CLASSICAL MECHANICS

SOLUTIONS MANUAL

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Contents

Chapter One

The algebra and calculus of vectors

Problem 1.1

In terms of the standard basis set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 4\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}$.

(i) Find $3\boldsymbol{a} + 2\boldsymbol{b} - 4\boldsymbol{c}$ and $|\boldsymbol{a} - \boldsymbol{b}|^2$.
(ii) Find $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $\boldsymbol{a} \cdot \boldsymbol{b}$. Deduce the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
(iii) Find the component of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ and in the direction of $\boldsymbol{b}$.
(iv) Find $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{b}\times\boldsymbol{c}$ and $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})$.
(v) Find $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ and $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ and verify that they are equal. Is the set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ right- or left-handed?
(vi) By evaluating each side, verify the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$.

Solution

(i)

$$\begin{aligned} 3\boldsymbol{a} + 2\boldsymbol{b} - 4\boldsymbol{c} &= 3(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) + 2(3\boldsymbol{i} - 4\boldsymbol{k}) - 4(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}) \\ &= 8\boldsymbol{i} + 17\boldsymbol{j} - 26\boldsymbol{k}. \blacksquare \end{aligned}$$

$$\begin{aligned} |\boldsymbol{a} - \boldsymbol{b}|^2 &= (\boldsymbol{a} - \boldsymbol{b})\cdot(\boldsymbol{a} - \boldsymbol{b}) \\ &= (-\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\cdot(-\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) \\ &= (-1)^2 + (-1)^2 + 2^2 = 6. \blacksquare \end{aligned}$$

(ii)

$$\begin{aligned} |\boldsymbol{a}|^2 &= \boldsymbol{a}\cdot\boldsymbol{a} \\ &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\cdot(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) \\ &= 2^2 + (-1)^2 + (-2)^2 = 9. \end{aligned}$$

Hence $|\boldsymbol{a}| = 3$. ■

$$\begin{aligned} |\boldsymbol{b}|^2 &= \boldsymbol{b}\cdot\boldsymbol{b} \\ &= (3\boldsymbol{i} - 4\boldsymbol{k})\cdot(3\boldsymbol{i} - 4\boldsymbol{k}) \\ &= 3^2 + (-4)^2 = 25. \end{aligned}$$

Hence $|\boldsymbol{b}| = 5$. ■

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b} &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\cdot(3\boldsymbol{i} - 4\boldsymbol{k}) \\ &= \big(2\times 3\big) + \big((-1)\times 0\big) + \big((-2)\times(-4)\big) \\ &= 14. \blacksquare \end{aligned}$$

The angle α between $\boldsymbol{a}$ and $\boldsymbol{b}$ is then given by

$$\begin{aligned}\cos\alpha &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\\ &= \frac{14}{3\times 5} = \frac{14}{15}.\end{aligned}$$

Thus $\alpha = \tan^{-1}\frac{14}{15}$. ■

(iii) The component of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ is

$$\begin{aligned}\boldsymbol{c}\cdot\widehat{\boldsymbol{a}} &= \boldsymbol{c}\cdot\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\right)\\ &= (\boldsymbol{i}-5\boldsymbol{j}+3\boldsymbol{k})\cdot\left(\frac{2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}}{|2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}|}\right)\\ &= \frac{\big(1\times 2\big)+\big((-5)\times(-1)\big)+\big(3\times(-2)\big)}{3}\\ &= \frac{1}{3}.\ ■\end{aligned}$$

The component of $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$ is

$$\begin{aligned}\boldsymbol{c}\cdot\widehat{\boldsymbol{b}} &= \boldsymbol{c}\cdot\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\\ &= (\boldsymbol{i}-5\boldsymbol{j}+3\boldsymbol{k})\cdot\left(\frac{3\boldsymbol{i}-4\boldsymbol{k}}{|3\boldsymbol{i}-4\boldsymbol{k}|}\right)\\ &= \frac{\big(1\times 3\big)+\big((-5)\times 0\big)+\big(3\times(-4)\big)}{5}\\ &= -\frac{9}{5}.\ ■\end{aligned}$$

(iv)

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b} &= (2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})\times(3\boldsymbol{i}-4\boldsymbol{k})\\ &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -1 & -2\\ 3 & 0 & -4\end{vmatrix}\\ &= \big(4-0\big)\boldsymbol{i}-\big((-8)-(-6)\big)\boldsymbol{j}+\big(0-(-3)\big)\boldsymbol{k}\\ &= 4\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}.\ ■\end{aligned}$$

$$\begin{aligned}\boldsymbol{b}\times\boldsymbol{c} &= (3\boldsymbol{i} - 4\boldsymbol{k})\times(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 3 & 0 & -4\\ 1 & -5 & 3\end{vmatrix}\\ &= \big(0-20\big)\boldsymbol{i} - \big(9-(-4)\big)\boldsymbol{j} + \big((-15)-0\big)\boldsymbol{k}\\ &= -20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k}. \blacksquare\end{aligned}$$

Hence

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c}) &= (4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})\times(-20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k})\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 4 & 2 & 3\\ -20 & -13 & -15\end{vmatrix}\\ &= \big((-30)-(-39)\big)\boldsymbol{i} - \big((-60)-(-60)\big)\boldsymbol{j} + \big((-52)-(-40)\big)\boldsymbol{k}\\ &= 9\boldsymbol{i} - 12\boldsymbol{k}. \blacksquare\end{aligned}$$

(v)

$$\begin{aligned}\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\cdot(-20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k})\\ &= \big(2\times(-20)\big) + \big((-1)\times(-13)\big) + \big((-2)\times(-15)\big)\\ &= 3.\end{aligned}$$

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} &= (4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})\cdot(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})\\ &= \big(4\times 1\big) + \big(2\times(-5)\big) + \big(3\times 3\big)\\ &= 3.\end{aligned}$$

These values are equal and this **verifies the identity**

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}.$$

Since $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is *positive*, the set $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$ must be **right-handed**. ■

(vi) The **left side** of the identity is

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\times(-20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k})\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -1 & -2\\ -20 & -13 & -15\end{vmatrix}\\ &= \big(15-26\big)\boldsymbol{i} - \big((-30)-40\big)\boldsymbol{j} + \big((-26)-20\big)\boldsymbol{k}\\ &= -11\boldsymbol{i} + 70\boldsymbol{j} - 46\boldsymbol{k}.\end{aligned}$$

Since

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} &= \Big(\big(2\times 1\big) + \big((-1)\times(-5)\big) + \big((-2)\times 3\big)\Big)\boldsymbol{b}\\
&= \boldsymbol{b}\\
&= 3\boldsymbol{i} - 4\boldsymbol{k},
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c} &= \Big(\big(2\times 3\big) + \big((-1)\times 0\big) + \big((-2)\times(-4)\big)\Big)\boldsymbol{c}\\
&= 14\boldsymbol{c} = 14(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})\\
&= 14\boldsymbol{i} - 70\boldsymbol{j} + 42\boldsymbol{k},
\end{aligned}
$$

the **right side** of the identity is

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} &= (3\boldsymbol{i} - 4\boldsymbol{k}) - (14\boldsymbol{i} - 70\boldsymbol{j} + 42\boldsymbol{k})\\
&= -11\boldsymbol{i} + 70\boldsymbol{j} - 46\boldsymbol{k}.
\end{aligned}
$$

Thus the right and left sides are equal and this **verifies the identity**. ■

Problem 1.2

Find the angle between any two diagonals of a cube.

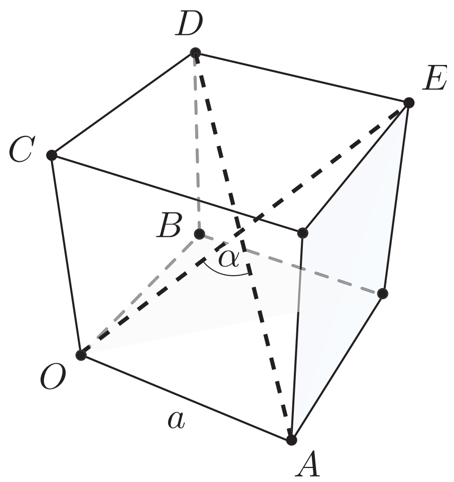

FIGURE 1.1 Two diagonals of a cube.

Solution

Figure 1.1 shows a cube of side a; OE and AD are two of its diagonals. Let O be the origin of position vectors and suppose the points A, B and C have position vectors $a\boldsymbol{i}$, $a\boldsymbol{j}$, $a\boldsymbol{k}$ respectively. Then the line segment $\overrightarrow{OE}$ represents the vector

$$a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}$$

and the line segment $\overrightarrow{AD}$ represents the vector

$$(a\boldsymbol{j} + a\boldsymbol{k}) - a\boldsymbol{i} = -a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}.$$

Let α be the angle between OE and AD. Then

$$\begin{aligned}\cos\alpha &= \frac{(a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k})\cdot(-a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k})}{|a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}||-a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}|}\\ &= \frac{-a^2 + a^2 + a^2}{(\sqrt{3}a)(\sqrt{3}a)} = \frac{1}{3}.\end{aligned}$$

Hence the **angle between the diagonals** is $\cos^{-1}\frac{1}{3}$, which is approximately $70.5°$. ■

Problem 1.3

$ABCDEF$ is a regular hexagon with centre O which is also the origin of position vectors. Find the position vectors of the vertices C, D, E, F in terms of the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ of A and B.

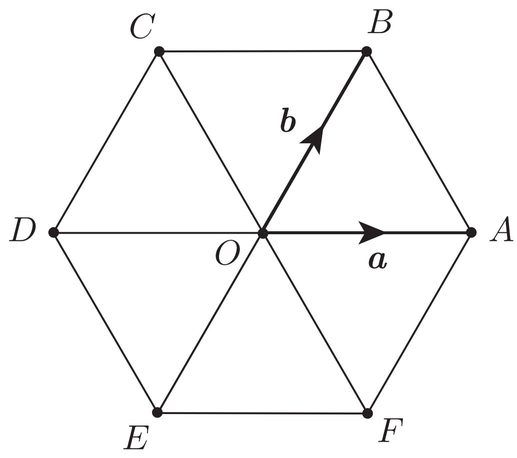

FIGURE 1.2 $ABCDEF$ is a regular hexagon.

Solution

(i) The position vector $\boldsymbol{c}$ is represented by the line segment $\overrightarrow{OC}$ which has the same magnitude and direction as the line segment $\overrightarrow{AB}$. Hence

$$\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a}. \blacksquare$$

(ii) The position vector $\boldsymbol{d}$ is represented by the line segment $\overrightarrow{OD}$ which has the same magnitude as, but *opposite* direction to, the line segment $\overrightarrow{OA}$. Hence

$$\boldsymbol{d} = -\boldsymbol{a}. \blacksquare$$

(iii) The position vector $\boldsymbol{e}$ is represented by the line segment $\overrightarrow{OE}$ which has the same magnitude as, but *opposite* direction to, the line segment $\overrightarrow{OB}$. Hence

$$\boldsymbol{e} = -\boldsymbol{b}. \blacksquare$$

(iv) The position vector $\boldsymbol{f}$ is represented by the line segment $\overrightarrow{OF}$ which has the same magnitude as, but *opposite* direction to, the line segment $\overrightarrow{AB}$. Hence

$$\boldsymbol{e} = -(\boldsymbol{b} - \boldsymbol{a}) = \boldsymbol{a} - \boldsymbol{b}. \blacksquare$$

Problem 1.4

Let $ABCD$ be a general (skew) quadrilateral and let P, Q, R, S be the mid-points of the sides AB, BC, CD, DA respectively. Show that $PQRS$ is a parallelogram.

Solution

Let the points A, B, C, D have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ relative to some origin O. Then the position vectors of the points P, Q, R, S are given by

$$\boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}), \quad \boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}), \quad \boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d}), \quad \boldsymbol{s} = \tfrac{1}{2}(\boldsymbol{d} + \boldsymbol{a}).$$

Now the line segment $\overrightarrow{PQ}$ represents the vector

$$\boldsymbol{q} - \boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) - \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{c} - \boldsymbol{a}),$$

and the line segment $\overrightarrow{SR}$ represents the vector

$$\boldsymbol{r} - \boldsymbol{s} = \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d}) - \tfrac{1}{2}(\boldsymbol{d} + \boldsymbol{a}) = \tfrac{1}{2}(\boldsymbol{c} - \boldsymbol{a}).$$

The lines PQ and SR are therefore parallel. Similarly, the lines QR and PS are parallel. The quadrilateral $PQRS$ is therefore a **parallelogram**. ■

Problem 1.5

In a general tetrahedron, lines are drawn connecting the mid-point of each side with the mid-point of the side opposite. Show that these three lines meet in a point that bisects each of them.

Solution

Let the vertices of the tetrahedron be A, B, C, D and suppose that these points have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ relative to some origin O. Then X, the mid-point of AB, has position vector

$$\boldsymbol{x} = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}),$$

and Y, the mid-point of CD, has position vector

$$\boldsymbol{y} = \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d}).$$

Hence the mid-point of XY has position vector

$$\tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) = \tfrac{1}{2}\Big(\tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) + \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d})\Big) = \tfrac{1}{4}\big(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}\big).$$

The mid-points of the other two lines that join the mid-points of opposite sides of the tetrahedron are found to have the same position vector. These three points are therefore coincident. Hence *the three lines that join the mid-points of opposite sides of the tetrahedron meet in a point that bisects each of them*. ■

Problem 1.6

Let $ABCD$ be a general tetrahedron and let P, Q, R, S be the median centres of the faces opposite to the vertices A, B, C, D respectively. Show that the lines AP, BQ, CR, DS all meet in a point (called the *centroid* of the tetrahedron), which divides each line in the ratio 3:1.

Solution

Let the vertices of the tetrahedron be A, B, C, D and suppose that these points have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ respectively, relative to some origin O. Then P, the median centre of the face BCD has position vector

$$\boldsymbol{p} = \tfrac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}).$$

The point that divides the line AP in the ratio 3:1 therefore has position vector

$$\frac{\boldsymbol{a} + 3\boldsymbol{p}}{4} = \tfrac{1}{4}\big(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}\big).$$

The corresponding points on the other three lines that join the vertices of the tetrahedron to the median centres of the opposite faces are all found to have the same position vector. These four points are therefore coincident. Hence *the four lines that join the vertices of the tetrahedron to the median centres of the opposite faces meet in a point that divides each line in the ratio 3:1*. It is the same point as was constructed in Problem 1.5. ■

Problem 1.7

A number of particles with masses $m_1, m_2, m_3, \ldots$ are situated at the points with position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots$ relative to an origin O. The centre of mass G of the particles is defined to be the point of space with position vector

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

Show that if a different origin O' were used, this definition would still place G at the same point of space.

Solution

Suppose the line segment $\overrightarrow{OO'}$ (that connects the two origins) represents the vector $\boldsymbol{a}$. Then $\boldsymbol{r}'_1, \boldsymbol{r}'_2, \boldsymbol{r}'_3, \ldots$, the position vectors of the masses relative to the origin O', are given by the triangle law of addition to be

$$\boldsymbol{r}'_i = \boldsymbol{r}_i - \boldsymbol{a}.$$

The position vector of the centre of mass measured relative to O' is defined to be

$$\begin{aligned}
\boldsymbol{R}' &= \frac{m_1\boldsymbol{r}'_1 + m_2\boldsymbol{r}'_2 + m_3\boldsymbol{r}'_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \\
&= \frac{m_1(\boldsymbol{r}_1 - \boldsymbol{a}) + m_2(\boldsymbol{r}_2 - \boldsymbol{a}) + m_3(\boldsymbol{r}_3 - \boldsymbol{a}) + \cdots}{m_1 + m_2 + m_3 + \cdots} \\
&= \left(\frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}\right) - \boldsymbol{a} \\
&= \boldsymbol{R} - \boldsymbol{a}.
\end{aligned}$$

By the triangle law of addition, this defines the **same point of space** as before. ■

Problem 1.8

Prove that the three perpendiculars of a triangle are concurrent.

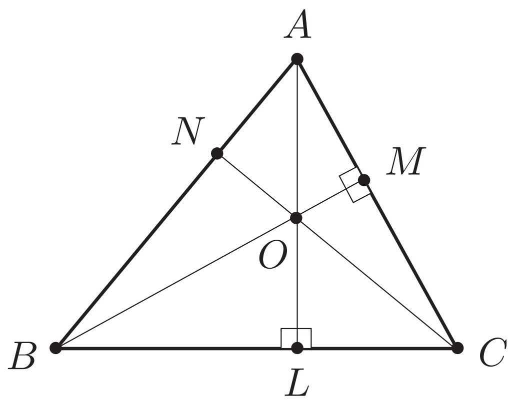

FIGURE 1.3 AL and BM are two of the perpendiculars of the triangle ABC.

Solution

Let ABC be the triangle and construct the perpendiculars AL and BM from A and B; let O be their point of intersection. Now construct the line CO and extend it to meet AB in the point N. We wish to show that CN is perpendicular to AB.

Suppose the points A, B, C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ relative to O. Then, since AL is perpendicular to BC, we have

$$\boldsymbol{a} \cdot (\boldsymbol{c} - \boldsymbol{b}) = 0,$$

and, since BM is perpendicular to CA, we have

$$\boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) = 0.$$

On adding these equalities, we obtain

$$\boldsymbol{c} \cdot (\boldsymbol{a} - \boldsymbol{b}) = 0,$$

which shows that the line CN is **perpendicular** to the side AB. ■

Problem 1.9

If $\boldsymbol{a}_1 = \lambda_1 \boldsymbol{i} + \mu_1 \boldsymbol{j} + \nu_1 \boldsymbol{k}$, $\boldsymbol{a}_2 = \lambda_2 \boldsymbol{i} + \mu_2 \boldsymbol{j} + \nu_2 \boldsymbol{k}$, $\boldsymbol{a}_3 = \lambda_3 \boldsymbol{i} + \mu_3 \boldsymbol{j} + \nu_3 \boldsymbol{k}$, where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis, show that

$$\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

Deduce that cyclic rotation of the vectors in a triple scalar product leaves the value of the product unchanged.

Solution

Since

$$\boldsymbol{a}_2 \times \boldsymbol{a}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}$$

$$= \boldsymbol{i} \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} \lambda_2 & \nu_2 \\ \lambda_3 & \nu_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix},$$

it follows that

$$\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \left(\lambda_1 \boldsymbol{i} + \mu_1 \boldsymbol{j} + \nu_1 \boldsymbol{k}\right) \cdot \left(\boldsymbol{i} \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} \lambda_2 & \nu_2 \\ \lambda_3 & \nu_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix} \right)$$

$$= \lambda_1 \begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} - \mu_1 \begin{vmatrix} \lambda_2 & \nu_2 \\ \lambda_3 & \nu_3 \end{vmatrix} + \nu_1 \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}. \blacksquare$$

Since the value of this determinant is unchanged a cyclic rotation of its rows, it follows that *the value of a triple scalar product is unchanged by a cyclic rotation of its vectors*. ■

Problem 1.10

By expressing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ in terms of a suitable standard basis, prove the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$.

Solution

The algebra in this solution is reduced by selecting a special basis set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ so that

$$\begin{aligned}\boldsymbol{a} &= a_1\boldsymbol{i},\\ \boldsymbol{b} &= b_1\boldsymbol{i} + b_2\boldsymbol{j},\\ \boldsymbol{c} &= c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}.\end{aligned}$$

Such a choice is always possible. Then

$$\begin{aligned}\boldsymbol{b}\times\boldsymbol{c} &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ b_1 & b_2 & 0\\ c_1 & c_2 & c_3\end{vmatrix}\\ &= \big(b_2c_3 - 0\big)\boldsymbol{i} - \big(b_1c_3 - 0\big)\boldsymbol{j} + \big(b_1c_2 - b_2c_1\big)\boldsymbol{k}\\ &= b_2c_3\,\boldsymbol{i} - b_1c_3\,\boldsymbol{j} + \big(b_1c_2 - b_2c_1\big)\boldsymbol{k}\end{aligned}$$

and hence the **left side** of the identity is

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & 0 & 0\\ b_2c_3 & -b_1c_3 & b_1c_2 - b_2c_1\end{vmatrix}\\ &= \big(0 - 0\big)\boldsymbol{i} - \big(a_1(b_1c_2 - b_2c_1) - 0\big)\boldsymbol{j} + \big(a_1(-b_1c_3) - 0\big)\boldsymbol{k}\\ &= a_1(b_2c_1 - b_1c_2)\boldsymbol{j} - a_1b_1c_3\,\boldsymbol{k}.\end{aligned}$$

The **right side** of the identity is

$$\begin{aligned}(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} &= (a_1c_1)\boldsymbol{b} - (a_1b_1)\boldsymbol{c}\\ &= a_1c_1\big(b_1\boldsymbol{i} + b_2\boldsymbol{j}\big) - a_1b_1\,(c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k})\\ &= a_1(b_2c_1 - b_1c_2)\boldsymbol{j} - (a_1b_1c_3)\boldsymbol{k}.\end{aligned}$$

Thus the right and left sides are equal and **this proves the identity**. ■

Problem 1.11

Prove the identities

(i) $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$

(ii) $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = [\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]\boldsymbol{d}$

(iii) $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{0}$ (Jacobi's identity)

Solution

(i)

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) &= \boldsymbol{a}\cdot\big(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})\big)\\ &= \boldsymbol{a}\cdot\big((\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d}\big)\\ &= (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}).\ \blacksquare\end{aligned}$$

(ii)

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) &= \big((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{d}\big)\boldsymbol{c} - \big((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\big)\boldsymbol{d}\\ &= [\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]\boldsymbol{d}.\ \blacksquare\end{aligned}$$

(iii)

$$\begin{aligned}&\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})\\ &= \big((\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\big) + \big((\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}\big) + \big((\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}\big)\\ &= \big(\boldsymbol{c}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{c}\big)\boldsymbol{a} + \big(\boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{c}\cdot\boldsymbol{a}\big)\boldsymbol{b} + \big(\boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b}\big)\boldsymbol{c}\\ &= \boldsymbol{0}.\ \blacksquare\end{aligned}$$

Problem 1.12 *Reciprocal basis*

Let $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ be any basis set. Then the corresponding **reciprocal basis** $\{\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*\}$ is defined by

$$\boldsymbol{a}^* = \frac{\boldsymbol{b}\times\boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}, \quad \boldsymbol{b}^* = \frac{\boldsymbol{c}\times\boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}, \quad \boldsymbol{c}^* = \frac{\boldsymbol{a}\times\boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}.$$

(i) If $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis, show that $\{\boldsymbol{i}^*, \boldsymbol{j}^*, \boldsymbol{k}^*\} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

(ii) Show that $[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*] = 1/[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$. Deduce that if $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a right handed set then so is $\{\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*\}$.

(iii) Show that $\{(\boldsymbol{a}^*)^*, (\boldsymbol{b}^*)^*, (\boldsymbol{c}^*)^*] = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$.

(iv) If a vector $\boldsymbol{v}$ is expanded in terms of the basis set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ in the form

$$\boldsymbol{v} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c},$$

show that the coefficients λ, μ, ν are given by $\lambda = \boldsymbol{v}\cdot\boldsymbol{a}^*$, $\mu = \boldsymbol{v}\cdot\boldsymbol{b}^*$, $\nu = \boldsymbol{v}\cdot\boldsymbol{c}^*$.

Solution

(i) If $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis, then

$$\begin{aligned}\boldsymbol{i}^* &= \frac{\boldsymbol{j}\times\boldsymbol{k}}{\boldsymbol{i}\cdot(\boldsymbol{j}\times\boldsymbol{k})} \\ &= \frac{\boldsymbol{i}}{\boldsymbol{i}\cdot\boldsymbol{i}} = \frac{\boldsymbol{i}}{1} \\ &= \boldsymbol{i}.\end{aligned}$$

Similar arguments hold for $\boldsymbol{j}^*$ and $\boldsymbol{k}^*$ and hence $\{\boldsymbol{i}^*, \boldsymbol{j}^*, \boldsymbol{k}^*\} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. ■

(ii)

$$\begin{aligned}[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*] &= \boldsymbol{a}^*\cdot\left(\boldsymbol{b}^*\times\boldsymbol{c}^*\right) \\ &= \boldsymbol{a}^*\cdot\left(\frac{\boldsymbol{c}\times\boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\times\frac{\boldsymbol{a}\times\boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\right) \\ &= \frac{\boldsymbol{a}^*}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]^2}\cdot\Big(((\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b})\,\boldsymbol{a} - ((\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{a})\,\boldsymbol{b}\Big) \\ &= \frac{\boldsymbol{b}\times\boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]^3}\cdot\Big([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} - \boldsymbol{0}\Big) \\ &= \frac{1}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}.\end{aligned}$$

If $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a right-handed basis set, then $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is positive. It follows that $[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*]$ must also be positive and therefore also **right-handed**. ■

(iii)

$$
\begin{aligned}
(\boldsymbol{a}^*)^* &= \frac{\boldsymbol{b}^* \times \boldsymbol{c}^*}{[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*]} \\
&= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] \left(\frac{\boldsymbol{c} \times \boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \times \frac{\boldsymbol{a} \times \boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right) \\
&= \frac{1}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \Big(((\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b})\, \boldsymbol{a} - ((\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{a})\, \boldsymbol{b} \Big) \\
&= \frac{1}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \Big([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} - \boldsymbol{0} \Big) \\
&= \boldsymbol{a}.
\end{aligned}
$$

Similar arguments hold for $(\boldsymbol{b}^*)^*$ and $(\boldsymbol{c}^*)^*$ and hence $\{(\boldsymbol{a}^*)^*, (\boldsymbol{b}^*)^*, (\boldsymbol{c}^*)^*\} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$. ■

(iv) Suppose $\boldsymbol{v}$ is expanded in terms of the basis set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ in the form

$$\boldsymbol{v} = \lambda \boldsymbol{a} + \mu \boldsymbol{b} + \nu \boldsymbol{c}.$$

On taking the scalar product of this equation with $\boldsymbol{a}^*$, we obtain

$$
\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{a}^* &= \lambda \boldsymbol{a} \cdot \boldsymbol{a}^* + \mu \boldsymbol{b} \cdot \boldsymbol{a}^* + \nu \boldsymbol{c} \cdot \boldsymbol{a}^* \\
&= \lambda \boldsymbol{a} \cdot \left(\frac{\boldsymbol{b} \times \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right) + \mu \boldsymbol{b} \cdot \left(\frac{\boldsymbol{b} \times \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right) + \nu \boldsymbol{c} \cdot \left(\frac{\boldsymbol{b} \times \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right) \\
&= \lambda + 0 + 0 \\
&= \lambda.
\end{aligned}
$$

Hence $\lambda = \boldsymbol{v} \cdot \boldsymbol{a}^*$, and, by similar arguments, $\mu = \boldsymbol{v} \cdot \boldsymbol{b}^*$ and $\mu = \boldsymbol{v} \cdot \boldsymbol{c}^*$. ■

Problem 1.13

Lamé's equations The directions in which X-rays are strongly scattered by a crystal are determined from the solutions $\boldsymbol{x}$ of Lamé's equations, namely

$$\boldsymbol{x}\cdot\boldsymbol{a} = L, \qquad \boldsymbol{x}\cdot\boldsymbol{b} = M, \qquad \boldsymbol{x}\cdot\boldsymbol{c} = N,$$

where $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ are the basis vectors of the crystal lattice, and L, M, N are *any* integers. Show that the solutions of Lamé's equations are

$$\boldsymbol{x} = L\,\boldsymbol{a}^* + M\,\boldsymbol{b}^* + N\,\boldsymbol{c}^*,$$

where $\{\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*\}$ is the reciprocal basis to $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$.

Solution

Let us seek solutions of Lamé's equations in the form

$$\boldsymbol{x} = \lambda\,\boldsymbol{a}^* + \mu\,\boldsymbol{b}^* + \nu\,\boldsymbol{c}^*,$$

where $\{\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*\}$ is the **reciprocal basis** corresponding to the lattice basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$. On substituting this expansion into Lamé's equations, we find that $\lambda = L$, $\mu = M$ and $\nu = N$. The only **solution of Lamé's equations** (corresponding to given values of L, M, N) is therefore

$$\boldsymbol{x} = L\,\boldsymbol{a}^* + M\,\boldsymbol{b}^* + N\,\boldsymbol{c}^*. \blacksquare$$

Problem 1.14

If $\boldsymbol{r}(t) = (3t^2 - 4)\boldsymbol{i} + t^3\boldsymbol{j} + (t+3)\boldsymbol{k}$, where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a constant standard basis, find $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$. Deduce the time derivative of $\boldsymbol{r} \times \dot{\boldsymbol{r}}$.

Solution

If $\boldsymbol{r}(t) = (3t^2 - 4)\boldsymbol{i} + t^3\boldsymbol{j} + (t+3)\boldsymbol{k}$, then

$$\begin{aligned} \dot{\boldsymbol{r}} &= 6t\,\boldsymbol{i} + 3t^2\boldsymbol{j} + \boldsymbol{k}, \\ \ddot{\boldsymbol{r}} &= 6\boldsymbol{i} + 6t\,\boldsymbol{j}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\left(\boldsymbol{r} \times \dot{\boldsymbol{r}}\right) &= \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} \\ &= \boldsymbol{0} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3t^2 - 4 & t^3 & t+3 \\ 6 & 6t & 0 \end{vmatrix} \\ &= -6t(t+3)\boldsymbol{i} + 6(t+3)\boldsymbol{j} + 12t(t^2-2)\boldsymbol{k}. \blacksquare \end{aligned}$$

Problem 1.15

The vector $\boldsymbol{v}$ is a function of the time t and $\boldsymbol{k}$ is a constant vector. Find the time derivatives of (i) $|\,\boldsymbol{v}\,|^2$, (ii) $(\boldsymbol{v}\cdot\boldsymbol{k})\,\boldsymbol{v}$, (iii) $[\,\boldsymbol{v},\dot{\boldsymbol{v}},\boldsymbol{k}\,]$.

Solution

(i)

$$\begin{aligned}\frac{d}{dt}|\,\boldsymbol{v}\,|^2 &= \frac{d}{dt}\big(\boldsymbol{v}\cdot\boldsymbol{v}\big)\\ &= \dot{\boldsymbol{v}}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\dot{\boldsymbol{v}}\\ &= 2\,\boldsymbol{v}\cdot\dot{\boldsymbol{v}}.\ \blacksquare\end{aligned}$$

(ii)

$$\begin{aligned}\frac{d}{dt}\big((\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{v}\big) &= \big(\dot{\boldsymbol{v}}\cdot\boldsymbol{k}+\boldsymbol{v}\cdot\dot{\boldsymbol{k}}\big)\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{k})\dot{\boldsymbol{v}}\\ &= (\dot{\boldsymbol{v}}\cdot\boldsymbol{k})\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{k})\dot{\boldsymbol{v}}.\ \blacksquare\end{aligned}$$

(iii)

$$\begin{aligned}\frac{d}{dt}[\,\boldsymbol{v},\dot{\boldsymbol{v}},\boldsymbol{k}\,] &= [\,\dot{\boldsymbol{v}},\dot{\boldsymbol{v}},\boldsymbol{k}\,]+[\,\boldsymbol{v},\ddot{\boldsymbol{v}},\boldsymbol{k}\,]+[\,\boldsymbol{v},\dot{\boldsymbol{v}},\dot{\boldsymbol{k}}\,]\\ &= 0+[\,\boldsymbol{v},\ddot{\boldsymbol{v}},\boldsymbol{k}\,]+0\\ &= [\,\boldsymbol{v},\ddot{\boldsymbol{v}},\boldsymbol{k}\,].\ \blacksquare\end{aligned}$$

Problem 1.16

Find the unit tangent vector, the unit normal vector and the curvature of the circle $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$ at the point with parameter θ.

Solution

Let $\boldsymbol{i}$, $\boldsymbol{j}$ be unit vectors in the directions Ox, Oy respectively. Then the vector form of the equation for the circle is

$$\boldsymbol{r} = a\cos\theta\,\boldsymbol{i} + a\sin\theta\,\boldsymbol{j}.$$

Then

$$\frac{d\boldsymbol{r}}{d\theta} = -a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j}$$

and so

$$\left|\frac{d\boldsymbol{r}}{d\theta}\right| = a.$$

The **unit tangent vector** to the circle is therefore

$$\boldsymbol{t}(\theta) = \frac{d\boldsymbol{r}}{d\theta}\bigg/\left|\frac{d\boldsymbol{r}}{d\theta}\right| = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}. \blacksquare$$

By the chain rule,

$$\frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}/d\theta}{ds/d\theta} = \frac{d\boldsymbol{t}/d\theta}{|d\boldsymbol{r}/d\theta|} = \frac{-\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}}{a}.$$

Hence the **unit normal vector** and **curvature** of the circle are given by

$$\boldsymbol{n}(\theta) = -\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}, \qquad \kappa(\theta) = \frac{1}{a}. \blacksquare$$

The **radius of curvature** of the circle is a. ■

Problem 1.17

Find the unit tangent vector, the unit normal vector and the curvature of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$ at the point with parameter θ.

Solution

Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors in the directions Ox, Oy, Oz respectively. Then the vector form of the equation for the helix is

$$\boldsymbol{r} = a\cos\theta\,\boldsymbol{i} + a\sin\theta\,\boldsymbol{j} + b\theta\,\boldsymbol{k}.$$

Then

$$\frac{d\boldsymbol{r}}{d\theta} = -a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j} + b\,\boldsymbol{k}$$

and so

$$\left|\frac{d\boldsymbol{r}}{d\theta}\right| = \left(a^2 + b^2\right)^{1/2}.$$

The **unit tangent vector** to the helix is therefore

$$\boldsymbol{t}(\theta) = \frac{d\boldsymbol{r}}{d\theta}\Big/\left|\frac{d\boldsymbol{r}}{d\theta}\right| = \frac{-a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j} + b\,\boldsymbol{k}}{\left(a^2 + b^2\right)^{1/2}}. \blacksquare$$

By the chain rule,

$$\frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}/d\theta}{ds/d\theta} = \frac{d\boldsymbol{t}/d\theta}{|d\boldsymbol{r}/d\theta|} = \frac{-a\cos\theta\,\boldsymbol{i} - a\sin\theta\,\boldsymbol{j}}{a^2 + b^2}.$$

Hence the **unit normal vector** and **curvature** of the helix are given by

$$\boldsymbol{n}(\theta) = -\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}, \qquad \kappa(\theta) = \frac{a}{a^2 + b^2} \blacksquare$$

The **radius of curvature** of the helix is $\left(a^2 + b^2\right)/a$. ■

Problem 1.18

Find the unit tangent vector, the unit normal vector and the curvature of the parabola $x = ap^2$, $y = 2ap$, $z = 0$ at the point with parameter p.

Solution

Let $\boldsymbol{i}$, $\boldsymbol{j}$ be unit vectors in the directions Ox, Oy respectively. Then the vector form of the equation for the parabola is

$$\boldsymbol{r} = ap^2\boldsymbol{i} + 2ap\boldsymbol{j}.$$

Then

$$\frac{d\boldsymbol{r}}{dp} = 2ap\boldsymbol{i} + 2a\boldsymbol{j} \qquad \text{and} \qquad \left|\frac{d\boldsymbol{r}}{dp}\right| = 2a\left(p^2+1\right)^{1/2}.$$

The **unit tangent vector** to the parabola is therefore

$$\begin{aligned}\boldsymbol{t}(p) &= \frac{d\boldsymbol{r}}{dp}\Bigg/\left|\frac{d\boldsymbol{r}}{dp}\right| \\ &= \frac{p\boldsymbol{i} + \boldsymbol{j}}{\left(p^2+1\right)^{1/2}}. \blacksquare\end{aligned}$$

By the chain rule,

$$\begin{aligned}\frac{d\boldsymbol{t}}{ds} &= \frac{d\boldsymbol{t}/dp}{ds/dp} = \frac{d\boldsymbol{t}/dp}{|d\boldsymbol{r}/dp|} \\ &= \frac{1}{2a\left(p^2+1\right)^{1/2}}\left(\frac{\boldsymbol{i}}{\left(p^2+1\right)^{1/2}} - \frac{p(p\boldsymbol{i}+\boldsymbol{j})}{\left(p^2+1\right)^{3/2}}\right) \\ &= \frac{\boldsymbol{i} - p\boldsymbol{j}}{2a\left(p^2+1\right)^{2}}.\end{aligned}$$

Hence the **unit normal vector** and **curvature** of the parabola are given by

$$\boldsymbol{n}(\theta) = \frac{\boldsymbol{i} - p\boldsymbol{j}}{\left(p^2+1\right)^{1/2}} \qquad \kappa(\theta) = \frac{1}{2a\left(p^2+1\right)^{3/2}}. \blacksquare$$

The **radius of curvature** of the parabola is $2a\left(p^2+1\right)^{3/2}$. ■

Chapter Two

Velocity, acceleration and scalar angular velocity

Problem 2.1

A particle P moves along the x-axis with its displacement at time t given by $x = 6t^2 - t^3 + 1$, where x is measured in metres and t in seconds. Find the velocity and acceleration of P at time t. Find the times at which P is at rest and find its position at these times.

Solution

Since the displacement of P at time t is

$$x = 6t^2 - t^3 + 1,$$

the **velocity** of P at time t is given by

$$v = \frac{dx}{dt} = 12t - 3t^2 \text{ m s}^{-1},$$

and the **acceleration** of P at time t is given by

$$a = \frac{dv}{dt} = 12 - 6t \text{ m s}^{-2}.$$

P is instantaneously **at rest** when $v = 0$, that is, when

$$12t - 3t^2 = 0.$$

This equation can be written in the form

$$3t(4 - t) = 0$$

and its solutions are therefore $t = 0$ s and $t = 4$ s.

When $t = 0$ s, the displacement of P is $x = 6(0^2) - 0^3 + 1 = 1$ m and when $t = 4$ s, the displacement of P is $x = 6(4^2) - 4^3 + 1 = 33$ m. ■

Problem 2.2

A particle P moves along the x-axis with its acceleration a at time t given by

$$a = 6t - 4 \text{ m s}^{-2}.$$

Initially P is at the point $x = 20$ m and is moving with speed 15 m s^{-1} in the negative x-direction. Find the velocity and displacement of P at time t. Find when P comes to rest and its displacement at this time.

Solution

Since the acceleration of P at time t is given to be

$$a = 6t - 4,$$

the velocity v of P at time t must satisfy the ODE

$$\frac{dv}{dt} = 6t - 4.$$

Integrating with respect to t gives

$$v = 3t^2 - 4t + C,$$

where C is a constant of integration. The initial condition that $v = -15$ when $t = 0$ gives

$$-15 = 3(0^2) - 4(0) + C,$$

from which $C = -15$. Hence the **velocity** of P at time t is

$$v = 3t^2 - 4t - 15 \text{ m s}^{-1}.$$

By writing $v = dx/dt$ and integrating again, we obtain

$$x = t^3 - 2t^2 - 15t + D,$$

where D is a second constant of integration. The initial condition that $x = 20$ when $t = 0$ gives

$$20 = 0^3 - 2(0^2) - 15(0) + D,$$

from which $D = 20$. Hence the **displacement** of P at time t is

$$x = t^3 - 2t^2 - 15t + 20 \text{ m.}$$

P comes to rest when $v = 0$, that is, when

$$3t^2 - 4t - 15 = 0.$$

This equation can be written in the form

$$(t - 3)(3t + 5) = 0$$

and its solutions are therefore $t = 3$ s and $t = -\frac{5}{3}$ s. The time $t = -\frac{5}{3}$ s is *before* the motion started and is therefore not a permissible solution. It follows that P **comes to rest** only when $t = 3$ s. The **displacement** of P at this time is

$$x = 3^3 - 2(3^2) - 15(3) + 20 = -16 \text{ m.} \blacksquare$$

Problem 2.3 *Constant acceleration formulae*

A particle P moves along the x-axis with *constant* acceleration a in the positive x-direction. Initially P is at the origin and is moving with velocity u in the positive x-direction. Show that the velocity v and displacement x of P at time t are given by

$$v = u + at, \qquad x = ut + \tfrac{1}{2}at^2,$$

and deduce that

$$v^2 = u^2 + 2ax.$$

In a standing quarter mile test, the Suzuki Bandit 1200 motorcycle covered the quarter mile (from rest) in 11.4 seconds and crossed the finish line doing 116 miles per hour. Are these figures consistent with the assumption of constant acceleration?

