i\widehat{C}I_4)$. Therefore, the $2N \times 2N$

matrix generators of the corresponding Clifford–Dirac algebra over the field of real numbers can be found by the simple redefinition

$$\tilde{\Gamma}^1_{2N} \equiv i\Gamma^1_{2N}, \quad \tilde{\Gamma}^2_{2N} \equiv i\Gamma^3_{2N}, \quad \tilde{\Gamma}^3_{2N} \equiv \widehat{C}I_{2N}, \tag{5.39}$$

$$\tilde{\Gamma}^4_{2N} \equiv i\widehat{C}I_{2N}, \quad \tilde{\Gamma}^5_{2N} \equiv i\Gamma^0_{2N}, \quad \tilde{\Gamma}^6_{2N} \equiv -\Gamma^2_{2N},$$

of the matrices $(i\Gamma^0_{2N}, i\Gamma^1_{2N}, \Gamma^2_{2N}, i\Gamma^3_{2N}, \widehat{C}I_{2N}, i\widehat{C}I_{2N})$, where Γ^μ_{2N} are given in (5.32), (5.33).

Matrices (5.39) together with the matrix $\tilde{\Gamma}^7_{2N} \equiv \tilde{\Gamma}^1_{2N}\tilde{\Gamma}^2_{2N}\tilde{\Gamma}^3_{2N}\tilde{\Gamma}^4_{2N}\tilde{\Gamma}^5_{2N}\tilde{\Gamma}^6_{2N} = \Gamma^4_{2N}$ satisfy the anti-commutation relations of the Clifford–Dirac algebra generators in the form

$$\tilde{\Gamma}^A_{2N}\tilde{\Gamma}^B_{2N} + \tilde{\Gamma}^B_{2N}\tilde{\Gamma}^A_{2N} = 2g^{AB}; \tag{5.40}$$

$$A, B = \overline{1,7}, \quad (g^{AB}) = (+ + + + - - -).$$

As well as in (5.27) among the generators of (5.40) only the 4+2=6 matrices (5.39) are independent and form the basis of the algebra. Therefore, the found above algebra over the field of real numbers is defined as $C\ell^{\mathbb{R}}(4,2)$ and the dimension of this algebra is $2^6 = 64$.

Useful realization of (5.39), (5.40) is given in terms of completely anti-Hermitian generators

$$\Gamma^1_{2N}, \Gamma^2_{2N}, \Gamma^3_{2N}, \Gamma^4_{2N} = \Gamma^0_{2N}\Gamma^1_{2N}\Gamma^2_{2N}\Gamma^3_{2N}, \tag{5.41}$$

$$\Gamma^5_{2N} = \Gamma^1_{2N}\Gamma^3_{2N}\widehat{C}, \Gamma^6_{2N} = i\Gamma^1_{2N}\Gamma^3_{2N}\widehat{C}, \Gamma^7_{2N} = i\Gamma^0_{2N},$$

where matrices $\{\Gamma^\mu_{2N}\}$ are given in (5.32), (5.33). Matrices (5.41) obey the anti-commutation relations of 64-dimensional $C\ell^{\mathbb{R}}(0,6)$ algebra in the form

$$\Gamma^A_{2N}\Gamma^B_{2N} + \Gamma^B_{2N}\Gamma^A_{2N} = -2\delta^{AB}, \quad A, B = \overline{1,7}. \tag{5.42}$$

Note that here as well only 6 operators are independent generators: $\Gamma^4_{2N} = -i\Gamma^7_{2N}\Gamma^1_{2N}\Gamma^2_{2N}\Gamma^3_{2N}$.

Operators (5.41) generate also the 28 orts $s^{\bar{A}\bar{B}}$:

$$s^{\bar{A}\bar{B}} = \{s^{AB} = \frac{1}{4}[\Gamma^A_{2N}, \Gamma^B_{2N}], s^{A8} = -s^{8A} = \frac{1}{2}\Gamma^A_{2N}\}, \tag{5.43}$$

where $\bar{A}, \bar{B} = \overline{1,8}$ and generators $s^{\bar{A}\bar{B}}$ satisfy the commutation relations of SO(8) algebra

$$[s^{\bar{A}\bar{B}}, s^{\bar{C}\bar{D}}] = \delta^{\bar{A}\bar{C}}s^{\bar{B}\bar{D}} + \delta^{\bar{C}\bar{B}}s^{\bar{D}\bar{A}} + \delta^{\bar{B}\bar{D}}s^{\bar{A}\bar{C}} + \delta^{\bar{D}\bar{A}}s^{\bar{C}\bar{B}}. \tag{5.44}$$

Of course, the matrix representation (5.43) of the SO(8) algebra is considered over the field of real numbers.

Note that SO(8) algebra is not any subalgebra of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$. It is impossible in principle. Indeed, the mathematical object $C\ell^{\mathbb{R}}(0,6)$ belongs to the Clifford algebras and mathematical object SO(8) belongs to the Lie algebras (see the consideration in Chapter 1).

As the consequences of the equalities

$$\Gamma_{2N}^4 \equiv \prod_{\mu=0}^{3} \Gamma_{2N}^{\mu} \to \prod_{\bar{\mu}=0}^{4} \Gamma_{2N}^{\bar{\mu}} = -I, \tag{5.45}$$

known from the standard Clifford–Dirac algebra $C\ell^{\mathbb{C}}(1,3)$, and the anticommutation relations (5.42), in $C\ell^{\mathbb{R}}(0,6)$ algebra for the matrices Γ_{2N}^A (5.41) the following extended equalities are valid:

$$\Gamma_{2N}^7 \equiv -\prod_{\underline{A}=1}^{6} \Gamma_{2N}^{\underline{A}} \to \prod_{A=1}^{7} \Gamma_{2N}^A = I, \quad \Gamma_{2N}^5 \Gamma_{2N}^6 = i. \tag{5.46}$$

The relationship (5.43) between the Clifford–Dirac algebra $C\ell^{\mathbb{R}}(0,6)$ and algebra SO(8) is similar to the relationship between the standard Clifford–Dirac algebra $C\ell^{\mathbb{C}}(1,3)$ and algebra SO(3,3) found in [61, 62]; the relationship between $C\ell^{\mathbb{C}}(1,3)$ and SO(1,5) see in the Chapter 1.

Note that subalgebra SO(6)$\subset$SO(8) is the algebra of invariance of the Dirac equation in the FW representation [3, 4]. The 15 elements of this SO(6) algebra are given by

$$\left\{ s^{\underline{AB}} \right\}, \quad \underline{A}, \underline{B} = \overline{1, 6}, \tag{5.47}$$

where

$$\left\{ s^{\underline{AB}} \right\} = \left\{ s^{\underline{AB}} \equiv \frac{1}{4} \left[\Gamma_{2N}^{\underline{A}}, \Gamma_{2N}^{\underline{B}} \right] \right\}. \tag{5.48}$$

Now only the first 6 matrices

$$\left\{ \Gamma_{2N}^{\underline{A}} \right\} = \left\{ \Gamma_{2N}^1, \Gamma_{2N}^2, \Gamma_{2N}^3, \Gamma_{2N}^4, \Gamma_{2N}^5, \Gamma_{2N}^6 \right\} \tag{5.49}$$

from the set (5.41) play the role of generating operators in the constructions (5.48), (5.49).

Here Γ_{2N}^4 is independent. Therefore, in application to Clifford algebras operators (5.41) generate the 64-dimensional $C\ell^{\mathbb{R}}(0,6)$ algebra as well.

The maximal pure matrix algebra of invariance of the FW equation (5.32) is the 31-dimensional algebra $SO(6) \oplus i\Gamma^0_{2N} \cdot SO(6) \oplus i\Gamma^0_{2N}$.

The additional possibilities, which are open by the 28 orts of the algebra $SO(8)$ in comparison with 15 orts of the well-known algebra $SO(1,5)$, are principal in description of the Bose states in the framework of the Dirac theory [85–91]. The algebra $SO(8)$ includes two independent spin s=1/2 SU(2) subalgebras $(s^1 \equiv \frac{1}{2}\Gamma^2_{2N}\Gamma^3_{2N}, s^2 \equiv \frac{1}{2}\Gamma^3_{2N}\Gamma^1_{2N}, s^3 \equiv \frac{1}{2}\Gamma^1_{2N}\Gamma^2_{2N})$ and $(\check{s}^1 \equiv \frac{1}{2}\Gamma^5_{2N}\Gamma^6_{2N}, \check{s}^2 \equiv \frac{1}{2}\Gamma^6_{2N}\Gamma^4_{2N}, \check{s}^3 \equiv \frac{1}{2}\Gamma^4_{2N}\Gamma^5_{2N})$ (in the anti-Hermitian form), whereas the $SO(1,5)$ algebra includes only one set of SU(2) generators given by the elements $(s^1 \equiv \frac{1}{2}\Gamma^2_{2N}\Gamma^3_{2N}, s^2 \equiv \frac{1}{2}\Gamma^3_{2N}\Gamma^1_{2N}, s^3 \equiv \frac{1}{2}\Gamma^1_{2N}\Gamma^2_{2N})$. Moreover, the algebra $SO(6)$, which is the algebra of invariance of the FW equation (5.32), includes these two independent SU(2) subalgebras as well. As long as these two spin s=1/2 SU(2) sets of generators commute between each other, their combination gives the generators of spin s=1 representation of SU(2) algebra. Such SU(2) algebra is the building element in construction of spin s=1 Lorentz and Poincaré algebras, with respect to which the FW equation (5.32) is invariant.

The transition to the RCQM is given by the transformation (5.28), (5.29). Thus, in the quantum-mechanical representation the seven Γ matrices (5.41) (in the terms of standard $\Gamma^{\bar{u}}_{2N}$ matrices (5.32), (5.33)) have the form (5.36) together with

$$\bar{\Gamma}^5_{2N} \equiv \Gamma^1_{2N}\Gamma^3_{2N}\widehat{C}, \quad \bar{\Gamma}^6_{2N} \equiv -i\Gamma^2_{2N}\Gamma^4_{2N}\widehat{C}, \quad \bar{\Gamma}^7_{2N} \equiv i; \tag{5.50}$$

$$\bar{\Gamma}^1_{2N}\bar{\Gamma}^2_{2N}\bar{\Gamma}^3_{2N}\bar{\Gamma}^4_{2N}\bar{\Gamma}^5_{2N}\bar{\Gamma}^6_{2N}\bar{\Gamma}^7_{2N} = I_{2N},$$

and satisfy the anti-commutation relations of the Clifford–Dirac algebra $C\ell^{\mathbb{R}}(0,6)$ representation in the following form

$$\bar{\Gamma}^A_{2N}\bar{\Gamma}^B_{2N} + \bar{\Gamma}^B_{2N}\bar{\Gamma}^A_{2N} = -2\delta^{AB}, \quad A, B = \overline{1,7}. \tag{5.51}$$

The RCQM representation of the algebra $SO(8)$ is given by

$$\bar{s}^{\bar{A}\bar{B}} = \{\bar{s}^{AB} = \frac{1}{4}[\bar{\Gamma}^A_{2N}, \bar{\Gamma}^B_{2N}], \bar{s}^{A8} = -\bar{s}^{8A} = \frac{1}{2}\bar{\Gamma}^A_{2N}\}, \tag{5.52}$$

$$\bar{A}, \bar{B} = \overline{1,8},$$

where the matrices $\bar{\Gamma}^A_{2N}$ are given in (5.36), (5.50) and generators $\bar{s}^{\bar{A}\bar{B}}$ satisfy the commutation relations

$$[\bar{s}^{\bar{A}\bar{B}}, \bar{s}^{\bar{C}\bar{D}}] = \delta^{\bar{A}\bar{C}}\bar{s}^{\bar{B}\bar{D}} + \delta^{\bar{C}\bar{B}}\bar{s}^{\bar{D}\bar{A}} + \delta^{\bar{B}\bar{D}}\bar{s}^{\bar{A}\bar{C}} + \delta^{\bar{D}\bar{A}}\bar{s}^{\bar{C}\bar{B}}. \tag{5.53}$$

The RCQM representation of the algebra SO(6)$\subset$SO(8) is given by

$$\left\{ \bar{s}^{\underline{A}\underline{B}} \right\}, \quad \underline{A}, \underline{B} = \overline{1,6}, \tag{5.54}$$

where

$$\left\{ \bar{s}^{\underline{A}\underline{B}} \right\} = \left\{ \bar{s}^{\underline{A}\underline{B}} \equiv \frac{1}{4} \left[\bar{\Gamma}_{2N}^{\underline{A}}, \bar{\Gamma}_{2N}^{\underline{B}} \right] \right\}. \tag{5.55}$$

The maximal pure matrix algebra of invariance of the Schrödinger–Foldy equation (5.11) is the 31-dimensional algebra a_{31} =SO(6)$\oplus i \cdot$SO(6)$\oplus i$.

In the RCQM representation two independent sets of generators of SU(2) subalgebra of SO(6) algebra, with respect to which the Schrödinger–Foldy equation (5.11) is invariant, have the form

$$\bar{s}^1 \equiv \frac{1}{2}\bar{\Gamma}_{2N}^2\bar{\Gamma}_{2N}^3, \, \bar{s}^2 \equiv \frac{1}{2}\bar{\Gamma}_{2N}^3\bar{\Gamma}_{2N}^1, \, \bar{s}^3 \equiv \frac{1}{2}\bar{\Gamma}_{2N}^1\bar{\Gamma}_{2N}^2, \tag{5.56}$$

$$\bar{\bar{s}}^1 \equiv \frac{1}{2}\bar{\Gamma}_{2N}^5\bar{\Gamma}_{2N}^6, \, \bar{\bar{s}}^2 \equiv \frac{1}{2}\bar{\Gamma}_{2N}^6\bar{\Gamma}_{2N}^4, \, \bar{\bar{s}} \equiv \frac{1}{2}\bar{\Gamma}_{2N}^4\bar{\Gamma}_{2N}^5, \tag{5.57}$$

where the matrices $\bar{\Gamma}_{2N}^{\underline{A}}$ are given in (5.36), (5.50). The situation here is similar to the FW representation. As long as these two spin s=1/2 SU(2) sets of generators commute between each other, their combination gives the generators of spin s=1 representation of SU(2) algebra. Such SU(2) algebra is the building element in construction of spin s=1 Lorentz and Poincaré algebras, with respect to which the Schrödinger–Foldy equation (5.11) is invariant. Note that formulas (5.57), (5.57) are valid for the N-dimensional multiplet but only for the spin s = 1/2. The arbitrary spin cannot be written in terms of Γ_{2N} matrices as (5.56), (5.57).

The partial case of 4 component formalism and 4×4 γ matrices algebra follows from the above given consideration after corresponding substitutions 2N=4, etc.

An important fact is that in Pauli–Dirac representation the 31-dimensional algebra SO(6)$\oplus i\Gamma_{2N}^0 \cdot$SO(6)$\oplus i\Gamma_{2N}^0$ still is the algebra of invariance of the Dirac equation (or 2N component Dirac-like equation). The only difference is that in Pauli–Dirac representation the operators $s^{\underline{A}\underline{B}}$ and gamma matrices $\Gamma_{2N}^{\underline{A}}$ are not pure matrix. They are the nonlocal pseudo-differential operators, which contain the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$. For the standard 4 component Dirac equation the corresponding gamma matrices are given by (1.22)–(1.25) in Chapter 1.

Another interesting fact is that operator of the Dirac equation with Hamiltonian $H \equiv \gamma^0 \overrightarrow{\gamma} \cdot \widehat{\overrightarrow{p}} + \gamma^0 m - e^2/|\overrightarrow{x}|$ commutes with all 31 generators of the $SO(6) \oplus i\gamma^0 \cdot SO(6) \oplus i\gamma^0$ algebra, which elements are given in terms of gamma matrices from (1.22)–(1.25). Therefore, the relativistic hydrogen atom has wide additional symmetries given by the 31 nontrivial generators from $SO(6) \oplus i\gamma^0 \cdot SO(6) \oplus i\gamma^0$. Moreover, the relativistic hydrogen atom has additional spin 1 symmetries, which for the free Dirac equation are known from [47–53] and which are the consequences of the algebra $SO(6) \oplus i\gamma^0 \cdot SO(6) \oplus i\gamma^0$. Indeed, more wide symmetries than SO(4) spin 1/2 symmetry of V. Fock [63] or further symmetries found in [64–68] (see the brief analysis in [69]) can be found on the basis of this 31-dimensional algebra.

Note that consideration of this axiom above is given as the simple generalization of the correct spin 1/2 particle-antiparticle doublet formalism. In reality the dimensions of Clifford–Dirac (and SO(n)) algebras for the higher spin cases $2(2s+1) \geq 8$ can be more higher then considered above 64-dimensional $C\ell^{\mathbb{R}}(4,2)$ and $C\ell^{\mathbb{R}}(0,6)$ algebras (and SO(8) algebra). The reason of this situation is the possibility to construct for $2(2s+1) \geq 8$ cases the additional Γ matrices obeying the Clifford–Dirac anticommutation relations. The concrete examples of such additional Γ matrices can be found in [70, 71].

Visual and demonstrative description of matrix representations of $C\ell^{\mathbb{R}}(0,6)$ and SO(8) algebras in terms of 4×4 gamma matrices can be found in [72, 73] and in Chapter 1.

5.5.9. On the Main and Additional Conservation Laws

Similarly to the nonrelativistic quantum mechanics *the conservation laws are found in the form of quantum-mechanical mean values of the operators, which commute with the operator of the equation of motion.*

The important physical consequence of the assertion about the relativistic invariance is the fact that 10 integral dynamical variables of the doublet

$$(P_\mu, J_{\mu\nu}) \equiv \int d^3x f^\dagger(t, \overrightarrow{x})(\widehat{p}_\mu, \widehat{j}_{\mu\nu}) f(t, \overrightarrow{x}) = \text{Const} \qquad (5.58)$$

do not depend on time, i.e., they are the constants of motion for this doublet. In (5.58) and below in the text of this axiom the wave function is chosen as (5.2), (5.12) and generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ are given in (5.20), (5.21)].

Note that the external and internal degrees of freedom for the free arbitrary spin particle-antiparticle doublet are independent. Therefore, the operator $\overrightarrow{s}$

(5.14) commutes not only with the operators $\widehat{\vec{p}}$, $\vec{x}$, but also with the orbital part $\widehat{m}_{\mu\nu}$ of the total angular momentum operator. And both operators $\vec{s}$ and $\widehat{m}_{\mu\nu}$ commute with the operator $i\partial_t - \sqrt{-\Delta + m^2}$ of the equation (5.11). Therefore, besides the 10 main (consequences of the 10 Poincaré generators) conservation laws (5.58), 12 additional constants of motion exist for the free arbitrary spin particle-antiparticle doublet. These additional conservation laws are the consequences of the operators of the following observables (index 2N is omitted):

$$s_j, \quad \check{s}_\ell = \frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega}+m}, \quad \widehat{m}_{\ell n} = x_l\widehat{p}_n - x_n\widehat{p}_\ell, \quad \widehat{m}_{0\ell} = -\widehat{m}_{l0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\},$$
(5.59)

part of which are given and explained in (5.25). Here $s_j = s_{\ell n}$ are given in (5.14).

Thus, the following assertions can be proved. In the space $H^A = \{A\}$ of the quantum-mechanical amplitudes the 10 main conservation laws (5.58) have the form

$$(P_\mu, J_{\mu\nu}) = \int d^3 k A^\dagger(\vec{k})(\check{p}_\mu, \check{j}_{\mu\nu})A(\vec{k}), \quad A(\vec{k}) \equiv \begin{vmatrix} a^1(\vec{k}) \\ a^2(\vec{k}) \\ \cdot \\ \cdot \\ a^{2N}(\vec{k}) \end{vmatrix}, \quad (5.60)$$

where the density generators of $\mathcal{P}^A$, $(\check{p}_\mu, \check{j}_{\mu\nu})$ from (5.60) are given by

$$\check{p}_0 = p_0 = \omega, \; \check{p}_\ell = p_\ell = k_\ell, \; \check{j}_{\ell n} = \tilde{x}_\ell k_n - \tilde{x}_n k_\ell + s_{\ell n}, \quad (5.61)$$

$$\check{j}_{0\ell} = -\frac{1}{2}\{\tilde{x}_\ell, \omega\} - (\check{s}_\ell \equiv \frac{s_{\ell n}k_n}{\omega+m}); \quad (\tilde{x}_l = -i\frac{\partial}{\partial k^l}). \quad (5.62)$$

In the formula (5.60) $A(\vec{k})$ is a 2N-column of amplitudes.

Note that the operators (5.60)–(5.62) satisfy the Poincaré commutation relations in the manifestly covariant form (3.4).

It is evident that the 12 additional conservation laws

$$(M_{\mu\nu}, S_{\ell n}, \check{S}_\ell) \equiv \int d^3 x f^\dagger(t, \vec{x})(\widehat{m}_{\mu\nu}, s_{\ell n}, \check{s}_\ell)f(t, \vec{x}) \quad (5.63)$$

generated by the operators (5.59), are the separate terms in the expressions (5.60)–(5.62) of principal (main) conservation laws.

The 22 above given conservation laws are valid for an arbitrary spin (in the case s=0 we have only 10 conservation laws generated by operators (5.25)).

Additional set of conservation laws is given by 31 quantities

$$(A_{31}) \equiv \int d^3x f^\dagger(t, \overrightarrow{x})(\widehat{a}_{31}) f(t, \overrightarrow{x}) = \text{Const}, \qquad (5.64)$$

or in terms of quantum-mechanical amplitudes

$$(A_{31}) = \int d^3k A^\dagger(\overrightarrow{k})(\widehat{a}_{31}) A(\overrightarrow{k}). \qquad (5.65)$$

Here $(\widehat{a}_{31})$ are the pure matrix generators of the algebra $A_{31} = \text{SO}(6) \oplus i \cdot \text{SO}(6) \oplus i$.

The conservation law of charge for the singlet has the form $q_\text{s} = e \int d^3x f^\dagger(t, \overrightarrow{x}) f(t, \overrightarrow{x})$ and in the case of particle-antiparticle doublet this quantity is given by $q_\text{d} = e \int d^3x f^\dagger(t, \overrightarrow{x}) g f(t, \overrightarrow{x})$, where operator g is presented in (5.14). In terms of quantum-mechanical amplitudes the corresponding conservation laws are given by $q_\text{s} = \int d^3k A^\dagger(\overrightarrow{k}) A(\overrightarrow{k})$, $q_\text{d} = \int d^3k A^\dagger(\overrightarrow{k}) g A(\overrightarrow{k})$.

In the case of spin s=1/2 the conservation law of spin is given twice: in the set (5.63) and in the set (5.65). For cases s>1/2 the sets (5.63) and (5.65) are completely different.

5.5.10. On the Stationary Complete Sets of Operators

Let us consider now the outstanding role of the different *complete sets* of operators from the algebra of observables A_S. If one does not appeal to the complete sets of operators, then the solutions of the the Schrödinger-Foldy equation (5.11) are linked directly only with the Sturm-Liouville problem for the energy operator (5.3). In this case one comes to so-called "degeneration" of solutions. Recall that for an arbitrary complete sets of operators the notion of degeneration is absent in the Sturm-Liouville problem (see, e.g., [42]): only one state vector corresponds to any one point of the common spectrum of a complete set of operators. To wit, for a complete set of operators there is a one to one correspondence between any point of the common spectrum and an eigenvector.

The *stationary complete sets* play the special role among the complete sets of operators. Recall that the stationary complete set is the set of all functionally independent mutually commuting operators, each of which commutes with the operator of energy (in our case with the operator (5.3)). The examples of the stationary complete sets in $H^{3,2N}$ are given by $(\widehat{\overrightarrow{p}}, s_z \equiv s^3_\text{2N}, g)$, $(\overrightarrow{p}, \overrightarrow{s}_\text{2N} \cdot \overrightarrow{p}, g)$,

ets; g is the charge sign operator. The set $(\overrightarrow{x}, s_z, g)$ is an example of non-stationary complete set. The $\overrightarrow{x}$-realization (5.8) of the space $\mathrm{H}^{3,2\mathrm{N}}$ and of quantum-mechanical Schrödinger-Foldy equation (5.11) are related just to this complete set.

For the goals of this paper the stationary complete set $(\widehat{\overrightarrow{p}}, s_z \equiv s_{2\mathrm{N}}^3, g)$ is chosen. The series of partial examples of equations on eigenvectors and eigenvalues for this stationary complete set is given in this chapter below. Consider here as an example the spin s=3/2 particle-antiparticle doublet case. Corresponding equations on eigenvectors and eigenvalues are given by

$$\widehat{\overrightarrow{p}}\, e^{-ikx}\mathrm{d}_{\bar{\mathrm{A}}} = \overrightarrow{k}\, e^{-ikx}\mathrm{d}_{\bar{\mathrm{A}}}, \quad \bar{\mathrm{A}} = \overline{1,8}, \tag{5.66}$$

$$s_8^3\mathrm{d}_1 = \frac{3}{2}\mathrm{d}_1,\, s_8^3\mathrm{d}_2 = \frac{1}{2}\mathrm{d}_2,\, s_8^3\mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,\, s_8^3\mathrm{d}_4 = -\frac{3}{2}\mathrm{d}_4, \tag{5.67}$$

$$s_8^3\mathrm{d}_5 = -\frac{3}{2}\mathrm{d}_5,\, s_8^3\mathrm{d}_6 = -\frac{1}{2}\mathrm{d}_6,\, s_8^3\mathrm{d}_7 = \frac{1}{2}\mathrm{d}_7,\, s_8^3\mathrm{d}_8 = \frac{3}{2}\mathrm{d}_8,$$

$$g\mathrm{d}_1 = -\mathrm{d}_1,\, g\mathrm{d}_2 = -\mathrm{d}_2,\, g\mathrm{d}_3 = -\mathrm{d}_3,\, g\mathrm{d}_4 = -\mathrm{d}_4, \tag{5.68}$$

$$g\mathrm{d}_5 = +\mathrm{d}_5,\, g\mathrm{d}_6 = +\mathrm{d}_6,\, g\mathrm{d}_7 = +\mathrm{d}_7,\, g\mathrm{d}_8 = +\mathrm{d}_8,$$

where the Cartesian orts $\{\mathrm{d}_{\bar{\mathrm{A}}}\}$ are given by $\{\mathrm{d}_{\bar{\mathrm{A}}}\} = \{\delta_{\bar{\mathrm{A}}}^{\bar{\mathrm{B}}}\}$, $\bar{\mathrm{A}}, \bar{\mathrm{B}} = \overline{1,8}$.

Comparison of the given in equations (5.67) spin s=3/2 case with the spin s=2 case (formulas (194) in [38] and (5.166) here in this chapter below) shows that the general form is out of clear visualization. Thus, the appealing to such general form is absent here.

The interpretation of the amplitudes in the general solution (5.12) follows from equations of (5.66)–(5.68) type for the fixed value of arbitrary spin. In general, the functions $a^{2\mathrm{N}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes. Thus, the first half of functions $a^{2\mathrm{N}}(\overrightarrow{k})$ is the momentum-spin amplitudes of the particle with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(-)$ and corresponding spin projections, respectively. Further, the second (bottom) half of functions $a^{2\mathrm{N}}(\overrightarrow{k})$ is the momentum-spin amplitudes of the antiparticle with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(+)$ and opposite spin projections, respectively.

Thus, the conclusion about the fermionic (or bosonic) spin features of the solution (5.12) (i.e., the interpretation of the solution (5.12)) follows from the

equations on eigenvectors and eigenvalues (of (5.66)–(5.68) type) and the above given interpretation of the amplitudes.

5.5.11.   On the Solutions of the Schrödinger-Foldy Equation

Let us consider the Schrödinger-Foldy equation (5.11) general solution related to the stationary complete sets $(\overrightarrow{\widehat{p}}, s_z \equiv s^3, g)$, where s^3 is given in (5.14). The fundamental solutions of the equation (5.11), which are the eigen solutions of this stationary complete sets, are given by the relativistic de Broglie waves:

$$\varphi_{\vec{k}2N}(t, \overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-i\omega t + i\vec{k}\vec{x}} d_{2N}, \qquad (5.69)$$

where d_{2N} are the orts of 2N-dimensional Cartesian basis (the Cartesian orts are the common eigenvectors for the operators (s_z, g)). The explicit form of N-dimensional Cartesian basis is given in formulas (5.13).

Vectors (5.69) are the generalized solutions of the equation (5.11). These solutions do not belong to the quantum-mechanical space $H^{3,2N}$, i.e., they are not realized in the nature. Nevertheless, the solutions (5.69) are the complete ortonormalized orts in the rigged Hilbert space $S^{3,2N} \subset H^{3,2N} \subset S^{3,2N*}$. In symbolic form the conditions of ortonormalization and completeness are given by

$$\int d^3x \varphi_{\vec{k}\alpha}^{\dagger}(t, \overrightarrow{x}) \varphi_{\vec{k}'\alpha'}(t, \overrightarrow{x}) = \delta(\overrightarrow{k} - \overrightarrow{k}') \delta_{\alpha\alpha'}, \qquad (5.70)$$

$$\int d^3k \sum_{\alpha=1}^{2N} \varphi_{\vec{k}\alpha}^{\beta}(t, \overrightarrow{x}) \varphi_{\vec{k}\alpha}^{*\beta'}(t, \overrightarrow{x}') = \delta(\overrightarrow{x} - \overrightarrow{x}') \delta_{\beta\beta'}. \qquad (5.71)$$

The functional forms of these conditions are omitted because of their bulkiness.

In the rigged Hilbert space $S^{3,2N} \subset H^{3,2N} \subset S^{3,2N*}$ an arbitrary solution of the equation (5.11) can be decomposed in terms of fundamental solutions (5.69). Furthermore, for the solutions $f \in S^{3,2N} \subset H^{3,2N}$ the expansion (5.12) is, (i) mathematically well-defined in the framework of the standard differential and integral calculus, (ii) if in the expansion (5.12) a state $f \in S^{3,2N} \subset H^{3,2N}$, then the amplitudes $a^{2N}(\overrightarrow{k})$ in (5.12) belong to the set of the Schwartz test functions over $R_{\vec{k}}^3$. Therefore, they have the unambiguous physical sense of the amplitudes of probability distributions over the eigenvalues of the stationary complete sets $(\overrightarrow{\widehat{p}}, s_z, g)$. Moreover, the complete set of quantum-mechanical amplitudes unambiguously determine the corresponding representation of the space $H^{3,2N}$ (in

this case it is the $(\overrightarrow{k}, s_z, g)$-representation), which vectors have the harmonic time dependence

$$\widetilde{f}(t, \overrightarrow{k}) = e^{-i\omega t} A(\overrightarrow{k}), \quad A(\overrightarrow{k}) \equiv \begin{vmatrix} a^1(\overrightarrow{k}) \\ a^2(\overrightarrow{k}) \\ \cdot \\ \cdot \\ a^{2N}(\overrightarrow{k}) \end{vmatrix}, \quad (5.72)$$

i.e., are the states with the positive sign of the energy ω.

The similar assertion is valid for the expansions of the states $f \in \mathrm{H}^{3,2N}$ over the basis states, which are the eigenvectors of an arbitrary stationary complete sets. Therefore, the corresponding representation of the space $\mathrm{H}^{3,2N}$, which is related to such expansions, is often called as the generalized Fourier transformation.

By the way, the $\overrightarrow{x}$-realization (5.8) of the states space is associated with the non-stationary complete set of operators $(\overrightarrow{x}, s_z, g)$. Therefore, the amplitudes $f^{2N}(t, \overrightarrow{x}) = \mathrm{d}^\dagger_{2N} f(t, \overrightarrow{x}) = U(t) f(0, \overrightarrow{x})$ of the probability distribution over the eigenvalues of this complete set depend on time t non-harmonically.

5.5.12. The Axiom on the Mean Value of the Operators of Observables

Note that any apparatus can not fulfill the absolutely precise measurement of a value of the physical quantity having continuous spectrum. Therefore, the customary quantum-mechanical axiom about the possibility of "precise" measurement, for example, of the coordinate (or another quantity with the continuous spectrum), which is usually associated with the corresponding "reduction" of the wave-packet, can be revisited. This assertion for the values with the continuous spectrum can be replaced by the axiom that only the mean value of the operator of observable (or the corresponding complete set of observables) is the experimentally observed for $\forall f \in \mathrm{H}^{3,N}$. Such axiom, without any loss of generality of consideration, unambiguously justifies the using of the subspace $\mathrm{S}^{3,N} \subset \mathrm{H}^{3,N}$ as an approximative space of the physically realizable states of the considered object. This axiom as well does not enforce the application of the conception of the ray in $\mathrm{H}^{3,N}$ (the set of the vectors $e^{i\alpha} f$ with an arbitrary-fixed real number α) as the state of the object. Therefore, the mapping $(a, \omega) \rightarrow U(a, \omega)$ in the formulas (5.20)–(5.22) for the $\mathcal{P}$-representations

in $S^{3,N} \subset H^{3,N}$ is an unambiguous. Such axiom actually removes the problem of the wave packet "reduction", which discussion started from the well-known von Neumann monograph [6]. Therefore, the subjects of the discussions of all "paradoxes" of quantum mechanics, a lot of attention to which was paid in the past century, are removed also.

The important conclusion about the RCQM is as follows. The consideration of all aspects of this model is given on the basis of using only such conceptions and quantities, which have the direct relation to the experimentally observable physical quantities of this "elementary" physical system.

5.5.13. On the Principles of Heredity and the Correspondence

The explicit forms (5.58)–(5.63) of the main and additional conservation laws demonstrate evidently that the model of RCQM satisfies the principles of the heredity and the correspondence with the non-relativistic classical and quantum theories. The deep analogy between RCQM and these theories for the physical system with the finite number degrees of freedom (where the values of the free dynamical conserved quantities are additive) is also evident.

Our new way of the Dirac equation derivation (section 9 in [38]) has been started from the RCQM of the spin s=(1/2,1/2) particle-antiparticle doublet.

5.5.14. On the Second Quantization

This axiom is external (not internal) in the RCQM. It is necessary for quantum field theory. The formalism of RCQM is complete without this axiom. The reader can see the brief consideration in the end of subsection 5.7.2 below in the details of the example of spin s=(1/2,1/2) particle-antiparticle doublet.

5.5.15. On the Physical Interpretation

The physical interpretation always is the final step in the arbitrary model of the physical reality formulation. In the beginning of the interpretation is better to recall the different formulations of the quantum theory collected, e.g., in [7].

The above considered model of physical reality is called the canonical quantum mechanics due to the principles of the heredity and correspondence with nonrelativistic Schrödinger quantum mechanics [6].

The above considered canonical quantum mechanics is called the relativistic canonical quantum mechanics due to its invariance with respect to the corresponding representations of the Poincaré group $\mathcal{P}$.

The above considered RCQM describes the spin (s,s) particle-antiparticle doublet due to the corresponding eigenvalues in the equations like (5.66)–(5.68) for the stationary complete set of operators and the explicit forms of the Casimir operators (5.23), (5.24) of the corresponding Poincaré group $\mathcal{P}$ representation, with respect to which the dynamical equation of motion is invariant.

The axioms of this section eventually need to be reconciled with three levels of description used in this paper: RCQM, canonical FW and Dirac models. Nevertheless, this interesting problem cannot be considered in few pages. The readers of this paper can compare the axioms of RCQM given above with the main principles of the Dirac model given in B. Thaller's monograph [74] on the high mathematical level.

5.6. Examples of Particle Singlets

5.6.1. Spin s=0

A brief scheme of the relativistic canonical quantum mechanics of the single spin s=0 boson is considered.

The Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = f^1, \tag{5.73}$$

where $f(x)$ is one component function. It is the spinless Salpeter equation.

The space of the states is as follows

$$S^{3,1} \subset H^{3,1} \subset S^{3,1*}. \tag{5.74}$$

The general solution of the Schrödinger–Foldy equation (5.73) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k a^1(\overrightarrow{k}) e^{-ikx}, \tag{5.75}$$

where notations

$$kx \equiv \widetilde{\omega} t - \overrightarrow{k}\overrightarrow{x}, \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \tag{5.76}$$

are used.

The equation on eigenvalue of the momentum operator has the form $\widehat{\overrightarrow{p}}\,e^{-ikx} = \overrightarrow{k}\,e^{-ikx}$.

The function $a^1(\overrightarrow{k})$ is the quantum-mechanical momentum amplitude of the spinless boson.

The Schrödinger–Foldy equation (5.73) and the set $\{f\}$ of its solutions (5.75) are invariant with respect to the irreducible unitary spin $s = 0$ representation (5.22) $(\widehat{j}_{\mu\nu} = \widehat{m}_{\mu\nu})$ of the Poincaré group $\mathcal{P}$. The corresponding generators of the Lie algebra of the Poincaré group $\mathcal{P}$ are given by (5.25).

The validity of this assertion is verified by the three steps considered after the formula (5.22) above. The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_2, \quad W = 0. \tag{5.77}$$

Hence, above a brief consideration of the RCQM foundations of the particle with the mass $m > 0$ and the spin $s = 0$ (spinless boson) has been given.

5.6.2. Spin s=1/2

A brief scheme of the relativistic canonical quantum mechanics of the single spin s=1/2 fermion is considered.

The Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \end{vmatrix}. \tag{5.78}$$

The space of the states is as follows

$$\mathrm{S}^{3,2} \subset \mathrm{H}^{3,2} \subset \mathrm{S}^{3,2*}. \tag{5.79}$$

The generators of the SU(2)-spin have an explicit form

$$\overrightarrow{s} = \frac{1}{2}\overrightarrow{\sigma}, \quad \left[s^j, s^\ell\right] = i\varepsilon^{j\ell n} s^n, \tag{5.80}$$

where $\overrightarrow{\sigma}$ are the standard Pauli matrices

$$\sigma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma^2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \tag{5.81}$$

The Casimir operator is given by

$$\overrightarrow{s}^2 = \frac{3}{4}\mathrm{I}_2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\mathrm{I}_2, \quad \mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \tag{5.82}$$

The general solution of the Schrödinger–Foldy equation (5.78) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3 k\, e^{-ikx} \left[a_+^-(\overrightarrow{k})\mathrm{d}_1 + a_-^-(\overrightarrow{k})\mathrm{d}_2 \right], \tag{5.83}$$

where notations (5.76) are used. The orts of the 2-dimensional Cartesian basis have the form

$$\mathrm{d}_1 = \left| \begin{matrix} 1 \\ 0 \end{matrix} \right|, \mathrm{d}_2 = \left| \begin{matrix} 0 \\ 1 \end{matrix} \right|. \tag{5.84}$$

The solution (5.83) is associated with the stationary complete set of operators $\widehat{\overrightarrow{p}}$, $s^3 = s_z$, $g = -e$ of the momentum, spin projection and sign of the charge of the spin 1/2 fermion, respectively.

The equations on the eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$, spin projection operator $s^3 = \frac{1}{2} \left| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right|$ and sign of the charge operator g are given by

$$\widehat{\overrightarrow{p}}\, e^{-ikx}\mathrm{d}_r = \overrightarrow{k}\, e^{-ikx}\mathrm{d}_r, \quad r = 1,2,$$

$$s^3 \mathrm{d}_1 = \frac{1}{2}\mathrm{d}_1, \quad s^3 \mathrm{d}_2 = -\frac{1}{2}\mathrm{d}_2, \tag{5.85}$$

$$g\mathrm{d}_1 = -e\mathrm{d}_1, \quad g\mathrm{d}_2 = -e\mathrm{d}_2.$$

The interpretation of the amplitudes $a_+^-(\overrightarrow{k})$, $a_-^-(\overrightarrow{k})$ of the general solution (5.83) follows from equations (5.85). The functions $a_+^-(\overrightarrow{k})$, $a_-^-(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the fermion with charge -e and the spin projection eigenvalues +1/2 and -1/2, respectively.

The Schrödinger–Foldy equation (5.78) and the set $\{f\}$ of its solutions (5.83) are invariant with respect to the irreducible unitary spin $s = 1/2$ representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 2×2 matrix-differential generators are given by (5.20), (5.21), where the spin 1/2 SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.80).

The validity of this assertion is verified by the three steps considered after the formula (5.22) above. The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_2, \tag{5.86}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)m^2 I_2, \tag{5.87}$$

where I_2 is given in (5.82).

Hence, above a brief consideration of the RCQM foundations of the particle with the mass $m > 0$ and the spin $s = 1/2$ (fermion with spin $s = 1/2$) has been given.

5.6.3. Spin s=1

A brief scheme of the relativistic canonical quantum mechanics of the single spin s=1 boson is considered.

The Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \end{vmatrix}. \tag{5.88}$$

The space of the states is as follows

$$S^{3,3} \subset H^{3,3} \subset S^{3,3*}. \tag{5.89}$$

The generators of the SU(2)-spin in the most spread explicit form are given by

$$s^1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad s^2 = \frac{i}{\sqrt{2}} \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}, \quad s^3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}. \tag{5.90}$$

It is easy to verify that the commutation relations

$$\left[s^j, s^\ell \right] = i\varepsilon^{j\ell n} s^n \tag{5.91}$$

of the SU(2)-algebra are valid.

The Casimir operator for this representation of the SU(2)-algebra is given by

$$\overrightarrow{s}^2 = 2I_3 = 1(1+1)I_3, \quad I_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{5.92}$$

The general solution of the Schrödinger–Foldy equation (5.88) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{-ikx} \left[c^1(\overrightarrow{k}) d_1 + c^2(\overrightarrow{k}) d_2 + c^3(\overrightarrow{k}) d_3 \right], \tag{5.93}$$

where notations (5.76) are used. The orts of the 3-dimensional Cartesian basis have the form

$$d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}, d_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}. \tag{5.94}$$

The solution (5.93) is associated with the stationary complete set $\widehat{\overrightarrow{p}}$, $s^3 = s_z$ of the momentum and spin projection operators of the spin s=1 boson, respectively.

The equations on the eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$ and spin projection operator $s^3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$ are given by

$$\widehat{\overrightarrow{p}}e^{-ikx}d_j = \overrightarrow{k}e^{-ikx}d_j; \quad s^3d_1 = 1d_1, s^3d_2 = 0, s^3d_3 = -1d_3. \tag{5.95}$$

The interpretation of the amplitudes $c^j(\overrightarrow{k})$ in (5.93) follows from equations (5.95). The functions $c^1(\overrightarrow{k}), c^2(\overrightarrow{k}), c^3(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the boson with the spin projection eigenvalues +1, 0 and -1, respectively.

The Schrödinger–Foldy equation (5.88) and the set $\{f\}$ of its solutions (5.93) are invariant with respect to the irreducible unitary spin $s = 1$ representation (5,22) of the Poincaré group $\mathcal{P}$. The corresponding 3×3 matrix-differential generators are given by (5.20), (5.21), whereas the spin 1 SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.90).

The validity of this assertion is verified by the three steps already given after formula (5.22) above. The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_3, \tag{5.96}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = 1(1+1)m^2 I_3, \tag{5.97}$$

where I_3 is given in (5.92).

Hence, above a brief consideration of the RCQM foundations of the particle with the mass $m > 0$ and the spin $s = 1$ (boson with spin $s = 1$) has been given.

5.6.4. Spin s=3/2

A brief scheme of the relativistic canonical quantum mechanics of the single spin s=3/2 fermion is considered.

The Schrödinger–Foldy equation is given by

$$
i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \end{vmatrix}. \tag{5.98}
$$

The space of the states is as follows

$$
S^{3,4} \subset H^{3,4} \subset S^{3,4*}. \tag{5.99}
$$

The generators of the SU(2)-spin in the most spread explicit form are given by

$$
s^1 = \frac{1}{2} \begin{vmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix}, \quad s^2 = \frac{i}{2} \begin{vmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix},
$$

$$
s^3 = \frac{1}{2} \begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix}. \tag{5.100}
$$

It is easy to verify that the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n} s^n$ of the SU(2)-algebra are valid.

The Casimir operator for this representation of the SU(2)-algebra is given by

$$
\vec{s}^2 = \frac{15}{4} I_4 = \frac{3}{2}\left(\frac{3}{2} + 1\right) I_4, \quad I_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{5.101}
$$

The general solution of the Schrödinger–Foldy equation (37) is given by

$$
f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} \left[b^1(\vec{k})d_1 + b^2(\vec{k})d_2 + b^3(\vec{k})d_3 + b^4(\vec{k})d_4\right], \tag{5.102}
$$

where notations (5.76) are used. The orts of 4-dimensional Cartesian basis have the form

$$\mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix},\ \mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix},\ \mathrm{d}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix},\ \mathrm{d}_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{5.103}$$

The solution (5.102) is associated with the stationary complete set of operators $\widehat{\overrightarrow{p}}, s^3 = s_z, g = -e$ of the momentum, spin projection and sign of the charge of spin s=3/2 fermion, respectively.

The equations on the eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$, spin projection operator $s^3 = \frac{1}{2} \begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{vmatrix}$ and sign of the charge operator $g = -e$ are given by

$$\widehat{\overrightarrow{p}}\, e^{-ikx}\mathrm{d}_{\bar{\alpha}} = \overrightarrow{k}\, e^{-ikx}\mathrm{d}_{\bar{\alpha}}, \quad \bar{\alpha} = 1,2,3,4,$$

$$s^3\mathrm{d}_1 = \frac{3}{2}\mathrm{d}_1,\ s^3\mathrm{d}_2 = \frac{1}{2}\mathrm{d}_2,\ s^3\mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,\ s^3\mathrm{d}_4 = -\frac{3}{2}\mathrm{d}_4. \tag{5.104}$$

$$g\mathrm{d}_{\bar{\alpha}} = -e\mathrm{d}_{\bar{\alpha}}.$$

The interpretation of the amplitudes $b^{\bar{\alpha}}(\overrightarrow{k})$ in (5.102) follows from equations (5.104). The functions $b^1(\overrightarrow{k}), b^2(\overrightarrow{k}), b^3(\overrightarrow{k}), b^4(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the fermion with the spin projection eigenvalues $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, respectively.

The Schrödinger–Foldy equation (5.98) and the set $\{f\}$ of its solutions (5.102) are invariant with respect to the irreducible unitary spin $s = 3/2$ representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 4×4 matrix-differential generators are given by (5.20). (5.21), where the spin 3/2 SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.100).

The validity of this assertion is verified by the three steps already explained after the formula (5.22). The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_4, \tag{5.105}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = \frac{3}{2}\left(\frac{3}{2}+1\right) m^2 \mathrm{I}_4, \tag{5.106}$$

where I_4 is given in (5.101).

Hense, above a brief consideration of the RCQM foundations of the particle with the mass $m > 0$ and the spin $s = 3/2$ has been given.

5.6.5. Spin s=2

A brief scheme of the relativistic canonical quantum mechanics of the single spin s=2 boson is considered.

The Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \\ f^5 \end{vmatrix}. \tag{5.107}$$

The space of the states is as follows

$$S^{3,5} \subset H^{3,5} \subset S^{3,5*}. \tag{5.108}$$

The generators of the SU(2)-spin in the most spread explicit form are given by

$$s^1 = \frac{1}{2} \begin{vmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{vmatrix}, \quad s^2 = \frac{i}{2} \begin{vmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{vmatrix},$$
$$\tag{5.109}$$

$$s^3 = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix}.$$

It is easy to verify that the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n} s^n$ of the SU(2)-algebra are valid.

The Casimir operator for this representation of the SU(2)-algebra is given by

$$\vec{s}^2 = 6I_5 = 2(2+1)I_5, \tag{5.110}$$

where I_5 is the 5×5 unit matrix.

The general solution of the Schrödinger–Foldy equation (5.107) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k\, e^{-ikx} \left[g^1(\overrightarrow{k})d_1 + g^2(\overrightarrow{k})d_2 + \ldots + g^5(\overrightarrow{k})d_5 \right], \quad (5.111)$$

where notations (5.76) are used. The orts of the 5-dimensional Cartesian basis have the form

$$d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, d_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} d_5 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \quad (5.112)$$

The solution (5.111) is associated with the stationary complete set $\widehat{\overrightarrow{p}}, s^3 = s_z$ of the momentum and spin projection operators of the spin 2 boson, respectively.

The equations on eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$ and spin pro-

jection operator $s^3 = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix}$ are given by

$$\widehat{\overrightarrow{p}}\, e^{-ikx} d_A = \overrightarrow{k}\, e^{-ikx} d_A,$$

$$s^3 d_1 = 2d_1, \ s^3 d_2 = 1d_2, \ s^3 d_3 = 0, \ s^3 d_4 = -1d_4, \ s^3 d_5 = -2d_5. \quad (5.113)$$

The interpretation of the amplitudes $g^A(\overrightarrow{k})$, $A = \overline{1,5}$ in (5.111) follows from the equations (5.113). The functions $g^1(\overrightarrow{k}), g^2(\overrightarrow{k}), g^3(\overrightarrow{k}, g^4(\overrightarrow{k}), g^5(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the boson with the spin $s = 2$, mass $m > 0$ and with the spin projection eigenvalues $2, 1, 0, -1, -2$, respectively.

The Schrödinger–Foldy equation (5.107) and the set $\{f\}$ of its solutions (5.111) are invariant with respect to the irreducible unitary spin $s = 2$ representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 5×5 matrix-differential generators are given by (5.20), (5.21), whereas the spin 2 SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.109).

The validity of this assertion is verified by the three steps, which already are described after the formula (5.22). The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_5, \tag{5.114}$$

$$W = w^\mu w_\mu = m^2 \vec{s}^2 = 2(2+1)m^2 \mathrm{I}_5, \tag{5.115}$$

where I_5 is the 5×5 unit matrix.

Hence, above a brief consideration of the RCQM foundations of the particle with the mass $m > 0$ and the spin $s = 2$ has been given.

5.7.   Examples of Particle-Antiparticle Doublets

5.7.1.   Spin s=(0,0) Particle-Antiparticle Doublet

A brief scheme of the relativistic canonical quantum mechanics of the bosonic particle-antiparticle doublet with spin s=0 is considered.

The Schrödinger–Foldy equation is given by

$$i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x), \quad f = \begin{vmatrix} f^1 \\ f^2 \end{vmatrix}. \tag{5.116}$$

The space of the states is as follows

$$\mathrm{S}^{3,2} \subset \mathrm{H}^{3,2} \subset \mathrm{S}^{3,2*}, \tag{5.117}$$

i.e., is the direct sum of two spaces (5.74).

The general solution of the Schrödinger–Foldy equation (5.73) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} \left[a^1(\vec{k}) \mathrm{d}_1 + a^2(\vec{k}) \mathrm{d}_2 \right], \tag{5.118}$$

where the orts of two dimensional Cartesian basis have the form (5.84) and the notations (5.76) are used.

The equation on eigenvalues of the momentum operator $\widehat{\vec{p}}$ has the form $\widehat{\vec{p}} e^{-ikx} \mathrm{d}_r = \vec{k} e^{-ikx} \mathrm{d}_r, \quad r = 1, 2,$

The function $a^1(\vec{k})$ is the quantum-mechanical momentum amplitude of the spinless boson. The function $a^2(\vec{k})$ is the quantum-mechanical momentum amplitude of the corresponding antiparticle.

The Schrödinger–Foldy equation (5.116) and the set $\{f\}$ of its solutions (5.118) are invariant with respect to the reducible unitary spin $s = 0$ representation (5.22) $(\widehat{j}_{\mu\nu} = \widehat{m}_{\mu\nu})$ of the Poincaré group $\mathcal{P}$. The corresponding generators of the Lie algebra of the Poincaré group $\mathcal{P}$ are given by (5.25).

The validity of this assertion is verified by the three steps considered after the formula (5.22) above. The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^{\mu}\widehat{p}_{\mu} = m^2 I_2, \quad W = 0. \tag{5.119}$$

Hence, above a brief consideration of the RCQM foundations of the bosonic particle-antiparticle doublet with the mass $m > 0$ and the spin $s = 0$ has been given.

5.7.2. Spin s=(1/2,1/2) Particle-Antiparticle Doublet

A brief scheme of the relativistic canonical quantum mechanics of the fermionic spin s=(1/2,1/2) particle-antiparticle doublet is considered.

For the fermionic spin s=(1/2,1/2) particle-antiparticle doublet the system of two 2-component equations (5.78) should be used. Therefore, the corresponding Schrödinger–Foldy equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \end{vmatrix}. \tag{5.120}$$

Here the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is the relativistic analog of the energy operator (Hamiltonian) of nonrelativistic quantum mechanics.

Thus, for the fermionic spin s=(1/2,1 /2) particle-antiparticle doublet the system of two 2-component equations $(i\partial_0 - \widehat{\omega})f(x) = 0$ and $(i\partial_0 - \widehat{\omega})f(x) = 0$ is used. Therefore, the corresponding Schrödinger–Foldy equation is given by (5.120), where the 4-component wave function is the direct sum of the particle and antiparticle wave functions, respectively. Due to the historical tradition of the physicists the antiparticle wave function is put in the down part of the 4-column.

The general solution of the Schrödinger–Foldy equation of motion (5.120) has the form

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} =$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{-ikx}\left[a_+^-(\overrightarrow{k})\mathrm{d}_1+a_-^-(\overrightarrow{k})\mathrm{d}_2+a_-^+(\overrightarrow{k})\mathrm{d}_3+a_+^+(\overrightarrow{k})\mathrm{d}_4\right], \quad (5.121)$$

where the orts $\{\mathrm{d}_{\tilde{\alpha}}\}$ of the 4-component Cartesian basis are given in (5.103) and the notations (5.76) are used.

The comparison of the general solution (5.121) of the Schrödinger–Foldy equation with the general solution (4.28) of the FW equation demonstrates the advantages of (5.121) in the quantum-mechanical interpretation. Indeed, here in the framework of RCQM the positive and negative frequency terms are absent in the solution. Therefore, the negative energies, which are the consequences of corresponded frequency part of the solution, are absent as well.

The space of the states in the RCQM of the spin s=(1/2,1/2) particle-antiparticle doublet is the rigged Hilbert space

$$S^{3,4} \subset H^{3,4} \subset S^{3,4*}, \quad (5.122)$$

i.e., is the direct sum of two spaces (5.79). Here the Hilbert space $H^{3,4}$ in its $\overrightarrow{x}$-realization is defined as

$$H^{3,4}=L_2(R^3)\otimes C^{\otimes 4}=\{f=(f^{\alpha}):R^3\to C^{\otimes 4},\int d^3x|f(t,\overrightarrow{x})|^2<\infty\}, \quad (5.123)$$

of complex-valued 4-component square-integrable functions of $x \in R^3 \subset M(1,3)$ (similarly, in momentum, $\overrightarrow{p}$-realization). Here $\overrightarrow{x}$ and $\widehat{\overrightarrow{p}}$ are the operators of canonically conjugated dynamical variables of the spin s=(1/2,1/2) particle-antiparticle doublet, and the vectors f, $\tilde{f}$ in $\overrightarrow{x}$- and $\overrightarrow{p}$-realizations are linked by the 3-dimensional Fourier transformation (the variable t is the parameter of time-evolution).

Thus, in the model under consideration information about the positive and equal masses of the particle and antiparticle is inserted. Further, information that the observer sees the antiparticle as the mirror reflection of the particle is also inserted. Therefore, the charge of the antiparticle should be opposite in sign to that of the particle, and the spin projection of the antiparticle should be opposite in sign to the spin projection of the particle.

Therefore, according to these principles, the corresponding SU(2)-spin generators are taken in the form

$$s^1=\frac{1}{2}\begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{vmatrix}, \quad s^2=\frac{1}{2}\begin{vmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{vmatrix}, \quad s^3=\frac{1}{2}\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{vmatrix}.$$

$$(5.124)$$

The matrices (5.124) can also be presented in the following useful form

$$\vec{s} = \frac{1}{2} \left| \begin{matrix} \vec{\sigma} & 0 \\ 0 & -C\vec{\sigma}C \end{matrix} \right|,$$

(5.125)

where $\vec{\sigma}$ are the standard Pauli matrices (5.81), C is the operator of complex conjugation, the operator of involution in $H^{3,4}$, I_2 is explained in (5.82).