Solution

When the acceleration a is a *constant*, the ODE

$$\frac{dv}{dt} = a$$

integrates to give

$$v = at + C,$$

where C is a constant of integration. The initial condition $v = u$ when $t = 0$ gives

$$u = 0 + C,$$

from which $C = u$. Hence the **velocity** of P at time t is given by

$$v = u + at. \qquad (1)$$

On writing $v = dx/dt$ and integrating again, we obtain

$$x = ut + \tfrac{1}{2}at^2 + D,$$

where D is a second constant of integration. The initial condition $x = 0$ when $t = 0$ gives $D = 0$ so that the **displacement** of P at time t is given by

$$x = ut + \tfrac{1}{2}at^2. \qquad (2)$$

From equation (1),

$$\begin{aligned} v^2 &= (u + at)^2 \\ &= u^2 + 2uat + a^2t^2 \\ &= u^2 + 2a\left(ut + \tfrac{1}{2}at^2\right) \\ &= u^2 + 2ax, \end{aligned}$$

on using equation (1). We have thus obtained the relation

$$v^2 = u^2 + 2ax. \tag{3}$$

In the notation used above, the results of the Bandit test run were

$$\begin{aligned} u &= 0, & v &= 116 \text{ mph } (= 170 \text{ ft s}^{-1}), \\ x &= 1320 \text{ ft}, & t &= 11.4 \text{ s}, \end{aligned}$$

in Imperial units.

Suppose that the Bandit does have constant acceleration a. Then formula (1) gives

$$170 = 0 + 11.4\,a,$$

from which $a = 14.9$ ft s^{-2}. However, formula (2) gives

$$1320 = 0 + \tfrac{1}{2}a(11.4)^2$$

from which $a = 20.3$ ft s^{-2}. These two values for a do not agree and so the Bandit must have had **non constant acceleration**. ■

Problem 2.4

The trajectory of a charged particle moving in a magnetic field is given by

$$\boldsymbol{r} = b\cos\Omega t\,\boldsymbol{i} + b\sin\Omega t\,\boldsymbol{j} + ct\,\boldsymbol{k},$$

where b, Ω and c are positive constants. Show that the particle moves with constant speed and find the magnitude of its acceleration.

Solution

Since the position vector of P at time t is

$$\boldsymbol{r} = b\cos\Omega t\,\boldsymbol{i} + b\sin\Omega t\,\boldsymbol{j} + ct\,\boldsymbol{k},$$

the **velocity** of P at time t is given by

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -\Omega b\sin\Omega t\,\boldsymbol{i} + \Omega b\cos\Omega t\,\boldsymbol{j} + c\,\boldsymbol{k},$$

and the **acceleration** of P at time t is given by

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -\Omega^2 b\cos\Omega t\,\boldsymbol{i} - \Omega^2 b\sin\Omega t\,\boldsymbol{j}.$$

It follows that

$$\begin{aligned}|\boldsymbol{v}|^2 &= (-\Omega b\sin\Omega t)^2 + (\Omega b\cos\Omega t)^2 + c^2\\ &= \Omega^2 b^2\left(\sin^2\Omega t + \cos^2\Omega t\right) + c^2\\ &= \Omega^2 b^2 + c^2.\end{aligned}$$

Hence $|\boldsymbol{v}| = \left(\Omega^2 b^2 + c^2\right)^{1/2}$, which is a constant.

Furthermore,

$$\begin{aligned}|\boldsymbol{a}|^2 &= (-\Omega^2 b\cos\Omega t)^2 + (-\Omega^2 b\sin\Omega t)^2\\ &= \Omega^4 b^2\left(\cos^2\Omega t + \sin^2\Omega t\right)\\ &= \Omega^4 b^2.\end{aligned}$$

Hence $|\boldsymbol{a}| = \Omega^2 b$, which is also a constant. ■

Problem 2.5 *Acceleration due to rotation and orbit of the Earth*

A body is at rest at a location on the Earth's equator. Find its acceleration due to the Earth's rotation. [Take the Earth's radius at the equator to be 6400 km.]

Find also the acceleration of the Earth in its orbit around the Sun. [Take the Sun to be fixed and regard the Earth as a particle following a circular path with centre the Sun and radius 15×10^{10} m.

Solution

(i) The distance travelled by a body on the equator in one rotation of the Earth is $2\pi R$, where R is the Earth's radius. This distance is traversed in one day. The **speed** of the body is therefore

$$v = \frac{2\pi \times 6,400,000}{24 \times 60 \times 60} = 465 \text{ m s}^{-1},$$

in S.I. units. The **acceleration** of the body is directed towards the centre of the Earth and has magnitude

$$a = \frac{v^2}{R} = 0.034 \text{ m s}^{-2}. \blacksquare$$

(ii) The distance travelled by the Earth in one orbit of the Sun is $2\pi R$, where R is now the radius of the Earth's *orbit*. This distance is traversed in one year. The **speed** of the Earth in its orbit is therefore

$$v = \frac{2\pi \left(15 \times 10^{10}\right)}{365 \times 24 \times 60 \times 60} = 3.0 \times 10^4 \text{ m s}^{-1},$$

in S.I. units. The **acceleration** of the Earth is directed towards the Sun and has magnitude

$$a = \frac{v^2}{R} = 0.0060 \text{ m s}^{-2}. \blacksquare$$

Problem 2.6

An insect flies on a spiral trajectory such that its polar coordinates at time t are given by

$$r = be^{\Omega t}, \qquad \theta = \Omega t,$$

where b and Ω are positive constants. Find the velocity and acceleration vectors of the insect at time t, and show that the angle between these vectors is always $\pi/4$.

Solution

The **velocity** of the insect at time t is given by

$$\begin{aligned}\boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}\\ &= \left(\Omega b e^{\Omega t}\right)\widehat{\boldsymbol{r}} + \left(\Omega b e^{\Omega t}\right)\widehat{\boldsymbol{\theta}}\end{aligned}$$

and the **acceleration** of the insect at time t is given by

$$\begin{aligned}\boldsymbol{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}}\\ &= \left(\Omega^2 b e^{\Omega t} - \Omega^2 b e^{\Omega t}\right)\widehat{\boldsymbol{r}} + \left(0 + 2\Omega^2 b e^{\Omega t}\right)\widehat{\boldsymbol{\theta}}\\ &= 2\Omega^2 b e^{\Omega t}\,\widehat{\boldsymbol{\theta}}.\end{aligned}$$

It follows that

$$|\boldsymbol{v}| = \sqrt{2}\,\Omega b e^{\Omega t} \qquad \text{and} \qquad |\boldsymbol{a}| = 2\Omega^2 b e^{\Omega t}.$$

The **angle** α between $\boldsymbol{v}$ and $\boldsymbol{a}$ is then given by

$$\begin{aligned}\cos\alpha &= \frac{\boldsymbol{v}\cdot\boldsymbol{a}}{|\boldsymbol{v}||\boldsymbol{a}|}\\ &= \frac{2\Omega^3 b^2 e^{2\Omega t}}{\left(\sqrt{2}\,\Omega b e^{\Omega t}\right)\left(2\Omega^2 b e^{\Omega t}\right)}\\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

Hence the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ is always $\pi/4$. ■

Problem 2.7

A racing car moves on a circular track of radius b. The car starts from rest and its *speed* increases at a constant rate α. Find the angle between its velocity and acceleration vectors at time t.

Solution

Since the car has speed $v = \alpha t$ at time t, its **velocity** is

$$\boldsymbol{v} = v\,\widehat{\boldsymbol{\theta}} = \alpha t\,\widehat{\boldsymbol{\theta}}$$

and its **acceleration** is

$$\boldsymbol{a} = \left(-\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}} = \left(-\frac{\alpha^2 t^2}{b}\right)\widehat{\boldsymbol{r}} + \alpha\,\widehat{\boldsymbol{r}}.$$

The **angle** β between $\boldsymbol{v}$ and $\boldsymbol{a}$ is then given by

$$\begin{aligned}\cos\beta &= \frac{\boldsymbol{v}\cdot\boldsymbol{a}}{|\boldsymbol{v}||\boldsymbol{a}|}\\ &= \frac{\alpha^2 t}{\alpha t\left(\dfrac{\alpha^4 t^4}{b^2} + \alpha^2\right)^{1/2}}\\ &= \frac{b}{\left(b^2 + \alpha^2 t^4\right)^{1/2}}.\end{aligned}$$

The angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ at time t is therefore

$$\beta = \cos^{-1}\left(\frac{b}{\left(b^2 + \alpha^2 t^4\right)^{1/2}}\right). \blacksquare$$

Problem 2.8

A particle P moves on a circle with centre O and radius b. At a certain instant the speed of P is v and its acceleration vector makes an angle α with PO. Find the magnitude of the acceleration vector at this instant.

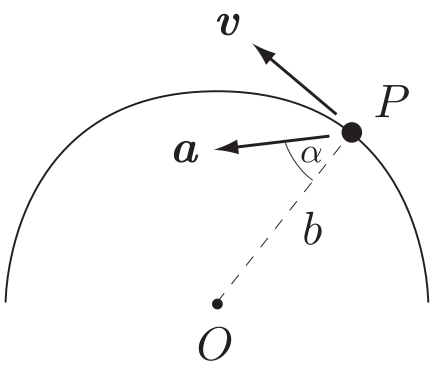

FIGURE 2.1 The velocity and acceleration vectors of the particle P.

Solution

In the standard notation, the **velocity** and **acceleration** vectors of P have the form

$$\boldsymbol{v} = v\widehat{\boldsymbol{\theta}},$$
$$\boldsymbol{a} = -\frac{v^2}{b}\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}},$$

where v is the *circumferential velocity* of P.

Consider the component of $\boldsymbol{a}$ in the direction $\overrightarrow{PO}$. This can be written in the geometrical form $|\boldsymbol{a}|\cos\alpha$ and also in the algebraic form $\boldsymbol{a}\cdot(-\widehat{\boldsymbol{r}})$. Hence

$$\begin{aligned}|\boldsymbol{a}|\cos\alpha &= \boldsymbol{a}\cdot(-\widehat{\boldsymbol{r}})\\ &= \left(-\frac{v^2}{b}\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}}\right)\cdot(-\widehat{\boldsymbol{r}})\\ &= \frac{v^2}{b}.\end{aligned}$$

It follows that

$$|\boldsymbol{a}| = \frac{v^2}{b\cos\alpha}. \blacksquare$$

Problem 2.9 ✻

A bee flies on a trajectory such that its polar coordinates at time t are given by

$$r = \frac{bt}{\tau^2}(2\tau - t) \qquad \theta = \frac{t}{\tau} \qquad (0 \le t \le 2\tau),$$

where b and τ are positive constants. Find the velocity vector of the bee at time t.

Show that the least speed achieved by the bee is b/τ. Find the acceleration of the bee at this instant.

Solution

The **velocity** vector of the bee is given by

$$\begin{aligned} \boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= \frac{2b}{\tau^2}(\tau - t)\,\widehat{\boldsymbol{r}} + \frac{bt}{\tau^3}(2\tau - t)\,\widehat{\boldsymbol{\theta}}. \end{aligned}$$

It follows that

$$\begin{aligned} |\boldsymbol{v}|^2 &= \frac{4b^2}{\tau^4}(\tau - t)^2 + \frac{b^2t^2}{\tau^6}(2\tau - t)^2 \\ &= \frac{b^2}{\tau^6}\left(t^4 - 4\tau t^3 + 8\tau^2 t^2 - 8\tau^3 t + 4\tau^4\right), \end{aligned}$$

after some simplification.

To find the maximum value of $|\boldsymbol{v}|$, consider the time derivative of $|\boldsymbol{v}|^2$.

$$\begin{aligned} \frac{d}{dt}|\boldsymbol{v}|^2 &= \frac{b^2}{\tau^6}\left(4t^3 - 12\tau t^2 + 16\tau^2 t - 8\tau^3\right) \\ &= \frac{4b^2}{\tau^6}(t - \tau)\left(t^2 - 2\tau t + 2\tau^2\right). \end{aligned}$$

The expression $t^2 - 2\tau t + 2\tau^2$ is always positive and hence

$$\frac{d}{dt}|\boldsymbol{v}|^2 \begin{cases} < 0 & \text{for} \quad t < \tau, \\ = 0 & \text{for} \quad t = \tau, \\ > 0 & \text{for} \quad t > \tau. \end{cases}$$

Hence $|\boldsymbol{v}|$ achieves its minimum value when $t = \tau$. At this instant,

$$|\boldsymbol{v}| = \frac{b}{\tau},$$

which is therefore the **minimum speed** of the bee.

The **acceleration** vector of the bee at time t is given by

$$\begin{aligned}\boldsymbol{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= \left(-\frac{2b}{\tau^2} - \frac{bt}{\tau^4}(2\tau - t)\right)\widehat{\boldsymbol{r}} + \left(0 + \frac{4b}{\tau^3}(\tau - t)\right)\widehat{\boldsymbol{\theta}} \\ &= -\frac{3b}{\tau^2}\widehat{\boldsymbol{r}},\end{aligned}$$

when $t = \tau$. Hence, when the speed of the bee is a minimum,

$$|\boldsymbol{a}| = \frac{3b}{\tau^2}. \blacksquare$$

Problem 2.10 ✻ *A pursuit problem: Daniel and the Lion*

The luckless Daniel (D) is thrown into a circular arena of radius a containing a lion (L). Initially the lion is at the centre O of the arena while Daniel is at the perimeter. Daniel's strategy is to run with his maximum speed u around the perimeter. The lion responds by running at its maximum speed U in such a way that it remains on the (moving) radius OD. Show that r, the distance of L from O, satisfies the differential equation

$$\dot{r}^2 = \frac{u^2}{a^2}\left(\frac{U^2a^2}{u^2} - r^2\right).$$

Find r as a function of t. If $U \geq u$, show that Daniel will be caught, and find how long this will take.

Show that the path taken by the lion is a circle. For the special case in which $U = u$, sketch the path taken by the lion and find the point of capture.

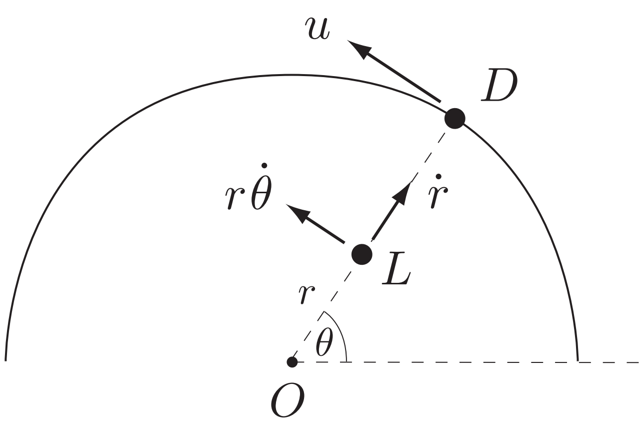

FIGURE 2.2 Daniel D is pursued by the lion L. The lion remains on the rotating radius OD.

Solution

Let the lion have polar coordinates r, θ as shown in Figure 2.2. Then the **velocity** vector of the lion is

$$\begin{aligned}\boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}\\ &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(\frac{ur}{a}\right)\widehat{\boldsymbol{\theta}},\end{aligned}$$

since the lion remains on the radius OD which is rotating with angular velocity $\dot{\theta} = u/a$. Since the lion is running with speed U, it follows that

$$\dot{r}^2 + \left(\frac{ur}{a}\right)^2 = U^2,$$

which can be written in the form

$$\dot{r}^2 = \frac{u^2}{a^2}\left(\frac{U^2a^2}{u^2} - r^2\right).$$

This is the **equation** satisfied by the radial coordinate $r(t)$.

On taking square roots and selecting the positive sign, we obtain

$$\dot{r} = \frac{u}{a}\left(\frac{U^2a^2}{u^2} - r^2\right)^{1/2},$$

which is a separable first order ODE. Separation gives

$$\begin{aligned}\frac{ut}{a} &= \int \left(\frac{U^2a^2}{u^2} - r^2\right)^{-1/2} dr\\ &= \sin^{-1}\left(\frac{ur}{Ua}\right) + C,\end{aligned}$$

where C is a constant of integration. The initial condition $r = 0$ when $t = 0$ gives $C = 0$ so that

$$\frac{ut}{a} = \sin^{-1}\left(\frac{ur}{Ua}\right),$$

that is,

$$r = \frac{Ua}{u}\sin\left(\frac{ut}{a}\right).$$

This is the **solution** for r as a function of t.

Daniel will be caught when $r = a$, that is, when

$$\sin\left(\frac{ut}{a}\right) = \frac{u}{U}.$$

If $U \geq u$, this equation has the real solution

$$t = \frac{a}{u}\sin^{-1}\frac{u}{U}$$

and so **Daniel will be caught** after this time.

Since $\theta = ut/a$, the polar equation of the **path** of the lion is

$$r = \frac{Ua}{u} \sin\theta.$$

In order to recognise this equation as a circle, we express it in Cartesian coordinates. This is made easier if both sides are multiplied by r. The equation then becomes

$$x^2 + y^2 = \left(\frac{Ua}{u}\right) y,$$

the standard form of which is

$$x^2 + \left(y - \frac{Ua}{2u}\right)^2 = \left(\frac{Ua}{2u}\right)^2.$$

This is a **circle** with centre at $(0, Ua/2u)$ and radius $Ua/2u$. Note that the lion does not traverse the full circle. Daniel will be caught when the lion has traversed an arc of length $(Ua/u)\sin^{-1}(u/U)$.

For the special case in which $U = u$ (that is, the lion and Daniel have the same speed) the path of the lion is

$$x^2 + \left(y - \tfrac{1}{2}a\right)^2 = \left(\tfrac{1}{2}a\right)^2,$$

which is a circle with centre at $(0, \frac{1}{2}a)$ and radius $\frac{1}{2}a$. Daniel will be caught when the lion has traversed half of this circle, as shown in Figure 2.3. The point of capture is $(0, a)$. ■

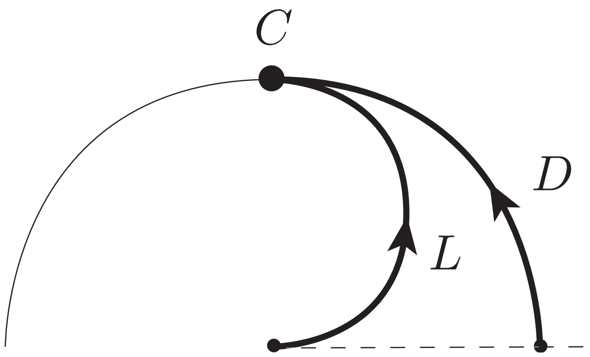

FIGURE 2.3 The paths of Daniel and the lion when $U = u$. C is the point of capture.

Problem 2.11 *General motion with constant speed*

A particle moves along any path in three-dimensional space with *constant speed*. Show that its velocity and acceleration vectors must always be perpendicular to each other. [*Hint*. Differentiate the formula $\boldsymbol{v} \cdot \boldsymbol{v} = v^2$ with respect to t.]

Solution

If P moves with constant speed v, its **velocity** vector $\boldsymbol{v}$ satisfies the equation

$$\boldsymbol{v} \cdot \boldsymbol{v} = v^2$$

at all times. On differentiating this equation with respect to t, we obtain

$$\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \dot{\boldsymbol{v}} = 0,$$

that is,

$$\boldsymbol{a} \cdot \boldsymbol{v} = 0,$$

where $\boldsymbol{a}$ $(= \dot{\boldsymbol{v}})$ is the **acceleration** vector of P. Hence the velocity and acceleration vectors of P must always be perpendicular to each other. ■

Problem 2.12

A particle P moves so that its position vector $\boldsymbol{r}$ satisfies the differential equation

$$\dot{\boldsymbol{r}} = \boldsymbol{c} \times \boldsymbol{r},$$

where $\boldsymbol{c}$ is a constant vector. Show that P moves with constant speed on a circular path. [*Hint*. Take the dot product of the equation first with $\boldsymbol{c}$ and then with $\boldsymbol{r}$.]

Solution

First we take the scalar product of the equation

$$\dot{\boldsymbol{r}} = \boldsymbol{c} \times \boldsymbol{r},$$

with $\boldsymbol{r}$. This gives

$$\begin{aligned} \boldsymbol{r} \cdot \dot{\boldsymbol{r}} &= \boldsymbol{r} \cdot (\boldsymbol{c} \times \boldsymbol{r}) \\ &= 0. \end{aligned}$$

This equation can be integrated with respect to t to give

$$\boldsymbol{r} \cdot \boldsymbol{r} = R^2,$$

where R is a positive constant. The motion of P is therefore restricted to the surface of a **sphere** $\mathcal{S}$ with centre O and radius R.

Second we take the scalar product of the equation

$$\dot{\boldsymbol{r}} = \boldsymbol{c} \times \boldsymbol{r},$$

with $\boldsymbol{c}$. This gives

$$\begin{aligned} \dot{\boldsymbol{r}} \cdot \boldsymbol{c} &= (\boldsymbol{c} \times \boldsymbol{r}) \cdot \boldsymbol{c} \\ &= 0. \end{aligned}$$

This equation can be integrated with respect to t to give

$$\boldsymbol{r} \cdot \boldsymbol{c} = \text{constant},$$

which can be expressed in the more standard form

$$\boldsymbol{r} \cdot \widehat{\boldsymbol{c}} = p,$$

where $\widehat{\boldsymbol{c}}$ is a *unit* vector parallel to $\boldsymbol{c}$ and p is a positive constant. The motion of P is therefore restricted to a **plane** $\mathcal{P}$ perpendicular to the vector $\boldsymbol{c}$ whose perpendicular distance from O is p.

It follows that P must move on the **circle** $\mathcal{C}$ that is the intersection of the sphere $\mathcal{S}$ and the plane $\mathcal{P}$. The axis $\{O, \boldsymbol{c}\}$ passes through the centre of the circle and is perpendicular to its plane.

Finally, if $\dot{\boldsymbol{r}} = \boldsymbol{v}$ and $\ddot{\boldsymbol{r}} = \boldsymbol{a}$, then

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{v}) &= 2\boldsymbol{v}\cdot\boldsymbol{a}\\ &= 2\boldsymbol{v}\cdot(\boldsymbol{c}\times\boldsymbol{v})\\ &= 0.\end{aligned}$$

Hence $\boldsymbol{v}\cdot\boldsymbol{v}$ is constant and so P moves along the circle $\mathcal{C}$ with **constant speed**. ■

Problem 2.13

A large truck with double rear wheels has a brick jammed between two of its tyres which are 4 ft in diameter. If the truck is travelling at 60 mph, find the maximum speed of the brick and the magnitude of its acceleration. [Express the acceleration as a multiple of $g = 32$ ft s^{-2}.]

Solution

From the theory of the rolling wheel (see the book pp. 38–40), the **maximum speed** of the brick is 120 mph and occurs when the brick is in its highest position.

The acceleration of the brick is the same as that measured in a reference frame moving with the truck. (In other words, we can disregard the *translational* motion of the wheel.) In Imperial units, the **acceleration** of the brick has magnitude

$$\begin{aligned} |\boldsymbol{a}| &= \frac{v^2}{b} \\ &= \frac{88^2}{2} = 3,872 \text{ ft s}^{-2} \\ &= 121g. \blacksquare \end{aligned}$$

Problem 2.14

A particle is sliding along a smooth radial grove in a circular turntable which is rotating with constant angular speed Ω. The distance of the particle from the rotation axis at time t is observed to be

$$r = b \cosh \Omega t$$

for $t \geq 0$, where b is a positive constant. Find the speed of the particle (relative to a fixed reference frame) at time t, and find the magnitude and direction of the acceleration.

Solution

Relative to a fixed reference frame, the polar coordinates of the particle at time t are

$$\begin{aligned} r &= b \cosh \Omega t \\ \theta &= \Omega t. \end{aligned}$$

The **velocity** vector of P is therefore

$$\begin{aligned} \boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= (\Omega b \sinh \Omega t)\,\widehat{\boldsymbol{r}} + (\Omega b \cosh \Omega t)\,\widehat{\boldsymbol{\theta}}. \end{aligned}$$

The **speed** of P is therefore given by

$$\begin{aligned} |\boldsymbol{v}|^2 &= \Omega^2 b^2 \sinh^2 \Omega t + \Omega^2 b^2 \cosh^2 \Omega t \\ &= \Omega^2 b^2 \cosh 2\Omega t. \end{aligned}$$

The **acceleration** vector of P is

$$\begin{aligned} \boldsymbol{v} &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= \left(\Omega^2 b \cosh \Omega t - \Omega^2 b \cosh \Omega t\right)\widehat{\boldsymbol{r}} + \left(0 + 2\Omega^2 b \sinh \Omega t\right)\widehat{\boldsymbol{\theta}} \\ &= \left(2\Omega^2 b \sinh \Omega t\right)\widehat{\boldsymbol{\theta}}. \end{aligned}$$

The acceleration of P is therefore $2\Omega^2 b \sinh \Omega t$ in the circumferential direction. ■

Problem 2.15

Book Figure 2.11 shows an eccentric circular cam of radius b rotating with constant angular velocity ω about a fixed pivot O which is a distance e from the centre C. The cam drives a valve which slides in a straight guide. Find the maximum speed and maximum acceleration of the valve.

Solution

The **displacement** x of the face of the valve from C is

$$\begin{aligned} x &= b + e\cos\theta \\ &= b + e\cos\omega t. \end{aligned}$$

The **velocity** v and **acceleration** a of the valve are therefore

$$\begin{aligned} v &= \frac{dx}{dt} = -\omega e \sin\omega t, \\ a &= \frac{dv}{dt} = -\omega^2 e\cos\omega t. \end{aligned}$$

Thus the **maximum speed** of the valve is ωe and the **maximum acceleration** is $\omega^2 e$. ■

Problem 2.16

Book Figure 2.12 shows a piston driving a crank OP pivoted at the end O. The piston slides in a straight cylinder and the crank is made to rotate with constant angular velocity ω. Find the distance OQ in terms of the lengths b, c and the angle θ. Show that, when b/c is small, OQ is given approximately by

$$OQ = c + b\cos\theta - \frac{b^2}{2c}\sin^2\theta,$$

on neglecting $(b/c)^4$ and higher powers. Using this approximation, find the maximum acceleration of the piston.

Solution

The **distance** OQ is given by

$$OQ = b\cos\theta + c\cos\phi,$$

where ϕ is the angle $O\widehat{Q}P$. By an application of the sine rule in the triangle OPQ,

$$\frac{\sin\phi}{b} = \frac{\sin\theta}{c}$$

so that $\sin\phi = (b/c)\sin\theta$ and

$$\cos\phi = \left(1 - \frac{b^2}{c^2}\sin^2\theta\right)^{1/2}.$$

Hence

$$\begin{aligned} OQ &= b\cos\theta + c\left(1 - \frac{b^2}{c^2}\sin^2\theta\right)^{1/2} \\ &= b\cos\theta + c\left(1 - \frac{b^2}{2c^2}\sin^2\theta + O\left(\frac{b}{c}\right)^4\right) \\ &= c + b\cos\theta - \frac{b^2}{2c}\sin^2\theta \end{aligned}$$

on neglecting $(b/c)^4$ and higher powers. In this approximation, the **displacement** OQ at time t is given by

$$\begin{aligned} x &= c + b\cos\omega t - \frac{b^2}{2c}\sin^2\omega t \\ &= c + b\cos\omega t - \frac{b^2}{4c}(1 - \cos 2\omega t). \end{aligned}$$

The **acceleration** of the piston at time t is therefore

$$a = \frac{d^2x}{dt^2} = -\omega^2 b \cos \omega t - \frac{b^2}{c} \cos 2\omega t$$

and the **maximum** value achieved by $|a|$ is

$$\omega^2 b \left(1 + \frac{b}{c}\right). \blacksquare$$

Problem 2.17

Book Figure 2.13 shows an epicyclic gear arrangement in which the ‘Sun’ gear $\mathcal{G}_1$ of radius b_1 and the ‘ring’ gear $\mathcal{G}_2$ of inner radius b_2 rotate with angular velocities ω_1, ω_2 respectively about their fixed common centre O. Between them they grip the ‘planet’ gear $\mathcal{G}$, whose centre C moves on a circle centre O. Find the circumferential velocity of C and the angular velocity of the planet gear $\mathcal{G}$. If O and C were connected by an arm pivoted at O, what would be the angular velocity of the arm?

Solution

Let v be the velocity of the centre C of the planet gear $\mathcal{G}$ and let ω be its angular velocity. Note that the radius b of the planet gear is $\frac{1}{2}(b_2 - b_1)$. Then the rolling condition at the point of contact of $\mathcal{G}$ and the Sun gear $\mathcal{G}_1$ gives

$$\begin{aligned}\omega_1 b_1 &= v - \omega b\\ &= v - \tfrac{1}{2}\omega(b_2 - b_1).\end{aligned}$$

The rolling condition at the point of contact of $\mathcal{G}$ and the ring gear $\mathcal{G}_2$ gives

$$\begin{aligned}\omega_2 b_2 &= v + \omega b\\ &= v + \tfrac{1}{2}\omega(b_2 - b_1).\end{aligned}$$

It follows that the planet gear has **velocity**

$$v = \tfrac{1}{2}(\omega_1 b_1 + \omega_2 b_2)$$

and **angular velocity**

$$\omega = \frac{\omega_2 b_2 - \omega_1 b_1}{b_2 - b_1}.$$

If O and C are connected by an arm pivoted at O, the length L of the arm is

$$L = b_1 + \tfrac{1}{2}(b_2 - b_1) = \tfrac{1}{2}(b_1 + b_2)$$

and the **angular velocity** Ω of the arm satisfies the equation $\Omega L = v$. Hence

$$\Omega = \frac{v}{L} = \frac{\omega_1 b_1 + \omega_2 b_2}{b_1 + b_2}. \blacksquare$$

Problem 2.18

Book Figure 2.14 shows a straight rigid link of length a whose ends contain pins P, Q that are constrained to move along the axes OX, OY. The displacement x of the pin P at time t is prescribed to be $x = b\sin\Omega t$, where b and Ω are positive constants with $b < a$. Find the angular velocity ω and the speed of the centre C of the link at time t.

Solution

Let θ be the angle between the rod and the negative x-axis. Then the angular velocity of the rod (as shown in book Figure 2.14) is $\omega = -\dot{\theta}$. The angle θ is related to the displacement x by the formula $x = a\cos\theta$ from which it follows that $\dot{x} = -(a\sin\theta)\,\dot{\theta}$. Hence

$$\begin{aligned}\omega &= \frac{\dot{x}}{a\sin\theta}\\ &= \frac{\Omega b\cos\Omega t}{a\sin\theta} = \frac{\Omega b\cos\Omega t}{\left(a^2 - a^2\cos^2\theta\right)^{1/2}}\\ &= \frac{\Omega b\cos\Omega t}{\left(a^2 - b^2\sin^2\Omega t\right)^{1/2}}.\end{aligned}$$

This is the **angular velocity** of the rod at time t.

Let the centre C of the link have coordinates (X, Y). Then

$$\begin{aligned}X &= \tfrac{1}{2}a\cos\theta,\\ Y &= \tfrac{1}{2}a\sin\theta,\end{aligned}$$

and so

$$\begin{aligned}\dot{X} &= -\left(\tfrac{1}{2}a\sin\theta\right)\dot{\theta},\\ \dot{Y} &= \left(\tfrac{1}{2}a\cos\theta\right)\dot{\theta}.\end{aligned}$$

Hence

$$\begin{aligned}\dot{X}^2 + \dot{Y}^2 &= \tfrac{1}{4}\left(a^2\sin^2\theta\right)\dot{\theta}^2 + \tfrac{1}{4}\left(a^2\cos^2\theta\right)\dot{\theta}^2\\ &= \tfrac{1}{4}a^2\dot{\theta}^2\\ &= \frac{\Omega^2 a^2 b^2\cos^2\Omega t}{4\left(a^2 - b^2\sin^2\Omega t\right)}.\end{aligned}$$

The **speed** of C at time t is therefore

$$\frac{\Omega ab|\cos \Omega t|}{2\left(a^2 - b^2 \sin^2 \Omega t\right)^{1/2}}. \blacksquare$$

Problem 2.19

An aircraft is to fly from a point A to an airfield B 600 km due north of A. If a steady wind of 90 km/h is blowing from the north-west, find the direction the plane should be pointing and the time taken to reach B if the cruising speed of the aircraft in still air is 200 km/h.

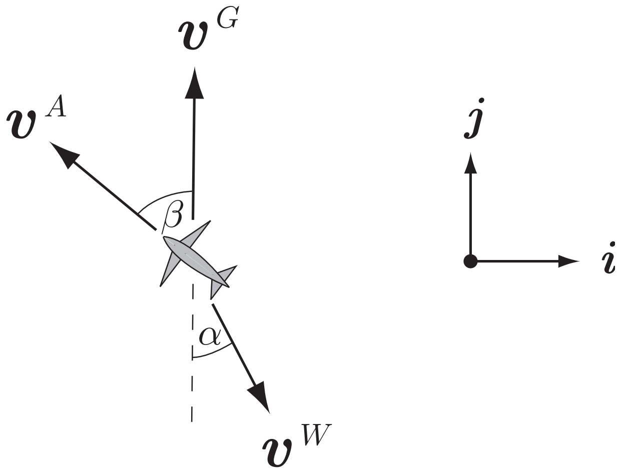

FIGURE 2.4 The ground velocity $\boldsymbol{v}^G$ of the aircraft is the sum of its air velocity $\boldsymbol{v}^A$ and the wind velocity $\boldsymbol{v}^W$.

Solution

Let $\boldsymbol{v}^A$ be the velocity of the aircraft relative to the surrounding air, let $\boldsymbol{v}^G$ be its ground velocity and let $\boldsymbol{v}^W$ be the wind velocity; v^A, v^G and v^W denote the corresponding *speeds*.

In still air, the aircraft can cruise with speed v^A in any direction. When a steady wind is blowing, this remains true when the aircraft is observed from a frame *moving with the wind*. Hence, the ground velocity $\boldsymbol{v}^G$ of the aircraft is given by

$$\boldsymbol{v}^G = \boldsymbol{v}^A + \boldsymbol{v}^W. \tag{1}$$

The situation in the present problem is shown in Figure 2.4. The speeds v^A and v^W (and the angle α) are given, and we wish to choose the angle β so that the velocity $\boldsymbol{v}^G$ points north. Let the unit vectors $\{\boldsymbol{i}, \boldsymbol{j}\}$ be as shown, with $\boldsymbol{i}$ pointing east and $\boldsymbol{j}$ pointing north. Then, on taking components of equation (1) in the $\boldsymbol{i}$- and

$\boldsymbol{j}$-directions, we obtain

$$\begin{aligned} 0 &= -v^A \sin\beta + v^W \sin\alpha, \\ v^G &= v^A \cos\beta - v^W \cos\alpha. \end{aligned}$$

The first equation shows that the **aircraft heading** β is

$$\sin\beta = \left(\frac{v^W}{v^A}\right)\sin\alpha,$$

and the second equation then determines the **ground speed** v^G.

In the present problem, $v^A = 200$ km h^{-1}, $v^W = 90$ km h^{-1} and $\alpha = 45°$. It follows that the heading β must be

$$\beta = \sin^{-1}\left(\frac{90}{200} \times \frac{1}{\sqrt{2}}\right) \approx 18.6°,$$

and that the ground speed v^G is

$$v^G = 200\cos\beta - 90\cos\alpha \approx 126 \text{ km h}^{-1}.$$

The **time taken** to fly to a destination 600 km to the north is therefore 600/126 hours = 4 h 46 m. ■

Problem 2.20

An aircraft takes off from a horizontal runway with constant speed U, climbing at a constant angle α to the horizontal. A car is moving on the runway with constant speed u directly towards the front of the aircraft. The car is distance a from the aircraft at the instant of take-off. Find the distance of closest approach of the car and aircraft. [Don't try this one at home.]

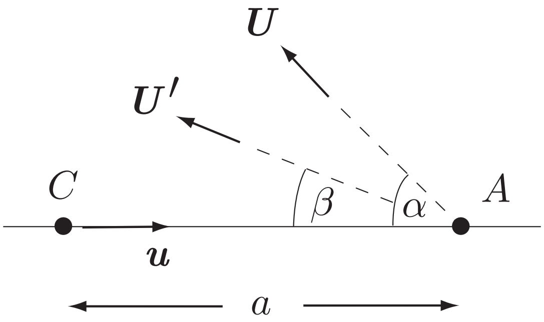

FIGURE 2.5 The velocities of the aircraft and the car are $\boldsymbol{U}$ and $\boldsymbol{u}$ respectively. At the instant of take-off, the car is at C and the aircraft at A.

Solution

Let $\boldsymbol{U}$ be the velocity of the aircraft and $\boldsymbol{u}$ the velocity of the car; U and u are the corresponding *speeds*. Let $\boldsymbol{U}'$ be the velocity of the aircraft in a frame *moving with the car*. Then

$$\boldsymbol{U} = \boldsymbol{u} + \boldsymbol{U}'. \tag{1}$$

Hence

$$\begin{aligned} |\boldsymbol{U}'|^2 &= \boldsymbol{U}' \cdot \boldsymbol{U}' \\ &= (\boldsymbol{U} - \boldsymbol{u}) \cdot (\boldsymbol{U} - \boldsymbol{u}) \\ &= U^2 + u^2 - 2\boldsymbol{U} \cdot \boldsymbol{u} \\ &= U^2 + u^2 + 2u \cos\alpha. \end{aligned}$$

Also, on taking components of equation (1) in the vertical direction, we obtain

$$U' \sin\beta = U \sin\alpha,$$

where the angles α and β are shown in Figure 2.5. Hence

$$\sin\beta = \frac{U\sin\alpha}{U'} = \frac{U\sin\alpha}{\left(U^2 + u^2 + 2u\cos\alpha\right)^{1/2}}.$$

The distance of **closest approach** of the car and the aircraft is $a\sin\beta$, that is,

$$\frac{aU\sin\alpha}{\left(U^2 + u^2 + 2u\cos\alpha\right)^{1/2}}. \blacksquare$$

Problem 2.21 ✻

An aircraft has cruising speed v and a flying range (out and back) of R_0 in still air. Show that, in a north wind of speed u ($u < v$) its range in a direction whose true bearing from north is θ is given by

$$\frac{R_0(v^2 - u^2)}{v(v^2 - u^2 \sin^2\theta)^{1/2}}.$$

What is the maximum value of this range and in what directions is it attained?