In the choice of the spin (5.124)=(5.125) the principle of correspondence and heredity with the FW representation is used, where the particle-antiparticle doublet spin operator is given by

$$\vec{s}_{FW} = \frac{1}{2} \left| \begin{matrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{matrix} \right|.$$

(5.126)

The link between the spins (5.125) and (5.126) is given by the transformation operator v (2.49) (note that transformation v is valid for the case of anti-Hermitian form of spins (5.125), (5.126)).

The spin matrices (5.125), (5.126) satisfy the commutation relations $\left[s^j, s^l\right] = i\varepsilon^{jln}s^n$, $\varepsilon^{123} = +1$, of the algebra of SU(2) group, where ε^{jln} is the Levi-Civita tensor and $s^j = \varepsilon^{jln}s_{\ell n}$ are the Hermitian 4×4 matrices (5.125), (5.126) – the generators of a 4-dimensional reducible representation of the spin group SU(2) (universal covering of the SO(3)⊂SO(1,3) group).

The Casimir operator for the RCQM representation of SU(2) spin given in (5.125) has the form

$$\vec{s}^2 = \frac{3}{4}I_4 = \frac{1}{2}(\frac{1}{2} + 1)I_4,$$

(5.127)

where I_4 is 4×4 unit matrix.

The RCQM spin (5.124)=(5.125) in quantum-mechanical interpretation has the advantage in comparison with the FW spin (5.126). Indeed, for the spin (5.124)=(5.125) the spin projection of the antiparticle is opposite in sign to the spin projection of the particle, i.e., here, contrary to the FW representation, the observer sees the antiparticle as the mirror reflection of the particle.

Further, information that the observer sees the antiparticle as the mirror reflection of the particle is also inserted. Therefore, the charge of the antiparticle should be opposite in sign to that of the particle, and the spin projection of the antiparticle should be opposite in sign to the spin projection of the particle. The stationary complete set of operators can be chosen in the terms of operators $\widehat{\vec{p}}, s^3 = s_z, g = -e$ of the momentum, spin projection and sign of the charge of

the spin 1/2 particle-antiparticle doublet, respectively. The equations on eigen vectors and eigenvalues of the operators of this stationary complete set have the form

$$\widehat{\overrightarrow{p}}\,e^{-ikx}\mathrm{d}_{\bar{\alpha}} = \overrightarrow{k}\,e^{-ikx}\mathrm{d}_{\bar{\alpha}}, \quad \bar{\alpha} = 1,2,3,4, \tag{5.128}$$

$$s^3\mathrm{d}_1 = \frac{1}{2}\mathrm{d}_1,\, s^3\mathrm{d}_2 = -\frac{1}{2}\mathrm{d}_2,\, s^3\mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,\, s^3\mathrm{d}_4 = \frac{1}{2}\mathrm{d}_4, \tag{5.129}$$

$$g\mathrm{d}_1 = -e\mathrm{d}_1,\, g\mathrm{d}_2 = -e\mathrm{d}_2,\, g\mathrm{d}_3 = e\mathrm{d}_3,\, g\mathrm{d}_4 = e\mathrm{d}_4, \tag{5.130}$$

where the Cartesian orts $\{\mathrm{d}_{\bar{\alpha}}\}$ are given in (5.103).

The interpretation of the amplitudes in the general solution (5.121) follows from equations (5.128)–(5.130). Thus, the functions $a_+^-(\overrightarrow{k}),\, a_-^-(\overrightarrow{k})$ are the momentum-spin amplitudes of the particle (e.g., electron) with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(-e)$ and spin projections $(\frac{1}{2}, -\frac{1}{2})$, respectively. Further, the functions $a_-^+(\overrightarrow{k}),\, a_+^+(\overrightarrow{k})$ in (5.121) are the momentum-spin amplitudes of the antiparticle (e.g., positron) with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(+e)$ and spin projections $(-\frac{1}{2}, \frac{1}{2})$, respectively.

Thus, the conclusion about the fermionic spin s=(1/2,1/2) features of solution (5.121) (i.e., the interpretation of the solution (5.121)) follows from the equations (5.128)–(5.130) and the above given interpretation of the amplitudes.

The relativistic invariance of the model (implementation of the special relativity is ensured by the proof of the invariance of the Schrödinger–Foldy equation (5.120) with respect to the unitary representation (5.22) of the universal covering $\mathcal{P} \supset \mathcal{L}$=SL(2,C) of the proper ortochronous Poincaré group $\mathrm{P}_+^\uparrow = \mathrm{T}(4)\times)\mathrm{L}_+^\uparrow \supset \mathrm{L}_+^\uparrow$. Here $\mathcal{L}$ = SL(2,C) is the universal covering of proper ortochronous Lorentz group $\mathrm{L}_+^\uparrow$.

The generators of the fermionic $(\mathcal{P}^\mathrm{f})$ representation of the group $\mathcal{P}$, with respect to which the Schrödinger–Foldy equation (5.120) is invariant, are given by (5.20), (5.21) in the $\overrightarrow{x}$-realization of the space $\mathrm{H}^{3,4}$ (5.123) and by

$$p_0 = \widetilde{\omega},\, p_\ell = k_\ell,\, \widetilde{j}_{\ell n} = \widetilde{x}_\ell k_n - \widetilde{x}_n k_\ell + s_{\ell n}; \quad (\widetilde{x}_\ell = -i\widetilde{\partial}_\ell,\, \widetilde{\partial}_\ell \equiv \frac{\partial}{\partial k^\ell}), \tag{5.131}$$

$$\widetilde{j}_{0\ell} = -\widetilde{j}_{\ell 0} = tk_l - \frac{1}{2}\{\widetilde{x}_\ell, \widetilde{\omega}\} - \left(\frac{s_{\ell n}k_n}{\widetilde{\omega}+m} \equiv \breve{s}_\ell\right), \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \tag{5.132}$$

in the momentum $\overrightarrow{k}$-realization $\widetilde{H}^{3,4}$ of the doublet states space, respectively. The explicit form of the spin terms $s_{\ell n}$ in the expressions (5.20), (5.21), (5.131), (5.132), which is used for the spin s=(1/2,1/2) particle-antiparticle doublet, is given in the definition (5.124).

Despite the manifestly non-covariant forms (5.20), (5.21), (5.131), (5.132) of the $\mathcal{P}^{\text{f}}$-generators, they satisfy the commutation relations of the $\mathcal{P}$ algebra in the manifestly covariant form (3.4).

The $\mathcal{P}^{\text{f}}$-representation of the group $\mathcal{P}$ in the space $H^{3,4}$ (5.123) is given by a converged in this space exponential series (5.22) or, in the momentum space $\widetilde{H}^{3,4}$, by a corresponding exponential series given in terms of the generators (5.131), (5.132).

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_4, \tag{5.133}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)m^2 \mathrm{I}_4, \tag{5.134}$$

where $\overrightarrow{s}$ is given in (5.124)=(5.125) and I_4 is 4×4 unit matrix.

Hense, above a brief consideration of the RCQM foundations of the particle-antiparticle fermionic doublet with the mass $m > 0$ and the spin s=(1/2,1/2) has been given.

Finally, consider briefly *the program of the canonical quantization* of the spin $s = (1/2, 1/2)$ RCQM model. Note that the expression for the total energy P_0 plays a special role in the procedure of a so called "second quantization". In the RCQM spin $s = (1/2, 1/2)$ doublet model, as it is evident from the expression for the P_0 in (5.60) in terms of the charge sign-momentum-spin amplitudes

$$P_0^{\text{QM}} = \int d^3 k \omega \left(\left| a_{\text{r}}^- (\overrightarrow{k}) \right|^2 + \left| a_{\text{f}}^+ (\overrightarrow{k}) \right|^2 \right) \geq m > 0,$$

the energy is positive. The same assertion is valid for the amplitudes related to the arbitrary-fixed stationary complete set of operators. Furthermore, the operator $\widehat{P}_0$ of the quantized energy corresponding to above given expression for P_0^{QM} is a positive-valued operator. The explicit form of the operator $\widehat{P}_0$ follows from the expression for P_0^{QM} after the anticommutation quantization of the amplitudes

$$\left\{ \widehat{a}_{\bar{\alpha}}(\overrightarrow{k}), \widehat{a}_{\bar{\beta}}^\dagger(\overrightarrow{k}) \right\} = \delta_{\bar{\alpha}\bar{\beta}} \delta\left(\overrightarrow{k} - \overrightarrow{k}' \right)$$

(other operators anticommute) and their substitution $a^{\mp} \to \widehat{a}^{\mp}$ into the formula for P_0^{QM}. Note that the quantized amplitudes determine the Fock space $\mathcal{H}^{\text{F}}$ (over the quantum-mechanical space $\text{H}^{3,4}$). Moreover, the operators of dynamical variables $\widehat{P}_{\mu}, \widehat{J}_{\mu\nu}$ in $\mathcal{H}^{\text{F}}$, which are expressed according to formulae (5.60) in terms of the operator amplitudes $\widehat{a}_{\tilde{\alpha}}(\overrightarrow{k}), \widehat{a}_{\tilde{\beta}}^{\dagger}(\overrightarrow{k})$, automatically have the form of "normal products" and satisfy the commutation relations (3.4) of the $\mathcal{P}$ group in the Fock space $\mathcal{H}^{\text{F}}$. Operators $\widehat{P}_{\mu}, \widehat{J}_{\mu\nu}$ determine the corresponding unitary representation in $\mathcal{H}^{\text{F}}$. Other details are not the subject of this consideration.

5.7.3. Spin s=(1,1) Particle-Antiparticle Bosonic Doublet

A brief scheme of the relativistic canonical quantum mechanics of the 6-component spin s=(1,1) particle-antiparticle bosonic doublet is considered.

For the bosonic spin s=(1,1) particle-antiparticle doublet the system of two 3-component equations (5.88) should be used. Therefore, the corresponding Schrödinger–Foldy equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \\ f^5 \\ f^6 \end{vmatrix}, \tag{5.135}$$

where the pseudo-differential operator $\widehat{\omega}$ is given in (5.3)–(5.5). In (5.135), the 6-component wave function is the direct sum of the particle and antiparticle wave functions, whereas the particle wave function has the form (5.93). According to the quantum-mechanical tradition the antiparticle wave function is put into the bottom part of the 6-column.

Therefore, the general solution of the Schrödinger–Foldy equation (5.135) has the form

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} =$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{-ikx} \left[c^1(\overrightarrow{k})\text{d}_1 + c^2(\overrightarrow{k})\text{d}_2 + ... + c^5(\overrightarrow{k})\text{d}_5 + c^6(\overrightarrow{k})\text{d}_6 \right], \tag{5.136}$$

where the orts of the 6-component Cartesian basis are given by

$$
d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, d_5 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, d_6 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{5.137}
$$

Hence, the space of the states in the RCQM of the spin s=(1,1) particle-antiparticle doublet is the rigged Hilbert space

$$
S^{3,6} \subset H^{3,6} \subset S^{3,6*}, \tag{5.138}
$$

i.e., is the direct sum of two spaces (5.89).

Thus, in the model under consideration information about the positive and equal masses of the particle and antiparticle is inserted. Further, information that the observer sees the antiparticle as the mirror reflection of the particle is also inserted. Therefore, the charge of the antiparticle should be opposite in sign to that of the particle, and the spin projection of the antiparticle should be opposite in sign to the spin projection of the particle.

Therefore, according to these principles, the corresponding SU(2)-spin generators are taken in the form

$$
s^1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix}, \quad s^2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{vmatrix},
$$

$$
s^3 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}. \tag{5.139}
$$

The matrices (5.139) can also be presented in the following useful form

$$
\vec{s} = \begin{vmatrix} \vec{s}_3 & 0 \\ 0 & -C\vec{s}_3 C \end{vmatrix}, \tag{5.140}
$$

where the 3×3-matrices $\vec{s}_3$ are given in (5.90) and C is the operator of complex conjugation.

It is easy to verify that for the operators (5.139), (5.140) the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n} s^n$ of the SU(2)-algebra are valid.

The Casimir operator for this reducible representation of the SU(2)-algebra is given by

$$\vec{s}^2 = 2 I_6 = 1(1+1) I_6, \tag{5.141}$$

where I_6 is the 6×6- unit matrix.

The solution (5.136) is associated with the stationary complete set $\widehat{\vec{p}}, s^3 = s_z$ of the momentum and spin projection operators of the spin s=(1,1) bosonic particle-antiparticle doublet.

The equations on the momentum operator eigenvalues have the form

$$\widehat{\vec{p}} e^{-ikx} d_A = \vec{k} e^{-ikx} d_A, \quad A = \overline{1,6}, \tag{5.142}$$

and the equations on the spin projection operator $s^3 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$ eigenvalues are given by

$$s^3 d_1 = d_1, \, s^3 d_2 = 0, \, s^3 d_3 = -d_3, \, s^3 d_4 = -d_4, \, s^3 d_5 = 0, \, s^3 d_6 = d_6. \tag{5.143}$$

Therefore, the functions $c^1(\vec{k}), c^2(\vec{k}), c^3(\vec{k})$ in the solution (5.136) are the momentum-spin amplitudes of the particle (boson) with the momentum $\widehat{\vec{p}}$ and spin projection eigenvalues (+1, 0, -1), respectively, and the functions $c^4(\vec{k}), c^5(\vec{k}), c^6(\vec{k})$ are the momentum-spin amplitudes of the antiparticle with the momentum $\widehat{\vec{p}}$ and spin projection eigenvalues (-1, 0, +1), respectively.

Note that for the bosons having charge ($W^\mp$ bosons) the stationary complete set includes the sign charge operator and the additional equation on eigenvectors and eigenvalues of this operator is valid. As soon as the W^+ boson is considered as the antiparticle for the W^- boson, the sign charge operator is determined in the form $g = -\Gamma_6^0 = \begin{vmatrix} -I_3 & 0 \\ 0 & I_3 \end{vmatrix}$. The corresponding equations on eigenvalues are given by $gd_1 = -ed_1, \, gd_2 = -ed_2, \, gd_3 = -ed_3, \, gd_4 = ed_4, \, gd_5 = ed_5, \, gd_6 =$

ed_6. In this case the functions $c^1(\overrightarrow{k})$, $c^2(\overrightarrow{k})$, $c^3(\overrightarrow{k})$ in the solution (5.136) are the charge-momentum-spin amplitudes of the particle (boson) with the charge $-e$, momentum $\widehat{\overrightarrow{p}}$ and spin projection eigenvalues (+1, 0, -1), respectively, and the functions $c^4(\overrightarrow{k})$, $c^5(\overrightarrow{k})$, $c^6(\overrightarrow{k})$ are the charge-momentum-spin amplitudes of the antiparticle with the charge $+e$, momentum $\widehat{\overrightarrow{p}}$ and spin projection eigenvalues (-1, 0, +1), respectively.

The Schrödinger–Foldy equation (5.135) and the set $\{f\}$ of its solutions (5.136) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 6×6 matrix-differential generators are given by (5.20), (5.21), whereas the spin s=(1,1) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.139), (5.140).

The proof of this assertion is fulfilled by the three steps already described after formula (5.22). The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^{\mu} \widehat{p}_{\mu} = m^2 \mathrm{I}_6, \tag{5.144}$$

$$W = w^{\mu} w_{\mu} = m^2 \overrightarrow{s}^{\,2} = 1\,(1+1)\,m^2 \mathrm{I}_6, \tag{5.145}$$

where I_6 is the 6×6 unit matrix.

Hense, above a brief consideration of the RCQM foundations of the particle-antiparticle doublet with the mass $m > 0$ and the spin s=(1,1) has been given. In the limit m=0 this model describes the particular case of the photon-antiphoton doublet.

5.7.4.   Spin s=(3/2,3/2) Particle-Antiparticle Fermionic Doublet

A brief scheme of the relativistic canonical quantum mechanics of the 8-component fermionic spin s=(3/2,3/2) particle-antiparticle doublet is considered.

This model is constructed in complete analogy with the RCQM of the 4-component spin s=(1/2,1/2) particle-antiparticle doublet, which is given in the subsection 5.7.2. Moreover, the principles of constructing and describing such particle-antiparticle multiplet within the framework of the RCQM are in a complete analogy with the principles of the describing and the constructing of the spin s=(1,1) particle-antiparticle doublet considered in subsection 5.7.3. The difference is only in the dimensions of the corresponding spaces and matrices.

Therefore, the details can be omitted. The model below is useful for the Σ-hyperon description.

The 8-component fermionic spin s=(3/2,3/2) particle-antiparticle doublet is constructed as the direct sum of the two spin s=3/2 singlets. The spin s=3/2 singlet was considered in the subsection 5.6.4.

The most important fact is that here the link with the Dirac-like equation is similar to that between the spin s=(1/2,1/2) particle-antiparticle doublet and the standard 4-component Dirac equation. Therefore, the spin s=(3/2,3/2) particle-antiparticle doublet is of special interest.

The Schrödinger–Foldy equation and the space of states are given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \\ f^5 \\ f^6 \\ f^7 \\ f^8 \end{vmatrix}, \tag{5.146}$$

(operator $\widehat{\omega}$ is given in (5.3)–(5.5))

$$S^{3,8} \subset H^{3,8} \subset S^{3,8*}. \tag{5.147}$$

The general solution of the equation (5.146) for the spin s=(3/2,3/2) particle-antiparticle doublet is given by

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k\, e^{-ikx} b^A(\overrightarrow{k}) d_A, \quad A = \overline{1,8}, \tag{5.148}$$

where the orts of the 8-component Cartesian basis have the form

$$d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix},$$

$$
d_5 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_6 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, d_7 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, d_8 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}.
\tag{5.149}
$$

The generators of the corresponding SU(2)-spin that satisfy the commutation relations (5.91) of the SU(2) algebra are as follows

$$
\vec{s}_8 = \begin{vmatrix} \vec{s} & 0 \\ 0 & -C\vec{s}C \end{vmatrix},
\tag{5.150}
$$

where CI_4 is the diagonal 4×4 operator of the complex conjugation and the matrices $\vec{s}$ for the single spin s=3/2 particle are given in (5.100). In the explicit form the SU(2) spin operators (5.150) are given by

$$
s_8^1 = \frac{1}{2} \cdot \begin{vmatrix}
0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{3} & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0
\end{vmatrix},
$$

$$
s_8^2 = \frac{i}{2} \cdot \begin{vmatrix}
0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{3} & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0
\end{vmatrix},
\tag{5.151}
$$

$$s_8^3 = \frac{1}{2} \begin{vmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{vmatrix}.$$

The Casimir operator has the form of the following 8×8 diagonal matrix

$$\overrightarrow{s}^2 = \frac{15}{4} I_8 = \frac{3}{2} \left(\frac{3}{2} + 1 \right) I_8, \tag{5.152}$$

where I_8 is the 8×8 unit matrix.

The stationary complete set of operators is given by

$$g = \begin{vmatrix} -I_4 & 0 \\ 0 & I_4 \end{vmatrix}, \quad \widehat{p}^j = -i\partial_j, \quad s_8^3 = \frac{1}{2} \begin{vmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{vmatrix},$$

$$\tag{5.153}$$

where g is the charge sign operator, $\widehat{\overrightarrow{p}} = (\widehat{p}^j)$ is the momentum operator and $s_8^3 = s_z$ is the operator of the spin (5.151) projection on the axis z.

The equations on the eigenvalues of the operators g, $\widehat{\overrightarrow{p}}$, $s_8^3 = s_z$ have the form

$$gd_1 = -d_1, \, gd_2 = -d_2, \, gd_3 = -d_3, \, gd_4 = -d_4,$$

$$gd_5 = +d_5, \, gd_6 = +d_6, \, gd_7 = +d_7, \, gd_8 = +d_8, \tag{5.154}$$

$$\widehat{\overrightarrow{p}} e^{-ikx} d_A = \overrightarrow{k} e^{-ikx} d_A, \quad A = \overline{1,8}, \tag{5.155}$$

$$s_8^3 d_1 = \frac{3}{2} d_1, \, s_8^3 d_2 = \frac{1}{2} d_2, \, s_8^3 d_3 = -\frac{1}{2} d_3, \, s_8^3 d_4 = -\frac{3}{2} d_4,$$

$$s_8^3 d_5 = -\frac{3}{2} d_5, \, s_8^3 d_6 = -\frac{1}{2} d_6, \, s_8^3 d_7 = \frac{1}{2} d_7, \, s_8^3 d_8 = \frac{3}{2} d_8. \tag{5.156}$$

Therefore, the functions $b^1(\vec{k}), b^2(\vec{k}), b^3(\vec{k}), b^4(\vec{k})$ in solution (5.148) are the momentum-spin amplitudes of the massive fermion with the spin s=3/2 and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively; $b^5(\vec{k}), b^6(\vec{k})$, $b^7(\vec{k}), b^8(\vec{k})$ are the momentum-spin amplitudes of the antiparticle (antifermion) with the spin s=3/2 and the spin projection $(-3/2, -1/2, 1/2, 3/2)$, respectively.

The Schrödinger–Foldy equation (5.146) (and the set {f} of its solutions (5.148)) is invariant with respect to the reducible unitary fermionic representation (5.22) of the Poincaré group $\mathcal{P}$, whose Hermitian 8×8 matrix-differential generators are given by (5.20), (5.21), where the spin s=(3/2,3/2) SU(2) generators $\vec{s} = (s^{\ell n})$ are given in (5.150), (5.151).

The proof is similar to that given after formula (5.22). The Casimir operators of this reducible fermionic spin s=(3/2,3/2) representation of the group $\mathcal{P}$ have the form

$$p^2 = \widehat{p}^{\mu}\widehat{p}_{\mu} = m^2 \mathrm{I}_8, \qquad (5.156)$$

$$W = w^{\mu}w_{\mu} = m^2 \vec{s}_8^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)m^2 \mathrm{I}_8, \qquad (5.157)$$

where I_8 is the 8×8 unit matrix.

Thus, above the foundations of the RCQM of the 8-component spin s=(3/2,3/2) particle-antiparticle doublet are considered briefly. It is the basis for the transition to the covariant local field theory of the spin s=(3/2,3/2) particle-antiparticle doublet given below.

5.7.5. Spin s=(2,2) Particle-Antiparticle Bosonic Doublet

A brief scheme of the relativistic canonical quantum mechanics of the 10-component spin s=(2,2) particle-antiparticle bosonic doublet is considered.

This model is constructed in complete analogy with the RCQM of the 4-component spin s=(1/2,1/2) particle-antiparticle doublet (subsection 5.7.2), spin s=(1,1) particle-antiparticle doublet (subsection 5.7.3) and spin s=(3/2,3/2) particle-antiparticle doublet (subsection 5.7.4). The most close analogy is with the model of bosonic spin s=(1,1) particle-antiparticle doublet (subsection 5.7.3).

For the bosonic spin s=(2,2) particle-antiparticle doublet the system of two 5-component equations (5.107) should be used. Therefore, the corresponding

Schrödinger–Foldy equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \\ f^5 \\ f^6 \\ f^7 \\ f^8 \\ f^9 \\ f^{10} \end{vmatrix}, \tag{5.158}$$

where the pseudo-differential operator $\widehat{\omega}$ is given in (5.3)–(5.5). In (5.158), the 10-component wave function is the direct sum of the particle and antiparticle wave functions, whereas the particle wave function has the form (5.109). According to the quantum-mechanical tradition the antiparticle wave function is put into the bottom part of the 10-column.

Therefore, the general solution of the Schrödinger–Foldy equation (5.158) has the form

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k\, e^{-ikx} g^{\text{A}}(\overrightarrow{k})\mathrm{d}_{\text{A}}, \quad \text{A} = \overline{1,10}, \tag{5.159}$$

where the orts of the 10-component Cartesian basis are given by

$$\mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \mathrm{d}_5 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix},$$

$$d_6 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_7 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_8 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, d_9 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, d_{10} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{5.160}$$

Hence, the space of the states in the RCQM of the spin s=(2,2) particle-antiparticle doublet is the rigged Hilbert space

$$S^{3,10} \subset H^{3,10} \subset S^{3,10*}, \tag{5.161}$$

i.e., is the direct sum of two spaces (5.108).

Thus, in the model under consideration information about the positive and equal masses of the particle and antiparticle is inserted. Further, information that the observer sees the antiparticle as the mirror reflection of the particle is also inserted. Therefore, the charge of the antiparticle should be opposite in sign to that of the particle (in the case of the existence of the charge), and the spin projection of the antiparticle should be opposite in sign to the spin projection of the particle.

Therefore, according to these principles the corresponding SU(2)-spin generators are taken in the form

$$\overrightarrow{s}_{10} = \begin{vmatrix} \overrightarrow{s}_5 & 0 \\ 0 & -C\overrightarrow{s}_5 C \end{vmatrix}, \tag{5.162}$$

where the 5×5-matrices $\overrightarrow{s}_5$ are given in (5.109) and CI_5 is 5×5 diagonal matrix operator of complex conjugation.

It is easy to verify that for the operators (5.162) the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n} s^n$ of the SU(2)-algebra are valid.

In the explicit form the SU(2) spin operators (5.162) are given by

$$
s_{10}^1 = \frac{1}{2}
\begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{6} & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & -\sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0
\end{vmatrix},
$$

$$
s_{10}^2 = \frac{i}{2}
\begin{vmatrix}
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{6} & 0 & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & -\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{vmatrix},
$$

$$(5.163)$$

$$
s_{10}^3 =
\begin{vmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{vmatrix}.
$$

The Casimir operator for this reducible representation of the SU(2)-algebra is given by

$$\overrightarrow{s}^2 = 6\mathrm{I}_{10} = 2\,(2+1)\,\mathrm{I}_{10}, \tag{5.164}$$

where I_{10} is the 10×10 unit matrix.

The solution (5.159) is associated with the stationary complete set $\widehat{\overrightarrow{p}}$, $s^3 = s_z$ of the momentum and spin projection operators of the spin s=(2,2) bosonic particle-antiparticle doublet.

The equations on the momentum operator eigenvalues have the form

$$\widehat{\overrightarrow{p}} e^{-ikx} d_A = \overrightarrow{k} e^{-ikx} d_A, \quad A = \overline{1,10}, \tag{5.165}$$

and the equations on the spin projection operator s_{10}^3 (5.163) eigenvalues are given by

$$s_{10}^3 d_1 = 2d_1, \, s_{10}^3 d_2 = d_2, \, s_{10}^3 d_3 = 0, \, s_{10}^3 d_4 = -d_4, \, s_{10}^3 d_5 = -2d_5,$$

$$s_{10}^3 d_6 = -2d_6, \, s_{10}^3 d_7 = -d_7, \, s_{10}^3 d_8 = 0, \, s_{10}^3 d_9 = d_9, \, s_{10}^3 d_{10} = 2d_{10}. \tag{5.166}$$

Therefore, the functions $g^1(\overrightarrow{k})$, $g^2(\overrightarrow{k})$, $g^3(\overrightarrow{k})$, $g^4(\overrightarrow{k})$, $g^5(\overrightarrow{k})$ in the solution (5.159) are the momentum-spin amplitudes of the particle (boson) with the momentum $\widehat{\overrightarrow{p}}$ and spin projection eigenvalues (+2, +1, 0, -1, -2), respectively, and the functions $g^6(\overrightarrow{k})$, $g^7(\overrightarrow{k})$, $g^8(\overrightarrow{k})$, $g^9(\overrightarrow{k})$, $g^{10}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle with the momentum $\widehat{\overrightarrow{p}}$ and spin projection eigenvalues (-2, -1, 0, +1, +2), respectively.

The Schrödinger–Foldy equation (5.158) and the set {f} of its solutions (5.159) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 10×10 matrix-differential generators are given by (5.20), (5.21), whereas the spin s=(2,2) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.162), (5.163).

The proof of this assertion is fulfilled by the three steps already explained after the formula (5.22). The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_{10}, \tag{5.167}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_{10}^2 = 2(2+1) m^2 I_{10}, \tag{5.168}$$

where I_{10} is the 10×10 unit matrix.

Hense, above a brief consideration of the RCQM foundations of the particle-antiparticle doublet with the mass $m > 0$ and the spin s=(2,2) has been given. In

the limit m=0 this model describes the partial case of the graviton-antigraviton doublet. The hypothesis about the massive graviton and the problems of the gravity in general are out of this consideration.

5.8.   Examples of Particle Multiplets

5.8.1.   Spin s=(1,0) Particle Multiplet

A brief scheme of the relativistic canonical quantum mechanics of the spin s=(1,0) bosonic multiplet is considered.

In this case both partners of the multiplet are the ordinary particles. Therefore, specification of the antiparticle, which is the content of the previous section, is absent. Hence, the model of the spin s=(1,0) multiplet is constructed as the ordinary direct sum of the spin s=1 and spin s=0 singlets. The last one is described by the one component Schrödinger–Foldy equation also called the spinless Salpeter equation [13, 26, 36]. As shown in [50–53], this s=(1,0) multiplet is directly linked with the spin s=(1/2,1/2) particle-antiparticle doublet (in particular, with the $e^- e^+$ doublet).

Thus, the corresponding Schrödinger–Foldy equation has the form (5.98), where the s=0 contribution is taken as the f^4 component of the column. This equation is similar to that for the spin s=(1/2,1/2) particle-antiparticle doublet. Therefore, equation (5.98) for the case s=(1,0) should be considered in the same space of the states

$$S^{3,4} \subset H^{3,4} \subset S^{3,4*}, \tag{5.169}$$

where the Schrödinger–Foldy equation for the spin s=(1/2,1/2) particle-antiparticle doublet is determined. Moreover, equation of motion for s=(1,0) multiplet and the rigged Hilbert space (5.169) are similar to equation and the space of states for the spin s=3/2 particle singlet.

The general solution of the Schrödinger–Foldy equation (5.98) for the case s=(1,0) is given by

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} \left[c^1(\overrightarrow{k})\mathrm{d}_1 + c^2(\overrightarrow{k})\mathrm{d}_2 + c^3(\overrightarrow{k})\mathrm{d}_3 + c^4(\overrightarrow{k})\mathrm{d}_4 \right],$$
$$\tag{5.170}$$

where notations (5.76) are used. The orts of the 4-dimensional Cartesian basis have the form (5.103).

The generators of the corresponding SU(2)-spin that satisfy the commutation relations (5.91) of the SU(2) algebra are given by

$$s^1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad s^2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad s^3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

(5.171)

The Casimir operator for this reducible representation of the SU(2)-algebra is given by

$$\overrightarrow{s}^2 = 2 \begin{vmatrix} I_3 & 0 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 1(1+1)I_3 & 0 \\ 0 & 0 \end{vmatrix},$$

(5.172)

where I_3 is 3×3 unit matrix.

The stationary complete set of operators is given by $\widehat{\overrightarrow{p}}$, $s^3 = s_z$ The equations on the eigenvalues of the spin projection operator $s^3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$

have the form

$$s^3 d_1 = d_1, \, s^3 d_2 = 0, \, s^3 d_3 = -d_3, \, s^3 d_4 = 0.$$

(5.173)

The equation on the eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$ is the same as in (5.128).

Interpretation of the amplitudes $c^\alpha(\overrightarrow{k})$ in the solution (162) follows from equations (5.128) and (5.173). The functions $c^1(\overrightarrow{k}), c^2(\overrightarrow{k}), c^3(\overrightarrow{k})$, are the quantum-mechanical momentum-spin amplitudes of the boson with the spin s=1 and the eigenvalues of the spin projection $1, 0, -1$, respectively. The function $c^4(\overrightarrow{k})$ is the amplitude of the spinless boson.

The Schrödinger–Foldy equation (5.98) and the set {f} of its solutions (5.170) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 4×4 matrix-differential generators are given by (5.20), (5.21), where the spin s=(1,0) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.171).

The proof is fulfilled similarly to that given after formula (5.22). The Casimir operators of this reducible bosonic representation of the group $\mathcal{P}$ have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_4,$$

(5.174)

$$W = w^\mu w_\mu = m^2 \vec{s}^{\,2} = m^2 \begin{vmatrix} 1(1+1)\,I_3 & 0 \\ 0 & 0 \end{vmatrix}, \tag{5.175}$$

where I_3 and I_4 are the 3×3 and 4×4 unit matrices, respectively.

Hense, above a brief consideration of the RCQM foundations of the particle multiplet with mass $m > 0$ and the spin s=(1,0) is given.

5.8.2.   Spin s=(1,0,1,0) Particle-Antiparticle Multiplet

A brief scheme of the relativistic canonical quantum mechanics of the 8-component bosonic spin s=(1,0,1,0) particle-antiparticle multiplet is considered.

The 8-component bosonic spin s=(1,0,1,0) particle-antiparticle multiplet is constructed as the direct sum of the two spin s=(1,0) multiplets. The spin s=(1,0) multiplet was considered in the previous section. The principles of constructing and describing such particle-antiparticle multiplet within the framework of the RCQM are in a complete analogy with the principles of the describing and the constructing of the spin s=(1,1) particle-antiparticle doublet considered in subsection 5.7.3. Therefore, the details can be omitted.

The most important fact is that here the link with the Dirac-like equation is similar to that between the spin s=(1/2,1/2) particle-antiparticle doublet and the standard 4-component Dirac equation demonstrated in subsection 5.7.2. Therefore, the spin s=(1,0,1,0) particle-antiparticle multiplet is of special interest. It is much more useful than the spin s=(1,0) particle multiplet.

Thus, the Schrödinger–Foldy equation has the form

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \\ f^5 \\ f^6 \\ f^7 \\ f^8 \end{vmatrix}, \tag{5.176}$$

where the operator $\widehat{\omega}$ is given in (5.3)–(5.5).

The space of the states is as follows

$$S^{3,8} \subset H^{3,8} \subset S^{3,8*}. \tag{5.177}$$

The general solution of the Schrödinger–Foldy equation (5.176) is given by

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} =$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} \left[c^1(\overrightarrow{k})d_1 + c^2(\overrightarrow{k})d_2 + \ldots + c^7(\overrightarrow{k})d_5 + c^8(\overrightarrow{k})d_6 \right], \quad (5.178)$$

where the orts of the 8-component Cartesian basis have the form (5.149)

The explicit form of the generators of the corresponding SU(2)-spin that satisfy the commutation relations (5.91) of the SU(2) algebra is as follows

$$\overrightarrow{s}_8 = \begin{vmatrix} \overrightarrow{s} & 0 \\ 0 & -C\overrightarrow{s}C \end{vmatrix}, \quad (5.179)$$

where CI_4 is the diagonal 4×4 operator of the complex conjugation and components of $\overrightarrow{s}$ are given in (5.171).

The Casimir operator is given by the following 8×8 diagonal matrix

$$\overrightarrow{s}^2 = 2 \begin{vmatrix} I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1(1+1)I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1(1+1)I_3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad (5.180)$$

where I_3 is the 3×3 unit matrix.

The stationary complete set of operators is given by $\widehat{\overrightarrow{p}}, s_8^3 = s_z$ and the equations on the eigenvalues of the operator $s_8^3 = s_z$ have the form

$$s_8^3 d_1 = d_1, \; s_8^3 d_2 = 0, \; s_8^3 d_3 = -d_3, \; s_8^3 d_4 = 0,$$

$$s_8^3 d_5 = -d_5, \; s_8^3 d_6 = 0, \; s_8^3 d_7 = d_7, \; s_8^3 d_8 = 0. \quad (5.181)$$

The equation on eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$ is known from (5.155).

Therefore, the functions $c^1(\overrightarrow{k}), c^2(\overrightarrow{k}), c^3(\overrightarrow{k})$ in solution (5.178) are the momentum-spin amplitudes of the massive boson with the spin s=1 and the spin projection $(1, 0, -1)$, respectively, $c^4(\overrightarrow{k})$ is the amplitude of the spinless boson; $c^5(\overrightarrow{k}), c^6(\overrightarrow{k}), c^7(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antiboson) with the spin s=1 and the spin projection $(-1, 0, 1)$, respectively, $c^8(\overrightarrow{k})$ is the amplitude of the spinless antiboson.

The Schrödinger–Foldy equation (5.176) (and the set {f} of its solutions (5.178)) is invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$, whose Hermitian 8×8 matrix-differential generators are given by (5.20), (5.21), where the spin s=(1,0,1,0) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.179).

The proof is similar to that given after formula (5.22). The Casimir operators of this reducible bosonic spin s=(1,0,1,0) representation of the group $\mathcal{P}$ have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_8, \tag{5.182}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_8^2 = m^2 \begin{vmatrix} 1(1+1)\mathrm{I}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1(1+1)\mathrm{I}_3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{5.183}$$

where I_8 and I_3 are the 8×8 and 3×3 unit matrices, respectively.

Thus, above the foundations of the RCQM of the 8-component multiplet of two bosons with the spins s=(1,0) and their antiparticle doublet are considered briefly. It is the basis for the transition to the covariant local field theory of the spin s=(1,0,1,0) particle-antiparticle multiplet given below.

5.8.3. Spin s=(2,0,2,0) Particle-Antiparticle Multiplet

A brief scheme of the relativistic canonical quantum mechanics of the 12-component spin s=(2,0,2,0) particle-antiparticle bosonic multiplet is considered.

This model is constructed in complete analogy with the RCQM of spin s=(1,0,1,0) particle-antiparticle multiplet (subsection 5.8.2). The 12-component bosonic spin s=(2,0,2,0) particle-antiparticle multiplet is constructed as the direct sum of the two spin s=(2,0) multiplets. The principles of constructing and describing such particle-antiparticle multiplet within the framework of the RCQM are in a complete analogy with the principles of principles considered in the previous sections. Therefore, the details can be omitted.

The most important fact is that here the link with the Dirac-like equation is similar to that between the spin s=(1/2,1/2) particle-antiparticle doublet and the standard 4-component Dirac equation. Therefore, the spin s=(2,0,2,0) particle-antiparticle multiplet is of special interest. It is much more useful than the spin s=(1,0) or (2,0) particle multiplets.

Therefore, the corresponding Schrödinger–Foldy equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f^{12} \end{vmatrix}, \tag{5.184}$$

where the pseudo-differential operator $\widehat{\omega}$ is given in (5.3)–(5.5). In (5.184), the 12-component wave function is the direct sum of the particle and antiparticle wave functions. According to the quantum-mechanical tradition the wave function of two antiparticles is put into the bottom part of the 12-column.

Therefore, the general solution of the Schrödinger–Foldy equation (5.184) has the form

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k\, e^{-ikx} g^{\text{B}}(\overrightarrow{k})d_{\text{B}}, \quad \text{B} = \overline{1,12}, \tag{5.185}$$

where the orts of the 12-component Cartesian basis are given by

$$d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \ldots, \quad d_{12} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{5.186}$$

Hence, the space of the states in the RCQM of the spin s=(2,0,2,0) particle-antiparticle doublet is the rigged Hilbert space

$$S^{3,12} \subset H^{3,12} \subset S^{3,12*}, \tag{5.187}$$

i.e., is the direct sum of 4 corresponding spaces.

The corresponding SU(2)-spin generators are taken in the form

$$\overrightarrow{s}_{12} = \begin{vmatrix} \overrightarrow{s}_6 & 0 \\ 0 & -C\overrightarrow{s}_6 C \end{vmatrix}, \tag{5.188}$$

where the 6×6-matrices $\overrightarrow{s}_6$ are given by

$$s_6^1 = \frac{1}{2}\begin{vmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \quad s_6^2 = \frac{i}{2}\begin{vmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$s_6^3 = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \tag{5.189}$$

and CI_6 is 6×6 diagonal matrix operator of complex conjugation.

It is easy to verify that for the operators (5.188) the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n}s^n$ of the SU(2)-algebra are valid.

The Casimir operator for this reducible representation of the SU(2)-algebra is given by

$$\overrightarrow{s}_{12}^2 = 6\begin{vmatrix} I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{5.190}$$

where I_5 is the 5×5 unit matrix.

The solution (5.185) is associated with the stationary complete set $\widehat{\overrightarrow{p}}$, $s_{12}^3 = s_z$ of the momentum and spin projection operators of the spin s=(2,0,2,0) bosonic particle-antiparticle multiplet.

The equations on the momentum operator eigenvalues have the form

$$\widehat{\overrightarrow{p}}\, e^{-ikx} d_B = \overrightarrow{k}\, e^{-ikx} d_B, \quad B = \overline{1,12}, \tag{5.191}$$

and the equations on the spin projection operator s_{12}^3 (5.188) eigenvalues are given by

$$s_{12}^3 d_1 = 2d_1, \, s_{12}^3 d_2 = d_2, \, s_{12}^3 d_3 = 0, \, s_{12}^3 d_4 = -d_4, \, s_{12}^3 d_5 = -2d_5, \, s_{12}^3 d_6 = 0,$$

$$s_{12}^3 d_7 = -2d_7, \, s_{12}^3 d_8 = -d_8, \, s_{12}^3 d_9 = 0, \, s_{12}^3 d_{10} = d_{10}, \, s_{12}^3 d_{11} = 2d_{11}, \, s_{12}^3 d_{12} = 0.$$

$$(5.192)$$

Therefore, the functions $g^1(\overrightarrow{k}), g^2(\overrightarrow{k}), g^3(\overrightarrow{k}), g^4(\overrightarrow{k}), g^5(\overrightarrow{k})$ in the solution (5.185) are the momentum-spin amplitudes of the particle (boson) with the momentum $\widehat{\overrightarrow{p}}$, spin s=2 and spin projection eigenvalues (+2, +1, 0, -1, -2), respectively, $g^6(\overrightarrow{k})$ is the amplitude of the spinless boson; the functions $g^7(\overrightarrow{k}), g^8(\overrightarrow{k}), g^9(\overrightarrow{k}), g^{10}(\overrightarrow{k}), g^{11}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antiboson) with the momentum $\widehat{\overrightarrow{p}}$, spin s=2 and spin projection eigenvalues (-2, -1, 0, +1, +2), respectively, $g^{12}(\overrightarrow{k})$ is the amplitude of the spinless antiboson.

The Schrödinger–Foldy equation (5.184) and the set $\{f\}$ of its solutions (5.185) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 12×12 matrix-differential generators are given by (5.20), (5.21), whereas the spin s=(2,0,2,0) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (5.188).

The proof of this assertion is fulfilled by the three steps already given in section 2 after formula (5.22). The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_{12}, \tag{5.193}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_{12}^2 = m^2 \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{5.194}$$

where I_{12} and I_5 are the 12×12 and 5×5 unit matrices, respectively.

Hense, above a brief consideration of the RCQM foundations of the particle-antiparticle multiplet with the mass $m > 0$ and the spin s=(2,0,2,0) has been given. In the limit m=0 this model describes the partial case of corresponding massless particle-antiparticle multiplet.

5.8.4.   Spin s=(2,1,2,1) Particle-Antiparticle Multiplet

A brief scheme of the relativistic canonical quantum mechanics of the 16-component bosonic spin s=(2,1,2,1) particle-antiparticle multiplet is considered.

The 16-component bosonic spin s=(2,1,2,1) particle-antiparticle multiplet is constructed as the direct sum of the two spin s=(2,1) multiplets. The principles of constructing and describing such particle-antiparticle multiplet within the framework of the RCQM are in a complete analogy with the principles of principles considered in the previous sections. Therefore, the details can be omitted.

The most important fact is that here the link with the Dirac-like equation is similar to that between the spin s=(1/2,1/2) particle-antiparticle doublet and the standard 4-component Dirac equation. Therefore, the spin s=(2,1,2,1) particle-antiparticle multiplet is of special interest. It is much more useful than the spin s=(1,0), (2,0), (2,1) particle multiplets.

Thus, the Schrödinger–Foldy equation has the form

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \begin{vmatrix} f^1 \\ f^2 \\ . \\ . \\ . \\ f^{16} \end{vmatrix}, \tag{5.195}$$

where the operator $\widehat{\omega}$ is given in (5.3)–(5.5).

The space of the states is given by

$$S^{3,16} \subset H^{3,16} \subset S^{3,16*}. \tag{5.196}$$

The general solution of the Schrödinger–Foldy equation (5.195) is given by

$$f(x) = \begin{vmatrix} f_{\text{part}} \\ f_{\text{antipart}} \end{vmatrix} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{-ikx} b^{\widetilde{A}}(\overrightarrow{k}) d_{\widetilde{A}}, \quad \widetilde{A} = \overline{1,16}, \tag{5.197}$$

where the orts of the 16-component Cartesian basis have the form

$$
d_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \; d_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \; \ldots, \; d_{16} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{5.198}
$$

The explicit form of the generators of the corresponding SU(2)-spin that satisfy the commutation relations (5.91) of the SU(2) algebra is as follows

$$
\vec{s}_{16} = \begin{vmatrix} \vec{s}_8 & 0 \\ 0 & -C\vec{s}_8 C \end{vmatrix}, \tag{5.199}
$$

where the 8×8-matrices $\vec{s}_8$ are given by

$$
s_8^1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix},
$$

$$s_8^2 = \frac{i}{\sqrt{2}} \begin{vmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}, \tag{5.200}$$

$$s_8^3 = \begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix},$$

and CI_8 is 8×8 diagonal matrix operator of complex conjugation.

The Casimir operator is given by the following 16×16 diagonal matrix

$$\vec{s}_{16}^{\,2} = \begin{vmatrix} 6I_5 & 0 & 0 & 0 \\ 0 & 2I_3 & 0 & 0 \\ 0 & 0 & 6I_5 & 0 \\ 0 & 0 & 0 & 2I_3 \end{vmatrix} = \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 1(1+1)I_3 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 1(1+1)I_3 \end{vmatrix},$$

$$\tag{5.201}$$

where I_5 and I_3 are 5×5 and 3×3 unit matrices, respectively.

The stationary complete set of operators is given by $\overrightarrow{p}$, $s_{16}^3 = s_z$ and the equations on the eigenvalues of the operators $\overrightarrow{p}$ and $s_{16}^3 = s_z$ have the form

$$\widehat{\overrightarrow{p}} e^{-ikx} d_{\widetilde{A}} = \overrightarrow{k} e^{-ikx} d_{\widetilde{A}}, \quad \widetilde{A} = \overline{1, 16}, \tag{5.202}$$

$$s_{16}^3 d_1 = 2d_1,\; s_{16}^3 d_2 = d_2,\; s_{16}^3 d_3 = 0,\; s_{16}^3 d_4 = -d_4,$$

$$s_{16}^3 d_5 = -2d_5,\; s_{16}^3 d_6 = d_6,\; s_{16}^3 d_7 = 0,\; s_{16}^3 d_8 = -d_8. \tag{5.203}$$

$$s_{16}^3 d_9 = -2d_9,\; s_{16}^3 d_{10} = -d_{10},\; s_{16}^3 d_{11} = 0,\; s_{16}^3 d_{12} = d_{12},$$

$$s_{16}^3 d_{13} = 2d_{13},\; s_{16}^3 d_{14} = -d_{14},\; s_{16}^3 d_{15} = 0,\; s_{16}^3 d_{16} = d_{16}.$$

Therefore, the functions $b^1(\overrightarrow{k}) - b^5(\overrightarrow{k})$ in solution (5.197) are the momentum-spin amplitudes of the massive boson with the spin s=2 and the eigenvalues of the spin projection $(2,1,0,-1,-2)$, respectively, $b^6(\overrightarrow{k}) - b^8(\overrightarrow{k})$ are the amplitudes of the massive boson with the spin s=1 and the eigenvalues of the spin projection $(1,0,-1)$, respectively.; $b^9(\overrightarrow{k}) - b^{13}(\overrightarrow{k})$ in solution (5.197) are the momentum-spin amplitudes of the massive antiboson with the spin s=2 and the eigenvalues of the spin projection $(-2,-1,0,1,2)$, respectively, $b^{14}(\overrightarrow{k}) - b^{16}(\overrightarrow{k})$ are the amplitudes of the massive antiboson with the spin s=1 and the eigenvalues of the spin projection $(-1,0,1)$, respectively.

The Schrödinger–Foldy equation (5.195) (and the set $\{f\}$ of its solutions (5.197)) is invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$, whose Hermitian 16×16 matrix-differential generators are given by (5.20), (5.21), where the spin s=(2,1,2,1) SU(2) generators $\overrightarrow{s}_{16} = (s_{16}^{\ell n})$ are given in (5.199).

The proof is similar to that given in text after formula (5.22). The Casimir operators of this reducible bosonic spin s=(2,1,2,1) representation of the group $\mathcal{P}$ have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_{16}, \qquad (5.204)$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_8^2 = m^2 \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 1(1+1)I_3 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 1(1+1)I_3 \end{vmatrix},$$

$$(5.205)$$

where I_{16}, I_5 and I_3 are 16×16, 5×5 and 3×3 unit matrices8, respectively.

Thus, above the foundations of the RCQM of the 16-component multiplet of two bosons with the spins s=(2,1) and their antiparticle doublet are considered briefly. It is the basis for the transition to the covariant local field theory of the spin s=(2,1,2,1) particle-antiparticle multiplet given below.

5.9. On the Transition to the Nonrelativistic Canonical Quantum Mechanics of the Arbitrary Mass and Spin

In the nonrelativistic limit the Schrödinger–Foldy equation (5.11) is transformed into the ordinary Schrödinger equation

$$\left(i\partial_0 - \frac{\overrightarrow{p}^2}{2m} \right) \psi(x) = 0, \tag{5.206}$$

for the N-component wave function (in the case of singlet)

$$\psi \equiv \text{column}(\psi^1, \psi^2, ..., \psi^N), \quad N = 2s + 1. \tag{5.207}$$

Each of the above considered equations of motion of the RCQM (2-, 3-, 4-, 5-, 6-, 8-, 10-, 12-, 16-component) is transformed into the equation (5.206) with the corresponding number of the components.

Moreover, here the SU(2) spin operator is the same as in the RCQM and is given by the formulas (5.14), (5.17). Therefore, here as in the RCQM, the SU(2) generators for the spin s=1/2 are given in (5.80), (5.81), for the spin s=1 in (29), for the spin s=1 in (5.90), for the spin s=3/2 in (5.100), for the spin s=2 in (5.109), e.t.c. for the doublet and multiplet SU(2) spins.

Therefore, the equation (5.206) is invariant with respect to same representations of the SU(2) group, with respect to which the relativistic equation (5.11) is invariant. The difference is in the application of the Galilean group and its representations instead of the Poincaré $\mathcal{P}$ group and its representations.

For the models with interaction it is much more easier to solve the equation (5.206) with interaction potential $V(x)$ instead of pseudo-differential equation (5.11). Moreover, the solutions of the equation (5.206) with interaction can be useful for the obtaining of the corresponding solutions of the equation (5.11).

Thus, the equation (5.206) can be useful not only itself but for the different approximations of the relativistic equation (5.11) as well.

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Chapter 6

Relativistic Field Theory of Arbitrary Spin in Canonical Foldy–Wouthuysen Type Representation

The role of the Foldy–Wouthuysen (FW) representation in modern quantum field theory is considered in the Chapter 4. After the start in [1, 2] the approach was developed for the case of presence of external field given by different potentials, the FW transformation for an arbitrary field was investigated, the different problems, which are actual in field theory, are considered in the FW representation [3–14]. In the paper [15] the quantum electrodynamics in the FW representation was constructed.

In our program of synthesis of field equations on the basis of start from relativistic canonical quantum mechanica (RCQM) the FW type representation for arbitrary spin is the intermediate stage between the RCQM and manifestly covariant field theory. The link between these two representations is given by the operator transformation suggested in our publications, see, e.g., [16–20]. Indeed, the chain of our synthesis program can be demonstrated as *RCQM → FW → manifestly covariant equations*. Therefore, as a first step we derive here the FW model starting from RCQM.

6.1.   General Formalism of Arbitrary Spin

The operator v_{2N}, which transform the RCQM of the arbitrary spin particle-antiparticle multiplet into the corresponding canonical particle-antiparticle field, is given in (5.28). The operator v_{2N} (5.28) translates any operator from RCQM into canonical field FW representation and vice versa. The corresponded formulas are presented in (5.29). Note that this transition is valid only for the operators taking in anti-Hermitian form, see the subsection 5.5.8 of the Chapter 5. The possibility of using the anti-Hermitian operators is explained in [21, 22].

Thus, the equation of motion for the free non-interacting particle-antiparticle fermionic doublet of arbitrary spin in the FW representation (the FW type equation of arbitrary spin) has the form

$$(i\partial_0 - \Gamma^0_{2N}\widehat{\omega})\phi(x) = 0, \tag{6.1}$$

where $\Gamma^0_{2N} \equiv \sigma^3_{2N} = \begin{vmatrix} I_N & 0 \\ 0 & -I_N \end{vmatrix}$, $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$, $N = 2s + 1$, and vice verso.

The operator of the FW equation (6.1) is derived here from the the operator of the Schrödinger–Foldy equation (5.11) in anti-Hermitian form

$$v_{2N}(\partial_0 + i\widehat{\omega})v_{2N} \rightarrow (\partial_0 + i\Gamma^0_{2N}\widehat{\omega}). \tag{6.2}$$

Further, the operator v_{2N} (5.28) translates $\phi = v_{2N}f$, $f = v_{2N}\phi$, the solution (5.12) of the Schrödinger–Foldy equation (5.11) into the solution (5.31) of the FW equation .

Thus, the formulas of the FW representations formalism are found here from the corresponding formulas of RCQM with the help of the operator v_{2N} (5.28) on the basis of its properties (5.28), (5.29). For the general form of arbitrary spin canonical particle-antiparticle field the equation of motion of the FW type is found in the form (6.1). The general solution has the form (5.31), where $a^N(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the particle and $a^{\check{N}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the antiparticle, {d} is 2N-component Cartesian basis.

It is evident from (5.31) that the model under consideration is not quantum mechanics. Indeed, contrary to (5.12) the solution (5.31) contains positive and negative frequency terms and, as a consequence, equation (6.1) is dealing with positive and negative energies (contrary to equation (5.11)).

The spin operator, which follows from RCQM spin of (5.14), has the form

$$\overrightarrow{s}'_{2N} = \begin{vmatrix} \overrightarrow{s}_N & 0 \\ 0 & \overrightarrow{s}_N \end{vmatrix}, \quad N = 2s + 1, \tag{6.3}$$

where $\overrightarrow{s}_N$ are $N \times N$ generators of arbitrary spin irreducible representations of SU(2) algebra, which satisfy the commutation relations (5.15). In the terms of Γ_{2N} matrices (5.32), (5.33) the spin operator (6.3) has the form (5.38) (note that the form (5.38) is valid only in the dimensions where the gamma matrices exist).

The equation on eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$ now changed and has the form

$$\widehat{\overrightarrow{p}} e^{-ikx} d_N = \overrightarrow{k} e^{-ikx} d_N, \quad \widehat{\overrightarrow{p}} e^{ikx} d_{\check{N}} = -\overrightarrow{k} e^{ikx} d_{\check{N}}, \tag{6.4}$$

where $N = 1, 2, ..., N, \quad \check{N} = N+1, N+2, ..., 2N$. The interpretation of the amplitudes in the general solution (5.31) follows from the equations (6.4) and corresponding equations on eigenvalues of the operators of spin projection and sign of the charge, which here have similar to the RCQM forms.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (6.1) and the set $\{\phi\}$ of its solutions (5.31) are invariant, are given by

$$\widehat{p}^0 = \Gamma^0_{2N} \widehat{\omega} \equiv \Gamma^0_{2N} \sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \tag{6.5}$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s^{\ell n}_{2N} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{2N},$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2} \Gamma^0_{2N} \{x^\ell, \widehat{\omega}\} + \Gamma^0_{2N} \frac{(\overrightarrow{s}_{2N} \times \overrightarrow{p})^\ell}{\widehat{\omega} + m}, \tag{6.6}$$

where arbitrary spin SU(2) generators $\overrightarrow{s}_{2N} = (s^{\ell n}_{2N})$ have the form (6.3), Γ^0_{2N} is given in (5.32). The explicit forms (6.5), (6.6) of the generators $(\widehat{p}^0, \widehat{p}^\ell, \widehat{j}^{\ell n}, \widehat{j}^{0\ell})$ are found here from the generators (5.20), (5.21) on the basis of the transformation given by the operator v_{2N} (5.28), (5.29).