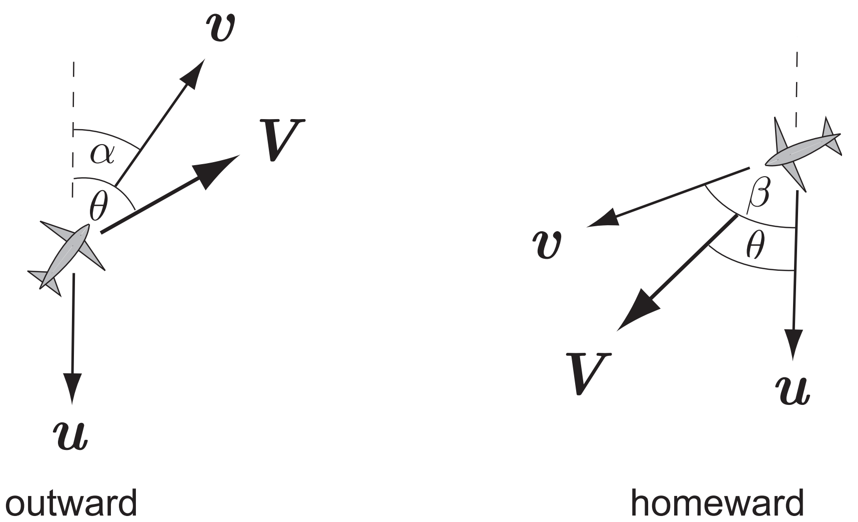

FIGURE 2.6 The outward and homeward journeys of an aircraft in a steady wind.

Solution

Outward leg

The outward leg is shown in Figure 2.6 (left). Note that v ($= |\boldsymbol{v}|$) and θ are given but the aircraft bearing α is unknown. The ground velocity $\boldsymbol{V}$ is given by

$$\boldsymbol{V} = \boldsymbol{v} + \boldsymbol{u}$$

and hence

$$\begin{aligned} v^2 &= \boldsymbol{v} \cdot \boldsymbol{v} \\ &= (\boldsymbol{V} - \boldsymbol{u}) \cdot (\boldsymbol{V} - \boldsymbol{u}) \\ &= V^2 + u^2 - 2\boldsymbol{V} \cdot \boldsymbol{u} \\ &= V^2 + u^2 + 2Vu\cos\theta. \end{aligned}$$

The **outward ground speed** V therefore satisfies the equation

$$V^2 + 2Vu\cos\theta - (v^2 - u^2) = 0$$

and, on selecting the positive root, we find that

$$V^{\text{out}} = -u\cos\theta + \left(v^2 - u^2\sin^2\theta\right)^{1/2}.$$

Homeward leg

The homeward leg is shown in Figure 2.6 (right). The quantities v and θ are the same as before and β is unknown. The ground speed is still given by

$$\boldsymbol{V} = \boldsymbol{v} + \boldsymbol{u},$$

but the velocities $\boldsymbol{V}$ and $\boldsymbol{v}$ are *not* the same as on the outward leg. By proceeding in the same way as before, we find that the **homeward ground speed** satisfies the equation

$$V^2 - 2Vu\cos\theta - (v^2 - u^2) = 0$$

and, on selecting the positive root, we find that

$$V^{\text{back}} = u\cos\theta + \left(v^2 - u^2\sin^2\theta\right)^{1/2}.$$

The range

The range R is restricted by the flying time which must not exceed $2R_0/v$. Since the times taken to fly out and back are R/V^{out} and R/V^{back} respectively, R is determined by

$$\frac{R}{V^{\text{out}}} + \frac{R}{V^{\text{back}}} = \frac{2R_0}{v},$$

that is

$$R = \frac{2R_0 V^{\text{out}} V^{\text{back}}}{v\,(V^{\text{out}} + V^{\text{back}})}.$$

On substituting in the values that we have obtained for V^{out} and V^{back}, we find that the **flying range** in the direction whose true bearing from north is θ is given by

$$R = \frac{R_0(v^2 - u^2)}{v(v^2 - u^2 \sin^2\theta)^{1/2}}.$$

This range takes its maximum value when $\theta = \pm\frac{1}{2}\pi$ (that is, in directions at right angles to the wind). In these cases, the range is

$$R_0\left(1 - \frac{u^2}{v^2}\right)^{1/2},$$

which is still less than the range in still air. ■

Chapter Three

Newton's laws of motion and the law of gravitation

Problem 3.1

Four particles, each of mass m, are situated at the vertices of a regular tetrahedron of side a. Find the gravitational force exerted on any one of the particles by the other three.

Three uniform rigid spheres of mass M and radius a are placed on a horizontal table and are pressed together so that their centres are at the vertices of an equilateral triangle. A fourth uniform rigid sphere of mass M and radius a is placed on top of the other three so that all four spheres are in contact with each other. Find the gravitational force exerted on the upper sphere by the three lower ones.

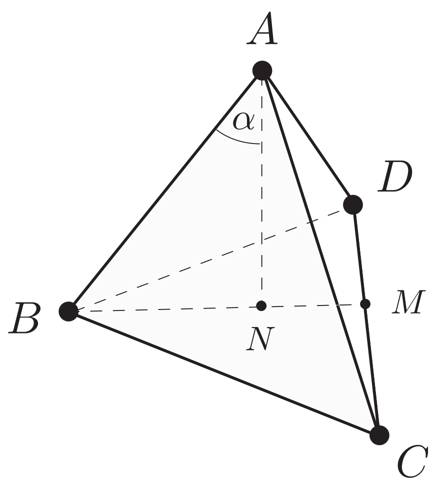

FIGURE 3.1 The particles A, B C and D each have mass m and are located at the vertices of a regular tetrahedron.

Solution

By the law of gravitation, each of the particles B, C and D attracts the particle A with a force of magnitude

$$\frac{m^2G}{a^2}.$$

By symmetry, the resultant force $\boldsymbol{F}$ points in the direction AN and its magnitude F can be found by summing the components of the contributing forces in this direction. Hence

$$F = 3\left(\frac{m^2G\cos\alpha}{a^2}\right),$$

where α is the angle shown in Figure 3.1. The angle α can be found by elementary geometry. By an application of Pythagoras,

$$BM^2 = BC^2 - CM^2 = a^2 - \left(\tfrac{1}{2}a\right)^2 = \tfrac{3}{4}a^2,$$

and, since N is the centroid of the triangle BCD,

$$BN = \tfrac{2}{3}BM = \frac{a}{\sqrt{3}}.$$

A second application of Pythagoras then gives

$$AN^2 = AB^2 - BN^2 = a^2 - \left(\frac{a}{\sqrt{3}}\right)^2 = \tfrac{2}{3}a^2.$$

and so

$$\cos\alpha = \frac{AN}{AB} = \sqrt{\tfrac{2}{3}}.$$

Hence the **resultant force** exerted on particle A by the particles B, C and D acts in the direction AN and has magnitude

$$F = \frac{\sqrt{6}\,m^2G}{a^2}.$$

Since the four balls are spherically symmetric masses, their gravitational effect is the same as if each one were replaced by a particle of mass M at its centre. These four 'particles' form a regular tetrahedron of side $2a$. Hence, the gravitational force exerted on the upper ball by the three lower ones acts vertically downwards and has magnitude

$$\frac{\sqrt{6}\,m^2G}{4a^2}. \blacksquare$$

Problem 3.2

Eight particles, each of mass m, are situated at the corners of a cube of side a. Find the gravitational force exerted on any one of the particles by the other seven.

Deduce the total gravitational force exerted on the four particles lying on one face of the cube by the four particles lying on the opposite face.

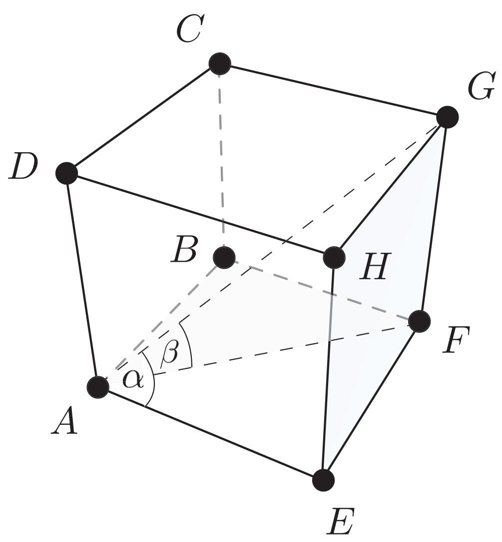

FIGURE 3.2 The particles $A, \ldots, H$ each have mass m and are located at the vertices of a cube.

Solution

By the law of gravitation, each of the particles $B, C, \ldots, H$ attracts the particle A with a force of magnitude

$$\frac{m^2 G}{R^2},$$

where R is the distance between them. By symmetry, the resultant force $\boldsymbol{F}$ points in the direction AG and so its magnitude F can be found by summing the components of the contributing forces in this direction. Hence

$$F = 3\left(\frac{m^2 G}{a^2}\right)\cos\alpha + 3\left(\frac{m^2 G}{\left(\sqrt{2}a\right)^2}\right)\cos\beta + \left(\frac{m^2 G}{\left(\sqrt{3}a\right)^2}\right)$$

where α. β are the angles shown in Figure 3.2. By using elementary geometry, it is easily found that

$$\cos\alpha = \sqrt{\frac{1}{3}}, \quad \text{and} \quad \cos\beta = \sqrt{\frac{2}{3}}.$$

Hence the **resultant force** exerted on particle A by the particles $B, C, \ldots H$ acts in the direction AG and has magnitude

$$F = \frac{m^2G}{a^2}\left(\sqrt{3} + \sqrt{\frac{3}{2}} + \frac{1}{3}\right).$$

By symmetry, the resultant force exerted by the particles E, F, G H on the particles A, B, C, D points in the direction AE and its magnitude F' can be found by summing the components of the contributing forces in this direction. Now the resultant force that *all* the other particles exert on particle A has magnitude F and the component of *this* force in the direction AE is $F\cos\alpha$. Since the the forces that B, C, D exert on A have zero component in this direction, $F\cos\alpha$ is equal to the resultant force that E, F, G, H exert on A, resolved in the direction AE. It follows that the resultant force exerted by the particles E, F, G H on the particles A, B, C, D points in the direction AE and has magnitude

$$F' = 4F\cos\alpha = \frac{4m^2G}{a^2}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right). \blacksquare$$

Problem 3 . 3

A uniform rod of mass M and length $2a$ lies along the interval $[-a, a]$ of the x-axis and a particle of mass m is situated at the point $x = x'$. Find the gravitational force exerted by the rod on the particle.

Two uniform rods, each of mass M and length $2a$, lie along the intervals $[-a, a]$ and $[b - a, b + a]$ of the x-axis, so that their centres are a distance b apart ($b > 2a$). Find the gravitational forces that the rods exert upon each other.

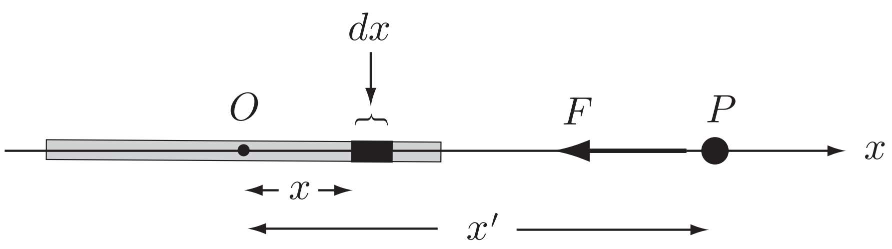

FIGURE 3.3 The rod and the particle.

Solution

Consider the element $[x,\ x + dx]$ of the rod which has mass $M\ dx/2a$ and exerts an attractive force of magnitude

$$\frac{m(M\ dx/2a)G}{(x' - x)^2}$$

on P (see Figure 3.3). We must now sum these contributions but, since the rod is a continuous distribution of mass, this sum becomes an integral. The **resultant force** F_1 exerted by the rod is therefore given by

$$\begin{aligned} F_1 &= \frac{mMG}{2a} \int_{-a}^{a} \frac{dx}{(x' - x)^2} \\ &= \frac{mMG}{2a} \left[\frac{1}{x' - x} \right]_{x=-a}^{x=a} \\ &= \frac{mMG}{2a} \left(\frac{1}{x' - a} - \frac{1}{x' + a} \right) \\ &= \frac{mMG}{x'^2 - a^2}. \end{aligned}$$

Consider the element $[x',\ x'+dx']$ of the second rod which has mass $M\,dx'/2a$. The force exerted by the first rod on this element is towards O and of magnitude

$$\frac{(M\ dx/2a)MG}{(x'-x)^2}.$$

We must now sum these contributions but, since the second rod is also a continuous distribution of mass, this sum also becomes an integral. The **resultant force** F_2 that each rod exerts on the other is therefore given by

$$\begin{aligned}F_2 &= \frac{M^2G}{2a}\int_{b-a}^{b+a}\frac{dx'}{x'^2-a^2}\\ &= \frac{M^2G}{4a^2}\int_{b-a}^{b+a}\left(\frac{1}{x'-a}-\frac{1}{x'+a}\right)dx'\\ &= \frac{M^2G}{4a^2}\Big[\ln(x'-a)-\ln(x'+a)\Big]_{b-a}^{b+a}\\ &= \frac{M^2G}{4a^2}\ln\left(\frac{b^2}{b^2-4a^2}\right). \blacksquare\end{aligned}$$

Problem 3.4

A uniform rigid disk has mass M and radius a, and a uniform rigid rod has mass M' and length b. The rod is placed along the axis of symmetry of the disk with one end in contact with the disk. Find the forces necessary to pull the disk and rod apart. [*Hint.* Make use of the solution in the 'disk' example.]

Solution

Let the axis Oz be perpendicular to the disk with O at the centre, and suppose that the rod occupies the interval $0 \le z \le b$. Consider the element $[z, z + dz]$ of the rod which has mass $M' dz/b$. The force exerted by the disk on this element acts towards O and has magnitude

$$\frac{2MM'G}{a^2 b}\left(1 - \frac{z}{\left(z^2 + a^2\right)^{1/2}}\right) dz,$$

on using the result of Example 3.6. We must now sum these contributions but, since the rod is a continuous distribution of mass, this sum becomes an integral. The **resultant force** F that the disk exerts on the rod is therefore given by

$$\begin{aligned} F &= \frac{2MM'G}{a^2 b} \int_0^b \left(1 - \frac{z}{\left(z^2 + a^2\right)^{1/2}}\right) dz \\ &= \frac{2MM'G}{a^2 b} \left[z - \left(z^2 + a^2\right)^{1/2} \right]_0^b \\ &= \frac{2MM'G}{a^2 b} \left(a + b - \left(a^2 + b^2\right)^{1/2} \right). \end{aligned}$$

This is the **force needed** to pull the rod and the disk apart. ■

Problem 3.5

Show that the gravitational force exerted on a particle *inside* a hollow symmetric sphere is zero. [*Hint*. The proof is the same as for a particle *outside* a symmetric sphere, except in one detail.]

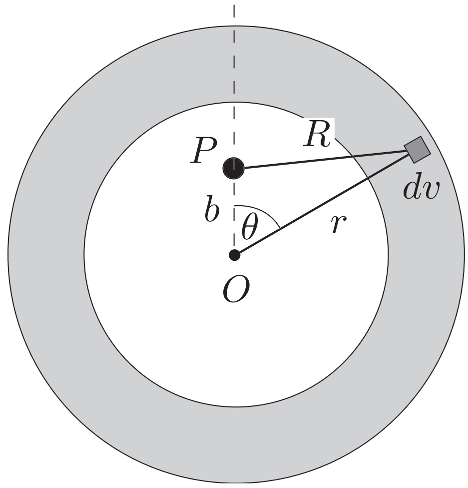

FIGURE 3.4 The particle P is inside a hollow gravitating sphere.

Solution

The only difference that occurs when the particle P is *inside* a hollow sphere is that the polar coordinate r is now always *greater* than the distance b (see Figure 3.4). The range of the variable R is then $r - b \le R \le r + b$ and the integral over R (see the book , p. 66) is replaced by

$$\begin{aligned}\int_{r-b}^{r+b}\left(1+\frac{b^2-r^2}{R^2}\right)dR &= \int_{r-b}^{r+b}\left(1-\frac{r^2-b^2}{R^2}\right)dR\\ &= \left[R+\frac{r^2-b^2}{R}\right]_{R=r-b}^{R=r+b}\\ &= \Big((r+b)+(r-b)\Big)-\Big((r-b)+(r+b)\Big)\\ &= 0.\end{aligned}$$

(When P is outside the sphere, the corresponding value is $4r$.) It follows that the *gravitational force exerted on a particle inside a hollow sphere is zero*. ■

Problem 3.6

A narrow hole is drilled through the centre of a *uniform* sphere of mass M and radius a. Find the gravitational force exerted on a particle of mass m which is inside the hole at a distance r from the centre.

Solution

In this solution, we will neglect the material that was removed from the sphere to make the hole. When the particle P is a distance r from the centre, it is

(i) exterior to a uniform *solid* sphere of radius r and mass $(r/a)^3 M$, and

(ii) interior to a uniform *hollow* sphere with inner radius r and outer radius a.

The solid sphere exerts the same force as that of a particle of mass $(r/a)^3 M$ located at the centre, while (from Problem 3.5) the hollow sphere exerts no resultant force. The **resultant force** exerted on P when it is inside the hole is therefore

$$\frac{m\left((r/a)^3 M\right) G}{r^2} + 0 = \left(\frac{mMG}{a^3}\right) r.$$

This force acts towards the centre and is proportional to r. Hence, in the absence of any other forces (such as air resistance or boiling lava!), P will perform **simple harmonic oscillations** in the hole. Note that this result applies only to a *uniform* sphere. ■

Problem 3.7

A symmetric sphere, of radius a and mass M, has its centre a distance b ($b > a$) from an infinite plane containing a uniform distribution of mass σ per unit area. Find the gravitational force exerted on the sphere.

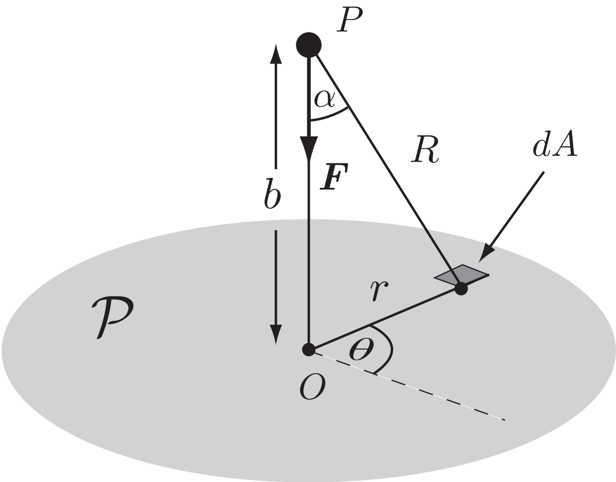

FIGURE 3.5 The particle P is distance b from a unform gravitating plane.

Solution

This is basically the same problem as that in Example 3.6 where the disk now has infinite radius (and therefore infinite mass). Despite this, the gravitational force it exerts on P is still finite.

The force on the sphere is the same as that on a particle P of mass M located at its centre. Consider the element dA of the plane shown in Figure 3.5. This element has mass σdA and attracts the sphere with a force whose component perpendicular to the plane is

$$\left(\frac{M(\sigma dA)G}{R^2}\right)\cos\alpha.$$

We must now sum these contributions but, since the plane is a continuous distribution of mass, this sum becomes an integral. The **resultant force** F that the plane exerts on P is therefore given by

$$F = M\sigma G \int_{\mathcal{P}} \frac{\cos\alpha}{R^2}\, dA,$$

where $\mathcal{P}$ is the region occupied by the mass distribution.

This integral is most easily evaluated using polar coordinates. In this case $dA = (dr)(r\,d\theta) = r\,dr\,d\theta$, and the integrand becomes

$$\frac{\cos\alpha}{R^2} = \frac{R\cos\alpha}{R^3} = \frac{b}{(r^2+b^2)^{3/2}},$$

where b is the distance of P from the plane. The ranges of integration for r, θ are $0 \le r \le \infty$ and $0 \le \theta \le 2\pi$. We thus obtain

$$F = bM\sigma G \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{(r^2+b^2)^{3/2}}\right) r\,dr\,d\theta.$$

Since the integrand is independent of θ, the θ-integration is trivial leaving

$$\begin{aligned} F &= 2\pi bM\sigma G \int_{r=0}^{r=\infty} \frac{r\,dr}{(r^2+b^2)^{3/2}} \\ &= 2\pi bM\sigma G \left[-(r^2+b^2)^{-1/2}\right]_{r=0}^{r=\infty} \\ &= -2\pi bM\sigma G\left(0-\frac{1}{b}\right) \\ &= 2\pi M\sigma G. \end{aligned}$$

This is the **gravitational force** exerted on the sphere. It seems strange that this force is independent of the distance b, but this is because the attracting mass distribution is an *infinite plane*. ■

Problem 3.8 ✻

Two uniform rigid hemispheres, each of mass M and radius a are placed in contact with each other so as to form a complete sphere. Find the forces necessary to pull the hemispheres apart.

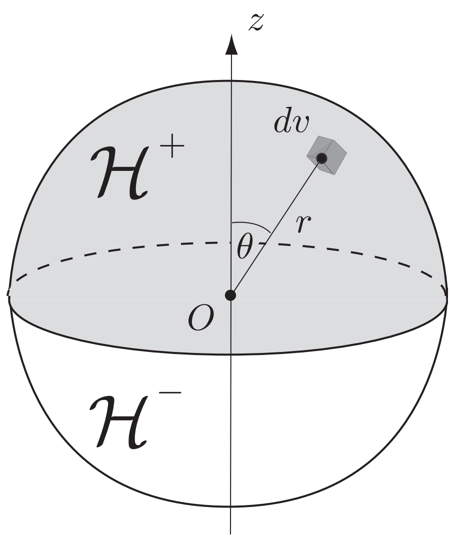

FIGURE 3.6 The solid hemispheres $\mathcal{H}^+$ and $\mathcal{H}^-$ attract each other.

Solution

Let the two hemispheres $\mathcal{H}^+$ and $\mathcal{H}^-$ be as shown in Figure 3.6. We wish to calculate the force F that $\mathcal{H}^+$ exerts on $\mathcal{H}^-$. Since $\mathcal{H}^+$ exerts no resultant force upon *itself*, F is equal to the force that $\mathcal{H}^+$ exerts on the *whole sphere* of mass $2M$. It is tempting to say that this is equal to the force that $\mathcal{H}^+$ exerts on a particle of mass $2M$ located at O. However, this is not true since the mass of $\mathcal{H}^+$ lies *inside the whole sphere*. We must therefore proceed in the same manner as in Problem 3.6.

Consider the volume element dv of $\mathcal{H}^+$. This has mass ρdv, where ρ is the constant density. This element attracts the whole sphere with a force which is the same as if the sphere were replaced by a particle of mass $(r/a)^3(2M)$ at its centre. The component of this force in the z-direction is

$$\left(\frac{2M(r/a)^3(\rho dv)G}{r^2}\right)\cos\theta.$$

We must now sum these contributions but, since $\mathcal{H}^+$ is a continuous distribution of mass, this sum becomes an integral. The **resultant force** F is therefore given by

$$F = \frac{2M\rho G}{a^3} \int_{\mathcal{H}+} r \cos\theta \, dv.$$

This integral is most easily evaluated using spherical polar coordinates r, θ, ϕ. In this case $dv = (dr)(r\,d\theta)(r \sin\theta\, d\phi) = r^2 \sin\theta \, dr\, d\theta\, d\phi$, and the integral becomes

$$\begin{aligned}
F &= \frac{2M\rho G}{a^3} \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=2\pi} r^3 \sin\theta \cos\theta \, dr\, d\theta\, d\phi \\
&= \frac{4\pi M\rho G}{a^3} \left(\int_{r=0}^{r=a} r^3\, dr \right) \left(\int_{\theta=0}^{\theta=\pi/2} \sin\theta\cos\theta\, d\theta \right) \\
&= \frac{4\pi M\rho G}{a^3} \left(\tfrac{1}{4} a^4 \right) \left[-\tfrac{1}{4} \cos 2\theta \right]_0^{\pi/2} \\
&= \tfrac{1}{2} \pi M \rho a G.
\end{aligned}$$

Finally, on using the relation $M = \frac{2}{3}\pi a^3 \rho$, we find that the **resultant force** that $\mathcal{H}^+$ exerts on $\mathcal{H}^-$ is

$$F = \frac{3M^2 G}{4a^2}.$$

This is the **force needed** to pull the hemispheres apart. ■

Chapter Four

Problems in particle dynamics

Problem 4.1

Two identical blocks each of mass M are connected by a light inextensible string and can move on the surface of a *rough* horizontal table. The blocks are being towed at constant speed in a straight line by a rope attached to one of them. The tension in the tow rope is T_0. What is the tension in the connecting string? The tension in the tow rope is suddenly increased to $4T_0$. What is the instantaneous acceleration of the blocks and what is the instantaneous tension in the connecting string?

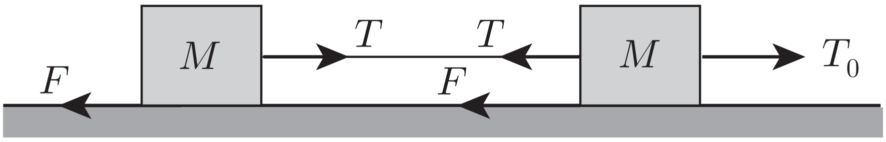

FIGURE 4.1 The two blocks are linked together and towed by a force T_0.

Solution

Let the tension in the connecting string be T and the frictional force acting on each block be F (see Figure 4.1). The two frictional forces are equal because the blocks are physically identical and are travelling at the same speed.

(i) Suppose first that the whole system is moving at *constant speed*. Then the blocks have zero acceleration and their **equations of motion** are therefore

$$T_0 - T - F = 0,$$
$$T - F = 0.$$

Hence

$$T = \tfrac{1}{2}T_0 \quad \text{and} \quad F = \tfrac{1}{2}T_0.$$

The **tension** in the connecting string is therefore $\frac{1}{2}T_0$. ■

(ii) Suppose that the tension in the tow rope is increased to $4T_0$ and that the system then has acceleration a at time t. The **equations of motion** for the two blocks then become

$$4T_0 - T - F = Ma,$$
$$T - F = Ma.$$

At any instant, the two blocks have the same speed and so the two frictional forces do remain *equal*. However, we have no right to suppose that, as the speed of the system increases, the frictional forces will remain *constant*. But, at the instant that the tension in the tow rope is increased, the speed is (as yet) unchanged and it will still be true that $F = \frac{1}{2}T_0$. Thus, *at this instant*, we have

$$\begin{aligned} \tfrac{7}{2}T_0 - T &= Ma, \\ T - \tfrac{1}{2}T_0 &= Ma. \end{aligned}$$

Hence

$$T = 2T_0 \quad \text{and} \quad a = \frac{3T_0}{2M}.$$

Hence the *instantaneous* acceleration of the blocks is $3T_0/2M$ and the *instantaneous* tension in the connecting string is $2T_0$. (If it happens to be true that F is independent of the speed of the blocks, these values will remain constant in the subsequent motion.) ■

Problem 4.2

A body of mass M is suspended from a fixed point O by an inextensible uniform rope of mass m and length b. Find the tension in the rope at a distance z below O. The point of support now begins to rise with acceleration $2g$. What now is the tension in the rope?

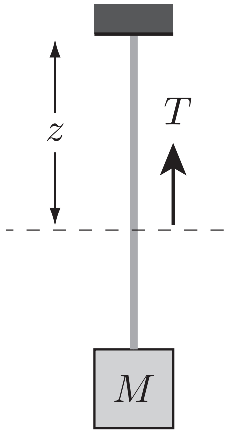

FIGURE 4.2 The block of mass M is suspended from a support by a uniform rope of mass m and length b.

Solution

Consider the motion of $\mathcal{S}'$, which is that part of the system that lies *below* the horizontal plane shown dashed in Figure 4.2. This consists of the block of mass M and a segment of the rope of length $b - z$ and mass $m(b - z)/b$. Then:

(i) When the system is in **equilibrium**, the acceleration is zero and the **equation of motion** for $\mathcal{S}'$ is

$$T - Mg - m\left(1 - \frac{z}{b}\right)g = 0,$$

where T $(= T(z))$ be the tension in the rope at depth z. Hence, the **tension** in the rope is

$$T = Mg + m\left(1 - \frac{z}{b}\right)g.$$

This tension takes its **maximum value** of $(M + m)g$ at $z = 0$. ■

(ii) When the support is accelerating upwards with acceleration $2g$, the **equation of motion** for $\mathcal{S}'$ becomes

$$T - Mg - m\left(1 - \frac{z}{b}\right)g = \left(Mg + m\left(1 - \frac{z}{b}\right)\right)(2g).$$

The **tension** in the rope is therefore

$$T = 3Mg + 3m\left(1 - \frac{z}{b}\right)g,$$

which is three times the static value. ■

Problem 4.3

Two uniform lead spheres each have mass 5000 kg and radius 47 cm. They are released from rest with their centres 1 m apart and move under their mutual gravitation. Show that they will collide in *less* than 425 s. [$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.]

Solution

The uniform spheres may be replaced by particles of mass 5000 kg which are released from rest a distance 1 m apart. We wish to know how long it takes for each particle to move a distance 3 cm.

Strictly speaking, this is a problem with non-constant accelerations, but we may find an *upper bound* for the time taken by replacing the non-constant acceleration of each particle by its *initial value*. By the inverse square law, the subsequent accelerations will be greater than this so that the true time will be less than that calculated by this approximation. The **initial acceleration** of each particle is given by the inverse square law to be

$$a = \frac{mG}{R^2} = \frac{5000 \times 6.67 \times 10^{-11}}{1^2} = 3.335 \times 10^{-7} \text{ m s}^{-2}.$$

If the particles moved with this constant acceleration, their displacements after time t would be given by the constant acceleration formula $x = \frac{1}{2}at^2$. The time τ taken for each sphere to move a distance 3 cm would then be

$$\tau = \left(\frac{2 \times 0.03}{3.335 \times 10^{-7}} \right)^{1/2} = 424.2 \text{ s}.$$

Hence, (allowing a little for rounding error) the **spheres will collide** in *less* than 425 s. ■

Problem 4.4

The block in Figure 4.2 is sliding down the inclined surface of a fixed wedge. This time the frictional force F exerted on the block is given by $F = \mu N$, where N is the normal reaction and μ is a positive constant. Find the acceleration of the block. How do the cases $\mu < \tan\alpha$ and $\mu > \tan\alpha$ differ?

Solution

We will make use of the results on p. 79 of the book. The **equations of motion** for the block are

$$\begin{aligned} Mg\sin\alpha - F &= M\frac{dv}{dt}, \\ N - Mg\cos\alpha &= 0. \end{aligned}$$

In the present problem, we are given that F and N are related by $F = \mu N$, where μ is a positive constant. It follows that v satisfies the equation

$$\frac{dv}{dt} = (\sin\alpha - \mu\cos\alpha)\, g.$$

Assuming that the block is moving at all, this is its **acceleration**; it may be positive or negative. If $\mu > \tan\alpha$, the block will always come to rest and will then *remain at rest*. If $\mu < \tan\alpha$, the block may come to rest (v may be negative initially), but will then slide down the plane. ■

Problem 4.5

A stuntwoman is to be fired from a cannon and projected a distance of 40 m over level ground. What is the least projection speed that can be used? If the barrel of the cannon is 5 m long, show that she will experience an acceleration of at least $4g$ in the barrel. [Take $g = 10\ \mathrm{m\,s^{-2}}$.]

Solution

We will make use of the projectile results on p. 89 of the book. In the absence of air resistance, the least projection speed will be needed when the barrel is inclined at 45° to the horizontal. In this case, the horizontal range R is given by

$$R = \frac{u^2}{g},$$

in the standard notation. Hence, the stuntwoman must be **launched with speed**

$$u = (Rg)^{1/2} = (40 \times 10)^{1/2} = 20\ \mathrm{m\,s^{-1}}.$$

Suppose that the acceleration of the stuntwoman in the barrel is a constant a. Then the constant acceleration formula $v^2 - u^2 = 2ax$ shows that a is given by

$$a = \frac{v^2 - u^2}{2x} = \frac{20^2 - 0^2}{2 \times 5} = 40\ \mathrm{m\,s^{-2}} = 4g.$$

This is the stuntwoman's **acceleration** in the barrel, provided that it is constant. If her acceleration is not constant, then there will be times at which it is less than $4g$ and other times at which it is greater than $4g$. In all cases then, an acceleration of $4g$ will be experienced by the stuntwoman. ■

Problem 4.6

In an air show, a pilot is to execute a circular loop at the speed of sound ($340\ \mathrm{m\,s^{-1}}$). The pilot may black out if his acceleration exceeds $8g$. Find the radius of the smallest circle he can use. [Take $g = 10\ \mathrm{m\,s^{-2}}$.]

Solution

The **acceleration** a of the pilot is given by

$$a = \frac{v^2}{R} = \frac{340^2}{R},$$

where R is the radius of the circle in metres. If a is not to exceed $8g$, then R must satisfy

$$R \geq \frac{340^2}{8 \times 10} = 1445\ \mathrm{m}.$$

This is the radius of **smallest circle** the pilot can use. This is nearly a mile, which is surprisingly large. ■

Problem 4.7

A body has terminal speed V when falling in still air. What is its terminal velocity (relative to the ground) when falling in a steady horizontal wind with speed U?

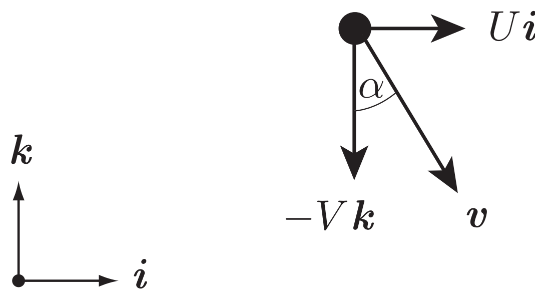

FIGURE 4.3 Terminal velocity in a steady wind.

Solution

In still air, the body has terminal speed V, which means that the equation of motion for the body velocity $\boldsymbol{v}$ has the constant solution $\boldsymbol{v} = -V\boldsymbol{k}$ (see Figure 4.3). When a steady horizontal wind $U\boldsymbol{i}$ is present, let us view the motion of the body from a frame $\mathcal{F}'$ *moving with the wind*. This is an inertial frame in which the air is at rest. It follows that the equation of motion for the apparent body velocity $\boldsymbol{v}'$ has the constant solution $\boldsymbol{v}' = -V\boldsymbol{k}$. When viewed from the fixed frame, this solution becomes $\boldsymbol{v} = U\boldsymbol{i} - V\boldsymbol{k}$. *This is a constant solution for the body velocity* $\boldsymbol{v}$ *when the wind is present*. It represents a **terminal velocity** with speed $\left(U^2 + V^2\right)^{1/2}$ inclined at an angle $\tan^{-1}(U/V)$ to the downward vertical, as shown in Figure 4.3. ■

Problem 4.8 *Cathode ray tube*

A particle of mass m and charge e is moving along the x-axis with speed u when it passes between two charged parallel plates. The plates generate a uniform electric field $E_0\boldsymbol{j}$ in the region $0 \le x \le b$ and no field elsewhere.Find the angle through which the particle is deflected by its passage between the plates. [The cathode ray tube uses this arrangement to deflect the electron beam.]

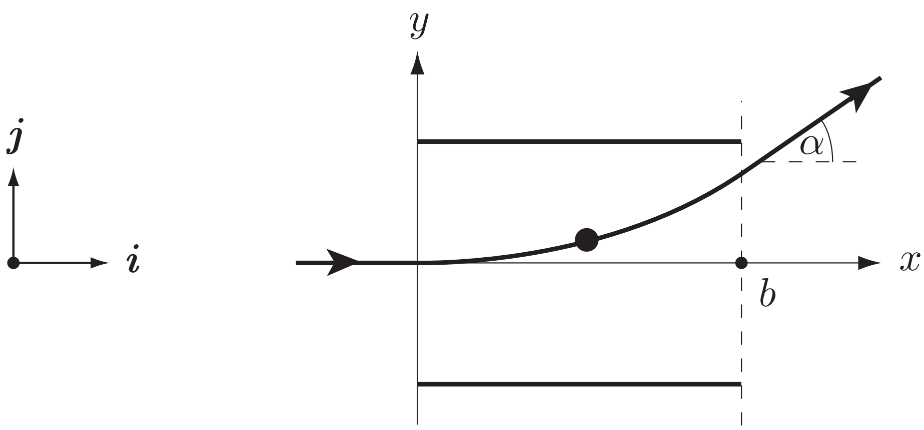

FIGURE 4.4 A charged particle moves through a region in which there is a uniform electric field.

Solution

While the particle is between the plates it experiences the force $eE_0\,\boldsymbol{j}$. Its **equation of motion** in this region is therefore

$$m\frac{d\boldsymbol{v}}{dt} = eE_0\,\boldsymbol{j}\,.$$

If we now write $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$ and take components of this equation in the $\boldsymbol{i}$- and $\boldsymbol{j}$-directions, we obtain the two scalar equations of motion

$$\frac{dv_x}{dt} = 0, \qquad \frac{dv_y}{dt} = \frac{eE_0}{m}.$$

Simple integrations then give

$$v_x = C, \qquad v_y = \left(\frac{eE_0}{m}\right)t + D,$$

where C and D are constants of integration. Suppose that the particle is at the origin when $t = 0$. Then the initial conditions $v_x = u$ and $v_y = 0$ when $t = 0$ imply that $C = u$ and $D = 0$ so that the **velocity** components of the particle are given by

$$v_x = u, \qquad v_y = \left(\frac{eE_0}{m}\right) t.$$

The position of the particle at time t can now be found by integrating the expressions for v_x, v_y and applying the initial conditions $x = 0$ and $y = 0$ when $t = 0$. This gives

$$x = u\,t, \qquad y = \left(\frac{eE_0}{2m}\right) t^2,$$

which is the **trajectory** of the particle. On eliminating the time t between these two equations, the **path** of the particle is found to be

$$y = \left(\frac{eE_0}{2mu^2}\right) x^2.$$

The angle through which the particle is deflected by its passage between the plates is the angle α shown in Figure 4.4. Since

$$\tan\alpha = \left.\frac{dy}{dx}\right|_{x=b} = \frac{ebE_0}{mu^2},$$

it follows that the **deflection angle** is $\tan^{-1}(ebE_0/mu^2)$. ■

Problem 4.9

An object is dropped from the top of a building and is in view for time τ while passing a window of height h some distance lower down. How high is the top of the building above the top of the window?

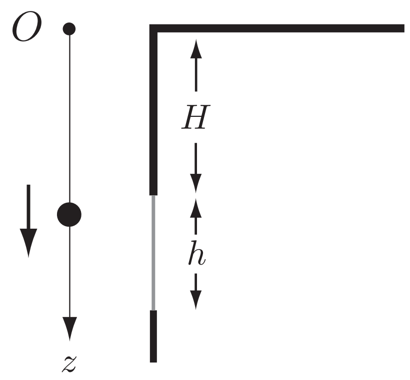

FIGURE 4.5 The body is released from the top of the building and falls past the window.

Solution

Let the axis Oz point vertically downwards, where O is the point from which the body is released. Then the displacement of the body after time t is

$$z = \tfrac{1}{2} g t^2.$$

It follows that

$$H = \tfrac{1}{2} g T^2,$$
$$H + h = \tfrac{1}{2} g (T + \tau)^2,$$

where H is the height of the top of the building above the top of the window, and T is the time taken for the body to fall this distance. We are asked to find H, but it is easier to find T first. On subtracting the first of these equations from the second, we obtain

$$h = \tfrac{1}{2} g \left(2T\tau + \tau^2\right),$$

from which it follows that

$$T = \frac{h}{g\tau} - \tfrac{1}{2}\tau.$$

On substituting this value for T into the equation for H, we find that the **height** of the top of the building above the top of the window is

$$H = \frac{1}{8g\tau^2}\left(2h - g\tau^2\right)^2. \blacksquare$$

Problem 4.10

A particle P of mass m moves under the gravitational attraction of a mass M fixed at the origin O. Initially P is at a distance a from O when it is projected with the *critical* escape speed $(2MG/a)^{1/2}$ directly away from O. Find the distance of P from O at time t, and confirm that P escapes to infinity.