Note that together with the generators (6.5), (6.6) another set of 10 operators commutes with the operator of equation (6.1), satisfies the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (6.5) together with the orbital

parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (6.5), (6.6), respectively. In another way this set follows from the set (5.25) after the transformation v_{2N} (5.28), (5.29).

The explicit form of this set of generators, which is very useful for the case $s = 0$ and arbitrary dimensions, is given by

$$\widehat{p}^0 = \Gamma^0_{2N}\widehat{\omega} \equiv \Gamma^0_{2N}\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \tag{6.7}$$

$$\widehat{m}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell, \quad \widehat{m}^{0\ell} = -\widehat{m}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\Gamma^0_{2N}\left\{x^\ell, \widehat{\omega}\right\}. \tag{6.8}$$

The calculation of the corresponding Casimir operators $p^2 = \widehat{p}^\mu \widehat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin completes the brief description of the model.

6.2. Examples of Particle-Antiparticle Doublets

6.2.1. The Example of Spin s=(0,0) Particle-Antiparticle Doublet

The below considered formalism follows from the general formalism of arbitrary spin presented above in section 6.1 after the substitution s=0. On the other hand this example of FW spin s=(0,0) particle-antiparticle doublet follows from the consideration of spin s=(0,0) particle-antiparticle doublet in RCQM presented here in the Chapter 5 in subsection 5.7.1.

The RCQM Schrödinger–Foldy equation for spin s=(0,0) particle-antiparticle doublet is given by (5.116), i.e., it is 2-component equation. The partial case of operator v_{2N} (5.28), which gives the link between the spin s=(0,0) particle-antiparticle doublet in RCQM (subsection 5.7.1) and FW representation, has the form

$$v_1 = \begin{vmatrix} 1 & 0 \\ 0 & C \end{vmatrix}, \quad v_1^{-1} = v_1^\dagger = v_1, \tag{6.9}$$

where C is the operator of complex conjugation, $C\phi = \phi^*$.

The corresponding FW type equation of canonical field theory is given by

$$(i\partial_0 - \sigma^3 \widehat{\omega})\phi(x) = 0; \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}. \tag{6.10}$$

The general solution is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^1(\overrightarrow{k}) d_1 + e^{ikx} a^{*2}(\overrightarrow{k}) d_2 \right]. \qquad (6.11)$$

The interpretation of the amplitudes in the general solution (6.11) follows from the equation on eigenvalues of the momentum operator $\widehat{\overrightarrow{p}}$:

$$\widehat{\overrightarrow{p}} e^{-ikx} d_1 = \overrightarrow{k} e^{-ikx} d_1, \quad \widehat{\overrightarrow{p}} e^{ikx} d_2 = -\overrightarrow{k} e^{ikx} d_2. \qquad (6.12)$$

The function $a^1(\overrightarrow{k})$ is the amplitude of the spinless boson with momentum $\widehat{\overrightarrow{p}}$ and the function $a^{*2}(\overrightarrow{k})$ is the amplitude of corresponding antiparticle. The quantum-mechanical interpretation of the amplitudes $a^1(\overrightarrow{k}), a^2(\overrightarrow{k})$ is given in the subsection 5.7.1 of the Chapter 5.

The Poincaré group $\mathcal{P}$ generators, with respect to which the equation (6.9) and the set $\{\phi\}$ of its solutions (6.10) are invariant, have the form

$$\hat{p}^0 = \sigma^3 \widehat{\omega} \equiv \sigma^3 \sqrt{-\Delta + m^2}, \quad \hat{p}^\ell = -i\partial_\ell, \qquad (6.13)$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell, \quad \widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\sigma^3 \left\{ x^\ell, \widehat{\omega} \right\}. \qquad (6.14)$$

Generators (6.13), (6.14) are the partial 2×2 matrix form of operators (6.5), (6.6) taken with the spin terms equal to zero, or the partial 2×2 matrix form of operators (6.7), (6.8).

6.2.2. The Example of Spin s=(1/2,1/2) Particle-Antiparticle Doublet

This example is s the most studied. It was a starting point of the investigations of that kind [1, 2]. The spin s=(1/2,1/2) particle-antiparticle doublet in the FW representation is considered here in Chapters 2 and 5, especially in the Chapter 4. Therefore, below the brief description is given.

Two different ways of deriving the FW model of the spin s=(1/2,1/2) particle-antiparticle doublet can be applied. This model is the partial case of the considered in section 6.1 general formalism of arbitrary spin particle-antiparticle doublet in the FW representation. Moreover, the special formalism of the spin s=(1/2,1/2) particle-antiparticle doublet in the FW representation can be derived from the similar formalism in the frame of RCQM.

The FW equation for the free non-interacting spin s=(1/2,1/2) particle-antiparticle doublet is presented in (4.21), the general solution of this equation has the form (4.28). The operator v (2.49) transforms the RCQM Schrödinger–Foldy equation (2.35) for spin s=(1/2,1/2) particle-antiparticle doublet into the FW equation (4.21). According to the transformation v (2.49) the general solution (2.41) of the Schrödinger–Foldy equation (2.35) is transformed into the general solution (4.28) of the FW equation (4.21). Furthermore, the FW spin operator (2.48) is the consequence of the transformation v (2.49) application to the RCQM spin operator $\overrightarrow{s}$ from (2.47).

Note that the amplitudes $a^{\alpha}(\overrightarrow{k})$, $\bar{\alpha} = 1,2,3,4$, in (4.28) are the same as in solution $f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx}[a^1(\overrightarrow{k})d_1 + a^2(\overrightarrow{k})d_2 + a^3(\overrightarrow{k})d_3 + a^4(\overrightarrow{k})d_4]$ of the Schrödinger–Foldy equation (2.35) and the orts $\{d_{\bar{\alpha}}\}$ of the Cartesian basis are given by (4.29). Note that transformation v (2.49) does not change the orts $\{d_{\bar{\alpha}}\}$.

The Casimir operator for the SU(2) spin $s^{\ell n} \equiv \frac{i}{4}[\gamma^{\ell}, \gamma^n]$ (or $\overrightarrow{s}_{\text{FW}}$ (2.48)) has the form

$$\overrightarrow{s}^2 = \frac{3}{4}I_4 = \frac{1}{2}\left(\frac{1}{2}+1\right)I_4, \tag{6.15}$$

where I_4 is the 4×4 unit matrix. This operator also can be found by the transition v (2.49) from the corresponding Casimir operator in the RCQM.

The interpretation of the amplitudes in the general solution (4.28) follows from the equations on eigen vectors and eigenvalues

$$\widehat{\overrightarrow{p}}e^{-ikx}d_r = \overrightarrow{k}e^{-ikx}d_r, r = (1,2); \quad \widehat{\overrightarrow{p}}e^{ikx}d_{\check{r}} = -\overrightarrow{k}e^{ikx}d_{\check{r}}, \check{r} = (3,4), \tag{6.16}$$

(4.45) and (2.46) of the operators $\widehat{\overrightarrow{p}}$, $s^3 = s_z$, $g = -e$ of the momentum, spin projection and sign of the charge of the spin 1/2 FW particle-antiparticle doublet, respectively.

Thus, the functions $a^1(\overrightarrow{k})$, $a^2(\overrightarrow{k})$ are the momentum-spin amplitudes of the particle (e.g., electron) with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(-e)$ and spin projections $(\frac{1}{2}, -\frac{1}{2})$, respectively. The functions $a^{*3}(\overrightarrow{k})$, $a^{*4}(\overrightarrow{k})$ are the complex conjugated functions to the quantum-mechanical amplitudes $a^3(\overrightarrow{k})$, $a^4(\overrightarrow{k})$ of the standard quantum-mechanical sense, which was considered in the Chapter 5 in subsection 5.7.2 in the text after the formula (5.130).

An attempt to interpret the functions $a^{*3}(\overrightarrow{k})$, $a^{*4}(\overrightarrow{k})$ inside the FW model leads to the result as follows. The functions $a^{*3}(\overrightarrow{k})$, $a^{*4}(\overrightarrow{k})$ in (4.28) are the

momentum-spin amplitudes of the antiparticle (e.g., positron) with the momentum $-\overrightarrow{\widehat{p}}$, sign of the charge $(+e)$ and spin projections $(\frac{1}{2}, -\frac{1}{2})$, respectively.

The explicit form of the generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (4.21) and the set $\{\phi\}$ of its solutions (4.28) are invariant, can be found both as the partial 4×4 matrix form of generators (6.5), (6.6) and as the v (2.49) consequences of the corresponding spin $s = (1/2, 1/2)$ RCQM generators.

Thus, the generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (4.21) and the set $\{\phi\}$ of its solutions (4.28) are invariant, are given by

$$\widehat{p}^0 = \gamma^0 \widehat{\omega} \equiv \gamma^0 \sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \tag{6.17}$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s_{\text{FW}}^{\ell n} \equiv \widehat{m}^{\ell n} + s_{\text{FW}}^{\ell n},$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\gamma^0 \left\{ x^\ell, \widehat{\omega} \right\} + \gamma^0 \frac{(\overrightarrow{s}_{\text{FW}} \times \overrightarrow{\widehat{p}})^\ell}{\widehat{\omega} + m}, \tag{6.18}$$

where FW spin SU(2) generators $\overrightarrow{s}_{\text{FW}} = (s_{\text{FW}}^{\ell n})$ have the form (2.48), γ^0 is given in (1.5).

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_4, \tag{6.19}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_{\text{FW}}^2 = \frac{1}{2}\left(\frac{1}{2} + 1\right) m^2 \mathrm{I}_4, \tag{6.20}$$

where $\overrightarrow{s}_{\text{FW}}$ is given in (2.48) and I_4 is 4×4 unit matrix.

Below some additional important features of the FW model are considered. Here all operators are taken in anti-Hermitian form, which is necessary for the transition from the RCQM to the FW model.

The FW equation with anti-Hermitian Hamiltonian has the form

$$\left(\partial_0 + i\gamma^0 \widehat{\omega}\right) \phi(x) = 0; \quad \phi = vf \equiv \left| \begin{array}{c} \varphi_- \\ \varphi_+^* \end{array} \right| \in \mathrm{H}^{3,4}. \tag{6.21}$$

The quantum-mechanical sense of the object ϕ in the FW representation is as follows. The equation (6.21) is a system of two 2-component equations

$$(\partial_0 + i\widehat{\omega})\varphi_-(x) = 0, \quad (\partial_0 - i\widehat{\omega})\varphi_+^*(x) = 0. \tag{6.22}$$

The first equation is the equation for the wave function φ_- of the particle (electron) and the second is the one for the function φ_+^*, being the complex conjugate to the wave function φ_+, of the antiparticle (positron).

The action of the operator v (2.49) on arbitrary RCQM (or arbitrary FW) operator is given by

$$vq_{\text{cf}}^{\text{anti}-\text{Herm}}v = q_{\text{qm}}^{\text{anti}-\text{Herm}}, \quad vq_{\text{qm}}^{\text{anti}-\text{Herm}}v = q_{\text{cf}}^{\text{anti}-\text{Herm}}. \tag{6.23}$$

Here $q_{\text{qm}}^{\text{anti}-\text{Herm}}$ is an arbitrary operator from the RCQM of the 4-component particle-antiparticle doublet in the anti-Hermitian form, e.g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion (2.35), the RCQM spin operator $\overrightarrow{s}$ from (2.47) taken in anti-Hermitian form, etc., $q_{\text{cf}}^{\text{anti}-\text{Herm}}$ is an arbitrary operator from the canonical field theory of the 4-component particle-antiparticle doublet in the anti-Hermitian form.

Under the transformations (6.23) the $\mathcal{P}^{\text{f}}$-generators (5.20), (5.21) *(taken with the spin term (2.47) and in the prime anti-Hermitian form)* become the prime $\mathcal{P}^{\phi}$-generators ($\mathcal{P}$-symmetries of the FW equation (4.21))

$$p_0 = -i\gamma_0\widehat{\omega}, \ p_\ell = \partial_\ell, \ j_{\ell n} = x_\ell\partial_n - x_n\partial_\ell + s_{\ell n} \equiv m_{\ell n} + s_{\ell n}, \tag{6.24}$$

$$j_{0\ell} = t\partial_\ell + \frac{i}{2}\gamma_0\left\{x_l, \widehat{\omega}\right\} + \gamma_0\frac{s_{\ell n}p_n}{\widehat{\omega}+m}, \tag{6.25}$$

where

$$s_{\ell n} = s^{\ell n} \equiv \frac{1}{4}\left[\gamma^\ell, \gamma^n\right], \tag{6.26}$$

and γ^μ are the standard Dirac matrices (1.5) (in the Pauli–Dirac representation).

Therefore, the $\mathcal{P}^{\phi}$-generators (6.24), (6.25) of the $\mathcal{P}^{\phi}$-representation in H^{3,4} (as well as the operators $q^{\phi} = vq^{\text{f}}v = vq_{\text{qm}}^{\text{anti}-\text{Herm}}v$ of the algebra of all observable physical quantities in the FW model of the fermionic doublet) are the functions generated by the 10 operators

$$\overrightarrow{x} = (x^j) \in \text{R}^3, \ \overrightarrow{p} = -\nabla, \ \overrightarrow{s} \equiv (s_{23}, s_{31}, s_{12}), \ \gamma^0, \tag{6.27}$$

where $s_{\ell n}$ are given in (6.26). The physical sense of these operators (as well as of the functions q^{ϕ} from them) follows from the physical sense of the corresponding quantum-mechanical operators $(\overrightarrow{x}, \overrightarrow{p}\,\overrightarrow{s}, \gamma^0)$ (and q^{f}), which are verified by the principles of the heredity and correspondence with non-relativistic quantum and classical theories.

Note that in the subsection 5.7.2 the general solution of the Schrödinger–Foldy equation (5.120) for the spin $s = (1/2, 1/2)$ particle-antiparticle doublet is presented in the form (5.121), where for the amplitudes the notations $a_+^-(\overrightarrow{k}), a_-^-(\overrightarrow{k}), a_-^+(\overrightarrow{k}), a_+^+(\overrightarrow{k})$ are applied. Such notations are very useful for the interpretation of the amplitudes in the case of spin $s = (1/2, 1/2)$ particle-antiparticle doublet. Interpretation now is demonstrative and evident. Nevertheless, for the generalization for arbitrary spin the notations $a^1(\overrightarrow{k}), a^2(\overrightarrow{k}), a^3(\overrightarrow{k}), a^4(\overrightarrow{k})$ are much better.

Further, similarly to the RCQM in the FW model the additional conservation laws also exist together with the main 10 Poincaré conservation quantities

$$(p_\mu, j_{\mu\nu})^\phi \rightarrow (P_\mu, J_{\mu\nu})^\phi = \int d^3x \phi^\dagger(x) i(p_\mu, j_{\mu\nu})^\phi \phi(x). \qquad (6.28)$$

The 12 additional conservation laws, which were considered in subsection 5.5.9, also exist and can be very easy calculated here. Naturally, due to non-unitarity of the operator v from (2.49) the explicit form of the conservation laws (6.28) does not coincide with the quantum-mechanical quantities (5.58). It is evident from the expression (6.28) in the terms of quantum-mechanical amplitudes

$$(P_\mu, J_{\mu\nu})^\phi = \int d^3k A^{\phi\dagger}(\overrightarrow{k})(\widetilde{p}_\mu, \widetilde{j}_{\mu\nu})^\phi A^\phi(\overrightarrow{k}), \qquad (6.29)$$

where $A^\phi(\overrightarrow{k})$ has the form $A^\phi(\overrightarrow{k}) \equiv \begin{vmatrix} a_+^- \\ a_-^- \\ a_-^{*+} \\ a_+^{*+} \end{vmatrix}$, $(\widetilde{p}_\mu, \widetilde{j}_{\mu\nu})^\phi$ are given by

$$\widetilde{p}_0 = \gamma^0 \widetilde{\omega}, \ \widetilde{p}_\ell = \gamma^0 k_\ell, \ \widetilde{j}_{\ell n} = \widetilde{x}_\ell k_n - \widetilde{x}_n k_\ell + \widehat{s}_{\ell n}, \qquad (6.30)$$

$$\widetilde{j}_{0\ell} = -\widetilde{j}_{\ell 0} = -\frac{1}{2}\{\widetilde{x}_\ell, \widetilde{\omega}\} + \gamma^0(\widecheck{s}_\ell \equiv \frac{\widehat{s}_{\ell n} k_n}{\widetilde{\omega} + m}), \qquad (6.31)$$

and the definitions $(\widetilde{x}_\ell = -i\widetilde{\partial}_\ell, \ \widetilde{\partial}_\ell \equiv \frac{\partial}{\partial k^\ell})$, $\widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}$ are used.

For example, the total energy of the field ϕ, instead of the expression P_0^{QM} in spin $s = (1/2, 1/2)$ particle-antiparticle doublet RCQM, has the form

$$P_0 = \int d^3k \omega \left(a_r^{*-}(\overrightarrow{k})a_r^-(\overrightarrow{k}) - a_{\widecheck{r}}^+(\overrightarrow{k})a_{\widecheck{r}}^{*+}(\overrightarrow{k}) \right), \quad r = (1,2), \widecheck{r} = (3,4),$$
$$(6.32)$$

which is not positively defined. In this sense, the FW model (whose quantum-mechanical content is unambiguous) in fact is not the quantum-mechanical model for the spin s=(1/2,1/2) particle-antiparticle doublet. Therefore, in the procedure of "canonical quantization" of the field ϕ on the basis of anticommutation relations

$$\left\{ \widehat{a}_{\bar{\alpha}}(\overrightarrow{k}), \widehat{a}_{\bar{\beta}}^{\dagger}(\overrightarrow{k}) \right\} = \delta_{\bar{\alpha}\bar{\beta}} \delta\left(\overrightarrow{k} - \overrightarrow{k}' \right) \tag{6.33}$$

an *additional axiom* is applied in this model for the quantized field $\widehat{\phi}$. According to this axiom the definition of the operators of the physical quantities in the Fock space $\mathcal{H}^{\mathrm{F}}$ is extended by taking them only in the form of "normal products" with respect to the operator amplitudes $\widehat{a}_{\mathrm{r}}^-(\overrightarrow{k})$, $\widehat{a}_{\mathrm{r}}^{*-}(\overrightarrow{k})$, $\widehat{a}_{\mathrm{f}}^+(\overrightarrow{k})$, $\widehat{a}_{\mathrm{f}}^{*+}(\overrightarrow{k})$. It is easy to verify that the operators of 10 main conserved quantities of the "quantized field" $\widehat{\phi}$ in the form of normal products coincide with the corresponding expressions in the "second quantized" RCQM model of the Fermionic doublet

$$: (\widehat{P}_{\mu}, \widehat{J}_{\mu\nu})^{\phi} := \int d^3 k \widehat{A}^{\dagger}(\overrightarrow{k})(\breve{p}_{\mu}, \breve{j}_{\mu\nu})\widehat{A}(\overrightarrow{k}), \tag{6.34}$$

where $\widehat{A}(\overrightarrow{k}) \equiv \begin{vmatrix} \widehat{a}_+^- \\ \widehat{a}_-^- \\ \widehat{a}_-^+ \\ \widehat{a}_+^+ \end{vmatrix}$ and $(\breve{p}_{\mu}, \breve{j}_{\mu\nu})$ are given in (5.61), (5.62).

6.2.3.   The Example of Spin s=(1,1) Particle-Antiparticle Doublet

The model based on the canonical field equation for the 6-component spin s=(1,1) particle-antiparticle doublet is under consideration.

The RCQM of the spin s=(1,1) particle-antiparticle doublet is considered in subsection 5.7.3. Transition to the canonical field theory is fulfilled in a complete analogy with the method suggested in section 6.1 and subsection 6.2.2. The difference is only in the explicit form of the transition operator, which is now given by the 6×6 diagonal matrix with the 3×3 operator of the complex conjugation in the bottom part.

Thus, the Schrödinger–Foldy equation (5.135) and its solution (5.136) are linked with the canonical field theory equation

$$(i\partial_0 - \Gamma_6^0 \widehat{\omega})\phi(x) = 0; \quad \Gamma_6^0 = \begin{vmatrix} \mathrm{I}_3 & 0 \\ 0 & -\mathrm{I}_3 \end{vmatrix}, \, \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \tag{6.35}$$

and its solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} c^{\ell}(\overrightarrow{k}) d_{\ell} + e^{ikx} c^{*\breve{\ell}}(\overrightarrow{k}) d_{\breve{\ell}} \right], \quad \ell = 1,2,3, \breve{\ell} = 4,5,6,$$

(6.36)

with the help of the following operator

$$v_6 = \begin{vmatrix} I_3 & 0 \\ 0 & CI_3 \end{vmatrix}; \quad v_6^{-1} = v_6^{\dagger} = v_6, \ v_6 v_6 = I_6,$$

(6.37)

where the Cartesian orts $(d_{\ell}, d_{\breve{\ell}})$ are given in (5.137), C is the operator of the complex conjugation and I_3 is the 3×3 unit matrix.

The operator v_6 (6.37) transforms an arbitrary operator $\hat{q}_{qm}$ of the 6-component RCQM into the corresponding operator $\hat{q}_{cf}$ of the canonical field theory and *vice verso*:

$$v_6 \hat{q}_{qm}^{anti-Herm} v_6 = \hat{q}_{cf}^{anti-Herm},$$

(6.38)

$$v_6 \hat{q}_{cf}^{anti-Herm} v_6 = \hat{q}_{qm}^{anti-Herm},$$

(6.39)

where the operators $\hat{q}_{qm}$ and $\hat{q}_{cf}$ must be taken in the anti-Hermitian form. The corresponding link between solutions (5.136) and (6.36) is as follows

$$\phi = v_6 f, \quad f = v_6 \phi.$$

(6.40)

Note that formulas (6.38), (6.39) are valid for the anti-Hermitian (prime) operators only. For our goals we often use the operators in the anti-Hermitian form (see also the comments in Chapter 3). The mathematical correctness of such choice and the physical interpretation are explained in the books [21, 22]. Return to the Hermitian operators is very easy.

The important examples of the transitions (6.38), (6.39) are the transformations of the operators of equations (5.135), (6.35)

$$v_6 (\partial_0 + i\widehat{\omega}) v_6 = \partial_0 + i\Gamma_6^0 \widehat{\omega},$$

(6.41)

$$v_6 (\partial_0 + i\Gamma_6^0 \widehat{\omega}) v_6 = \partial_0 + i\widehat{\omega},$$

(6.42)

and of the SU(2) spin operators. The spin operators of the canonical field theory, found from the RCQM SU(2) spin (5.139) on the basis of the transformation

(6.37), satisfy the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n} s^n$ and have the following form

$$\overrightarrow{s} = \left| \begin{array}{cc} \overrightarrow{s}_3 & 0 \\ 0 & \overrightarrow{s}_3 \end{array} \right|, \tag{6.43}$$

where the 3×3 spin s=1 SU(2) generators are denoted as $\overrightarrow{s}_3$ and are given in (5.90). The corresponding Casimir operator is given by

$$\overrightarrow{s}^2 = 2\mathrm{I}_6 = 1(1+1)\mathrm{I}_6, \tag{6.44}$$

where I_6 is the 6×6- unit matrix.

The stationary complete set of operators is given by the operators $\overrightarrow{p}$, $s^3 = s_z$ of the momentum and spin projection. The equations on the eigenvectors and eigenvalues of the operators $\overrightarrow{p}$ and $s^3 = s_z$ from (6.43) have the form

$$\widehat{\overrightarrow{p}}\, e^{-ikx} \mathrm{d}_\ell = \overrightarrow{k}\, e^{-ikx} \mathrm{d}_\ell, \ell = 1,2,3; \quad \widehat{\overrightarrow{p}}\, e^{ikx} \mathrm{d}_{\breve{\ell}} = -\overrightarrow{k}\, e^{ikx} \mathrm{d}_{\breve{\ell}}, \breve{\ell} = 4,5,6;$$

$$s^3 \mathrm{d}_1 = \mathrm{d}_1, s^3 \mathrm{d}_2 = 0, s^3 \mathrm{d}_3 = -\mathrm{d}_3, s^3 \mathrm{d}_4 = \mathrm{d}_4, s^3 \mathrm{d}_5 = 0, s^3 \mathrm{d}_6 = -\mathrm{d}_6, \tag{6.45}$$

and determine the interpretation of the amplitudes in the general solution (6.36). Note that the direct quantum-mechanical interpretation of the amplitudes $c^\ell(\overrightarrow{k})$, $c^{*\ell}(\overrightarrow{k})$ in solution (6.36) should be taken from the quantum-mechanical equations (5.142), (5.143) and is given in subsection 5.7.3.

The relativistic invariance of the canonical field equation (6.35) follows from the corresponding invariance of the Schrödinger–Foldy equation (5.135) and transformation (6.37)–(6.40) (for the anti-Hermitian operators). The explicit form of the corresponding generators follows from of the explicit form the generators (5.20), (5.21) with the spin matrices (5.139), (5.140) and transformation (6.37)–(6.39).

Thus, the canonical field equation (6.35) and the set $\{\phi\}$ of its solutions (6.36) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$, whose Hermitian 6×6 matrix-differential generators are given by

$$\widehat{p}^0 = \Gamma_6^0 \widehat{\omega} \equiv \Gamma_6^0 \sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s^{\ell n} \equiv \widehat{m}^{\ell n} + s^{\ell n}, \tag{6.46}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\Gamma_6^0 \left\{ x^\ell, \widehat{\omega} \right\} + \Gamma_6^0 \frac{(\overrightarrow{s} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega} + m}, \tag{6.47}$$

where the spin s=(1,1) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ have the form (6.43) and matrix Γ_6^0 is given in (6.35).

It is easy to prove by the direct verification that generators (6.46), (6.47) commute with the operator $(i\partial_0 - \Gamma_6^0 \widehat{\omega})$ of the canonical field equation (6.35) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$.

The corresponding Casimir operators are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_6, \tag{6.48}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = 1\,(1+1)\,m^2 \mathrm{I}_6, \tag{6.49}$$

where I_6 is the 6×6 unit matrix. Note that the difference between the Casimir operators (6.48), (6.49) and the corresponding expressions in RCQM, see the formulae (5.144), (5.145) above, is only in the explicit form of the operator $\overrightarrow{s}$.

Thus, due to the eigenvalues in equations (6.45), positive and negative frequencies form of the solution (6.36) and the Bargman–Wigner analysis of the Casimir operators (6.48), (6.49), one can come to a conclusion that equation (6.35) describes the canonical field (the bosonic particle-antiparticle doublet) with the spins s=(1,1) and $m > 0$. Transition to the $m = 0$ limit leads to the canonical field equation for the photon-antiphoton field.

6.2.4.   The Example of Spin s=(3/2,3/2) Particle-Antiparticle Doublet

The model based on the canonical field equation for the 8-component spin s=(3/2,3/2) particle-antiparticle fermionic doublet is under consideration.

The start of this derivation is given in subsection 5.7.4, where the RCQM of the 8-component fermionic spin s=(3/2,3/2) particle-antiparticle doublet is considered. The second step in our synthesis of covariant particle equations is the transition from the Schrödinger–Foldy equation (5.146) to the canonical field equation. This step is possible only for the anti-Hermitian form of the operators. Nevertheless, the resulting operators can be chosen in the standard Hermitian form and do not contain the operator C of complex conjugation. The last step of the transition from the canonical field equation to the covariant local field equation is fulfilled in analogy with the FW transformation in next Chapter below.

The transition from the Schrödinger–Foldy equation (5.146) and general solution (5.148) to the corresponded FW type equation and its solution is given by the operator

$$
v_8 = \begin{vmatrix} I_4 & 0 \\ 0 & CI_4 \end{vmatrix}, \quad v_8^{-1} = v_8^{\dagger} = v_8, \quad v_8 v_8 = I_8, \tag{6.50}
$$

$$
\phi = v_8 f, \quad f = v_8 \phi, \tag{6.51}
$$

$$
v_8 \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}} v_8 = \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}, \tag{6.52}
$$

$$
v_8 \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}} v_8 = \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}. \tag{6.53}
$$

Here $\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the RCQM of the 8-component particle-antiparticle doublet in the anti-Hermitian form, e.g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion, the operator of spin (5.150)=(5.151), etc., $\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 8-component particle-antiparticle doublet in the anti-Hermitian form, CI_4 is the 4×4 operator of complex conjugation.

Thus, the canonical field equation for the 8-component spin s=(3/2,3/2) fermionic particle-antiparticle doublet (8 component analogy of the FW equation) is found from the the Schrödinger–Foldy equation (5.146) on the basis of the transformation v_{2N} (5.28), taken above in 8×8 partial form, and is given by.

$$
(i\partial_0 - \Gamma_8^0 \widehat{\omega})\phi(x) = 0, \quad \phi = \begin{vmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \\ \phi^5 \\ \phi^6 \\ \phi^7 \\ \phi^8 \end{vmatrix}, \tag{6.54}
$$

where

$$
\widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \Gamma_8^0 = \begin{vmatrix} I_4 & 0 \\ 0 & -I_4 \end{vmatrix}, \quad I_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{6.55}
$$

The general solution of this equation in the case of spin s=(3/2,3/2) fermionic particle-antiparticle doublet is found with the help of the transformation v_8 (6.50), (6.51) from the general solution (5.148) of the Schrödinger–Foldy equation (5.146) and is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} b^A(\overrightarrow{k}) d_A + e^{ikx} b^{*B}(\overrightarrow{k}) d_B \right], \tag{6.56}$$

where $A = \overline{1,4}$, $B = \overline{5,8}$, the orts of the 8-component Cartesian basis are given in (5.149) and the quantum-mechanical interpretation of the amplitudes $(b^A(\overrightarrow{k}), b^{*B}(\overrightarrow{k}))$ is given according to (5.154)–(5.156).

The SU(2) spin operators, which satisfy the commutation relations (5.91) and commute with the operator $(i\partial_0 - \Gamma_8^0 \widehat{\omega})$ of the equation of motion (6.54), are derived from the corresponding RCQM operators (5.150), (5.151) on the basis of transformations (6.50), (6.52). These canonical field spin operators are given by

$$\overrightarrow{s}_8 = \begin{vmatrix} \overrightarrow{s} & 0 \\ 0 & \overrightarrow{s} \end{vmatrix}, \quad \overrightarrow{s}_8^2 = \frac{3}{2}\left(\frac{3}{2}+1\right) I_8, \tag{6.57}$$

where the 4×4 operators $\overrightarrow{s}$ are given in (5.100). In the explicit form the SU(2) spin operators (6.57) are given by

$$s_8^1 = \frac{1}{2} \begin{vmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \end{vmatrix},$$

$$s_8^2 = \frac{i}{2} \begin{vmatrix} 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \end{vmatrix}, \tag{6.58}$$

$$s_8^3 = \frac{1}{2} \begin{vmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{vmatrix}.$$

The stationary complete set of operators is given by the operators $g, \widehat{\overrightarrow{p}}, s_8^3 = s_z$ of the charge sign, momentum and spin projection, respectively. The equations on the eigen vectors and eigenvalues of the spin projection operator $s_8^3 = s_z$ from (6.58) have the form

$$s_8^3 d_1 = \frac{3}{2} d_1, \; s_8^3 d_2 = \frac{1}{2} d_2, \; s_8^3 d_3 = -\frac{1}{2} d_3, \; s_8^3 d_4 = -\frac{3}{2} d_4,$$

$$s_8^3 d_5 = \frac{3}{2} d_5, \; s_8^3 d_6 = \frac{1}{2} d_6, \; s_8^3 d_7 = -\frac{1}{2} d_7, \; s_8^3 d_8 = -\frac{3}{2} d_8. \qquad (6.59)$$

The equations on eigenvalues of the operators $g, \widehat{\overrightarrow{p}}$ are similar as in the above given examples of FW particle-antiparticle doublets. Therefore, the functions $b^1(\overrightarrow{k}), b^2(\overrightarrow{k}), b^3(\overrightarrow{k}), b^4(\overrightarrow{k})$ in solution (6.59) are the momentum-spin amplitudes of the massive fermion with the spin s=3/2 and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively; $b^{*5}(\overrightarrow{k}), b^{*6}(\overrightarrow{k}), b^{*7}(\overrightarrow{k}), b^{*8}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antifermion) with the spin s=3/2 and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively.

Note that direct quantum-mechanical interpretation of the amplitudes in solution (6.56) should be given in the framework of the RCQM. Such interpretation is already given in subsection 6.2.4 in paragraph after equations (5.156).

The generators of the reducible unitary fermionic spin s=(3/2,3/2) doublet representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (6.54) and the set $\{\phi\}$ of its solutions (6.56) are invariant, are derived from the RCQM set of generators (5.20), (5.21) with the spin (5.150), (5.151) on the basis of the transformations (6.50), (6.52).

Thus, the canonical field equation (6.54) and the set $\{\phi\}$ of its solutions (6.56) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$, whose Hermitian 8×8 matrix-differential gen-

erators are given by

$$\widehat{p}^0 = \Gamma_8^0 \widehat{\omega} \equiv \Gamma_8^0 \sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s^{\ell n} \equiv \widehat{m}^{\ell n} + s^{\ell n}, \tag{6.60}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\Gamma_8^0 \left\{ x^\ell, \widehat{\omega} \right\} + \Gamma_8^0 \frac{(\overrightarrow{s} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega} + m}, \tag{6.61}$$

where the spin s=(3/2,3/2) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ have the form (6.57)=(6.58) and matrix Γ_8^0 is given in (6.55).

It is easy to prove by the direct verification that the generators (6.60), (6.61) commute with the operator $(i\partial_0 - \Gamma_8^0 \widehat{\omega})$ of the canonical field equation (6.54) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$. The Casimir operators for the representation (6.60), (6.61) with SU(2) spin (6.57)=(6.58) are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_8, \tag{6.62}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_8^2 = \frac{3}{2}\left(\frac{3}{2} + 1\right)\mathrm{I}_8. \tag{6.63}$$

Thus, due to the eigenvalues in equations (6.59), positive and negative frequencies form of solution (6.56) and the Bargman–Wigner analysis of the Casimir operators (6.62), (6.63) one can come to a conclusion that equation (6.54) describes the 8-component canonical field (the fermionic particle-antiparticle doublet) with the spins s=(3/2,3/2) and $m > 0$.

6.2.5.   The Example of Spin s=(2,2) Particle-Antiparticle Doublet

The model based on the canonical field equation for the 10-component spin s=(2,2) particle-antiparticle bosonic doublet is under consideration.

This model is constructed in complete analogy for the consideration of the spin s=(1,1) particle-antiparticle doublet given in subsection 6.2.3.

The RCQM of the spin s=(2,2) particle-antiparticle doublet is considered in subsection 5.7.5. Transition to the canonical field theory is fulfilled in a complete analogy with the above applied method. The difference is only in

the explicit form of the transition operator, which is now given by the 10×10 diagonal matrix with the 5×5 operator of the complex conjugation in the bottom part.

Thus, the Schrödinger–Foldy equation (5.158) and its solution (5.159) are linked with the canonical field theory equation

$$(i\partial_0 - \Gamma^0_{10}\widehat{\omega})\phi(x) = 0, \quad \Gamma^0_{10} = \begin{vmatrix} I_5 & 0 \\ 0 & -I_5 \end{vmatrix}, \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \qquad (6.64)$$

and its solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^A(\overrightarrow{k}) d_A + e^{ikx} g^{*B}(\overrightarrow{k}) d_B \right], \qquad (6.65)$$

$$A = 1,2,3,4,5, \quad B = 6,7,8,9,10,$$

with the help of the following operator

$$v_{10} = \begin{vmatrix} I_5 & 0 \\ 0 & CI_5 \end{vmatrix}, \quad v_{10}^{-1} = v_{10}^{\dagger} = v_{10}, \, v_{10}v_{10} = I_{10}, \qquad (6.66)$$

where the Cartesian orts (d_A, d_B) are given in (5,160), C is the operator of the complex conjugation and I_5 is the 5×5 unit matrix.

The operator v_{10} (6.66) transforms an arbitrary operator $\hat{q}_{qm}$ of the 10-component RCQM from subsection 5.7.5 into the corresponding operator $\hat{q}_{cf}$ of the canonical field theory and *vice verso*:

$$v_{10}\hat{q}_{qm}^{anti-Herm} v_{10} = \hat{q}_{cf}^{anti-Herm}, \qquad (6.67)$$

$$v_{10}\hat{q}_{cf}^{anti-Herm} v_{10} = \hat{q}_{qm}^{anti-Herm}, \qquad (6.68)$$

where the operators $\hat{q}_{qm}$ and $\hat{q}_{cf}$ must be taken in the anti-Hermitian form. The corresponding link between solutions (5.159) and (6.65) is as follows

$$\phi = v_{10}f, \quad f = v_{10}\phi. \qquad (6.69)$$

Note that formulae (6.67), (6.68) are valid for the anti-Hermitian (prime) operators only. For the goals stated in our previous papers [16–20], we often use the operators in the anti-Hermitian form (see also the comments in Chapter 3). The mathematical correctness of such choice and the physical interpretation

are explained in the books [21, 22]. Return to the Hermitian operators is very easy.

The important examples of the transitions (6.67), (6.68) are the transformations of the operators of equations (5.158), (6.64)

$$v_{10}(\partial_0 + i\widehat{\omega})v_{10} = \partial_0 + i\Gamma^0_{10}\widehat{\omega}, \tag{6.70}$$

$$v_{10}(\partial_0 + i\Gamma^0_{10}\widehat{\omega})v_{10} = \partial_0 + i\widehat{\omega}, \tag{6.71}$$

and of the SU(2) spin operators. The spin operators of the canonical field theory found from the corresponding RCQM SU(2) spin (5.163) on the basis of the transformation (6.66) satisfy the commutation relations (5.91) and have the following form

$$\overrightarrow{s}_{10} = \left| \begin{array}{cc} \overrightarrow{s}_5 & 0 \\ 0 & \overrightarrow{s}_5 \end{array} \right|, \tag{6.72}$$

where the 5×5 spin s=2 SU(2) generators are denoted as $\overrightarrow{s}_5$ and are given in (5.109). The corresponding Casimir operator is given by

$$\overrightarrow{s}^2_{10} = 6I_{10} = 2(2+1)I_{10}, \tag{6.73}$$

where I_{10} is the 10×10- unit matrix.

The stationary complete set of operators is given by the operators $\widehat{\overrightarrow{p}}, s^3_{10} = s_z$ of the momentum and spin projection. The equations on the eigenvectors and eigenvalues of the operators $\widehat{\overrightarrow{p}}$ and $s^3_{10} = s_z$ from (6.72) have the form

$$\widehat{\overrightarrow{p}}e^{-ikx}d_A = \overrightarrow{k}e^{-ikx}d_A, A = \overline{1,5}, \quad \widehat{\overrightarrow{p}}e^{ikx}d_B = -\overrightarrow{k}e^{ikx}d_B, B = \overline{6,10},$$

$$s^3_{10}d_1 = 2d_1, s^3_{10}d_2 = d_2, s^3_{10}d_3 = 0, s^3_{10}d_4 = -d_4, s^3_{10}d_5 = -2d_5,$$

$$s^3_{10}d_6 = 2d_6, s^3_{10}d_7 = d_7, s^3_{10}d_8 = 0, s^3_{10}d_9 = -d_9, s^3_{10}d_{10} = -2d_{10}, \tag{6.74}$$

and determine the interpretation of the amplitudes in the general solution (6.65). Note that the direct quantum-mechanical interpretation of the amplitudes $g^A(\overrightarrow{k}), g^B(\overrightarrow{k})$ in solution (6.65) should be taken from the quantum-mechanical equations (5.165), (5.166) and is given in subsection 5.7.5.

The relativistic invariance of the canonical field equation (6.64) follows from the corresponding invariance of the Schrödinger–Foldy equation (5.158) and transformation (6.66)–(6.69) (for the anti-Hermitian operators). The explicit form of the corresponding generators follows from of the explicit form

the generators (5.20), (5.21) with the spin matrices (5.162), (5.163) and transformation (6.66), (6,67).

Thus, the canonical field equation (6.64) and the set $\{\phi\}$ of its solutions (6.65) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$, whose Hermitian 10×10 matrix-differential generators are given by

$$\widehat{p}^0 = \Gamma_{10}^0 \widehat{\omega} \equiv \Gamma_{10}^0 \sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s_{10}^{\ell n} \equiv \widehat{m}^{\ell n} + s_{10}^{\ell n}, \tag{6.75}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\Gamma_{10}^0 \left\{ x^\ell, \widehat{\omega} \right\} + \Gamma_{10}^0 \frac{(\overrightarrow{s}_{10} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega} + m}, \tag{6.76}$$

where the spin s=(2,2) SU(2) generators $\overrightarrow{s}_{10} = (s_{10}^{\ell n})$ have the form (6.72) and Γ_{10}^0 matrix is given in (6.64).

It is easy to prove by the direct verification that generators (6.75), (6.76) commute with the operator $(i\partial_0 - \Gamma_6^0 \widehat{\omega})$ of the canonical field equation (6.64) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$.

The corresponding Casimir operators are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_{10}, \tag{6.77}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_{10}^2 = 2(2+1) m^2 \mathrm{I}_{10}, \tag{6.78}$$

where I_{10} is the 10×10 unit matrix. Note that the difference between the Casimir operators (6.77), (6.78) and the corresponding expressions in RCQM, see the formulas (5.174), (5.175) above, is only in the explicit form of the operator $\overrightarrow{s}_{10}$.

Thus, due to the eigenvalues in equations (6,74), positive and negative frequencies form of the solution (6.65) and the Bargman–Wigner analysis of the Casimir operators (6.77), (6.78), one can come to a conclusion that equation (6.64) describes the canonical field (the bosonic particle-antiparticle doublet) with the spins s=(2,2) and $m > 0$. Transition to the $m = 0$ limit leads to the canonical field equation for the graviton-antigraviton field (if the graviton is massless ?).

Final transition to the covariant local field theory in the case of this 10-component doublet contains the specific features and is the subject of further investigations. The result of such transition can be the Duffin–Kemmer–Petiau equation or the 10-component Dirac-like equation for the spin s=(2,2) and $m > 0$ classical field. The resulting covariant local field theory equation can be found in an analogy with the consideration of the spin s=(1,1) doublet.

6.3. Examples of Particle-Antiparticle Multiplets

6.3.1. The Example of Spin s=(1,0,1,0) Particle-Antiparticle Multiplet

The model based on the canonical field equation for the 10-component spin s=(1,0,1,0) particle-antiparticle bosonic multiplet is under consideration.

In this section, the canonical FW type field model of the spin s=(1,0) bosonic multiplet and the corresponding spin s=(1,0) antiparticle multiplet is constructed. The start of this derivation is given in subsection 5.8.2, where the RCQM model of the 8-component bosonic spin s=(1,0,1,0) particle-antiparticle multiplet is given. The analogy with the case of the spin s=(3/2,3/2) model of the subsection 6.2.4 is used. Some formulas are similar. The case of the spin s=(3/2,3/2) is 8-component as well. Therefore, the details can be omitted.

The canonical field equation of spin s=(1,0,1,0) particle-antiparticle multiplet is the same as in (6.54). The link with the corresponded RCQM is given by (6.50)–(6.53). The general solution of equation (6.54) for s=(1,0,1,0) is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} c^{\mathrm{A}}(\overrightarrow{k}) \mathrm{d_A} + e^{ikx} c^{*\mathrm{B}}(\overrightarrow{k}) \mathrm{d_B} \right], \tag{6.79}$$

where $\mathrm{A} = \overline{1,4}$, $\mathrm{B} = \overline{5,8}$ and the orts of the 8-component Cartesian basis are given in (5.149).

The SU(2) spin operators, which satisfy the commutation relations (5.91) and commute with the operator $(i\partial_0 - \Gamma_8^0 \widehat{\omega})$ of the equation of motion (6.54), are derived from the RCQM operators (5.179) on the basis of transformations (6.50), (6.52). These canonical field spin operators are given by

$$\overrightarrow{s}_8 = \left| \begin{array}{cc} \overrightarrow{s} & 0 \\ 0 & \overrightarrow{s} \end{array} \right|, \tag{6.80}$$

where the 4×4 operators $\overrightarrow{s}$ are given in (5.171). The corresponding Casimir operator is given by the 8×8 diagonal matrix (5.180). Note that this Casimir operator is the same both for the spin (5.179) and for the spin (6.80).

The stationary complete set of operators is given by the operators $\widehat{\overrightarrow{p}}$, $s_8^3 = s_z$ of the momentum and spin projection. The equations on the eigenvectors and eigenvalues of the spin projection operator $s_8^3 = s_z$ from (6.80) have the form

$$s_8^3 d_1 = d_1, \ s_8^3 d_2 = 0, \ s_8^3 d_3 = -d_3, \ s_8^3 d_4 = 0,$$

$$s_8^3 d_5 = d_5, \ s_8^3 d_6 = 0, \ s_8^3 d_7 = -d_7, \ s_8^3 d_8 = 0. \tag{6.81}$$

Therefore, the functions $c^1(\overrightarrow{k}), c^2(\overrightarrow{k}), c^3(\overrightarrow{k})$ in solution (6.79) are the momentum-spin amplitudes of the massive boson with the spin s=1 and the spin projection $(1, 0, -1)$, respectively, $c^4(\overrightarrow{k})$ is the amplitude of the spinless boson; $c^{*5}(\overrightarrow{k}), c^{*6}(\overrightarrow{k}), c^{*7}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antiboson) with the spin s=1 and the spin projection $(1, 0, -1)$, respectively, $c^{*8}(\overrightarrow{k})$ is the amplitude of the spinless antiboson.

Note that direct quantum-mechanical interpretation of the amplitudes in solution (6.79) should be given in the framework of the RCQM. Such interpretation is already given in subsection 5.8.2 in paragraph after equations (5.181).

The generators of the reducible unitary bosonic spin s=(1,0,1,0) multiplet representation (5.22) of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (6.54) and the set $\{\phi\}$ of its solutions (6.79) are invariant, are derived from the RCQM set of generators (5.20), (5.21) with the spin (5.179) on the basis of the transformations (6.50), (6.52). These Hermitian 8×8 matrix-differential generators are given by

$$\widehat{p}^0 = \Gamma_8^0 \widehat{\omega} \equiv \Gamma_8^0 \sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s_8^{\ell n} \equiv \widehat{m}^{\ell n} + s_8^{\ell n}, \tag{6.82}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\Gamma_8^0 \left\{x^\ell, \widehat{\omega}\right\} + \Gamma_8^0 \frac{(\overrightarrow{s}_8 \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega} + m}, \tag{6.83}$$

where the spin s=(1,0,1,0) SU(2) generators $\overrightarrow{s}_8 = (s_8^{\ell n})$ have the form (6.80) and matrix Γ_8^0 is given in (6.55).

It is easy to prove by the direct verification that the generators (6.82), (6.83) commute with the operator $(i\partial_0 - \Gamma_8^0 \widehat{\omega})$ of the canonical field equation (6.54)

and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$. The Casimir operators for the representation (6.82), (6,83) are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_8, \qquad (6.84)$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_8^2 = m^2 \begin{vmatrix} 1(1+1)\mathrm{I}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1(1+1)\mathrm{I}_3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad (6.85)$$

where I_8 and I_3 are 8×8 and 3×3 unit matrices, respectively.

Thus, due to the eigenvalues in equations (6.81), positive and negative frequencies form of solution (6.79) and the Bargman–Wigner analysis of the Casimir operators (6.84), (6.85) one can come to a conclusion that equation (6.54) describes the 8-component canonical field (the bosonic particle-antiparticle multiplet) with the spins s=(1,0,1,0) and $m > 0$. Transition to the $m = 0$ limit leads to the canonical field equation for the 8-component (photon massless spinless boson)-(antiphoton- massless spinless antiboson) field.

Note that in subsection 6.2.4 the same equation (6.54) was proved to describe another object, i.e., the fermionic spin s=(3/2,3/2) particle-antiparticle doublet. Therefore, the equation (6.54) is good example in order to explain the property of the Fermi-Bose duality of relativistic particle equations, which is discussed in our last years publications, see, e.g., the papers [23, 24].

The investigation that only two concepts, *the equation of motion and the representation of the Poincaré group, under which the equation is invariant*, determine the physical interpretation of equation was presented first in [25]. We wrote there *"One of the conclusions of the investigation presented here is that a field equation itself does not answer the question, what kind of particle (Bose or Fermi) is described by this equation. To answer this question one needs to find all the representations of the Poincaré group, under which the equation is invariant. If more than one such Poincaré representation is found, including the representations with integer and half-integer spins, then the given equation describes both Bose and Fermi particles, and both quantization types (Bose and Fermi) of the field function, obeying this equation, satisfy the micro-causality condition."*

6.3.2. The Example of Spin s=(2,0,2,0) Particle-Antiparticle Multiplet

The model based on the canonical field equation for the 12-component spin s=(2,0,2,0) particle-antiparticle bosonic multiplet is under consideration.

In this subsection, the united canonical field model of the spin s=(2,0) bosonic multiplet and the corresponding spin s=(2,0) antiparticle multiplet is constructed. The complete analogy with the case of the spin s=(1,0,1,0) model of subsection 6.3.1 is used.

The start of this derivation is given in subsection 5.8.3, where the RCQM of the 12-component bosonic spin s=(2,0,2,0) particle-antiparticle multiplet is considered.

Thus, the corresponding canonical field equation is found in the form

$$(i\partial_0 - \Gamma_{12}^0 \widehat{\omega})\phi(x) = 0, \quad \phi = \begin{vmatrix} \phi^1 \\ \phi^2 \\ \cdot \\ \cdot \\ \cdot \\ \phi^{12} \end{vmatrix}, \tag{6.86}$$

where

$$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \Gamma_{12}^0 = \begin{vmatrix} I_6 & 0 \\ 0 & -I_6 \end{vmatrix}, \quad I_6 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}. \tag{6.87}$$

The general solution of equation (6.86) is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^{\mathrm{A}}(\overrightarrow{k}) \mathrm{d_A} + e^{ikx} g^{*\mathrm{B}}(\overrightarrow{k}) \mathrm{d_B} \right], \tag{6.88}$$

where $A = \overline{1,6}$, $B = \overline{5,12}$ and the orts of the 12-component Cartesian basis are given in (5.186).

The transition from the Schrödinger–Foldy equation (5.184) and its solution (5.185) to canonical field equation (6.86) and solution (6.88) is given by the

operator

$$v_{12} = \begin{vmatrix} I_6 & 0 \\ 0 & CI_6 \end{vmatrix}, \quad v_{12}^{-1} = v_{12}^{\dagger} = v_{12}, \, v_{12}v_{12} = I_{12}, \tag{6.89}$$

$$\phi = v_{12}f, \quad f = v_{12}\phi, \tag{6.90}$$

$$v_{12}\hat{q}_{\text{qm}}^{\text{anti-Herm}}v_{12} = \hat{q}_{\text{cf}}^{\text{anti-Herm}}, \tag{6.91}$$

$$v_{12}\hat{q}_{\text{cf}}^{\text{anti-Herm}}v_{12} = \hat{q}_{\text{qm}}^{\text{anti-Herm}}. \tag{6.92}$$

Here $\hat{q}_{\text{qm}}^{\text{anti-Herm}}$ is an arbitrary operator from the RCQM of the 12-component particle-antiparticle doublet in the anti-Hermitian form, e.g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion, the operator of spin (5.188), etc., $\hat{q}_{\text{cf}}^{\text{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 12-component particle-antiparticle doublet in the anti-Hermitian form, CI_6 is the 6×6 operator of complex conjugation.

The SU(2) spin operators, which satisfy the commutation relations $\left[s^j, s^\ell\right] = i\varepsilon^{j\ell n}s^n$ and commute with the operator $(i\partial_0 - \Gamma_{12}^0\widehat{\omega})$ of the equation of motion (6.86), are derived from the RCQM operators (5.188) on the basis of transformations (6.89), (6.91). These canonical field spin operators are given by

$$\overrightarrow{s}_{12} = \begin{vmatrix} \overrightarrow{s}_6 & 0 \\ 0 & \overrightarrow{s}_6 \end{vmatrix}, \tag{6.93}$$

where the 6×6 operators $\overrightarrow{s}_6$ are given in (5.189). The corresponding Casimir operator is given by the 12×12 diagonal matrix (5.190). Note that this Casimir operator is the same both for the spin (5.188) and for the spin (6.93).

The stationary complete set of operators is given by the operators $\widehat{\overrightarrow{p}}, s_{12}^3 = s_z$ of the momentum and spin projection. The equations on the eigenvectors and eigenvalues of the spin projection operator $s_{12}^3 = s_z$ from (6.93) have the form

$$s_{12}^3 d_1 = 2d_1, \, s_{12}^3 d_2 = d_2, \, s_{12}^3 d_3 = 0, \, s_{12}^3 d_4 = -d_4,$$

$$s_{12}^3 d_5 = -2d_5, \, s_{12}^3 d_6 = 0, \, s_{12}^3 d_7 = 2d_7, \, s_{12}^3 d_8 = d_8, \tag{6.94}$$

$$s_{12}^3 d_9 = 0, \, s_{12}^3 d_{10} = -d_{10}, \, s_{12}^3 d_{11} = -2d_{11}, \, s_{12}^3 d_{12} = 0.$$

Therefore, the functions $g^1(\overrightarrow{k}), g^2(\overrightarrow{k}), g^3(\overrightarrow{k}), g^4(\overrightarrow{k}), g^5(\overrightarrow{k})$ in solution (6.88) are the momentum-spin amplitudes of the massive boson with the spin

s=2 and the spin projection $(2,1,0,-1,-2)$, respectively, $g^6(\overrightarrow{k})$ is the amplitude of the spinless boson; $g^7(\overrightarrow{k})$, $g^8(\overrightarrow{k})$, $g^9(\overrightarrow{k})$, $g^{10}(\overrightarrow{k})$, $g^{11}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antiboson) with the spin s=2 and the spin projection $(2,1,0,-1,-2)$, respectively, $g^{12}(\overrightarrow{k})$ is the amplitude of the spinless antiboson.

Note that direct quantum-mechanical interpretation of the amplitudes in solution (6.88) should be given in the framework of the RCQM. Such interpretation is already given in subsection 5.8.3 in paragraph after equations (5.192).

The generators of the reducible unitary bosonic spin s=(2,0,2,0) multiplet representation (5.22) of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (6.86) and the set $\{\phi\}$ of its solutions 6.88) are invariant, are derived from the RCQM set of generators (5.20), (5.21) with the spin (6.93) on the basis of the transformations (6.89), (6.91). These Hermitian 12×12 matrix-differential generators are given by

$$\widehat{p}^0 = \Gamma^0_{12}\widehat{\omega} \equiv \Gamma^0_{12}\sqrt{-\Delta+m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell\widehat{p}^n - x^n\widehat{p}^\ell + s^{\ell n}_{12} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{12}, \tag{6.95}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\Gamma^0_{12}\left\{x^\ell,\widehat{\omega}\right\} + \Gamma^0_{12}\frac{(\overrightarrow{s}_{12} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega}+m}, \tag{6.96}$$

where the spin s=(2,0,2,0) SU(2) generators $\overrightarrow{s}_{12} = (s^{\ell n}_{12})$ have the form (6.93) and matrix Γ^0_{12} is given in (6.87).