Solution

By symmetry, the motion of P takes place in a straight line through O. By the law of gravitation, the **equation of motion** is

$$m\frac{dv}{dt} = -\frac{mMG}{r^2},$$

where r is the distance OP. Since

$$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt} = v\frac{dv}{dr},$$

this can be written in the form

$$v\frac{dv}{dr} = -\frac{MG}{r^2},$$

which is a first order separable ODE for v as a function of r. Separation gives

$$\int v\, dv = -MG \int \frac{dr}{r^2},$$

so that

$$\tfrac{1}{2}v^2 = \frac{MG}{r} + C,$$

where C is the integration constant. On applying the initial condition $v = (2MG/a)^{1/2}$ when $r = a$, we find that $C = 0$. It follows that the **velocity** of P when its displacement is r is given by

$$v^2 = \frac{2MG}{r}.$$

To find r as a function of t, we write $v = dr/dt$ and solve the ODE

$$\left(\frac{dr}{dt}\right)^2 = \frac{2Mg}{r}.$$

After taking the positive square root of each side (dr/dt is certainly positive initially), the equation separates to give

$$\int r^{1/2} dr = (2MG)^{1/2} \int dt.$$

Hence

$$\tfrac{2}{3} r^{3/2} = (2MG)^{1/2} t + D,$$

where D is an integration constant. On applying the initial condition $r = a$ when $t = 0$, we find that $D = \frac{2}{3} a^{3/2}$ and, after some simplification, the **displacement** of P at time t is found to be

$$r = \left(a^{3/2} + \tfrac{3}{2} (2MG)^{1/2} t \right)^{2/3}.$$

It is evident that the right side of this expression tends to infinity with t and hence the **particle escapes**. ■

Problem 4.11

A particle P of mass m is attracted towards a fixed origin O by a force of magnitude $m\gamma/r^3$, where r is the distance of P from O and γ is a positive constant. [It's gravity Jim, but not as we know it.] Initially, P is at a distance a from O, and is projected with speed u directly away from O. Show that P will escape to infinity if $u^2 > \gamma/a^2$.

For the case in which $u^2 = \gamma/(2a^2)$, show that the maximum distance from O achieved by P in the subsequent motion is $\sqrt{2}a$, and find the time taken to reach this distance.

Solution

By symmetry, the motion of P takes place in a straight line through O. From the specified law of attraction, the **equation of motion** is

$$m\frac{dv}{dt} = -\frac{m\gamma}{r^3},$$

where r is the distance OP. Since

$$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt} = v\frac{dv}{dr},$$

this can be written in the form

$$v\frac{dv}{dr} = -\frac{\gamma}{r^3},$$

which is a first order separable ODE for v as a function of r. Separation gives

$$\int v\,dv = -\gamma \int \frac{dr}{r^3},$$

so that

$$\tfrac{1}{2}v^2 = \frac{\gamma}{2r^2} + C,$$

where C is the integration constant. On applying the initial condition $v = u$ when $r = a$, we find that $C = \frac{1}{2}u^2 - \frac{1}{2}\gamma/a^2$. It follows that the **velocity** of P when its displacement is r is given by

$$v^2 = \frac{\gamma}{r^2} + \left(u^2 - \frac{\gamma}{a^2}\right).$$

Suppose first that $u^2 > \gamma/a^2$. Then we can write

$$u^2 - \frac{\gamma}{a^2} = V^2,$$

where V is a positive constant. Then

$$\begin{aligned} v^2 &= \frac{\gamma}{r^2} + V^2 \\ &> V^2. \end{aligned}$$

Hence v always exceeds V and the **particle escapes**.

Suppose now that $u^2 = \gamma/(2a^2)$. The formula for v then becomes

$$v^2 = \frac{\gamma}{r^2} - \frac{\gamma}{2a^2}.$$

In this case, v becomes zero when

$$\frac{\gamma}{r^2} - \frac{\gamma}{2a^2} = 0,$$

that is, when $r = \sqrt{2}a$. The **maximum distance** from O achieved by the particle is therefore $\sqrt{2}a$.

To find the time taken to reach this distance, we write $v = dr/dt$ and solve the ODE

$$\left(\frac{dr}{dt}\right)^2 = \frac{\gamma}{r^2} - \frac{\gamma}{2a^2}.$$

After taking the positive square root of each side ($dr/dt \geq 0$ in this outward motion), the equation separates to give

$$\int_a^{\sqrt{2}a} \frac{r}{\left(2a^2 - r^2\right)^{1/2}}\, dr = \left(\frac{\gamma}{2a^2}\right)^{1/2} \int_0^\tau dt.$$

Here we have introduced the initial and final conditions directly into the limits of integration; τ is the elapsed time. Hence, the **time taken** for the particle to achieve its maximum distance is given by

$$\begin{aligned} \tau &= \left(\frac{\gamma}{2a^2}\right)^{-1/2} \left[-\left(2a^2 - r^2\right)^{1/2}\right]_a^{\sqrt{2}a} \\ &= \left(\frac{\gamma}{2a^2}\right)^{-1/2} a \\ &= \left(\frac{2}{\gamma}\right)^{1/2} a^2. \blacksquare \end{aligned}$$

Problem 4.12

If the Earth were suddenly stopped in its orbit, how long would it take for it to collide with the Sun? [Regard the Sun as a *fixed* point mass. You may make use of the formula for the period of the Earth's orbit.]

Solution

By symmetry, the motion takes place in a straight line through the Sun. From the law of gravitation, the **equation of motion** is

$$m\frac{dv}{dt} = -\frac{mMG}{r^2},$$

where M is the mass of the Sun, m is the mass of the Earth and r is the distance of the Earth from the Sun. Since

$$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt} = v\frac{dv}{dr},$$

this can be written in the form

$$v\frac{dv}{dr} = -\frac{MG}{r^2},$$

which is a first order separable ODE for v as a function of r. Separation gives

$$\int v\,dv = -MG\int \frac{dr}{r^2},$$

so that

$$\tfrac{1}{2}v^2 = \frac{MG}{r} + C,$$

where C is the integration constant. On applying the initial condition $v = 0$ when $r = R$, where R is the radius of the Earth's orbit, we find that $C = -MG/R$. It follows that the **velocity** of the Earth when it is distance r from the Sun is given by

$$v^2 = 2MG\left(\frac{1}{r} - \frac{1}{R}\right).$$

To find the time taken for the Earth to reach the Sun, we write $v = dr/dt$ and solve the ODE

$$\left(\frac{dr}{dt}\right)^2 = 2MG\left(\frac{1}{r} - \frac{1}{R}\right).$$

After taking the square root of each side (and remembering that $dr/dt < 0$ in this motion), the equation separates to give

$$\int_R^0 \left(\frac{r}{R-r}\right)^{1/2} dr = -\left(\frac{2MG}{R}\right)^{1/2} \int_0^T dt.$$

Here we have introduced the initial and final conditions directly into the limits of integration; T is the elapsed time. Hence, the **time taken** for the Earth to reach the Sun is

$$T = \left(\frac{R}{2MG}\right)^{1/2} \int_0^R \left(\frac{r}{R-r}\right)^{1/2} dr.$$

This integral can be evaluated by making the substitution $r = R\sin^2\theta$. Then $dr = 2R\sin\theta\cos\theta\, d\theta$ and

$$\begin{aligned}
T &= \left(\frac{R}{2MG}\right)^{1/2} \int_0^{\pi/2} \left(\frac{\sin^2\theta}{1-\sin^2\theta}\right)^{1/2} (2R\sin\theta\cos\theta)\, d\theta \\
&= \left(\frac{R^3}{2MG}\right)^{1/2} \int_0^{\pi/2} 2\sin^2\theta\, d\theta = \left(\frac{R^3}{2MG}\right)^{1/2} \int_0^{\pi/2} (1-\cos 2\theta)\, d\theta \\
&= \left(\frac{R^3}{2MG}\right)^{1/2} \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} \\
&= \tfrac{1}{2}\pi \left(\frac{R^3}{2MG}\right)^{1/2}.
\end{aligned}$$

We could substitute the numerical data directly into this formula, but it is neater to observe that T is related to τ, the period of the Earth's orbit (before it was brought to rest!). Since $\tau^2 = 4\pi^2 R^3/MG$, it follows that T is given by the simple formula

$$T = \frac{\tau}{4\sqrt{2}}.$$

For the Earth, this is 65 days. ■

Problem 4.13

A particle P of mass m slides on a smooth horizontal table. P is connected to a second particle Q of mass M by a light inextensible string which passes through a small smooth hole O in the table, so that Q hangs below the table while P moves on top. Investigate motions of this system in which Q remains at rest vertically below O, while P describes a circle with centre O and radius b. Show that this is possible provided that P moves with constant speed u, where $u^2 = Mgb/m$.

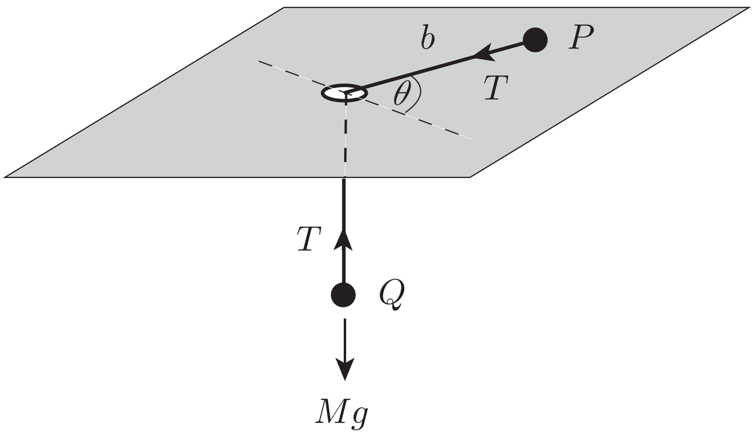

FIGURE 4.6 The particle P slides on the table while particle Q hangs below.

Solution

Since the particle Q is at rest, the resultant force acting on it is zero and so the **tension** T in the string must be equal to Mg. Now consider the motion of P. The polar **equations of motion** are

$$m\left(0 - b\dot{\theta}^2\right) = -Mg,$$
$$m\left(b\ddot{\theta} + 0\right) = 0.$$

The second equation shows that $b\dot{\theta} = u$, where u is a constant that we can identify as the *circumferential velocity* of P. The first equation then requires that

$$\frac{mu^2}{b} = Mg.$$

Hence, circular motions of any radius b are possible provided that P moves with

constant speed $(Mgb/m)^{1/2}$. ■

Problem 4.14

A light pulley can rotate freely about its axis of symmetry which is fixed in a horizontal position. A light inextensible string passes over the pulley. At one end the string carries a mass $4m$, while the other end supports a second light pulley. A second string passes over this pulley and carries masses m and $4m$ at its ends. The whole system undergoes planar motion with the masses moving vertically. Find the acceleration of each of the masses.

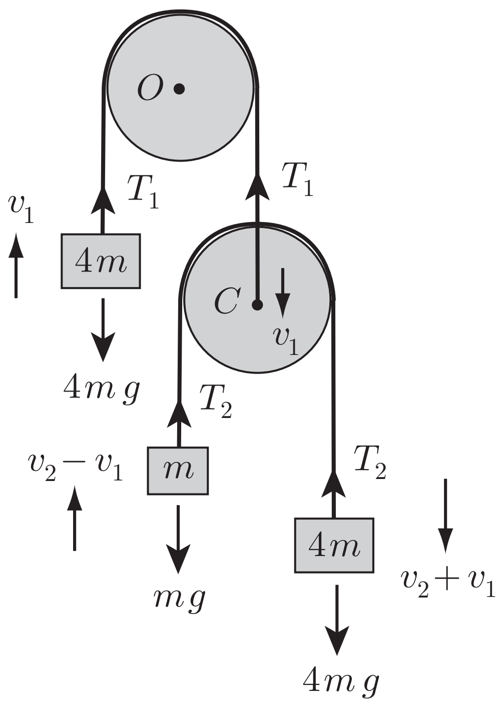

FIGURE 4.7 The double Attwood machine.

Solution

The system is shown in Figure 4.7. Let v_1 be the upward velocity of the mass $4m$, which is the same as the downward velocity of the centre C of the moving pulley. Let v_2 be the upward velocity of the mass m *measured relative to* C; this is the same as the downward velocity of the (lower) mass $4m$ relative to C. The corresponding true velocities are $v_2 - v_1$ and $v_2 + v_1$ repectively. Note that, since the pulleys are light, the strings have constant tensions T_1 and T_2 respectively. The **equations of**

motion for the three masses are then

$$4m\frac{dv_1}{dt} = T_1 - 4mg, \qquad (1)$$

$$m\frac{d}{dt}(v_2 - v_1) = T_2 - mg \qquad (2)$$

$$4m\frac{d}{dt}(v_2 + v_1) = 4mg - T_2. \qquad (3)$$

Let $a_1 = dv_1/dt$ and $a_2 = dv_2/dt$. We then have the four unknowns a_1, a_2, T_1, T_2, but only three equations. However, an additional equation is provided by the 'equation of motion' of the moving pulley. Since this pulley is of negligible mass, the *resultant force acting upon it must be zero, no matter how it is moving*. It follows that

$$T_1 - 2T_2 = 0. \qquad (4)$$

On eliminating T_1 and T_2 from equations (1)–(4), we find that the accelerations a_1, a_2 satisfy the equations

$$3a_1 + 5a_2 = 3g,$$
$$3a_1 - a_2 = -g,$$

from which it follows that

$$a_1 = -\tfrac{1}{9}g, \qquad a_2 = \tfrac{2}{3}g.$$

Hence the three masses have **accelerations** $-\frac{1}{9}g$, $\frac{7}{9}g$ and $\frac{5}{9}g$ respectively. Note that, somewhat surprisingly, the (upper) mass $4m$ accelerates *downwards*. ■

Problem 4.15

A particle P of mass m can slide along a *smooth* rigid straight wire. The wire has one of its points fixed at the origin O, and is made to rotate in the (x, y)-plane with angular speed Ω. By using the vector equation of motion of P in polar co-ordinates, show that r, the distance of P from O, satisfies the equation

$$\ddot{r} - \Omega^2 r = 0,$$

and find a second equation involving N, where $N\widehat{\boldsymbol{\theta}}$ is the force the wire exerts on P. [Ignore gravity in this question.]

Initially, P is at rest (relative to the wire) at a distance a from O. Find r as a function of t in the subsequent motion, and deduce the corresponding formula for N.

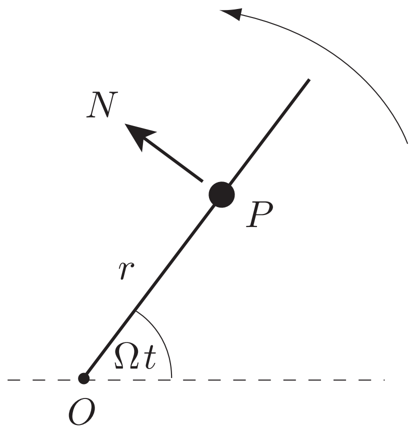

FIGURE 4.8 The particle P slides along the rotating wire.

Solution

Since the wire is *smooth*, the reaction N that it exerts on P must always be perpendicular to the wire, as shown in Figure 4.8. The polar **equations of motion** for P are therefore

$$m\left(\ddot{r} - r\Omega^2\right) = 0,$$
$$m\left(0 + 2\dot{r}\Omega\right) = N,$$

on using the fact that $\dot{\theta} = \Omega$ and $\ddot{\theta} = 0$. Hence r satisfies the equation

$$\ddot{r} - \Omega^2 r = 0$$

and the reaction of the wire is given by $N = 2m\Omega\dot{r}$. The equation for r is a second order linear ODE with constant coefficients . Its general solution can be written in the form

$$r = A\cosh\Omega t + B\sinh\Omega t,$$

where A, B are arbitrary constants. The initial conditions $r = a$ and $\dot{r} = 0$ when $t = 0$ imply that $A = a$ and $B = 0$ so that the **position** of P at time t is given by

$$r = a\cosh\Omega t.$$

On using this expression for r in the formula for N, the **reaction of the wire** at time t is found to be

$$N = 2ma\Omega^2\sinh\Omega t. \blacksquare$$

Problem 4.16

A body of mass m is projected with speed u in a medium that exerts a resistance force of magnitude (i) $mk|\boldsymbol{v}|$, or (ii) $mK|\boldsymbol{v}|^2$, where k and K are positive constants and $\boldsymbol{v}$ is the velocity of the body. Gravity can be ignored. Determine the subsequent motion in each case. Verify that the motion is bounded in case (i), but not in case (ii).

Solution

(i) Suppose that the motion starts from the origin and takes place along the positive x-axis. Then the **equation of motion** is

$$m\frac{dv}{dt} = -mkv,$$

where $v = \dot{x}$. This is a separable first order ODE for v. On separating, we obtain

$$\int \frac{dv}{v} = -k \int dt,$$

that is,

$$\ln v = -kt + C,$$

where C is an integration constant. The initial condition $v = u$ when $t = 0$ gives $C = \ln u$ and hence the **velocity** of the body at time t is

$$v = u e^{-kt}.$$

To find the displacement of the body, we write $v = dx/dt$ and integrate again. This gives

$$x = -\frac{u}{k} e^{-kt} + D,$$

where D is a second integration constant. The initial condition $x = 0$ when $t = 0$ gives $D = u/k$ and hence the **displacement** of the body at time t is

$$x = \frac{u}{k}\left(1 - e^{-kt}\right).$$

As t tends to infinity, the negative exponential e^{-kt} tends to zero and so x tends to u/k. Hence x tends to a *finite* limit and the **motion is bounded**. ■

(ii) Suppose that the motion starts from the origin and takes place along the positive x-axis. Then the **equation of motion** is

$$m\frac{dv}{dt} = -mKv^2,$$

where $v = \dot{x}$. This is a separable first order ODE for v. On separating, we obtain

$$\int \frac{dv}{v^2} = -K \int dt,$$

that is,

$$-\frac{1}{v} = -Kt + C,$$

where C is an integration constant. The initial condition $v = u$ when $t = 0$ gives $C = -1/u$ and hence the **velocity** of the body at time t is

$$v = \frac{u}{Kut + 1}.$$

To find the displacement of the body, we write $v = dx/dt$ and integrate again. This gives

$$x = \frac{1}{K}\ln(Kut + 1) + D,$$

where D is a second integration constant. The initial condition $x = 0$ when $t = 0$ gives $D = 0$ and hence the **displacement** of the body at time t is

$$x = \frac{1}{K}\ln(Kut + 1).$$

As t tends to infinity, x tends to infinity and so the **motion is unbounded**. ■

Problem 4.17

A body is projected vertically upwards with speed u and moves under uniform gravity in a medium that exerts a resistance force proportional to the square of its speed and in which the body's terminal speed is V. Find the maximum height above the starting point attained by the body and the time taken to reach that height.

Show also that the speed of the body when it returns to its starting point is $uV/(V^2+u^2)^{1/2}$. [*Hint*. The equations of motion for ascent and descent are different.]

Solution

Suppose that the medium exerts a resistance force of magnitude mKv^2 on the body, where K is a positive constant. Then, if the body were falling vertically downwards with its terminal speed V, its acceleration would be zero and so

$$mg - mKV^2 = 0.$$

Hence, the **terminal speed** is related to the resistance constant K by the formula

$$V^2 = \frac{g}{K}.$$

Upward motion Suppose that the upward motion starts from the origin and takes place along the z-axis, which is pointing vertically upwards. Then the equation of motion is

$$m\frac{dv}{dt} = -mg - mKv^2,$$

where $v = \dot{z}$. On intoducing the terminal speed V instead of K, this equation becomes

$$\frac{dv}{dt} = -g\left(1 + \frac{v^2}{V^2}\right),$$

which is the **equation for upwards motion**.

This is a separable first order ODE for v as a function of the time t. On separating, we obtain

$$\int \frac{dv}{v^2+V^2} = -\frac{g}{V^2}\int dt,$$

that is,

$$\frac{1}{V}\tan^{-1}\frac{v}{V} = -\frac{g}{V^2}t + C,$$

where C is an integration constant. The initial condition $v = u$ when $t = 0$ gives

$$C = \frac{1}{V}\tan^{-1}\frac{u}{V}$$

and hence

$$t = \frac{V}{g}\left(\tan^{-1}\frac{u}{V} - \tan^{-1}\frac{v}{V}\right).$$

We could invert this expression to find a formula for the upward velocity v at time t, but this manipulation is not neccessary. Since $v = 0$ at the highest point, the **time τ taken to reach the maximum height** is given immediately by

$$\tau = \frac{V}{g}\left(\tan^{-1}\frac{u}{V}\right). \blacksquare$$

To find the maximum height itself, we will begin again with a modified form of the equation of motion. Since

$$\frac{dv}{dt} = \frac{dv}{dz}\times\frac{dz}{dt} = v\frac{dv}{dz},$$

the equation of motion can be written in the **modified form**

$$v\frac{dv}{dz} = -g\left(1 + \frac{v^2}{V^2}\right),$$

which a separable first order ODE for v as a function of the height z. On separating, we obtain

$$\int \frac{v\,dv}{v^2+V^2} = -\frac{g}{V^2}\int dz,$$

that is,

$$\tfrac{1}{2}\ln\left(v^2+V^2\right) = -\frac{g}{V^2}z + D,$$

where D is a second integration constant. The initial condition $v = u$ when $z = 0$ gives

$$D = \tfrac{1}{2}\ln\left(u^2+V^2\right)$$

and hence

$$z = \frac{V^2}{2g} \ln\left(\frac{u^2 + V^2}{v^2 + V^2}\right).$$

We could invert this expression to find a formula for the upward velocity v at height z, but this manipulation is not neccessary. Since $v = 0$ at the highest point, the **maximum height** H is given immediately by

$$H = \frac{V^2}{2g} \ln\left(1 + \frac{u^2}{V^2}\right). \blacksquare$$

Downward motion In the downward motion, it is best to take new axes with O at the highest point and Oz pointing vertically *downwards*. The equation of motion is then

$$m\frac{dv}{dt} = mg - mKv^2,$$

where v $(= \dot{z})$ is the *downwards* velocity of the body. On introducing the terminal speed V instead of K, this becomes

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{V^2}\right),$$

which is the **equation for downwards motion**. We will take this equation in the **modified form**

$$v\frac{dv}{dz} = g\left(1 - \frac{v^2}{V^2}\right),$$

which is a separable first order ODE for v as a function of z. Separation gives

$$\int \frac{v\,dv}{V^2 - v^2} = \frac{g}{V^2}\int dz,$$

that is,

$$-\tfrac{1}{2}\ln\left(V^2 - v^2\right) = \frac{g}{V^2}z + E,$$

where E is a third integration constant. The initial condition $v = 0$ when $z = 0$ gives

$$E = -\tfrac{1}{2}\ln V^2$$

and hence

$$z = \frac{V^2}{2g} \ln\left(\frac{V^2}{V^2 - v^2}\right).$$

The body returns to its starting point when $z = H$, that is, when

$$\frac{V^2}{2g} \ln\left(\frac{V^2}{V^2 - v^2}\right) = \frac{V^2}{2g} \ln\left(1 + \frac{u^2}{V^2}\right).$$

This equation solves quite easily for v to give

$$v = \frac{uV}{\left(u^2 + V^2\right)^{1/2}}.$$

This is the downward **velocity** of the body on its return to its starting point. ■

Problem 4.18 ✻

A body is released from rest and moves under uniform gravity in a medium that exerts a resistance force proportional to the square of its speed and in which the body's terminal speed is V. Show that the time taken for the body to fall a distance h is

$$\frac{V}{g}\cosh^{-1}\left(e^{gh/V^2}\right).$$

In his famous (but probably apocryphal) experiment, Galileo dropped different objects from the top of the tower of Pisa and timed how long they took to reach the ground. If Galileo had dropped two iron balls, of 5 mm and 5 cm radius respectively, from a height of 25 m, what would the descent times have been? Is it likely that this difference could have been detected? [Use the quadratic law of resistance with $C = 0.8$. The density of iron is 7500 kg m^{-3}.]

Solution

Suppose that the motion starts from the origin and takes place along the z-axis, where Oz points vertically *downwards*. Suppose also that the medium exerts a resistance force of magnitude mKv^2 on the body, where K is a positive constant. Then the equation of motion is

$$m\frac{dv}{dt} = mg - mKv^2,$$

where v $(= \dot{z})$ is the *downwards* velocity of the body.

In particular, if the body were falling with its terminal speed V, its acceleration would be zero and so

$$mg - mKV^2 = 0.$$

Hence the **terminal speed** V is related to the resistance constant K by the formula

$$V^2 = \frac{g}{K}.$$

On intoducing the terminal speed V instead of K, we obtain

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{V^2}\right),$$

which is the **equation for downwards motion**. This is a separable first order ODE for v as a function of the time t. On separating, we obtain

$$\int \frac{dv}{V^2 - v^2} = \frac{g}{V^2}\int dt.$$

Hence

$$\frac{g}{V^2}t = \int \frac{dv}{V^2 - v^2}$$
$$= \frac{1}{2V}\int \left(\frac{1}{V-v} + \frac{1}{V+v}\right) dv$$
$$= \frac{1}{2V}\ln\left(\frac{V+v}{V-v}\right) + C,$$

where C is an integration constant. The initial condition $v = 0$ when $t = 0$ gives $C = 0$ and hence

$$t = \frac{V}{2g}\ln\left(\frac{V+v}{V-v}\right).$$

This formula can be now be inverted to give v as a function of t. After some manipulation, we find that

$$v = V\tanh\frac{gt}{V}.$$

This is the **velocity** of the body at time t.

To find the displacement z of the body, we write $v = dz/dt$ and integrate again. This gives

$$z = V\int \tanh\frac{gt}{V}\,dt$$
$$= V\int \frac{\sinh(gt/V)}{\cosh(gt/V)}\,dt$$
$$= \frac{V^2}{g}\ln\left(\cosh\frac{gt}{V}\right) + D,$$

where D is a second integration constant. The initial condition $z = 0$ when $t = 0$ gives $D = 0$ and hence the downward **displacement** of the body at time t is

$$z = \frac{V^2}{g}\ln\left(\cosh\frac{gt}{V}\right).$$

This formula can be inverted to find t as a function of z. After some manipulation, we find that the **time** τ **taken** for the body to fall a distance h is

$$\tau = \frac{V}{g}\cosh^{-1}\left(e^{gh/V^2}\right).$$

To calculate the descent times in Galileo's experiment, we must first find the terminal speeds of the two iron balls. If a ball is falling with its terminal speed V, then $D = mg$, where D is the drag on the ball and m is its mass. [In this problem, the buoyancy of the air is negligible.] With the quadratic law of resistance, this requires that

$$C\rho a^2 V^2 = \left(\tfrac{4}{3}\pi a^3\right)\rho' g,$$

where C is the drag coefficient of the sphere, a is its radius, and ρ, ρ' are densities of air and iron respectively. The **terminal speed** of the ball is therefore given by

$$V = \left(\frac{4\pi\rho' g a}{3C\rho}\right)^{1/2}.$$

On using the data given in the problem (and Table 1 for the density of air), we can now calculate the terminal speeds and hence the **descent times** of the balls. We find that

(i) the ball of radius 5 mm has a terminal speed of 40 $\mathrm{m\,s^{-1}}$ and a descent time of 2.32 s, and

(ii) the ball of radius 5 cm has a terminal speed of 127 $\mathrm{m\,s^{-1}}$ and a descent time of 2.27 s.

Thus the larger ball arrives first but the time difference is too small to have been observed by Galileo. ■

Problem 4.19

A body is projected vertically upwards with speed u and moves under uniform gravity in a medium that exerts a resistance force proportional to the fourth power its speed and in which the body's terminal speed is V. Find the maximum height above the starting point attained by the body.

Deduce that, however large u may be, this maximum height is always less than $\pi V^2/4g$.

Solution

Suppose that the medium exerts a resistance force of magnitude mKv^4 on the body, where K is a positive constant. Then, if the body were falling vertically downwards with its terminal speed V, its acceleration would be zero and so

$$mg - mKV^4 = 0.$$

Hence, the **terminal speed** is related to the resistance constant K by the formula

$$V^4 = \frac{g}{K}.$$

Suppose that the upward motion starts from the origin and takes place along the z-axis, which is pointing vertically upwards. Then the equation of motion is

$$m\frac{dv}{dt} = -mg - mKv^4,$$

where $v = \dot{z}$. On intoducing the terminal speed V instead of K, this equation becomes

$$\frac{dv}{dt} = -g\left(1 + \frac{v^4}{V^4}\right),$$

which is the **equation for upwards motion**. To find the maximum height, we will use the modified form of this equation. Since

$$\frac{dv}{dt} = \frac{dv}{dz} \times \frac{dz}{dt} = v\frac{dv}{dz},$$

the equation of motion can be written in the **modified form**

$$v\frac{dv}{dz} = -g\left(1 + \frac{v^4}{V^4}\right),$$

which is a separable first order ODE for v as a function of the height z. On separating, we obtain

$$\int \frac{v\,dv}{v^4 + V^4} = -\frac{g}{V^4}\int dz.$$

To perform the integral over v, we make the substitution $w = v^2$. Then $dw = 2v dv$ and we find that

$$\begin{aligned} -\frac{g}{V^4}\,z &= \frac{1}{2}\int \frac{dw}{w^2 + V^4} \\ &= \frac{1}{2V^2}\tan^{-1}\frac{w}{V^2} + C \\ &= \frac{1}{2V^2}\tan^{-1}\frac{v^2}{V^2} + C, \end{aligned}$$

where C is an integration constant. The initial condition $v = u$ when $t = 0$ gives

$$C = -\frac{1}{2V^2}\tan^{-1}\frac{u^2}{V^2}$$

and hence

$$z = \frac{V^2}{2g}\left(\tan^{-1}\frac{u^2}{V^2} - \tan^{-1}\frac{v^2}{V^2}\right).$$

We could invert this expression to find a formula for the upward velocity v at height z, but this manipulation is not neccessary. Since $v = 0$ at the highest point, the **maximum height** attained by the body is given immediately by

$$\frac{V^2}{2g}\tan^{-1}\frac{u^2}{V^2}.$$

No matter how large u may be, $\tan^{-1}(u^2/V^2)$ is always less than $\frac{1}{2}\pi$. Hence, for any projection speed, the height reached is always less than $\pi V^2/4g$. ■

Problem 4.20 *Millikan's experiment*

A microscopic spherical oil droplet, of density ρ and unknown radius, carries an unknown electric charge. The droplet is observed to have terminal speed v_1 when falling vertically in air of viscosity μ. When a uniform electric field E_0 is applied in the vertically upwards direction, the same droplet was observed to move *upwards* with terminal speed v_2. Find the charge on the droplet. [Use the low Reynolds number approximation for the drag.]

Solution

If the droplet is simply falling through air with terminal speed v_1, then $D = mg$, where D is the drag on the droplet and m is its mass. [In this problem, the buoyancy of the air is negligible.] On using the Stokes formula for D, we obtain

$$6\pi a\mu v_1 = \left(\tfrac{4}{3}\pi a^3\right)\rho' g,$$

where a is the radius of the droplet, μ is the viscosity of air, and ρ' is the density of the oil. [Since the droplet is *not* a rigid body, one may wonder why the Stokes formula can be used. Stokes's analysis can be generalised to the case of a *liquid* sphere. This analysis shows that there is a correction to Stokes's formula of order $O(\mu/\mu')$, where μ' is the viscosity of the *oil*. The ratio μ/μ' is about 10^{-4} for air/oil and so (fortunately) the correction is negligible.] The **radius** of the droplet is therefore

$$a = 3\left(\frac{\mu v_1}{2\rho' g}\right)^{1/2}.$$

Suppose that the droplet is now subject to an upwards electric field E_0 and is rising with terminal speed v_2. Then $eE_0 = mg + D$, where e is the (positive) charge on the droplet, m is its mass and D is the drag. On using the Stokes formula again, we obtain

$$\begin{aligned} eE_0 &= \left(\tfrac{4}{3}\pi a^3\right)\rho' g + 6\pi a\mu v_2 \\ &= 6\pi a\mu v_1 + 6\pi a\mu v_2 \\ &= 6\pi a\mu\,(v_1 + v_2)\,. \end{aligned}$$

Hence the **charge** carried by the droplet is

$$e = \frac{6\pi a\mu\,(v_1 + v_2)}{E_0},$$

where $a = 3(\mu v_1/2\rho' g)^{1/2}$. ■

Problem 4.21

A mortar gun, with a maximum range of 40 m on level ground, is placed on the edge of a vertical cliff of height 20 m overlooking a horizontal plain. Show that the horizontal range R of the mortar gun is given by

$$R = 40\left\{\sin\alpha + \left(1 + \sin^2\alpha\right)^{\frac{1}{2}}\right\}\cos\alpha,$$

where α is the angle of elevation of the mortar above the horizontal. [Take $g = 10\ \mathrm{m\,s^{-2}}$.]

Evaluate R (to the nearest metre) when $\alpha = 45°$ and $35°$ and confirm that $\alpha = 45°$ does not yield the maximum range.

Solution

Suppose that the motion starts from the origin and takes place in the (x, z)-plane, where Oz points vertically upwards. The path of the shell is then

$$z = x\tan\alpha - \left(\frac{g}{2u^2\cos^2\alpha}\right)x^2,$$

where u is the muzzle speed and α is elevation angle of the gun (see the book p.89). Suppose that the plain is distance h below the cliff. Then the shell lands when $z = -h$, that is, when

$$-h = x\tan\alpha - \left(\frac{g}{2u^2\cos^2\alpha}\right)x^2.$$

The x-coordinate of the landing point therefore satisfies the equation

$$x^2 - \left(\frac{2u^2\sin\alpha\cos\alpha}{g}\right)x - \frac{2hu^2\cos^2\alpha}{g} = 0.$$

The **range** R of the mortar is the *positive* root of this equation, namely,

$$R = R_0\left[\sin\alpha + \left(\sin^2\alpha + \frac{2gh}{u^2}\right)^{1/2}\right]\cos\alpha,$$

where $R_0 = u^2/g$ is the maximum range of the mortar on *level* ground.

From the data in the problem, $R_0 = 40$ m, $h = 20$ m, $g = 10\ \mathrm{m\,s^{-2}}$ and so

$$R = 40\left[\sin\alpha + \left(\sin^2\alpha + 1\right)^{1/2}\right]\cos\alpha \quad \text{m}.$$

When $\alpha = 45°$, $R = 55$ m, and when $\alpha = 35°$, $R = 57$ m, correct to the nearest metre. Thus $\alpha = 45°$ does not yield the maximum range in this problem. ■

Problem 4.22

It is required to project a body from a point on level ground in such a way as to clear a thin vertical barrier of height h placed at distance a from the point of projection. Show that the body will just skim the top of the barrier if

$$\left(\frac{ga^2}{2u^2}\right)\tan^2\alpha - a\tan\alpha + \left(\frac{ga^2}{2u^2} + h\right) = 0,$$

where u is the speed of projection and α is the angle of projection above the horizontal.

Deduce that, if the above trajectory is to exist for some α, then u must satisfy

$$u^4 - 2ghu^2 - g^2a^2 \geq 0.$$

Find the least value of u that satisfies this inequality.

For the special case in which $a = \sqrt{3}h$, show that the minimum projection speed necessary to clear the barrier is $(3gh)^{\frac{1}{2}}$, and find the projection angle that must be used.

Solution

Suppose that the motion starts from the origin and takes place in the (x, z)-plane, where Oz points vertically upwards. Then the path of the body is

$$z = x\tan\alpha - \left(\frac{g}{2u^2\cos^2\alpha}\right)x^2,$$

where u is the projection speed and α is the angle between the direction of projection and the positive x-axis (see the book p.89). If the path just skims the top of the barrier, then u and α must satisfy the equation

$$h = a\tan\alpha - \left(\frac{g}{2u^2\cos^2\alpha}\right)a^2.$$

On using the trigonometric identity $\sec^2\alpha = 1+\tan^2\alpha$, this condition can be written in the form

$$ga^2\tan^2\alpha - 2au^2\tan\alpha + \left(ga^2 + 2hu^2\right) = 0,$$

which is a quadratic equation in the variable $\tan\alpha$. A path skimming the barrier will exist if this equation has *real* roots for $\tan\alpha$. The condition for real roots is

$$u^4 \geq g\left(ga^2 + 2hu^2\right),$$

which can be written in the form

$$\left(u^2 - gh\right)^2 \geq g^2 \left(a^2 + h^2\right).$$

Hence, a path skimming the top of the barrier will exist if the projection speed u satisfies the inequality

$$u^2 \geq gh + \left(a^2 + h^2\right)^{1/2}.$$

For the special case in which $a = \sqrt{3}h$, this condition on u becomes

$$u^2 \geq 3gh.$$

The corresponding value(s) of α are found by solving the quadratic equation for $\tan\alpha$. For the critical case in which $u^2 = 3gh$, the equation for $\tan\alpha$ becomes

$$\tan^2\alpha - 2\sqrt{3}\tan\alpha + 3 = 0,$$

that is,

$$\left(\tan\alpha - \sqrt{3}\right)^2 = 0.$$

Hence (in the critical case) only one projection angle is possible, namely $\alpha = \tan^{-1}\sqrt{3} = 60°$. ■

Problem 4.23

A particle is projected from the origin with speed u in a direction making an angle α with the horizontal. The motion takes place in the (x, z)-plane, where Oz points vertically upwards. If the projection speed u is fixed, show that the particle can be made to pass through the point (a, b) for some choice of α if (a, b) lies *below* the parabola

$$z = \frac{u^2}{2g}\left(1 - \frac{g^2x^2}{u^4}\right).$$

This is called the **parabola of safety**. Points *above* the parabola are 'safe' from the projectile.

An artillery shell explodes on the ground throwing shrapnel in all directions with speeds of up to 30 m s^{-1}. A man is standing at an open window 20 m above the ground in a building 60 m from the blast. Is he safe? [Take $g = 10$ m s^{-2}.]

Solution

This is the same as Problem 4.22 except that now the projection speed u is fixed from the start.

Suppose that the motion starts from the origin and takes place in the (x, z)-plane, where Oz points vertically upwards. Then the path of the body is

$$z = x\tan\alpha - \left(\frac{g}{2u^2\cos^2\alpha}\right)x^2,$$

where u is the given projection speed and α is the angle between the direction of projection and the positive x-axis (see the book p.89). If the path passes through the point (a, b), then a, b and α must satisfy the equation

$$b = a\tan\alpha - \left(\frac{g}{2u^2\cos^2\alpha}\right)a^2.$$

On using the trigonometric identity $\sec^2\alpha = 1+\tan^2\alpha$, this condition can be written in the form

$$ga^2\tan^2\alpha - 2au^2\tan\alpha + \left(ga^2 + 2bu^2\right) = 0,$$

which is a quadratic equation in the variable $\tan\alpha$. A path through the point (a, b) will exist if this equation has *real* roots for $\tan\alpha$. The condition for real roots is

$$u^4 \geq g\left(ga^2 + 2bu^2\right),$$

which can be written as a condition on the coordinate b in the form

$$b \le \frac{u^2}{2g}\left(1 - \frac{g^2a^2}{u^4}\right).$$

Hence, a path through (a, b) will exist if (a, b) lies *below* the parabola

$$z = \frac{u^2}{2g}\left(1 - \frac{g^2x^2}{u^4}\right).$$

This is the **parabola of safety**.

From the data given in the problem, $u = 30\ \mathrm{m\,s^{-1}}$ and $g = 10\ \mathrm{m\,s^{-2}}$ so that the parabola of safety is

$$z = 45\left(1 - \frac{x^2}{8100}\right).$$

The window is at the point $(60, 20)$ which lies *below* this parabola. It follows that the **man is not safe**. ■

Problem 4.24

A projectile is fired from the top of a conical mound of height h and base radius a. What is the least projection speed that will allow the projectile to clear the mound? [*Hint*. Make use of the parabola of safety.]

A mortar gun is placed on the summit of a conical hill of height 60 m and base diameter 160 m. If the gun has a muzzle speed of 25 m s^{-1}, can it shell anywhere on the hill? [Take $g = 10$ m s^{-2}.]

Solution

Suppose that the motion starts from the origin and takes place in the (x, z)-plane, where Oz points vertically upwards. If the projection speed is u, then the **parabola of safety** is

$$z = \frac{u^2}{2g}\left(1 - \frac{g^2x^2}{u^4}\right).$$

The foot of the mound can be reached if the point $(a, -h)$ lies below this parabola, that is, if

$$-h \le \frac{u^2}{2g}\left(1 - \frac{g^2a^2}{u^4}\right).$$

Hence the foot of the mound can be reached by the projectile if its projection speed u satisfies the condition

$$u^4 + 2gh\,u^2 - g^2a^2 \ge 0.$$

This inequality can be written in the form

$$\left(u^2 + gh\right)^2 \ge g^2\left(a^2 + h^2\right).$$

Hence, a **path clearing the mound** will exist if the projection speed u satisfies the condition

$$u^2 > g\left(a^2 + h^2\right)^{1/2} - gh.$$

From the data given in the problem, $a = 80$ m, $h = 60$ m and $g = 10$ m s^{-2}. All points on the hill can therefore be reached if the muzzle speed u satisfies

$$u^2 \ge 10\left(80^2 + 60^2\right)^{1/2} - 600 = 400,$$

that is, if $u \ge 20$ m s^{-1}. The actual muzzle speed of 25 m s^{-1} is therefore more than enough to shell anywhere on the hill. ■

Problem 4.25

An artillery gun is located on a plane surface inclined at an angle β to the horizontal. The gun is aligned with the line of steepest slope of the plane. The gun fires a shell with speed u in the direction making an angle α with the (upward) line of steepest slope. Find where the shell lands.

Deduce the maximum ranges R^U, R^D, up and down the plane, and show that

$$\frac{R^U}{R^D} = \frac{1 - \sin\beta}{1 + \sin\beta}.$$

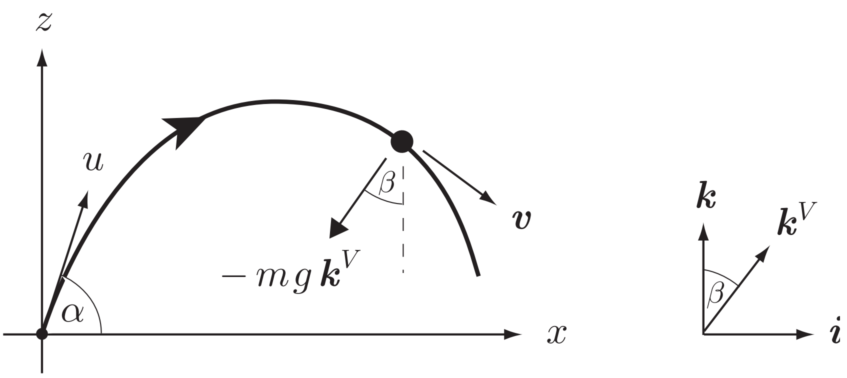

FIGURE 4.9 Projectile motion under uniform gravity on a plane inclined at angle β to the horizontal. The unit vector $\boldsymbol{k}^V$ points *vertically upwards*.

Solution

Suppose that the motion starts from the origin and takes place in the (x, z)-plane, where Ox points up the line of steepest slope of the plane and Oz is perpendicular to the plane; $\boldsymbol{i}$ and $\boldsymbol{j}$ are the corresponding unit vectors. Note that the upward vertical is inclined at angle β to the axis Oz (see Figure 4.9).

The vector **equation of motion** for the shell is

$$m\frac{d\boldsymbol{v}}{dt} = -mg\boldsymbol{k}^V,$$

where $\boldsymbol{k}^V$ is the unit vector pointing vertically upwards. The initial condition is $\boldsymbol{v} = (u\cos\alpha)\boldsymbol{i} + (u\sin\alpha)\boldsymbol{k}$ when $t = 0$. If we now write $\boldsymbol{v} = v_x\boldsymbol{i} + v_z\boldsymbol{k}$ and take components of this equation (and initial condition) in the $\boldsymbol{i}$- and $\boldsymbol{k}$-directions, we obtain the two scalar equations of motion

$$\frac{dv_x}{dt} = -g\sin\beta,$$
$$\frac{dv_z}{dt} = -g\cos\beta,$$

with the respective initial conditions $v_x = u\cos\alpha$ and $v_z = u\sin\alpha$ when $t = 0$. Simple integrations then give the components of the shell **velocity** at time t to be

$$v_x = u\cos\alpha - g\sin\beta\, t,$$
$$v_z = u\sin\alpha - g\cos\beta\, t.$$

The **position** of the particle at time t can now be found by integrating the expressions for v_x, v_z and applying the initial conditions $x = 0$ and $z = 0$ when $t = 0$. This gives

$$x = u\cos\alpha\, t - \tfrac{1}{2}g\sin\beta\, t^2,$$
$$z = u\sin\alpha\, t - \tfrac{1}{2}g\cos\beta\, t^2.$$

The shell lands when $z = 0$ again, that is, when

$$t = \frac{2u\sin\alpha}{g\cos\beta}.$$

The value of x at this instant is

$$\begin{aligned} x &= u\cos\alpha\, t - \tfrac{1}{2}g\sin\beta\, t^2, \\ &= u\cos\alpha\left(\frac{2u\sin\alpha}{g\cos\beta}\right) - \tfrac{1}{2}g\sin\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2 \\ &= \frac{u^2}{g\cos^2\beta}\Big(\sin 2\alpha\cos\beta - \sin\beta(1-\cos 2\alpha)\Big) \\ &= \frac{u^2}{g\cos^2\beta}\Big(\sin(2\alpha+\beta) - \sin\beta\Big). \end{aligned}$$

On allowing the elevation α of the gun to vary in the range $0 < \alpha < \pi$, we see that the landing point of the shell varies in the range

$$\frac{u^2}{g\cos^2\beta}\big(-1-\sin\beta\big) \le x \le \frac{u^2}{g\cos^2\beta}\big(1-\sin\beta\big).$$

It follows that the R^U and R^D, the **ranges of the shell** up and down the plane, are given by

$$R^U = \frac{u^2}{g\cos^2\beta}\big(1-\sin\beta\big)$$
$$R^D = \frac{u^2}{g\cos^2\beta}\big(1+\sin\beta\big).$$

In particular, the ratio of the two ranges is

$$\frac{R^U}{R^D} = \frac{1-\sin\beta}{1+\sin\beta}. \blacksquare$$

Problem 4.26

Show that, when a particle is projected from the origin in a medium that exerts *linear* resistance, its position vector at time t has the general form

$$\boldsymbol{r} = -\alpha(t)\boldsymbol{k} + \beta(t)\boldsymbol{u},$$

where $\boldsymbol{k}$ is the vertically upwards unit vector and $\boldsymbol{u}$ is the *velocity* of projection. Deduce the following results:

(i) A number of particles are projected simultaneously from the same point, with the same speed, but in *different directions*. Show that, at each later time, the particles all lie on the surface of a sphere.
(ii) A number of particles are projected simultaneously from the same point, in the same direction, but with *different speeds*. Show that, at each later time, the particles all lie on a straight line.
(iii) Three particles are projected simultaneously in a completely general manner. Show that the plane containing the three particles remains parallel to some fixed plane.

Solution

Suppose that the motion starts from the origin and takes place in the (x, z)-plane, where Oz points vertically upwards; $\boldsymbol{i}$ and $\boldsymbol{k}$ are the corresponding unit vectors. The solution to the projectile problem with linear resistance has been obtained in the book on p. 90. The position of the body at time t was found to be

$$x = \frac{u\cos\alpha}{K}\left(1 - e^{-Kt}\right),$$

$$z = \frac{Ku\sin\alpha + g}{K^2}\left(1 - e^{-Kt}\right) - \frac{g}{K}\,t,$$

where K is the resistance constant, u is the projection speed and α is the angle between the direction of projection and the positive x-axis. The **position vector** of the body at time t is therefore

$$\begin{aligned}
\boldsymbol{r} &= x\boldsymbol{i} + z\boldsymbol{k} \\
&= \frac{u\cos\alpha}{K}\left(1 - e^{-Kt}\right)\boldsymbol{i} + \left(\frac{Ku\sin\alpha + g}{K^2}\left(1 - e^{-Kt}\right) - \frac{g}{K}\,t\right)\boldsymbol{k} \\
&= \frac{1}{K}\left(1 - e^{-Kt}\right)\left(u\cos\alpha\,\boldsymbol{i} + u\sin\alpha\,\boldsymbol{k}\right) - \left(Kt - \left(1 - e^{-Kt}\right)\right)\frac{g}{K^2}\boldsymbol{k} \\
&= -\alpha(t)\boldsymbol{k} + \beta(t)\boldsymbol{u},
\end{aligned}$$

where

$$\alpha(t) = e^{-Kt} - 1 + Kt,$$
$$\beta(t) = \frac{1}{K}\left(1 - e^{-Kt}\right),$$

and $\boldsymbol{u}$ $(= u\cos\alpha\,\boldsymbol{i} + u\sin\alpha\,\boldsymbol{k})$ is the velocity of projection.

It follows that, if a number of particles P_1, P_2, ... are *simultaneously* projected from O with velocities $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ..., their position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ... at time t are given by the formula

$$\boldsymbol{r}_i = -\alpha(t)\boldsymbol{k} + \beta(t)\boldsymbol{u}_i.$$

The geometrical interpretation of this formula is shown in Figure 4.10.

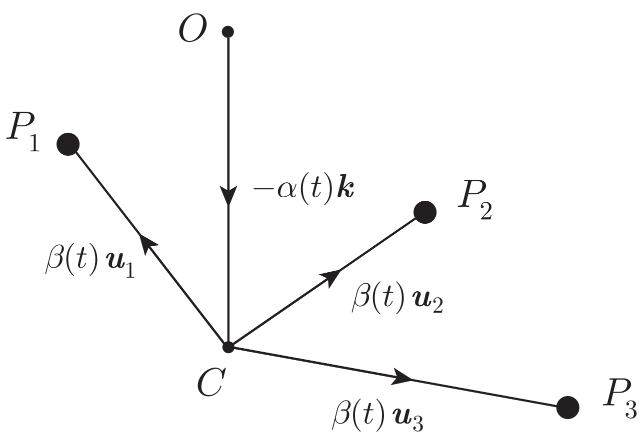

FIGURE 4.10 The positons of three of the particles at time t.

(i) If the particles all have the *same initial speed* u, then the distances CP_1, CP_2, ... are all equal to $u\beta(t)$. Hence the particles all lie on a sphere with centre C and radius $u\beta(t)$. Note that this sphere is both falling and expanding. ■

(ii) If the initial velocities of the particles are all *parallel*, then the line segments $\overrightarrow{CP_1}$, $\overrightarrow{CP_1}$, ... are all parallel. Hence the particles all lie on a straight line through C. Note that this line is falling but remains 'parallel to itself'. ■

(iii) Consider three particles A, B, C with initial velocities $\boldsymbol{u}^A$, $\boldsymbol{u}^B$, $\boldsymbol{u}^C$. Then their position vectors at time t are

$$\begin{aligned} \boldsymbol{a} &= -\alpha(t)\,\boldsymbol{k} + \beta(t)\,\boldsymbol{u}^A, \\ \boldsymbol{b} &= -\alpha(t)\,\boldsymbol{k} + \beta(t)\,\boldsymbol{u}^B, \\ \boldsymbol{c} &= -\alpha(t)\,\boldsymbol{k} + \beta(t)\,\boldsymbol{u}^C. \end{aligned}$$

Hence, the vector $(\boldsymbol{b} - \boldsymbol{a})\times(\boldsymbol{c} - \boldsymbol{a})$, which is *normal* to the plane ABC, is given by

$$(\boldsymbol{b} - \boldsymbol{a})\times(\boldsymbol{c} - \boldsymbol{a}) = \beta(t)^2 \left(\boldsymbol{u}^B - \boldsymbol{u}^A\right)\times\left(\boldsymbol{u}^C - \boldsymbol{u}^A\right).$$

This vector has constant direction and so the plane ABC remains ‘parallel to itself’. ■

Problem 4.27

A body is projected in a steady horizontal wind and moves under uniform gravity and *linear* air resistance. Show that the influence of the wind is the same as if the magnitude and direction of gravity were altered. Deduce that it is possible for the body to return to its starting point. What is the shape of the path in this case?

Solution

Let the unit vector $\boldsymbol{k}$ point vertically upwards and let $U\,\boldsymbol{i}$ be the constant wind velocity. Suppose the motion is viewed from a reference frame moving with the wind. In this frame, the air is still and the equation of motion for the *apparent* velocity $\boldsymbol{v}'$ is

$$m\frac{d\boldsymbol{v}'}{dt} = -mg\,\boldsymbol{k} - mK\boldsymbol{v}',$$

where K is the linear resistance constant. This equation can be written in the form

$$\frac{d\boldsymbol{v}'}{dt} = -g\left(\boldsymbol{k} + \frac{\boldsymbol{v}'}{V}\right),$$

where V is the terminal speed of the body *in still air*. Since the true velocity $\boldsymbol{v}$ is related to the apparent velocity $\boldsymbol{v}'$ by

$$\boldsymbol{v} = \boldsymbol{v}' + U\,\boldsymbol{i}\,,$$

it follows that the **equation of motion** for the true velocity is

$$\frac{d\boldsymbol{v}}{dt} = -g\left(\boldsymbol{k} + \frac{\boldsymbol{v} - U\,\boldsymbol{i}}{V}\right).$$

This equation can be written in the form

$$\frac{d\boldsymbol{v}}{dt} = -g^*\left(\boldsymbol{k}^* + \frac{\boldsymbol{v}}{V^*}\right).$$

where

$$g^* = g\left(1 + \frac{U^2}{V^2}\right)^{1/2}, \quad \boldsymbol{k}^* = \frac{V\,\boldsymbol{k} - U\,\boldsymbol{i}}{\left(U^2 + V^2\right)^{1/2}}, \quad V^* = \left(U^2 + V^2\right)^{1/2}.$$

This is the same as the equation with no wind, except that g, $\boldsymbol{k}$ and V are replaced by g^*, $\boldsymbol{k}^*$ and V^*. The quantities g^* and $-\boldsymbol{k}^*$ can be regarded as the magnitude and direction of **modified gravity**, and V^* is the **modified terminal speed**. This terminal speed is consistent with that calculated in Problem 4.7.

The body will return to its starting point if it is projected in the direction of $\boldsymbol{k}^*$. In this case, the path is a straight line inclined into the wind from the vertical by an angle $\tan^{-1}(U/V)$. ■

Problem 4.28

The radius of the Moon's approximately circular orbit is 384,000 km and its period is 27.3 days. Estimate the mass of the Earth. [$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.] The actual mass is 5.97×10^{24} kg. What is the main reason for the error in your estimate?

An artificial satellite is to be placed in a circular orbit around the Earth so as to be 'geostationary'. What must the radius of its orbit be? [The period of the Earth's rotation is 23 h 56 m, *not* 24 h. Why?]

Solution

Example 4.8 in the book solves the problem of a body moving in a circular orbit about a *fixed* gravitating mass. The period τ of the motion was found to be

$$\tau^2 = \frac{4\pi^2 R^3}{MG},$$

where R is the radius of the orbit and M is the fixed mass.

(i) If the radius and period of a circular orbit are known, the **gravitating mass** M can be found from the formula

$$M = \frac{4\pi^2 R^3}{G\tau^2}.$$

In the orbit of the moon about the Earth, $R = 384,000$ km and $\tau = 27.3$ days. The calculated value of the **mass of the Earth** is then

$$M = \frac{4\pi^2 \times (3.84 \times 10^8)^3}{(6.67 \times 10^{-11}) \times (27.3 \times 1436 \times 60)^2}$$
$$= 6.06 \times 10^{24} \text{ kg}.$$

This figure overestimates the actual mass of the Earth, which is 5.97×10^{24} kg. Most of this small error arises because we have ignored the motion of the Earth induced by the Moon. ■

(ii) If we need to produce a satellite orbit that has a given period τ, then the **orbit radius** R must be taken to be

$$R = \left(\frac{MG\tau^2}{4\pi^2}\right)^{1/3}.$$

For a geostationary satellite, $M = 5.97 \times 10^{24}$ kg and $\tau = 1436$ min. The calculated **radius of the geostationary orbit** is then

$$R = \left(\frac{(5.97 \times 10^{24}) \times (6.67 \times 10^{-11}) \times (1436 \times 60)^2}{4\pi^2} \right)^{1/3}$$
$$= 4.23 \times 10^7 \text{ m}$$
$$= 42,300 \text{ km.} \blacksquare$$

Problem 4.29 *Conical pendulum*

A particle is suspended from a fixed point by a light inextensible string of length a. Investigate 'conical motions' of this pendulum in which the string maintains a constant angle α with the downward vertical. Show that, for any acute angle α, a conical motion exists and that the particle speed u is given by $u^2 = ag \sin\alpha \tan\alpha$.

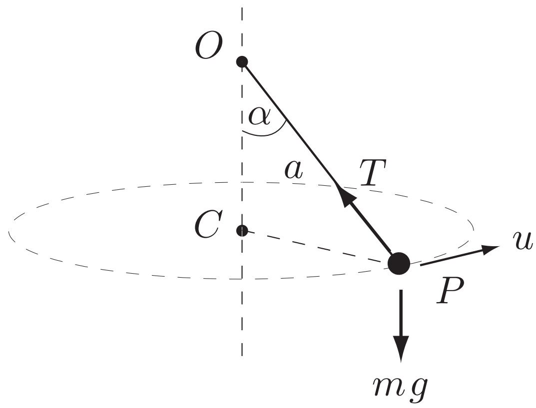

FIGURE 4.11 The conical pendulum.

Solution

Suppose the pendulum is in conical motion with the string inclined at angle α to the downward vertical (see Figure 4.11). Let the speed of the mass be u. Then the vertical component of the **equation of motion** gives

$$0 = T \cos\alpha - mg,$$

the component in the direction $\overrightarrow{PC}$ gives

$$m\left(\frac{u^2}{a \sin\alpha}\right) = T \sin\alpha,$$

and the component in the direction of motion is satisfied identically if u is constant. Hence $T = mg/\cos\alpha$ and a **conical motion at angle** α **is possible** if the speed of the mass is given by

$$u^2 = ag \sin\alpha \tan\alpha. \blacksquare$$

Problem 4.30

A particle of mass m is attached to the highest point of a *smooth* rigid sphere of radius a by a light inextensible string of length $\pi a/4$. The particle moves in contact with the outer surface of the sphere, with the string taut, and describes a horizontal circle with constant speed u. Find the reaction of the sphere on the particle and the tension in the string. Deduce the maximum value of u for which such a motion could take place. What will happen if u exceeds this value?

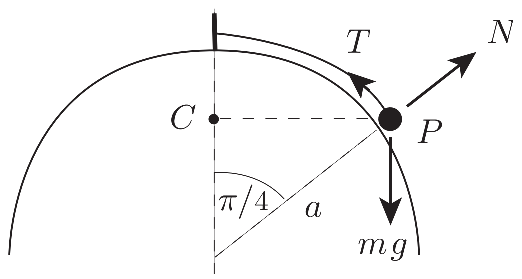

FIGURE 4.12 The 'conical' pendulum on a sphere.

Solution

The system is shown in Figure 4.12. The vertical component of the **equation of motion** gives

$$0 = \frac{T}{\sqrt{2}} + \frac{N}{\sqrt{2}} - mg,$$

the component in the direction $\overrightarrow{PC}$ gives

$$m\left(\frac{\sqrt{2}\,u^2}{a}\right) = \frac{T}{\sqrt{2}} - \frac{N}{\sqrt{2}},$$

and the component in the direction of motion is satisfied identically if u is constant. On solving these simultaneous equations, we find that

$$T = \frac{mg}{\sqrt{2}} + \frac{mu^2}{a}, \qquad N = \frac{mg}{\sqrt{2}} - \frac{mu^2}{a}.$$

The motion as described is possible provided that $N \geq 0$; otherwise the particle will leave the sphere. Thus the **particle remains on the sphere** if the speed u of the mass satisfies the inequality

$$u^2 \leq \frac{ag}{\sqrt{2}}. \blacksquare$$

Problem 4.31

A particle of mass m can move on a *rough* horizontal table and is attached to a fixed point on the table by a light inextensible string of length b. The resistance force exerted on the particle is $-mK\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the particle. Initially the string is taut and the particle is projected horizontally, at right angles to the string, with speed u. Find the angle turned through by the string before the particle comes to rest. Find also the tension in the string at time t.

Solution

Let the fixed point on the table be the origin O, and let r, θ be the plane polar coordinates of the particle. Then $r = b$ and the **velocity** and **acceleration** of the particle are given by

$$\boldsymbol{v} = v\,\widehat{\boldsymbol{\theta}},$$
$$\boldsymbol{a} = -\left(\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}},$$

where $v\ (= b\dot{\theta})$ is the circumferential velocity. The **equation of horizontal motion** is therefore

$$m\left[-\left(\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\theta}\right] = -mK\left(v\,\widehat{\boldsymbol{\theta}}\right) - T\,\widehat{\boldsymbol{r}},$$

where T is the tension in the string. This vector equation is equivalent to the two scalar equations

$$\frac{mv^2}{b} = T,$$
$$\dot{v} + Kv = 0.$$

The general solution of the ODE for v is

$$v = C\,e^{-Kt},$$

where C is an integration constant. The initial condition $v = u$ when $t = 0$ gives $C = u$ and so the **circumferential velocity** of the particle at time t is

$$v = ue^{-Kt}.$$

On making use of this formula for v, the **tension** in the string at time t is found to be

$$T = \left(\frac{mu^2}{b}\right)e^{-2Kt}.$$

To find the angle turned by the string at time t, we write $v = b(d\theta/dt)$ and integrate again. This gives

$$\theta = -\left(\frac{u}{Kb}\right) e^{-Kt} + D,$$

where D is a second integration constant. The initial condition $\theta = 0$ when $t = 0$ gives $D = u/Kb$ so that the **angle turned** by the string at time t is

$$\theta = \frac{u}{Kb}\left(1 - e^{-Kt}\right).$$

The particle never actually comes to rest, but, as t tends to infinity, v *tends* to zero and θ *tends* to the value u/Kb. ■

Problem 4.32 *Mass spectrograph*

A stream of particles of various masses, all carrying the same charge e, is moving along the x-axis in the positive x-direction. When the particles reach the origin they encounter an electronic 'gate' which allows only those particles with a specified speed V to pass. These particles then move in a uniform magnetic field B_0 acting in the z-direction. Show that each particle will execute a semicircle before meeting the y-axis at a point which depends upon its mass. [This provides a method for determining the masses of the particles.]

Solution

The **equation of motion** for a particle of mass m and charge e moving in the uniform magnetic field $B_0\boldsymbol{k}$ is

$$m\frac{d\boldsymbol{v}}{dt} = e\boldsymbol{v}\times(B_0\boldsymbol{k}),$$

which can be written in the form

$$\frac{d\boldsymbol{v}}{dt} = \Omega\boldsymbol{v}\times\boldsymbol{k},$$

where $\Omega = eB_0/m$. This vector equation is equivalent to the three scalar equations

$$\frac{dv_x}{dt} = \Omega v_y, \qquad \frac{dv_y}{dt} = -\Omega v_x, \qquad \frac{dv_z}{dt} = 0.$$

It follows that v_y satisfies the equation

$$\frac{d^2v_y}{dt^2} + \Omega^2 v_y = 0.$$

This second order linear ODE has the general solution

$$v_y = C\cos\Omega t + D\sin\Omega t,$$

where C and D are arbitrary constants. The corresponding expression for v_x is then

$$v_x = C\sin\Omega - D\cos\Omega t.$$

The initial conditions $v_x = V$ and $v_y = 0$ when $t = 0$ give $C = 0$ and $D = -V$ and so the **velocity components** v_x and v_y at time t are given by

$$v_x = V\cos\Omega t,$$
$$v_y = -V\sin\Omega t.$$

The velocity component v_z is easily shown to be zero.

To find the position of the particle at time t, we write $v_x = dx/dt$, $v_y = dy/dt$, $v_z = dz/dt$ and integrate again. This gives

$$x = \frac{V}{\Omega} \sin \Omega t + E,$$

$$y = \frac{V}{\Omega} \cos \Omega t + F,$$

$$z = G,$$

where E, F, G are integration constants. The initial conditions $x = y = z = 0$ when $t = 0$ give $E = 0$, $F = -V/\Omega$, $G = 0$ and so the **position** of the particle at time t is given by

$$x = \frac{V}{\Omega} \sin \Omega t,$$

$$y = -\frac{V}{\Omega}\big(1 - \cos \Omega t\big),$$

$$z = 0.$$

Thus the particle moves on a circle with centre at $(0, V/\Omega)$ and radius V/Ω.

The particle next meets the y-axis when $t = \pi/\Omega$; by this time, the particle will have executed a *semi-circle*. The meeting point is at $(0, Y)$, where

$$Y = -\frac{2V}{\Omega} = -\frac{2mV}{eB_0}.$$

The distance of this point from O is thus *proportional to the mass* m of the particle and this provides a method for measuring particle masses. ■

Problem 4.33 *The magnetron*

An electron of mass m and charge $-e$ is moving under the combined influence of a uniform electric field $E_0\boldsymbol{j}$ and a uniform magnetic field $B_0\boldsymbol{k}$. Initially the electron is at the origin and is moving with velocity $u\,\boldsymbol{i}$. Show that the trajectory of the electron is given by

$$x = a(\Omega t) + b\sin\Omega t, \qquad y = b(1-\cos\Omega t), \qquad z = 0,$$

where $\Omega = eB_0/m$, $a = E_0/\Omega B_0$ and $b = (uB_0 - E_0)/\Omega B_0$. Use computer assistance to plot typical paths of the electron for the cases $a < b$, $a = b$ and $a > b$. [The general path is called a *trochoid*, which becomes a *cycloid* in the special case $a = b$. Cycloidal motion of electrons is used in the **magnetron** vacuum tube, which generates the microwaves in a microwave oven.]

Solution

The **equation of motion** of the electron is

$$m\frac{d\boldsymbol{v}}{dt} = -eE_0\boldsymbol{j} - e\boldsymbol{v}\times(B_0\boldsymbol{k}),$$

which can be written in the form

$$\frac{d\boldsymbol{v}}{dt} = -\left(\frac{\Omega E_0}{B_0}\right)\boldsymbol{j} - \Omega\,\boldsymbol{v}\times\boldsymbol{k},$$

where $\Omega = eB_0/m$. This vector equation is equivalent to the three scalar equations

$$\frac{dv_x}{dt} = -\Omega v_y, \qquad \frac{dv_y}{dt} = -\frac{\Omega E_0}{B_0} + \Omega v_x, \qquad \frac{dv_z}{dt} = 0.$$

It follows that v_y satisfies the equation

$$\frac{d^2 v_y}{dt^2} + \Omega^2 v_y = 0.$$

This second order linear ODE has the general solution

$$v_y = C\cos\Omega t + D\sin\Omega t,$$

where C and D are arbitrary constants. The corresponding expression for v_x is then

$$v_x = \frac{E_0}{B_0} - C\sin\Omega t + D\cos\Omega t.$$

The initial conditions $v_x = u$ and $v_y = 0$ when $t = 0$ give $C = 0$ and $D = u - (E_0/B_0)$ and so the **velocity components** v_x and v_y at time t are given by

$$
\begin{aligned}
v_x &= \left(u - \frac{E_0}{B_0}\right)\cos\Omega t + \frac{E_0}{B_0},\\
v_y &= \left(u - \frac{E_0}{B_0}\right)\sin\Omega t.
\end{aligned}
$$

The velocity component v_z is easily shown to be zero.

To find the position of the particle at time t, we write $v_x = dx/dt$, $v_y = dy/dt$, $v_z = dz/dt$ and integrate again. This gives

$$
\begin{aligned}
x &= \frac{1}{\Omega}\left(u - \frac{E_0}{B_0}\right)\sin\Omega t + \left(\frac{E_0}{B_0}\right)t + E,\\
y &= -\frac{1}{\Omega}\left(u - \frac{E_0}{B_0}\right)\cos\Omega t + F,\\
z &= G,
\end{aligned}
$$

where E, F, G are integration constants. The initial conditions $x = y = z = 0$ when $t = 0$ give

$$
E = 0, \qquad F = \frac{1}{\Omega}\left(u - \frac{E_0}{B_0}\right), \qquad G = 0,
$$

and so the position of the particle at time t is given by

$$
\begin{aligned}
x &= \frac{1}{\Omega}\left(u - \frac{E_0}{B_0}\right)\sin\Omega t + \left(\frac{E_0}{B_0}\right)t,\\
y &= \frac{1}{\Omega}\left(u - \frac{E_0}{B_0}\right)\big(1 - \cos\Omega t\big),\\
z &= 0.
\end{aligned}
$$

This is the **trajectory of the particle**; the path is called a **trochoid**. It can be written in the more compact form

$$
\begin{aligned}
x &= b\sin\Omega t + a\Omega t,\\
y &= b\big(1 - \cos\Omega t\big),\\
z &= 0,
\end{aligned}
$$

where

$$a = \frac{E_0}{\Omega B_0} \qquad \text{and} \qquad b = \frac{1}{\Omega}\left(u - \frac{E_0}{B_0}\right).$$

Three examples of trochoidal motion (corresponding to the cases $a < b$, $a = b$ and $a > b$) are shown in Figure 4.13. ■

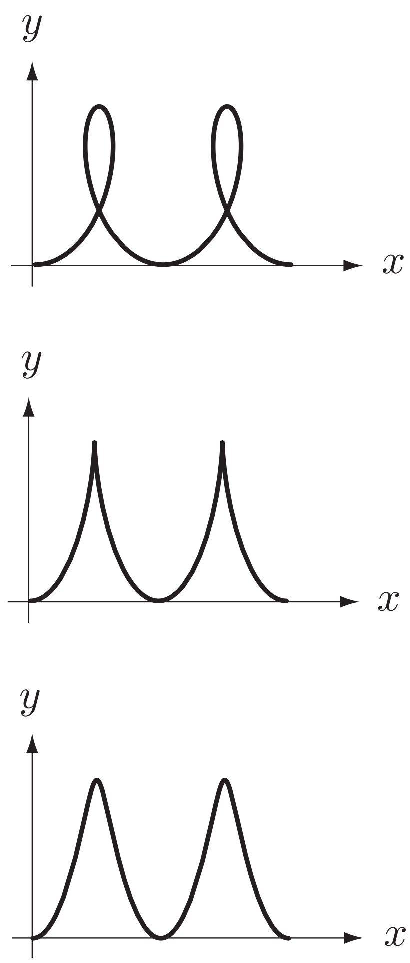

FIGURE 4.13 Three examples of trochoidal motion (two cycles of each are shown):
Top: $a < b$.
Centre: $a = b$ (the cycloid),
Bottom: $a > b$.

Chapter Five

Linear oscillations

and normal modes

Problem 5.1

A certain oscillator satisfies the equation

$$\ddot{x} + 4x = 0.$$

Initially the particle is at the point $x = \sqrt{3}$ when it is projected towards the origin with speed 2. Show that, in the subsequent motion,

$$x = \sqrt{3}\cos 2t - \sin 2t.$$

Deduce the amplitude of the oscillations. How long does it take for the particle to first reach the origin?

Solution

The **general solution** of the equation of motion is

$$x = A\cos 2t + B\sin 2t,$$

where A, B are arbitrary constants. The **initial conditions** $x = \sqrt{3}$ and $\dot{x} = -2$ when $t = 0$ give $A = \sqrt{3}$ and $B = -1$ respectively. The **motion of the particle** is therefore given by

$$x = \sqrt{3}\cos 2t - \sin 2t.$$

The **amplitude** of the oscillations is therefore $\left((\sqrt{3})^2 + (-1)^2\right)^{1/2} = 2.$

The particle is at the origin when

$$\sqrt{3}\cos 2t - \sin 2t = 0,$$

that is, when

$$\tan 2t = \sqrt{3}.$$

This *first* occurs when $t = \frac{1}{6}\pi$. ■

Problem 5.2

When a body is suspended from a fixed point by a certain linear spring, the angular frequency of its vertical oscillations is found to be Ω_1. When a different linear spring is used, the oscillations have angular frequency Ω_2. Find the angular frequency of vertical oscillations when the two springs are used together (i) in parallel, and (ii) in series. Show that the first of these frequencies is at least twice the second.

Solution

In each case, we are being asked to find the effective strength of the composite spring.

(i) **Springs in parallel**
Let x be the *common extension* of the springs and let the tensions be T_1, T_2 respectively. Then the total restoring force is $T_1 + T_2$. The **effective strength** α^P of the springs in parallel is then

$$\begin{aligned}\alpha^P &= \frac{T_1 + T_2}{x}\\ &= \frac{T_1}{x} + \frac{T_2}{x}\\ &= \alpha_1 + \alpha_2\\ &= m\Omega_1^2 + m\Omega_2^2.\end{aligned}$$

Hence the **angular frequency** Ω^P when the body is suspended from springs in parallel is given by

$$m\left(\Omega^P\right)^2 = m\Omega_1^2 + m\Omega_2^2,$$

that is,

$$\Omega^P = \left(\Omega_1^2 + \Omega_2^2\right)^{1/2}. \blacksquare$$

(ii) **Springs in series**
Let T be the *common tension* of the two springs and let the extensions be x_1, x_2 respectively. Then the total extension is $x_1 + x_2$. The **effective strength**

α^S of the springs in series is then

$$\begin{aligned}\alpha^S &= \frac{T}{x_1 + x_2} \\ &= \frac{T}{(T/\alpha_1) + (T/\alpha_2)} \\ &= \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2} \\ &= \frac{m\Omega_1^2\Omega_2^2}{\Omega_1^2 + \Omega_2^2}.\end{aligned}$$

Hence the **angular frequency** Ω^S when the body is suspended from springs in series is given by

$$m\left(\Omega^S\right)^2 = \frac{m\Omega_1^2\Omega_2^2}{\Omega_1^2 + \Omega_2^2},$$

that is,

$$\Omega^S = \frac{\Omega_1\Omega_2}{\left(\Omega_1^2 + \Omega_2^2\right)^{1/2}}. \blacksquare$$

From the above formulae, it follows that

$$\begin{aligned}\frac{\Omega^P}{\Omega^S} &= \frac{\Omega_1^2 + \Omega_2^2}{\Omega_1\Omega_2} \\ &= \frac{(\Omega_1 - \Omega_2)^2 + 2\Omega_1\Omega_2}{\Omega_1\Omega_2} \\ &= \frac{(\Omega_1 - \Omega_2)^2}{\Omega_1\Omega_2} + 2 \\ &\geq 2,\end{aligned}$$

since $(\Omega_1 - \Omega_2)^2/\Omega_1\Omega_2$ is *positive*. Hence, whatever the values of Ω_1, Ω_2, it is always true that

$$\Omega^P \geq 2\Omega^S. \blacksquare$$

Problem 5.3

A particle of mass m moves along the x-axis and is acted upon by the restoring force $-m(n^2 + k^2)x$ and the resistance force $-2mk\dot{x}$, where n, k are positive constants. If the particle is released from rest at $x = a$, show that, in the subsequent motion,

$$x = \frac{a}{n} e^{-kt}(n \cos nt + k \sin nt).$$

Find how far the particle travels before it next comes to rest.

Solution

The **equation of motion** for the particle is

$$m\ddot{x} = -m(n^2 + k^2)x - 2mk\dot{x},$$

that is,

$$\ddot{x} + 2k\dot{x} + \left(n^2 + k^2\right)x = 0.$$

The solution procedure is the same as that on pp.109–110 of the book. We seek solutions of the form $x = e^{\lambda t}$. Then λ must satisfy the equation

$$\lambda^2 + 2k\lambda + \left(n^2 + k^2\right) = 0,$$

the roots of which are $\lambda = -k \pm in$. We have thus found the pair of complex solutions

$$x = e^{-kt}e^{\pm int},$$

which form a basis for the space of complex solutions. The real and imaginary parts of the first complex solution are

$$x = \begin{cases} e^{-kt} \cos nt \\ e^{-kt} \sin nt \end{cases}$$

and these functions form a basis for the space of real solutions. The **general real solution** of the equation of motion is therefore

$$x = e^{-kt}\left(A \cos nt + B \sin nt\right),$$

where A and B are real arbitrary constants. The initial condition $x = a$ when $t = 0$ gives $A = a$, and the condition $\dot{x} = 0$ when $t = 0$ then gives $B = ak/n$. The **motion of the particle** is therefore given by

$$x = \frac{a}{n} e^{-kt}(n \cos nt + k \sin nt).$$

The particle is (instantaneously) at rest when $\dot{x} = 0$. On using the above formula for x, we find that

$$\dot{x} = -\frac{a}{n}\left(n^2 + k^2\right) e^{-kt} \sin nt,$$

which is zero when $\sin nt = 0$. This next happens when $t = \pi/n$ and, at this instant, the particle is at the point

$$x = -a\, e^{-\pi k/n}.$$

Since the motion starts at the point $x = a$, the particle therefore travels a **distance**

$$a\left(1 + e^{-\pi k/n}\right)$$

before it next comes to rest. ■

Problem 5.4

An overdamped harmonic oscillator satisfies the equation

$$\ddot{x} + 10\dot{x} + 16x = 0.$$

At time $t = 0$ the particle is projected from the point $x = 1$ towards the origin with speed u. Find x in the subsequent motion.

Show that the particle will reach the origin at some later time t if

$$\frac{u-2}{u-8} = e^{6t}.$$

How large must u be so that the particle will pass through the origin?

Solution

The equation of motion is solved in the standard manner by seeking solutions of the form $x = e^{\lambda t}$. Then λ must satisfy the equation

$$\lambda^2 + 10\lambda + 16 = 0,$$

the roots of which are $\lambda = -2, -8$. We have thus found the pair of solutions

$$x = \begin{cases} e^{-2t}, \\ e^{-8t}. \end{cases}$$

The **general solution** of the equation of motion is therefore

$$x = Ae^{-2t} + Be^{-8t},$$

where A and B are arbitrary constants. The initial conditions $x = 1$ and $\dot{x} = -u$ when $t = 0$ give the equations

$$\begin{aligned} A + B &= 1, \\ 2A + 8B &= u, \end{aligned}$$

from which it follows that $A = -\frac{1}{6}(u-8)$, $B = \frac{1}{6}(u-2)$. The **motion of the particle** is therefore given by

$$x = \tfrac{1}{6}(u-2)e^{-8t} - \tfrac{1}{6}(u-8)e^{-2t}.$$

The particle is at the origin at time t if

$$\tfrac{1}{6}(u-2)e^{-8t} - \tfrac{1}{6}(u-8)e^{-2t} = 0,$$

that is, if

$$e^{6t} = \frac{u-2}{u-8}.$$

Such a value of t will exist if u is such that $F(u) > 1$, where

$$F = \frac{u-2}{u-8}.$$

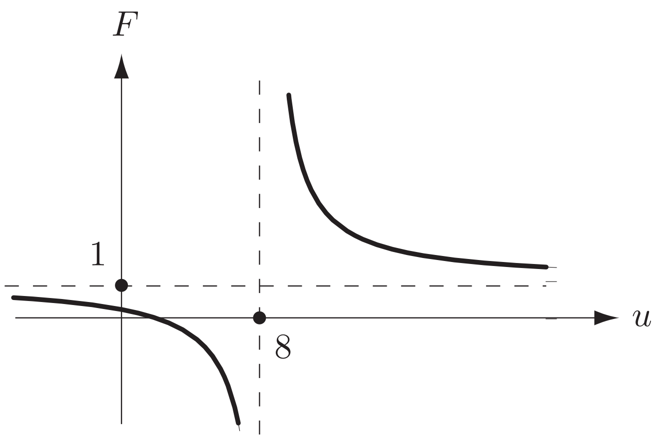

FIGURE 5.1 The function $F(u)$.