It is easy to prove by the direct verification that the generators (6.95), (6.96) commute with the operator $(i\partial_0 - \Gamma^0_{12}\widehat{\omega})$ of the canonical field equation (6.86) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$. The Casimir operators for the representation (6.95), (6.96) are given by

$$p^2 = \widehat{p}^\mu\widehat{p}_\mu = m^2\mathrm{I}_{12}, \tag{6.97}$$

$$W = w^\mu w_\mu = m^2\overrightarrow{s}^2_{12} = m^2\begin{vmatrix} 2(2+1)\mathrm{I}_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2(2+1)\mathrm{I}_5 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{6.98}$$

where I_{12} is 12×12 unit matrix and I_5 is 5×5 unit matrix.

Thus, due to the eigenvalues in equations (6.94), positive and negative frequencies form of solution (6.88) and the Bargman–Wigner analysis of the Casimir operators (6.97), (6.98) one can come to a conclusion that equation (6.86) describes the 12-component canonical field (the bosonic particle-antiparticle doublet) with the spins s=(2,0,2,0) and $m > 0$. Transition to the $m = 0$ limit leads to the canonical field equation for the 12-component (graviton massless spinless boson)-(antigraviton- massless spinless antiboson) field.

6.3.3. The Example of Spin s=(2,1,2,1) Particle-Antiparticle Multiplet

The model based on the canonical field equation for the 16-component spin s=(2,1,2,1) particle-antiparticle bosonic multiplet is under consideration.

In this section, the field model of the spin s=(2,1) bosonic multiplet and the corresponding spin s=(2,1) antiparticle multiplet is constructed. The complete analogy with the case of the spin s=(1,0,1,0) model of subsection 6.3.1 and the spin s=(2,0,2,0) model of the previous section is used.

The start of this derivation is given in subsection 5.8.4, where the RCQM of the 16-component bosonic spin s=(2,1,2,1) particle-antiparticle multiplet is considered. The second step is the transition from the Schrödinger–Foldy equation (5.195) to the canonical field equation. This step, as shown above, is possible only for the anti-Hermitian form of the operators. Nevertheless, the resulting operators can be chosen in the standard Hermitian form and do not contain the operator C of complex conjugation. The last step of the transition from the canonical field equation to the covariant local field equation is fulfilled in analogy with the FW transformation and is considered in next chapter.

Thus, the corresponding canonical field equation is found in the form

$$(i\partial_0 - \Gamma_{16}^0 \widehat{\omega})\phi(x) = 0, \quad \phi = \begin{vmatrix} \phi^1 \\ \phi^2 \\ . \\ . \\ . \\ \phi^{16} \end{vmatrix}, \tag{6.99}$$

where

$$\widehat{\omega} \equiv \sqrt{-\Delta+m^2}, \quad \Gamma_{16}^0 = \begin{vmatrix} I_8 & 0 \\ 0 & -I_8 \end{vmatrix}, \quad I_8 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}. \tag{6.100}$$

The general solution of equation (6.99) is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^{A}(\overrightarrow{k}) d_{A} + e^{ikx} g^{*B}(\overrightarrow{k}) d_{B} \right], \tag{6.101}$$

where $A = \overline{1,8}$, $B = \overline{9,16}$ and the orts of the 16-component Cartesian basis are given in (5.198).

The transition from the Schrödinger–Foldy equation (5.195) and its solution (5.197) to canonical field equation (6.99) and solution (6.101) is given by the operator

$$v_{16} = \begin{vmatrix} I_8 & 0 \\ 0 & CI_8 \end{vmatrix}, \quad v_{16}^{-1} = v_{16}^{\dagger} = v_{16}, v_{16}v_{16} = I_{16}, \tag{6.102}$$

$$\phi = v_{16}f, \quad f = v_{16}\phi, \tag{6.103}$$

$$v_{16}\hat{q}_{\text{qm}}^{\text{anti}-\text{Herm}}v_{16} = \hat{q}_{\text{cf}}^{\text{anti}-\text{Herm}}, \tag{6.104}$$

$$v_{16}\hat{q}_{\text{cf}}^{\text{anti}-\text{Herm}}v_{16} = \hat{q}_{\text{qm}}^{\text{anti}-\text{Herm}}. \tag{6.105}$$

Here $\hat{q}_{\text{qm}}^{\text{anti}-\text{Herm}}$ is an arbitrary operator from the RCQM of the 16-component particle-antiparticle doublet in the anti-Hermitian form, e.g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion, the operator of spin (5.199), etc., $\hat{q}_{\text{cf}}^{\text{anti}-\text{Herm}}$ is an arbitrary operator from the canonical field theory of the 16-component particle-antiparticle doublet in the anti-Hermitian form, CI_8 is the 8×8 operator of complex conjugation.

The SU(2) spin operators, which satisfy the commutation relations (5.91) and commute with the operator $(i\partial_0 - \Gamma^0_{16}\widehat{\omega})$ of the equation of motion (6.99), are derived from the RCQM operators (5.199) on the basis of transformations (6.102), (6.104). These canonical field spin operators are given by

$$\overrightarrow{s}_{16} = \left| \begin{array}{cc} \overrightarrow{s}_8 & 0 \\ 0 & \overrightarrow{s}_8 \end{array} \right|, \tag{6.106}$$

where the 8×8 operators $\overrightarrow{s}_8$ are given in (5.200). The corresponding Casimir operator is given by the 16×16 diagonal matrix (5.201). Note that this Casimir operator is the same both for the spin (5.199) and for the spin (6.106).

The stationary complete set of operators is given by the operators $\widehat{\overrightarrow{p}}$, $s^3_{16} = s_z$ of the momentum and spin projection. The equations on the eigenvectors and eigenvalues of the spin projection operator $s^3_{16} = s_z$ from (6.106) have the form

$$s^3_{16}d_1 = 2d_1, \ s^3_{16}d_2 = d_2, \ s^3_{16}d_3 = 0, \ s^3_{16}d_4 = -d_4,$$

$$s^3_{16}d_5 = -2d_5, \ s^3_{16}d_6 = d_6, \ s^3_{16}d_7 = 0, \ s^3_{16}d_8 = -d_8, \tag{6.107}$$

$$s^3_{16}d_9 = 2d_9, \ s^3_{16}d_{10} = d_{10}, \ s^3_{16}d_{11} = 0, \ s^3_{16}d_{12} = -d_{12},$$

$$s^3_{16}d_{13} = -2d_{13}, \ s^3_{16}d_{14} = d_{14}, \ s^3_{16}d_{15} = 0, \ s^3_{16}d_{16} = -d_{16}.$$

Therefore, the functions $g^1(\overrightarrow{k}), g^2(\overrightarrow{k}), g^3(\overrightarrow{k}), g^4(\overrightarrow{k}), g^5(\overrightarrow{k})$ in solution (6.101) are the momentum-spin amplitudes of the massive boson with the spin s=2 and the spin projection $(2,1,0,-1,-2)$, respectively, $g^6(\overrightarrow{k}), g^7(\overrightarrow{k}), g^8(\overrightarrow{k})$ are the momentum-spin amplitudes of the massive boson with the spin s=1 and the spin projection $(1,0,-1)$; the functions $g^9(\overrightarrow{k}), g^{10}(\overrightarrow{k}), g^{11}(\overrightarrow{k}), g^{12}(\overrightarrow{k}), g^{13}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antiboson) with the spin s=2 and the spin projection $(2,1,0,-1,-2)$, respectively, $g^{14}(\overrightarrow{k}), g^{15}(\overrightarrow{k}), g^{16}(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (massive antiboson) with the spin s=1 and the spin projection $(1,0,-1)$.

Note that direct quantum-mechanical interpretation of the amplitudes in solution (6.101) should be given in the framework of the RCQM. Such interpretation is already given in subsection 5.8.4 in paragraph after equations (5.203).

The generators of the reducible unitary bosonic spin s=(2,1,2,1) multiplet representation (5.22) of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (6.99) and the set $\{\phi\}$ of its solutions (6.101) are invariant, are

derived from the RCQM set of generators (5.20), (5.21) with the spin (6.106) on the basis of the transformations (6.102), (6.104). These Hermitian 16×16 matrix-differential generators are given by

$$\widehat{p}^0 = \Gamma^0_{16}\widehat{\omega} \equiv \Gamma^0_{16}\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n \widehat{p}^\ell + s^{\ell n}_{16} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{16}, \tag{6.108}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\Gamma^0_{16}\left\{x^\ell, \widehat{\omega}\right\} + \Gamma^0_{16}\frac{(\overrightarrow{s}_{16} \times \overrightarrow{p})^\ell}{\widehat{\omega} + m}, \tag{6.109}$$

where the spin s=(2,1,2,1) SU(2) generators $\overrightarrow{s}_{16} = (s^{\ell n}_{16})$ have the form (6.106) and matrix Γ^0_{16} is given in (6.100).

It is easy to prove by the direct verification that the generators (6.108), (6.109) commute with the operator $(i\partial_0 - \Gamma^0_{16}\widehat{\omega})$ of the canonical field equation (6.99) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$. The Casimir operators for the representation (6.108), (6.109) are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_{16}, \tag{6.110}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2_{16} = m^2 \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 1(1+1)I_3 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 1(1+1)I_3 \end{vmatrix}, \tag{6.111}$$

where I_{16} is 16×16 unit matrix, I_5 is 5×5 unit matrix and I_3 is 3×3 unit matrix, respectively.

Thus, due to the eigenvalues in equations (6.107), positive and negative frequencies form of solution (6.101) and the Bargman–Wigner analysis of the Casimir operators (6.110), (6.111) one can come to a conclusion that equation (6.99) describes the 16-component canonical field (the bosonic particle-antiparticle doublet) with the spins s=(2,1,2,1) and $m > 0$. Transition to the $m = 0$ limit leads to the canonical field equation for the 16-component (graviton - photon)-(antigraviton - antiphoton) field.

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Chapter 7

Covariant Equations of Arbitrary Spin

In this chapter the final step of our program of synthesis of arbitrary spin co-variant field equations is presented. The link between the canonical Foldy–Wouthuysen (FW) type of arbitrary spin equations and manifestly covariant field equations is presented. For the case of spin s=(1/2,1/2) particle-antiparticle doublet this link is given by the well-known inverse FW transformation [1, 2] (see the brief comment in subsection 2.2.9). In subsection 2.4.3 the direct link (2.38)–(2.40) between the relativistic canonical quantum mechanics (RCQM) and the Dirac equation is considered. It is useful to investigate another link, in which the FW type representation for arbitrary spin is the intermediate stage between the RCQM and manifestly covariant field theory, see, e.g., the consideration in [3–6]. Such link is useful due to the independent role of the canonical FW type representation in the field theory. Indeed, interesting partial examples of the doublets and multiplets described above in the Chapter 6 demonstrate the role of consideration on the level of canonical FW type field model. We suggest to pay attention for the spin s=(1,1) and spin s=(2,2) particle-antiparticle bosonic doublets described above in the subsections 6.2.3 and 6.2.5, respectively. The canonical FW type equations of motion (6.35) and (6.64) has been derived.

The chain of our synthesis program can be demonstrated as *RCQM → FW → manifestly covariant equations*. Therefore, here below the final step, where we derive the Dirac-like manifestly covariant equation of arbitrary spin together with partial cases of fixed spin particle-antiparticle doublets and multiplets. Note that transition from the FW to the Dirac-like model does not need the

appealing to the anti-Hermitian form of operators (which is necessary for the link between the RCQM and FW models).

7.1. Brief Analysis of the Existing Covariant Equations for an Arbitrary Spin

One of the goals of [3–6] (and this book) is the link between the RCQM of an arbitrary spin and the different approaches to the covariant local field theory of an arbitrary spin. Surely, at least the brief analysis of the existing covariant equations for an arbitrary spin should be presented.

Note that in [3–6] and here only the first-order particle and the field equations (together with their canonical nonlocal pseudo-differential representations) are considered. The second order equations (like the Klein–Gordon–Fock equation) are not the subject of this investigation. Thus, the Bhabha, Bargman–Wigner, Pauli–Fierz and many other such type equations are under consideration.

Different approaches to the description of the field theory of an arbitrary spin can be found in [2, 7–24]. Again, as with the relativistic equation for the electron, the Paul Dirac [7] was the first. Here and in [3–6] only the approach started in [2] is the basis for further application. Other results given in [7–24] are not used here.

Note only some general deficiencies of the known equations for arbitrary spin. The consideration of the partial cases, when the substitution of the fixed value of spin is fulfilled, is not successful in all cases. For example, for the spin $s > 1$ existing equations have the redundant components and should be complemented by some additional conditions. Indeed, the known equations [8, 9, 25] for the spin s=3/2 (and their confirmation in [26]) should be essentially complemented by the additional conditions. The main difficulty in the models of an arbitrary spin is the interaction between the fields of higher-spin. Even the quantization of higher-spin fields generated the questions. These and other deficiencies of the known equations for higher-spin are considered in [27–45] (a brief review of deficiencies see in [37]).

Equations suggested in [3–6] and considered below here are free of these deficiencies. The start of such consideration was given from [2], where the main foundations of the RCQM were formulated. In the texts of [3–6] and here the results of [2] are generalized and extended. The operator link between the results

of [1] and [2] (between the canonical FW type field theory and the RCQM) is suggested. Note that the cases s=3/2 and s=2 are not presented in [2], especially in explicit demonstrative forms. The results of [3–6] formally look like as the given in [11, 13]. Nevertheless, the difference is evident. Indeed, in [11, 13] the Bargman–Wigner equations and their Foldy–Wouthuysen representations have been considered. Our approach [3–6] is completely different in comparison with Bargman–Wigner approach.

Indeed, equations suggested in [3–6] do not contain redundant components. The equations from [3–6] have been derived directly from the well-defined equations of RCQM, where every component is well-defined. Therefore, the partial differential equations for arbitrary spin found in [3–6] are without redundant components. It is the advantage of equations from [3–6] in comparison with equations considered in [7–24].

Indeed, the Rarita–Schwinger equation for spin s=3/2 contains 16 components, whereas our equation [3–6] for spin s=(3/2,3/2) particle-antiparticle doublet contains 8 components. The Bargman–Wigner equation [11, 13] in partial case s=3/2 has 12 components. Bhabha itself [46] have analyzed the partial case s=3/2 for his equation [10]. He have found [46] that in this case his equation [10] coincides with the Rarita–Schwinger equation, i.e., has 16 components. Therefore, in [3–6] the new equation for arbitrary spin has been suggested.

Even this brief analysis makes us sure in the prospects of the investigations started in [3–6]. The description of the arbitrary spin field models [7–46] is not the solved problem today.

7.2. General Formalism of Arbitrary Spin

Consider the results of the approach suggested in [3–6].

The operator, which transform the canonical (FW type) model of the arbitrary spin particle-antiparticle field into the corresponding locally covariant particle-antiparticle field, is the generalized FW operator and is given by

$$V^{\mp} = \frac{\mp\overrightarrow{\Gamma}_{2N} \cdot \widehat{\overrightarrow{p}} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}, \quad V^- = (V^+)^\dagger, \tag{7.1}$$

$$V^-V^+ = V^+V^- = I_{2N}, \quad N = 2s+1,$$

where Γ_{2N}^j are known from (5.33) and Σ_N^j are the $N \times N$ Pauli matrices.

Of course, for the matrices Γ^μ_{2N} (5.32), (5.33) (together with the matrix $\Gamma^4_{2N} \equiv \Gamma^0_{2N}\Gamma^1_{2N}\Gamma^2_{2N}\Gamma^3_{2N}$) the relations (5.27) are valid.

For the general form of arbitrary spin locally covariant particle-antiparticle field the Dirac-like equation of motion follows from the equation (6.1) after the transformation (7.1) and is given by

$$\left[i\partial_0 - \Gamma^0_{2N}(\overrightarrow{\Gamma}_{2N} \cdot \widehat{\overrightarrow{p}} + m) \right] \psi(x) = 0. \tag{7.2}$$

We must be careful in choosing the explicit form of Γ^μ_{2N} matrices in (7.1) and (7.2). For our purposes of synthesis of covariant particle equations these matrices can not be chosen as

$$\Gamma^\mu_{2N} = \begin{vmatrix} \Gamma^\mu_N & 0 \\ 0 & \Gamma^\mu_N \end{vmatrix}. \tag{7.3}$$

Only the form

$$\Gamma^0_{2N} \equiv \sigma^3_{2N} = \begin{vmatrix} I_N & 0 \\ 0 & -I_N \end{vmatrix}, \quad \Gamma^j_{2N} = \begin{vmatrix} 0 & \Sigma^j_N \\ -\Sigma^j_N & 0 \end{vmatrix}, \quad j = 1,2,3, \tag{7.4}$$

is possible. Indeed, in the canonical FW type equation (6.1) for the arbitrary spin the matrix $\Gamma^0_{2N} \equiv \sigma^3_{2N}$ from (7.4) (not from (7.3)) is used.

Further, there is a degree of freedom in the choice of Σ^j_N matrices in (7.4). This freedom started from the case of $4 \times 4 \Sigma^j_4$ matrices, which can be chosen in both forms

$$\Sigma^j_4 = \begin{vmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{vmatrix}, \tag{7.5}$$

where $\{\sigma^j\}$ are the standard Pauli matrices (1.6), and

$$\Sigma^1_4 = \begin{vmatrix} 0 & I_2 \\ I_2 & 0 \end{vmatrix}, \quad \Sigma^2_4 = \begin{vmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{vmatrix}, \quad \Sigma^3_4 = \begin{vmatrix} I_2 & 0 \\ 0 & -I_2 \end{vmatrix}. \tag{7.6}$$

Note that in formulas (7.1) and in all formulas before the end of the section the values of N are only even. Therefore, the canonical field equation (6.1) describes the larger number of multiplets then the generalized Dirac equation (7.2).

The formulas (7.2) and (7.7)–(7.11) below are found from the corresponding formulas (5.31), (5.32), (6.3), (6.5), (6.6) of canonical field model on the basis of the operator (7.1).

The general solution has the form

$$\psi(x) = V^-\phi(x) =$$

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^{N}(\overrightarrow{k}) v_N^-(\overrightarrow{k}) + e^{ikx} a^{*\check{N}}(\overrightarrow{k}) v_{\check{N}}^+(\overrightarrow{k}) \right], \qquad (7.7)$$

where amplitudes and notation $\check{N}$ are the same as in (5.31); $\left\{ v_N^-(\overrightarrow{k}), v_{\check{N}}^+(\overrightarrow{k}) \right\}$ are 2N-component Dirac basis spinors with properties of ortonormalization and completeness similar to 4-component Dirac spinors from [47].

The spin operator is given by

$$\overrightarrow{s}_D = V^- \overrightarrow{s}_{2N} V^+, \qquad (7.8)$$

where operator $\overrightarrow{s}_{2N}$ is known from (6.3). The explicit forms of few partial cases of spin operators (7.8) are given in formulae (259)–(261), (284)–(286), (359) of [48] for the particle-antiparticle multiplets s=(1,0,1,0), s=(3/2,3/2), s=(2,1,2,1), respectively.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant field equation (7.2) and the set $\{\psi\}$ of its solutions (7.7) are invariant, have the form

$$\hat{p}^0 = \Gamma_{2N}^0 (\overrightarrow{\Gamma}_{2N} \cdot \widehat{\overrightarrow{p}} + m), \quad \hat{p}^\ell = -i\partial_\ell, \qquad (7.9)$$

$$\hat{j}^{\ell n} = x_D^\ell \hat{p}^n - x_D^n \hat{p}^\ell + s_D^{\ell n} \equiv \widehat{m}^{\ell n} + s_D^{\ell n},$$

$$\hat{j}^{0\ell} = -\hat{j}^{\ell 0} = x^0 \hat{p}^\ell - \frac{1}{2}\left\{ x_D^\ell, \hat{p}^0 \right\} + \frac{\hat{p}^0 (\overrightarrow{s}_D \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega}(\widehat{\omega} + m)}, \qquad (7.10)$$

where the spin matrices $\overrightarrow{s}_D = (s_D^{\ell n})$ are given in (7.8) and the operator $\overrightarrow{x}_D$ has the form

$$\overrightarrow{x}_D = \overrightarrow{x} + \frac{i\overrightarrow{\Gamma}_{2N}}{2\widehat{\omega}} - \frac{\overrightarrow{s}_{2N}^\Gamma \times \widehat{\overrightarrow{p}}}{\widehat{\omega}(\widehat{\omega} + m)} - \frac{i\widehat{\overrightarrow{p}}(\overrightarrow{\Gamma}_{2N} \cdot \widehat{\overrightarrow{p}})}{2\widehat{\omega}^2(\widehat{\omega} + m)}, \qquad (7.11)$$

where specific spin-like matrices $\overrightarrow{s}_{2N}^\Gamma$ are given by

$$\overrightarrow{s}_{2N}^\Gamma \equiv \left(s_{2N}^1, s_{2N}^2, s_{2N}^3 \right) = \frac{i}{2} (\Gamma_{2N}^2 \Gamma_{2N}^3, \Gamma_{2N}^3 \Gamma_{2N}^1, \Gamma_{2N}^1 \Gamma_{2N}^2) \qquad (7.12)$$

with Γ_{2N}^j given in (7.4).

Note that for corresponding partial cases of $\overrightarrow{x}_D$ in [48] ((267) for s=(1,0,1,0), once more (267) but for s=(3/2,3/2), (333) for s=(2,0,2,0), (363) for s=(2,1,2,1)) the corresponding $\overrightarrow{s}_{2N}^{\Gamma}$ are given by

$$s = (1,0,1,0) \text{ and } s = (3/2,3/2):$$

$$\overrightarrow{s}_8^{\Gamma} \equiv \left(s_8^1, s_8^2, s_8^3\right) = \frac{i}{2}(\Gamma_8^2\Gamma_8^3, \Gamma_8^3\Gamma_8^1, \Gamma_8^1\Gamma_8^2), \qquad (7.13)$$

for $\overrightarrow{x}_D$ [48] (267), where the Γ_8^j matrices are given in (253),

$$s = (2,0,2,0):$$

$$\overrightarrow{s}_{12}^{\Gamma} \equiv \left(s_{12}^1, s_{12}^2, s_{12}^3\right) = \frac{i}{2}(\Gamma_{12}^2\Gamma_{12}^3, \Gamma_{12}^3\Gamma_{12}^1, \Gamma_{12}^1\Gamma_{12}^2), \qquad (7.14)$$

for $\overrightarrow{x}_D$ [48] (333), where the Γ_{12}^j matrices are given in (324),

$$s = (2,1,2,1):$$

$$\overrightarrow{s}_{16}^{\Gamma} \equiv \left(s_{16}^1, s_{16}^2, s_{16}^3\right) = \frac{i}{2}(\Gamma_{16}^2\Gamma_{16}^3, \Gamma_{16}^3\Gamma_{16}^1, \Gamma_{16}^1\Gamma_{16}^2), \qquad (7.15)$$

for $\overrightarrow{x}_D$ [48] (363), where the Γ_{16}^j matrices are given in (353).

These partial cases can be found below in this Chapter in next sections as well.

It is easy to verify that the generators (7.9), (7.10) for any N commute with the operator of equation (7.2), and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group. The last step in the brief description of the model is the calculation of the Casimir operators $p^2 = \hat{p}^\mu \hat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin.

7.3. The Dirac Model of Spin s=(1/2,1/2) Particle-Antiparticle Doublet

The below considered formalism follows from the general formalism of arbitrary spin presented above in section 7.1 after the substitution corresponded to spin s=(1/2,1/2) formulas. On the other hand this example of spin s=(1/2,1/2) particle-antiparticle doublet description follows from the consideration of spin s=(1/2,1/2) particle-antiparticle doublet in canonical FW representation described here in the Chapter 6 in subsection 6.2.2. Hence, below the well-known

Dirac model of the spin s=(1/2,1/2) particle-antiparticle doublet is considered as an consequence of more fundamental model of the physical reality. Note that description of spin s=(1/2,1/2) particle-antiparticle doublet, which is considered on three different levels in subsections 5.7.2, 6.2.2 and here, is the main testing example and the generalization on arbitrary spin follows from this formalism.

Thus, on the basis of the inverse FW transformation $V^{\pm}$ (4.23) from the FW equation (4.21) the Dirac equation (4.22) is derived. Relations between the solutions of the Dirac and FW equations are given by (4.24). The solution of the free Dirac equation is derived in the form (4.30), where the notations are given by (4.31). The relationship between the FW spin operator $\vec{s}$ (4.14) and Dirac spin has the form (4.27).

It is easy to verify that spin (4.27) commutes with the operator $i\partial_0 - (\vec{\alpha} \cdot \widehat{\vec{p}} + \beta m)$ of the equation (4.22) and satisfies the SU(2) commutation relations. Moreover, it is easy to verify that the Dirac spinors (4.31) satisfy the equations on eigen vectors and eigenvalues of the spin operator (4.27) in the form (4.40) similar to equations (4.45) from the FW representation. Furthermore, equations (4.40) follow from equations (4.45) and the transformation $V^{\pm}$ (4.23).

The generators of the Lie algebra of the Poincaré group representation, with respect to which the Dirac equation is invariant, are found as a partial case of explicit forms of the expressions (7.9), (7.10). The partial case of 4×4 standard Dirac gamma matrices (1.5) is chosen, the Dirac spin $\vec{s}_{\mathrm{D}} = (s_{\mathrm{D}}^{\ell n})$ is given in (4.27), the position operator $\vec{x}_{\mathrm{D}}$ has the form (4.38) and the FW spin $\vec{s}_{2\mathrm{N}}^{\Gamma} \equiv \vec{s}_{\mathrm{FW}}$ is known as (2.48)=(4.14). Thus, this partial spin s=(1/2,1/2) case of (7.9), (7.10) has the form

$$\widehat{p}^0 = \gamma^0(\vec{\gamma} \cdot \widehat{\vec{p}} + m), \quad \widehat{p}^\ell = -i\partial_\ell, \tag{7.16}$$

$$\widehat{j}^{\ell n} = x_{\mathrm{D}}^\ell \widehat{p}^n - x_{\mathrm{D}}^n \widehat{p}^\ell + s_{\mathrm{D}}^{\ell n} \equiv \widehat{m}^{\ell n} + s_{\mathrm{D}}^{\ell n},$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\left\{ x_{\mathrm{D}}^\ell, \widehat{p}^0 \right\} + \frac{\widehat{p}^0(\vec{s}_{\mathrm{D}} \times \widehat{\vec{p}})^\ell}{\widehat{\omega}(\widehat{\omega}+m)}. \tag{7.17}$$

Further, the substitution of the values (4.27), (4.38) and (2.48) into the formulas (7.16), (7.17), together with the gamma matrix Clifford–Dirac algebra anti-commutation relations application, allows to simplify essentially the explicit forms of these generators.

Finally, the assertion is valid. The Dirac equation (4.22) and the set $\{\psi\}$ of its solutions (4.30) are invariant with respect to the induced fermionic rep-

resentation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 4×4 matrix-differential generators are given by

$$p^0 = H_{\mathrm{D}} = \overrightarrow{\alpha} \cdot \widehat{\overrightarrow{p}} + \beta m, \quad p^\ell = -i\partial_\ell, \quad j^{\ell n} = x^\ell p^n - x^n p^\ell + s^{\ell n} \equiv m^{\ell n} + s^{\ell n},$$

$$(7.18)$$

$$j^{0\ell} = x^0 p^\ell - \frac{1}{2}\left\{ x^\ell, H_{\mathrm{D}} \right\} = x^0 p^\ell - x^\ell p^0 + s^{0\ell},$$

whereas the SU(2) spin matrices are given by

$$\overrightarrow{s} = (s^{23}, s^{31}, s^{12}) = s_{\ell n} = s^{\ell n} \equiv \frac{i}{4}\left[\gamma^\ell, \gamma^n\right] \tag{7.19}$$

and the matrices of the spinor $L_+^\uparrow = \mathrm{SO}(1,3)$ representation of the Lorentz group have the form

$$s^{\mu\nu} \equiv \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]. \tag{7.20}$$

In the set of solutions $\{\psi\}$ (4.30) of the Dirac equation (4.22), realization (7.18) coincides with the covariant one

$$p^\mu = i\partial^\mu, \quad j^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + s^{\mu\nu}, \tag{7.21}$$

in which the generators have the form of the local Lie operators.

For the goals of this manuscript the important conclusion is as follows. The realization (7.18) is the direct consequence of generators (6.17), (6.18) from the FW representation. Generators (7.18) are found from those (6.17), (6.18) by the FW transformation $V^\pm$ (4.23). Moreover, generators (7.18) are the consequence of the RCQM generators (5.20), (5.21) with the spin members (5.14)=(5.124). The operator of such resulting transformation is given by V (2.38)–(2.40) and is valid only for the anti-Hermitian operators.

It is easy to verify by the direct calculations that both generators (7.18) and generators (7.21) commute with the operator $i\partial_0 - (\overrightarrow{\alpha} \cdot \widehat{\overrightarrow{p}} + \beta m)$ of the Dirac equation (4.22). Furthermore, both sets of operators (7.18) and (7.21) satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$.

The results of the Casimir operators calculation for realization (7.18) coincide with the calculations of the invariant operators (6.19), (6.20). It is the consequence of the link $V^\pm$ (4.23) between the sets of generators (6.17), (6.18) and (7.18). For realization (7.21) given by the local Lie generators, the Casimir operators have the form

$$p^2 = p^\mu p_\mu = \partial^\mu \partial_\mu I_4, \tag{7.22}$$

$$W = w^\mu w_\mu = m^2 \vec{s}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\partial^\mu\partial_\mu I_4. \tag{7.23}$$

In the set of solutions $\{\psi\}$ (4.30) of the Dirac equation (4.22) (more exactly, in the set of solutions of the Klein–Gordon–Fock equation) the Casimir operators (7.22), (7.23) of the realization (7.18) coincide with the invariant operators (6.19), (6.20) for the sets of generators (6.17), (6.18).

Moreover, in the manifold $\psi \subset H^{3,4}$ of the solutions (4.30) of the Dirac equation (4.22), the operators (7.21) coincide with the generators (7.18). Therefore, the $\mathcal{P}^\psi$-representation determined by the generators (7.18) is called induced. As a consequence of this fact, for example, the following equalities are valid for the 10 main dynamical variables of the Dirac spinor field ψ

$$(p_\mu, j_{\mu\nu}) \rightarrow (P_\mu, J_{\mu\nu})^\psi \equiv$$

$$\int d^3x \overline{\psi} i(p_\mu, j_{\mu\nu})\psi = \int d^3x \overline{\psi} i(\breve{p}_\mu, \breve{j}_{\mu\nu})\psi = (P_\mu, J_{\mu\nu})^\phi, \tag{7.24}$$

where the conservation laws $(P_\mu, J_{\mu\nu})^\phi$ are given by formula (6.28) and in terms of the amplitudes $a_r^-(\vec{k}), a_r^{*-}(\vec{k}), a_f^+(\vec{k}), a_f^{*+}(\vec{k})$ are given by formula (6.29). This means that the procedure of canonical quantization of the field ψ is reduced to the corresponding procedure of the field ϕ quantization.

Note that the $\mathcal{P}$-operators (7.21) are the functions of 14 independent "generated" operators $x_\mu, \partial_\mu, \widehat{s}_{\mu\nu}$. Further, the $\mathcal{P}$-generators (7.18) are the functions of 12 independent operators $x_\ell, \partial_\ell, \widehat{s}_{\mu\nu}$. Nevertheless, only the operator $\vec{p} = -\nabla$ has the physical sense of the quantum-mechanical Fermi doublet momentum operator among the above mentioned 14 independent operators. As it is proven in [1], the operators $\vec{x} = (x^\ell)$ and $\widehat{s}_{\mu\nu}$ essentially used in the constructions (7.18), (7.21), have no physical sense of the quantum-mechanical operators of the fermionic doublet coordinate and the SU(2)-spin. This fact evidently demonstrates the validity of the assertion that the standard Dirac local model (in the ordinary version) is not the *quantum-mechanical model* of the fermionic doublet at all.

The axioms of section 5.5 in Chapter 5 eventually need to be reconciled with three levels of description used in this paper: RCQM, FW and Dirac equations. Nevertheless, this interesting problem cannot be considered in few pages. The simple level comparison of RCQM, FW and Dirac models given above is very brief and not complete. Only the main features are considered briefly. The

extended version will be given in next investigation. The readers of this book can compare the axioms of RCQM (section 5.5) with the main principles of the Dirac model given in B. Thaller's monograph [49].

The important conclusion from this subsection is the group-theoretical and quantum-mechanical interpretation of the Dirac equation. It is shown that its relation to the spinor field (spin s = (1/2,1/2) particle-antiparticle doublet) with the spin s=(1/2,1/2) and $m > 0$ follows from equations (4.40) and the Bargman–Wigner analysis of the fermionic representations (7.18), (7.21) on the basis of the Casimir operators (6.19), (6.20) and (7.22), (7.23), respectively. Only such incomplete (in comparison with the interpretation given in subsection 5.7.2) fermionic quantum-mechanical interpretation of the Dirac equation is possible. The important external step in this interpretation is given in the section 6.2.2 and here. It is the direct link between the Dirac model and the spin s=(1/2,1/2) particle-antiparticle doublet RCQM, where the quantum-mechanical interpretation is complete and evident.

In our papers [50–52] after the construction of the bosonic spin s=(1,0) representations of the Poincaré group $\mathcal{P}$, with respect to which the Dirac equation (4.22) is invariant, the similar procedure of interpretation was used. On this basis in [50–52], the relation of the Dirac equation to the description of bosonic field (spin s=(1,0) particle multiplet) of spins s=1, s=0 and $m > 0$ was proved. Similarly to the above given fermionic interpretation such bosonic interpretation of the Dirac model is incomplete as well. And again the important external step in such interpretation is the direct link between the spin s=(1,0), or spin s=(1,1), RCQM and the corresponding Dirac, or Dirac-like equations. This link is given in next chapters. Recal that in the framework of corresponding RCQM the quantum-mechanical interpretation is complete and evident. Below in the next sections the methods of interpretation developed here are applied to other covariant models of higher-spin particle-antiparticle doublets.

Second important conclusion from subsections 5.7.2, 6.2.2 and this one is the evident demonstration of the link between the non-local RCQM of the spin s =(1/2,1/2) particle-antiparticle doublet and the covariant local spinor Dirac model. In other words, in the subsections 5.7.2, 6.2.2 and here the derivation of the Dirac equation from the RCQM Schrödinger–Foldy equation for the spin s=(1/2,1/2) particle-antiparticle doublet is suggested (a review of other methods of the Dirac equation derivation see in Chapter 2).

Below in similar way the Dirac-like equations for other particle-antiparticle doublets of higher spins are derived. The start from the corresponding models

of RCQM is given as well. Thus, here the foundations of the synthesis of covariant particle equations on the basis of the start from the RCQM are formulated. Hence, the author's small addition to the Foldy's synthesis of simplest covariant particle equations is suggested.

7.4. Covariant Field Equation for the 8-Component Spin s=(1,0,1,0) Bosonic Particle-Antiparticle Multiplet

Considered below example of local covariant field model is useful for the derivation of the Maxwell-like equations with nonzero mass.

The operator of the transition to the covariant local field theory representation (the 8×8 analogy of the 4×4 FW transformation operator (4.23)) is given by

$$V^{\mp} = \frac{\mp \overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}, \quad V^- = (V^+)^\dagger, \quad V^-V^+ = V^+V^- = I_8, \quad (7.25)$$

$$\psi = V^- \phi, \quad \phi = V^+ \psi, \quad (7.26)$$

$$\hat{q}_D = V^- \hat{q}_{CF} V^+, \quad \hat{q}_{CF} = V^+ \hat{q}_D V^-, \quad (7.27)$$

where $\hat{q}_D$ is an arbitrary operator (both in the Hermitian and anti-Hermitian form) in the covariant local field theory representation. The inverse transformation is valid as well.

Thus, on the basis of transformation (7.25)–(7.27) the 8-component Dirac-like equation is found from the canonical field equation (6.54) in the form

$$\left[i\partial_0 - \Gamma_8^0 (\overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}} + m) \right] \psi(x) = 0. \quad (7.28)$$

In transition operator (7.25) and in the field equation (7.28) the Γ_8^μ matrices are given by

$$\Gamma_8^0 = \begin{vmatrix} I_4 & 0 \\ 0 & -I_4 \end{vmatrix}, \quad \Gamma_8^j = \begin{vmatrix} 0 & \Sigma^j \\ -\Sigma^j & 0 \end{vmatrix}, \quad (7.29)$$

where Σ^j are the 4×4 Pauli matrices

$$\Sigma^j = \begin{vmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{vmatrix}, \quad (7.30)$$

and σ^j are the standard 2×2 Pauli matrices (1.6).

The matrices Σ^j satisfy the similar commutation relations as the 2×2 Pauli matrices (1.6) and have other similar properties. The matrices Γ_8^μ (7.29) satisfy the anticommutation relations of the Clifford–Dirac algebra in the form

$$\Gamma_8^\mu \Gamma_8^\nu + \Gamma_8^\nu \Gamma_8^\mu = 2g^{\mu\nu}. \tag{7.31}$$

It is evident that equation (7.28) is not the ordinary direct sum of the two Dirac equations. Therefore, it is not the complex Dirac–Kahler equation [53]. Moreover, it is not the standard Dirac–Kahler equation [54].

The solution of equation (7.28) is derived from solution (6.79) of this equation in the canonical representation (6.54) on the basis of transformation (7.25), (7.26) and is given by

$$\psi(x) = V^- \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} c^A(\overrightarrow{k}) v_A^-(\overrightarrow{k}) + e^{ikx} c^{*B}(\overrightarrow{k}) v_B^+(\overrightarrow{k}) \right], \tag{7.32}$$

where $A = \overline{1,4}$, $B = \overline{5,8}$ and the 8-component spinors $(v_A^-(\overrightarrow{k}), v_B^+(\overrightarrow{k}))$ are given by

$$v_1^-(\overrightarrow{k}) = N \begin{vmatrix} \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1+ik^2 \\ 0 \\ 0 \end{vmatrix}, \quad v_2^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ k^1-ik^2 \\ -k^3 \\ 0 \\ 0 \end{vmatrix},$$

$$v_3^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1+ik^2 \end{vmatrix}, \quad v_4^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ k^1-ik^2 \\ -k^3 \end{vmatrix},$$

$$v_5^+(\overrightarrow{k}) = N \begin{vmatrix} k^3 \\ k^1 + ik^2 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \\ 0 \\ 0 \end{vmatrix}, v_6^+(\overrightarrow{k}) = N \begin{vmatrix} k^1 - ik^2 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \\ 0 \end{vmatrix}, \tag{7.33}$$

$$v_7^+(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ k^3 \\ k^1 + ik^2 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \end{vmatrix}, v_8^+(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ k^1 - ik^2 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \end{vmatrix},$$

where

$$N \equiv \frac{1}{\sqrt{2\widetilde{\omega}(\widetilde{\omega} + m)}}, \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}. \tag{7.34}$$

The spinors (7.33) are derived from the orts $\{d_\alpha\}$ of the Cartesian basis (5.149) with the help of the transformation (7.25), (7.26). The spinors (7.33) satisfy the relations of the orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors, see, e.g., [47]. In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $\left[s_{8D}^j, s_{8D}^\ell\right] = i\varepsilon^{j\ell n} s_{8D}^n$ and commute with the operator $\left[i\partial_0 - \Gamma_8^0(\overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}} + m)\right]$ of equation (7.28), are derived from the pure matrix operators (6.80) with the help of operator (7.25), (7.27). The explicit form of these SU(2) generators is given by

$$s_{8D}^1 = \frac{1}{2\sqrt{2}\widehat{\omega}\Omega} \begin{vmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 \\ 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\ 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 \end{vmatrix}, \tag{7.35}$$

where the matrix elements have the form

$$11 = 2p^{13}, \ 12 = 2p^1 z^* m\Omega, \ 13 = p^3 z^*, \ 14 = z^{*2},$$

$$15 = 2ip^2 \Omega, \ 16 = -2p^3 \Omega, \ 17 = -z\Omega, \ 18 = 0,$$

$$21 = 2p^1 zm\Omega, \ 22 = -2p^{13}, \ 23 = \Omega^2 - p^{33}, \ 24 = -p^3 z^*,$$

$$25 = 2p^3 \Omega, \ 26 = -2ip^2 \Omega, \ 27 = 2p^3 \Omega, \ 28 = z^* \Omega,$$

$$31 = p^3 z, \ 32 = \Omega^2 - p^{33}, \ 33 = 0, \ 34 = 0,$$

$$35 = z\Omega, \ 36 = -2p^3 \Omega, \ 37 = 0, \ 38 = 0,$$

$$41 = z^2, \ 42 = -p^3 z, \ 43 = 0, \ 44 = 0, \ 45 = 0, \ 46 = -z\Omega, \ 47 = 0, \ 48 = 0,$$

$$51 = -2ip^2 \Omega, \ 52 = 2p^3 \Omega, \ 53 = z^* \Omega, \ 54 = 0$$

$$55 = 2p^{13}, \ 56 = 2p^1 z^* m\Omega, \ 57 = p^3 z^*, \ 58 = z^{*2},$$

$$61 = -2p^3 \Omega, \ 62 = 2ip^2 \Omega, \ 63 = -2p^3 \Omega, \ 64 = -z^* \Omega,$$

$$65 = 2p^1 zm\Omega, \ 66 = -2p^{13}, \ 67 = \Omega^2 - p^{33}, \ 68 = -p^3 z^*,$$

$$71 = -z\Omega, \ 72 = 2p^3 \Omega, \ 73 = 0, \ 74 = 0, \ 75 = p^3 z, \ 76 = \Omega^2 - p^{33}, \ 77 = 0, \ 78 = 0,$$

$$81 = 0, \ 82 = z\Omega, \ 83 = 0, \ 84 = 0, \ 85 = z^2, \ 86 = -p^3 z, \ 87 = 0, \ 88 = 0,$$

$$s^2_{8D} = \frac{i}{2\sqrt{2}\hat{\omega}\Omega} \begin{vmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 \\ 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\ 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 \end{vmatrix}, \tag{7.36}$$

where the matrix elements are given by

$$11 = -2ip^{33}, \ 12 = -2p^3 z^* m\Omega, \ 13 = -p^3 z^*, \ 14 = -z^{*2},$$

$$15 = -2p^1 \Omega, \ 16 = 2p^3 \Omega, \ 17 = z^* \Omega, \ 18 = 0$$

$$21 = 2p^2 zm\Omega, \ 22 = 2ip^{23}, \ 23 = -\Omega^2 + p^{33}, \ 24 = p^3 z^*,$$

$$25 = 2p^3 \Omega, \ 26 = 2p^1 \Omega, \ 27 = -2p^3 \Omega, \ 28 = -z^* \Omega,$$

$$31 = p^3 z, \, 32 = \Omega^2 - p^{33}, \, 33 = 0, \, 34 = 0,$$

$$35 = z\Omega, \, 36 = -2p^3\Omega, \, 37 = 0, \, 38 = 0,$$

$$41 = z^2, \, 42 = -p^3 z, \, 43 = 0, \, 44 = 0, \, 45 = 0, \, 46 = -z\Omega, \, 47 = 0, \, 48 = 0,$$

$$51 = 2p^1\Omega, \, 52 = -2p^3\Omega, \, 53 = -z^*\Omega, \, 54 = 0,$$

$$55 = -2ip^{23}, \, 56 = -2p^2 z^* m\Omega, \, 57 = -p^3 z^*, \, 58 = -z^{*2},$$

$$61 = -2p^3\Omega, \, 62 = -2p^1\Omega, \, 63 = 2p^3\Omega, \, 64 = z^*\Omega,$$

$$65 = 2p^2 zm\Omega, \, 66 = 2ip^{23}, \, 67 = -\Omega^2 + p^{33}, \, 68 = p^3 z^*,$$

$$71 = -z\Omega, \, 72 = 2p^3\Omega, \, 73 = 0, \, 74 = 0,$$

$$75 = p^3 z, \, 76 = \Omega^2 - p^{33}, \, 77 = 0, \, 78 = 0,$$

$$81 = 0, \, 82 = z\Omega, \, 83 = 0, \, 84 = 0, \, 85 = z^2, \, 86 = -p^3 z, \, 87 = 0, \, 88 = 0,$$

$$s_{8D}^3 = \frac{1}{2\widehat{\omega}\Omega} \begin{vmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 \\ 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\ 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 \end{vmatrix}, \tag{7.37}$$

where the matrix elements have the form

$$11 = \Omega^2 + p^{33}, \, 12 = p^3 z^*, \, 13 = 0, \, 14 = 0, \, 15 = 0, \, 16 = z^*\Omega, \, 17 = 0, \, 18 = 0,$$

$$21 = p^3 z, \, 22 = p^{11} + p^{22}, \, 23 = 0, \, 24 = 0, \, 25 = -z\Omega, \, 26 = 0, \, 27 = 0, \, 28 = 0,$$

$$31 = 0, \, 32 = 0, \, 33 = -\Omega^2 - p^{33}, \, 34 = -p^3 z^*, \, 35 = 0, \, 36 = 0, \, 37 = 0, \, 38 = -z^*\Omega,$$

$$41 = 0, \, 42 = 0, \, 43 = -p^3 z, \, 44 = -p^{11} - p^{22}, \, 45 = 0, \, 46 = 0, \, 47 = z\Omega, \, 48 = 0,$$

$$51 = 0, \, 52 = -z^*\Omega, \, 53 = 0, \, 54 = 0, \, 55 = \Omega^2 + p^{33}, \, 56 = p^3 z^*, \, 57 = 0, \, 58 = 0,$$

$$61 = z\Omega, \, 62 = 0, \, 63 = 0, \, 64 = 0, \, 65 = p^3 z, \, 66 = p^{11} + p^{22}, \, 67 = 0, \, 68 = 0,$$

$$71 = 0, \, 72 = 0, \, 73 = 0, \, 74 = z^*\Omega, \, 75 = 0, \, 76 = 0, \, 77 = -\Omega^2 - p^{33}, \, 78 = -p^3 z^*,$$

$$81 = 0, \, 82 = 0, \, 83 = -z\Omega, \, 84 = 0, \, 85 = 0, \, 86 = 0, \, 87 = -p^3 z, \, 88 = -p^{11} - p^{22}.$$

Above, in the formulae for matrix elements of the spin operators (7.35)–(7.37) the following notations are used

$$p^1 zm\Omega \equiv p^1 z + m\Omega, \; p^1 z^* m\Omega \equiv p^1 z^* + m\Omega,$$

$$p^2 zm\Omega \equiv -ip^2 z + m\Omega, \; p^2 z^* m\Omega \equiv ip^2 z^* + m\Omega,$$

$$p^{12} \equiv p^1 p^2, \; p^{13} \equiv p^1 p^3, \; p^{23} \equiv p^2 p^3, \; p^{11} \equiv p^1 p^1, \; p^{22} \equiv p^2 p^2, \; p^{33} \equiv p^3 p^3,$$
$$(7.38)$$

$$\Omega \equiv \omega + m, \; \omega \equiv \sqrt{\overrightarrow{p} + m^2}, z \equiv p^1 + ip^2, z^* \equiv p^1 - ip^2, z^{*2} \equiv (z^*)^2.$$

The equations on eigen vectors and eigenvalues of the operator s_{8D}^3 (7.37) follow from the equations (6.81) and the transformation (7.25)–(7.27). In addition to it, the action of the operator s_{8D}^3 (7.37) on the spinors $(v_A^-(\overrightarrow{k}), v_B^+(\overrightarrow{k}))$ (7.33) also leads to the result

$$s_{8D}^3 v_1^-(\overrightarrow{k}) = v_1^-(\overrightarrow{k}), \; s_{8D}^3 v_2^-(\overrightarrow{k}) = 0, \; s_{8D}^3 v_3^-(\overrightarrow{k}) = -v_3^-(\overrightarrow{k}), \; s_{8D}^3 v_4^-(\overrightarrow{k}) = 0,$$
$$(7.39)$$
$$s_{8D}^3 v_5^+(\overrightarrow{k}) = v_5^+(\overrightarrow{k}), \; s_{8D}^3 v_6^+(\overrightarrow{k}) = 0, \; s_{8D}^3 v_7^+(\overrightarrow{k}) = -v_7^+(\overrightarrow{k}), \; s_{8D}^3 v_8^+(\overrightarrow{k}) = 0.$$

In order to verify equations (7.39) the following identity

$$(\widetilde{\omega} + m)^2 + (\overrightarrow{k})^2 = 2\widetilde{\omega}(\widetilde{\omega} + m), \qquad (7.40)$$

should be used. In the case $v_B^+(\overrightarrow{k})$ in the expression for $s_{8D}^3(\overrightarrow{k})$ (7.37) the substitution $\overrightarrow{k} \to -\overrightarrow{k}$ is fulfilled.

The equations (7.39) determine the interpretation of the amplitudes in the solution (7.32). This interpretation is similar to the given above in subsection 6.3.1 in the paragraph after equations (6.81). Nevertheless, the direct quantum-mechanical interpretation of the amplitudes should be given in the framework of the RCQM, see the subsection 5.8.2 above.

The explicit form of the $\mathcal{P}$-generators of the bosonic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (7.28) and the set $\{\psi\}$ of its solutions (7.32) are invariant, is derived from the generators (6.82), (6.83) on the basis of the transformation (7.25), (7.27). The corresponding generators are given by

$$\widehat{p}^0 = \Gamma_8^0(\overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}} + m), \quad \widehat{p}^\ell = -i\partial_\ell, \quad \widehat{j}^{\ell n} = x_D^\ell \widehat{p}^n - x_D^n \widehat{p}^\ell + s_{8D}^{\ell n} \equiv \widehat{m}^{\ell n} + s_{8D}^{\ell n},$$
$$(7.41)$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\left\{x_D^\ell, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\overrightarrow{s}_{8D} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega}(\widehat{\omega}+m)}, \qquad (7.42)$$

where the spin matrices $\overrightarrow{s}_{8D} = (s_{8D}^{\ell n})$ are given in (7.35)–(7.37) and the operator $\overrightarrow{x}_D$ has the form

$$\overrightarrow{x}_D = \overrightarrow{x} + \frac{i\overrightarrow{\Gamma}_8}{2\widehat{\omega}} - \frac{\overrightarrow{s}_8^\Gamma \times \widehat{\overrightarrow{p}}}{\widehat{\omega}(\widehat{\omega}+m)} - \frac{i\widehat{\overrightarrow{p}}(\overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}})}{2\widehat{\omega}^2(\widehat{\omega}+m)}, \qquad (7.43)$$

where the specific spin-like matrices $\overrightarrow{s}_8^\Gamma$ are given by

$$\overrightarrow{s}_8^\Gamma \equiv \frac{i}{2}(\Gamma_8^2\Gamma_8^3, \Gamma_8^3\Gamma_8^1, \Gamma_8^1\Gamma_8^2) \qquad (7.44)$$

with Γ_8^j determined in (7.29).

It is easy to verify that the generators (7.41), (7.42) commute with the operator $\left[i\partial_0 - \Gamma_8^0(\overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}} + m)\right]$ of equation (7.28), satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group and the corresponding Casimir operators are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_8, \qquad (7.45)$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_{8D}^2 = m^2 \begin{vmatrix} 1(1+1)I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1(1+1)I_3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad (7.46)$$

where I_8 and I_3 are 8×8 and 3×3 unit matrices, respectively.

As it was already explained in details in the previous sections, the conclusion that equation (7.28) describes the bosonic particle-antiparticle multiplet of the spin s=(1,0,1,0) and mass $m > 0$ (and its solution (7.32) is the bosonic field of the above mentioned spin and nonzero mass) follows from the analysis of equations (7.39) and the Casimir operators (7.45), (7.46). Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin s=(1,0,1,0) particle-antiparticle multiplet, where the quantum-mechanical interpretation is direct and evident. Therefore, the bosonic spin s=(1,0,1,0) properties of equation (7.28) are proved. In particular, the equation (7.28) can be used for the $W^\mp$-boson.

Contrary to the above found bosonic properties of the equation (7.28), the fermionic properties of this equation are evident. The fact that equation (7.28) describes the multiplet of two fermions with the spin s=1/2 and two antifermions with that spin can be proved much more easier then the above bosonic consideration. The proof is similar to that given in the standard 4-component Dirac model (subsections 5.7.2, 6.2.2 and section 7.3 above). Moreover, it is easy to show that equation (7.28) can describe the spin s=(3/2,3/2) particle-antiparticle doublet and the corresponding spinor field (section 7.6 below). Therefore, equation (7.28) has more extended property of the Fermi–Bose duality then the standard Dirac equation [50–52].

7.5. Maxwell-Like Equations for the Spin s=(1,0) Bosonic Doublet with Nonzero Mass

In the partial case of $m = 0$ equation (7.28) describes (among the other possibilities) the system of the 8-component *(photon massless spinless boson)-(antiphoton masslees spinless antiboson)* field. The 4-component photon massless spinless boson subsystem was described in [55–58]. In other possible interpretation [55–58], it is the system of the Maxwell equations with gradient type current and charge densities (the magnetic gradient type current and charge densities also are included). It was shown in [55–58] that this 4-component model can describe the relativistic hydrogen spectrum (i.e., has the coupled finite solutions) and the longitudinal electromagnetic waves. The 8-component model given here above has the additional possibilities related to the antiphoton presence. The antiphoton can be considered as the photon with opposite sign of helicity (or right-handed in comparison with left-handed photon).

The link between the standard 4-component Dirac equation and the original Maxwell equations is the direct consequence of the substitutions like

$$\psi = \begin{vmatrix} E^3 + iH^0 \\ E^1 + iE^2 \\ iH^3 + E^0 \\ -H^2 + iH^1 \end{vmatrix}, \tag{7.47}$$

or

$$\psi = \begin{vmatrix} \vec{E} - i\vec{H} \\ E^0 - iH^0 \end{vmatrix}, \tag{7.48}$$

into the massless Dirac equation. In formulae (7.47), (7.48) the functions $\overrightarrow{E} = (E^1, E^2, E^3)$, $\overrightarrow{H} = (H^1, H^2, H^3)$ are the field strengths of electromagnetic field and the functions E^0, H^0 are the corresponding characteristics of the spinless bosonic field. The results of substitutions like (7.47), (7.48) into the 4-component spinors have been investigated step by step by many authors, see, e.g., [59–75].

The link between the 8-component Dirac-like equation (7.28) and the original Maxwell equations is the direct consequence of the substitution

$$\psi = \begin{vmatrix} E^3 + iH^0 \\ E^1 + iE^2 \\ iH^3 + E^0 \\ -H^2 + iH^1 \\ -iH^3 - E^0 \\ H^2 - iH^1 \\ -E^3 - iH^0 \\ -E^1 - iE^2 \end{vmatrix} \tag{7.49}$$

into equation (7.28). The resulting system of Maxwell-like equations is given by

$$\partial_0 E^0 + \partial_j E^j \mp mH^3 = 0,$$

$$\partial_0 E^1 - (\operatorname{curl}\overrightarrow{H})^1 + \partial_1 E^0 \mp mE^2 = 0,$$

$$\partial_0 E^2 - (\operatorname{curl}\overrightarrow{H})^2 + \partial_2 E^0 \pm mE^1 = 0,$$

$$\partial_0 E^3 - (\operatorname{curl}\overrightarrow{H})^3 + \partial_3 E^0 \mp mH^0 = 0,$$

$$\partial_0 H^0 + \partial_j H^j \pm mE^3 = 0, \tag{7.50}$$

$$\partial_0 H^1 + (\operatorname{curl}\overrightarrow{E})^1 + \partial_1 H^0 \mp mH^2 = 0,$$

$$\partial_0 H^2 + (\operatorname{curl}\overrightarrow{E})^2 + \partial_2 H^0 \pm mH^1 = 0,$$

$$\partial_0 H^3 + (\operatorname{curl}\overrightarrow{E})^3 + \partial_3 H^0 \pm mE^0 = 0.$$

Thus, we have found here the slightly generalized Maxwell equations, known e.g., from [55–58] as massless, with nonzero mass.