The graph of F is shown in Figure 5.1. The condition $F > 1$ is satisfied if $u > 8$, but not otherwise. Hence the particle will **pass through the origin** if $u > 8$. ■

Problem 5.5

A damped oscillator satisfies the equation

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = 0$$

where K and Ω are positive constants with $K < \Omega$ (under-damping). At time $t = 0$ the particle is released from rest at the point $x = a$. Show that the subsequent motion is given by

$$x = ae^{-Kt}\left(\cos\Omega_D t + \frac{K}{\Omega_D}\sin\Omega_D t\right),$$

where $\Omega_D = (\Omega^2 - K^2)^{1/2}$.

Find all the turning points of the function $x(t)$ and show that the ratio of successive maximum values of x is $e^{-2\pi K/\Omega_D}$.

A certain damped oscillator has mass 10 kg, period 5 s and successive maximum values of its displacement are in the ratio 3 : 1. Find the values of the spring and damping constants α and β.

Solution

By using the method given on p.110 of the book, the **general solution** of the equation of motion is found to be

$$x = e^{-Kt}\big(A\cos\Omega_D t + B\sin\Omega_D t\big),$$

where $\Omega_D = (\Omega^2 - K^2)^{1/2}$. The corresponding formula for $\dot{x}$ is

$$\dot{x} = e^{-Kt}\Big(\big(\Omega_D B - KA\big)\cos\Omega_D t - \big(\Omega_D A + KB\big)\sin\Omega_D t\Big).$$

The **initial conditions** $x = a$ and $\dot{x} = 0$ when $t = 0$ give $A = a$ and $B = Ka/\Omega_D$. The **motion of the body** is therefore given by

$$x = ae^{-Kt}\left(\cos\Omega_D t + \frac{K}{\Omega_D}\sin\Omega_D t\right).$$

The turning points of the function $x(t)$ occur when $\dot{x} = 0$, where $\dot{x}$ is given by

$$\begin{aligned}\dot{x} &= -ae^{-Kt}\left(\Omega_D + \frac{K^2}{\Omega_D}\right)\sin\Omega_D t\\ &= -a\left(\frac{\Omega^2}{\Omega_D}\right)e^{-Kt}\sin\Omega_D t.\end{aligned}$$

This is zero when $\sin \Omega_D t = 0$, that is, when $t = 0, \pi/\Omega_D, 2\pi/\Omega_D, \ldots$. The **maxima** of x occur when $t = 0, 2\pi/\Omega_D, 4\pi/\Omega_D, \ldots$, and the values of x at these maxima are

$$a, \quad ae^{-2\pi K/\Omega_D}, \quad ae^{-4\pi K/\Omega_D},$$

and so on. The **ratio of successive maximum values** of x is therefore $e^{-2\pi K/\Omega_D}$.

Suppose we have a damped oscillator with period τ and for which the successive maximum values of its displacement are in the ratio $\gamma : 1$. Then

$$\frac{2\pi}{\Omega_D} = \tau,$$
$$e^{-2\pi K/\Omega_D} = \frac{1}{\gamma},$$

where $\Omega_D = \left(\Omega^2 - K^2\right)^{1/2}$. It then follows that

$$K = \frac{1}{\tau}\ln\gamma, \qquad \Omega^2 = \frac{4\pi^2 + \left(\ln\gamma\right)^2}{\tau^2}.$$

On using the values $\tau = 5\,s$ and $\gamma = 3$ given in the question, we find that $K = 0.22\ \mathrm{s}^{-1}$ and $\Omega = 1.28\ \mathrm{s}^{-1}$ approximately. The **spring constant** α and **damping constant** β therefore have the approximate values

$$\alpha = m\Omega^2 = 16.3\ \mathrm{N\,m^{-1}},$$
$$\beta = 2mK = 4.4\ \mathrm{N\,s\,m^{-1}}. \blacksquare$$

Problem 5.6 *Critical damping*

Find the general solution of the damped SHM equation for the special case of critical damping, that is, when $K = \Omega$. Show that, if the particle is initially released from rest at $x = a$, then the subsequent motion is given by

$$x = ae^{-\Omega t}(1 + \Omega t).$$

Sketch the graph of x against t.

Solution

When $K = \Omega$, the **equation of motion** becomes

$$\ddot{x} + 2\Omega\dot{x} + \Omega^2 x = 0.$$

This equation is solved in the standard manner by seeking solutions of the form $x = e^{\lambda t}$. Then λ must satisfy the equation

$$\lambda^2 + 2\Omega\lambda + \Omega^2 = 0,$$

which has the *repeated root* $\lambda = -\Omega$. In this special case, the functions

$$x = \begin{cases} e^{-\Omega t}, \\ te^{-\Omega t}. \end{cases}$$

are a pair of solutions. The **general solution** of the equation of motion is therefore

$$x = e^{-\Omega t}(A + Bt),$$

where A and B are arbitrary constants.

The **initial conditions** $x = a$ and $\dot{x} = 0$ when $t = 0$ give $A = a$ and $B = a\Omega$. The **motion of the particle** is therefore given by

$$x = ae^{-\Omega t}(1 + \Omega t).$$

The graph of x is shown in Figure 5.2. Qualitatively, it is indistinguishable from the corresponding over-damped problem. ■

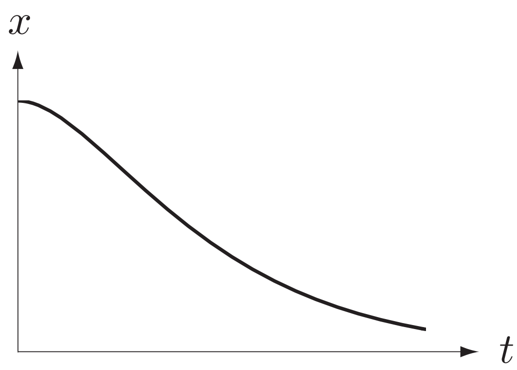

FIGURE 5.2 The graph of x against t in Problem 5.6.

Problem 5.7 ✻ *Fastest decay*

The oscillations of a galvanometer satisfy the equation

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = 0.$$

The galvanometer is released from rest with $x = a$ and we wish to bring the reading permanently within the interval $-\epsilon a \le x \le \epsilon a$ as quickly as possible, where ϵ is a small positive constant. What value of K should be chosen? One possibility is to choose a sub-critical value of K such that the first minimum point of $x(t)$ occurs when $x = -\epsilon a$. [Sketch the graph of $x(t)$ in this case.] Show that this can be acheived by setting the value of K to be

$$K = \Omega \left[1 + \left(\frac{\pi}{\ln(1/\epsilon)} \right)^2 \right]^{-1/2}.$$

If K has this value, show that the time taken for x to reach its first minimum is approximately $\Omega^{-1} \ln(1/\epsilon)$ when ϵ is small.

Solution

By using the method given on p.110 of the book, the **general solution** of the equation of motion is found to be

$$x = e^{-Kt}\big(A \cos \Omega_D t + B \sin \Omega_D t\big),$$

where $\Omega_D = (\Omega^2 - K^2)^{1/2}$. The corresponding formula for $\dot{x}$ is

$$\dot{x} = e^{-Kt}\Big(\big(\Omega_D B - KA\big) \cos \Omega_D t - \big(\Omega_D A + KB\big) \sin \Omega_D t\Big).$$

The **initial conditions** $x = a$ and $\dot{x} = 0$ when $t = 0$ give $A = a$ and $B = Ka/\Omega_D$. The **motion of the galvanometer** is therefore given by

$$x = ae^{-Kt} \left(\cos \Omega_D t + \frac{K}{\Omega_D} \sin \Omega_D t \right).$$

The stationary points of the function $x(t)$ occur when $\dot{x} = 0$, where $\dot{x}$ is given by

$$\begin{aligned} \dot{x} &= -ae^{-Kt} \left(\Omega_D + \frac{K^2}{\Omega_D} \right) \sin \Omega_D t \\ &= -a \left(\frac{\Omega^2}{\Omega_D} \right) e^{-Kt} \sin \Omega_D t. \end{aligned}$$

This is zero when $\sin \Omega_D t = 0$, that is, when $t = 0, \pi/\Omega_D, 2\pi/\Omega_D, \ldots$. The **first minimum** of the function $x(t)$ thus occurs when $t = \pi/\Omega_D$, and the value of x at this instant is

$$x = -ae^{-\pi K/\Omega_D}.$$

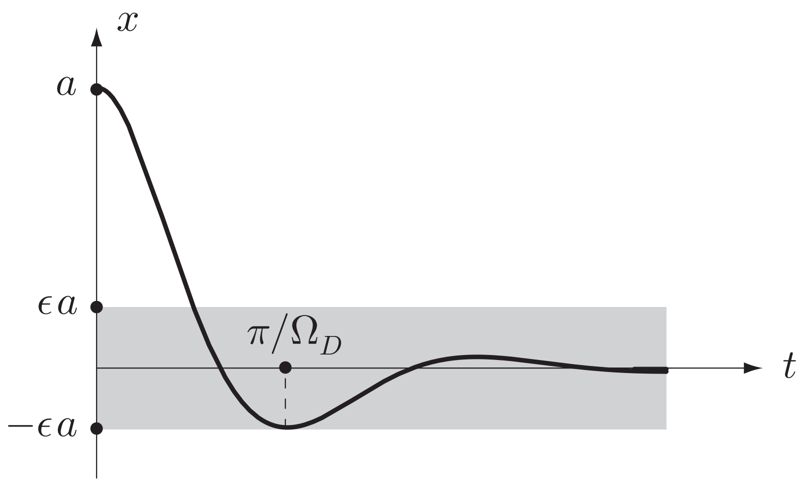

FIGURE 5.3 The damping constant K is chosen so that the first turning point of the motion lies on the line $x = -\epsilon a$.

The suggestion in the question is to select K so that this first minimum of x occurs when $x = -\epsilon a$; this is shown in Figure 5.3. For this to happen, K must be chosen to satisfy the equation

$$e^{-\pi K/\Omega_D} = \epsilon,$$

that is,

$$\frac{\pi K}{\left(\Omega^2 - K^2\right)^{1/2}} = \ln(1/\epsilon).$$

On solving this equation, we find that the **required value of** K is

$$K = \Omega \left(1 + \frac{\pi^2}{\left(\ln(1/\epsilon)\right)^2}\right)^{-1/2}.$$

When ϵ is small, this is *just less* than critical damping.

The time taken to reach the first minimum value of x is π/Ω^D where

$$\Omega_D^2 = \Omega^2 - K^2$$
$$= \Omega^2 - \Omega^2\left(1 + \frac{\pi^2}{\big(\ln(1/\epsilon)\big)^2}\right)^{-1}$$
$$= \frac{\pi^2\Omega^2}{\big(\ln(1/\epsilon)\big)^2 + \pi^2},$$

after a little algebra. The **time** τ **taken** for x to reach its first minimum is therefore

$$\tau = \frac{\ln(1/\epsilon)}{\Omega}\left(1 + \frac{\pi^2}{\big(\ln(1/\epsilon)\big)^2}\right)^{1/2}.$$

When ϵ tends to zero, $1/\ln(1/\epsilon)$ also tends to zero and τ is given approximately by

$$T = \frac{\ln(1/\epsilon)}{\Omega}.$$

With this choice of K, the galvanometer settles down remarkably quickly. For example, if $\epsilon = 10^{-4}$, then $\tau \approx 9.7/\Omega$, which is less than two periods of the undamped galvanometer. ■

Problem 5.8

A block of mass M is connected to a second block of mass m by a linear spring of natural length $8a$. When the system is in equilibrium with the first block on the floor, and with the spring and second block vertically above it, the length of the spring is $7a$. The upper block is then pressed down until the spring has half its natural length and is then resleased from rest. Show that the lower block will leave the floor if $M < 2m$. For the case in which $M = 3m/2$, find when the lower block leaves the floor.

Solution

Since the spring provides the restoring force mg when its extension is $-a$, the **spring constant** α is given by

$$\alpha = \frac{mg}{a}.$$

Let x be the upwards displacement of the upper block, *measured from its equilibrium position*. Then, providing that the lower block does not leave the floor, the **equation of motion** of the upper block is

$$m\ddot{x} = -\alpha x,$$

that is

$$\ddot{x} + \omega^2 x = 0,$$

where $\omega = (g/a)^{1/2}$. The **general solution** of this SHM equation is

$$x = A\cos\omega t + B\sin\omega t,$$

where A and B are arbitrary constants. The **initial conditions** $x = -3a$ and $\dot{x} = 0$ when $t = 0$ give $A = -3a$ and $B = 0$. The **motion of the upper block** is therefore given by

$$x = -3a\cos\omega t,$$

where $\omega = (g/a)^{1/2}$.

At time t, the extension of the spring is $x - a$ and the **tension** T is therefore

$$T = \alpha(x-a) = \frac{mg}{a}(x-a) = -mg(3\cos\omega t + 1).$$

This tension also acts *upwards* on the lower block. The lower block will therefore remain in place so long as $T \le Mg$, that is,

$$-mg(3\cos\omega t + 1) \le Mg.$$

Since $\cos\omega t$ lies in the range $[-1, 1]$, the left side of this inequality lies in the range $[-4mg, 2mg]$. Hence, the **lower block will never move** if $M \ge 2m$. If $M < 2m$, the lower block will leave the floor when

$$-mg(3\cos\omega t + 1) = Mg.$$

For the special case in which $M = \frac{3}{2}m$, this condition reduces to

$$6\cos\omega t = -5$$

so that the **lower block leaves the floor** when

$$t = \left(\frac{a}{g}\right)^{1/2} \cos^{-1}\left(-\tfrac{5}{6}\right) \approx 2.56 \left(\frac{a}{g}\right)^{1/2}. \blacksquare$$

Problem 5.9

A block of mass 2 kg is suspended from a fixed support by a spring of strength $2000\ \mathrm{N\,m^{-1}}$. The block is subject to the vertical driving force $36 \cos pt$ N. Given that the spring will yield if its extension exceeds 4 cm, find the range of frequencies that can safely be applied. [Take $g = 10\ \mathrm{m\,s^{-2}}$.]

Solution

Let x be the downward displacement of block (in metres), *measured from the equilibrium position*. Then the **equation of motion** of the block is

$$2\,\frac{d^2x}{dt^2} = -2000x + 36\cos pt,$$

that is

$$\ddot{x} + 1000x = 18\cos pt.$$

When damping is absent, it is not neccessary to use the complex method to find the driven response. One can simply seek a response of the form

$$x^D = A\cos pt,$$

where the constant A is to be determined. On substituting this form of solution into the equation of motion, we find that

$$A = \frac{18}{1000 - p^2}$$

so that the **driven response** of the block is

$$x^D = \frac{18\cos pt}{1000 - p^2}.$$

The **amplitude** a of the driven response is therefore

$$a = \frac{18}{|1000 - p^2|}.$$

In the equilibrium position, the force applied to the spring is 20 N and so the extension is $\frac{1}{100}$ m. The *maximum* extension Δ of the spring in the driven motion is therefore given by

$$\Delta = \frac{1}{100} + \frac{18}{|1000 - p^2|} \quad \text{metres.}$$

The spring is safe if $\Delta \le \frac{4}{100}$, that is, if

$$\frac{1800}{|1000 - p^2|} \le 3.$$

There are two cases:

(i) If $p^2 < 1000$, then the spring is safe if

$$1800 \le 3(1000 - p^2),$$

that is, if $p \le 20$.

(ii) If $p^2 > 1000$, then the spring is safe if

$$1800 \le 3(p^2 - 1000),$$

that is, if $p \ge 40$.

Hence, the **spring is safe** if **either** (i) $p \le 20$ rads s^{-1} **or** (ii) if $p \ge 40$ rads s^{-1}. ■

Problem 5.10

A driven oscillator satisfies the equation

$$\ddot{x} + \Omega^2 x = F_0 \cos[\Omega(1+\epsilon)t],$$

where ϵ is a positive constant. Show that the solution that satisfies the initial conditions $x = 0$ and $\dot{x} = 0$ when $t = 0$ is

$$x = \frac{F_0}{\epsilon(1+\frac{1}{2}\epsilon)\Omega^2} \sin \tfrac{1}{2}\epsilon\Omega t \sin \Omega(1+\tfrac{1}{2}\epsilon)t.$$

Sketch the graph of this solution for the case in which ϵ is small.

Solution

First we find the **driven response** x^D. When damping is absent, it is not neccessary to use the complex method. One can simply seek a response of the form

$$x^D = A\cos\Omega(1+\epsilon)t,$$

where the constant A is to be determined. On substituting this form of solution into the equation of motion, we find that

$$A = \frac{F_0}{\Omega^2 - \Omega^2(1+\epsilon)^2} = -\frac{F_0}{\epsilon(2+\epsilon)\Omega^2}$$

so that the **driven response** of the block is

$$x^D = -\frac{F_0\cos\Omega(1+\epsilon)t}{\epsilon(2+\epsilon)\Omega^2}.$$

Next we find the **complementary function** x^{CF}. This is the general solution of the corresponding undriven equation

$$\frac{d^2x}{dt^2} + \Omega^2 x = 0,$$

which is known to be

$$x^{CF} = A\cos\Omega t + B\sin\Omega t,$$

where A and B are arbitrary constants. The **general solution** of the equation of motion is therefore

$$\begin{aligned} x &= x^{CF} + x^D \\ &= A\cos\Omega t + B\sin\Omega t - \frac{F_0\cos\Omega(1+\epsilon)t}{\epsilon(2+\epsilon)\Omega^2}. \end{aligned}$$

It remains to choose A and B so that the **initial conditions** are satisfied. The condition $x = 0$ when $t = 0$ gives

$$A = \frac{F_0}{\epsilon(2+\epsilon)\Omega^2}$$

and the condition $\dot{x} = 0$ when $t = 0$ gives $B = 0$. The **required solution** satisfying the given initial conditions is therefore

$$\begin{aligned} x &= \frac{F_0 \cos \Omega t}{\epsilon(2+\epsilon)\Omega^2} - \frac{F_0 \cos \Omega(1+\epsilon)t}{\epsilon(2+\epsilon)\Omega^2} \\ &= \frac{F_0}{\epsilon(1+\frac{1}{2}\epsilon)\Omega^2} \sin \tfrac{1}{2}\epsilon\Omega t \, \sin \Omega(1+\tfrac{1}{2}\epsilon)t. \end{aligned}$$

Figure 5.4 shows the graph of a typical solution when ϵ is small. The slow modulation in the amplitude of the oscillations is the phenomenon known as **beats**. ■

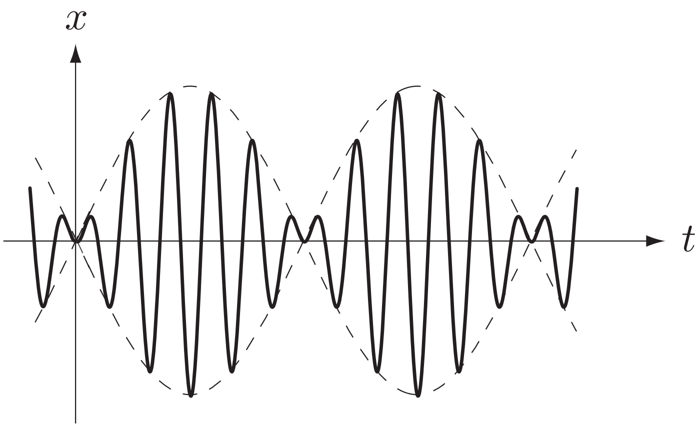

FIGURE 5.4 The solution to Problem 5.10 when $\epsilon = 0.2$.

Problem 5.11

Book Figure 5.12 shows a simple model of a car moving with constant speed c along a gently undulating road with profile $h(x)$, where $h'(x)$ is small. The car is represented by a chassis which keeps contact with the road, connected to an upper mass m by a spring and a damper. At time t the upper mass has displacement $y(t)$ above its equilibrium level. Show that, under suitable assumptions, y satisfies a differential equation of the form

$$\ddot{y} + 2K\dot{y} + \Omega^2 y = 2Kch'(ct) + \Omega^2 h(ct)$$

where K and Ω are positive constants.

Suppose that the profile of the road surface is given by $h(x) = h_0 \cos(px/c)$, where h_0 and p are positive constants. Find the amplitude a of the *driven* oscillations of the upper mass.

The vehicle designer adjusts the damper so that $K = \Omega$. Show that

$$a \le \frac{2}{\sqrt{3}}\, h_0,$$

whatever the values of the consants Ω and p.

Solution

Since the undulations in the road are small, we may suppose that the horizontal displacement of the car at time t is simply given by $x = ct$. Then the extension Δ of the spring at time t is

$$\Delta = y - h(ct)$$

and

$$\dot{\Delta} = \dot{y} - ch'(ct).$$

The **equation of motion** for the vertical oscillations of the car is therefore

$$m\ddot{y} = -\alpha\Delta - \beta\dot{\Delta},$$

where m is the suspended mass of the car, α is the spring constant, and β is the resistance constant. On writing $\alpha = m\Omega^2$ and $\beta = 2mK$, the equation of motion takes the form

$$\ddot{y} = -\Omega^2\big(y - h(ct)\big) - 2K\big(\dot{y} - ch'(ct)\big),$$

that is,

$$\ddot{y} + 2K\dot{y} + \Omega^2 y = 2Kch'(ct) + \Omega^2 h(ct).$$

When the road surface has the profile $h(x) = h_0 \cos(px/c)$, this equation becomes

$$\ddot{y} + 2K\dot{y} + \Omega^2 y = -2h_0 Kp \sin pt + h_0\Omega^2 \cos pt.$$

To find the **driven response** excited by these undulations, consider the complex equation

$$\ddot{y} + 2K\dot{y} + \Omega^2 y = h_0 \left(2iKp + \Omega^2\right) e^{ipt}.$$

On seeking a solution of this equation of the form $x = Ce^{ipt}$, we find that the **complex amplitude** C of the driven oscllations is

$$C = \frac{h_0 \left(2iKp + \Omega^2\right)}{\Omega^2 - p^2 + 2iKp}.$$

The **amplitude** a **of the driven oscillations** is therefore

$$\begin{aligned} a &= |C| \\ &= \frac{h_0|2iKp + \Omega^2|}{|\Omega^2 - p^2 + 2iKp|} \\ &= h_0 \left(\frac{4K^2p^2 + \Omega^4}{\left(\Omega^2 - p^2\right)^2 + 4K^2p^2}\right)^{1/2}. \end{aligned}$$

In the special case in which $K = \Omega$, this formula can be written in the form

$$\frac{a^2}{h_0^2} = \frac{4u + 1}{(u + 1)^2},$$

where $u = p^2/\Omega^2$. To find the maximum value of a (considered as a function of p with Ω fixed), we must find the maximum value attained by the function

$$F = \frac{4u + 1}{(u + 1)^2}$$

when u is *positive*. Now

$$F' = \frac{4(1+u)^2 - 2(1+4u)(1+u)}{(1+u)^4}$$

$$= \frac{2(1-2u)}{(1+u)^3}$$

$$\begin{cases} > 0 & \text{for} \quad 0 \le u < \frac{1}{2}, \\ = 0 & \text{for} \quad u = \frac{1}{2}, \\ < 0 & \text{for} \quad u > \frac{1}{2}. \end{cases}$$

It follows that the maximum value of the function $F(u)$ in the interval $0 \le u < \infty$ is $F(\frac{1}{2}) = \frac{4}{3}$. Hence, whatever the values of the frequencies Ω and p,

$$\frac{a^2}{h_0^2} \le \frac{4}{3}$$

and so

$$a \le \frac{2}{\sqrt{3}}\, h_0. \blacksquare$$

Problem 5.12 *Solution by Fourier series*

A driven oscillator satisfies the equation

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = F(t),$$

where K and Ω are positive constants. Find the driven response of the oscillator to the saw tooth' input, that is, when $F(t)$ is given by

$$F(t) = F_0\, t \qquad\qquad (-\pi < t < \pi)$$

and $F(t)$ is periodic with period 2π. [It is a good idea to sketch the graph of the function $F(t)$.]

Solution

Figure 5.5 shows the graph of the 'saw tooth' function $F(t)$.

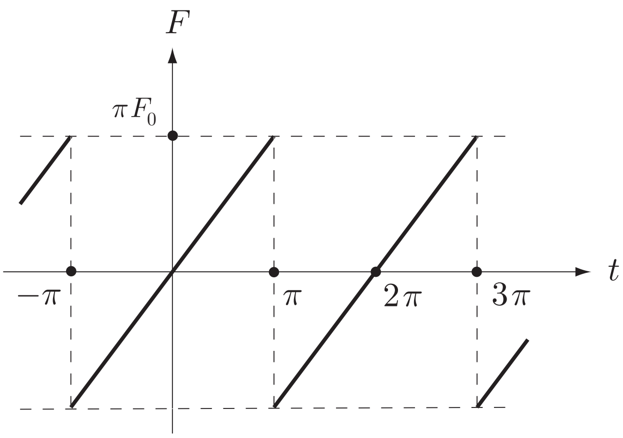

FIGURE 5.5 The 'saw tooth' function $F(t)$.

The first step is to find the Fourier series of the function $F(t)$. This function has period 2π and so the Fourier formulae on p.117 of the book apply. The coefficient a_n is given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\, dt \\ &= \frac{F_0}{\pi}\int_{-\pi}^{\pi} t\cos nt\, dt \\ &= 0. \end{aligned}$$

The last step follows since the integrand is an *odd* function of t and the range of integration is symmetrical about $t = 0$; the contributions from the intervals $[-\pi, 0]$ and $[0, \pi]$ therefore cancel.

In the same way,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin nt\, dt \\ &= \frac{F_0}{\pi}\int_{-\pi}^{\pi} t\sin nt\, dt \\ &= \frac{2F_0}{\pi}\int_{0}^{\pi} t\sin nt\, dt, \end{aligned}$$

since this time the contributions from the intervals $[-\pi, 0]$ and $[0, \pi]$ are equal. Hence

$$\begin{aligned} b_n &= \frac{2F_0}{\pi}\left[t\left(\frac{-\cos nt}{n}\right)\right]_0^{\pi} - \frac{2F_0}{\pi}\int_0^{\pi}(1)\left(\frac{-\cos nt}{n}\right)dt \\ &= \frac{2F_0}{\pi n}\Big(-\pi(-1)^n - 0\Big) + \frac{2F_0}{\pi n}\int_0^{\pi}\cos nt\, dt \\ &= \frac{2F_0(-1)^{n+1}}{n} + \frac{2F_0}{\pi n^2}\Big[\sin nt\Big]_{t=0}^{t=\pi} \\ &= \frac{2F_0(-1)^{n+1}}{n} + 0 \\ &= \frac{2F_0(-1)^{n+1}}{n}. \end{aligned}$$

Hence the **Fourier series** of the function $F(t)$ is

$$F(t) = \sum_{n=1}^{\infty}\frac{2F_0(-1)^{n+1}}{n}\sin nt.$$

The next step is to find the driven response of the oscillator to the forcing term $b_n \sin nt$. That is, we need the particular integral of the equation

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = b_n\sin nt.$$

The complex counterpart of this equaton is

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = b_n e^{int}$$

for which the particular integral is ce^{int}, where the complex amplitude c is given by

$$c = \frac{b_n}{\Omega^2 - n^2 + 2iKn}.$$

The particular integral of the real equation is then given by

$$\Im\left(\frac{b_n e^{int}}{\Omega^2 - n^2 + 2iKn}\right) = b_n\left(\frac{(\Omega^2 - n^2)\sin nt + 2Kn\cos nt}{(\Omega^2 - n^2)^2 + 4K^2n^2}\right).$$

Finally we add together these separate responses to find the **driven response** of the oscillator to the force $F(t)$. On inserting the value of the coefficient b_n, this gives

$$x = 2F_0\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{(\Omega^2 - n^2)\sin nt + 2Kn\cos nt}{(\Omega^2 - n^2)^2 + 4K^2n^2}\right).$$

Problem 5.13

A particle of mass m is connected to a fixed point O on a smooth horizontal table by a linear elastic string of natural length $2a$ and strength $m\Omega^2$. Initially the particle is released from rest at a point on the table whose distance from O is $3a$. Find the period of the resulting oscillations.

Solution

This problem has the feature that, when the distance of the particle from O is less than $2a$, the string goes slack and *exerts no force* on the particle. Hence, the **restoring force is non-linear**. It is convenient to split the motion into a number of parts, in each of which the equation of motion is linear.

(i) Suppose that the motion takes place along the axis Ox. Then the particle is initially at rest at the point $x = 3a$. In this position the string is taut and a motion begins. The **equation of motion** is

$$m\frac{d^2x}{dt^2} = -m\Omega^2(x - 2a),$$

which can be written in the form

$$\ddot{y} + \Omega^2 y = 0,$$

where $y = x - 2a$. This equation holds while $y \geq 0$. The solution corresponding to the initial conditions $y = a$ and $\dot{y} = 0$ is

$$y = a\cos\Omega t.$$

This is an SHM with amplitude a and period $2\pi/\Omega$. Thus, after a quarter of an oscillation, the particle reaches $y = 0$ (that is, $x = +2a$) moving with speed $a\Omega$ in the negative x-direction. The time that elapses during this part of the motion is therefore a quarter of a period, that is, $\pi/2\Omega$.

(ii) This part of the motion begins with the particle at $x = +2a$ and moving with speed $a\Omega$ in the negative x-direction. The string is slack and the particle continues to move with speed $a\Omega$ until it reaches the point $x = -2a$. The time that elapses during this part of the motion is $4a/a\Omega = 4/\Omega$.

(iii) This part of the motion begins with the particle at $x = -2a$ and moving with speed $a\Omega$ in the negative x-direction. The string becomes taut and the **equation of motion** is

$$m\frac{d^2x}{dt^2} = -m\Omega^2(x + 2a),$$

which can be written in the form

$$\ddot{z} + \Omega^2 z = 0,$$

where $z = x + 2a$. This equation holds while $z \leq 0$. The solution corresponding to the initial conditions $z = 0$ and $\dot{z} = -a\Omega$ is

$$z = -a \sin \Omega t,$$

where the initial time has been reset to zero. This is also an SHM with amplitude a and period $2\pi/\Omega$. Thus the particle executes half an oscillation of the SHM and returns to $z = 0$ (that is $x = -2a$) with speed $a\Omega$ in the positive x-direction. The time that elapses during this part of the motion is therefore half a period, that is, π/Ω.

(iv) This part of the motion begins with the particle at $x = -2a$ and moving with speed $a\Omega$ in the positive x-direction. The string is slack and the particle continues to move with speed $a\Omega$ until it reaches the point $x = +2a$. The time that elapses during this part of the motion is $4a/a\Omega = 4/\Omega$.

(v) This part of the motion begins with the particle at $x = +2a$ and moving with speed $a\Omega$ in the positive x-direction. The string becomes taut and the **equation of motion** is

$$m\frac{d^2x}{dt^2} = -m\Omega^2(x - 2a),$$

which can be written in the form

$$\ddot{y} + \Omega^2 y = 0,$$

where $y = x - 2a$. This equation holds while $y \geq 0$. The solution corresponding to the initial conditions $y = 0$ and $\dot{y} = a\Omega$ is

$$y = a \sin \Omega t,$$

where the initial time has again been reset to zero. This is an SHM with amplitude a and period $2\pi/\Omega$. Thus the particle comes to rest at $y = a$ (that is, $x = 3a$) after a quarter of an oscillation. The time that elapses during this part of the motion is therefore $\pi/2\Omega$.

The particle has thus come to rest at its starting point and the whole cycle is then repeated indefinitely. Hence, **the motion is periodic** and the **period** τ is given by

$$\tau = \frac{\pi}{2\Omega} + \frac{4}{\Omega} + \frac{\pi}{\Omega} + \frac{4}{\Omega} + \frac{\pi}{2\Omega} = \frac{2\pi + 8}{\Omega}. \blacksquare$$

Problem 5.14 *Coulomb friction*

The displacement x of a spring mounted mass under the action of Coulomb friction satisfies the equation

$$\ddot{x} + \Omega^2 x = \begin{cases} -F_0 & \dot{x} > 0 \\ F_0 & \dot{x} < 0 \end{cases}$$

where Ω and F_0 are positive constants. If $|x| > F_0/\Omega^2$ when $\dot{x} = 0$, then the motion continues; if $|x| \le F_0/\Omega^2$ when $\dot{x} = 0$, then the motion ceases. Initially the body is released from rest with $x = 9F_0/2\Omega^2$. Find where it finally comes to rest. How long was the body in motion?

Solution

This Problem has the feature that the **resistance force is non-linear**. It is convenient to split the motion into a number of parts, in each of which the equation of motion is linear.

First leg
On the first leg, the block is initially at rest at the point $x = 9F_0/\Omega^2$. In this position $\Omega^2|x| > F_0$ and a motion begins. The **equation of motion** is

$$\ddot{x} + \Omega^2 x = +F_0,$$

which is the SHM equation with a constant right hand side. The particular integral is the constant F_0/Ω^2 and the **general solution** is

$$x = A\cos\Omega t + B\sin\Omega t + \frac{F_0}{\Omega^2}.$$

The **initial conditions** $x = 9F_0/2\Omega^2$ and $\dot{x} = 0$ when $t = 0$ give $A = 7F_0/2\Omega^2$ and $B = 0$. The **motion of the block** is therefore given by

$$x = \frac{7F_0}{\Omega^2}\cos\Omega t + \frac{F_0}{\Omega^2}.$$

This solution holds until the block next comes to rest. Since

$$\dot{x} = -\frac{7F_0}{2\Omega}\sin\Omega t,$$

this happens when $t = \pi/\Omega$. At this instant, the block is at the point $x = -5F_0/2\Omega^2$.

Second leg
On the second leg, the block is initially at rest at the point $x = -5F_0/2\Omega^2$. In this position $\Omega^2|x| > F_0$ and a motion begins. The **equation of motion** is now

$$\ddot{x} + \Omega^2 x = -F_0.$$

The particular integral is the constant $-F_0/\Omega^2$ and the **general solution** is

$$x = A\cos\Omega t + B\sin\Omega t - \frac{F_0}{\Omega^2}.$$

If we now reset the initial time to zero, the **initial conditions** $x = -5F_0/\Omega^2$ and $\dot{x} = 0$ when $t = 0$ give $A = -3F_0/2\Omega^2$ and $B = 0$. The **motion of the body** is therefore given by

$$x = -\frac{3F_0}{2\Omega^2}\cos\Omega t - \frac{F_0}{\Omega^2}.$$

This solution holds until the block next comes to rest. Since

$$\dot{x} = \frac{3F_0}{2\Omega}\sin\Omega t,$$

this happens when $t = \pi/\Omega$. At this instant, the block is at the point $x = F_0/2\Omega^2$.

Third leg
On the third leg, the block is initially at rest at the point $x = F_0/2\Omega^2$. In this position $\Omega^2|x| < F_0$ and **no motion takes place**.

Hence, the **block comes to permanent rest** at the point $x = +F_0/2\Omega^2$. The **time τ for which the block was in motion** is given by

$$\tau = \frac{\pi}{\Omega} + \frac{\pi}{\Omega} = \frac{2\pi}{\Omega}. \blacksquare$$

Problem 5.15

A partially damped oscillator satisfies the equation

$$\ddot{x} + 2\kappa\,\dot{x} + \Omega^2 x = 0,$$

where Ω is a positive constant and κ is given by

$$\kappa = \begin{cases} 0 & x < 0 \\ K & x > 0 \end{cases}$$

where K is a positive constant such that $K < \Omega$. Find the period of the oscillator and the ratio of successive maximum values of x.

Solution

This problem has the feature that the **resistance force is non-linear**. It is convenient to split the motion into a number of parts, in each of which the equation of motion is linear.

(i) Suppose that the particle is initially at the origin and is moving with speed u_1 in the positive x-direction. [We must have *some* initial conditions.] The particle immediately *enters* the resisting medium and the **equation of motion** is

$$\ddot{x} + 2K\dot{x} + \Omega^2 x = 0.$$

This equation holds while $x \geq 0$. The solution of this damped SHM equation corresponding to the initial conditions $x = 0$ and $\dot{x} = u_1$ is

$$x = \frac{u_1}{\Omega_D} e^{-Kt} \sin \Omega_D t,$$

where $\Omega_D = \left(\Omega^2 - K^2\right)^{1/2}$. Thus the particle returns to the origin after time π/Ω_D moving with speed u_2 in the negative x-direction, where

$$u_2 = u_1 e^{-\pi K/\Omega_D}.$$

(ii) This part of the motion begins with the particle at the origin and moving with speed u_2 in the negative x-direction. The particle immediately *leaves* the resisting medium and the **equation of motion** is

$$\ddot{x} + \Omega^2 x = 0.$$

This equation holds while $x \le 0$. The solution of this SHM equation corresponding to the initial conditions $x = 0$ and $\dot{x} = -u_2$ is

$$x = -\frac{u_2}{\Omega} \sin \Omega t,$$

where the initial time has been reset to zero. Thus the particle returns to the origin after time π/Ω moving with speed u_2 in the positive x-direction.

This completes the first oscillation. The only difference between the second oscillation and the first is that the initial condition $\dot{x} = u_1$ is now replaced by $\dot{x} = u_2$. This change affects the *amplitude* of the second cycle, but not its period. Hence, there are infinitely many **periodic oscillations** and the **period** τ is given by

$$\tau = \frac{\pi}{\Omega_D} + \frac{\pi}{\Omega} = \pi \left(\frac{1}{\Omega_D} + \frac{1}{\Omega} \right). \blacksquare$$

Problem 5.16

A particle P of mass $3m$ is suspended fron a fixed point O by a light linear spring with strength α. A second particle Q of mass $2m$ is in turn suspended from P by a second spring of the same strength. The system moves in the vertical straight line through O. Find the normal frequencies and the form of the normal modes for this system. Write down the form of the general motion.

Solution

Let x, y be the downward displacements of the particles P, Q *measured from their equilibrium positions*. Then the extensions of the springs are x and $y - x$ respectively. The **equations of motion** for P and Q are therefore

$$3m\ddot{x} = -\alpha x + \alpha(y - x),$$
$$2m\ddot{y} = -\alpha(y - x),$$

which can be written in the form

$$3\ddot{x} + 2n^2 x - n^2 y = 0,$$
$$2\ddot{y} - n^2 x + n^2 y = 0,$$

where $n^2 = \alpha/m$.

These equations have **normal mode** solutions of the form

$$x = A\cos(\omega t - \gamma),$$
$$y = B\cos(\omega t - \gamma),$$

when the simultaneous linear equations

$$(2n^2 - 3\omega^2)A - n^2 B = 0,$$
$$-n^2 A + (n^2 - 2\omega^2)B = 0,$$

have a *non-trivial* solution for the amplitudes A, B. The condition for this is

$$\det\begin{pmatrix} 2n^2 - 3\omega^2 & -n^2 \\ -n^2 & n^2 - 2\omega^2 \end{pmatrix} = 0.$$

On simplification, this gives

$$6\omega^4 - 7n^2\omega^2 + n^4 = 0, \qquad (1)$$

a quadratic equation in the variable ω^2. This equation factorises and the roots are found to be

$$\omega_1^2 = \tfrac{1}{6}n^2, \qquad\qquad \omega_2^2 = n^2.$$

Hence there are **two normal modes** with **normal frequencies** $n/\sqrt{6}$ and n respectively.