The system of equations (7.50) contains 16 equations for the 8 real functions $E^\alpha = (E^0, E^1, E^2, E^3)$, $H^\alpha = (H^0, H^1, H^2, H^3)$. The subsystem of 8 equations with upper sign near mass member is the consequence of the four upper

equations from (7.28), while the subsystem of 8 equations with lower sign near mass member is the consequence of the four lower equations from (7.28). The situation is similar to the Dirac equation, wherever from the form of 8 equations with $\pm m$

$$(i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu + m) = -\partial_0\partial_0 + \Delta - m^2 \tag{7.51}$$

the system of 4 equations with $-m$, i.e., $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ for 4 functions is taken. In complete analogy here from the equations (7.50) the system of 8 equations with upper sign near mass for 8 functions $E^\alpha = (E^0, E^1, E^2, E^3)$, $H^\alpha = (H^0, H^1, H^2, H^3)$ is taken. It is the system (7.50) with upper sign near mass.

The system of equations (7.50) (upper sign near mass) describes the bosonic particle multiplet of the spin s=(1,0) and mass $m > 0$ (the coupled electromagnetic and scalar fields in the case of zero mass). Contrary to (7.28), here (in this form of equation) the relation to the Maxwell electrodynamics is evident. In partial case $m = 0$ equations (7.50) (8 real or 4 complex equations) coincide with the Maxwell equations with gradient-like current and charge densities considered in [55–58].

Furtermore, the partial case $E^0 = H^0 = 0$ in (7.50) leads to the system of equations

$$\partial_j E^j - mH^3 = 0,$$

$$\partial_0 E^1 - (\text{curl}\,\overrightarrow{H})^1 - mE^2 = 0,$$

$$\partial_0 E^2 - (\text{curl}\,\overrightarrow{H})^2 + mE^1 = 0,$$

$$\partial_0 E^3 - (\text{curl}\,\overrightarrow{H})^3 = 0,$$

$$\partial_j H^j + mE^3 = 0, \tag{7.52}$$

$$\partial_0 H^1 + (\text{curl}\,\overrightarrow{E})^1 - mH^2 = 0,$$

$$\partial_0 H^2 + (\text{curl}\,\overrightarrow{E})^2 + mH^1 = 0,$$

$$\partial_0 H^3 + (\text{curl}\,\overrightarrow{E})^3 = 0.$$

Equations (7.52) can be considered as the generalized electrodynamics for the electromagnetic field of the photon with mass. In general, equations (7.52) describes the field of arbitrary boson with spin s=1 and mass $m > 0$.

The system of equations (7.52) is derived from the equations (5.135), (6.35) for the spin s=(1,1) particle-antiparticle doublet in RCQM and FW representations, respectively, similarly to the way, in which the system of equations

(7.28), and further (7.50) (with upper sign near mass), is derived from the equations (5.176), (6.54) for the 8-component spin s=(1,0,1,0) bosonic particle-antiparticle multiplet (in RCQM and FW representations, respectively). The corresponding 6 component transformation of FW type will be the subject of special consideration. Hence, the system (7.52) is the direct consequence of the equations (5.135), (6.35) for the spin s=(1,1) particle-antiparticle doublet.

Moreover, equations (7.50) with upper sign near mass are linked directly with the RCQM equation (5.98) for the spin s=(1,0) multiplet. This link is based on the anti-Hermitian transformations, essential application of complex conjugation operator C and the corresponding equations for eigenvectors lead to complex number eigenvalues. Therefore, such links and transformations will be given in special publication.

Finally, the substitution $m = 0$ into the system of equations (7.52) leads to the Maxwell equations

$$\operatorname{div}\overrightarrow{E} = 0, \quad \partial_0\overrightarrow{E} - \operatorname{curl}\overrightarrow{H} = 0, \quad \operatorname{div}\overrightarrow{H} = 0, \quad \partial_0\overrightarrow{H} + \operatorname{curl}\overrightarrow{E} = 0, \quad (7.53)$$

for free electromagnetic field. Thus, the link between the RCQM and the Maxwell electrodynamics is finished.

7.6. Covariant Field Equation without Redundant Components for the 8-Component Spin s=(3/2,3/2) Fermionic Particle-Antiparticle Doublet

The model is constructed in complete analogy with the consideration in section 7.4.

The start of this derivation is given in subsection 5.7.4, where the RCQM of the 8-component fermionic spin s=(3/2,3/2) particle-antiparticle doublet is considered. The second step is the transition from the Schrödinger–Foldy equation (5.146) to the canonical field equation. This step, as shown in subsection 6.2.4, is possible only for the anti-Hermitian form of the operators. Nevertheless, the resulting operators can be chosen in the standard Hermitian form and do not contain the operator C of complex conjugation. The last step of the transition from the canonical field equation to the covariant local field equation is fulfilled in analogy with the FW transformation (4.23) with the help of the transformation (7.25).

The operator of the transition to the covariant local field theory representation (the 8×8 analogy of the 4×4 FW transformation operator (4.23)) is the same as in the section 7.4 and is given by (7.25)–(7.27). Thus, on the basis of transformation (7.25)–(7.27) the 8-component Dirac-like equation is found from the canonical field equation (6.54) in the form

$$\left[i\partial_0 - \Gamma_8^0 (\vec{\Gamma}_8 \cdot \widehat{\vec{p}} + m) \right] \psi(x) = 0, \tag{7.54}$$

where the 8×8 gamma matrices are given in (7.29). This equation only formally coincide with the field equation (7.28) for spin s=(1,0,1,0) bosonic multiplet. Below we demonstrate that this equation can describe the spin s=(3/2,3/2) particle-antiparticle doublet as well.

Note that equation (7.54) is not the ordinary direct sum of the two Dirac equations. Therefore, it is not the complex Dirac–Kahler equation [53]. Moreover, it is not the standard Dirac–Kahler equation [54].

The solution of equation (7.54) is derived from solution (6.56) of this equation in the canonical representation (6.54) on the basis of transformation (7.25), (7.26) and is given by

$$\psi(x) = V^- \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} b^A(\vec{k}) v_A^-(\vec{k}) + e^{ikx} b^{*B}(\vec{k}) v_B^+(\vec{k}) \right], \tag{7.55}$$

where $A = \overline{1,4}$, $B = \overline{5,8}$ and the 8-component spinors $(v_A^-(\vec{k}), v_B^+(\vec{k}))$ are given in (7.33).

The spinors (7.33) are derived from the orts $\{d_\alpha\}$ of the Cartesian basis (5.149) with the help of the transformation (7.25), (7.26). The spinors (7.33) satisfy the relations of the orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors, see, e.g., [47].

The direct quantum-mechanical interpretation of the amplitudes in solution (7.55) should be given in the framework of the RCQM. Such interpretation is already given in subsection 5.7.4 in paragraph after equations (5.156).

In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $\left[s_{8D}^j, s_{8D}^\ell \right] = i\varepsilon^{j\ell n} s_{8D}^n$ and commute with the operator $\left[i\partial_0 - \Gamma_8^0 (\vec{\Gamma}_8 \cdot \vec{p} + m) \right]$ of equation (7.54), are derived from the pure matrix operators (6.57) with the help of transition operator (7.25),

(7.27) $\vec{s}_{8D} = V^- \vec{s}_8 V^+$. The explicit forms of these s=(3/2,3/2) SU(2) genera-tors are given by:

$$
s^1_{8D} = k \begin{vmatrix}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\
51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 \\
61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 \\
71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\
81 & 82 & 83 & 84 & 85 & 86 & 87 & 88
\end{vmatrix},
\tag{7.56}
$$

where the matrix elements are as follows

$$11 = \sqrt{3}p^{13}, \; 12 = \sqrt{3}(p^1 z^* + m\Omega), \; 13 = p^3 z^*, \; 14 = z^{*2},$$

$$15 = i\sqrt{3}p^2\Omega, \; 16 = -\sqrt{3}p^3\Omega, \; 17 = -\Omega z^*, \; 18 = 0,$$

$$21 = \sqrt{3}(p^1 z + m\Omega), \; 22 = -\sqrt{3}p^{13}, \; 23 = 2m\Omega + p^{11}_{22}, \; 24 = -p^3 z^*,$$

$$25 = \sqrt{3}p^3\Omega, \; 26 = -i\sqrt{3}p^2\Omega, \; 27 = 2p^3\Omega, \; 28 = z^*\Omega,$$

$$31 = p^3 z, \; 32 = 2m\Omega + p^{11}_{22}, \; 33 = \sqrt{3}p^{13}, \; 34 = \sqrt{3}(p^1 z^* + m\Omega),$$

$$35 = 2\Omega z, \; 36 = -2p^3\Omega, 37 = \sqrt{3}p^1\Omega, \; 38 = -\sqrt{3}p^3\Omega,$$

$$41 = z^2, \; 42 = -p^3 z, 43 = \sqrt{3}(p^1 z + m\Omega), \; 44 = -\sqrt{3}p^{13},$$

$$45 = 0, \; 46 = -\Omega z, \; 47 = \sqrt{3}p^3\Omega, 48 = -i\sqrt{3}p^2\Omega,$$

$$51 = -i\sqrt{3}\Omega, \; 52 = \sqrt{3}p^3\Omega, \; 53 = \Omega z^*, \; 54 = 0$$

$$55 = \sqrt{3}p^{13}, \; 56 = \sqrt{3}(p^1 z^* + m\Omega), \; 57 = p^3 z^*, \; 58 = z^{*2},$$

$$61 = -\sqrt{3}p^3\Omega, \; 62 = i\sqrt{3}p^2\Omega, \; 63 = -2p^3\Omega, \; 64 = -\Omega z^*,$$

$$65 = \sqrt{3}(p^1 z + m\Omega), \; 66 = -\sqrt{3}p^{13}, \; 67 = 2m\Omega + p^{11}_{22}, \; 68 = -p^3 z^*,$$

$$71 = -\Omega z, \; 72 = 2p^3\Omega, \; 73 = -i\sqrt{3}p^2\Omega, \; 74 = \sqrt{3}p^3\Omega,$$

$$75 = p^3 z^*, 76 = 2m\Omega^2 + p^{11}_{22}, \; 77 = \sqrt{3}p^{13}, \; 78 = \sqrt{3}(p^1 z^* + m\Omega),$$

$$81 = 0, \; 82 = \Omega z, \; 83 = -\sqrt{3}p^3\Omega, \; 84 = i\sqrt{3}p^2\Omega,$$

$$85 = z^2, \; 86 = -p^3 z, \; 87 = \sqrt{3}(p^1 z + m\Omega), \; 88 = -\sqrt{3}p^{13},$$

$$s^2_{8D} = k \begin{vmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 \\ 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\ 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 \end{vmatrix}, \tag{7.57}$$

where the matrix elements are given by

$$11 = \sqrt{3}p^{23},\ 12 = -i\sqrt{3}(m\Omega + ip^2 z^*),\ 13 = --ip^3 z^*,\ 14 = -iz^{*2},$$

$$15 = -i\sqrt{3}p^1\Omega,\ 16 = i\sqrt{3}p^3\Omega,\ 17 = iz^*\Omega,\ 18 = 0,$$

$$21 = i\sqrt{3}(m\Omega - ip^2 z),\ 22 = \sqrt{3}p^{23},\ 23 = -i(\Omega^2 - p^{33}),\ 24 = ip^3 z^*,$$

$$25 = i\sqrt{3}p^3\Omega,\ 26 = i\sqrt{3}p^1\Omega,\ 27 = -2ip^3\Omega,\ 28 = -iz^*\Omega,$$

$$31 = ip^3 z,\ 32 = i(\Omega^2 - p^{33}),\ 33 = \sqrt{3}p^{23},\ 34 = -i\sqrt{3}(m\Omega + ip^2 z^*),$$

$$35 = iz\Omega,\ 36 = -2ip^3\Omega,\ 37 = -i\sqrt{3}p^1\Omega,\ 38 = i\sqrt{3}p^3\Omega,$$

$$41 = iz^2,\ 42 = -ip^3 z,\ 43 = i\sqrt{3}(m\Omega - ip^2 z),\ 44 = -\sqrt{3}p^{23},$$

$$45 = 0,\ 46 = -iz\Omega,\ 47 = i\sqrt{3}p^3\Omega,\ 48 = i\sqrt{3}p^1\Omega,$$

$$51 = i\sqrt{3}p^1\Omega,\ 52 = -i\sqrt{3}p^3\Omega,\ 53 = -2iz^*\Omega,\ 54 = 0,$$

$$55 = \sqrt{3}p^{23},\ 56 = -i\sqrt{3}(m\Omega + ip^2 z^*),\ 57 = -ip^3 z^*,\ 58 = -iz^{*2},$$

$$61 = -i\sqrt{3}p^3\Omega,\ 62 = -i\sqrt{3}p^1\Omega,\ 63 = 2ip^3\Omega,\ 64 = iz^*\Omega,$$

$$65 = i\sqrt{3}(m\Omega - ip^2 z),\ 66 = -\sqrt{3}p^{23},\ 67 = -i(\Omega^2 - p^{33}),\ 68 = ip^3 z^*,$$

$$71 = -iz\Omega,\ 72 = 2ip^3\Omega,\ 73 = i\sqrt{3}p^1\Omega,\ 74 = -i\sqrt{3}p^3\Omega,$$

$$75 = ip^3 z,\ 76 = i(\Omega^2 - p^{33}),\ 77 = \sqrt{3}p^{23},\ 78 = -i\sqrt{3}(m\Omega + ip^2 z^*),$$

$$81 = 0,\ 82 = iz\Omega,\ 83 = -i\sqrt{3}p^3\Omega,\ 84 = -i\sqrt{3}p^1\Omega,$$

$$85 = iz^2,\ 86 = -ip^3 z,\ 87 = i\sqrt{3}(m\Omega - ip^2 z),\ 88 = -\sqrt{3}p^{23},$$

$$s_{8D}^3 = k \begin{vmatrix} 11 & 12 & 0 & 0 & 0 & 16 & 0 & 0 \\ 21 & 22 & 0 & 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 33 & 34 & 0 & 0 & 0 & 38 \\ 0 & 0 & 43 & 44 & 0 & 0 & 47 & 0 \\ 0 & 52 & 0 & 0 & 55 & 56 & 0 & 0 \\ 61 & 0 & 0 & 0 & 65 & 66 & 0 & 0 \\ 0 & 0 & 0 & 74 & 0 & 0 & 77 & 78 \\ 0 & 0 & 83 & 0 & 0 & 0 & 87 & 88 \end{vmatrix}, \qquad (7.58)$$

where the nonzero matrix elements have the form

$$11 = 3\omega\Omega - p_{22}^{11},\ 12 = p^3 z^*,\ 16 = z^*\Omega,$$

$$21 = p^3 z,\ 22 = \omega\Omega + p_{22}^{11},\ 25 = -z\Omega,$$

$$33 = -\omega\Omega - p_{22}^{11},\ 34 = p^3 z^*,\ 38 = z^*\Omega,$$

$$43 = p^3 z,\ 44 = -3\omega\Omega + p_{22}^{11},\ 47 = -z\Omega,$$

$$52 = -z^*\Omega,\ 55 = 3\omega\Omega - p_{22}^{11},\ 56 = p^3 z^*,$$

$$61 = z\Omega,\ 65 = p^3 z,\ 66 = \omega\Omega + p_{22}^{11},$$

$$74 = -z^*\Omega,\ 77 = -\omega\Omega - p_{22}^{11},\ 78 = p^3 z^*,$$

$$83 = z\Omega,\ 87 = p^3 z,\ 88 = -3\omega\Omega + p_{22}^{11}.$$

Above, in the formulalas (7.56)–(7.58) and in corresponding matrix elements the following notations are used

$$k \equiv \frac{1}{2\omega\Omega}, \qquad p_{22}^{11} \equiv p^1 p^1 + p^2 p^2. \qquad (7.59)$$

Other notations are already given in (7.38).

The equations on eigenvectors and eigenvalues of the operator s_{8D}^3 (7.58) follow from the equations (6.59) and the transformation (7.25)–(7.27). In addition to it, the action of the operator s_{8D}^3 (7.58) on the spinors $(v_A^-(\overrightarrow{k}), v_B^+(\overrightarrow{k}))$ (7.33) also leads to the result

$$s_{8D}^3 v_1^-(\overrightarrow{k}) = \frac{3}{2} v_1^-(\overrightarrow{k}),\ s_{8D}^3 v_2^-(\overrightarrow{k}) = \frac{1}{2} v_2^-(\overrightarrow{k}),$$

$$s_{8D}^3 v_3^-(\overrightarrow{k}) = -\frac{1}{2} v_3^-(\overrightarrow{k}),\ s_{8D}^3 v_4^-(\overrightarrow{k}) = -\frac{3}{2} v_4^-(\overrightarrow{k}),$$

$$s_{8D}^3 v_5^+(\overrightarrow{k}) = \frac{3}{2}v_5^+(\overrightarrow{k}), \, s_{8D}^3 v_6^+(\overrightarrow{k}) = \frac{1}{2}v_6^+(\overrightarrow{k}), \quad (7.60)$$

$$s_{8D}^3 v_7^+(\overrightarrow{k}) = -\frac{1}{2}v_7^+(\overrightarrow{k}), \, s_{8D}^3 v_8^+(\overrightarrow{k}) = -\frac{3}{2}v_8^+(\overrightarrow{k}).$$

In order to verify equations (7.60) the identity (7.40) is used. In the case $v_B^+(\overrightarrow{k})$ in the expression $s_{8D}^3(\overrightarrow{k})$ (7.58) the substitution $\overrightarrow{k} \to -\overrightarrow{k}$ is fulfilled.

The equations (7.60) (together with corresponding equations on eigen values of the sign of the charge and momentum operators) determine the interpretation of the amplitudes in solution (7.55). This interpretation is similar to the given above in the paragraph after equations (5.156). Nevertheless, the direct quantum-mechanical interpretation of the amplitudes should be made in the framework of the RCQM, see the subsection 5.7.4 above.

The explicit form of the $\mathcal{P}$-generators of the fermionic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (7.54) and the set $\{\psi\}$ of its solutions (7.55) are invariant, is derived from the generators (6.60), (6.61) with SU(2) spin (6.58) on the basis of the trasformation (7.25), (7.27). The corresponding generators are given by (7.41), (7.42) where the spin matrices $\overrightarrow{s}_{8D} = (s_{8D}^{\ell n})$ are given in (7.56)–(7.58) and the operator $\overrightarrow{x}_D$ has the form (7.43) with the specific spin-like matrices $\overrightarrow{s}_8^\Gamma$ (7.44).

It is easy to verify that the generators (7.41), (7.42), taken with SU(2) spin (7.56)–(7.58), commute with the operator $\left[i\partial_0 - \Gamma_8^0(\overrightarrow{\Gamma}_8 \cdot \widehat{\overrightarrow{p}} + m)\right]$ of equation (7.54), satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group and the corresponding Casimir operators are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_8, \quad (7.61)$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}_{8D}^2 = \frac{3}{2}\left(\frac{3}{2}+1\right)m^2 \mathrm{I}_8. \quad (7.62)$$

As it was already explained in details in the previous sections, the conclusion that equation (7.54) describes the local field of fermionic particle-antiparticle doublet of the spin s=(3/2,3/2) and mass $m > 0$ (and its solution (7.55) is the local fermionic field of the above mentioned spin and nonzero mass) follows from the analysis of equations (7.60) and the Casimir operators (7.61), (7.62).

Hence, the equation (7.54) describes the spin s=(3/2,3/2) particle-antiparticle doublet on the same level, on which the standard 4-component Dirac

equation describes the spin s=(1/2,1/2) particle-antiparticle doublet. Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin s=(3/2,3/2) particle-antiparticle doublet, where the quantum-mechanical interpretation is direct and evident. Therefore, the fermionic spin s=(3/2,3/2) properties of equation (7.54) are proved.

Contrary to the bosonic spin s=(1,0,1,0) properties of the equation (7.54) (section 7.4), the fermionic spin s=(1/2,1/2,1/2,1/2) properties of this equation are evident. The fact that equation (7.54) describes the multiplet of two fermions with the spin s=1/2 and two antifermions with that spin can be proved much more easier then the above given consideration. The proof is similar to that given in the standard 4-component Dirac model. The detailed consideration can be found in subsections 5.7.2, 6.2.2 and section 7.3. Therefore, equation (7.54) has more extended property of the Fermi–Bose duality then the standard Dirac equation [50–52]. This equation has the property of the Fermi–Bose triality. The property of the Fermi–Bose triality of the manifestly covariant equation (7.54) means that this equation describes on equal level (i) the spin s=(1/2,1/2,1/2,1/2) multiplet of two spin s=(1/2,1/2) fermions and two spin s=(1/2,1/2) antifermions, (ii) the spin s=(1,0,1,0) multiplet of the vector and scalar bosons together with their antiparticles, (iii) the spin s=(3/2,3/2) particle-antiparticle doublet.

It is evident that equation (7.54) is new in comparison with the Pauli–Fierz [8, 9], Rarita–Schwinger [25] and confirmed by Davydov [26] equations for the spin s=3/2 particle. Contrary to 16-component equations from [9, 25, 26] equation (7.54) is 8-component and does not need any additional condition. Only formally equation (7.54) looks like to have some similar features with the Bargman–Wigner equation [11] for arbitrary spin, when the spin value is taken 3/2. The transformation $V_8^{\mp} = \frac{\mp \vec{\Gamma}_8 \cdot \vec{p} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}$ looks like the transformation of Pursey [13] in the case of s=3/2. Nevertheless, the difference is clear. Equation (7.54) is not the ordinary direct sum of the two Dirac equations. Furthermore, the given here model is derived from the first principles of RCQM (not from the FW type representation of the canonical field theory). Our consideration is original and new. The link with corresponding RCQM, the proof of the symmetry properties and relativistic invariance, the well defined spin operator (7.56)–(7.58), the features of the Fermi–Bose duality (triality) of the equation (7.54), the interaction with electromagnetic field and many other characteristics are suggested firstly.

Finally, the Bargman–Wigner equation [11] in the case s=3/2 has 12 components. The comparison with Bhabha equation is evident as well. Bhabha have proved itself [46] that his equation in the case s=3/2 coincide with Rarita–Schwinger equation, i.e., has 16 components.

Interaction, quantization and Lagrange approach in the above given spin s=(3/2,3/2) model are completely similar to the Dirac 4-component theory and standard quantum electrodynamics. For example, the Lagrange function of the system of interacting 8-component spinor and electromagnetic fields (in the terms of 4-vector potential $A^\mu(x)$) is given by

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\left(\overline{\psi}(x)\Gamma_8^\mu\frac{\partial\psi(x)}{\partial x^\mu} - \frac{\partial\overline{\psi}(x)}{\partial x^\mu}\Gamma_8^\mu\psi(x)\right) - \qquad (7.63)$$

$$m\overline{\psi}(x)\psi(x) + q\overline{\psi}(x)\Gamma_8^\mu\psi(x)A_\mu(x),$$

where $\overline{\psi}(x)$ is the independent Lagrange variable and $\overline{\psi} = \psi^\dagger\Gamma_8^0$ in the space of solutions $\{\psi\}$. In Lagrangian (7.63) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor in the terms of potentials, which play the role of variational variables in this Lagrange approach.

Therefore, the covariant local quantum field theory model for the interacting particles with spin s=3/2 and photons can be constructed in complete analogy to the construction of the modern quantum electrodynamics. This model can be useful for the investigations of processes with interacting hyperons and photons.

7.7. Covariant Field Equation for the 12-Component Spin s=(2,0,2,0) Bosonic Particle-Antiparticle Multiplet

The constructed below model can be useful as a variant of the gravity.

The operator of the transition to the covariant local field theory representation (the 12×12 analogy of the 4×4 inverse FW transformation operator (4.23)) is given by

$$V^\mp = \frac{\mp\overrightarrow{\Gamma}_{12}\cdot\widehat{\overrightarrow{p}} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}, \quad V^-V^+ = V^+V^- = \mathrm{I}_{12}, \quad V^- = (V^+)^\dagger, \quad (7.64)$$

$$\psi = V^-\phi, \phi = V^+\psi, \qquad (7.65)$$

$$\hat{q}_{\mathrm{D}} = V^- \hat{q}_{\mathrm{CF}} V^+, \ \hat{q}_{\mathrm{CF}} = V^+ \hat{q}_{\mathrm{D}} V^-, \tag{7.66}$$

where $\hat{q}_{\mathrm{D}}$ is an arbitrary operator (both in the Hermitian and anti-Hermitian form) in the covariant local field theory representation. The inverse transformation is valid as well.

Thus, on the basis of transformation (7.64)–(7.66) the 12-component Dirac-like equation is found from the canonical field equation (6.86) in the form

$$\left[i\partial_0 - \Gamma^0_{12}(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}} + m) \right] \psi(x) = 0. \tag{7.67}$$

In transformation operator (7.64) and in equation (7.67) the Γ^μ_{12} matrices are given by

$$\Gamma^0_{12} = \begin{vmatrix} I_6 & 0 \\ 0 & -I_6 \end{vmatrix}, \quad \Gamma^j_{12} = \begin{vmatrix} 0 & \Sigma^j_6 \\ -\Sigma^j_6 & 0 \end{vmatrix}, \tag{7.68}$$

where Σ^j_6 are the 6×6 Pauli matrices

$$\Sigma^1_6 = \begin{vmatrix} 0 & I_3 \\ I_3 & 0 \end{vmatrix}, \quad \Sigma^2_6 = \begin{vmatrix} 0 & -iI_3 \\ iI_3 & 0 \end{vmatrix}, \quad \Sigma^3_6 = \begin{vmatrix} I_3 & 0 \\ 0 & -I_3 \end{vmatrix}, \tag{7.69}$$

and I_3 is the 3×3 unit matrix.

The matrices Σ^j_6 satisfy the similar commutation relations as the 2×2 Pauli matrices (1.6) and have other similar properties. The matrices Γ^μ_{12} (7.68) satisfy the anticommutation relations of the Clifford–Dirac algebra in the form

$$\Gamma^\mu_{12}\Gamma^\nu_{12} + \Gamma^\nu_{12}\Gamma^\mu_{12} = 2g^{\mu\nu}. \tag{7.70}$$

The solution of equation (7.67) is derived from solution (6.88) of this equation in the canonical representation (6.86) on the basis of transformation (7.64), (7.65) and is given by

$$\psi(x) = V^-\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^{\mathrm{A}}(\overrightarrow{k}) v_{\mathrm{A}}^-(\overrightarrow{k}) + e^{ikx} g^{*\mathrm{B}}(\overrightarrow{k}) v_{\mathrm{B}}^+(\overrightarrow{k}) \right], \tag{7.71}$$

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where $A = \overline{1,6}$, $B = \overline{7,12}$ and the 12-component spinors $(v_A^-(\overrightarrow{k}), v_B^+(\overrightarrow{k}))$ are given by

$$
v_1^-(\overrightarrow{k}) = N \begin{vmatrix} \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ 0 \\ 0 \\ k^1+ik^2 \\ 0 \\ 0 \end{vmatrix}, \quad
v_2^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ 0 \\ 0 \\ k^1+ik^2 \\ 0 \end{vmatrix},
$$

$$
v_3^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ 0 \\ 0 \\ k^1+ik^2 \end{vmatrix}, \quad
v_4^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ k^1-ik^2 \\ 0 \\ 0 \\ -k^3 \\ 0 \\ 0 \end{vmatrix},
$$

$$
v_5^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ k^1-ik^2 \\ 0 \\ 0 \\ -k^3 \\ 0 \end{vmatrix}, \quad
v_6^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ k^1-ik^2 \\ 0 \\ 0 \\ -k^3 \end{vmatrix}, \qquad (7.72)
$$

$$v_7^+(\vec{k}) = N \begin{vmatrix} k^3 \\ 0 \\ 0 \\ k^1 + ik^2 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad v_8^+(\vec{k}) = N \begin{vmatrix} 0 \\ k^3 \\ 0 \\ 0 \\ k^1 + ik^2 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \\ 0 \\ 0 \end{vmatrix},$$

$$v_9^+(\vec{k}) = N \begin{vmatrix} 0 \\ 0 \\ k^3 \\ 0 \\ 0 \\ k^1 + ik^2 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \\ 0 \end{vmatrix}, \quad v_{10}^+(\vec{k}) = N \begin{vmatrix} k^1 - ik^2 \\ 0 \\ 0 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \\ 0 \end{vmatrix},$$

$$v_{11}^+(\vec{k}) = N \begin{vmatrix} 0 \\ k^1 - ik^2 \\ 0 \\ 0 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \end{vmatrix}, \quad v_{12}^+(\vec{k}) = N \begin{vmatrix} 0 \\ 0 \\ k^1 - ik^2 \\ 0 \\ 0 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \end{vmatrix},$$

where N is given in (7.34).

The spinors (7.72) are derived from the orts $\{d\}$ of the Cartesian basis (5.186) with the help of the transformation (7.64), (7.65). The spinors (7.72) satisfy the relations of the orthonormalization and completeness similar to the

corresponding relations for the standard 4-component Dirac spinors, see, e.g., [47].

In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $\left[s^{j}_{12D}, s^{\ell}_{12D}\right] = i\varepsilon^{j\ell n} s^{n}_{12D}$ and commute with the operator $\left[i\partial_0 - \Gamma^0_{12}(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}} + m)\right]$ of equation (7.67), are derived from the pure matrix operators (6.93) with the help of operator (7.64), (7.66):

$$\overrightarrow{s}_{12D} = V^{-}\overrightarrow{s}_{12}V^{+}. \tag{7.73}$$

The explicit form can be presented in analogy with previous sections.

The equations on eigenvectors and eigenvalues of the operator s^3_{12D} (7.73) follow from the equations (6.94) and the transformation (7.64)–(7.66). In addition to it, the action of the operator s^3_{12D} (7.73) on the spinors $(v^-_A(\overrightarrow{k}), v^+_B(\overrightarrow{k}))$ (7.72) also leads to the result

$$s^3_{12D}v^-_1(\overrightarrow{k}) = 2v^-_1(\overrightarrow{k}), \, s^3_{12D}v^-_2(\overrightarrow{k}) = v^-_2(\overrightarrow{k}),$$

$$s^3_{12D}v^-_3(\overrightarrow{k}) = 0, \, s^3_{12D}v^-_4(\overrightarrow{k}) = -v^-_4(\overrightarrow{k}),$$

$$s^3_{12D}v^-_5(\overrightarrow{k}) = -2v^-_5(\overrightarrow{k}), \, s^3_{12D}v^-_6(\overrightarrow{k}) = 0, \tag{7.74}$$

$$s^3_{12D}v^+_7(\overrightarrow{k}) = 2v^+_7(\overrightarrow{k}), \, s^3_{12D}v^+_8(\overrightarrow{k}) = v^+_8(\overrightarrow{k}),$$

$$s^3_{12D}v^+_9(\overrightarrow{k}) = 0, \, s^3_{12D}v^+_{10}(\overrightarrow{k}) = -v^+_{10}(\overrightarrow{k}),$$

$$s^3_{12D}v^+_{11}(\overrightarrow{k}) = -2v^+_{11}(\overrightarrow{k}), \, s^3_{12D}v^+_{12}(\overrightarrow{k}) = 0.$$

In order to verify the equations (7.74) the identity (7.40) should be used. In the case $v^+_B(\overrightarrow{k})$ in the expression $s^3_{12D}(\overrightarrow{k})$ from (7.73) the substitution $\overrightarrow{k} \rightarrow -\overrightarrow{k}$ is fulfilled.

The equations (7.74) determine the interpretation of the amplitudes in solution (7.71). This interpretation is similar to the given above in the corresponding FW representation in the paragraph after equations (6.94). Nevertheless, the direct quantum-mechanical interpretation of the amplitudes should be made in the framework of the RCQM, see the subsection 5.8.3 above.

The explicit form of the $\mathcal{P}$-generators of the bosonic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (7.67) and the set $\{\psi\}$ of its solutions (7.71) are invariant, is derived from the generators (6.95),

(6.96) on the basis of the tramsformation (7.64), (7.66). The corresponding generators are given by

$$\widehat{p}^0 = \Gamma^0_{12}(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}} + m), \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x^\ell_D \widehat{p}^n - x^n_D \widehat{p}^\ell + s^{\ell n}_{12D} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{12D}, \tag{7.75}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\left\{x^\ell_D, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\overrightarrow{s}_{12D} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{7.76}$$

where the spin matrices $\overrightarrow{s}_{12D} = (s^{\ell n}_{12D})$ are given in (7.73) and the operator $\overrightarrow{x}_D$ has the form

$$\overrightarrow{x}_D = \overrightarrow{x} + \frac{i\overrightarrow{\Gamma}_{12}}{2\widehat{\omega}} - \frac{\overrightarrow{s}_{12} \times \widehat{\overrightarrow{p}}}{\widehat{\omega}(\widehat{\omega}+m)} - \frac{i\widehat{\overrightarrow{p}}(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}})}{2\widehat{\omega}^2(\widehat{\omega}+m)}, \tag{7.77}$$

where the specific spin-like matrices $\overrightarrow{s}^\Gamma_{12}$ are given by

$$\overrightarrow{s}^\Gamma_{12} \equiv \frac{i}{2}(\Gamma^2_{12}\Gamma^3_{12}, \Gamma^3_{12}\Gamma^1_{12}, \Gamma^1_{12}\Gamma^2_{12}) \tag{7.78}$$

with Γ^j_{12} determined in (7.68).

It is easy to verify that the generators (7.75), (7.76) commute with the operator $\left[i\partial_0 - \Gamma^0_{12}(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}} + m)\right]$ of equation (7.67), satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group and the corresponding Casimir operators are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_{12}, \tag{7.79}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2_{12D} = m^2 \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{7.80}$$

where I_{12} is 12×12 and I_5 is 5×5 unit matrix.

As it was already explained in details in the previous sections, the conclusion that equation (7.67) describes the bosonic particle-antiparticle multiplet of the spin s=(2,0,2,0) and mass $m > 0$ (and its solution (7.71) is the bosonic field of the above mentioned spin and nonzero mass) follows from the analysis of equations (7.74) and the Casimir operators (7.79), (7.80). Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin s=(2,0,2,0) particle-antiparticle multiplet, where the quantum-mechanical interpretation is direct and evident. Therefore, the bosonic spin s=(2,0,2,0) properties of equation (7.67) are proved.

7.8. Covariant Field Equation for the 16-Component Spin s=(2,1,2,1) Bosonic Particle-Antiparticle Multiplet

The model below can be useful for the approaches to gravity.

The operator of the transition to the covariant local field theory representation (the 16×16 analogy of the 4×4 FW transformation operator (4.23)) is given by

$$V^{\mp} = \frac{\mp \overrightarrow{\Gamma}_{16} \cdot \widehat{\overrightarrow{p}} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}, \quad V^- = (V^+)^{\dagger}, \quad V^- V^+ = V^+ V^- = I_{16}, \quad (7.81)$$

$$\psi = V^- \phi, \phi = V^+ \psi, \tag{7.82}$$

$$\hat{q}_{\mathrm{D}} = V^- \hat{q}_{\mathrm{CF}} V^+, \hat{q}_{\mathrm{CF}} = V^+ \hat{q}_{\mathrm{D}} V^-, \tag{7.83}$$

where $\hat{q}_{\mathrm{D}}$ is an arbitrary operator (both in the Hermitian and anti-Hermitian form) in the covariant local field theory representation. The inverse transformation is valid as well.

Thus, on the basis of transformation (7.81)–(7.83) the 16-component Dirac-like equation is found from the canonical field equation (6.99) in the form

$$\left[i\partial_0 - \Gamma^0_{16}(\overrightarrow{\Gamma}_{16} \cdot \widehat{\overrightarrow{p}} + m) \right] \psi(x) = 0. \tag{7.84}$$

In operator (7.81) and in equation (7.84) the Γ_{16}^μ matrices are given by

$$\Gamma_{16}^0 = \begin{vmatrix} I_8 & 0 \\ 0 & -I_8 \end{vmatrix}, \quad \Gamma_{16}^j = \begin{vmatrix} 0 & \Sigma_8^j \\ -\Sigma_8^j & 0 \end{vmatrix}, \quad (7.85)$$

where Σ_8^j are the 8×8 Pauli matrices

$$\Sigma_8^j = \begin{vmatrix} \sigma^j & 0 & 0 & 0 \\ 0 & \sigma^j & 0 & 0 \\ 0 & 0 & \sigma^j & 0 \\ 0 & 0 & 0 & \sigma^j \end{vmatrix}, \quad (7.86)$$

and the standard Pauli matrices σ^j are given in (1.6).

The matrices Σ_8^j satisfy the similar commutation relations as the 2×2 Pauli matrices (1.6) and have other similar properties. The matrices Γ_{16}^μ (7.85) satisfy the anticommutation relations of the Clifford–Dirac algebra in the form

$$\Gamma_{16}^\mu \Gamma_{16}^\nu + \Gamma_{16}^\nu \Gamma_{16}^\mu = 2g^{\mu\nu}. \quad (7.87)$$

The solution of equation (7.84) is derived from solution (6.101) of this equation in the canonical representation (6.99) on the basis of transformation (7.81), (7.82) and is given by

$$\psi(x) = V^- \phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} g^A(\overrightarrow{k}) v_A^-(\overrightarrow{k}) + e^{ikx} g^{*B}(\overrightarrow{k}) v_B^+(\overrightarrow{k}) \right], \quad (7.88)$$

where $A = \overline{1,8}, B = \overline{9,16}$ and the 16-component spinors $(v_A^-(\overrightarrow{k}), v_B^+(\overrightarrow{k}))$ are given by

$$v_1^-(\overrightarrow{k}) = N \begin{vmatrix} \tilde{\omega} + m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1 + ik^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad v_2^-(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ \tilde{\omega} + m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^1 - ik^2 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix},$$

$$
\mathrm{v}_3^-(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1+ik^2 \\ 0 \\ 0 \\ 0 \\ 0
\end{vmatrix}, \quad
\mathrm{v}_4^-(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^1-ik^2 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0
\end{vmatrix},
$$

$$
\mathrm{v}_5^-(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1+ik^2 \\ 0 \\ 0
\end{vmatrix}, \quad
\mathrm{v}_6^-(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^1-ik^2 \\ -k^3 \\ 0 \\ 0
\end{vmatrix},
$$

$$
\mathrm{v}_7^-(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1+ik^2
\end{vmatrix}, \quad
\mathrm{v}_8^-(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^1-ik^2 \\ -k^3
\end{vmatrix}, \tag{7.89}
$$

$$
v_9^+(\overrightarrow{k}) = N
\begin{vmatrix}
k^3 \\
k^1 + ik^2 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\widetilde{\omega} + m \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{vmatrix}
, \quad
v_{10}^+(\overrightarrow{k}) = N
\begin{vmatrix}
k^1 - ik^2 \\
-k^3 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\widetilde{\omega} + m \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{vmatrix}
,
$$

$$
v_{11}^+(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\
0 \\
k^3 \\
k^1 + ik^2 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\widetilde{\omega} + m \\
0 \\
0 \\
0 \\
0
\end{vmatrix}
, \quad
v_{12}^+(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\
0 \\
k^1 - ik^2 \\
-k^3 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\widetilde{\omega} + m \\
0 \\
0 \\
0 \\
0
\end{vmatrix}
,
$$

$$
v_{13}^+(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\
0 \\
0 \\
0 \\
k^3 \\
k^1 + ik^2 0 \\
0 \\
0 \\
0 \\
0 \\
\widetilde{\omega} + m \\
0 \\
0 \\
0
\end{vmatrix}
, \quad
v_{14}^+(\overrightarrow{k}) = N
\begin{vmatrix}
0 \\
0 \\
0 \\
0 \\
k^1 - ik^2 \\
-k^3 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\widetilde{\omega} + m \\
0 \\
0
\end{vmatrix}
,
$$

$$v^+_{15}(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1 + ik^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \\ 0 \end{vmatrix}, \quad v^+_{16}(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^1 - ik^2 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega} + m \end{vmatrix},$$

where N and $\widetilde{\omega}$ are given in (7.34).

The spinors (7.89) are derived from the orts {d} of the Cartesian basis (5.198) with the help of the transformation (7.81), (7.82). The spinors (7.89) satisfy the relations of the orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors, see, e.g., [47].

In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $\left[s^j_{16D}, s^\ell_{16D} \right] = i\varepsilon^{j\ell n} s^n_{16D}$ and commute with the operator $\left[i\partial_0 - \Gamma^0_{16}(\overrightarrow{\Gamma}_{16} \cdot \widehat{\overrightarrow{p}} + m) \right]$ of equation (7.84), are derived from the pure matrix operators (6.106) with the help of transformation operator (7.81), (7.83):

$$\overrightarrow{s}_{16D} = V^- \overrightarrow{s}_{16} V^+. \tag{7.90}$$

The explicit form can be presented in analogy with previous sections. The third component of this spin has the explicit form

$$s^3_{16D} = \frac{1}{2\omega\Omega} \cdot |16 \times 16|, \tag{7.91}$$

where the nonzero matrix elements a, b of 16×16 matrix are given by

$$1, 1 = 4\omega\Omega - p^{11}_{22}, \; 1, 2 = p^3 z^*, \; 1, 10 = z^*\Omega,$$
$$2, 1 = p^3 z, \; 2, 2 = 2\omega\Omega + p^{11}_{22}, \; 2, 9 = -z\Omega,$$
$$3, 3 = -p^{11}_{22}, \; 3, 4 = p^3 z^*, \; 3, 12 = z^*\Omega,$$
$$4, 3 = p^3 z, \; 4, 4 = \Omega^2 - p^{33}, \; 4, 11 = -z\Omega,$$

$$5,5 = -4\omega\Omega + 3p_{22}^{11}, \; 5,6 = -3p^3 z^*, \; 5,14 = -3z^*\Omega,$$

$$6,5 = -3p^3 z, \; 6,6 = 2\omega\Omega - 3p_{22}^{11}, \; 6,13 = 3z\Omega,$$

$$7,7 = -p_{22}^{11}, \; 7,8 = p^3 z^*, \; 7,16 = z^*\Omega,$$

$$8,7 = p^3 z, \; 8,8 = -\Omega^2 - p^{33}, \; 8,15 = -z\Omega,$$

$$9,2 = -z^*\Omega, \; 9,9 = 4\omega\Omega - p_{22}^{11}, \; 9,10 = p^3 z^*,$$

$$10,1 = z\Omega, \; 10,9 = p^3 z, \; 10,10 = 2\omega\Omega + p_{22}^{11},$$

$$11,4 = -z^*\Omega, \; 11,11 = -p_{22}^{11}, \; 11,12 = p^3 z^*,$$

$$12,3 = z\Omega, \; 12,11 = p^3 z, \; 12,12 = -\Omega^2 - p^{33},$$

$$13,6 = 3z^*\Omega, \; 13,13 = -4\omega\Omega + 3p_{22}^{11}, \; 13,14 = -3p^3 z^*,$$

$$14,5 = -3z\Omega, \; 14,13 = -3p^3 z, \; 14,14 = 2\omega\Omega - 3p_{22}^{11},$$

$$15,8 = -z^*\Omega, \; 15,15 = -p_{22}^{11}, \; 15,16 = p^3 z^*,$$

$$16,7 = z\Omega, \; 16,15 = p^3 z, \; 16,16 = -\Omega^2 - p^{33}.$$

Here in matrix elements the notations (7.38), (7.59) are used.

The equations on eigenvectors and eigenvalues of the operator s_{16D}^3 (7.91) follow from the equations (6.107) and the transformation (7.81)–(7.83). In addition to it, the action of the operator s_{16D}^3 (7.91) on the spinors $(v_A^-(\vec{k}), v_B^+(\vec{k}))$ (7.89) also leads to the result

$$s_{16D}^3 v_1^-(\vec{k}) = 2v_1^-(\vec{k}), \quad s_{16D}^3 v_2^-(\vec{k}) = v_2^-(\vec{k}),$$

$$s_{16D}^3 v_3^-(\vec{k}) = 0, \quad s_{16D}^3 v_4^-(\vec{k}) = -v_4^-(\vec{k}),$$

$$s_{16D}^3 v_5^-(\vec{k}) = -2v_5^-(\vec{k}), \quad s_{16D}^3 v_6^-(\vec{k}) = v_6^-(\vec{k}),$$

$$s_{16D}^3 v_7^-(\vec{k}) = 0, \quad s_{16D}^3 v_8^-(\vec{k}) = -v_8^-(\vec{k}), \qquad (7.92)$$

$$s_{16D}^3 v_9^+(\vec{k}) = 2v_9^+(\vec{k}), \quad s_{16D}^3 v_{10}^+(\vec{k}) = v_{10}^+(\vec{k}),$$

$$s_{16D}^3 v_{11}^+(\vec{k}) = 0, \quad s_{16D}^3 v_{12}^+(\vec{k}) = -v_{12}^+(\vec{k}),$$

$$s_{16D}^3 v_{13}^+(\vec{k}) = -2v_{13}^+(\vec{k}), \quad s_{16D}^3 v_{14}^+(\vec{k}) = v_{14}^+(\vec{k}),$$

$$s_{16D}^3 v_{15}^+(\vec{k}) = 0, \quad s_{16D}^3 v_{16}^+(\vec{k}) = -v_{16}^+(\vec{k}).$$

In order to verify equations (7.92) the identity (7.40) should be used. In the case $v_B^+(\overrightarrow{k})$ in the expression $s_{12D}^3(\overrightarrow{k})$ (7.91) the substitution $\overrightarrow{k} \to -\overrightarrow{k}$ is made.

The equations (7.92) determine the interpretation of the amplitudes in solution (7.88). This interpretation is similar to the given above in the FW representation in the paragraph after equations (6.107). Nevertheless, the direct quantum-mechanical interpretation of the amplitudes should be made in the framework of the RCQM, see the subsection 5.8.4 above.

The explicit form of the $\mathcal{P}$-generators of the bosonic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (7.84) and the set $\{\psi\}$ of its solutions (7.88) are invariant, is derived from the FW type generators (6.108), (6.109) on the basis of the tramsformation (7.81), (7.83). The corresponding generators are given by

$$\widehat{p}^0 = \Gamma_{16}^0(\overrightarrow{\Gamma}_{16} \cdot \widehat{\overrightarrow{p}} + m), \quad \widehat{p}^\ell = -i\partial_\ell,$$

$$\widehat{j}^{\ell n} = x_D^\ell \widehat{p}^n - x_D^n \widehat{p}^\ell + s_{16D}^{\ell n} \equiv \widehat{m}^{\ell n} + s_{16D}^{\ell n}, \tag{7.93}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0 \widehat{p}^\ell - \frac{1}{2}\left\{x_D^\ell, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\overrightarrow{s}_{16D} \times \widehat{\overrightarrow{p}})^\ell}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{7.94}$$

where the spin matrices $\overrightarrow{s}_{16D} = (s_{16D}^{\ell n})$ are given in (7.90), (7.91), and the operator $\overrightarrow{x}_D$ has the form

$$\overrightarrow{x}_D = \overrightarrow{x} + \frac{i\overrightarrow{\Gamma}_{16}}{2\widehat{\omega}} - \frac{\overrightarrow{s}_{16}^{\Gamma} \times \widehat{\overrightarrow{p}}}{\widehat{\omega}(\widehat{\omega}+m)} - \frac{i\widehat{\overrightarrow{p}}(\overrightarrow{\Gamma}_{16} \cdot \widehat{\overrightarrow{p}})}{2\widehat{\omega}^2(\widehat{\omega}+m)}, \tag{7.95}$$

where the specific spin-like matrices $\overrightarrow{s}_{16}^{\Gamma}$ are given by

$$\overrightarrow{s}_{16}^{\Gamma} \equiv \frac{i}{2}(\Gamma_{16}^2\Gamma_{16}^3, \Gamma_{16}^3\Gamma_{16}^1, \Gamma_{16}^1\Gamma_{16}^2) \tag{7.96}$$

with Γ_{16}^j determined in (7.85).

It is easy to verify that the generators (7.93), (7.94) commute with the operator $\left[i\partial_0 - \Gamma_{16}^0(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}} + m)\right]$ of equation (7.84), satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group and the corresponding Casimir operators are given by

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 I_{16}, \tag{7.97}$$

$$W = w^{\mu}w_{\mu} = m^2 \vec{s}^2_{16D} = m^2 \begin{vmatrix} 2(2+1)I_5 & 0 & 0 & 0 \\ 0 & 1(1+1)I_3 & 0 & 0 \\ 0 & 0 & 2(2+1)I_5 & 0 \\ 0 & 0 & 0 & 1(1+1)I_3 \end{vmatrix},$$

(7.98)

where I_{16}, I_5 and I_3 are 16×16, 5×5 and 3×3 unit matrices, respectively.

As it was already explained in details in the previous sections, the conclusion that equation (7.84) describes the bosonic particle-antiparticle multiplet of the spin s=(2,1,2,1) and mass $m > 0$ (and its solution (7.88) is the bosonic field of the above mentioned spin and nonzero mass) follows from the analysis of equations (7.92) and the Casimir operators (7.97), (7.98). Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin s=(2,1,2,1) particle-antiparticle multiplet, where the quantum-mechanical interpretation is direct and evident. Therefore, the bosonic spin s=(2,1,2,1) properties of equation (7.84) are proved.

It is easy to prove that equation (7.84) describes four spin s=(1/2,1/2) fermionic particle-antiparticle doublets, two spin s=(3/2,3/2) fermionic particle-antiparticle doublets and two spin s=(1,0,1,0) bosonic particle-antiparticle multiplets as well. Therefore, this equation has the extended property of Fermi–Bose *duality*. Such characteristic can be called the *quadro* Fermi–Bose property.

7.9. Natural Application to the Gravity

The most direct application for the gravity is the derivation of the field equation for the spin s=(2,2) particle-antiparticle bosonic doublet. Indeed, it is the field model for the graviton-antigraviton doublet both in general case of nonzero particle masses and in the limit $m = 0$. Above on the level of canonical field this model is given in the subsection 6.2.5. In particular the canonical Foldy–Wouthuysen type field equation and its solution are given by (6.64) and (6.65).

The only small difficulty is as follows. The covariant field theory equation for the spin s=(2,2) particle-antiparticle doublet can not be presented in terms of Γ matrices. Indeed, the 10×10 Clifford–Dirac Γ matrices are not exist as an mathematical objects. Therefore, one need to look for the locally covariant field equations for spin s=(2,2) particle-antiparticle doublet not in the terms of Γ matrices.

Consider two ways of the derivation of the covariant field equations for spin s=(2,2) particle-antiparticle doublet. The first way is based on the application of the corresponding FW transformation directly to the canonical field equation (6.64). Note that there is a nonzero chance to find the appropriate transformation among a big number of so-called FW transformations for an arbitrary spin [76–85]. Unfortunately for our approach, many authors considered for the spin s=1 case the 4-vector potential (see, e.g., [85]) and not 3-component wave function $\overrightarrow{\psi} = \overrightarrow{E} - i\overrightarrow{E}$ as it is in [3–6, 48, 55–58, 70–73]. Therefore, the objects for spin s=2 case are different as well. Indeed, our approach for arbitrary spin equations differs from other known approaches.

Hence, below (see some details in [5]) another way of the derivation of the covariant field equations for spin s=(2,2) particle-antiparticle doublet is realized. It is easy to construct the formalism of the RCQM of the spin s=(2,0,2,0) particle-antiparticle multiplet. Further, the canonical FW type model of the spin s=(2,0,2,0) particle-antiparticle field can be formulated. Moreover, both models are presented above in subsections 5.8.3 and 6.3.2. After that the 12×12 operator transformation (7.64) should be applied for the canonical FW type equation of the spin s=(2,0,2,0) particle-antiparticle multiplet.

Found by the above considered way (or found as the partial 12×12 case of arbitrary spin equation (7.2)) covariant field equation for the spin s=(2,0,2,0) particle-antiparticle multiplet (7.67) is given by $\left[i\partial_0 - \Gamma_{12}^0(\overrightarrow{\Gamma}_{12} \cdot \widehat{\overrightarrow{p}} + m)\right]\psi(x) = 0$, where the matrices Γ_{12}^μ have the form (7.68). The general solution of the equation (7.67) is given by the formula (7.71).

In order to find the covariant field equations for spin s=(2,2) particle-antiparticle doublet one can put equal to zero the scalar components in (7.67). The place of the scalar components is evident from the corresponding quantum-mechanical consideration in subsection 5.8.3.

The similar way was demonstrated in section 7.5 to find the covariant Maxwell-like field equations for the spin s=(1,1) particle-antiparticle doublet from the known Dirac-like equation (7.28) for the spin s=(1,0,1,0) particle-antiparticle multiplet. Similarly to equation (7.50) the covariant field equation (7.99) (or (7.100)), written below for the spin s=(2,2) particle-antiparticle doublet, can be presented in terms of **curl**, **div** and **grad** operators as well.

Thus, the locally covariant field equations for 10 component spin s=(2,2)

particle-antiparticle doublet found by this way has the form

$$i\partial_0\psi^1 - p^3\psi^6 - m\psi^1 = 0,$$

$$i\partial_0\psi^2 - p^3\psi^7 + ip^2\psi^{10} - p^1\psi^{10} - m\psi^2 = 0,$$

$$i\partial_0\psi^3 - p^3\psi^8 + ip^2\psi^9 - p^1\psi^9 - m\psi^3 = 0,$$

$$i\partial_0\psi^4 + p^3\psi^9 - ip^2\psi^8 - p^1\psi^8 - m\psi^4 = 0,$$

$$i\partial_0\psi^5 + p^3\psi^{10} - ip^2\psi^7 - p^1\psi^7 - m\psi^5 = 0,$$

$$i\partial_0\psi^6 - p^3\psi^1 + m\psi^6 = 0,$$

$$i\partial_0\psi^7 - p^3\psi^2 + ip^2\psi^5 - p^1\psi^5 + m\psi^7 = 0, \qquad (7.99)$$

$$i\partial_0\psi^8 - p^3\psi^3 + ip^2\psi^4 - p^1\psi^4 + m\psi^8 = 0,$$

$$i\partial_0\psi^9 + p^3\psi^4 - ip^2\psi^3 - p^1\psi^3 + m\psi^9 = 0,$$

$$i\partial_0\psi^{10} + p^3\psi^5 - ip^2\psi^2 - p^1\psi^2 + m\psi^{10} = 0,$$

$$-ip^2\psi^1 - p^1\psi^1 = 0, \quad -ip^2\psi^6 - p^1\psi^6 = 0,$$

or the Dirac-like form

$$\left[i\partial_0 - \Gamma_8^0(\vec{\Gamma}_8 \cdot \vec{p} + m)\right]\psi(x) = 0, \qquad (7.100)$$

$$(i\partial_0 - \sigma^1 p^3 - \sigma^3 m)\chi = 0, \quad (p^1 + ip^2)\chi = 0,$$

where $\psi = \text{column}(\psi^1, \psi^2, ..., \psi^8)$, $\chi = \begin{vmatrix} \psi^9 \\ \psi^{10} \end{vmatrix}$, $\Gamma_8^0 = \begin{vmatrix} I_4 & 0 \\ 0 & -I_4 \end{vmatrix}$, $\Gamma_8^j = \begin{vmatrix} 0 & \Sigma_4^j \\ -\Sigma_4^j & 0 \end{vmatrix}$, and Σ_4^j are the 4×4 Pauli matrices $\Sigma_4^1 = \begin{vmatrix} 0 & I_2 \\ I_2 & 0 \end{vmatrix}$, $\Sigma_4^2 = \begin{vmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{vmatrix}$, $\Sigma_4^3 = \begin{vmatrix} I_2 & 0 \\ 0 & -I_2 \end{vmatrix}$. The forms (7.99), (7.100) are linked by the ordinary linear transformation. The form similar to (7.50) is found by the method similar to one presented in section 7.5.

7.10. Synthesis of Arbitrary Spin Covariant Particle Equations: Discussions, Conclusions, Perspectives

The program of synthesis of arbitrary spin covariant particle equations is presented here in Chapters 5, 6 and in this chapter. The program is finished in the section 7.9 above. Therefore, consider below discussions, conclusions and perspectives.

7.10.1. Interaction

The consideration in Chapters 5, 6 and here in general is about free non-interacting fields and particle states. Note at first that the free non-interacting fields and particle states are the physical reality of the same level as the interacting fields and corresponding particle states. Nevertheless, it is meant that the interaction between a fields can be easily introduced on the every step of consideration. One test model with interaction is considered in explicit form in the section 7.6 above. The interaction can not be the deficiency in these constructions and can be introduced in many places by the method similar to the formula (7.63).

Note at first that the free non-interacting fields and particle states are the physical reality of the same level as the interacting fields and corresponding particle states. The best well-known example is the free Maxwell electromagnetic field in the form of free electromagnetic waves. Therefore, the systematic investigation of free non-interacting fields and particle states on the different levels of the RCQM of arbitrary spin (based on the Schrodinger-Foldy equation), the relativistic canonical field theory of arbitrary spin (based on the FW equation and its generalizations), the locally covariant field theory of arbitrary spin (based on the Dirac equation and its generalizations) and, moreover, the introduction of the operator link between these theories is some result as well and can have some independent significance. The precise free field consideration is the necessary first step in all consequent field theory presentation, in axiomatic presentation as well. Further, it is well-known that when the free field models are well-defined the interaction between them can be introduced very easy in well-defined forms known from the literature.

The free particle RCQM today is the subject of other authors considerations as well. Note that the wave packet solutions of the Salpeter equation and their

unusual properties (rather recent papers [86, 87]) have been derived for the free non-interacting cases.

Moreover, for the standard Dirac theory the interaction is given in all handbooks on the relativistic quantum field theory and quantum electrodynamics. Such consideration can be added here very easy. Nevertheless, here one typical interaction is considered already as the test example in order to demonstrate the way of introduction of the interaction in corresponding models. For the spin s=(3/2,3/2) field the interaction with electromagnetic potentials was given in the section 7.6 in the formula (7.63). Thus, it is enough to explain the situation with interaction here. The interaction can not be the deficiency in these constructions and can be introduced in many places by the method similar to the formula (7.63).

Other concrete physical effects and the problem of description of (inter)-particle interactions in frames of the RCQM are described by other authors, see the references [86–108]. Moreover, the way of introducing the interaction into RCQM model is easy and direct (similarly to non-relativistic quantum mechanics) not a matter of fact what particle-antiparticle doublet is under consideration. After that the covariant field models with interaction can be found very easy according to the operator links suggested in the Chapter 6 and in this chapter above.

7.10.2. Application to the Discussion around the Antiparticle Negative Mass

The system of vertical and horizontal links between the RCQM and the field theory, which is proved above, has different useful applications. One of the fundamental applications is the participation in the discussion around the antiparticle negative mass. We emphasize that the model of the RCQM and the corresponding field theory do not need the appealing to the antiparticle negative mass concept [109–112]. It is natural due to the following reasons.

It is only the energy which depends on the mass. And the total energy together with the momentum is related to the external degrees of freedom, which are common and the same for the particle and antiparticle (for the electron and positron). The difference between the particle and the antiparticle consists only in internal degrees of freedom such as the spin $\overrightarrow{s}$ and the sign of the charge $g = -\gamma^0$. Thus, if in the RCQM the mass of the particle is taken positive then the mass of the antiparticle must be taken positive too.

On the other hand, a comprehensive analysis [110] of the Dirac equation for the doublet had led the authors of [110] to the concept of the negative mass of the antiparticle. Therefore, the consideration here gives the additional arguments that the Dirac model (or the Foldy–Wouthuysen model related to it) is not the quantum-mechanical one. Furthermore, in the problem of the relativistic hydrogen atom the use of the negative-frequency part $\psi^-(x) = e^{-i\omega t}\psi(\overrightarrow{x})$ of the spinor $\psi(x)$ in the "role of the quantum-mechanical object" is not valid. In this case neither $|\psi(\overrightarrow{x})|^2$, nor $\overline{\psi}(\overrightarrow{x})\psi(\overrightarrow{x})$ is the probability distribution density with respect to the eigenvalues of the Fermi-doublet coordinate operator. It is due to the fact [6] that in the Dirac model the $\overrightarrow{x}$ is not the experimentally observable Fermi-doublet coordinate operator.

The application of the RCQM can be useful for the analysis of the experimental situation found in [113]. Such analysis is interesting due to the fact that (as it is demonstrated here and in [3–6]) the RCQM is the most fundamental model of the particle-antiparticle doublet.

Another interesting application of the RCQM is inspired by [114], where the quantum electrodynamics is reformulated in the FW representation. The author of [114] essentially used the result of [110] on the negative mass of the antiparticle. Starting from the RCQM we are able not to appeal to the concept of the antiparticle negative mass.

7.10.3. The Partial Case of Zero Mass

All equations of motion considered here are valid for the partial case $m = 0$. The corresponding expressions are found by the transition $m \to 0$. This assertion is valid both for the RCQM and for the covariant local field theory.

In the case $m = 0$ the above considered equations of motion are invariant with respect to all representations of the SU(2) group, which are mentioned in this article. The SU(2) symmetry does not depend on the difference between $m = 0$ and $m \neq 0$.

The difference between $m = 0$ and $m \neq 0$ is observed only for the symmetry with respect to the Poincaré group $\mathcal{P}$. In the case of $m = 0$ equations of motion are invariant with respect to all Poincaré generators given above in the paper. Nevertheless, the Casimir operators are equal to zero in this case. Therefore, another class of Poincaré group representations, using the Casimir operator $\overrightarrow{s} \cdot \overrightarrow{p}$ related to the helicity of elementary particles, is preferable. Hence, in the case $m = 0$ the momentum-helicity basis vectors are preferable in comparison

with momentum-spin basis, which is applied in this paper.

After this warning the theory for $m = 0$ follows from the theory with $m \neq 0$.

7.10.4. Discussions, Conclusions, Perspectives

In the presented above text (and briefly in [3–6]) our experience in the time span 2002–2015 in the investigation of the spin s=1/2 and s=1 fields is applied for the first time to the higher spin cases s=3/2 and s=2. Thus, our "old" papers are augmented by the list of new results for higher spins and generalization for arbitrary spin. Moreover, here (the start has been given in [3–6]) the system of different vertical and horizontal links between the arbitrary spin particles descriptions on the levels of relativistic quantum mechanics, canonical field theory (of FW type) and locally covariant field theory is suggested.

Among the results of this paper the original method of derivation of the Dirac (and the Dirac-like equations for higher spins) is suggested (Chapters 5–7). In order to determine the place of this derivation among the other known methods in Chapter 2 the different ways of the Dirac equation derivation have been reviewed. Thus, a review of the different derivations of the Dirac equation demonstrates that presented in Chapters 5–7 general method is original and new. Here the Dirac equation is derived from the Schrödinger–Foldy equation (5.120) (in 4-component case) of the RCQM. The RCQM model of the spin s=(1/2,1/2) particle-antiparticle doublet has been considered in details in Chapter 5. Hence, the Dirac equation is derived from the more fundamental model of the same physical reality, which is presented by the RCQM of the spin s=(1/2,1/2) particle-antiparticle doublet. The analysis of the spin s=(1/2,1/2) particle-antiparticle doublet is the basis of our generalization for the higher spin models and for arbitrary spin.

One of the general fundamental conclusions is as follows. It is shown by corresponding comparison that customary FW representation can not give the complete quantum-mechanical description of the relativistic particle (or particle multiplet). The analysis of canonical FW representation is presented in the Chapter 4. Compare, e.g., the equations on eigenvectors and eigenvalues for the third component of the spin operator in each quantum-mechanical and canonical field theory model here above. It is useful also to compare the general solutions in RCQM and in field theory (for this purpose it is enough to consider the field theory in the canonical FW representation). Contrary to RCQM in FW representation the general solution consists of positive and negative frequency

parts. As a consequence contrary to RCQM in FW representation the energy has an indefinite sign. Hence, the complete quantum-mechanical description of the relativistic particle (or particle multiplet) can be given only in the framework of the RCQM. Therefore, the customary FW transformation is extended here to the form, which gives the direct link between the locally covariant field theory and the RCQM. Hence, such extended inverse FW transformation is used here to fulfill the synthesis of covariant particle equations. The start of such synthesis is given here from the RCQM and not from the canonical field theory in the representations of the FW type.

Comparison of RCQM and FW representation visualizes the role of the J. von Neumann axiomatic [115] in this presentation. Therefore, relation of J. von Neumann axiomatic [115] to the overall contents of the paper is direct and unambiguous. It is shown that among the above considered models only RCQM of arbitrary spin can be formulated in J. von Neumann's axiomatic, whereas canonical and covariant field theories can not be formulated in its framework.

The new operator links $v_{2N} = \begin{vmatrix} I_N & 0 \\ 0 & \hat{C}I_N \end{vmatrix}$, $N = 2s + 1$, found here between the RCQM of arbitrary spin and the canonical (FW type) field theory enabled ones to translate the result found in these models from one model to another. For example, the results of [90–108] from RCQM can be traslated into the canonical field theory. Contrary, the results of [104] from canonical field theory can be translated into the RCQM (for free non-interacting cases and in the form of anti-Hermitian operators). Note that operator (5.28) is not unitary but is well defined and having inverse operator.

The partial case of the Schrodinger-Foldy equation, when the wave function has only one component [116], is called the spinless Salpeter equation and is under consideration in many recent papers [86, 93, 96, 97, 99–101, 106–108]. The partial wave packet solutions of this equation are given in [86]. The solutions for free massless and massive particle on a line, massless particle in a linear potential, plane wave solution for a free particle (these solution is given here in (5.12) for N-component case), free massless particle in three dimensions have been considered. Further, in the paper [87] other time dependent wave packet solutions of the free spinless Salpeter equation are given. Taking into account the relation of such wave packets to the Levy process the spinless Salpeter equation (in one dimensional space) is called in [87] as the Levy-Schrodinger equation. The several examples of the characteristic behavior of such wave packets have been shown, in particular, of the multimodality arising in their evolutions: a fea-

ture at variance with the typical diffusive unimodality of both the corresponding Levy process densities and usual Schrodinger wave functions. Therefore, the interesting task is to extend such consideration to the equations of the N-component relativistic canonical quantum mechanics considered above and to use the links given here in order to transform wave packet solutions [86, 87] into the solutions of the equations of the locally covariant field theory.

In this chapter the original FW transformation [1] is used and slightly generalized for the many component cases. The improvement of the FW transformation [1] is the task of many authors from 1950 until today, see, e.g., the list of publications [76–85]. Nevertheless, this transformation for free case of non-interacting spin 1/2 particle-antiparticle doublet is not changed from 1950 (the year of first publication) until today. Alexander Silenko was successful in FW transformation for single particles with spin 0 [117], spin 1/2 [118] and spin 1 [119, 120] interacting with external electric, magnetic and other fields. In the case of non-interacting particle, when the external electric, magnetic and other external fields are equal to zero, all results of [85, 117–120] and other authors reduce to the earlier results [1, 76]. Therefore, the choice in this paper of the exact FW transformation from 1950 as the initial (and basic for further generalizations for arbitrary spin) is evident and well-defined. For our next publications, we have a plan to consider interacting fields and to use the results of [117–120] and recent results of other authors, which generalize the FW formulas in the case of interaction.

A few remarks should be added about the choice of the spin operator. Authors of recent paper [121] considered all spin operators for a Dirac particle satisfying some logical and group-theoretical conditions. The discussion of other spin operators proposed in the literature has been presented as well. As a result only one satisfactory operator has been chosen. This operator is equivalent to the Newton–Wigner spin operator and FW mean-spin operator. Contrary to such way the situation here is evident. Above the choice of the spin operator for spin s=1/2 particle-antiparticle doublet is unique. The explicit form for such operator follows directly from the main principles of the RCQM of spin s=1/2 particle-antiparticle doublet, which are formulated in the Chapter 5. Such operator from RCQM is given here in formula (5.124) taken for N=2. After that the links between the RCQM, FW representation and the Dirac model unambiguously give at first the FW spin (5.126) (with N=2) and finally the spin (4.27), which is called in literature as the FW mean-spin operator (for the spin s=(1/2,1/2) particle-antiparticle doublet). Therefore, the similar con-

sideration for the higher spin doublets gives unambiguously the well-defined higher spin operators, which are presented in this chapter above. These new mean-spin operators (7.35)–(7.37), (7.56)–(7.58), (7.91) for higher spin multiplets s=(1,0,1,0), s=(3/2,3/2) and s=(2,1,2,1) are the interesting independent results.

The goal of the paper [122] is a comprehensive analysis of the intimate relationship between jump-type stochastic processes (e.g., Levy flights) and nonlocal (due to integro-differential operators involved) quantum dynamics. Special attention is paid to the spinless Salpeter equation and the various wave packets, in particular to their radial expression in 3D. Furthermore, Foldy's approach [2] is used [122] to encompass free Maxwell theory, which however is devoid of any "particle" content. Links with the photon wave mechanics are explored. Consideration in the subsections 5.7.3, 5.8.1, 5.8.2, 6.2.3, 6.3.1, sections 7.4 and 7.5, demonstrates another links between the Maxwell equations and the RCQM. In the generalization of the Foldy's synthesis of covariant particle equations given above the Maxwell equations and their analogy for nonzero mass are related to the RCQM of spin s=(1,1) and spin s=(1,0,1,0) particle-antiparticle doublets. The electromagnetic field equations that follow from the corresponding RCQM equations are considered. The new electrodynamical equations containing the hypothetical antiphoton and massless spinless antiboson are introduced. The Maxwell-like equations for the boson with spin s=1 and $m > 0$ (W-boson) have been introduced as well. In other words the Maxwell equations for the field with nonzero mass are introduced.

The covariant consideration of arbitrary spin field theory given above contains the non-covariant representations of the Poincaré algebra. Nevertheless, it is not the deficiency of the given model. For the Poincaré group $\mathcal{P}$ generators of spin s=(1/2,1/2) the covariant form (7.21) is well-known, in which the generators have the form of the local Lie operators. In order to have the uniform consideration (7.9), (7.10) the Poincaré generators for spin s=(1,0,1,0), (3/2,3/2), (2,0,2,0), (2,1,2,1) fields are given in formulas (7.41), (7.42), (7.75), (7.76), (7.93), (7.94) in corresponding uniform forms of non-covariant operators in covariant theory. After further transformations of these generators sets in the direction of finding the covariant forms like (7.21) some sets of generators can be presented in the manifestly covariant forms. For other sets of generators covariant forms are extrinsic. Some sets of generators can be presented only in the forms, which are similar to given in [50–52], where the prime anti-Hermitian operators and specific eigen vectors – eigenvalues equations (with

imagine eigenvalues) are used, see, e.g., formula (21) in [50].

The second reason of the stop on the level (7.41), (7.42), (7.75), (7.76), (7.93), (7.94) is to conserve the important property of the Poincaré generators in the canonical FW type representation. Similarly to the FW type Poincaré generators in the sets (7.41), (7.42), (7.75), (7.76), (7.93), (7.94) both angular momenta (orbital and spin) commute with the operator of the Dirac-like equation of motion (7.2). Contrary to the generators (7.9), (7.10), in the covariant form (7.21) only total angular momentum, which is the sum of orbital and spin angular momenta, commutes with the Diracian.

The main point is as follows. The non-covariance is not the barrier for the relativistic invariance! Not a matter of fact that non-covariant objects such as the Lebesgue measure d^3x and the non-covariant Poincaré generators are explored, the model of locally covariant field theory of arbitrary spin presented in section 7.2 is a relativistic invariant in the following sense. The Dirac-like equation (7.2) and the set $\{\psi\}$ of its solutions (7.7) are invariant with respect to the reducible representation of the Poincaré group $\mathcal{P}$ the non-local and non-covariant generators of which are given by (7.9), (7.10). Indeed, the direct calculations visualize that generators (7.9), (7.10) commute with the operator of equation (7.2) and satisfy the commutation relations (3.4) of the Lie algebra of the Poincaré group $\mathcal{P}$.

The 8-component manifestly covariant equation (7.54) for the spin s=3/2 field is the s=3/2 analogy of the 4-component Dirac equation for the spin s=1/2 doublet. It is shown that synthesis of this equation from the relativistic canonical quantum mechanics of the spin s=3/2 particle-antiparticle doublet is completely similar to the synthesis of the Dirac equation from the relativistic canonical quantum mechanics of the spin s=1/2 particle-antiparticle doublet. The difference is only in the value of spin (3/2 and 1/2). On this basis and on the basis of the investigation of solutions and transformation properties with respect to the Poincaré group this new 8-component equation is suggested to be well defined for the description of spin s=3/2 fermions. Note that known Rarita-Schwinger (Pauli-Fierz) equation has 16 components and needs the additional condition.

The properties of the Fermi–Bose duality, triality and quadro Fermi–Bose properties of equations found have been discussed briefly.

Hence, the method of synthesis of manifestly covariant field equations on the basis of start from the relativistic canonical quantum mechanics of arbitrary spin is suggested. Its approbation on few principal examples is presented.

The main general conclusion is as follows. Among the three main models of

arbitrary spin (relativistic canonical quantum mechanics, canonical field theory and covariant field theory) considered here the relativistic canonical quantum mechanics is the best in rigorous quantum-mechanical description. The transition from the relativistic canonical quantum mechanics to the canonical field theory essentially worsens the quantum-mechanical description. And the final transition from the canonical field theory to the covariant field theory essentially worsens the quantum-mechanical description once more.

Therefore above the following results can be found: the covariant local field theory equations for spin s = (1,1) particle-antiparticle doublet, spin s = (1,0,1,0) particle antiparticle multiplet, spin s = (3/2,3/2) particle-antiparticle doublet, spin s = (2,2) particle-antiparticle doublet, spin s = (2,0,2,0) particle-antiparticle multiplet and spin s = (2,1,2,1) particle-antiparticle multiplet. The Maxwell-like equations for the boson with spin s = 1 and nonzero are considered as well. The general form describing arbitrary spin also is presented.

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Chapter 8

Link between the Stationary Dirac Equation with Nonzero Mass in External Field and the Stationary Maxwell Equations in Specific Medium

Here in order to extend the limits of classical theory application in the microworld some slight generalization of Maxwell electrodynamics is considered. It is shown that slightly generalized classical Maxwell electrodynamics can describe the inneratomic phenomena with the same success as relativistic quantum mechanics can do. Group-theoretical grounds for the description of fermionic states in terms of Bose formalism are presented briefly. The advantages of generalized electrodynamics in inneratomic region in comparison with standard Maxwell electrodynamics are demonstrated on testing example of hydrogen atom. We are able to obtain some results, which are impossible in the framework of standard Maxwell electrodynamics. The Sommerfeld–Dirac formula for the fine structure of the hydrogen atom spectrum is obtained on the basis of such Maxwell equations without appealing to the Dirac equation. The

Bohr postulates and the Lamb shift are proved to be the consequences of the equations under consideration. The relationship of the new model with the Dirac theory is investigated. Possible directions of unification of such electrodynamics with gravity are mentioned.

8.1. Introduction to the Chapter 8

There is no doubt that the Maxwell classical electrodynamics of macroworld (without any generalization) is sufficient for the description of electrodynamical phenomena in macro region. On the other hand it is well known that for micro phenomena (inneratomic region) the classical Maxwell electrodynamics (as well as the classical mechanics) cannot work and must be replaced by quantum theory. Trying to extend the limits of classical electrodynamics application to the inneratomic region we came to the conclusion that it is possible by means of generalization of standard Maxwell classical electrodynamics in the direction of the extension of its symmetry. We also use the relationships between the Dirac and Maxwell equations for these purposes. Furthermore, the relationships between relativistic quantum mechanics and classical microscopical electrodynamics of media are investigated. Such relationships are considered here not only from the mathematical point of view – they are used for construction of fundamentals of a non-quantum-mechanical model of microworld.

The foundations of non-quantum-mechanical model of microworld are given by model of atom on the basis of slightly generalized Maxwell's equations, i.e., in the framework of moderately extended classical microscopical electrodynamics of media. This model is free from probability interpretation and can explain many inner-atomic phenomena by means of classical physics. Despite the fact that we construct the classical model, for the purposes of such construction we use essentially the analogy with the Dirac equation and the results, which were achieved on the basis of this equation. Note also that electrodynamics is considered here in the terms of field strengths without appealing to the vector potentials as the primary (input) variables of the theory.

The first step in our consideration is the one-to-one relationship (and wide range analogy) between the Dirac equation and slightly generalized Maxwell equations [1, 2].

Our second step is the symmetry principle. On the basis of this principle we introduced in [3, 4] the most symmetrical form of generalized Maxwell equations, which now can describe both bosons and fermions because they have (see [3, 4]) both spin 1 and spin 1/2 symmetries. On the other hand, namely these equations are unitary connected with the Dirac equation. Therefore, we have one more important argument to suggest these equations to be the equations of specific inneratomic classical electrodynamics.

In our third step we refer to Sallhofer, who suggested in [5–8] the possibility of introduction of interaction with external field as the interaction with specific media (a new way of introduction of the interaction into the field equations). Nevertheless, our model of atom (and of electron) [1, 2] is essentially different from the Sallhofer's one.

On the basis of these three main ideas we are able to postulate the slightly generalized Maxwell equations as the equations for inneratomic classical electrodynamics which may work in atomic, nuclear and particle physics on the same level of success as the Dirac equation can do. Below we illustrate it considering hydrogen atom within the classical model.

The interest to the problem of relationship between the Dirac and Maxwell equations dates back to the time of creation of quantum mechanics [9–26]. But the authors of these papers during long time considered only the most simple example of free and massless Dirac equation. The interest to this relationship has grown due to the results [5–8], where the investigations of the case $m \neq 0$ and the interaction potential $\Phi \neq 0$ were started. Another approach was developed in [27–32], where the quadratic relations between the fermionic and bosonic amplitudes were found and used. In our above mentioned papers [1–4, 21–24] and herein we consider the linear relations between the fermionic and bosonic amplitudes. In [21] we have found the relationship between the symmetry properties of the Dirac and Maxwell equations, the complete set of 8 transformations linking these equations, the relationship between the conservation laws for the electromagnetic and spinor fields, the relationship between the Lagrangians for these fields. Here we summarize our previous results and give some new details of the intraatomic electrodynamics and its application to the hydrogen atom. The possibilities of unification with gravitation are briefly discussed.

8.2. Hydrogen Atom Model in Classical Electrodynamics

8.2.1. Hydrogen Solutions

Consider the slightly generalized Maxwell equations in a medium with specific form of sources:

$$\operatorname{curl}\overrightarrow{H} - \partial_0 \varepsilon \overrightarrow{E} = \overrightarrow{j}_e, \quad \operatorname{curl}\overrightarrow{E} + \partial_0 \mu \overrightarrow{H} = \overrightarrow{j}_{mag},$$
$$\operatorname{div}\varepsilon\overrightarrow{E} = \rho_e, \quad \operatorname{div}\mu\overrightarrow{H} = \rho_{mag,} \tag{8.1}$$

where $\overrightarrow{E}$ and $\overrightarrow{H}$ are the electromagnetic field strengths, ε and μ are the electric and magnetic permeabilities of the medium being the same as in the electro-dynamical hydrogen atom model of H. Sallhofer [5–8] (see formula (2.4) in Chapter 2, where $\Phi(\overrightarrow{x}) = -Ze^2/r$). We use the units: $\hbar = c = 1$, transition to the standard system is fulfilled by the substitution $\widetilde{\omega} \longrightarrow \hbar\widetilde{\omega}$, $m \longrightarrow mc^2$. The current and charge densities in equations (8.1) have the form

$$\overrightarrow{j_e} = \operatorname{grad}E^0, \quad \overrightarrow{j}_{mag} = -\operatorname{grad}H^0,$$
$$\rho_e = -\varepsilon\mu\partial_0 E^0 + \overrightarrow{E}\operatorname{grad}\varepsilon, \quad \rho_{mag} = -\varepsilon\mu\partial_0 H^0 + \overrightarrow{H}\operatorname{grad}\mu, \tag{8.2}$$

where E^0, H^0 is the pair of functions (two real scalar fields) generating the densities of gradient-like sources.

One can easily see that equations (8.1) are not ordinary electrodynamical equations known from the Maxwell theory. These equations have the additional terms which can be considered as the magnetic current and charge densities - in one possible interpretation, or equations (8.1) can be considered as the equations for compound system of electromagnetic $(\overrightarrow{E}, \overrightarrow{H})$ and scalar E^0, H^0 fields in another possible interpretation.

The reasons of our slight generalization of the classical Maxwell electrodynamics are the following.

1. The standard Maxwell electrodynamics cannot work in inneratomic region and its equations are not mathematically equivalent to any of quantum mechanical equations for electron (Schrödinger equation, Dirac equation, etc...)

2. The existence of direct relationship between the equations (8.1) and the Dirac equation for the massive particle in external electromagnetic field in the stationary case can be applied. Namely these equations were shown in papers

[2–4] to be unitary equivalent with such Dirac equation (see also section 8.3 below).

3. Equations (8.1) can be derived from the principle of maximally possible symmetry - these equations have both spin 1 and spin 1/2 Poincaré symmetries and in the limit of vanishing of the interaction with medium, where $\varepsilon = \mu = 1$, they represent [2–4] the maximally symmetrical form of the Maxwell equations. This fact means first of all that from the group-theoretical point of view of Wigner, Bargmann–Wigner (and of modern field theory in general) equations (8.1) can describe both the states of bosons and of fermions (for more details see section 8.4 below). As a consequence of this fact one can use these equations particularly for the description of the electron. On the other hand, this fact means that inneratomic classical electrodynamics of electron needs further (relatively to that having been done by Maxwell) symmetrization of Weber–Faraday equations of classical electromagnetic theory which leads to the maximally symmetrical form (8.1). Below we demonstrate the possibilities of the equations (8.1) in the description of testing example of hydrogen atom.

Here the equations (8.1) are solved directly by means of separation of variables method. It is useful to rewrite these equations in the mathematically equivalent form where the sources are maximally simple:

$$\operatorname{curl}\overrightarrow{H} - \varepsilon\partial_0\overrightarrow{E} = \overrightarrow{j}_e, \quad \operatorname{curl}\overrightarrow{E} + \mu\partial_0\overrightarrow{H} = \overrightarrow{j}_{mag},$$
$$\operatorname{div}\overrightarrow{E} = \widetilde{\rho}_e, \quad \operatorname{div}\overrightarrow{H} = \widetilde{\rho}_{mag}, \tag{8.3}$$

where

$$\overrightarrow{j}_e = \operatorname{grad}E^0, \quad \overrightarrow{j}_{mag} = -\operatorname{grad}H^0, \quad \widetilde{\rho}_e = -\mu\partial_0 E^0, \quad \widetilde{\rho}_{mag} = -\varepsilon\partial_0 H^0. \tag{8.4}$$

Consider the stationary solutions of equations (8.3). Assuming the harmonic time dependence for the functions E^0, H^0

$$E^0(t, \overrightarrow{x}) = E_A^0(\overrightarrow{x})\cos\widetilde{\omega}t + E_B^0(\overrightarrow{x})\sin\widetilde{\omega}t,$$
$$H^0(t, \overrightarrow{x}) = H_A^0(\overrightarrow{x})\cos\widetilde{\omega}t + H_B^0(\overrightarrow{x})\sin\widetilde{\omega}t, \tag{8.5}$$

we are looking for the solutions of equations (8.3) in the form

$$\overrightarrow{E}(t, \overrightarrow{x}) = \overrightarrow{E}_A(\overrightarrow{x})\cos\widetilde{\omega}t + \overrightarrow{E}_B(\overrightarrow{x})\sin\widetilde{\omega}t,$$
$$\overrightarrow{H}(t, \overrightarrow{x}) = \overrightarrow{H}_A(\overrightarrow{x})\cos\widetilde{\omega}t + \overrightarrow{H}_B(\overrightarrow{x})\sin\widetilde{\omega}t. \tag{8.6}$$

For the 16 time-independent amplitudes we obtain the following two non-linked subsystems

$$\text{curl}\overrightarrow{H_A} - \tilde{\omega}\varepsilon\overrightarrow{E_B} = \text{grad}E_A^0, \quad \text{curl}\overrightarrow{E_B} - \tilde{\omega}\mu\overrightarrow{H_A} = -\text{grad}H_B^0,$$
$$\text{div}\overrightarrow{E_B} = \tilde{\omega}\mu E_A^0, \quad \text{div}\overrightarrow{H}_A = -\tilde{\omega}\varepsilon H_B^0,$$
$$(8.7)$$

$$\text{curl}\overrightarrow{H_B} + \tilde{\omega}\varepsilon\overrightarrow{E_A} = \text{grad}E_B^0, \quad \text{curl}\overrightarrow{E_A} + \tilde{\omega}\mu\overrightarrow{H_B} = -\text{grad}H_A^0,$$
$$\text{div}\overrightarrow{E_A} = -\tilde{\omega}\mu E_B^0, \quad \text{div}\overrightarrow{H}_B = \tilde{\omega}\varepsilon H_A^0.$$
$$(8.8)$$

Below we consider only the first subsystem (8.7). It is quite enough because the subsystems (8,7) and (8.8) are connected with transformations

$$E \longrightarrow H, \quad H \longrightarrow -E, \quad \varepsilon E \longrightarrow \mu H, \quad \mu H \longrightarrow -\varepsilon E,$$
$$\varepsilon \longrightarrow \mu, \quad \mu \longrightarrow \varepsilon,$$
$$(8.9)$$

which are the generalizations of duality transformation of free electromagnetic field. Due to this fact the solutions of subsystem (8.8) can be easily obtained from the solutions of subsystem (8.7).

Furthermore, it is useful to separate equations (8.7) into the following subsystems:

$$\tilde{\omega}\varepsilon E_B^3 - \partial_1 H_A^2 + \partial_2 H_A^1 + \partial_3 E_A^0 = 0,$$
$$\tilde{\omega}\varepsilon H_B^0 + \partial_1 H_A^1 + \partial_2 H_A^2 + \partial_3 H_A^3 = 0,$$
$$-\tilde{\omega}\mu E_A^0 + \partial_1 E_B^1 + \partial_2 E_B^2 + \partial_3 E_B^3 = 0,$$
$$\tilde{\omega}\mu H_A^3 - \partial_1 E_B^2 + \partial_2 E_B^1 - \partial_3 H_B^0 = 0,$$
$$(8.10)$$

and

$$\tilde{\omega}\varepsilon E_B^1 - \partial_2 H_A^3 + \partial_3 H_A^2 + \partial_1 E_A^0 = 0,$$
$$\tilde{\omega}\varepsilon E_B^2 - \partial_3 H_A^1 + \partial_1 H_A^3 + \partial_2 E_A^0 = 0,$$
$$\tilde{\omega}\mu H_A^1 - \partial_2 E_B^3 + \partial_3 E_B^2 - \partial_1 H_B^0 = 0,$$
$$\tilde{\omega}\mu H_A^2 - \partial_3 E_B^1 + \partial_1 E_B^3 - \partial_2 H_B^0 = 0.$$
$$(8.11)$$

Assuming the spherical symmetry case, when $\Phi(\overrightarrow{x}) = \Phi(r)$, $r \equiv |\overrightarrow{x}|$, we are making the transition into the spherical coordinate system and looking for the solutions in the spherical coordinates in the form

$$(E, H)(\overrightarrow{r}) = R_{(E,H)}(r) f_{(E,H)}(\theta, \phi), \quad (8.12)$$

where $E \equiv \left(E^0, \overrightarrow{E}\right)$, $H \equiv \left(H^0, \overrightarrow{H}\right)$. We choose for the subsystem (8.10) the

d'Alembert Ansatz in the form

$$\bar{E}_A^0 = \bar{C}_{E_4}\, R_{H_4} P_{l_{H_4}}^{\bar{m}_4} e^{-i\bar{m}_4\phi},$$

$$\bar{E}_B^k = \bar{C}_{E_k}\, R_{E_k} P_{l_{E_k}}^{\bar{m}_k} e^{-i\bar{m}_k\phi},$$

$$\bar{H}_B^0 = \bar{C}_{H_4}\, R_{E_4} P_{l_{E_4}}^{\bar{m}_4} e^{-i\bar{m}_4\phi}, \quad k = 1,2,3. \tag{8.13}$$

$$\bar{H}_A^k = \bar{C}_{H_k}\, R_{H_k} P_{l_{H_k}}^{\bar{m}_k} e^{-i\bar{m}_k\phi},$$

We use the following representation for $\partial_1, \partial_2, \partial_3$ operators in spherical coordinates

$$\partial_1 C R P_l^m e^{\mp im\phi} = \frac{e^{\mp im\phi}C}{2l+1}\cos\phi\left(R_{,l+1}P_{l-1}^{m+1} - R_{,-l}P_{l+1}^{m+1}\right) + e^{\mp i(m-1)\phi}C\frac{m}{\sin\theta}P_l^m\frac{R}{r},$$

$$\partial_2 C R P_l^m e^{\mp im\phi} = \frac{e^{\mp im\phi}C}{2l+1}\sin\phi\left(R_{,l+1}P_{l-1}^{m+1} - R_{,-l}P_{l+1}^{m+1}\right) \mp e^{\mp i(m-1)\phi}C\frac{im}{\sin\theta}P_l^m\frac{R}{r},$$

$$\partial_3 C R P_l^m e^{\mp im\phi} = \frac{e^{\mp im\phi}C}{2l+1}\left(R_{,l+1}(l+m)P_{l-1}^m + R_{,-l}(l-m+1)P_{l+1}^m\right). \tag{8.14}$$

Substitutions (8.13) and (8.14) together with the assumptions

$$R_{E_\alpha} = R_E, \quad l_{E_\alpha} = l_E, \quad R_{H_\alpha} = R_H, \quad l_{H_\alpha} = l_H,$$

$$\bar{m}_1 = \bar{m}_2 = \bar{m}_3 - 1 = \bar{m}_4 - 1 = m,$$

$$\bar{C}_{H_1} = i\,\bar{C}_{H_2}, \quad \bar{C}_{E_2} = -i\,\bar{C}_{E_1}, \quad \bar{C}_{H_4} = -i\,\bar{C}_{E_3}, \quad \bar{C}_{H_3} = -i\,\bar{C}_{E_4},$$

$$\bar{C}_{H_2}^I = \bar{C}_{E_4}^I\,(l_H^I + m + 1), \quad \bar{C}_{E_3}^I = -\bar{C}_{E_4}^I \equiv \bar{C}^I, \tag{8.15}$$

$$\bar{C}_{E_1}^I = \bar{C}_{E_3}^I\,(l_E^I - m), \quad l_H^I = l_E^I - 1 \equiv l^I,$$

$$\bar{C}_{H_2}^{II} = -\bar{C}_{E_4}^{II}\,(l_H^{II} - m), \quad \bar{C}_{E_3}^{II} = -\bar{C}_{E_4}^{II} \equiv \bar{C}^{II},$$

$$\bar{C}_{E_1}^{II} = -\bar{C}_{E_3}^{II}\,(l_E^{II} + m + 1), \quad l_H^{II} = l_E^{II} + 1 \equiv l^{II}$$

into the subsystem (8.10) guarantee the separation of variables in these equations and lead to the pair of equations for two radial functions R_E, R_H (for the subsystem (8.11) the procedure is similar):

$$\varepsilon\tilde{\omega}R_E^I - R_{H,-l}^I = 0, \quad \mu\tilde{\omega}R_H^I + R_{E,l+2}^I = 0, \tag{8.16}$$

$$\varepsilon\tilde{\omega}R_E^{II} - R_{H,l+1}^{II} = 0, \quad \mu\tilde{\omega}R_H^{II} + R_{E,-l+1}^{II} = 0; \quad R_{,a} \equiv \left(\frac{d}{dr} + \frac{a}{r}\right)R. \tag{8.17}$$

For the case $\Phi = -ze^2/r$ the equations (8.16), (8.17) coincide exactly with the radial equations for the hydrogen atom of the Dirac theory and, therefore, the procedure of their solution is the same as in well-known monographs on relativistic quantum mechanics. It leads to the well-known Sommerfeld - Dirac formula for the fine structure of the hydrogen spectrum. We note only that here the discrete picture of energetic spectrum in the domain $0 < \tilde{\omega} < mc^2$ is guaranteed by the demand for the solutions of the radial equations (8.16), (8.17) to decrease on infinity (when $r \to \infty$). From the equations (8.16), (8.17) and this condition the Sommerfeld–Dirac formula

$$\tilde{\omega} = \omega_{nj}^{hyd} = \frac{mc^2}{\hbar \sqrt{1 + \frac{\alpha^2}{\left(n_r + \sqrt{k^2 - \alpha^2}\right)^2}}} \tag{8.18}$$

follows, where the notations of the Dirac theory (see, e.g., [33]) are used: $n_r = n - k$, $k = j + 1/2$, $\alpha = e^2/\hbar c$. Let us note once more that the result (8.18) is obtained here not from the Dirac equation, but from the Maxwell equations (8.1) with sources (8,2) in the medium (2.4).

Substituting (8.15) into (8.13) one can easy obtain the angular part of the hydrogen solutions for the $(\overrightarrow{E}, \overrightarrow{H}, E^0, H^0)$ field and calculate according to (8.2) the corresponding currents and charges. Let us write down the explicit form for the set of electromagnetic field strengths $(\overrightarrow{E}, \overrightarrow{H})$, which are the hydrogen solutions of equations (8.1), and also for the currents and charges generating these field strengths (the complete set of solutions is given in Appendices I, II):

$$\overrightarrow{E^I} = R_E^I \begin{vmatrix} (-l+m-1) P_{l+1}^m \cos m\phi \\ (l-m+1) P_{l+1}^m \sin m\phi \\ -P_{l+1}^{m+1} \cos(m+1)\phi \end{vmatrix}, \quad \overrightarrow{H^I} = R_H^I \begin{vmatrix} (l+m+1) P_l^m \sin m\phi \\ (l+m+1) P_l^m \cos m\phi \\ -P_l^{m+1} \sin(m+1)\phi \end{vmatrix},$$

$$\overrightarrow{j_e^I} = grad R_H^I P_l^{m+1} \cos(m+1)\phi, \quad \overrightarrow{j_{mag}^I} = -grad R_e^I P_{l+1}^{m+1} \sin(m+1)\phi,$$

$$\rho_e^I = -\left(\varepsilon R_E^I\right)_{,l+2} P_l^{m+1} \cos(m+1)\phi, \quad \rho_{mag}^I = -\left(\mu R_H^I\right)_{,-l} P_{l+1}^{m+1} \sin(m+1)\phi,$$

$$\tag{8.19}$$

$$\overrightarrow{E^{II}} = R_E^{II} \begin{vmatrix} (l+m) P_{l-1}^m \cos m\phi \\ (-l-m) P_{l-1}^m \sin m\phi \\ P_{l-1}^{m+1} \cos(m+1)\phi \end{vmatrix}, \quad \overrightarrow{H^{II}} = R_H^{II} \begin{vmatrix} (-l+m) P_l^m \sin m\phi \\ (-l+m) P_l^m \cos m\phi \\ -P_l^{m+1} \sin(m+1)\phi \end{vmatrix},$$

$$\overrightarrow{j_e^{II}} = grad R_H^{II} P_l^{m+1} \cos(m+1)\phi, \quad \overrightarrow{j_{mag}^{II}} = -grad R_E^{II} P_{l-1}^{m+1} \sin(m+1)\phi,$$

$$\rho_e^{II} = -\left(\varepsilon R_E^{II}\right)_{,-l+1} P_l^{m+1} \cos(m+1)\phi, \quad \rho_{mag}^{II} = -\left(\mu R_H^{II}\right)_{,l+1} P_{l-1}^{m+1} \sin(m+1)\phi.$$

$$\tag{8.20}$$

In the first possible interpretation the states of the hydrogen atom are described by the field strength functions $(\overrightarrow{E}, \overrightarrow{H}$ generated by the corresponding currents and charge densities).

It is evident from (8.1) that currents and charges in (8.19), (8.20) are generated by scalar fields (E^0, H^0). Corresponding to (8.19), (8.20) (E^0, H^0) solutions of equations (8.1) are as follows:

$$
\begin{array}{ll}
E^{I0} = R^I_H P^{m+1}_l \cos(m+1)\phi, & H^{I0} = R^I_E P^{m+1}_{l+1} \sin(m+1)\phi, \\
E^{II0} = R^{II}_H P^{m+1}_l \cos(m+1)\phi, & H^{II0} = R^{II}_E P^{m+1}_{l-1} \sin(m+1)\phi.
\end{array}
\tag{8.21}
$$

Thus, in the second possible interpretation the states of the hydrogen atom are described by these field strength functions $(\overrightarrow{E}, \overrightarrow{H}, E^0, H^0)$ of electromagnetic and scalar fields.

In (8.19), (8.20) for the current and charge densities the following notations are used

$$
\left(\varepsilon R^\beta_\alpha\right)_{,l+2} \equiv \varepsilon \left(\frac{d}{dr} + \frac{l+2}{r}\right) R^\beta_\alpha + R^\beta_\alpha \frac{d\varepsilon}{dr}, \quad etc.
\tag{8.22}
$$

The solutions of the second subsystem (8.8) follow from (8.19)–(8.21) after the application of transformation (8.9).

As in quantum theory, the numbers $n = 0, 1, 2, ...$; $j = k - \frac{1}{2} = l \mp \frac{1}{2}$ ($k = 1, 2, ..., n$) and $m = -l, -l+1, ..., l$ mark both the terms (8.18) and the corresponding exponentially decreasing field functions $\overrightarrow{E}, \overrightarrow{H}$ (and E^0, H^0) in (8.19)–(8.21), i.e., they mark the different discrete states of the classical electrodynamical field (and the densities of the currents and charges) which by definitions describes the corresponding states of hydrogen atom in the model under consideration.

It is evident from this example that the discreteness of the physical system states (and its characteristics such as energy, etc.) may be a consequence of both quantum systems (Schrödinger, Dirac) and the classical (Maxwell) equations for the given system. In the present case, this discreetness is caused by the properties of the medium, which are given by the electric and magnetic permeabilities (2.4).

Note that the radial equations (8.16), (8.17) cannot be obtained if one neglects the sources in equations (8.1), or one (electric or magnetic) of these sources. Moreover, in this case there is no solution effectively concentrated in atomic region.

8.2.2. On the Bohr's Postulates

Now we can show on the basis of this model that the assertions known as *Bohr's postulates are the consequences of equations (8.1) and of their classical interpretation*, i.e., these assertions can be derived from the model, there is no necessity to postulate them from beyond the framework of classical physics as it was in Bohr's theory. To derive the first Bohr's postulate one can calculate the generalized Pointing vector for the hydrogen solutions (8.19)–(8.21), i.e., for the compound system of stationary electromagnetic and scalar fields $(\vec{E}, \vec{H}, E^0, H^0)$

$$\vec{P}_{gen} = \int d^3x (\vec{E} \times \vec{H} - \vec{E} E^0 - \vec{H} H^0). \tag{8.23}$$

The straightforward calculations show that not only vector (8.23) is identically equal to zero but the Pointing vector itself and the term with scalar fields (E^0, H^0) are also identically equal to zero:

$$\vec{P} = \int d^3x (\vec{E} \times \vec{H}) \equiv 0, \quad \int d^3x (\vec{E} E^0 + \vec{H} H^0) \equiv 0. \tag{8.24}$$

This means that in stationary states hydrogen atom does not emit any Pointing radiation neither due to the electromagnetic $(\vec{E}, \vec{H})$ field, nor to the scalar (E^0, H^0) field. That is the mathematical proof of the first Bohr postulate.

Similar calculations of the energy for the same system (in formulae (8.19)–(8.21) the functions $(\vec{E}, \vec{H}, E^0, H^0)$ are taken in appropriate physical dimension, which is given by the formula (8.25) below)

$$P^0 = \frac{1}{2} \int d^3x \, \mathcal{E}^\dagger \mathcal{E} = \frac{1}{2} \int d^3x (\vec{E}^2 + \vec{H}^2 + E_0^2 + H_0^2) = \omega_{nj}^{hyd} \tag{8.25}$$

give a constant ω_{nl}, depending on n, l (or n, j) and independent of m. In our model this constant is to be identified with the parameter $\tilde{\omega}$ in equations (8.7), which in the stationary states of $(\vec{E}, \vec{H}, E^0, H^0)$ field appears to be equal to the Sommerfeld–Dirac value ω_{nj}^{hyd} (8.18).

By abandoning the $\hbar = c = 1$ system and putting arbitrary "A" in equations (8.1) instead of $\hbar$ we obtain final ω_{nj}^{hyd} with "A" instead of $\hbar$. Then the numerical value of $\hbar$ can be obtained by comparison of ω_{nj}^{hyd} containing "A" with the experiment. These facts complete the proof of the second Bohr postulate.

This result means that in this model the Bohr postulates are no longer postulates, but the direct consequences of the classical electrodynamic equation

(8.1). Moreover, together with the Dirac or Schrödinger equations we now have a new equation, which can be used to find the solutions to atomic spectroscopy problems. In contrast to the well-known equations of quantum mechanics, our equation is classical. Thus, we have verified equations (8.1,) introduced by us, in the test case of the hydrogen atom (the third motivation of our generalization of the Maxwell system).

8.2.3. Lamb Shift, the Electric Charge, and Briefly on Gravity

It is very useful to consider the Lamb shift in the approach presented here. This specific quantum electrodynamic effect (as modern theory asserts) can be described here in the framework of the classical electrodynamics of media. In order to obtain the Lamb shift one must add to $\Phi(\overrightarrow{x}) = -Ze^2/r$ in (2.4) the quasipotential (known, e.g., from [34], which follows, of course, from quantum electrodynamics)

$$-\frac{Ze^4}{60\pi^2 m^2}\delta(r) \tag{8.26}$$

and solve the equations (8.1)=(8.3) for this medium, similar to the procedure presented above. Finally one obtains the Lamb shift correction to the Sommerfeld–Dirac formula (8.18). Therefore, the Lamb shift can be interpreted as a pure classical electrodynamic effect. It may be considered a consequence of the polarization of the medium (2.4), and not a polarization of some abstract concept, such as the vacuum in quantum electrodynamics. This brief example demonstrates that our proposition can essentially extend the limits of application of classical theory in the microworld, which was the main purpose of this part of our investigations.

The Electric Charge

Due to the unitary connection to the Dirac theory (considered above) the electric charge is still a conserved quantity here in the same sense as in the Dirac model. It may be defined similarly to the Dirac theory, or be derived from it on the basis of the unitary relationship between the massless Dirac and slightly generalized Maxwell equations (see here below or in [35]).

Brief Hypothesis on Gravity

A unified theory of electromagnetic and gravitational phenomena may be constructed in the approach under consideration in the following way. The main primary equations again are written as (8.1) and gravity is treated as a medium in these equations, i.e., the electric ε and magnetic μ permeabilities of the medium are some functions of the gravitational potential

$$\varepsilon = \varepsilon(\Phi_{grav}), \quad \mu = \mu(\Phi_{grav}). \tag{8.27}$$

Gravity as a medium may generate all the phenomena that in standard Einstein gravity are generated by Riemann geometry. For example, the refraction of the light beam near a massive star is a typical medium effect in a unified model of electromagnetic and gravitational phenomena. The main idea is as follows. The gravitational interaction between massive objects may be represented as an interaction with some medium, much as the electromagnetic interaction between charged particles is considered in equations (8.1) here.

8.2.4. Interpretation

Being aware that few interpretations of quantum mechanics (e.g., Copenhagen, statistical, Feynman's, Everett's, transactional, see e.g., [36–38]) exist, we are far from thinking that here the interpretation can be the only one. But the main point is that now the classical interpretation (without probabilities) is possible.

Today we prefer the following interpretation of hydrogen atom in the approach, when one considers only the motion of electron in the external field of the nucleon. In our model the interacting field of the nucleon and electron is represented by the medium with permeabilities ε, μ given by formulae (2.4). The atomic electron is interpreted as the stationary electromagnetic-scalar wave $(\overrightarrow{E}, \overrightarrow{H}, E^0, H^0)$ in medium (2.4), i.e., as the stationary electromagnetic wave interacting with massless scalar fields (E^0, H^0), or with complex massless scalar field $\mathcal{E}^0 = E^0 - iH^0$ with spin $s = 0$.

In other words, the electron can be interpreted as an object having the structure consisting of a photon and a massless meson with zero spin connected, probably, with leptonic charge. The role of the massless scalar field is the following: it generates the densities of electric and magnetic currents and charges $(\rho, \overrightarrow{j})$, which are the secondary objects in such model. The mass is the secondary parameter too. *There is no electron as an input charged massive corpuscle in this model!* The mass and the charge of electron appear only outside such

atom according to the law of electromagnetic induction and its gravitational analogy. That is why no difficulties of Rutherford–Bohr model (about different models of atom see, e.g., [39]) of atom are present here! The Bohr postulates are shown to be the consequences of the model.

This interpretation is based on the hypothesis of bosonic nature of matter (on the speculation of the bosonic structure of fermions) according to which all the fermions can be constructed from different bosons (something like new SUSY theory). Of course, before the experiment intended to observe the structure of electron and before the registration of massless spinless meson it is only the hypothesis but based on the mathematics presented here. We note that such massless spinless boson has many similar features with the Higgs boson and the transition here from intraatomic (with high symmetry properties) to macroelectrodynamics (with loss of many symmetries) looks similarly to the symmetry breakdown mechanism.

The successors of magnetic monopole can try to develop here the monopole interpretation (see e.g., [40]) for the review and some ideas about monopole) – we note that there are few interesting possibilities of interpretation but we want to mark first of all the mathematical facts which are more important than different ways of interpretation.

8.3. The Unitary Relationship between the Relativistic Quantum Mechanics and Classical Electrodynamics in Medium

Hence, the above considered Maxwell type equations are proved to describe the fine structure of the hydrogen spectrum. Therefore, below we are able to formulate the link between the stationary Dirac equation with nonzero mass in external field and the stationary Maxwell equations in specific medium (2.4) modeling the inneratomic interaction.

Let us consider the connection between the stationary Maxwell equations

$$\operatorname{curl}\overrightarrow{H} - \tilde{\omega}\varepsilon\overrightarrow{E} = \operatorname{grad}E^0, \quad \operatorname{curl}\overrightarrow{E} - \tilde{\omega}\mu\overrightarrow{H} = -\operatorname{grad}H^0,$$
$$\operatorname{div}\overrightarrow{E} = \tilde{\omega}\mu E^0, \quad \operatorname{div}\overrightarrow{H} = -\tilde{\omega}\varepsilon H^0, \tag{8.28}$$

which follow from the system (8.7) after ommitting indices A, B, and the stationary Dirac equation following from the ordinary Dirac equation

$$\left(i\gamma^\mu\partial_\mu - m + \gamma^0\Phi\right)\Psi = 0, \quad \Psi \equiv (\Psi^\alpha), \quad \bar{\alpha} = 1,2,3,4, \tag{8.29}$$

with $m \neq 0$ and the interaction potential $\Phi \neq 0$. Assuming the ordinary time dependence

$$\Psi(x) = \Psi(\vec{x})e^{-i\widetilde{\omega}t} \implies \partial_0 \Psi(x) = -i\widetilde{\omega}\Psi(x), \qquad (8.30)$$

for the stationary states and using the standard Pauli–Dirac representation for the γ matrices, one obtains the following system of equations for the components Ψ^α of the spinor Ψ:

$$\begin{aligned}
-i\widetilde{\omega}\varepsilon\Psi^1 + (\partial_1 - i\partial_2)\Psi^4 + \partial_3\Psi^3 &= 0, \\
-i\widetilde{\omega}\varepsilon\Psi^2 + (\partial_1 + i\partial_2)\Psi^3 - \partial_3\Psi^4 &= 0, \\
-i\widetilde{\omega}\mu\Psi^3 + (\partial_1 - i\partial_2)\Psi^2 + \partial_3\Psi^1 &= 0, \\
-i\widetilde{\omega}\mu\Psi^4 + (\partial_1 + i\partial_2)\Psi^1 - \partial_3\Psi^2 &= 0,
\end{aligned} \qquad (8.31)$$

where ε and μ are the same as in (2.4). After substitution in equations (8.31) instead of Ψ the following column

$$\Psi = \text{column} \left| -H^0 + iE^3, -E^2 + iE^1, E^0 + iH^3, -H^2 + iH^1 \right|. \qquad (8.32)$$

one obtains Eqs. (8.28). A complete set of 8 such transformations can be obtained with the help of the Pauli–Gursey symmetry operators [41–43] similarly to [3, 21, 24].

It is useful to represent the right-hand side of (8.32) in terms of components of the complex function (2.15), where $\overrightarrow{\mathcal{E}} = \overrightarrow{E} - i\overrightarrow{H}$ is the well-known form for the electromagnetic field used by Majorana as far back as near 1930 (see, e.g., [10]), and $\mathcal{E}^0 = E^0 - iH^0$ is a complex scalar field. In these terms the connection between the spinor and electromagnetic (together with the scalar) fields has the form

$$\mathcal{E} = W\Psi, \quad \Psi = W^\dagger \mathcal{E}, \qquad (8.33)$$

where the unitary operator W is as follows:

$$W = \begin{vmatrix} 0 & iC_- & 0 & C_- \\ 0 & -C_+ & 0 & iC_+ \\ iC_- & 0 & C_- & 0 \\ iC_+ & 0 & C_+ & 0 \end{vmatrix}; \quad C_\mp \equiv \frac{1}{2}(C \mp 1), \quad C\Psi \equiv \Psi^*, \quad C\mathcal{E} \equiv \mathcal{E}^*. \qquad (8.34)$$

The unitarity of the operator (8.34) can be verified easily by taking into account that the equations

$$(AC)^\dagger = CA^\dagger, \quad aC = Ca^*, \quad (aC)^* = Ca \qquad (8.35)$$

hold for an arbitrary matrix A and a complex number a. We note that in the real algebra (i.e., the algebra over the field of real numbers) and in the Hilbert space of quantum mechanical amplitudes this operator has all properties of unitarity: $WW^{-1} = W^{-1}W = 1$, $W^{-1} = W^\dagger$, plus linearity.