Slow mode: In the slow mode we have $\omega^2 = n^2/6$ so that the linear equations for the amplitudes A, B become

$$\begin{aligned} \tfrac{3}{2}n^2 A - n^2 B &= 0, \\ -n^2 A + \tfrac{2}{3}n^2 B &= 0. \end{aligned}$$

These two equations are each equivalent to the single equation $3A = 2B$ so that we have the family of non-trivial solutions $A = 2\delta$, $B = 3\delta$, where δ can take any (non-zero) value. The **slow normal mode** therefore has the form

$$\begin{aligned} x &= 2\delta\cos(\omega_1 t - \gamma), \\ y &= 3\delta\cos(\omega_1 t - \gamma), \end{aligned}$$

where $\omega_1 = n/\sqrt{6}$ and the amplitude factor δ and phase factor γ can take any values. In the slow mode, the two particles always move in the *same* direction.

Fast mode: In the fast mode we have $\omega^2 = n^2$ and, by following the same procedure, we find that the form of the **fast normal mode** is

$$\begin{aligned} x &= \delta\cos(\omega_2 t - \gamma), \\ y &= -\delta\cos(\omega_2 t - \gamma), \end{aligned}$$

where $\omega_2 = n$ and the amplitude factor δ and phase factor γ can take any values. In the fast mode, the two particles always move in *opposite* directions.

The **general motion** is now the sum of the first normal mode (with amplitude factor δ_1 and phase factor γ_1) and the second normal mode (with amplitude factor δ_2 and phase factor γ_2). This gives

$$\begin{aligned} x &= 2\delta_1\cos(\omega_1 t - \gamma_1) + \delta_2\cos(\omega_2 t - \gamma_2), \\ y &= 3\delta_1\cos(\omega_1 t - \gamma_1) - \delta_2\cos(\Omega_2 t - \gamma_2). \blacksquare \end{aligned}$$

Problem 5.17

Two particles P and Q, each of mass m, are secured at the points of trisection of a light string that is stretched to tension T_0 between two fixed supports a distance $3a$ apart. The particles undergo small *transverse* oscillations perpendicular to the equlilibrium line of the string. Find the normal frequencies, the forms of the normal modes, and the general motion of this system. [Note that the forms of the modes could have been deduced from the symmetry of the system.] Is the general motion periodic?

Solution

This solution is obtained under the same simplifying assumptions as were made in Example 5.4 of the book.

Let x, y be the transverse displacements of the particles P, Q *from their equilibrium positions*. Then the **equations of transverse motion** for P and Q are

$$m\ddot{x} = -T_0\left(\frac{x}{a}\right) + T_0\left(\frac{y-x}{a}\right),$$
$$m\ddot{y} = -T_0\left(\frac{y-x}{a}\right) - T_0\left(\frac{y}{a}\right),$$

which can be written in the form

$$\ddot{x} + 2n^2x - n^2y = 0,$$
$$\ddot{y} - n^2x + 2n^2y = 0,$$

where $n^2 = T_0/ma$.

These equations have **normal mode** solutions of the form

$$x = A\cos(\omega t - \gamma),$$
$$y = B\cos(\omega t - \gamma),$$

when the simultaneous linear equations

$$(2n^2 - \omega^2)A - n^2B = 0,$$
$$-n^2A + (2n^2 - \omega^2)B = 0,$$

have a *non-trivial* solution for the amplitudes A, B. The condition for this is

$$\det\begin{pmatrix} 2n^2-\omega^2 & -n^2 \\ -n^2 & 2n^2-\omega^2 \end{pmatrix} = 0.$$

On simplification, this gives

$$\omega^4 - 4n^2\omega^2 + 3n^4 = 0, \tag{1}$$

a quadratic equation in the variable ω^2. This equation factorises and the roots are found to be

$$\omega_1^2 = n^2, \qquad \omega_2^2 = 3n^2.$$

Hence there are **two normal modes** with **normal frequencies** n and $\sqrt{3}n$ respectively.

Slow mode: In the slow mode we have $\omega^2 = n^2$ so that the linear equations for the amplitudes A, B become

$$\begin{aligned} n^2A - n^2B &= 0, \\ -n^2A + n^2B &= 0. \end{aligned}$$

These two equations are each equivalent to the single equation $A = B$ so that we have the family of non-trivial solutions $A = \delta$, $B = \delta$, where δ can take any (non-zero) value. The **slow normal mode** therefore has the form

$$\begin{aligned} x &= \delta \cos(\omega_1 t - \gamma), \\ y &= \delta \cos(\omega_1 t - \gamma), \end{aligned}$$

where $\omega_1 = n$ and the amplitude factor δ and phase factor γ can take any values. In the slow mode, the two particles always move in the *same* direction.

Fast mode: In the fast mode we have $\omega^2 = 3n^2$ and, by following the same procedure, we find that the form of the **fast normal mode** is

$$\begin{aligned} x &= \delta \cos(\omega_2 t - \gamma), \\ y &= -\delta \cos(\omega_2 t - \gamma), \end{aligned}$$

where $\omega_2 = \sqrt{3}n$ and the amplitude factor δ and phase factor γ can take any values. In the fast mode, the two particles always move in *opposite* directions.

The **general motion** is now the sum of the first normal mode (with amplitude factor δ_1 and phase factor γ_1) and the second normal mode (with amplitude factor δ_2 and phase factor γ_2). This gives

$$\begin{aligned} x &= \delta_1 \cos(\omega_1 t - \gamma_1) + \delta_2 \cos(\omega_2 t - \gamma_2), \\ y &= \delta_1 \cos(\omega_1 t - \gamma_1) - \delta_2 \cos(\Omega_2 t - \gamma_2). \blacksquare \end{aligned}$$

The general motion is periodic if ω_1/ω_2 is a rational number. In the present problem, $\omega_1/\omega_2 = 1/\sqrt{3}$, which is an *irrational* number. The general motion is therefore **not periodic**. ■

Problem 5.18

A particle P of mass $3m$ is suspended from a fixed point O by a light inextensible string of length a. A second particle Q of mass m is in turn suspended from P by a second string of length a. The system moves in a vertical plane through O. Show that the linearised equations of motion for *small* oscillations near the downward vertical are

$$4\ddot{\theta} + \ddot{\phi} + 4n^2\theta = 0,$$
$$\ddot{\theta} + \ddot{\phi} + n^2\phi = 0,$$

where θ and ϕ are the angles that the two strings make with the downward vertical, and $n^2 = g/a$. Find the normal frequencies and the forms of the normal modes for this system.

Solution

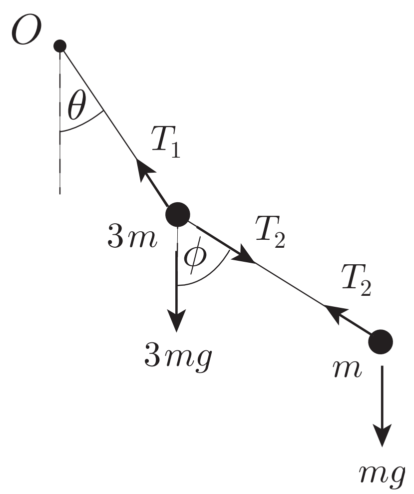

FIGURE 5.6 The double pendulum in Problem 5.18.

The system is shown in Figure 5.6. Let x_1, x_2 be the horizontal displacements of P, *Q from their equilibrium positions*, and let z_1, z_2 be the corresponding vertical displacements. Then the **exact equations of motion** for P and Q are

$$3m\ddot{x}_1 = T_2 \sin\phi - T_1 \sin\theta,$$
$$3m\ddot{z}_1 = T_1 \cos\theta - T_2 \cos\phi - 3mg,$$

$$m\ddot{x}_2 = -T_2 \sin\phi,$$
$$m\ddot{z}_2 = T_2 \cos\phi - mg,$$

where T_1, T_2 are the tensions in the strings. These are a complicated set of coupled *non-linear* equations in *four* unknowns. The situation simplifies greatly if we suppose the motion is small enough so that the squares of the angles θ, ϕ can be neglected. In this **linear approximation**, $x_1 = a\theta$, $x_2 = a\theta + a\phi$ and the vertical displacements z_1, z_2 are negligible. The equations of motion then simplify to give

$$3ma\ddot{\theta} = T_2\phi - T_1\theta,$$
$$0 = T_1 - T_2 - 3mg,$$
$$ma\left(\ddot{\theta} + \ddot{\phi}\right) = -T_2\phi,$$
$$0 = T_2 - mg.$$

Hence $T_1 = 4mg$ and $T_2 = mg$. [Thus, in the linear approximation, motions in the z-direction are negligible as are changes in the tensions.] The equations for the angles θ, ϕ can now be written in the form

$$3\ddot{\theta} + 4n^2\theta - n^2\phi = 0,$$
$$\ddot{\theta} + \ddot{\phi} + n^2\phi = 0,$$

where $n^2 = g/a$. These are a nice set of coupled *linear* equations in *two* unknowns. [They are not identical with the equations quoted in the question, but they are equivalent. The first equation in the question is just the sum of the two equations above.]

These equations have **normal mode** solutions of the form

$$\theta = A\cos(\omega t - \gamma),$$
$$\phi = B\cos(\omega t - \gamma),$$

when the simultaneous linear equations

$$(4n^2 - 3\omega^2)A - n^2 B = 0,$$
$$-\omega^2 A + (n^2 - \omega^2)B = 0,$$

have a *non-trivial* solution for the amplitudes A, B. The condition for this is

$$\det\begin{pmatrix} 4n^2 - 3\omega^2 & -n^2 \\ -\omega^2 & n^2 - \omega^2 \end{pmatrix} = 0.$$

On simplification, this gives

$$3\omega^4 - 8n^2\omega^2 + 4n^4 = 0, \qquad (1)$$

a quadratic equation in the variable ω^2. This equation factorises and the roots are found to be

$$\omega_1^2 = \tfrac{2}{3}n^2, \qquad \omega_2^2 = 2n^2.$$

Hence there are **two normal modes** with **normal frequencies** $\sqrt{\frac{2}{3}}\,n$ and $\sqrt{2}\,n$ respectively.

Slow mode: In the slow mode we have $\omega^2 = \frac{2}{3}n^2$ so that the linear equations for the amplitudes A, B become

$$\begin{aligned} 2n^2 A - n^2 B &= 0, \\ -\tfrac{2}{3}n^2 A + \tfrac{1}{3}n^2 B &= 0. \end{aligned}$$

These two equations are each equivalent to the single equation $2A = B$ so that we have the family of non-trivial solutions $A = \delta$, $B = 2\delta$, where δ can take any (non-zero) value. The **slow normal mode** therefore has the form

$$\begin{aligned} x &= \delta\cos(\omega_1 t - \gamma), \\ y &= 2\delta\cos(\omega_1 t - \gamma), \end{aligned}$$

where $\omega_1 = \sqrt{\frac{2}{3}}\,n$ and the amplitude factor δ and phase factor γ can take any values.

Fast mode: In the fast mode we have $\omega^2 = 2n^2$ and, by following the same procedure, we find that the form of the **fast normal mode** is

$$\begin{aligned} x &= \delta\cos(\omega_2 t - \gamma), \\ y &= -2\delta\cos(\omega_2 t - \gamma), \end{aligned}$$

where $\omega_2 = \sqrt{2}\,n$ and the amplitude factor δ and phase factor γ can take any values.
■

Chapter Six

Energy conservation

Problem 6.1

A particle P of mass 4 kg moves under the action of the force $\boldsymbol{F} = 4\boldsymbol{i} + 12t^2\boldsymbol{j}$ N, where t is the time in seconds. The initial velocity of the particle is $2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ m s^{-1}. Find the work done by $\boldsymbol{F}$, and the increase in kinetic energy of P, during the time interval $0 \le t \le 1$. What principle does this illustrate?

Solution

The **equation of motion** of the particle is

$$4\frac{d\boldsymbol{v}}{dt} = 4\boldsymbol{i} + 12t^2\boldsymbol{j},$$

which has the general solution

$$\boldsymbol{v} = t\boldsymbol{i} + t^3\boldsymbol{j} + \boldsymbol{C},$$

where $\boldsymbol{C}$ is the integration constant. The **initial condition** $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ when $t = 0$ gives $\boldsymbol{C} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and hence the **velocity** of the particle at time t is

$$\boldsymbol{v} = (t+2)\boldsymbol{i} + (t^3+1)\boldsymbol{j} + 2\boldsymbol{k}.$$

The **work** W **done** by the force during the time interval $0 \le t \le 1$ is therefore

$$\begin{aligned}
W &= \int_0^1 \boldsymbol{F}\cdot\boldsymbol{v}\,dt \\
&= \int_0^1 \big(4\boldsymbol{i} + 12t^2\boldsymbol{j}\big)\cdot\big((t+2)\boldsymbol{i} + (t^3+1)\boldsymbol{j} + 2\boldsymbol{k}\big)\,dt \\
&= \int_0^1 4(t+2) + 12t^2(t^3+1)\,dt \\
&= \Big[2t^6 + 4t^3 + 2t^2 + 8t\Big]_0^1 \\
&= 16 \text{ J}.
\end{aligned}$$

During the same time interval, the **increase** ΔT in the **kinetic energy** of the particle is

$$\begin{aligned}
\Delta T &= 2|\boldsymbol{v}(1)|^2 - 2|\boldsymbol{v}(0)|^2 \\
&= 2|3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}|^2 - 2|2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}|^2 \\
&= 34 - 18 = 16 \text{ J}.
\end{aligned}$$

This verifies the **energy principle** for the particle. ■

Problem 6.2

In a competition, a man pushes a block of mass 50 kg with constant speed $2\ \mathrm{m\,s^{-1}}$ up a smooth plane inclined at 30° to the horizontal. Find the rate of working of the man. [Take $g = 10\ \mathrm{m\,s^{-2}}$.]

Solution

The weight of the block is 500 N. Since the plane is *smooth*, the **force** that the man must apply to the block so that it moves up the plane with constant speed is $500 \sin 30° = 250$ N. The **rate of working** of the man is therefore $F \times v = 250 \times 2 = 500$ W. ■

Problem 6.3

An athlete putts a shot of mass 7 kg a distance of 20 m. Show that the athlete must do to at least 700 J of work to achieve this. [Ignore the height of the athlete and take $g = 10\,\mathrm{m\,s^{-2}}$.]

Solution

In order to project the shot a distance R, the *least* projection speed u that can be used is

$$u = (Rg)^{1/2}.$$

This is the projection speed needed when the elevation angle is 45°. Thus a putt of 20 m can be achieved with a projection speed of $10\sqrt{2}\,\mathrm{m\,s^{-1}}$. In this case, the **kinetic energy** of the shot at the instant of release is $\frac{1}{2} \times 7 \times 200 = 700$ J. By the **energy principle**, this increase in the kinetic energy of the shot is equal to the work done on the shot by the athlete. Hence the **work done by the athlete** is 700 J. If the shot were projected at any other angle of elevation, a bigger projection speed would be neccessary and the athlete would have to do *more* work. ■

Problem 6.4

Find the work needed to lift a satellite of mass 200 kg to a height of 2000 km above the Earth's surface. [Take the Earth to be spherically symmetric and of radius 6400 km. Take the surface value of g to be 9.8 m s^{-2}.]

Solution

Let the satellite have mass m and suppose the Earth is spherically symmetric with mass M and radius R. Then the force $\boldsymbol{F}$ that the Earth exerts on the satellite when it is distance r from the *centre* of the Earth is

$$F = -\left(\frac{mMG}{r^2}\right)\widehat{\boldsymbol{r}},$$

where $\widehat{\boldsymbol{r}}$ is the unit vector pointing radially outwards. When the satellite is at the Earth's surface, $r = R$ and $|\boldsymbol{F}| = mg$, where g is the *surface value* of the gravitational acceleration. Hence $MG = R^2 g$ and the formula for $\boldsymbol{F}$ can be written

$$\boldsymbol{F} = -mg\left(\frac{R}{r}\right)^2\widehat{\boldsymbol{r}}.$$

This is a conservative force field with potential energy

$$V = -\frac{mgR^2}{r}.$$

Suppose that the satellite is also subject to a force $\boldsymbol{G}$ and moves from a point A on the Earth's surface to a point B at height h. Then, by the **energy principle**,

$$W^{\boldsymbol{F}} + W^{\boldsymbol{G}} = T_{\boldsymbol{B}} - T_{\boldsymbol{A}},$$

where $W^{\boldsymbol{F}}$, $W^{\boldsymbol{G}}$ are the works done by the forces $\boldsymbol{F}$, $\boldsymbol{G}$ in this motion, and $T^{\boldsymbol{A}}$, $T^{\boldsymbol{B}}$ are the kinetic energies of the satellite at the points A, B. Now

$$\begin{aligned} W^{\boldsymbol{F}} &= V(A) - V(B) \\ &= -mgR^2\left(\frac{1}{R} - \frac{1}{R+h}\right) \\ &= \frac{mgRh}{R+h}. \end{aligned}$$

Hence

$$W^{\boldsymbol{G}} = \frac{mgRh}{R+h} + T_{\boldsymbol{B}} - T_{\boldsymbol{A}}.$$

In particular, if the satellite starts and finishes at rest, $T^A = T^B = 0$ and

$$W^{\boldsymbol{G}} = \frac{mgRh}{R+h}.$$

This is the work done by the force $\boldsymbol{G}$. On using the numerical data given in the question, we find that the **work done** by $\boldsymbol{G}$ in raising the satellite to a height of 2000 km is approximately 3.0×10^9 J.

Note. This calculation ignores the contribution to T_A made by the Earth's rotation. However, a quick calculation shows that T_A is about 2×10^7 J, which is relatively insignificant. Hence the rotation of the Earth can be safely disregarded.

On the other hand, if the force $\boldsymbol{G}$ is required to place the satellite in a *circular orbit* at height 2000 km, then T_B is approximately 4.8×10^9 J, which is larger than the work done against the Earth's gravity. ■

Problem 6.5

A particle P of unit mass moves on the positive x-axis under the force field

$$F = \frac{36}{x^3} - \frac{9}{x^2} \qquad (x > 0).$$

Show that each motion of P consists of either (i) a periodic oscillation between two extreme points, or (ii) an unbounded motion with one extreme point, depending upon the value of the total energy. Initially P is projected from the point $x = 4$ with speed 0.5. Show that P oscillates between two extreme points and find the period of the motion. [You may make use of the formula

$$\int_a^b \frac{x\,dx}{[(x-a)(b-x)]^{1/2}} = \frac{\pi(a+b)}{2}.]$$

Show that there is a single equilibrium position for P and that it is stable. Find the period of *small* oscillations about this point.

Solution

The **potential energy** of the force field $F(x)$ is

$$\begin{aligned} V &= -\int F(x)\,dx \\ &= -\int \left(\frac{36}{x^3} - \frac{9}{x^2}\right) dx \\ &= \frac{18}{x^2} - \frac{9}{x}, \end{aligned}$$

The **energy conservation** equation is then

$$\tfrac{1}{2}v^2 + V(x) = E,$$

where $v = \dot{x}$ and E is the constant total energy.

The graph of the function $V(x)$ is shown in Figure 6.1. The possible motions of the particle can be classified as follows:

(i) If $E < 0$ then the motion is a periodic oscillation between two extreme points.

(ii) If $E > 0$ then the motion is unbounded with one extreme point.

Consider now the motion arising from the initial condition $v = 0.5$ when $x = 4$. In this case, $E = -1$ and so the motion must be a periodic oscillation between two extreme points. At the extreme points, $v = 0$ and x must satisfy the equation

$$\frac{18}{x^2} - \frac{9}{x} = -1,$$

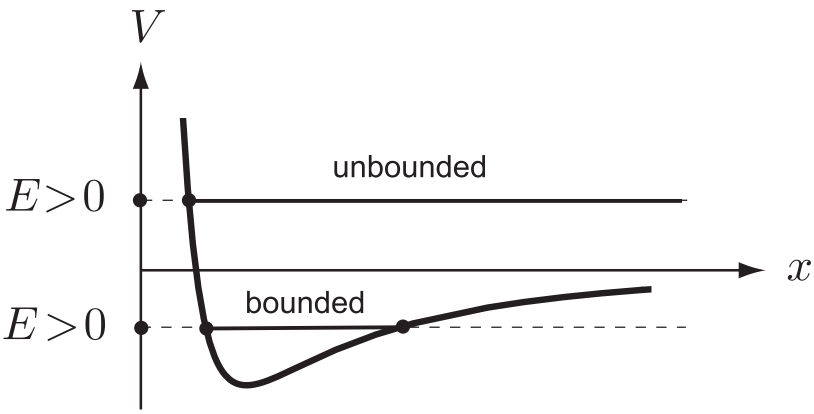

FIGURE 6.1 The potential energy function $V(x)$ in Problem 6.5.

that is,

$$x^2 - 9x + 18 = 0.$$

This quadratic equation factorises and the roots are $x = 3$ and $x = 6$. These are the **extreme points** of the motion.

To find the period τ, we must integrate the energy equation

$$\tfrac{1}{2}\dot{x}^2 + \frac{18}{x^2} - \frac{9}{x} = -1,$$

which can be written in the form

$$\dot{x}^2 = \frac{2}{x^2}(x-3)(6-x).$$

When the particle is moving to the *right*, we have

$$\frac{dx}{dt} = +\frac{\sqrt{2}}{x}\big((x-3)(6-x)\big)^{1/2},$$

which is a separable first order ODE for the function $x(t)$. On separating we obtain

$$\int_0^{\tau/2} dt = \frac{1}{\sqrt{2}}\int_3^6 \frac{x\,dx}{\big((x-3)(6-x)\big)^{1/2}}$$

so that the **period of the oscillations** is

$$\tau = \sqrt{2} \int_3^6 \frac{x\,dx}{\big((x-3)(6-x)\big)^{1/2}} = \frac{9\pi}{\sqrt{2}},$$

on using the formula given in the question.

The equilibrium positions of the particle are the stationary points of the function $V(x)$. Since

$$V' = -F = \frac{9}{x^2} - \frac{36}{x^3},$$

the only stationary point of V is at $x = 4$. This is the only **equilibrium position** of the particle. Now

$$\begin{aligned} V'' &= \frac{108}{x^4} - \frac{18}{x^3} \\ &= \frac{9}{64} \end{aligned}$$

when $x = 4$. Since $V''(4) > 0$, this equlibrium position is **stable**. The angular frequency Ω of small oscillatons about $x = 4$ is given by

$$\begin{aligned} \Omega &= \left(\frac{V''(4)}{m}\right)^{1/2} \\ &= \left(\frac{9/64}{1}\right)^{1/2} = \frac{3}{8}. \end{aligned}$$

The **period τ of small oscillations** about $x = 4$ is therefore

$$\tau = \frac{2\pi}{\Omega} = \frac{16\pi}{3}. \blacksquare$$

Problem 6.6

A particle P of mass m moves on the x-axis under the force field with potential energy $V = V_0(x/b)^4$, where V_0 and b are positive constants. Show that any motion of P consists of a periodic oscillation with centre at the origin. Show further that, when the oscillation has amplitude a, the period τ is given by

$$\tau = 2\sqrt{2}\left(\frac{m}{V_0}\right)^{1/2}\frac{b^2}{a}\int_0^1 \frac{d\xi}{(1-\xi^4)^{1/2}}.$$

[Thus, the larger the amplitude, the shorter the period!]

Solution

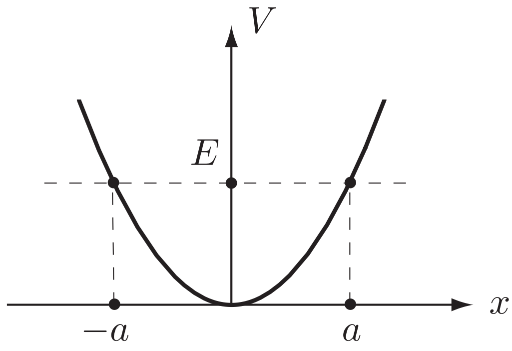

FIGURE 6.2 The potential energy function $V(x)$ in Problem 6.6.

The **energy conservation** equation for the particle is

$$\tfrac{1}{2}mv^2 + V_0\left(\frac{x}{b}\right)^4 = E,$$

where $v = \dot{x}$ and E is the constant total energy. The graph of the potential energy function $V(x)$ is shown in Figure 6.2. It is evident that *every* motion of the particle is a periodic oscillation that is symmetrical about the origin.

Consider an oscillating motion of amplitude a. In this case, $v = 0$ at $x = \pm a$ and so

$$E = V_0\left(\frac{x}{b}\right)^4$$

and the energy conservation equation becomes

$$\dot{x}^2 = \frac{2V_0}{mb^4}\left(a^4 - x^4\right).$$

To find the period τ, we must integrate the energy equation. When the particle is moving to the *right*,

$$\frac{dx}{dt} = +\left(\frac{2V_0}{mb^4}\right)^{1/2}\left(a^4 - x^4\right)^{1/2}$$

which is a separable first order ODE for the function $x(t)$. On separating we obtain

$$\left(\frac{2V_0}{mb^4}\right)^{1/2}\int_0^{\tau/4} dt = \int_0^a \frac{dx}{\left(a^4 - x^4\right)^{1/2}}$$

so that the **period of the oscillations** is

$$\begin{aligned}\tau &= 2\sqrt{2}\left(\frac{mb^4}{V_0}\right)^{1/2}\int_0^a \frac{dx}{\left(a^4 - x^4\right)^{1/2}}\\ &= 2\sqrt{2}\left(\frac{m}{V_0}\right)^{1/2}\frac{b^2}{a}\int_0^1 \frac{d\xi}{\left(1 - \xi^4\right)^{1/2}}\end{aligned}$$

on making the substitution $x = a\xi$. ■

[Just for the record,

$$\int_0^1 \frac{d\xi}{(1 - \xi^4)^{1/2}} = \frac{\sqrt{\pi}\,\Gamma(5/4)}{\Gamma(3/4)},$$

where $\Gamma(z)$ is the Gamma function. The numerical value of the integral is 1.31 approximately.]

Problem 6.7

A particle P of mass m, which is on the negative x-axis, is moving towards the origin with constant speed u. When P reaches the origin, it experiences the force $F = -Kx^2$, where K is a positive constant. How far does P get along the positive x-axis?

Solution

The **potential energy** of the force field F is

$$
\begin{aligned}
V &= -\int F\,dx = K\int x^2\,dx \\
&= \tfrac{1}{3}Kx^3.
\end{aligned}
$$

Hence, while the particle is in the region $x \geq 0$, its **energy conservation** equation is

$$\tfrac{1}{2}mv^2 + \tfrac{1}{3}Kx^3 = E,$$

where $v = \dot{x}$ and E is the constant total energy. Consider the motion arising from the initial condition $v = u$ when $x = 0$. In this case,

$$E = \tfrac{1}{2}mu^2$$

and the energy conservation equation becomes

$$\tfrac{1}{2}mv^2 + \tfrac{1}{3}Kx^3 = \tfrac{1}{2}mu^2.$$

The maximum value of x is attained when $v = 0$, that is, when x satisfies the equation

$$0 + \tfrac{1}{3}Kx^3 = \tfrac{1}{2}mu^2.$$

Hence the **farthest point** along the x-axis reached by the particle is

$$x = \left(\frac{3mu^2}{2K}\right)^{1/3}. \blacksquare$$

Problem 6.8

A particle P of mass m moves on the x-axis under the combined gravitational attraction of two particles, each of mass M, fixed at the points $(0, \pm a, 0)$ respectively. Example 3.4 shows that the force field $F(x)$ acting on P is given by

$$F = -\frac{2mMGx}{(a^2 + x^2)^{3/2}}.$$

Find the corresponding potential energy $V(x)$.

Initially P is released from rest at the point $x = 3a/4$. Find the maximum speed achieved by P in the subsequent motion.

Solution

The **potential energy** of the force field $F(x)$ is

$$\begin{aligned} V &= -\int F\,dx \\ &= 2mMG \int \frac{x\,dx}{(a^2 + x^2)^{3/2}} \\ &= -\frac{2mMG}{\left(a^2 + x^2\right)^{1/2}}. \end{aligned}$$

Hence the **energy conservation** equation for the particle is

$$\tfrac{1}{2}mv^2 - \frac{2mMG}{\left(a^2 + x^2\right)^{1/2}} = E$$

where $v = \dot{x}$ and E is the constant total energy. Consider now the motion arising from the initial condition $v = 0$ when $x = \frac{3}{4}a$. In this case,

$$\begin{aligned} E &= 0 - \frac{2mMG}{\left(a^2 + \frac{9}{16}a^2\right)^{1/2}} \\ &= -\frac{8mMG}{5a}, \end{aligned}$$

and the energy conservation equation is

$$\tfrac{1}{2}mv^2 = \frac{2mMG}{\left(a^2 + x^2\right)^{1/2}} - \frac{8mMG}{5a}.$$

The maximum value V of the speed $|v|$ is achieved when $x = 0$. Hence

$$\tfrac{1}{2}mV^2 = \frac{2mMG}{a} - \frac{8mMG}{5a}$$

and so

$$V = \left(\frac{4MG}{5a}\right)^{1/2}. \blacksquare$$

Problem 6.9

A particle P of mass m moves on the axis Oz under the gravitational attraction of a uniform circular disk of mass M and radius a. Example 3.6 shows that the force field $F(z)$ acting on P is given by

$$F = -\frac{2mMG}{a^2}\left[1 - \frac{z}{(a^2+z^2)^{1/2}}\right] \qquad (z > 0).$$

Find the corresponding potential energy $V(z)$ for $z > 0$.

Initially P is released from rest at the point $z = 4a/3$. Find the speed of P when it hits the disk.

Solution

The **potential energy** of the force field $F(z)$ is

$$\begin{aligned} V &= -\int F\,dx \\ &= \frac{2mMG}{a^2}\int\left(1 - \frac{z}{(a^2+z^2)^{1/2}}\right)dx \\ &= \frac{2mMG}{a^2}\left(z - (a^2+z^2)^{1/2}\right). \end{aligned}$$

Hence the **energy conservation** equation for the particle is

$$\tfrac{1}{2}mv^2 + \frac{2mMG}{a^2}\left(z - (a^2+z^2)^{1/2}\right) = E$$

where $v = \dot{z}$ and E is the constant total energy. Consider now the motion arising from the initial condition $v = 0$ when $z = \frac{4}{3}a$. In this case,

$$\begin{aligned} E &= 0 + \frac{2mMG}{a^2}\left(\frac{4a}{3} - \frac{5a}{3}\right) \\ &= -\frac{2mMG}{3a}, \end{aligned}$$

and the energy conservation equation is

$$\tfrac{1}{2}mv^2 = \frac{2mMG}{a^2}\left((a^2+z^2)^{1/2} - z\right) - \frac{2mMG}{3a}.$$

On substituting $z = 0$ into this formula, we find that the **speed of the particle** when it hits the disk is

$$\left(\frac{8MG}{3a}\right)^{1/2}. \blacksquare$$

Problem 6.10

A catapult is made by connecting a light elastic cord of natural length $2a$ and strength α between two fixed supports, which are distance $2a$ apart. A stone of mass m is placed at the center of the cord, which is pulled back a distance $3a/4$ and then released from rest. Find the speed with which the stone is projected by the catapult.

Solution

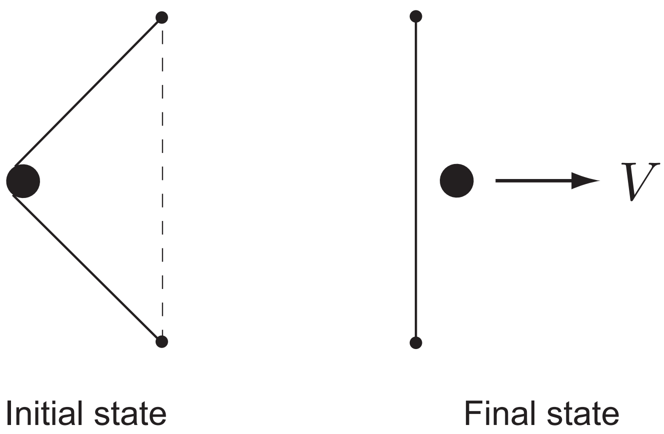

FIGURE 6.3 The catapult in Problem 6.10.

This is a 'before and after' problem. We do not obtain an equation of motion; instead we simply equate the initial and final values of the total energy.

Initial state: In the initial state, the stone is at rest and so its **kinetic energy** is zero. The main problem is to find the internal energy of the stretched elastic cord. Consider either of the the two equal segments that make up the cord. In the initial state, the length of the segment is

$$\left(a^2 + \left(\tfrac{3}{4}a\right)^2\right)^{1/2} = \tfrac{5}{4}a$$

so that its extension is $\frac{1}{4}a$. The strength of the cord is α, but the strength of the *segment* is 2α. (If the whole cord and the segment were both subjected to the same tension, the extension of the cord would be *twice* the extension of the segment.) Hence, the internal energy of the segment is $\frac{1}{2}(2\alpha)\left(\frac{1}{4}a\right)^2 = \frac{1}{16}\alpha a^2$. The **internal**

energy of the cord is therefore $\frac{1}{8}\alpha a^2$ and the **total energy** E is therefore

$$E = 0 + \tfrac{1}{8}\alpha a^2 = \tfrac{1}{8}\alpha a^2.$$

Final state: In the final state, the stone is moving with unknown speed V and so its **kinetic energy** is $\frac{1}{2}mV^2$. In the final state, each segment of the cord has its natural length and so the total **internal energy** is zero. The **total energy** E is therefore

$$E = \tfrac{1}{2}mV^2 + 0 = \tfrac{1}{2}mV^2.$$

Since the **total energy is conserved** in this problem, the initial and final values of E are equal. Hence

$$\tfrac{1}{2}mV^2 = \tfrac{1}{8}\alpha a^2$$

and the **speed** with which the stone is projected is therefore

$$V = \left(\frac{\alpha a^2}{4m}\right)^{1/2} \blacksquare$$

Problem 6.11

A light spring of natural length a is placed on a horizontal floor in the upright position. When a block of mass M is resting in equilibrium on top of the spring, the compression of the spring is $a/15$. The block is now lifted so that its underside is at height $3a/2$ above the floor and then released from rest. Find the compression of the spring when the block first comes to rest.

Solution

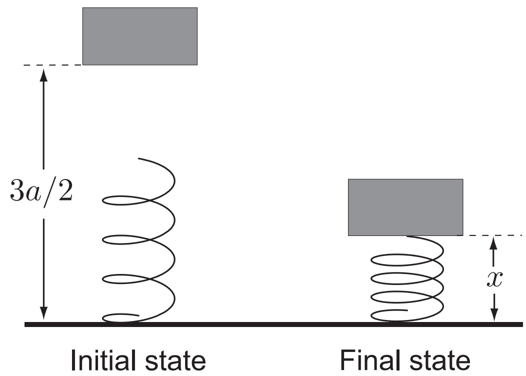

FIGURE 6.4 The system in Problem 6.11.

This is a 'before and after' problem. We do not obtain an equation of motion; instead we simply equate the initial and final values of the total energy.

Initial state: In the initial state, the block is at rest and the spring is unstretched. Hence the **kinetic energy** of the block and the **internal energy** of the spring are zero. The **gravitational potential energy** of the block is $Mg\left(\frac{3}{2}a + h\right)$, where M is the mass of the block and $2h$ is its thickness. Hence, the **total energy** E is

$$E = 0 + 0 + Mg\left(\tfrac{3}{2}a + h\right).$$

Final state: In the final state, the block is again at rest and so its **kinetic energy** is zero. The internal energy of the spring is $\frac{1}{2}\alpha(a - x)^2$, where α is its strength and x is the length to which it has been compressed when the block comes to rest. Since $\alpha = Mg/(\frac{1}{15}a) = 15Mg/a$, the **internal energy** of the spring is $\frac{15}{2}Mg(a-x)^2/a$. The **gravitational potential energy** of the block is $Mg(x + h)$. Hence, the **total energy** E is

$$E = 0 + \frac{15Mg}{2a}(a - x)^2 + Mg(x + h).$$

Since the **total energy is conserved** in this problem, the initial and final values of E are equal. Hence

$$Mg\left(\tfrac{3}{2}a+h\right)=\frac{15Mg}{2a}(a-x)^2+Mg(x+h),$$

which reduces to

$$15x^2-28ax+12a^2=0.$$

This quadratic equation factorises and its roots are $x=\frac{2}{3}a$ and $x=\frac{6}{5}a$. The second root is unphysical since it would require the block to come to rest *before* it had even met the spring. The **compression of the spring** when the block first comes to rest is therefore $a-\frac{2}{3}a=\frac{1}{3}a$. ■

Problem 6.12

A particle P carries a charge e and moves under the influence of the static magnetic field $\boldsymbol{B}(\boldsymbol{r})$ which exerts the force $\boldsymbol{F} = e\boldsymbol{v}\times\boldsymbol{B}$ on P, where $\boldsymbol{v}$ is the velocity of P. Show that P travels with constant *speed*.

Solution

Since $\boldsymbol{F} = e\boldsymbol{v}\times\boldsymbol{B}$, the rate at which $\boldsymbol{F}$ does work on the particle is

$$\boldsymbol{F}\cdot\boldsymbol{v} = e\left(\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\boldsymbol{v} = 0.$$

Thus $\boldsymbol{F}$ does no work and so *the kinetic energy of the particle is a constant of the motion*. Hence the particle moves with **constant speed** (but not neccessarily with constant *velocity*!) ■

Problem 6.13 ✻

A mortar shell is to be fired from level ground so as to clear a flat topped building of height h and width a. The mortar gun can be placed anywhere on the ground and can have any angle of elevation. What is the least projection speed that will allow the shell to clear the building? [*Hint* How is the reqired minimum projection speed changed if the mortar is raised to rooftop level?]

For the special case in which $h = \frac{1}{2}a$, find the optimum position for the mortar and the optimum elevation angle to clear the building.

Solution

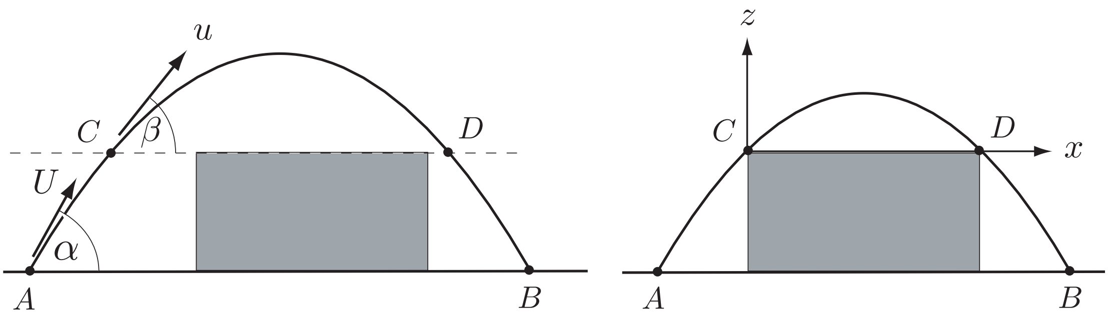

FIGURE 6.5 **Left**: A general trajectory that clears the building. **Right**: The optimum trajectory.