The operator (8.34) transforms the stationary Dirac equation

$$\left[(\tilde{\omega} - \Phi) \gamma^0 + i\gamma^k \partial_k - m \right] \Psi(\vec{x}) = 0 \tag{8.36}$$

from the standard representation (the Pauli–Dirac representation) into the bosonic representation

$$\left[(\tilde{\omega} - \Phi) \tilde{\gamma}^0 + i\tilde{\gamma}^k \partial_k - m \right] \mathcal{E}(\vec{x}) = 0. \tag{8.37}$$

Here the $\tilde{\gamma}^\mu$ matrices have the following unusual explicit form

$$\tilde{\gamma}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} C, \quad \tilde{\gamma}^1 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix},$$

$$\tilde{\gamma}^2 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad \tilde{\gamma}^3 = \begin{vmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{vmatrix} \tag{8.38}$$

in which $\tilde{\gamma}^0$ matrix explicitly contains operator C of complex conjugation. We call the representation (8.38) the bosonic representation of the γ matrices. In this representation the imaginary unit i is represented by the 4×4 matrix operator:

$$\tilde{i} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{vmatrix}. \tag{8.39}$$

Due to the unitarity of the operator (8.34) the $\tilde{\gamma}^\mu$ matrices still obey the Clifford–Dirac algebra

$$\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2g^{\mu\nu} \tag{8.40}$$

and have the same Hermitian properties as the Pauli–Dirac γ^μ matrices:

$$\tilde{\gamma}^{0\dagger} = \tilde{\gamma}^0, \quad \tilde{\gamma}^{k\dagger} = -\tilde{\gamma}^k. \tag{8.41}$$

Indeed, the formulae (8.38) give an exotic form of γ^μ matrices, which belong to the representation of the Clifford–Dirac algebra over the field of real numbers.

In the vector-scalar form the equation (8.37) is as follows

$$-i\mathrm{curl}\,\overrightarrow{\mathcal{E}} + [(\tilde{\omega}-\Phi)\,C-m]\,\overrightarrow{\mathcal{E}} = -\mathrm{grad}\,\mathcal{E}^0, \quad \mathrm{div}\,\overrightarrow{\mathcal{E}} = [(\tilde{\omega}-\Phi)\,C+m]\,\mathcal{E}^0.$$
(8.42)

Fulfilling the transition to the common real field strengths according to the formula $\mathcal{E}=E-iH$ and separating the real and imaginary parts we obtain equations (8.28) which are mathematically equivalent to the equations (8.1) in stationary case.

We emphasize that the only difference between the equation (8.37) in the case of description of fermions and in the case of bosons is the possibility of choosing γ^μ matrices: for the case of fermions these matrices may be chosen in arbitrary form (in each of representations of Pauli–Dirac, Majorana, Weyl, ...), in the case of the description of bosons the representation of γ^μ matrices and their explicit form *must be fixed* in the form (8.38). In the case of bosonic interpretation of equation (8.36) one must fixes the explicit form of γ^μ matrices (e.g., in standard Pauli–Dirac representation) and form of corresponding Ψ (8.32).

The mathematical facts considered here prove the one-to-one correspondence between the solutions of the stationary Dirac and the stationary Maxwell equations with 4-currents of gradient-like type. Hence, one can, using (8.32), write down the hydrogen solutions of the Maxwell equations (8.1) (or (8.3)) starting from the well-known hydrogen solutions of the Dirac equation (8.29), i.e., without special procedure of finding the solutions of the Maxwell equations, see [1].

In the case of zero mass the validity of the presented here link is evident without any proof.

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Chapter 9

Specific Case of Zero Mass

In the case of zero mass the Dirac equation is linked with the Maxwell equation. This fact and its consequences is the main content of the chapter below.

A slightly generalized classical Maxwell equations in a specific medium, modeling the relativistic atom, is considered in the previous Chapter. Here the link of this equation in the simplest case of free non-interacting electromagnetic-scalar field with the free massless Dirac equation is considered. The property of the Fermi-Bose duality of the massless Dirac equation is proved. The property of the Fermi-Bose duality of slightly generalized Maxwell system for electromagnetic-scalar field is proved as well. The fermionic symmetries, solutions, conservation laws and quantization procedure for such Maxwell equations are presented.

9.1. Slightly Generalized Maxwell Equations and Link with the Massless Dirac Equation

Consider briefly the case of absence of interaction of the compound field $(\overrightarrow{E},\overrightarrow{H},E^0,H^0)$ with media, i.e., the case $\varepsilon = \mu = 1$, and the symmetry properties of the corresponding equations. In this case equations for the system of electromagnetic and scalar fields $(\overrightarrow{E},\overrightarrow{H},E^0,H^0)$ have the form (2.13). The equations (2.13) are nothing more than the slightly generalized Maxwell equations ($\varepsilon = \mu = 1$) with gradient-like electric and magnetic sources $j^e_\mu = -\partial_\mu E^0$, $j^{mag}_\mu = -\partial_\mu H^0$ given in (2.14).

In terms of complex 4-component object $\mathcal{E} = E - iH$ from formula (2.15)

and in terms of complex tensor (2.16) equations (2.13) can be rewritten in the manifestly covariant forms (2.17), (2.18), respectively. The form (2.19), in which the generators $\overrightarrow{S} \equiv (S^j)$ (2.20) of irreducible representation D(1) of the group SU(2) are used, has been introduced as well.

The general solution of equations (2.13)=(2.17)=(2.18)=(2.19) is given by (2.21), see the details in subsection 2.4.2. Further, in subsection 2.4.2 other useful forms of such solution were considered. Furthermore, equations (2.13)=(2.17)=(2.18)=(2.19) are written in the form of massless Dirac equation $i\gamma^\mu\partial_\mu\psi(x) = 0$ (2.26) and in bosonic representation of this equation $\widetilde{\gamma}^\mu\partial_\mu\mathcal{E}(x) = 0$ (2.29) with gamma matrices (2.30).

The main assertions of consideration in subsection 2.4.2 are as follows. The operator links (2.27), (2.28) between the massless Dirac equation and the slightly generalized Maxwell equations are presented. This Maxwell system is derived from the massless Dirac equation and vice versa: the massless Dirac equation is derived from the equations (2.13)=(2.17)=(2.18)=(2.19). The complete set (2.34) of the corresponding transformations is presented.

Therefore, it is naturally to suppose the existence of Fermi symmetries of the slightly generalized Maxwell equations and the Bose symmetries of the massless Dirac equation. These non-ordinary symmetries were found in [1–5].

Thus, we introduce our generalization of the Maxwell equations on the basis of the Dirac equation (the first foundation of our generalization).

It is evident that there is no any relation of such generalization to other modifications of Maxwell's theory, i.e., to the Born–Infeld theory, Heisenberg–Euler approach, Podolsky's generalization, etc.

9.2. Bosonic Symmetries of the Massless Dirac Equation and Fermionic Symmetries of the Slightly Generalized Maxwell Equations

The massless Dirac equation is given by

$$i\gamma^\mu\partial_\mu\psi(x) = 0; \ x \equiv (x^\mu) \in R^4, \ \psi \equiv (\psi^\mu), \ \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \ \mu = 0, 1, 2, 3. \quad (9.1)$$

Consider two different representations of the Lorentz group SO(1,3) in the useful for us notations, which are associated with the massless Dirac equation

(9.1). The first one is determined by the operators

$$s_I^{\rho\sigma} \equiv \frac{1}{4} [\gamma^\rho, \gamma^\sigma] \in D\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right), \tag{9.2}$$

where $s_I^{\rho\sigma}$ are the generators of spinor representation $D\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ of SL(2,C) group. Let us define in terms of the six operators from the set (1.12) the SO(1,3) generators (1.13).

It is easy to verify that the operators (1,13) obey the same commutation relations as the generators (9.2) do, and, as a consequence, they form the another realization of the same spinor representation $D\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ of the SL(2,C) group. But, contrary to operators (9.2), generators (1.13) themselves (without any differential angular momentum part) are the symmetry operators of the massless Dirac equation (9.1), i.e., they leave this equation being invariant.

In order to consider the additional symmetries of the massless Dirac equation another representation (the so-called bosonic representation (2.30)) of the Dirac γ^μ matrices is preferable. This representation can be obtained from the γ^μ matrices (1.5) in the PD-representation by the unitary relationship $\gamma^\mu \longrightarrow \widetilde{\gamma}^\mu \equiv U^\dagger \gamma^\mu U$, where the unitary operator has the form (2.27), (2.28). Let us note that in the real algebra (i.e., the algebra over the field of real numbers) and in the Hilbert space of quantum mechanical amplitudes this operator has all properties of unitarity: $UU^{-1} = U^{-1}U = 1$, $U^{-1} = U^\dagger$, plus linearity.

The unitarity of the operator U (2.27), (2.28) is the reason for that the $\widetilde{\gamma}^\mu$ matrices obey the same relations (1.7) of the Clifford–Dirac algebra as the γ^μ matrices (1.5) in the standard PD-representation do. It is shown in subsection 2.4.2 that the massless Dirac equation $\widetilde{\gamma}^\mu \partial_\mu \mathcal{E}(x) = 0$ (2.29) can be considered not only as the equation for the spinor field but also as the equation for the bosonic field. Therefore the representation (2.29), (2.30) is called the bosonic representation (shortly: B-representation). The complete set of 8 transformations like (2.27), which relates generalized Maxwell equations (2.13) and massless Dirac equation (2.26), is presented in (2.34).

Operators (9.2), (1.13) satisfy the commutation relations of the Lie algebra of the Lorentz group SO(1,3) in the form:

$$[s^{\mu\nu}, s^{\rho\sigma}] = -g^{\mu\rho}s^{\nu\sigma} - g^{\rho\nu}s^{\sigma\mu} - g^{\nu\sigma}s^{\mu\rho} - g^{\sigma\mu}s^{\rho\nu}. \tag{9.3}$$

The transition in B-representation is described in subsection 2.4.2, the reasons of such transition are described there as well. The equation (2.29) is pre-

sented in the different forms (2.13), (2.17)–(2.19). In addition to the consideration in 2.4.2 the following information is useful

$$
\widetilde{\gamma}^4 = i, \ \widetilde{\gamma}^{01} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \ \widetilde{\gamma}^{02} = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix},
$$

$$
\widetilde{\gamma}^{03} = \begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \ \widetilde{\gamma}^{\mu 4} = -i\widetilde{\gamma}^{\mu}, \ \widetilde{\gamma}^{jk} = -i\varepsilon_{jkl}\widetilde{\gamma}^{0l}, \tag{9.4}
$$

where $\widetilde{\gamma}^{\mu\nu\cdots} \equiv \widetilde{\gamma}^{\mu}\widetilde{\gamma}^{\nu}\cdots$. The last relations in (9.3) are the transforms of the following PD-representation relations

$$
\gamma^{\mu 4} = -\gamma^4\gamma^{\mu}, \ \gamma^{jk} = \varepsilon^{jkl}\gamma^{0l4} \ (\gamma^4 = \gamma^{0123}). \tag{9.5}
$$

In the B-representation, the complex number i is represented by the following matrix operator

$$
\widetilde{i} = U^{\dagger}iU = i\Gamma, \ \Gamma \equiv \begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} = \Gamma^{\dagger} = \Gamma^{-1}, \ \Gamma^2 = 1, \tag{9.6}
$$

and the set (9.2) has the form

$$
\widetilde{s}_I^{\rho\sigma} = \frac{1}{4}[\widetilde{\gamma}^{\rho}, \widetilde{\gamma}^{\sigma}]: \quad \widetilde{s}_I^{jk} = -i\varepsilon_{jkl}\widetilde{s}_I^{0l}, \quad \widetilde{s}_I^{0l} = \frac{1}{2}\widetilde{\gamma}^{0l}, \quad \varepsilon^{123} \equiv +1. \tag{9.7}
$$

Let us write down the explicit form of the $s_{II}^{\rho\sigma}$ operators (1.13) in the B-representation:

$$
\widetilde{s}_{II}^{\rho\sigma} \equiv U^{\dagger}s_{PGI}^{\rho\sigma}U: \quad \widetilde{s}_{II}^{mn} = i\varepsilon_{jkl}\widetilde{s}_{II}^{0l},
$$

$$
\widetilde{s}_{II}^{01} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \ \widetilde{s}_{II}^{02} = \frac{1}{2}\begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \ \widetilde{s}_{II}^{03} = \frac{1}{2}\begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix},
$$

$$
\tag{9.8}
$$

Now we are able to introduce, in addition to the sets of generators (9.2), (1.13), the following two sets of matrices $s^{\rho\sigma}$:

$$\tilde{s}_{\text{III}}^{0k} = \tilde{s}_{\text{I}}^{0k} - \tilde{s}_{\text{II}}^{0k}, \quad \tilde{s}_{\text{III}}^{mn} = -\tilde{s}_{\text{I}}^{mn} + \tilde{s}_{\text{II}}^{mn}, \tag{9.9}$$

$$\tilde{s}_{\text{IV}}^{0k} = -\tilde{s}_{\text{I}}^{0k} - \tilde{s}_{\text{II}}^{0k}, \quad \tilde{s}_{\text{IV}}^{mn} = -\tilde{s}_{\text{I}}^{mn} + \tilde{s}_{\text{II}}^{mn}. \tag{9.10}$$

Lemma. The commutation relations (9.3) of the Lorentz group are valid for each set $s_{\text{I–IV}}^{\rho\sigma}$ of the $s^{\rho\sigma}$ matrices. The sets (9.2), (1.13) (or (9.7), (9.8)) are the generators of the same (spinor) representation $D\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ of the SL(2,C) group, the set (9.9) consists of the generators of the $D(0,1) \oplus (0,0)$ representation and the set (9.10) consists of the generators of the ireeducible vector $D\left(\frac{1}{2},\frac{1}{2}\right)$ representation of the same group.

PROOF. The fact that the matrices (9.2) (or in another representation (9.7)) are the generators of the spinor representation of the SL(2,C) group is well known. It is better to fulfill the proof of the nontrivial assertions of this lemma in the B-representation, where their validity can be seen directly from the explicit form of the operators $s^{\rho\sigma}$ even without the Casimir operators calculations. In fact, using the explicit forms of the matrices (9.7), (9.8) we find

$$\tilde{s}_{\text{II}}^{\rho\sigma} = C\tilde{s}_{\text{I}}^{\rho\sigma}C \Longleftrightarrow \tilde{s}_{\text{I}}^{\rho\sigma} = C\tilde{s}_{\text{II}}^{\rho\sigma}C, \tag{9.11}$$

$$\tilde{s}_{\text{IV}}^{01} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \; \tilde{s}_{\text{IV}}^{02} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \; \tilde{s}_{\text{IV}}^{03} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$
$$\tag{9.12}$$

$$\tilde{s}_{\text{IV}}^{mn} = \begin{vmatrix} s_{mn} & 0 \\ 0 & 0 \end{vmatrix} = \tilde{s}_{\text{III}}^{mn}; \quad \tilde{s}_{\text{III}}^{0k} = \begin{vmatrix} s_{0k} & 0 \\ 0 & 0 \end{vmatrix}, \tag{9.13}$$

where

$$s^{0k} = \frac{i}{2}\varepsilon^{kmn}s_{mn} = -s^{k0}, \quad \varepsilon^{123} = +1,$$

$$s^{23} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}, \quad s^{31} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad s^{12} = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$
$$\tag{9.14}$$

The unitarity of the C operator in relations (9.11), the direct calculation of correspondings commutators and the Casimir operators $s_\pm$ of the SL(2,C) group

$$s_\pm^2 = \frac{1}{2}(\tau_1 \pm i\tau_2), \quad \tau_1 \equiv -\frac{1}{2}S^{\mu\nu}s_{\mu\nu}, \quad \tau_2 \equiv -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}s_{\mu\nu}s_{\rho\sigma}, \tag{9.15}$$

complete the proof of the lemma. QED.

It is interesting to mark the following. Despite the fact that the matrices $\widehat{s}_{\mathrm{I}}^{\rho\sigma}$ and $\widehat{s}_{\mathrm{II}}^{\rho\sigma}$ are unitarily interconnected according to formulae (9.11), which in the PD-representation have the form

$$s_{\mathrm{I}}^{\rho\sigma} = \hat{C} s_{\mathrm{II}}^{\rho\sigma} \hat{C}, \quad \hat{C} \equiv UCU^\dagger = \begin{vmatrix} C & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \tag{9.16}$$

the matrices $s_{\mathrm{I}}^{\rho\sigma}$ (9.2) (or (9.7)), as well as the matrices (9.9), (9.10) (or (9.12), (9.13)), contrary to the matrix operators (1.13) (or (9.8)) being taken themselves are not the invariance transformations of the massless Dirac equation (9.1) (or (2.29)). It is evident because the C operator does not commute (or anticommute) with the Diracian $\gamma^\mu\partial_\mu$. Nevertheless, due to the validity of the relations

$$\left[s_{\mathrm{II}}^{\rho\sigma}, s_{\mathrm{I,III,IV}}^{\rho\sigma} \right] = 0, \quad \mu,\nu,\rho,\sigma = (0,1,2,3), \tag{9.17}$$

not only the generators $(\partial, \widehat{j}^l)$ of the well known spinor representation P^{S} of the Poincaré group, but also the following generators

$$j_{\mathrm{III,IV}}^{\rho\sigma} = m^{\rho\sigma} + s_{\mathrm{III,IV}}^{\rho\sigma} \tag{9.18}$$

are the transformations of invariance of the massless Dirac equation. It means the validity of the following assertion.

From the consideration in subsection 2.4.2 and from here one can come to the conclusion as follows. The equations (2.13)=(2.17)=(2.18)=(2.19)=(2.29) is the maximally symmetrical form among the generalized and non-generalized forms of Maxwell equations. Due to the fact that equations (2.17)=(2.18)=(2.19)=(2.29) are manifestly covariant vector, tensor-scalar and spinor forms of one and the same equation (2.13), respectively, the following theorem is valid.

Theorem. The slightly generalized Maxwell equations written as (2.13)=(2.17)=(2.18)=(2.19)=(2.29) are invariant with respect to the three different transformations, which are generated by three different representations P^V, P^{TS}, P^S of the Poincaré group P(1,3) given by the formulae

$$\begin{aligned}
\mathcal{E}(x) &\to \mathcal{E}^V(x) = \Lambda \mathcal{E}(\Lambda^{-1}(x-a)), \\
\mathcal{E}(x) &\to \mathcal{E}^{TS}(x) = F(\Lambda)\mathcal{E}(\Lambda^{-1}(x-a)), \\
\mathcal{E}(x) &\to \mathcal{E}^S(x) = S(\Lambda)\mathcal{E}(\Lambda^{-1}(x-a)),
\end{aligned} \tag{9.19}$$

where Λ is a vector $D(\frac{1}{2},\frac{1}{2})$, $F(\Lambda)$ is a tensor-scalar $D(0,1)\otimes(0,0)$ and $S(\Lambda)$ is a spinor representation $D(0,\frac{1}{2}) \otimes (\frac{1}{2},0)$ of SL(2,C) group. This means that the equations (2.13)=(2.17)=(2.18)=(2.19)=(2.29) have both spin 1 and spin 1/2 symmetries.

Proof. Let us write the infinitesimal transformations, following from (9.19), in the form

$$\mathcal{E}^{V,TS,S}(x) = \left(1 - a^\rho \partial_\rho - \frac{1}{2}\omega^{\rho\sigma} j_{\rho\sigma}^{V,TS,S}\right)\mathcal{E}(x). \tag{9.20}$$

Then the generators of the transformations (9.20) have the form

$$\partial_\rho = \frac{\partial}{\partial x^\rho}, \quad j_{\rho\sigma}^{V,TS,S} = x_\rho \partial_\sigma - x_\sigma \partial_\rho + s_{\rho\sigma}^{V,TS,S}, \tag{9.21}$$

where

$$(s_{\rho\sigma}^V)_v^\mu = \delta_\rho^\mu g_{\sigma v} - \delta_\sigma^\mu g_{\rho v}, \quad s_{\rho\sigma}^V \in D\left(\frac{1}{2},\frac{1}{2}\right), \tag{9.22}$$

$$s_{\rho\sigma}^{TS} = \left|\begin{array}{cc} s_{\rho\sigma} & 0 \\ 0 & 0 \end{array}\right| \in D(0,1)\oplus(0,0), \quad s_{\rho\sigma} = -s_{\sigma\rho}: \quad s_{0k} = -\frac{i}{2}\varepsilon^{kmn}s_{mn}, \tag{9.23}$$

where $s^{\rho\sigma}$ are given in (9.14) and

$$s_{\rho\sigma}^S = \frac{1}{4}[\tilde{\gamma}_\rho, \tilde{\gamma}_\sigma], \quad s_{\rho\sigma}^S \in D(0,\frac{1}{2}) \oplus (\frac{1}{2},0), \tag{9.24}$$

where $\tilde{\gamma}$ matrices in specific bosonic representation are given in (2.30) and satisfy the standard Clifford-Dirac algebra relations. Now the proof of the theorem is reduced to the verification that all the generators (9.21) obey the commutation relations of the P(1,3) algebra and commute with the operator of the generalized Maxwell equations (2.29). $\square$

Corollary. The transition (2.28) transforms the equations (2.13)=(2.17)=(2.18)=(2.19)=(2.29) into the massless Dirac equation (9.1) with matrices γ^μ in standard Pauli–Dirac representation. This means that massless Dirac equation has the same three different P^V, P^{TS}, P^S Poincaré symmetries as the slightly generalized Maxwell equations (2.29). QED

This result about the slightly generalized Maxwell equations (2.17)–(2.19) means the following. From group theoretical point of view these equations can describe both bosons and fermions. This means that one has direct group-theoretical grounds to apply these equations for the description of electron, as it is presented in Chapter 8.

A distinctive feature of the equation (2.17) for the system $\mathcal{E} = (\overrightarrow{\mathcal{E}}, \mathcal{E}^0)$ (i.e., for the system of interacting irreducible $(0,1)$ and $(0,0)$ fields) is the following. It is the manifestly covariant equation with minimal number of components, i.e., the equation without redundant components for this system.

Note that each of the three representations (9.19) of the P(1,3) group is a local one, because each matrix part of transformations (9.19) (matrices Λ, $F(\Lambda)$ and $S(\Lambda)$) does not depend on coordinates $x \in R^4$, and, consequently, the generators (9.21) belong to the Lie class of operators. Each of the transformations in (9.19) may be understood as connected with special relativity transformations in the space-time $R^4 = \{x\}$, i.e., with transformations in the manifold of inertial frame of references.

We emphasize that the equation (2.29) has the form of massless Dirac equation for fermions field. In such consideration of equation (2.29) the $\widetilde{\gamma}^\mu$ matrices may be chosen in arbitrary representation (e.g., in each of representations of Pauli-Dirac, Majorana, Weyl, ...). However, only in exotic representation (2.30) equation (2.29) is the Maxwell equation for the system of interacting electromagnetic $\overrightarrow{\mathcal{E}} = \overrightarrow{E} - i\overrightarrow{H}$ and scalar $\mathcal{E}^0 = E^0 - iH^0$ fields (therefore, we have called the representation (2.30) the bosonic one). Thus, if one considers the equation (2.29) as bosonic one, the representation of γ^μ matrices and their explicit form *must be fixed* in the form (2.30). In the case of bosonic interpretation of the equation (9.1) one must fixes the explicit form of γ^μ in standard Pauli–Dirac representation and fixes the form of Ψ as (2.27).

It follows from the equations (2.17)=(2.29) that the field $\mathcal{E} = (\overrightarrow{\mathcal{E}}, \mathcal{E}^0)$ is massless, i.e., $\partial^\nu \partial_\nu \mathcal{E}^\mu = 0$. Therefore it is interesting to note that neither P^V, nor P^{TS} symmetries cannot be extended to the local conformal C(1,3) symmetry. Only the spinor C^S representation of C(1,3) group, obtained from the local P^S representation, is the symmetry group for the slightly generalized

Maxwell equations (2.17). This fact is understandable: the electromagnetic field $\overrightarrow{\mathcal{E}} = \overrightarrow{E} - i\overrightarrow{H}$ obeying equations (2.17) is not free, it interacts with the scalar field $\mathcal{E}^0$.

Consider the particular case of standard (non-generalized) Maxwell equations, namely, the case of equations

$$\partial_0 \overrightarrow{\mathcal{E}} = i\operatorname{curl}\overrightarrow{\mathcal{E}} - \operatorname{grad}\mathcal{E}^0, \quad \operatorname{div}\overrightarrow{\mathcal{E}} = -\partial_0 \mathcal{E}^0, \tag{9.25}$$

or (2.17), without magnetic charge and current densities, i.e., the case when $H^0 = 0$ but $E^0 \neq 0$. The symmetry properties of such standard equations are strongly restricted in comparison with the generalized equations (2.17): they are invariant only with respect to tensor-scalar (spins 1 and 0) representation of Poincaré group defined by the corresponding representation D(0,1)⊕(0,0) of proper ortochronous Lorentz group SL(2,C). Another symmetries mentioned in the theorem are lost for this case too. The proof of this assertion follows from the fact that the vector $D\left(\frac{1}{2},\frac{1}{2}\right)$ and the spinor $D\left(0,\frac{1}{2}\right) \oplus \left(\frac{1}{2},0\right)$ transformations of $\mathcal{E} = (\overrightarrow{\mathcal{E}}, \mathcal{E}^0)$ mix the $\mathcal{E}^0$ and $\overrightarrow{\mathcal{E}}$ components of the field $\mathcal{E}$, and only the tensor-scalar $(0,1) \oplus (0,0)$ transformations do not mix them.

For the free Maxwell equation in vacuum without sources (the case $E^0 = H^0 = 0$) the losing of above mentioned symmetries is evident from the same reasons. Moreover, it is well known that such equations are invariant only with respect to tensor (spin 1) representations of Poincaré and conformal groups and with respect to dual transformation: $\overrightarrow{E} \to \overrightarrow{H}, \overrightarrow{H} \to -\overrightarrow{E}$. We have obtained the extended 32-dimensional Lie algebra [6, 7] (and the corresponding group) of invariance of free Maxwell equations, which is isomorphic to C(1,3)⊕C(1,3)⊕*dual* algebra. We were successful to prove it appealing not to Lie class of symmetry operators but to a more general, namely, to the simplest Lie–Backlund class of operators. The corresponding generalization of symmetries of equations (9.25)=(2.17) presented in the above theorem leads to a wide 246-dimensional Lie algebra in the class of first order Lie–Backlund operators. Many other additional symmetries of the massless Dirac equation can be found in [1, 2].

Therefore, the Maxwell equations (2.13)=(2.17)=(2.18)=(2.19)=(2.29) with electric and magnetic gradient-like sources have the maximally possible symmetry properties among the standard and generalized equations of classical electrodynamics.

Thus, we introduce our generalization of the Maxwell equations on the basis of symmetry principle (the second foundation of our generalization).

9.3. Link between the Fermionic and Bosonic Amplitudes

Finally, knowing the operator U (2.27), it is easy to obtain the relationship between the amplitudes $a^r(\overrightarrow{k})$, $b^r(\overrightarrow{k})$ determining the well known fermionic solution of the massless Dirac equation (in Pauli–Dirac representation), and the amplitudes $c^\alpha(\overrightarrow{k})$, determining the bosonic solution (2.21). The general solution of the massless Dirac equation (9.1) has the form

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k [a^r(\overrightarrow{k})v_r^-(\overrightarrow{k})e^{-ikx} + b^{*r}(\overrightarrow{k})v_r^+(\overrightarrow{k})e^{ikx}], \quad r = 1, 2,$$
(9.26)

where $kx \equiv \tilde{\omega}t - \overrightarrow{k}\overrightarrow{x}$, $\tilde{\omega} \equiv \sqrt{\overrightarrow{k}^2}$, and the 4-component Dirac spinors are given by

$$v_r^-(\overrightarrow{k}) = N \begin{vmatrix} \tilde{\omega}d_r \\ (\overrightarrow{\sigma} \cdot \overrightarrow{k})d_r \end{vmatrix}, \quad v_r^+(\overrightarrow{k}) = N \begin{vmatrix} (\overrightarrow{\sigma} \cdot \overrightarrow{k})d_r \\ \tilde{\omega}d_r \end{vmatrix};$$
(9.27)

$$N \equiv \frac{1}{\sqrt{2\tilde{\omega}}}, \quad d_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad d_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}.$$

The link between the fermionic and bosonic amplitudes from (2.21) and (9.26) is as follows:

$$a^1 = \frac{1}{2\tilde{\omega}} \left[i\sqrt{(\tilde{\omega} - k^3)(\tilde{\omega} + k^3)}(c^1 - c^2) - (\tilde{\omega} - k^3)c^3 + (\tilde{\omega} + k^3)c^4 \right], \quad (9.28)$$

$$a^2 = \frac{1}{2\tilde{\omega}} \left[-i(k^1 + ik^2)\left(\sqrt{\frac{\tilde{\omega} + k^3}{\tilde{\omega} - k^3}}c^1 + \sqrt{\frac{\tilde{\omega} - k^3}{\tilde{\omega} + k^3}}c^2 \right) + (k^1 + ik^2)(c^3 + c^4) \right],$$

$$b^1 = \frac{1}{2\tilde{\omega}} \left[i\sqrt{(\tilde{\omega} - k^3)(\tilde{\omega} + k^3)}(c^1 + c^2) + (\tilde{\omega} + k^3)c^3 + (\tilde{\omega} - k^3)c^4 \right],$$

$$b^2 = \frac{1}{2\tilde{\omega}} \left[i(k^1 + ik^2)\left(\sqrt{\frac{\tilde{\omega} - k^3}{\tilde{\omega} + k^3}}c^1 - \sqrt{\frac{\tilde{\omega} + k^3}{\tilde{\omega} - k^3}}c^2 \right) + (k^1 + ik^2)(c^3 - c^4) \right].$$

In terms of unitary operator V this formulae have the form:

$$\widehat{a} \equiv \begin{vmatrix} a^1 \\ a^2 \\ b^1 \\ b^2 \end{vmatrix} = \frac{1}{2\omega} \begin{vmatrix} i\sqrt{pq} & -p & -i\sqrt{pq} & q \\ -iz^*\sqrt{\frac{q}{p}} & z^* & -iz^*\sqrt{\frac{p}{q}} & z^* \\ i\sqrt{pq} & q & i\sqrt{pq} & p \\ iz\sqrt{\frac{p}{q}} & z & -iz\sqrt{\frac{q}{p}} & -z \end{vmatrix} \cdot \begin{vmatrix} c^1 \\ c^3 \\ c^2 \\ c^4 \end{vmatrix} = V \cdot \widehat{c}, \quad (9.29)$$

where $p = \tilde{\omega} - k^3$, $q = \tilde{\omega} + k^3$, $z = k^1 - ik^2$, $z^* = k^1 + ik^2$, $\tilde{\omega} \equiv \sqrt{\overrightarrow{k}^2}$. The operator V (the image of operator U (2.27) in the space of quantum-mechanical amplitudes $\widehat{c}$ and $\widehat{a}$, i.e., in the rigged Hilbert space $S^{3,4} \subset H^{3,4} \subset S^{3,4*}$, where $S^{3,4*}$ is the space of 4-component generalized Schwartz functions) is unitary one: $VV^{-1} = V^{-1}V = 1$, $V^{-1} = V^{\dagger}$, plus linearity.

Hence, the fermionic states may be constructed as linear combinations of bosonic states, namely, of the states of the coupled electromagnetic $\overrightarrow{\mathcal{E}} = \overrightarrow{E} - i\overrightarrow{H}$ and scalar $\mathcal{E}^0 = E^0 - iH^0$ fields. The inverse relationship between bosonic and fermionic states is also valid. We prefer the first possibility, which is new (bosonic) realization of the old idea (Thomson, Abraham, etc) of electromagnetic nature of mass and of material world. Thus, today on the basis of (8.25), (2.27), (9.28), (9.29) we may speak about more general conception of the bosonic field nature of material world.

On the basis of this link the relationship between quantized electromagnetic-scalar and massless spinor field was obtained. The possibility of both Bose and Fermi quantization types for electromagnrtic-scalar field (and, inversely, for the Dirac spinor field) was proved. We will not touch above the problems of quantization because above we are trying to demonstrate new possibilities of classical theory.

9.4. Two Types of Quantization

In the quantum theory of fields under consideration, the following connections are valid.

If bosonic creation-annihilation operators $c_k^{\dagger\bar{\alpha}}, c_k^{\bar{\alpha}}$ satisfy the standard commutation relations

$$\left[c_k^{\bar{\alpha}}, c_{k'}^{\dagger\bar{\beta}} \right] = \delta_{\bar{\beta}}^{\bar{\alpha}} \delta\left(\overrightarrow{k} - \overrightarrow{k}' \right); \quad \bar{\alpha}, \bar{\beta} = 1, 2, 3, 4, \qquad (9.30)$$

(the commutators of all other pairs are equal to zero), then the fermionic creation-annihilation operators $a^\dagger, b^\dagger, a, b$, connected with the bosonic operators by the transformation (9.29), or by inverse to (9.29) transformation, satisfy the following commutation (not anticommutation !) relations

$$
\begin{aligned}
\left[a_k^1, a_{k'}^{1\dagger} \right] &= - \left[b_k^{1\dagger}, b_{k'}^1 \right] = \tfrac{3\tilde{\omega}^2 + k^3 k^3}{\tilde{\omega}} \delta \left(\overrightarrow{k} - \overrightarrow{k}' \right), \\
\left[a_k^2, a_{k'}^{2\dagger} \right] &= - \left[b_k^{2\dagger}, b_{k'}^2 \right] = \tfrac{3\tilde{\omega}^2 - k^3 k^3}{\tilde{\omega}} \delta \left(\overrightarrow{k} - \overrightarrow{k}' \right), \\
\left[a_k^1, a_{k'}^{2\dagger} \right] &= - \left[b_k^{1\dagger}, b_{k'}^2 \right] = \tfrac{(k^1 - ik^2)k^3}{\tilde{\omega}} \delta \left(\overrightarrow{k} - \overrightarrow{k}' \right), \\
\left[a_k^1, b_{k'}^{2\dagger} \right] &= - \left[a_k^2, b_{k'}^{1\dagger} \right] = -(k^1 + ik^2) \delta \left(\overrightarrow{k} - \overrightarrow{k}' \right),
\end{aligned}
\tag{9.31}
$$

(the commutators of all other pairs are equal to zero).

Quite similarly the anticommutators of fermionic creation-annihilation operators $a^\dagger, b^\dagger, a, b$ are connected with the anticommutators of $c^\dagger, c$ operators.

Furthermore, the anticommutator(!)

$$
\left\{ \mathcal{E}(x), \mathcal{E}^\dagger(x') \right\} = U^\dagger \mathcal{D} \gamma^0 U D \equiv \widetilde{\gamma}^\mu \gamma^0 \partial_\mu D(x - x')
\tag{9.32}
$$

for the bosonic field $\mathcal{E}(x)$ is obtained from the anticommutator of the Heisenberg spinor (fermionic) field $\psi(x)$

$$
\left\{ \psi(x), \overline{\psi}(x') \right\} = \mathcal{D} D(x - x'), \quad \mathcal{D} \equiv \gamma^\mu \partial_\mu,
\tag{9.33}
$$

$$
D(x) \equiv \frac{i}{(2\pi)^3} \int \frac{d^3 k}{2\tilde{\omega}} (e^{-ikx} - e^{ikx}),
\tag{9.34}
$$

by substituting $\psi = U \mathcal{E}$ into formula (9.33).

Similarly, the commutator(!)

$$
\left[\psi(x), \overline{\psi}(x') \right] = U^\dagger \mathcal{M} U \gamma^0 D(x - x')
\tag{9.35}
$$

for the fermionic field $\psi(x)$ is obtained after the substitution of $\mathcal{E} = V\psi$ into the commutator

$$
\left[\mathcal{E}(x), \mathcal{E}^\dagger(x') \right] = \mathcal{M} D(x - x'),
\tag{9.36}
$$

which can be found for the bosonic field $\mathcal{E}(x)$ on the basis of relations (9.30). Here the notations

$$
\mathcal{M} D(x) \equiv \frac{i}{(2\pi)^3} \int \frac{d^3 k}{2\tilde{\omega}} M(\overrightarrow{k}) (e^{-ikx} - e^{ikx}),
\tag{9.37}
$$

$$M(\vec{k}) \equiv -i\tilde{\omega}^2 \left[e_1 e_1^\dagger + (e_3 + e_4)(e_3^\dagger + e_4^\dagger) \right], \tag{9.38}$$

are used.

Because of the unitary properties of the operator U and the starting point from the causal commutator (anticommutator) of the fields under consideration all the commutation (anticommutation) relations obtained here satisfy the microcausality condition. Let us underline that here we can find the causal commutation relations (9.35) (not anticommutation!) for the fermionic spinor Dirac field only for the $m = 0$ case.

9.5. Corresponding Lagrangians

Having the Lagrangian $\mathcal{L}^{(\Psi)}$ for the massless spinor field and using the connection (2.27), one obtains the Lagrangian $\mathcal{L}^{(\mathcal{E})}$ for the system of electromagnetic and scalar fields:

$$\mathcal{L}^{(\Psi)} \equiv \frac{i}{2} \left(\overline{\Psi} \gamma^\mu \Psi_{,\mu} - \overline{\Psi}_{,\mu} \gamma^\mu \Psi \right) = \\ -\frac{i}{2} \left(\mathcal{E}^\dagger \tilde{\gamma}^0 \Gamma \tilde{\gamma}^\mu \mathcal{E}_{,\mu} - \mathcal{E}_{,\mu}^\dagger \tilde{\gamma}^0 \Gamma \tilde{\gamma}^\mu \mathcal{E} \right) \equiv \mathcal{L}^{(\mathcal{E})}. \tag{9.39}$$

Of course, the Euler–Lagrange equation for the Lagrangian (9.39) coincides with the equation

$$i\Gamma\tilde{\gamma}^\mu \partial_\mu \mathcal{E} = 0; \quad \mathcal{E} = E - iH. \tag{9.40}$$

The Lagrange function for the system of interacting Dirac spinor field and electromagnetic-scalar field has the form

$$L = -\frac{i}{2} \left(\mathcal{E}^\dagger \tilde{\gamma}^0 \Gamma \tilde{\gamma}^\mu \mathcal{E}_{,\mu} - \mathcal{E}_{,\mu}^\dagger \tilde{\gamma}^0 \Gamma \tilde{\gamma}^\mu \mathcal{E} \right) + \frac{i}{2} \left(\overline{\Psi}(x) \gamma^\mu \frac{\partial \Psi(x)}{\partial x^\mu} - \frac{\partial \overline{\Psi}(x)}{\partial x^\mu} \gamma^\mu \Psi(x) \right) - \tag{9.41}$$
$$m\overline{\Psi}(x)\Psi(x) + e\overline{\Psi}(x)\gamma^\mu \Psi(x) \mathcal{E}_\mu(x).$$

The Lagrange approach based on the function (9.41) presents the analogy of the well-known Lagrange formalism in the terms of the vector-potentials $A^\mu(x)$ of the contemporary quantum electrodynamics. However, the non-trivial step is using of the electromagnetic-scalar field strengths $\mathcal{E}^\mu(x)$ as the Lagrange variables. Therefore, the based on (9.41) least action principle appeals to the electromagnetic-scalar variational variables $\mathcal{E}^\mu(x)$ and does not touch the potentials $A^\mu(x)$.

Note that reformulation of the quantum electrodynamics in the terms of field-strengths (without vector-potentials) is an interesting problem of modern quantum theory (see the results of [8–11] and the review therein). The approach (9.41) is a good solution. Indeed, it is proved above that the field $\mathcal{E}^{\mu}(x)$ can be considered as the P(1,3) vector. Therefore, on the bases of techniques similar to developed by Gupta–Bleuler [12, 13] one can formulate here the two-component photon model of quantum electrodynamics without redundant components. Furthermore, the 4-component model here, contrary to the $A^{\mu}(x)$ formalism, does not contain the non-physical "time" photons (longitudinal photons can be explained). Thus, the 4-component model of interacting electromagnetic-scalar and spinor fields has a good perspectives as well. Note that there are few well-defined possibilities to introduce interaction Lagrangian $L_{\mathrm{I}} = e\overline{\psi}(x)\gamma^{\mu}\psi(x)\mathcal{E}_{\mu}(x)$ in the physical dimension of $L_{\mathrm{I}} = e\overline{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x)$ and to reconcile the dimensions of all terms in (9.41).

9.6. Conclusion and Comments

9.6.1. Symmetry

One of the conclusions of our investigation presented here is that a field equation itself does not answer the question what kind of particles (Bose or Fermi) is described by this equation. To answer this question one needs to find all the representations of the Poincaré group under which the equation is invariant. If more than one such Poincaré representations are found, including the representations with integer and half-integer spins, then the given equation describes both Bose and Fermi particles, and both quantization types (Bose and Fermi) of the field function, obeying this equation, satisfy the microcausality condition. The strict group-theoretical ground of this assertion is the following: both slightly generalized Maxwell equations (with $\varepsilon = \mu = 1$) and massless Dirac equation for the free spinor field are invariant with respect to three different local representations of Poincaré group, namely the standard spinor, vector and tensor-scalar representations generating by the $D(0,\frac{1}{2}) \otimes (\frac{1}{2},0)$, $D(\frac{1}{2},\frac{1}{2})$, $D(0,1) \otimes (0,0)$ representations of the Lorentz SL(2,C) group, respectively.

Therefore, the pair "Dirac equation plus reducible, spins 1 and 0, representation" may describe a double bosonic system (photon plus boson).

The pair "Dirac equation plus spin 1/2 representation" may describe fermions (electron, neutrino, etc.).

The pair "generalized Maxwell equation plus spin 1 and 0 representation" may describe a double bosonic system (photon plus boson).

Finally, the pair "generalized Maxwell equation plus spin 1/2 representation" may describe fermions, e.g., the electron. This latter possibility is considered in this paper.

9.6.2. Model of Atom

Using the slightly generalized Maxwell classical electrodynamics (equations for the system of electromagnetic and scalar fields) and taking a spin 1/2 representation, we arrive at a model of the electron. Such model is based on the derivation of the hydrogen spectrum and Bohr's postulates (see Chapter 8 above). The electron can be interpreted as a system of electromagnetic and scalar fields (waves) in a specific medium (compound system of photon plus massless boson with spin equal to zero). The electron is a standing wave in the stationary case. Because it is a system of electromagnetic and scalar waves (not a charged corpuscle), it is free from the radiation difficulties of Rutherford's electron in electrodynamics. The charge here is a secondary quality, generated by interacting electromagnetic-scalar fields. The limit $m \neq 0$ interaction potential equal to zero produces the free electron. Thus, the electron can be constructed from bosons.

The simplest case $m = 0$, interaction potential equal to zero, is treated in detail in sections 9.3 and 9.4, where it is shown that amplitudes of fermionic states (or their creation-annihilation operators) are the linear combinations of amplitudes (or of creation-annihilation operators) of bosonic states. In this sense our model, where the electron is considered to be a compound system of photon plus mass-less spin-less boson (i.e., the electron's states are linear combinations of states of the electromagnetic-scalar field) has an analogy in modern quark models of hadrons. On the basis of our consideration we are able to construct fermionic states from bosonic states. Moreover, formula (8.25) expresses the mass of the atom in terms of bosonic (electromagnetic and scalar) field strengths! This is the basis for our hypothesis: the material world is bosonic in nature (more general than simple electromagnetic). Furthermore, changing the field strengths in (8.25) may cause a change in the mass of a material object like an atom. Can a new flying machine, with mass going to zero, be constructed on the basis of this phenomenon?

Thus, in the model of atom under consideration based on the equations of

Maxwell's electrodynamics, not of quantum mechanics, the atomic electron is interpreted as a classical stationary electromagnetic-scalar wave (the details of the interpretation see in subsection 8.2.4). That is why this model is essentially distinguished from the first electrodynamical hydrogen atom model suggested by Sallhofer, see, e.g., [14–16].

It is evident from the example of the hydrogen atom presented in section 8.2 that the discreteness of the physical system's states (and its characteristics, such as energy, etc.) may be a consequence of both quantum systems (Schrödinger, Dirac) and the classical (Maxwell) equations for the given system. In the present case, discreteness is caused by the properties of the medium, which are given by the electric and magnetic permeabilities (2.4).

The main conclusion from Chapters 8, 9 is the following. The unitary equivalence between the stationary Dirac equation and the stationary Maxwell equations with gradient-like currents and charges in a medium (2.4) offers the possibility of reformulating all the problems of atomic and nuclear physics (not just the problem of describing the hydrogen atom, which is only one example), which can be solved on the basis of the stationary Dirac equation, in the language of the classical electrodynamic stationary Maxwell equations. This means that our model for the stationary case is just as successful as conventional relativistic quantum mechanics. In the approach based on slightly generalized Maxwell equations, it is possible to solve other stationary problems of atomic physics without appealing to the Dirac equation or the probabilistic or Copenhagen interpretation.

Some non-stationary problems, e.g., the problem of transitions between stationary states caused by external perturbation, can probably be solved in terms of this electrodynamic model, just as this problem is now solved in terms of the stationary Schrödinger equation with corresponding perturbation.

9.6.3.   Interpretation of the Dirac ψ Function

A few words can be said about the interpretation of the Dirac ψ function. As follows from the consideration presented here, e.g., from the relationship (2.27), (9.28), the new interpretation of the Dirac ψ function can be suggested too: ψ function is the combination of the electromagnetic field strengths $\left(\overrightarrow{E}, \overrightarrow{H} \right)$ and two scalar fields $\left(E^0, H^0 \right)$ generating the electromagnetic sources, i.e., in this case the probability or Copenhagen interpretation of the function ψ is not necessary. Taking into account that many interpretations of quantum mechanics

(e.g., Copenhagen, statistical, Feynman's, Everett's, transactional [16–19]) exist, we are under no illusion that our interpretation should be the only one.

In our publications [3-5, 11] we have tried to develop the classical electrodynamic interpretation of the above facts, which is the main purpose of our investigation. Nevertheless, we have also emphasized [5] that a standard quantum mechanical Dirac (or spinor classical field-theoretical) interpretation is certainly also possible here. In this case the above facts only demonstrate in explicit form the classical electrodynamic aspect of the Dirac equation [5]. In other words, our equations may be considered (interpreted) as the Dirac equation for the classical (not quantum) spinor field in a specific electromagnetic representation. We have written one special paper [5] to admit this possibility, which may be more suitable for readers who are beyond the influence of Standard Model. Magnetic monopole enthusiasts may attempt to develop the monopole interpretation [20]: we note that there are few specific possibilities for interpretation. Thus, the new features that follow from our approach are:

(i) the classical interpretation,

(ii) a new equation and method in atomic and nuclear physics based on classical electrodynamics in an inneratomic medium as in (2.4),

(iii) hypothesis of the bosonic nature of matter (bosonic structure of femions),

(iv) application of classical theory extended further into the microworld,

(v) foundations of a unified model of electromagnetic and gravitational phenomena, in which gravitation is considered a medium in generalized equations,

(vi) the electron is described as a classical electromagnetic-scalar wave and is related to the corresponded equation of motion.

(vii) Lagrange approach for quantum electrodynamics in the terms of field strengths.

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Chapter 10

Fermi-Bose Duality of the Dirac Equation with Nonzero Mass

Fermi–Bose duality of the massless Dirac equation, which is considered in the previous Chapters 8, 9, is generalized here for the case of nonzero mass. This generalization is non-trivial. Corresponding derivations essentially appeal to the 64-dimensional gamma matrix representation of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ over the field of real numbers and the 28-dimensional gamma matrix representation of the algebra SO(8) (over the field of real numbers as well). Moreover, the start from the canonical Foldy–Wouthuysen (FW) representation is necessary and Fermi–Bose duality properties of the Dirac equation follow from the corresponding properties of the FW equation. The bosonic symmetries, solutions and conservation laws are considered.

10.1. Fermi–Bose Symmetries of the Dirac Equation with Nonzero Mass

Bosonic symmetries of the Dirac equation with nonzero mass, which existence is under consideration after our publications [1–6] in the years 2011–2015, are proved here on the basis of two different methods. The first one appeals to the 64-dimensional gamma matrix representation of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ over the field of real numbers and the 28-dimensional gamma matrix representation of the algebra SO(8) (over the field of real numbers as well). The second way of proof is based on the start from the relativistic canonical quantum me-

chanics (RCQM) of spin (1,0) particle multiplet and its relationship with the
Dirac equation, which is given by the extended FW transformation suggested
by us in 2014–2017. Both the Lorentz and Poincaré bosonic symmetries are
considered. The 31-dimensional algebra of invariance is found. The bosonic
solutions and conservation laws are found as well. The considered phenomenon
is called the fermion-boson duality of the Dirac equation according to P. Gar-
baczewski's titles suggested in [7–9].

10.1.1.   Motivation and Goals

It is well-known that the Dirac equation is invariant with respect to the transfor-
mations, which are determined by the spin $s = \frac{1}{2}$ representation of the Poincaré
group. On the basis of this fact the conclusion that the Dirac equation de-
scribes the spin $s = \frac{1}{2}$ fields and particles (fermions) is formulated. The repre-
sentations of the proper ortochronous Poincaré group (inhomogeneous Lorentz
group) have the principal importance.

Below we are able to demonstrate another hidden half of the Dirac equation
possibilities. We consider here the bosonic symmetries, solutions and conserva-
tion laws for the Dirac equation with nonzero mass.

Note that the Dirac equation taken itself (as well as every other field equa-
tion) does not carry the complete information on the subject what field (particle)
is described by this equation. The complete information is given only by the pair
of conceptions: equation and the transformation law of the field function (see
the examples in Chapters 7, 9). Therefore, the transformations, which are de-
termined not by the 1/2 eigen values of the particle spin operator, have a special
importance and physical meaning among the additional transformations, with
respect to which the Dirac equation is invariant.

Here two proofs of Bose properties of the Dirac equation with nonzero mass
are considered. The first one is based on the new mathematical objects. These
objects are the 64-dimensional gamma matrix representation of the Clifford al-
gebra $C\ell^{\mathbb{R}}(0,6)$ over the field of real numbers and the 28-dimensional gamma
matrix representation of the algebra SO(8) (over the field of real numbers as
well). These objects are considered in Chapter 1. The additional elements of
gamma matrix representations of these algebras unable us to prove the bosonic
symmetries of the Dirac equation and, as a consequence, to demonstrate the
bosonic solutions and conservation laws.

The second proof is based on the start from the RCQM of Bose spin (1,0)

particle multiplet and its link with the Dirac equation, which is given by the extended FW transformation.

In addition to the Dirac equation the γ matrix representations of the Clifford algebra are used widely for the multicomponent Dirac-like equations of higher and arbitrary spin such as the Bhabha, Bargman–Wigner, Rarita–Schwinger (for the field with the spin s=3/2), Iwanenko-Landau-Dirac–Kähler equations, etc. Thus, the application of the Clifford algebra in the quantum theory is more wide then the 4-component Dirac equation and corresponded spinor field. It is evident that the program of finding of more wide matrix representations then the 16-dimensional $C\ell^{\mathbb{C}}(1,3)$ and 15-dimensional SO(1,5) algebras representations is the interesting task (see Chapter 1). The corresponding matrix representations can be useful both in the known quantum field theory models and in the development of new approaches for higher spin.

Starting from the first steps of quantum mechanics and during the period of its growth many authors, see, e.g., [10–12] and the list of references in Chapter 8, have been investigated the Fermi-Bose duality of the massless spinor field and of the related electromagnetic field in the terms of field strengths. Such investigations were called the Maxwell–Dirac isomorphism [13], the relations between the Dirac and Maxwell equations [14], the mapping of the Maxwell formalism into the Dirac formalism [15], etc. Recent interest to the problem is known as well [16, 17], especially in graphene investigations [18].

For the most simple case of the massless Dirac equation the bosonic symmetries, solutions and conservation laws were found by us more then 15 years ago, see e.g., Chapters 8, 9 and the references therein. The corresponding fermionic properties of the slightly generalized original Maxwell equations were considered as well. The investigations of the general case of nonzero mass lead us to the new mathematical objects of Chapter 1. Only after introducing of the 64-dimensional gamma matrix representation of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ [1–6] over the field of real numbers (as well as the 28-dimensional gamma matrix representation of the real algebra SO(8) [19, 20]), considered here in Chapter 1, we were successful in extension of the results to the general case, when the mass in the Dirac equation is nonzero.

The interesting case for the symmetry analysis is the Dirac equation with nonzero mass and with nonzero Coulomb potential. In physical literature this case is called the symmetry of relativistic hydrogen atom. Here we pay for it a special attention.

10.1.2. Start from the Extended Clifford–Dirac Algebra and from the Foldy–Wouthuysen Representation

The first application of the matrix representations of the algebras $C\ell^{\mathbb{R}}(0,6)$ and SO(8) presented in Chapter 1 is the symmetry analysis (the search of groups and algebras with respect to which the equation is invariant) of the Dirac equation with nonzero mass. It is easy to understand that the FW representation [21, 22] is preferable for such analysis. Indeed, in this representation one must calculate the commutation relations of possible pure matrix symmetry operators from Table 3 in section 1.4 only with two elements of the FW equation (4.21) operator: they are γ^0 and i. After determining of the symmetries of the FW equation one can find the symmetries of the Dirac equation on the basis of the inverse FW transformation. Note that after such transformation only the small part of symmetry operators will be pure matrix, the main part of operators will contain the nonlocal operator $\widetilde{\omega} \equiv \sqrt{-\Delta + m^2}$ and the functions of it.

Furthermore, in the FW representation two subsets (s^{23}, s^{31}, s^{12}) and (s^{45}, s^{64}, s^{56}) of operators $s^{\widetilde{A}\widetilde{B}}$ from the Table 3 in Chapter 1 (i) determine two different sets of SU(2) spin 1/2 generators, (ii) commute between each other and (iii) commute with the operator of the Dirac equation in the FW representation. Therefore, we can use here the methods developed in Chapter 9 for the case $m = 0$. Moreover, as it was demonstrated in the Chapter 4 the spin operator of the fermionic particle-antiparticle doublet should be introduced in the FW representation. At least it better to do in the canonical FW and not in the Dirac local field representation.

The start of such investigations for $m \neq 0$ has been presented in [1–6], where on this bases the Bose-symmetries, Bose-solutions and Bose- conservation laws for the Dirac equation with nonzero mass have been found. Among the Bose-symmetries the important Lorentz and Poincaré algebras of invariance of the Dirac equation with nonzero mass were found.

10.1.3. Known Symmetries of the Relativistic Hydrogen Atom

Symmetry studies of the equations for the hydrogen atom originate from the non-relativistic case. The SO(4) symmetry of the non-relativistic Schrödinger equation for a hydrogen atom was found by V. Fock [23], see also [24].

Relativistic hydrogen atom is modeled here by the Dirac equation in the

external Coulomb field

$$\left(i\partial_0 - \widehat{H}\right)\psi(x) = 0; \quad \widehat{H} \equiv \gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 m - \frac{Ze^2}{|\vec{x}|}, \tag{10.1}$$

where $x \in \mathrm{M}(1,3)$, $\partial_\mu \equiv \partial/\partial x^\mu$, $Z = 1$, $\mathrm{M}(1,3)$ is the Minkowski space-time and 4-component function $\psi(x)$ belongs to rigged Hilbert space $\mathrm{S}^{3,4} \subset \mathrm{H}^{3,4} \subset \mathrm{S}^{3,4*}$ and $\hbar = c = 1$. The Dirac γ matrices are taken in the standard Dirac-Pauli representation (1.5).

Below we prove the Dirac equation in the external Coulomb field (10.1) to possess the symmetry determined by the 31 operators, which form the 31-dimensional algebra $\mathrm{SO}(6) \oplus i\gamma^0 \cdot \mathrm{SO}(6) \oplus i\gamma^0$. Two different fermionic $\mathrm{D}(1/2,0) \oplus (0,1/2)$ representations of the $\mathrm{SO}(1,3)$ algebra of the Lorentz group are found. Two different bosonic tensor-scalar $\mathrm{D}(1,0) \oplus (0,0)$ and vector $\mathrm{D}(1/2,1/2)$ representations of this algebra are found as well. The corresponding generators of the above mentioned algebras commute with the operator of the Dirac equation in the external Coulomb field (10.1), and, therefore, determine the hidden symmetries (algebras of invariance) of such Dirac equation.

At first we consider the known symmetries of the Dirac equation (10.1) after that we present the mathematical tools, which are necessary for our investigations and, finally, the list of different hidden symmetries of the relativistic hydrogen atom.

The first four constants of motion (symmetry operators, which commutes with the operator of the Dirac equation) for the equation (10.1) were found by P. Dirac in his paper [25], where the equation (10.1) has been derived and introduced. They are three components of the vector $\vec{J} = (J^1, J^2, J^3)$ of the total angular momentum

$$\vec{J} = \vec{L} + \vec{s}, \quad \vec{L} \equiv \vec{x} \times \vec{p}, \quad \vec{s} \equiv \frac{1}{2}\begin{vmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{vmatrix}, \tag{10.2}$$

where $\vec{L}$ is the orbital angular momentum, $\vec{s}$ is the spin-1/2 angular momentum, and the found by Dirac additional constant of motion K:

$$K = \gamma^0\left(2\vec{s}\cdot\vec{L} + 1\right), \quad K^2 = \vec{J}^2 + \frac{1}{4}. \tag{10.3}$$

Next symmetry operator is the Johnson–Lippman constant of motion [26]

$$D = 2\vec{s}\cdot\frac{\vec{x}}{|\vec{x}|} + \frac{1}{mZe^2}K\gamma^4\left(\widehat{H} - \gamma^0 m\right), \quad D^2 = 1 + \left(\frac{\widehat{H}^2}{m^2} - 1\right)\frac{K^2}{Z^2 e^4}, \tag{10.4}$$

which commutes with the operator $\left(i\partial_0 - \widehat{H}\right)$ of the Dirac equation and anti-commutes with the Dirac symmetry operator K of (10.3). Here the anti-Hermitian $\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$ instead of Hermitian γ^5 of other authors is used. Note that article [26] about excellent result (10.4) was published as a brief 1/10 of a journal page remark, containing only single formula for D from (10.4).

After that the way to SO(4) symmetry of the relativistic hydrogen atom was direct. This symmetry was found in [27] and [28], for the consideration in [29] see the comments of [30]. Thus, the SO(4) symmetry of the Dirac equation (10.1) for the hydrogen atom is given by the six operators

$$\vec{I} = \vec{J} + \vec{T}, \quad \vec{R} = \vec{J} - \vec{T}, \tag{10.5}$$

where $\vec{J}$ is known from (10.2) and components of $\vec{T} = (T^1, T^2, T^3)$ have the form

$$T^1 = \frac{D}{2\sqrt{D^2}}, \quad T^2 = \frac{iDK}{2\sqrt{D^2K^2}}, \quad T^3 = \frac{K}{2\sqrt{K^2}}. \tag{10.6}$$

In the paper [27] the object $\vec{T} = (T^1, T^2, T^3)$ (10.6) was called as Lentz spin-1/2 vector operator. The notations used in (10.6) are explained in (10.3) and (10.4) above.

Next symmetry is as follows. The Pauli–Gürsey operators [31] and [32]

$$s^{01} = \frac{i}{2}\gamma^2\widehat{C}, \quad s^{02} = \frac{1}{2}\gamma^2\widehat{C}, \quad s^{12} = -\frac{i}{2}, \tag{10.7}$$

where $\widehat{C}$ is the operator of complex conjugation, $\widehat{C}\psi = \psi^*$ (the operator of involution in the space $H^{3,4}$), determine according to [33] the SO(1,2)$\subset$SO(1,3) algebra of invariance of the Dirac equation in the form $(i\gamma^\mu\partial_\mu - m + \frac{e^2}{|\vec{x}|})\psi(x) = 0$.

In [34] the symmetry of relativistic hydrogen atom in the form of $gl(8,R)$ algebra has been found. The stationary case of the Dirac equation (10.1) was considered and the discrete transformation were used.

The author of [35] considered another problem. The quasi-potential two-particle model was presented for the description of spinless relativistic hydrogen atom. The O(4) symmetry and its breaking were investigated.

10.1.4. New Fermionic Symmetries of the Dirac Equation in External Coulomb Field

Below we present both the symmetries of the FW equation [21] in the external Coulomd field

$$(i\partial_0 - \gamma^0\omega + \frac{e^2}{|\vec{x}|})\phi(x) = 0; \quad \omega \equiv \sqrt{-\Delta + m^2}, \; x \in \mathrm{M}(1,3), \; \phi \in \left\{ \mathrm{S}^{3,4} \subset \mathrm{H}^{3,4} \subset \mathrm{S}^{3,4*} \right\}, \quad (10.8)$$

and, of course, the symmetries of the Dirac equation (10.1) in such external field as well.

Now we can start the consideration of the new symmetries of relativistic hydrogen atom. The beginning was suggested in [36, 37]. The fundamental assertions are as follows.

(i) The gamma matrix representation of the subalgebra SO(6) of the algebra SO(8) from Chapter 1, which is formed by the operators

$$\{s^{\check{A}\check{B}}\} = \{s^{\check{A}\check{B}} \equiv \frac{1}{4}[\gamma^{\check{A}}, \gamma^{\check{B}}]\}, \quad \check{A}, \check{B} = \overline{1,6}, \tag{10.9}$$

determines the algebra of invariance of the Dirac equation in the FW representation $(\partial_0 + i\gamma^0\omega - \frac{e^2}{|\vec{x}|})\phi(x) = 0$ (in (10.9) the six matrices $\{\gamma^{\check{A}}\} = \{\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6\}$ are known from (1.14)).

(ii) On the basis of SO(6) (10.9) the 31-dimensional gamma matrix representation of the Lie algebra SO(6)$\oplus i\gamma^0$SO(6)$\oplus i\gamma^0$ is constructed, which is formed by the elements from $C\ell^{\mathbb{R}}(0,6)$ and is the maximal pure matrix algebra of invariance of the Dirac equation in the Foldy–Wouthuysen representation $(\partial_0 + i\gamma^0\omega - \frac{e^2}{|\vec{x}|})\phi(x) = 0$.

(iii) The Dirac equation in external Coulomb field (10.1) is invariant with respect to the 31-dimensional gamma matrix representation of the algebra $\widetilde{\mathrm{SO}}(6) \oplus i\widetilde{\gamma^0}\widetilde{\mathrm{SO}}(6) \oplus i\widetilde{\gamma^0}$, where the representation of the algebra $\widetilde{\mathrm{SO}}(6)$ is given in the form of (10.9) with gamma operators (1.22)–(1.25). These formulas give the images of gamma matrices (1.14) in the Dirac representation after fulfilling the inverse FW transformation [21].

Thus, the $\widetilde{\mathrm{SO}}(6) \oplus i\widetilde{\gamma^0}\widetilde{\mathrm{SO}}(6) \oplus i\widetilde{\gamma^0}$ algebra is found from SO(6)$\oplus i\gamma^0$SO(6)$\oplus i\gamma^0$ on the basis of the inverse FW transformation [21]. For the Dirac equation only the part of this algebra is pure matrix, other elements contain the operator $\widetilde{\omega} \equiv \sqrt{-\Delta + m^2}$.

Consider the symmetries of the relativistic hydrogen atom with respect to the Lorentz group. On the basis of $C\ell^{\mathbb{R}}(0,6)$ and SO(8) we can determine two

different realizations of the $D(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$ representation of the Lie algebra of *universal covering* $\mathcal{L} = \mathrm{SL}(2,\mathbb{C})$ of the proper ortochronous Lorentz group $\mathrm{L}_+^\uparrow = \mathrm{SO}(1,3) = \{\Lambda = (\Lambda_\nu^\mu)\}$, with respect to which the equation $(\partial_0 + i\gamma^0\omega - \frac{e^2}{|\vec{x}|})\phi(x) = 0$ is invariant:

$$s_{\mathrm{I}}^{\mu\nu} = \{s_{\mathrm{I}}^{0k} = \frac{i}{2}\gamma^k\gamma^4, \quad s_{\mathrm{I}}^{km} = \frac{1}{4}[\gamma^k,\gamma^m]\}, \quad \gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \quad (k,m = \overline{1,3}), \tag{10.10}$$

$$s_{\mathrm{II}}^{\mu\nu} = \{s_{\mathrm{II}}^{01} = -\frac{i}{2}\gamma^2\hat{C}, \quad s_{\mathrm{II}}^{02} = -\frac{1}{2}\gamma^2\hat{C}, \quad s_{\mathrm{II}}^{03} = \frac{1}{2}\gamma^0, \tag{10.11}$$

$$s_{\mathrm{II}}^{23} = -\frac{1}{2}\gamma^0\gamma^2\hat{C}, \quad s_{\mathrm{II}}^{31} = \frac{i}{2}\gamma^0\gamma^2\hat{C}, \quad s_{\mathrm{II}}^{12} = -\frac{i}{2}\},$$

10.1.5. Bose Symmetries of the Relativistic Hydrogen Atom

Taking the combinations of operators (10.10), (10.11) we construct the generators of bosonic representations:

$$s_{\mathrm{TS}}^{\mu\nu} = \{s_{\mathrm{TS}}^{0k} = s_{\mathrm{I}}^{0k} + s_{\mathrm{II}}^{0k}, \quad s_{\mathrm{TS}}^{mn} = s_{\mathrm{I}}^{mn} + s_{\mathrm{II}}^{mn}\}, \tag{10.12}$$

$$s_{\mathrm{V}}^{\mu\nu} = \{s_{\mathrm{V}}^{0k} = -s_{\mathrm{I}}^{0k} + s_{\mathrm{II}}^{0k}, \quad s_{\mathrm{V}}^{mn} = s_{\mathrm{TS}}^{mn}\},$$

where $s_{\mathrm{TS}}^{\mu\nu}$ and $s_{\mathrm{V}}^{\mu\nu}$ are the generators of the tensor-scalar $D(1,0) \oplus (0,0)$ and irreducible vector $D(\frac{1}{2},\frac{1}{2})$ representations of the Lie algebra $\mathrm{SO}(1,3)$ of the Lorentz group $\mathcal{L}$ respectively, with respect to which the FW equation $(\partial_0 + i\gamma^0\omega - \frac{e^2}{|\vec{x}|})\phi(x) = 0$ is invariant.

Anti-Hermitian operators of every set (10.10)–(10.12) satisfy the commutation relations (9.3) of the Lie algebra $\mathrm{SO}(1,3)$ of the Lorentz group $\mathcal{L}$. For the application of anti-Hermitian operators in physics see [38, 39] and the Chapter 3.

For the Dirac equation in the space of Dirac spinors $\{\psi\}$ (i.e., in the Pauli–Dirac representation) the form of the generators of the tensor-scalar $D(1,0)\oplus(0,0)$ and irreducible vector $D(\frac{1}{2},\frac{1}{2})$ representations of the Lie algebra $\mathrm{SO}(1,3)$ of the Lorentz group $\mathcal{L}$ is similar to (10.12) (with (10.10), (10.11) but the gamma operators are in this case too much complicated and are given by (1.22)–(1.25). The images of the operators (10.10)–(10.12) in the Dirac representation determined by gamma operators (1.22)–(1.25) satisfy the commutation relations (9.3) as well.

In [5, 6] for the free Dirac and FW equations we used also the *bosonic representation* of (10.12), in which the Casimir operators are diagonal and the proof of Bose properties is most convenient. Such *bosonic representation* of (10.12) can be useful here as well. The corresponding transition operator is given by (1.32) and translates the gamma matrices (1.14) into the form (1.27)–(1.31). Further, transformation (1.32) translates s_{TS}^{mn} operators from (10.12) into the form

$$
\check{s}^{1} = \frac{1}{\sqrt{2}}
\begin{vmatrix}
0 & 0 & iC & 0 \\
0 & 0 & -C & 0 \\
-iC & C & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix},
\quad
\check{s}^{2} = \frac{1}{\sqrt{2}}
\begin{vmatrix}
0 & 0 & C & 0 \\
0 & 0 & -iC & 0 \\
-C & iC & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix},
\quad (10.13)
$$

$$
\check{s}^{3} =
\begin{vmatrix}
-i & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix},
\quad
\overrightarrow{s}^{2} = -1(1+1)
\begin{vmatrix}
I_{3} & 0 \\
0 & 0
\end{vmatrix},
$$

where the Bose character of the operators is demonstrated (here operator of complex conjugation C is not matrix).

Representation (10.13) is very useful for the link RCQM→FW→Dirac, which is used in this book for the synthesis of covariant particle equations. The reason is in special box structure of the operator $\check{s}^{3}$ in (10.13) and in relation to Cartesian basis.

Consider another bosonic representation of the generators (10.12), in which their Bose properties are evident. Transition to such *evident bosonic representation*, which corresponds to the cyclic basis, is performed by the operator

$$
W_{\text{B}} = \frac{1}{\sqrt{2}}
\begin{vmatrix}
0 & -1 & 0 & C \\
0 & i & 0 & iC \\
-1 & 0 & C & 0 \\
-1 & 0 & -C & 0
\end{vmatrix},
\quad
W_{\text{B}}^{-1} = \frac{1}{\sqrt{2}}
\begin{vmatrix}
0 & 0 & -1 & -1 \\
-1 & -i & 0 & 0 \\
0 & 0 & C & -C \\
C & iC & 0 & 0
\end{vmatrix},
$$

$$
(10.14)
$$

$$
W_{\text{B}}W_{\text{B}}^{-1} = W_{\text{B}}^{-1}W_{\text{B}} = 1.
$$

In such cyclic Bose-representation, the $\gamma^{\mu\text{B}} = W_{\text{B}}\gamma^{\mu}W_{\text{B}}^{-1}$ matrices contain not only the operator i, but also the operator C of complex conjugation:

$$
\gamma^{0\text{B}} =
\begin{vmatrix}
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{vmatrix},
\quad
\gamma^{1\text{B}} =
\begin{vmatrix}
0 & 0 & 0 & C \\
0 & 0 & iC & 0 \\
0 & -iC & 0 & 0 \\
-C & 0 & 0 & 0
\end{vmatrix},
\quad
\gamma^{2\text{B}} =
\begin{vmatrix}
0 & 0 & -iC & 0 \\
0 & 0 & 0 & C \\
iC & 0 & 0 & 0 \\
0 & -C & 0 & 0
\end{vmatrix},
$$

$$\gamma^{3B} = \begin{vmatrix} 0 & iC & 0 & 0 \\ -iC & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & -C & 0 \end{vmatrix}, i^B = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{vmatrix}, \hat{C}^B = \begin{vmatrix} C & 0 & 0 & 0 \\ 0 & -C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix},$$

$$(10.15)$$

(of course, they satisfy the Clifford–Dirac relations (1.7)). Calculation of operator constructions (10.12) needs also the explicit forms of operators i^B, $\hat{C}^B$, therefore we present them in (10.15) too. Now, together with $\gamma^{\mu B}$ matrices, they are not so simple as iI_4, $\hat{C}I_4$ in standard PD-representation (I_4 is 4×4 unit matrix). Nevertheless, we note, the Lorentz spin matrices $s_{\mu\nu}^B = W_B s_{\mu\nu}^{V,TS} W_B^{-1}$ in the cyclic Bose-representation do not contain the operator $\hat{C}$ and take the explicit forms known well for the matrix representations $D(1,0)\otimes(0,0)$ and $D(1/2,1/2)$ of the group $L \sim SO(1,3)$:

$$s_{12}^{V,TSB} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad s_{31}^{V,TSB} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad s_{23}^{V,TSB} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$s_{01}^{TSB} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad s_{02}^{TSB} = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad s_{03}^{TSB} = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$(10.16)$$

$$s_{01}^{VB} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad s_{02}^{VB} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad s_{03}^{VB} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}.$$

It is easy to verify that generators (10.16) commute with the operator $(\partial_0 + i^B\gamma^{0B}\widetilde{\omega} - \frac{e^2}{|\vec{x}|})\phi(x) = 0$ of the FW equation in the Bose-representation (10.15). The corresponding symmetries of the Dirac equation are obtained with the help of inverse FW transformation [21].

10.1.6. Bosonic Poincaré Symmetries of the Free Non-Interacting Dirac Equation

In the case when interaction is absent the set of symmetry operators for the Dirac equation is much more wide. In addition to above given symmetries we can determine bosonic Poincaré symmetry of the Dirac equation for the free spinor field.

The FW equation (4.21) is invariant not only with respect to the well-known standard spin 1/2 $\mathcal{P}^F$-representation (6.24), (6.25), but *also with respect to the canonical-type spin 1 representation (Bose representation) of the Poincaré group* $\mathcal{P}$, i.e., with respect to the unitary (in the set $\{\phi\}$ of solutions of equation (4.21)) $\mathcal{P}^B$-representation, which is determined by the primary generators

$$\widehat{p}_0 = \check{p}_0 = -i\gamma_0\widehat{\omega}, \quad \widehat{p}_n = \check{p}_n = \partial_n, \quad \widehat{j}_{ln} = x_l\partial_n - x_n\partial_l + s^I_{ln} + s^{II}_{ln}, \quad (10.17)$$

$$\widehat{j}_{0k} = x_0\partial_k + i\gamma_0\left\{x_k\widehat{\omega} + \frac{\partial_k}{2\widehat{\omega}} + \frac{[(\overrightarrow{s}^I + \overrightarrow{s}^{II}) \times \overrightarrow{\partial}]_k}{\widehat{\omega}+m}\right\},$$

where s^I_{ln} and s^{II}_{ln} are given in (10.10), (10.11), respectively, and $\overrightarrow{s}^{I,II} = (s_{23}, s_{31}, s_{12})^{I,II}$.

The **proof** is performed by the straightforward calculations of (i) the corresponding $\mathcal{P}$-commutators (4.2) between the generators (10.17), (ii) the commutators between generators (10.17) and operator $(\partial_0 + i\gamma_0\widehat{\omega})$, (iii) the Casimir operators of the Poincaré group for the generators (10.17). Acoording to the Bargman–Wigner classification of the $\mathcal{P}$-covariant fields, just these facts (especially (iii)) visualize the hidden Bose essence of the $\mathcal{P}^B$-representation, generated by the operators (10.17). For the $\mathcal{P}^B$-representation the explicit form of main Casimir operators is following

$$\widehat{p}^\mu\widehat{p}_\mu = m^2, \quad \mathcal{W}^B = w^\mu w_\mu = m^2(\overrightarrow{s}^{TS})^2, \quad (10.18)$$

(compare with (6.19), (6.20)), where $w^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\widehat{p}_\rho\widehat{j}_{\nu\sigma}$ and, after diagonalization carried out with the help of operator W_B (10.14),

$$(\overrightarrow{s}^{TS})^2 = (\overrightarrow{s}^{TS\,Bose})^2 = -1(1+1)\begin{vmatrix} I_3 & 0 \\ 0 & 0 \end{vmatrix}, \quad I_3 \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \quad (10.19)$$

It is easy to see that *the prime Dirac equation*

$$(\partial_0 + iH)\psi(x) = 0, \quad H \equiv \overrightarrow{\alpha} \cdot \widehat{\overrightarrow{p}} + \beta m, \quad \overrightarrow{\alpha} = \gamma^0\overrightarrow{\gamma}, \quad \beta = \gamma^0, \quad (10.20)$$

has all above mentioned spin 1 symmetries of the FW equation. The corresponding explicit forms of the generators q^{PD} in the manifold $\{\psi\}$ are obtained from the corresponding formulae (10.9), (10.12), (10.17) for the FW generators q^{FW} with the help of the inverse FW operator (4.23): $q^{PD} = V^- q^{FW} V^+$.

As a meaningful example, we present here the explicit form for *the spin 1 generators of* $\mathcal{P}^{\mathrm{B}}$ *symmetries of the Dirac equation*

$$\widehat{p}_0^{\mathrm{PD}} = \widetilde{p}_0 = -iH, \quad \widehat{p}_k^{\mathrm{PD}} = \partial_k, \quad \widehat{j}_{kl}^{\mathrm{PD}} = x_k\partial_l - x_l\partial_k + s_{kl} + \hat{s}_{kl}, \qquad (10.21)$$

$$\widehat{j}_{0k}^{\mathrm{PD}} = x_0\partial_k - x_k\widetilde{p}_0 + s_{0k} + \frac{\varepsilon_{k\ell n}\hat{s}_{0\ell}\partial_n}{\widehat{\omega}+m},$$

where ε_{kln} is the Levi-Chivitta tensor ($\varepsilon^{123} = +1$), and the operators $s_{\mu\nu}$, $\hat{s}_{\mu\nu} = V^- s_{\mu\nu}^{\mathrm{II}} V^+$ have the form

$$s_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right], \quad \mu, \nu = \overline{0,3},$$

$$\hat{s}_{\mu\nu} = \{\hat{s}_{01} = \frac{1}{2}i\gamma_2\hat{C}, \quad \hat{s}_{02} = -\frac{1}{2}\gamma_2\hat{C}, \quad \hat{s}_{03} = -\gamma^0\frac{\overrightarrow{\gamma}\cdot\overrightarrow{p}+m}{2\widehat{\omega}} = -\frac{H}{2\widehat{\omega}}, \quad (10.22)$$

$$\hat{s}_{12} = \frac{i}{2}, \quad \hat{s}_{31} = \frac{iH}{2\widehat{\omega}}\gamma_2\hat{C}, \quad \hat{s}_{23} = \frac{H}{2\widehat{\omega}}\gamma_2\hat{C}\}$$

(the part of the Lorentz spin operators from (10.22) is not pure matrix because they depends on the pseudodifferential operator $\widehat{\omega} \equiv \sqrt{-\Delta+m^2}$ well-defined in the space $S^{3,4}$). Of course, the Casimir operators for the $\mathcal{P}^{\mathrm{B}}$ generators (10.21) have the same final form (10.18), (10.19) as for the generators (10.17). The Dirac equation in the Bose representation of the γ-matrices ($\gamma^{\mu\mathrm{B}} = W_{\mathrm{B}}\gamma^\mu W_{\mathrm{B}}^{-1}$) is the Maxwell-type equation for a massive tensor-scalar field.

Note that the generators (10.17) without the additional terms s_{ln}^{II} (10.11) and $\vec{s}^{\mathrm{II}} = (s_{23}, s_{31}, s_{12})^{\mathrm{II}}$ directly coincide with the well-known generators (6.24), (6.25) of standard Fermi (spin 1/2) $\mathcal{P}^{\mathrm{F}}$ symmetries of the FW equation (similar situation occurs for the generators (10.21) taken without the terms including the operators $\hat{s}_{\mu\nu}$ from (10.22) – they coincide with the operators (4.15) of the induced $\mathcal{P}^{\mathrm{F}}$-representation (4.16)). These well-known forms determine the Fermi-case, while operators suggested here are related to the Bose interpretation of equations (4.21), (4.22), which is found here also to be possible. The only difference of our Fermi-case from the spin 1/2 generators in [21] is that we use the prime form of generators related to the real parameters of the Poincare group.

10.1.7. The Case of Zero Mass, Brief Remix

After the analysis of the arbitrary mass case presented here, the specific character of the zero mass case in Chapters 8, 9 becomes evident. For $m = 0$ the

analogs of the additional operators $\hat{s}_{\mu\nu}$ (10.22) are the local pure matrix Lie operators (see, e.g., formulae (9.9), (9.10), (9.12), (9.13)). Therefore, for the $m = 0$ case, the corresponding $\mathcal{P}^{B}$ generators are much simpler than those in (10.21) and not contain non-local terms (see (9.21)). Moreover, for the $m = 0$ case the appealing to the FW representation and extended Clifford–Dirac algebra is not necessary. In Chapter 9 all Bose symmetries of the massless Dirac equation are found in the standard Pauli–Dirac representation of the spinor field ψ on the basis of the ordinary Clifford–Dirac algebra and known [25, 26] PGI operators. Thus, the search for the additional bosonic symmetries of the massless Dirac equation is technically much easier. Therefore, our results in Chapters 9, 10, contain a lot of additional meaningful information in comparison with the results presented here for the nonzero mass. In Chapters 9, 10, the full consideration of the Fermi–Bose duality of the massless Dirac equation is given. Furthermore, the Fermi–Bose duality of the Maxwell equations with the gradient-type sources, which are the Bose partner of the massless Dirac equation, is investigated in details. Thus, it is evident that the application of the developed here methods to the case of $m = 0$ essentially extend the symmetry properties of the corresponding massless equations.

10.2.   Bosonic Solutions and Conservation Laws of the Dirac Equation with Nonzero Mass

10.2.1.   Bosonic Solutions of the Foldy–Wouthuysen and Dirac Equations

Here as the next step in the Fermi–Bose duality investigation we consider the bosonic solution of the Dirac (FW) equation. A bosonic solution of the FW equation (4.21) is found completely similarly to the procedure of construction of standard fermionic solution. We use essentially the proved above Bose symmetry of the FW equation (4.21), which is based on the Bose generators (10.13).

Thus, the bosonic solution is determined by some stationary diagonal complete set of operators of bosonic physical quantities for the spin s=(1,0)-multiplet in the FW representation, e.g., by the set momentum operator and spin projection operator taken in the Hermitian form. Corresponding spin projection

operator found from the bosonic set (10.13) is given by

$$\hat{s}^3 = i\check{s}^3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{10.23}$$

and the stationary diagonal complete set of operators consists of

$$(\widehat{\overrightarrow{p}} = -\nabla, \hat{s}^3). \tag{10.24}$$

The fundamental solutions of equation (4.21), which are the common eigen solutions of the bosonic complete set (10.24), have the form

$$\varphi_{\mathrm{kr}}^-(t, \overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-ikx} \mathrm{d_r}, \quad \varphi_{\mathrm{k\check{r}}}^+(t, \overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{ikx} \mathrm{d_{\check{r}}}, \tag{10.25}$$

where $kx = \omega t - \overrightarrow{k}\overrightarrow{x}$ and $\mathrm{d}_{\bar{\alpha}} = (\delta_{\bar{\alpha}}^{\bar{\beta}})$ are the Cartesian orts in the space $C^{\otimes 4} \subset H^{3,4}$, numbers $\mathrm{r} = (1,2)$, $\check{\mathrm{r}} = (3,4)$ mark the eigen values $(+1, -1, 0, \underline{0})$ of the operator $\hat{s}^3$ from (10.23).

The bosonic solutions of equation (4.21) are the generalized states belonging to the space $S^{3,4*}$; they form a complete orthonormalized system of bosonic states. Therefore, any bosonic physical state of the FW field ϕ from the dense in $H^{3,4}$ manifold $S^{3,4}$ (the general bosonic solution of the equation (4.21)) is uniquely presented in a form

$$\phi_{(1,0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k [\xi^{\mathrm{r}}(\overrightarrow{k}) \mathrm{d_r} e^{-ikx} + \xi^{*\check{\mathrm{r}}}(\overrightarrow{k}) \mathrm{d_{\check{r}}} e^{ikx}], \tag{10.26}$$

where $\xi(\overrightarrow{k})$ are the coefficients of expansion of the bosonic solution of the FW equation (4.21) with respect to the Cartesian basis $\mathrm{d}_{\bar{\alpha}} = (\delta_{\bar{\alpha}}^{\bar{\beta}})$ (5.103).

The interpretation of the amplitudes in general solution (10.26) follow from the equation on eigen vectors and eigenvalues of the operators (10.24)

$$\widehat{\overrightarrow{p}} e^{-ikx} \mathrm{d_r} = \overrightarrow{k} e^{-ikx} \mathrm{d_r}, \mathrm{r} = (1,2); \quad \widehat{\overrightarrow{p}} e^{ikx} \mathrm{d_{\check{r}}} = -\overrightarrow{k} e^{ikx} \mathrm{d_{\check{r}}}, \check{\mathrm{r}} = (3,4), \tag{10.27}$$

$$\hat{s}^3 \mathrm{d_1} = \mathrm{d_1}, \hat{s}^3 \mathrm{d_2} = -\mathrm{d_2}, \hat{s}^3 \mathrm{d_3} = 0, \hat{s}^3 \mathrm{d_4} = 0. \tag{10.28}$$

Thus, the functions $\xi^1(\overrightarrow{k}), \xi^2(\overrightarrow{k})$ are the momentum-spin amplitudes of the spin s=1 boson with the momentum $\widehat{\overrightarrow{p}}$ and spin projections $(1,-1)$, respectively. An attempt to interpret the functions $\xi^{*3}(\overrightarrow{k}), \xi^{*4}(\overrightarrow{k})$ inside the FW model leads to the result as follows. The function $\xi^{*3}(\overrightarrow{k})$ is the momentum-spin amplitude of the spin s=1 boson state with the momentum $-\overrightarrow{p}$ and spin projections 0. The function $\xi^{*4}(\overrightarrow{k})$ in (10.26) is the momentum-spin amplitude of the spinless boson with the momentum $-\widehat{\overrightarrow{p}}$. And if $\phi_{(1,0)}(x) \in S^{3,4}$, then the bosonic amplitudes $\xi(\overrightarrow{k})$ belong to the Schwartz complex-valued test function space too.

It is evident from the consideration in Chapters 4–7 that exact quantum-mechanical interpretation of the amplitudes should be given inside the quantum mechanics. Here it is the RCQM from Chapter 5. The RCQM of the spin s=(1,0) particle multiplet is considered in the subsection 5.8.1. The images of the operators (10.13) in RCQM have the form:

$$\breve{s}^1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & -1 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \breve{s}^2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad (10.29)$$

$$\breve{s}^3 = \begin{vmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}; \quad \left[\breve{s}^j, \breve{s}^\ell\right] = \varepsilon^{j\ell n} \breve{s}^n.$$

The transition from the canonical field FW representation (10.13) to the RCQM is fulfilled on the basis of operator v (2.49) (transformation (2.49) is valid for the prime anti-Hermitian operators). The quantum-mechanical Hermitian form of the operators (10.29) is as follows:

$$\breve{s}^1_{\text{QM}} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \breve{s}^2_{\text{QM}} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 1 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad (10.30)$$

$$\breve{s}^3_{\text{QM}} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}; \quad \left[\breve{s}^j_{\text{QM}}, \breve{s}^\ell_{\text{QM}}\right] = i\varepsilon^{j\ell n} \breve{s}^n_{\text{QM}}.$$

The general solution in RCQM is the consequence of the FW bosonic solution (10.26) and the transformation v (2.49) as well. It has the form

$$f_{(1,0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} \left[\xi^1(\overrightarrow{k})d_1 + \xi^2(\overrightarrow{k})d_2 + \xi^3(\overrightarrow{k})d_3 + \xi^4(\overrightarrow{k})d_4 \right].$$

(10.31)

Generators (10.30) are unitary related to the standard spin s=(1,0) SU(2) generators (5.171). Corresponding transformation is given by

$$\overset{\smile}{s}_{\text{QM}} = u \overrightarrow{s} u^{-1}; \quad u = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad u^{-1} = u^\dagger = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix};$$

(10.32)

$$uu^{-1} = u^{-1}u = 1.$$

The equations on eigen vectors and eigenvalues for the diagonal stationary complete set of operators have the form

$$\widehat{\overrightarrow{p}} e^{-ikx}d_{\bar{\alpha}} = \overrightarrow{k} e^{-ikx}d_{\bar{\alpha}}, \quad \bar{\alpha} = (1,2,3,4),$$

(10.33)

$$\overset{\smile}{s}^3_{\text{QM}}d_1 = d_1, \quad \overset{\smile}{s}^3_{\text{QM}}d_2 = -d_2, \quad \overset{\smile}{s}^3_{\text{QM}}d_3 = 0, \quad \overset{\smile}{s}^3_{\text{QM}}d_4 = 0.$$

(10.34)

Now we can give the quantum-mechanical interpretation of the amplitudes both in the RCQM solution (10.31) and in the canonical FW field solution (10.26) as well. The functions $\xi^1(\overrightarrow{k}), \xi^2(\overrightarrow{k}), \xi^3(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the spin s=1 boson with the momentum $\widehat{\overrightarrow{p}}$ and spin projections $(1,-1,0)$, respectively. The function $\xi^4(\overrightarrow{k})$ is the quantum-mechanical momentum-spin amplitude of the spinless boson with the momentum $\widehat{\overrightarrow{p}}$.

The main step in the proof of the bosonic nature of the solution (10.31) is the corresponding bosonic Poincaré symmetry of the set $\{f_{(1,0)}(x)\}$ of solutions (10.31).

Indeed, the Schrödinger–Foldy equation (5.98) and the set $\{f_{(1,0)}(x)\}$ of its solutions (10.31) are invariant with respect to the reducible unitary bosonic representation (5.22) of the Poincaré group $\mathcal{P}$. The corresponding 4×4 matrix-differential generators are given by (5.20), (5.21), where the spin s=(1,0) SU(2) generators $\overrightarrow{s} = (s^{\ell n})$ are given in (10.30).

The proofs of similar asserions are presented in the Chapter 5.

Moreover, the set $\{\phi_{(1,0)}(x)\}$ of solutions (10.26) is invariant with respect to the unitary bosonic representation of the group $\mathcal{P}$, which is determined by the generators (10.17) and the Casimir operators (10.18). Therefore, the Bargman–Wigner analysis of the Poincaré symmetry of the set $\{\phi_{(1,0)}(x)\}$ of solutions (10.26) completes the demonstration that it is the set of Bose states $\phi_{(1,0)}$ of the field ϕ, i.e., the s=(1,0) multiplet states. Hence, the existence of bosonic solutions of the FW equation is proved.

Note that the link between the fermionic amplitudes from the solutions (4.28), (4.30) and the bosonic amplitudes from (10.26) can be presented in the form of relations (42) in [6]. The formulas (42) from [6] are the consequences of application of operator (1.32) to the column of corresponding quantum-mechanical amplitudes.

The bosonic solution of the Dirac equation follows from the bosonic solution (10.26) of the FW equation $\psi = V^-\phi$ as a result of inverse FW transformation V^- from (4.24).

In the terms of bosonic quantum-mechanical momentum-spin amplitudes $\xi^\alpha(\overrightarrow{k})$ from (10.31) and (10.26) the bosonic spin (1,0) multiplet solution $\psi = V^-\phi$ of the Dirac equation (10.20) is given by

$$\psi_{(1,0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k [\xi^{\mathrm{r}}(\overrightarrow{k})\mathrm{v}_{\mathrm{r}} e^{-ikx} + \xi^{*\check{\mathrm{r}}}(\overrightarrow{k})\mathrm{v}_{\check{\mathrm{r}}} e^{ikx}], \quad \mathrm{r}=(1,2), \check{\mathrm{r}}=(3,4),$$
(10.35)

where the amplitudes and their quantum-mechanical interpretation are similar to (10.31), (10.26) and the 4-component spinors are the same as in the Dirac theory of fermionic doublet

$$\mathrm{v}_1(\overrightarrow{k}) = N \begin{vmatrix} \widetilde{\omega}+m \\ 0 \\ k^3 \\ k^1+ik^2 \end{vmatrix}, \quad \mathrm{v}_2(\overrightarrow{k}) = N \begin{vmatrix} 0 \\ \widetilde{\omega}+m \\ k^1-ik^2 \\ -k^3 \end{vmatrix}, \tag{10.36}$$

$$\mathrm{v}_3(\overrightarrow{k}) = N \begin{vmatrix} k^3 \\ k^1+ik^2 \\ \widetilde{\omega}+m \\ 0 \end{vmatrix}, \quad \mathrm{v}_4(\overrightarrow{k}) = N \begin{vmatrix} k^1-ik^2 \\ -k^3 \\ 0 \\ \widetilde{\omega}+m \end{vmatrix},$$

and notation N is as follows $N \equiv \dfrac{1}{\sqrt{2\widetilde{\omega}(\widetilde{\omega}+m)}}$.

Finally, the bosonic nature of the solution (10.35) is proved on the basis of bosonic representation (10.21) of the Poincaré group $\mathcal{P}$, with respect to which the Dirac equation (10.20) is invariant. Indeed, the Casimir operators for the representation (10.21) have the bosonic eigenvalues.

Note that the Bose solutions of the Dirac equation can be found without the framework of our program of *synthesis of arbitrary spin covariant particle equations without redundant components*, which is the main content of the Chapters 5–7. Nevertheless, above such solutions are found inside this program as well. It is possible only after special procedure of transition (10.32) to the specific representation (10.30).

The reason is as follows. The program of such synthesis is developed for the particle-antiparticle doublets. It is evident that the spin s=(1,0) multiplet is the object of another nature. Therefore, in Chapters 5–7 we appeal to the 8-component form of the Dirac equation in comparison with bosonic spin s=(1.0.1,0) particle-antiparticle doublet, where the proof of the Fermi–Bose duality is fulfilled without transitions like (10.32), (10.30), and is the standard procedure of the step by step synthesis *RCQM* → *FW* → *Dirac-like model*.

10.2.2. Lagrangian for the Foldy–Wouthuysen Equation

Before the Noether analysis of conservation laws we must consider the Lagrange approach (L-approach) for the spinor field $\phi(x)$ in the FW representation. The L-approach in this representation has been formulated first in [40, 41]. Representation of the operator $\widehat{\omega} \equiv \sqrt{-\triangle + m^2}$ in a form of the series over the Laplace operator $\triangle$ powers has been used. Author applied a nonstandard formulation of the least action principle in the terms of infinite order derivatives from the field functions. We have a questions for such consideration.

Therefore, we present in [42] and below briefly a well-defined L-approach for the spinor field in the FW representation, which is based on the standard formulation of the least action principle. The quantum-mechanical rigged Hilbert space (both in the coordinate and momentum realizations of this space) is used, but the start is well-defined from momentum realization. In such realization rigged Hilbert space is given by

$$\widetilde{S}^{3,4} \subset \widetilde{H}^{3,4} \subset \widetilde{S}^{3,4*}; \quad \widetilde{H}^{3,4} = L_2(R_k^3) \otimes C^{\otimes 4}$$

$$= \{\phi = (\phi^\mu) : R_k^3 \to C^{\otimes 4}; \int d^3k |f(t, \overrightarrow{k})|^2 < \infty\}\}, \tag{10.37}$$

Here R_k^3 is the momentum operator $\overrightarrow{p}$ spectrum, which is canonically conjugated to the coordinate $\overrightarrow{x}$, $([x^j, p^l] = i\delta^{jl})$. Corresponding $\overrightarrow{x}$-realization is connected to (10.37) by 3-dimensional Fourier transformation. The alternative use of both realizations is based on the principle of heredity with classical and non-relativistic quantum mechanics of single mass point and with the mechanics of continuous media. The Lagrange function and the action (in alternative $\overrightarrow{x}$ or $\overrightarrow{k}$-realizations) are constructed in complete analogy with their consideration in the classical mechanics of a system with finite number of freedom degrees $q = (q_1, q_2, \ldots)$. The difference is only in the fact that here the continuous variable $\overrightarrow{k} \in R_k^3$ is the carrier of freedom degrees.

In the $\overrightarrow{k}$-realization, where this analogy is maximally clear, the Lagrange function has a form

$$L = L(\widetilde{\phi}, \widetilde{\phi}^\dagger, \widetilde{\phi}_{,0}, \widetilde{\phi}^\dagger_{,0}) = \frac{i}{2}[\widetilde{\phi}^\dagger(\widetilde{\phi}_{,0} + i\gamma^0\widetilde{\omega}\widetilde{\phi}) - (\widetilde{\phi}^\dagger_{,0} - i\widetilde{\omega}\widetilde{\phi}^\dagger\gamma^0)\widetilde{\phi}], \quad (10.38)$$

and in the $\overrightarrow{x}$-realization this function can be found from (10.38) by the Fourier transformation. The Euler–Lagrange equations coincide with the FW equation in both realizations. For example, in the $\overrightarrow{k}$-realization the Euler–Lagrange equations

$$\frac{\delta W}{\delta\widetilde{\phi}^\dagger} \equiv \frac{\partial L}{\partial\widetilde{\phi}^\dagger} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\widetilde{\phi}^\dagger_{,0}} = 0, \quad \frac{\delta W}{\delta\widetilde{\phi}} \equiv \frac{\partial L}{\partial\widetilde{\phi}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\widetilde{\phi}_{,0}} = 0, \quad (10.39)$$

coincide with the FW equation for the vectors $\widetilde{\phi} \in \widetilde{H}^{3,4}$:

$$(i\partial_0 - \gamma^0\widetilde{\omega})\widetilde{\phi}(t, \overrightarrow{k}) = 0; \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \ \overrightarrow{k} \in R_k^3, \quad (10.40)$$

and with conjugated equation for $\widetilde{\phi}^\dagger$.

The well defined L-approach for the FW field becomes essentially actual problem after the construction in [43] of the quantum electrodynamics in the FW representation.

10.2.3. The Fermi–Bose Conservation Laws for the Spinor Field

Note briefly the existence of the Fermi–Bose conservation laws for the spinor field. It is preferable to calculate them in the FW (not local Pauli–Dirac) representation too. In FW representation the Fermi spin $\overrightarrow{s}$ from (4.14) (together

with the "boost spin") is the independent symmetry operator for the FW equation. The orbital angular momentum and pure Lorentz angular momentum (the carriers of external statistical degrees of freedom) are in this representations the independent symmetry operators too (one can find the corresponding independent spin and angular momentum symmetries in the Pauli–Dirac representation for the Dirac equation too, but the corresponding operators are essentially non-local). Hence, one obtains 10 Poincaré and 12 additional (3 spin, 3 pure Lorentz spin, 3 angular momentum, 3 pure angular momentum) conservation laws.

Therefore, in the FW representation one can find very easily the 22 fermionic and 22 bosonic conservation laws. Separation into bosonic and fermionic set is caused by the existence of the Fermi–Bose symmetries and solutions. Indeed, if substitution of bosonic $\mathcal{P}$ generators q (10.17) and bosonic solutions (10.26) into the Noether formula

$$Q = \int d^3x \phi^\dagger(x) q \phi(x) \tag{10.41}$$

is made, then automatically the bosonic conservation laws for s=(1,0)-multiplet are obtained. Of course, the standard substitution of corresponding well-known fermionic generators and solutions gives fermionic conservation laws.

We illustrate briefly the difference in fermionic and bosonic conservation laws on the example of corresponding spin conservation. For the fermionic spin (4.14) in FW model and bosonic spin (10.13), (10.16) the conservation laws are given by

$$S_{mn}^{\mathrm{F}} = \int d^3x \phi^\dagger(x) s_{mn} \phi(x) = \int d^3k A^\dagger(\vec{k}) s_{mn} A(\vec{k})), \tag{10.42}$$

$$S_{mn}^{\mathrm{B}} = \int d^3x \phi^\dagger(x) \breve{s}_{mn} \phi(x) = \int d^3k B^\dagger(\vec{k}) \breve{s}_{mn} B(\vec{k})), \tag{10.43}$$

where

$$A(\vec{k}) = \mathrm{column}(a^1, a^2, a^{*3}, a^{*4}), \tag{10.44}$$

$$B(\vec{k}) = \mathrm{column}(\xi^1, \xi^2, \xi^{*3}, \xi^{*4}). \tag{10.45}$$

We present these conservation laws in terms of quantum-mechanical Fermi and Bose amplitudes. Such explicit quantum-statistical form is common for all integral conserved quantities.

10.2.4.   Conclusion Remarks

Fermi–Bose duality of the Dirac equation with nonzero mass is demonstrated. The property of the Fermi–Bose duality of the Dirac equation is proved above on the three levels: bosonic symmetries, bosonic solutions and corresponding conservation laws. The role of the Clifford algebra $C\ell^{\mathbb{R}}(0,6)$ and corresponded 28-dimensional Lie algebra SO(8) in the proof of these assertions is demonstrated.

We do not consider here the Pauli principle. Our results are not related to this concept. In any case we do not change the main well-known postulates and theory of the Fermi–Dirac and Bose–Einstein statistics. Our results have another, new principal, meaning. In our approach the Fermi–Bose duality of the spinor field found on the level of amplitude relations is proved in another way on the examples of the existence of the bosonic symmetries (section 10.1), solutions and conservation laws (section 10.2) of the Dirac equation with nonzero mass. It opens new possibilities of the Dirac equation application for the description of bosonic states. The specific case of zero mass is presented in the chapters 8, 9.

Thus, the property of the Fermi–Bose duality of the Dirac equation does not break the Fermi statistics for fermions (with the Pauli principle) and Bose statistics for bosons (with Bose condensation). We also never mixed the Fermi and Bose statistics between each other. Our assertion is following. One can apply with equal success both Fermi and Bose statistics for one and the same Dirac equation and one and the same spinor field, i.e., the Dirac equation can describe both fermionic and bosonic states.

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Appendices

Appendix I. The Complete Set of the Hydrogen Solutions (Angular Part) of the Equations (8.1)

$$\overset{I_0}{\bar E_A} = \overset{I}{\bar C}\ R_H^I P_{l^I}^{m+1} \cos(m+1)\phi,$$

$$\overset{I_1}{\bar E_B} = -\overset{I}{\bar C}\ \left(l^I - m + 1\right) R_E^I P_{l^I+1}^m \cos m\phi,$$

$$\overset{I_2}{\bar E_B} = \overset{I}{\bar C}\ \left(l^I - m + 1\right) R_E^I P_{l^I+1}^m \sin m\phi,$$

$$\overset{I_3}{\bar E_B} = -\overset{I}{\bar C}\ R_E^I P_{l^I+1}^{m+1} \cos(m+1)\phi,$$

$$\overset{I_0}{\bar H_B} = \overset{I}{\bar C}\ R_E^I P_{l^I+1}^{m+1} \sin(m+1)\phi,$$

$$\overset{I_1}{\bar H_A} = \overset{I}{\bar C}\ \left(l^I + m + 1\right) R_H^I P_{l^I}^m \sin m\phi,$$

$$\overset{I_2}{\bar H_A} = \overset{I}{\bar C}\ \left(l^I + m + 1\right) R_H^I P_{l^I}^m \cos m\phi,$$

$$\overset{I_3}{\bar H_A} = -\overset{I}{\bar C}\ R_H^I P_{l^I}^{m+1} \sin(m+1)\phi,$$

$$\overset{II_0}{\bar E_A} = \overset{II}{\bar C}\ R_H^{II} P_{l^{II}}^{m+1} \cos(m+1)\phi,$$

$$\overset{II_1}{\bar E_B} = \overset{II}{\bar C}\ \left(l^{II} + m\right) R_E^{II} P_{l^{II}-1}^m \cos m\phi,$$

$$\overset{II_2}{\bar E_B} = -\overset{II}{\bar C}\ \left(l^{II} + m\right) R_E^{II} P_{l^{II}-1}^m \sin m\phi,$$

$$\overset{II_3}{\bar E_B} = -\overset{II}{\bar C}\ R_E^{II} P_{l^{II}-1}^{m+1} \cos(m+1)\phi,$$

$$\overset{II_0}{\bar H_B} = \overset{II}{\bar C}\ R_E^{II} P_{l^{II}-1}^{m+1} \sin(m+1)\phi,$$

$$\overset{II_1}{\bar H_A} = -\overset{II}{\bar C}\ \left(l^{II} - m\right) R_H^{II} P_{l^{II}}^m \sin m\phi,$$

$$\overset{II_2}{\bar H_A} = -\overset{II}{\bar C}\ \left(l^{II} - m\right) R_H^{II} P_{l^{II}}^m \cos m\phi,$$

$$\overset{II_3}{\bar H_A} = -\overset{II}{\bar C}\ R_H^{II} P_{l^{II}}^{m+1} \sin(m+1)\phi,$$

$$\overset{+I_0}{E_A} = \overset{+I}{C} \left(l^I + m + 1 \right) R_H^I P_{l^I}^m \cos m\phi,$$

$$\overset{+I_1}{E_B} = \overset{+I}{C} R_E^I P_{l^I+1}^{m+1} \cos(m+1)\phi,$$

$$\overset{+I_2}{E_B} = \overset{+I}{C} R_E^I P_{l^I+1}^{m+1} \sin(m+1)\phi,$$

$$\overset{+I_3}{E_B} = - \overset{+I}{C} (l^I - m + 1) R_E^I P_{l^I+1}^m \cos m\phi,$$

$$\overset{+I_0}{H_B} = - \overset{+I}{C} \left(l^I - m + 1 \right) R_E^I P_{l^I+1}^m \sin m\phi,$$

$$\overset{+I_1}{H_A} = \overset{+I}{C} R_H^I P_{l^I}^{m+1} \sin(m+1)\phi,$$

$$\overset{+I_2}{H_A} = - \overset{+I}{C} R_H^I P_{l^I}^{m+1} \cos(m+1)\phi,$$

$$\overset{+I_3}{H_A} = \overset{+I}{C} (l^I + m + 1) R_H^I P_{l^I}^m \sin m\phi,$$

$$\overset{+II_0}{E_A} = - \overset{+II}{C} \left(l^{II} - m \right) R_H^{II} P_{l^{II}}^m \cos m\phi,$$

$$\overset{+II_1}{E_B} = \overset{+II}{C} R_E^{II} P_{l^{II}-1}^{m+1} \cos(m+1)\phi,$$

$$\overset{+II_2}{E_B} = \overset{+II}{C} R_E^{II} P_{l^{II}-1}^{m+1} \sin(m+1)\phi,$$

$$\overset{+II_3}{E_B} = \overset{+II}{C} (l^{II} + m) R_E^{II} P_{l^{II}-1}^m \cos m\phi,$$

$$\overset{+II_0}{H_B} = \overset{+II}{C} \left(l^{II} + m \right) R_E^{II} P_{l^{II}-1}^m \sin m\phi,$$

$$\overset{+II_1}{H_A} = \overset{+II}{C} R_H^{II} P_{l^{II}}^{m+1} \sin(m+1)\phi,$$

$$\overset{+II_2}{H_A} = - \overset{+II}{C} R_H^{II} P_{l^{II}}^{m+1} \cos(m+1)\phi,$$

$$\overset{+II_3}{H_A} = - \overset{+II}{C} (l^{II} - m) R_H^{II} P_{l^{II}}^m \sin m\phi,$$

Appendix II. The Complete Set of Charges for the Equations (8.1)

$$\overset{-I}{\rho_e} = - \overset{-I}{C} P_{l^I}^{m+1} \cos(m+1)\phi \left(\varepsilon R_E^I \right)_{,l^I+2},$$

$$\overset{-II}{\rho_e} = - \overset{-II}{C} P_{l^{II}}^{m+1} \cos(m+1)\phi \left(\varepsilon R_E^{II} \right)_{,-l^{II}+1},$$

$$\overset{+I}{\rho_e} = - \overset{+I}{C} (l^I + m + 1) P_{l^I}^m \cos m\phi \left(\varepsilon R_E^I \right)_{,l^I+2},$$

$$\overset{+II}{\rho_e} = \overset{+II}{C} (l^{II} - m) P_{l^{II}}^m \cos m\phi \left(\varepsilon R_E^{II} \right)_{,-l^{II}+1},$$

$$\overset{-I}{\rho}_{mag} = -\overset{-I}{C}\, P^{m+1}_{l^I+1} \sin(m+1)\phi \left(\mu R^I_H\right)_{,-l^I},$$

$$\overset{-II}{\rho}_{mag} = -\overset{-II}{C}\, P^{m+1}_{l^{II}-1} \sin(m+1)\phi \left(\mu R^{II}_H\right)_{,l^{II}+1},$$

$$\overset{+I}{\rho}_{mag} = \overset{+I}{C}\, (l^I-m+1) P^m_{l^I+1} \sin m\phi \left(\mu R^I_H\right)_{,-l^I},$$

$$\overset{+II}{\rho}_{mag} = -\overset{+II}{C}\, (l^{II}+m) P^m_{l^{II}-1} \sin m\phi \left(\mu R^{II}_H\right)_{,l^{II}+1},$$

where the following notations are used

$$\left(\varepsilon R^\beta_\alpha\right)_{,l+2} \equiv \varepsilon \left(\frac{d}{dr} + \frac{l+2}{r}\right) R^\beta_\alpha + R^\beta_\alpha \frac{d\varepsilon}{dr}, \quad etc.$$

Appendix III. Verification of the Hydrogen Solutions

Our experience both in publications and conferences [1–7] shows that for the readers it is not so easy to believe in the possibility of using classical electrodynamics in medium instead of relativistic quantum mechanics, especially for such a fundamental problem as the description of hydrogen spectrum, which, as we can recall from the history (see, e.g., [8]), was the birthplace of quantum mechanics. So our proof here must be completely clear. The best thing in such situation is the direct verification made by direct substitution of solutions into the equations in order to demonstrate them being transformed into identities. Moreover, it can help the readers.

Let us make the substitution of solutions (8.19)–(8.21) into the equations (8.1). We verify here only the first component of the first curl-equation in (8.11) (and only for one set of the solutions)

$$\partial_2 H^3 - \partial_3 H^2 - \tilde\omega \varepsilon E^1 - \partial_1 E^0 = 0, \tag{10.45}$$

– the verification of another components is similar. We are using the specific formulae in spherical coordinates which can be easily found with the help of the well-known recurrence relations for spherical harmonics. They are:

$$\partial_1 \left[R^I_H P^{m+1}_l \cos(m+1)\phi\right] =$$
$$= \frac{1}{2l+1} \cos\phi \cos(m+1)\phi \left(R^I_{H,l+1} P^{m+2}_{l-1} - R^I_{H,-l} P^{m+2}_{l+1}\right) + \tag{10.45}$$
$$+ \frac{m+1}{\sin\theta} P^{m+1}_l \frac{R^I_H}{r} \cos m\phi,$$

$$\partial_2 \left[R_H^I P_l^{m+1} \sin(m+1)\phi \right] =$$
$$= \tfrac{1}{2l+1} \sin\phi \sin(m+1)\phi \left(R_{H,l+1}^I P_{l-1}^{m+2} - R_{H,-l}^I P_{l+1}^{m+2} \right) + \qquad (10.45)$$
$$+ \tfrac{m+1}{\sin\theta} P_l^{m+1} \tfrac{R_H^I}{r} \cos m\phi,$$

$$\partial_3 \left[(l+m+1) R_H^I P_l^m \cos m\phi \right] =$$
$$= (l+m+1) \tfrac{\cos m\phi}{2l+1} \left[(l+m) R_{H,l+1}^I P_{l-1}^m + (l-m+1) R_{H,-l}^I P_{l+1}^m \right]. \qquad (10.45)$$

With the help of these formulae and a simple trigonometric formula equation (1) is transformed to the form:

$$\cos m\phi \left[\begin{array}{c} \tilde\omega\varepsilon(l-m+1) R_E^I P_{l+1}^m - \\ -\tfrac{1}{2l+1} \left(R_{H,l+1}^I P_{l-1}^{m+2} - R_{H,-l}^I P_{l+1}^{m+2} \right) - 2\tfrac{m+1}{\sin\theta} P_l^{m+1} \tfrac{R_H^I}{r} - \\ -\tfrac{(l+m+1)}{2l+1} \left[(l+m) R_{H,l+1}^I P_{l-1}^m + (l-m+1) R_{H,-l}^I P_{l+1}^m \right] \end{array} \right] = 0. \qquad (10.45)$$

It is useful for a moment to refuse in equation (5) from the notations (8.17) and to apply the formula

$$2(2l+1)\tfrac{m+1}{\sin\theta} P_l^{m+1} = -(l+1) P_{l-1}^{m+2} - l P_{l+1}^{m+2} -$$
$$-(l+1)(l+m+1)(l+m) P_{l-1}^m - l(l-m)(l-m+1) P_{l+1}^m, \qquad (10.45)$$

which is the linear combination of the well-known recurrence relations for spherical harmonics. After that it is easy to come to the result

$$(l-m+1) P_{l+1}^m \cos(m\phi) \left[\varepsilon\omega R_E^I - R_{H,-l}^I \right] = 0, \qquad (10.45)$$

which consists of a nonzero expression and the first one of the radial Dirac equations (8.16). Q. E. D.

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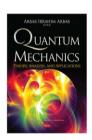
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