A typical trajectory that clears the building is shown in Figure 6.5 (left). The problem is to choose the projection point A and the elevation angle α so that the building can be cleared using the least value of U. Suppose the path first cuts the horizontal plane at rooftop level at C at which point the speed of the shell is u and the elevation angle is β. Then, by **energy conservation**, U and u are related by

$$U^2 = u^2 + 2gh.$$

Hence U^2 and u^2 *differ by a constant*. It follows that the original problem is equivalent to the problem of choosing C and β so that the building can be cleared using the least value of u. But the solution to this second problem is well known. This is solved by taking (i) C to be at the top corner of the building, (ii) the angle β to be 45°, and (iii) the speed u to be $(ga)^{1/2}$. This **optimum trajectory** is shown in Figure 6.5 (right). The value of the *initial* projection speed U in this trajectory is then given by

$$U^2 = u^2 + 2gh = ga + 2gh = g(a + 2h).$$

Hence the **least projection speed** that will allow the shell to clear the building is $\left(g(a+2h)\right)^{1/2}$.

To find the position of A and the elevation α, we must investigate the optimum trajectory in more detail. Take axes Cxz as shown in Figure 6.5 (right). Then the **optimum path** is

$$z = \frac{x}{a}(a-x).$$

This path intersects the ground when $z = -h$, that is, when

$$x^2 - ax - ah = 0.$$

The two roots of this quadratic equation are the coordinates of the points A, B in Figure 6.5 (right). From now on, we will work with the **special case** in which $h = \frac{1}{2}a$. In this case, the equation for x becomes

$$2x^2 - 2ax - a^2 = 0,$$

the roots of which are $x = \frac{1}{2}\left(1 \pm \sqrt{3}\right)a$. It follows that (in the special case when $h = \frac{1}{2}a$) the **mortar should be placed** a distance $\frac{1}{2}\left(\sqrt{3}-1\right)a$ from the wall of the building. The corresponding value of the elevation α is given by

$$\begin{aligned}\tan\alpha &= \left.\frac{dz}{dx}\right|_{x=\frac{1}{2}(1-\sqrt{3})a} \\ &= \left[1 - \frac{2x}{a}\right]_{x=\frac{1}{2}(1-\sqrt{3})a} \\ &= \sqrt{3}.\end{aligned}$$

Hence (in the special case when $h = \frac{1}{2}a$) the **elevation of the mortar** should be taken to be 60°. ■

Problem 6.14 ✻

An *earthed* conducting sphere of radius a is fixed in space, and a particle P, of mass m and charge q, can move freely outside the sphere. Initially P is a distance b ($> a$) from the centre O of the sphere when it is projected directly away from O. What must the projection speed be for P to escape to infinity? [Ignore electro*dynamic* effects. Use the method of images to solve the electro*static* problem.]

Solution

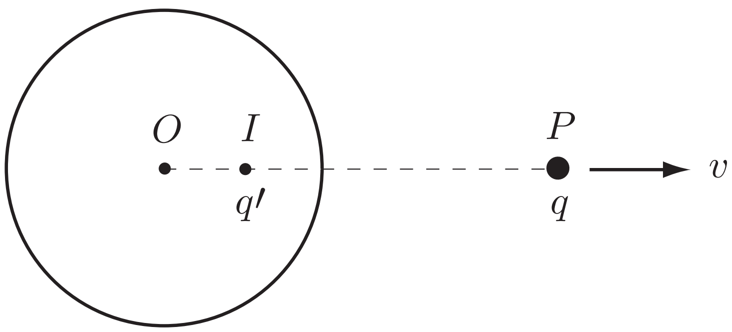

FIGURE 6.6 The charge q and its image charge q'.

The system is shown in Figure 6.6. When electro*dynamic* effects are neglected, the electric field outside the sphere is the same *as if* the sphere were removed and an 'image charge' q' placed at the 'image point' I, where $q' = -qa/r$ and $OI = a^2/r$. The outward **force** F experienced by P is therefore

$$F = \frac{qq'}{IP^2} = \frac{q\left(-\frac{qa}{r}\right)}{\left(r - \frac{a^2}{r}\right)^2}$$

$$= -\frac{q^2 ar}{\left(r^2 - a^2\right)^2}.$$

This formula is correct in cgs/electrostatic units.

The **potential energy** of this force field is

$$\begin{aligned} V &= -\int F\,dr \\ &= q^2 a \int \frac{r\,dr}{\left(r^2-a^2\right)^2} \\ &= -\frac{q^2 a}{2\left(r^2-a^2\right)}. \end{aligned}$$

The **energy conservation** equation for the particle is therefore

$$\tfrac{1}{2}mv^2 - \frac{q^2 a}{2\left(r^2-a^2\right)} = E,$$

where $v = \dot{r}$ and E is the constant total energy. Consider the motion that arises from the initial condition $v = u$ when $r = b$. In this case

$$E = \tfrac{1}{2}mu^2 - \frac{q^2 a}{2\left(b^2-a^2\right)}$$

and the energy conservation equation becomes

$$mv^2 = \left(mu^2 - \frac{q^2 a}{b^2-a^2}\right) + \frac{q^2 a}{r^2-a^2}.$$

The **condition for escape** is that the quantity in the brackets is *positive*, that is,

$$u^2 \ge \frac{q^2 a}{m\left(b^2-a^2\right)}. \blacksquare$$

Problem 6.15 ✻

An *uncharged* conducting sphere of radius a is fixed in space and a particle P, of mass m and charge q, can move freely outside the sphere. Initially P is a distance $b\ (> a)$ from the centre O of the sphere when it is projected directly away from O. What must the projection speed be for P to escape to infinity? [Ignore electro*dynamic* effects. Use the method of images to solve the electro*static* problem.]

Solution

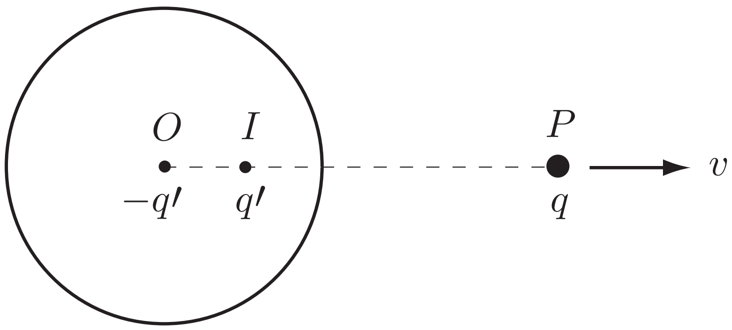

FIGURE 6.7 The charge q and its image charges q' and $-q'$.

The system is shown in Figure 6.7. When electro*dynamic* effects are neglected, the electric field outside the sphere is the same *as if* the sphere were removed and 'image charges' q' and $-q'$ were placed at the 'image points' I and O, where $q' = -qa/r$ and $OI = a^2/r$. The outward **force** F experienced by P is therefore

$$F = \frac{qq'}{IP^2} - \frac{qq'}{OP^2} = \frac{q\left(-\dfrac{qa}{r}\right)}{\left(r - \dfrac{a^2}{r}\right)^2} + \frac{q\left(\dfrac{qa}{r}\right)}{r^2}$$

$$= -\frac{q^2 ar}{\left(r^2 - a^2\right)^2} + \frac{q^2 a}{r^3}.$$

This formula is correct in cgs/electrostatic units.

The **potential energy** of this force field is

$$\begin{aligned} V &= -\int F\,dr \\ &= q^2 a \int \left(\frac{r}{\left(r^2 - a^2\right)^2} - \frac{1}{r^3} \right) dr \\ &= -\frac{q^2 a}{2\left(r^2 - a^2\right)} + \frac{q^2 a}{2r^2}. \end{aligned}$$

The **energy conservation** equation for the particle is therefore

$$\tfrac{1}{2} m v^2 - \frac{q^2 a}{2\left(r^2 - a^2\right)} + \frac{q^2 a}{2r^2} = E,$$

where $v = \dot{r}$ and E is the constant total energy. Consider the motion that arises from the initial condition $v = u$ when $r = b$. In this case

$$E = \tfrac{1}{2} m u^2 - \frac{q^2 a}{2\left(b^2 - a^2\right)} + \frac{q^2 a}{2b^2}$$

and the energy conservation equation becomes

$$m v^2 = \left(m u^2 - \frac{q^2 a}{b^2 - a^2} + \frac{q^2 a}{b^2} \right) + \frac{q^2 a}{r^2 - a^2}.$$

The **condition for escape** is that the quantity in the brackets is *positive*, that is,

$$u^2 \geq \frac{q^2 a^3}{m b^2 \left(b^2 - a^2\right)}. \blacksquare$$

Problem 6.16

A bead of mass m can slide on a smooth circular wire of radius a, which is fixed in a vertical plane. The bead is connected to the highest point of the wire by a light spring of natural length $3a/2$ and strength α. Determine the stability of the equilibrium position at the lowest point of the wire in the cases (i) $\alpha = 2mg/a$, and (ii) $\alpha = 5mg/a$.

Solution

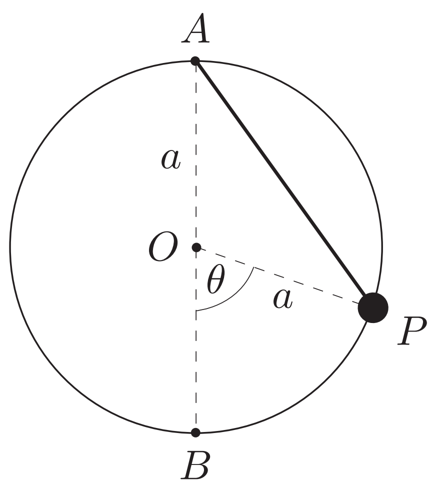

FIGURE 6.8 The system in Problem 6.16.

Since the wire is *smooth* the constraint force that it exerts on the particle does no work. Thus energy conservation holds in its standard form.

Let θ be the angle between the radius OP and the downwards vertical, as shown in Figure 6.8. The length of the spring is $2a\cos\frac{1}{2}\theta$ and its **internal energy** is therefore

$$\tfrac{1}{2}\alpha\left(2a\cos\tfrac{1}{2}\theta - \tfrac{3}{2}a\right)^2 = \frac{\alpha a^2}{8}\left(4\cos\tfrac{1}{2}\theta - 3\right)^2.$$

The **gravitational potential energy** of the particle is $-mga\cos\theta$. The **total potential energy** of the system is therefore

$$V = \frac{\alpha a^2}{8}\left(4\cos\tfrac{1}{2}\theta - 3\right)^2 - mga\cos\theta.$$

On differentiating, we find that

$$V' = \tfrac{3}{2}\alpha a^2 \sin\tfrac{1}{2}\theta + \left(mga - \alpha a^2\right)\sin\theta$$

and

$$V'' = \tfrac{3}{4}\alpha a^2 \cos\tfrac{1}{2}\theta + \left(mga - \alpha a^2\right)\cos\theta.$$

Hence

$$V'(0) = 0 \qquad \text{and} \qquad V''(0) = mga - \tfrac{1}{4}\alpha a^2.$$

This confirms that $\theta = 0$ is an equilibrium position of the particle and shows that the equilibrium there is *stable* when $\alpha < 4mg/a$ and *unstable* when $\alpha > 4mg/a$. Hence:

(i) When $\alpha = 2mg/a$, the equilibrium is **stable**.

(ii) When $\alpha = 5mg/a$, the equilibrium is **unstable**. ■

Problem 6.17

A smooth wire has the form of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$, where θ is a real parameter, and a, b are positive constants. The wire is fixed with the axis Oz pointing vertically upwards. A particle P, which can slide freely on the wire, is released from rest at the point $(a, 0, 2\pi b)$. Find the speed of P when it reaches the point $(a, 0, 0)$ and the time taken for it to do so.

Solution

Since the wire is *smooth* the constraint force that it exerts does no work. Hence energy conservation holds in its standard form. The **energy conservation equation** is therefore

$$\tfrac{1}{2}mv^2 + mgz = E,$$

where m is the mass of the particle, v is its speed and E is the constant total energy. The initial conditions $z = 2\pi b$ and $v = 0$ when $t = 0$ give $E = 2\pi mgb$ and the energy equation can be written

$$v^2 = 2g(2\pi b - z).$$

Hence, providing that the particle arrives at $z = 0$ at all, its **arrival speed** is $2(\pi gb)^{1/2}$, *whatever the shape of the wire*.

To find the time taken, we must investigate the motion in more detail. For the helical wire given,

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= \left(-a\sin\theta\,\dot{\theta}\right)^2 + \left(a\cos\theta\,\dot{\theta}\right)^2 + \left(b\dot{\theta}\right)^2 \\ &= \left(a^2 + b^2\right)\dot{\theta}^2, \end{aligned}$$

and the energy equation can be written

$$\left(a^2 + b^2\right)\dot{\theta}^2 = 2gb(2\pi - \theta).$$

Since θ is *decreasing* in this motion, we have

$$\left(a^2 + b^2\right)^{1/2}\dot{\theta} = -(2gb)^{1/2}(2\pi - \theta)^{1/2},$$

which is a separable first order ODE for θ. On separating, we obtain

$$\left(a^2 + b^2\right)^{1/2}\int_{2\pi}^{0}\frac{d\theta}{(2\pi - \theta)^{1/2}} = -(2gb)^{1/2}\int_0^{\tau} dt,$$

where τ is the duration of the motion. Hence

$$\begin{aligned}\tau &= \left(\frac{a^2+b^2}{2gb}\right)^{1/2}\int_0^{2\pi}\frac{d\theta}{(2\pi-\theta)^{1/2}}\\ &= \left(\frac{a^2+b^2}{2gb}\right)^{1/2}\left[-2(2\pi-\theta)^{1/2}\right]_0^{2\pi}\\ &= 2\left(\frac{\pi\left(a^2+b^2\right)}{gb}\right)^{1/2}.\end{aligned}$$

This is the **time taken** for the particle to reach the point $(a, 0, 0)$. ■

Problem 6.18

A smooth wire has the form of the parabola $z = x^2/2b$, $y = 0$, where b is a positive constant. The wire is fixed with the axis Oz pointing vertically upwards. A particle P, which can slide freely on the wire, is performing oscillations with x in the range $-a \le x \le a$. Show that the period τ of these oscillations is given by

$$\tau = \frac{4}{(gb)^{1/2}} \int_0^a \left(\frac{b^2 + x^2}{a^2 - x^2}\right)^{1/2} dx.$$

By making the substitution $x = a \sin\psi$ in the above integral, obtain a new formula for τ. Use this formula to find a two-term approximation to τ, valid when the ratio a/b is small.

Solution

Since the wire is *smooth* the constraint force that it exerts does no work. Hence energy conservation holds in its standard form. The **energy conservation equation** is therefore

$$\tfrac{1}{2}mv^2 + mgz = E,$$

where m is the mass of the particle, v is its speed and E is the constant total energy. In the present problem, $z = x^2/2b$ and

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{z}^2 \\ &= \dot{x}^2 + \left(\frac{x\dot{x}}{b}\right)^2 \\ &= \left(1 + \frac{x^2}{b^2}\right)\dot{x}^2, \end{aligned}$$

so that the energy equation becomes

$$\tfrac{1}{2}m\left(1 + \frac{x^2}{b^2}\right)\dot{x}^2 + \frac{mgx^2}{2b} = E.$$

Consider an oscillatory motion of amplitude a. In this case, $v = 0$ $x = \pm a$ and so $E = mga^2/2b$. The energy equation for this motion is therefore

$$\left(b^2 + x^2\right)\dot{x}^2 = gb\left(a^2 - x^2\right).$$

To find the period τ, we must integrate the energy equation. When the particle is moving to the *right*, we have

$$\left(b^2 + x^2\right)^{1/2} \frac{dx}{dt} = +(gb)^{1/2}\left(a^2 - x^2\right)^{1/2},$$

which is a separable first order ODE for the function $x(t)$. On separating we obtain

$$(gb)^{1/2}\int_0^{\tau/4} dt = \int_0^a \left(\frac{b^2+x^2}{a^2-x^2}\right)^{1/2} dx$$

so that the **period of the oscillations** is

$$\tau = \frac{4}{(gb)^{1/2}}\int_0^a \left(\frac{b^2+x^2}{a^2-x^2}\right)^{1/2} dx.$$

On making the substitution $x = a\sin\psi$ in the integral, this formula becomes

$$\tau = 4\left(\frac{b}{g}\right)^{1/2}\int_0^{\pi/2}\left(1+\frac{a^2}{b^2}\sin^2\psi\right)^{1/2} d\psi.$$

When the ratio a/b is small, the integrand can be expanded in the form

$$\left(1+\frac{a^2}{b^2}\sin^2\psi\right)^{1/2} = 1+\frac{a^2}{2b^2}\sin^2\psi + O\left(\frac{a^4}{b^4}\right)$$

from which it follows that

$$\begin{aligned}\tau &= 4\left(\frac{b}{g}\right)^{1/2}\int_0^{\pi/2}\left[1+\frac{a^2}{2b^2}\sin^2\psi + O\left(\frac{a^4}{b^4}\right)\right]d\psi\\ &= 4\left(\frac{b}{g}\right)^{1/2}\left[\frac{\pi}{2}+\frac{a^2}{2b^2}\left(\frac{\pi}{4}\right)+O\left(\frac{a^4}{b^4}\right)\right]\\ &= 2\pi\left(\frac{b}{g}\right)^{1/2}\left[1+\frac{a^2}{4b^2}+O\left(\frac{a^4}{b^4}\right)\right].\end{aligned}$$

This is the required **two term approximation** to the period, valid when the ratio a/b is small. ■

Problem 6.19 ✻

A smooth wire has the form of the cycloid $x = c(\theta + \sin\theta)$, $y = 0$, $z = c(1 - \cos\theta)$, where c is a positive constant and the parameter θ lies in the range $-\pi \le \theta \le \pi$. The wire is fixed with the axis Oz pointing vertically upwards. [Make a sketch of the wire.] A particle can slide freely on the wire. Show that the energy conservation equation is

$$(1 + \cos\theta)\,\dot{\theta}^2 + \frac{g}{c}(1 - \cos\theta) = \text{constant}.$$

A new parameter u is defined by $u = \sin\frac{1}{2}\theta$. Show that, in terms of u, the equation of motion for the particle is

$$\ddot{u} + \left(\frac{g}{4c}\right)u.$$

Deduce that the particle performs oscillations with period $4\pi(c/g)^{1/2}$, independent of the amplitude!

Solution

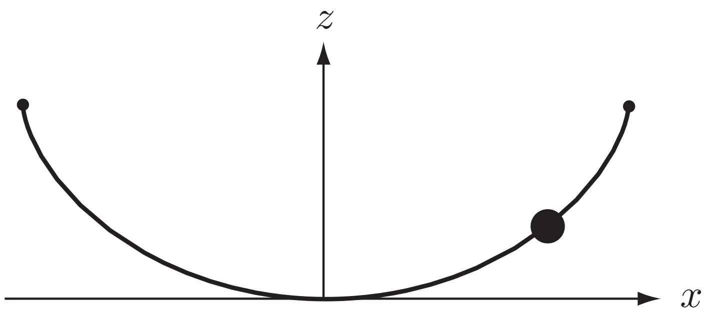

FIGURE 6.9 The cycloidal wire in Problem 6.19.

Since the wire is *smooth* the constraint force that it exerts does no work. Hence energy conservation holds in its standard form. The **energy conservation equation** is therefore

$$\tfrac{1}{2}mv^2 + mgz = E,$$

where m is the mass of the particle, v is its speed and E is the constant total energy.

In the present problem, $x = c(\theta + \sin\theta)$ and $z = c(1 - \cos\theta)$ so that

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{z}^2 \\ &= c^2(1+\cos\theta)^2\dot{\theta}^2 + c^2\sin^2\theta\,\dot{\theta}^2 \\ &= 2c^2\,(1+\cos\theta)\,\dot{\theta}^2. \end{aligned}$$

The energy equation then becomes

$$mc^2\,(1+\cos\theta)\,\dot{\theta}^2 + mgc(1-\cos\theta) = E,$$

which is the form required.

If we make the substitution $u = \sin\frac{1}{2}\theta$, then

$$\begin{aligned} 1 - \cos\theta &= 2\sin^2\tfrac{1}{2}\theta = 2u^2, \\ (1+\cos\theta)\dot{\theta}^2 &= 2\cos^2\tfrac{1}{2}\theta\,\dot{\theta}^2 = 8\dot{u}^2 \end{aligned}$$

and the energy equation becomes

$$8mc^2\,\dot{u}^2 + 2mgc\,u^2 = E.$$

This is actually the energy equation for the simple harmonic oscillator. On differentiating with respect to t, we obtain

$$\ddot{u} + \left(\frac{g}{4c}\right)u = 0,$$

which is the SHM equation with $\Omega^2 = g/4c$. Hence the **period** τ of the oscillations is

$$\tau = 4\pi\left(\frac{c}{g}\right)^{1/2},$$

independent of the amplitude. ■

Problem 6.20

A smooth horizontal table has a vertical post fixed to it which has the form of a circular cylinder of radius a. A light inextensible string is wound around the base of the post (so that it does not slip) and its free end of the string is attached to a particle that can slide on the table. Initially the unwound part of the string is taut and of length b. The particle is then projected with speed u at right angles to the string so that the string winds itself *on* to the post. How long does it take for the particle to hit the post?

Solution

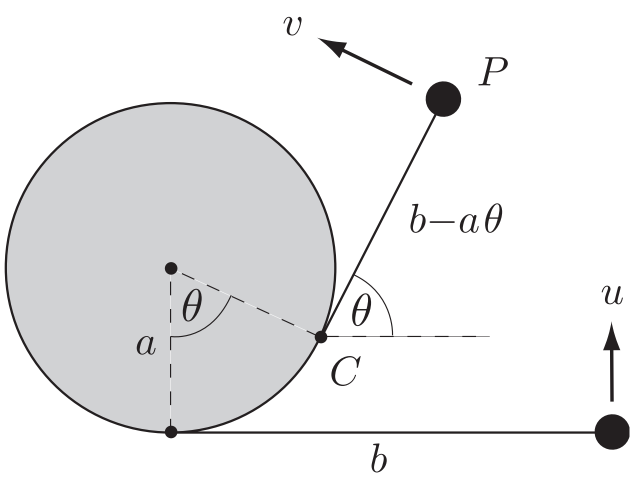

FIGURE 6.10 The system in Problem 6.20.

Since the string does not slip on the post, the points of the string that are in contact with the post are at rest. In particular, this applies to the point C shown in Figure 6.10. The free part of the string is therefore (instantaneously) rotating about C. The velocity of the particle is therefore perpendicular to CP and so the *tension in the string does no work*. The **energy conservation equation** for the particle is therefore

$$\tfrac{1}{2}mv^2 = E,$$

where m is the mass of the particle, v is its speed, and E is the constant total energy. The particle therefore moves with *constant speed*.

Let θ be the angle between the direction of the free string at time t and its initial direction, as shown in Figure 6.10. Since the length of the free string at time t is $b - a\theta$, it follows that $v = (b - a\theta)\dot{\theta}$. On using the initial condition $v = u$ when $t = 0$, the **energy conservation equation** becomes

$$(b - a\theta)\dot{\theta} = u.$$

This is a separable first order ODE for $\theta(t)$. On separating, we obtain

$$\int_0^{b/a} (b - a\theta)\, d\theta = u \int_0^{\tau} dt,$$

where τ is the time taken for the particle to hit the post. Hence

$$\begin{aligned} \tau &= \frac{1}{u} \int_0^{b/a} (b - a\theta)\, d\theta \\ &= \frac{1}{u} \left[b\theta - \tfrac{1}{2} a\theta^2 \right]_0^{b/a} \\ &= \frac{b^2}{2au}. \end{aligned}$$

This is the **time taken** for the particle to hit the post. ■

Problem 6.21

A heavy ball is suspended from a fixed point by a light inextensible string of length b. The ball is at rest in the equilibrium position when it is projected horizontally with speed $(7gb/2)^{1/2}$. Find the angle that the string makes with the upward vertical when the ball begins to leave its circular path. Show that, in the subsequent projectile motion, the ball returns to its starting point.

Solution

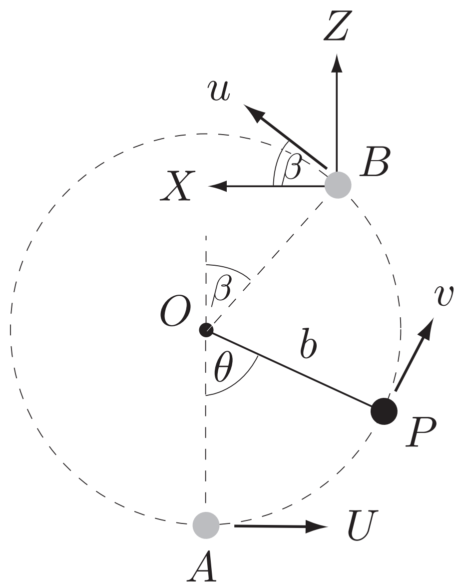

FIGURE 6.11 The system in Problem 6.21.

Since the tension in the string does no work, energy conservation holds in its standard form. The **energy conservation equation** is therefore

$$\tfrac{1}{2}mv^2 + mgz = E,$$

where m is the mass of the ball, v is its speed, z is its vertical displacement above O, and E is the constant total energy. In the present problem, $z = -b\cos\theta$ so that

$$\tfrac{1}{2}mv^2 - mgb\cos\theta = E.$$

The initial condition $v = (7gb/2)^{1/2}$ when $\theta = 0$ gives

$$E = \tfrac{7}{4}mgb - mgb = \tfrac{3}{4}mgb$$

and the energy equation becomes

$$v^2 = \tfrac{1}{2} g b (3 + 4\cos\theta).$$

The tension T in the string can be found by using the Second Law in reverse. Consider the component of the Second Law $\boldsymbol{F} = m\boldsymbol{a}$ in the direction $\overrightarrow{PO}$. This gives

$$T - mg\cos\theta = \frac{mv^2}{b},$$

and, on using the formula for v^2 provided by the energy equation, we find that

$$T = \tfrac{3}{2} mgb(1 + 2\cos\theta).$$

This formula holds *while the ball moves on the circular path*.

The **ball leaves the circle** when $T = 0$, that is, when $\theta = 120°$. The angle β shown in Figure 6.11 is therefore 60°. The **speed** u of the ball at this instant is given by the energy equation to be $u = (gb/2)^{1/2}$. The subsequent trajectory is given by standard projectile theory. In the coordinate system BXZ shown in Figure 6.11, the path of the ball is

$$\begin{aligned} Z &= \tan\beta\, X - \left(\frac{g}{2u^2\cos^2\beta}\right) X^2 \\ &= \sqrt{3}\, X - \frac{4X^2}{b}, \end{aligned}$$

on using the calculated values of β and u. The starting point A has coordinates

$$\begin{aligned} X &= b\sin\beta = \frac{\sqrt{3}\, b}{2}, \\ Z &= -b\cos\beta - b = -\frac{3b}{2}, \end{aligned}$$

and it is easily verified that this point does lie on the path of the ball. The ball therefore **returns to its starting point**. ■

Problem 6.22 ✻

A new *avant garde* mathematics building has a highly polished outer surface in the shape of a huge hemisphere of radius 40 m. The Head of Department, Prof. Oldfart, has his student, Vita Youngblood, hauled to the summit (to be photographed for publicity purposes) but a small gust of wind causes Vita to begin to slide down. Oldfart's displeasure is increased when Vita lands on (and severely damages) his car which is parked nearby. How far from the outer edge of the building did Oldfart park his car? Did he get what he deserved? (Happily, Vita escaped injury and found a new supervisor.)

Solution

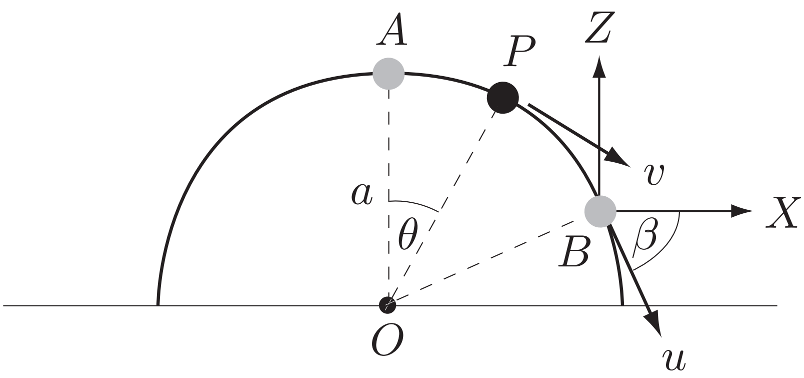

FIGURE 6.12 The system in Problem 6.22.

Since the surface of the building is *smooth*, the reaction that it exerts does no work. Hence energy conservation holds in its standard form. The **energy conservation equation** is therefore

$$\tfrac{1}{2}mv^2 + mgz = E,$$

where m and v are Vita's mass and speed, z is her height above the ground, and E is the constant total energy. Let θ be the angle shown in Figure 6.12. Then $z = a\cos\theta$ and we have

$$\tfrac{1}{2}mv^2 + mga\cos\theta = E.$$

The initial condition $v = 0$ when $\theta = 0$ gives

$$E = mga$$

so that the energy equation becomes

$$v^2 = 2ga(1 - \cos\theta).$$

The normal reaction R exerted by the roof can be found by using the Second Law in reverse. Consider the component of the Second Law $\boldsymbol{F} = m\boldsymbol{a}$ in the direction $\overrightarrow{PO}$. This gives

$$mg\cos\theta - R = \frac{mv^2}{b},$$

and, on using the formula for v^2 provided by the energy equation, we find that

$$R = mgb(3\cos\theta - 2).$$

This formula holds *while Vita remains in contact with the roof*.

Vita **leaves the roof** when $R = 0$, that is, when $\theta = \cos^{-1}\frac{2}{3}$. This is the angle β shown in Figure 6.11. Vita's **speed** u at this instant is given by the energy equation to be $u = (2ga/3)^{1/2}$. Her subsequent trajectory is given by standard projectile theory. In the coordinate system BXZ shown in Figure 6.11, her path is

$$\begin{aligned} Z &= -\tan\beta\, X - \left(\frac{g}{2u^2\cos^2\beta}\right)X^2 \\ &= -\frac{\sqrt{5}}{2}X - \frac{27X^2}{16a}, \end{aligned}$$

on using the calculated values of β and u. This path intersects the ground when $Z = -a\cos\beta$, that is, when

$$-\frac{2}{3}a = -\frac{\sqrt{5}}{2}X - \frac{27X^2}{16a}.$$

The X coordinate of the landing point therefore satisfies the quadratic equation

$$27X^2 + 8\sqrt{5}aX - \tfrac{32}{3}a^2 = 0$$

the roots of which are

$$X = \left(\frac{-4\sqrt{5} \pm 4\sqrt{23}}{27}\right)a.$$

The physically appropriate root is the *positive* one, X^+, which is $0.379a$ approximately. The distance of Vita's landing point from the wall of the building is therefore $a\sin\beta + X^+ - a = 0.125a$ approximately. When $a = 40$ m, this is approximately 5 m. Hence Oldfart's **car was parked** 5 m from the wall of the building. ■

Problem 6.23 ✻ ✻

A heavy ball is attached to a fixed point O by a light inextensible string of length $2a$. The ball is drawn back until the string makes an acute angle α with the downward vertical and is then released from rest. A thin peg is fixed a distance a vertically below O in the path of the string, as shown in book Figure 6.6. In a game of skill, the contestant chooses the value of α and wins a prize if the ball strikes the peg. Show that the winning value of α is approximately $86°$.

Solution

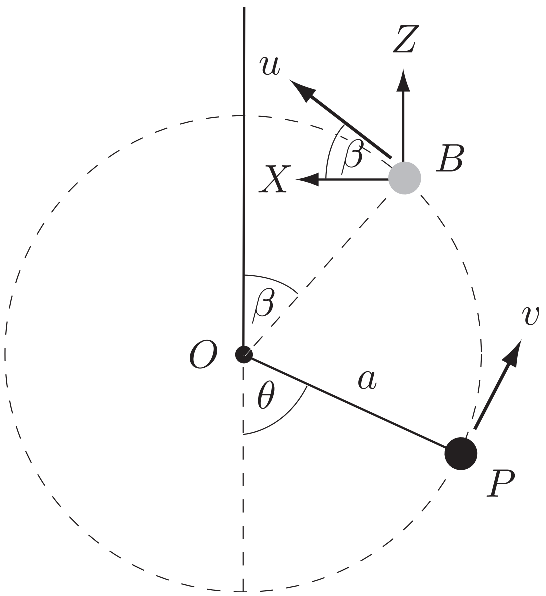

FIGURE 6.13 The system in Problem 6.23 after the string has met the peg.

Once the string has met the peg, the ball moves on a circular path of radius a, as shown in figure 6.13. Suppose that the ball leaves this circular path at the point B, where its speed is u and its direction of motion makes an angle β with the horizontal. At this instant, the tension in the string is zero. Then the Second Law, resolved in the direction $\overrightarrow{BO}$, gives

$$mg\cos\beta = \frac{mu^2}{a}.$$

Hence u and β are related by $u^2 = ga\cos\beta$. Once the string has become slack, the trajectory of the ball is given by standard projectile theory. In the coordinate system BXZ shown in Figure 6.13, the path of the ball is

$$\begin{aligned} Z &= \tan\beta\, X - \left(\frac{g}{2u^2\cos^2\beta}\right) X^2 \\ &= \tan\beta\, X - \left(\frac{1}{2a\cos^3\beta}\right) X^2. \end{aligned}$$

on using the relation $u^2 = ga\cos\beta$. If the ball is to hit the peg, we must have $Z = -a\cos\beta$ when $X = a\sin\beta$. This requires that

$$-\cos\beta = \sin\beta\tan\beta - \frac{\sin^2\beta}{2\cos^3\beta},$$

which reduces to the simple equation $3\cos^2\beta = 1$. The physically appropriate root of this equation is the *positive acute* angle $\beta = \cos^{-1}(1/\sqrt{3})$. This determines the angle β and, on making use of the relation $u^2 = ag\cos\beta$ again, the **speed** of the ball at B is found to be $u = \left(ag/\sqrt{3}\right)^{1/2}$.

The initial inclination α of the string can now be found by energy conservation. Since the tension in the string and the reaction of the peg do no work, energy conservation holds in its standard form. The **energy conservation equation** is therefore

$$\tfrac{1}{2}mv^2 + mgz = E,$$

where m is the mass of the ball, v is its speed, z is its vertical displacement above O, and E is the constant total energy. The initial condition $v = 0$ when $z = a-2a\cos\alpha$ gives $E = mga(1 - 2\cos\alpha)$ so that the energy equation becomes

$$v^2 + 2gz = 2ga(1 - 2\cos\alpha).$$

In particular, when the ball is at B, $z = a/\sqrt{3}$ and $v^2 = ag/\sqrt{3}$. Hence

$$\frac{ag}{\sqrt{3}} + \frac{2ga}{\sqrt{3}} = 2ga(1 - 2\cos\alpha)$$

from which it follows that

$$\cos\alpha = \tfrac{1}{4}(2 - \sqrt{3}).$$

Hence $\alpha = \cos^{-1}\left(\tfrac{1}{4}(2 - \sqrt{3})\right) = 86°$ approximately. ■

Chapter Seven

Orbits in a central field

including Rutherford scattering

Problem 7.1

A particle P of mass m moves under the repulsive inverse cube field $\boldsymbol{F} = (m\gamma/r^3)\,\widehat{\boldsymbol{r}}$. Initially P is at a great distance from O and is moving with speed V towards O along a straight line whose perpendicular distance from O is p. Find the equation satisfied by the apsidal distances. What is the distance of closest approach of P to O?

Solution

The (specific) **potential energy** corresponding to the force field $\boldsymbol{F} = (m\gamma/r^3)\,\widehat{\boldsymbol{r}}$ is

$$V = \frac{\gamma}{2r^2},$$

and, from the initial conditions, the energy and angular momentum constants are $E = \frac{1}{2}V^2$ and $L = pV$. The energy and angular momentum **conservation equations** are therefore

$$\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\gamma}{2r^2} = \tfrac{1}{2}V^2,$$
$$r^2\dot{\theta} = pV.$$

On eliminating $\dot{\theta}$ between these two equations, we obtain the **radial motion** equation

$$\tfrac{1}{2}\dot{r}^2 + V^* = \tfrac{1}{2}V^2,$$

where the **effective potential** V^* is given by

$$V^* = \left(\gamma + p^2V^2\right)\frac{1}{2r^2}.$$

Since $\dot{r} = 0$ at an apse, it follows that the apsidal distances satisfy the equation $V^*(r) = \frac{1}{2}V^2$, that is,

$$r^2 = p^2 + \frac{\gamma}{V^2}.$$

Hence the only **apsidal distance** is $r = \left(p^2 + \gamma/V^2\right)^{1/2}$.

The graph of the effective potential V^* is shown in Figure 7.1. It is evident that, whatever the values of the constants E and L, this unique apsidal distance is the *minimum* value achieved by r. The **distance of closest approach** $r^{\min}$ is therefore given by

$$r^{\min} = \left(p^2 + \frac{\gamma}{V^2}\right)^{1/2}. \blacksquare$$

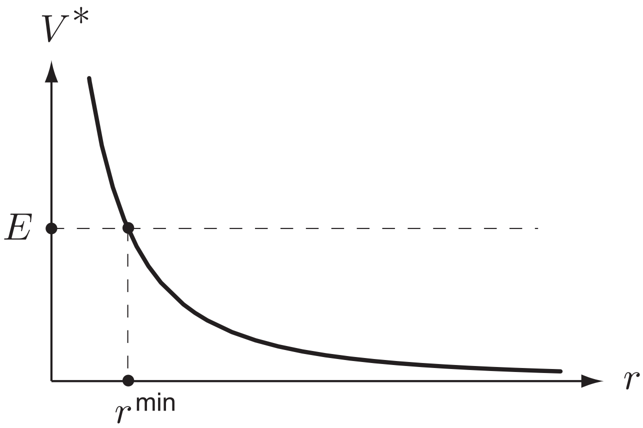

FIGURE 7.1 The effective potential V^* in Problem 7.1.

Problem 7.2

A particle P of mass m moves under the attractive inverse square field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$. Initially P is at a point C, a distance c from O, when it is projected with speed $(\gamma/c)^{1/2}$ in a direction making an acute angle α with the line OC. Find the apsidal distances in the resulting orbit.

Given that the orbit is an ellipse with O at a focus, find the semi-major and semi-minor axes of this ellipse.

Solution

The (specific) **potential energy** corresponding to the force field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$ is

$$V = -\frac{\gamma}{r},$$

and, from the initial conditions, the energy and angular momentum constants are given by

$$E = \frac{\gamma}{2c} - \frac{\gamma}{c} = -\frac{\gamma}{2c},$$

$$L = c\left(\frac{\gamma}{c}\right)^{1/2} \sin\alpha = (\gamma c)^{1/2} \sin\alpha.$$

The energy and angular momentum **conservation equations** are therefore

$$\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\gamma}{r} = -\frac{\gamma}{2c},$$

$$r^2\dot{\theta} = (\gamma c)^{1/2} \sin\alpha.$$

On eliminating $\dot{\theta}$ between these two equations, we obtain the **radial motion** equation

$$\tfrac{1}{2}\dot{r}^2 + V^* = -\frac{\gamma}{2c},$$

where the **effective potential** V^* is given by

$$V^* = -\frac{\gamma}{r} + \frac{\gamma c \sin^2\alpha}{2r^2}.$$

Since $\dot{r} = 0$ at an apse, it follows that the apsidal distances satisfy the equation $V^*(r) = -\gamma/2c$, that is,

$$r^2 - 2cr + c^2 \sin^2\alpha = 0.$$

The **apsidal distances** are therefore $r = c(1 \pm \cos\alpha)$. Since the initial value of r lies *between* these distances, it follows that r must oscillate in the range $c(1-\cos\alpha) \leq r \leq c(1+\cos\alpha)$. Hence the **least** and **greatest distances** from O achieved by the particle are

$$r^{\text{min}} = c(1-\cos\alpha),$$
$$r^{\text{max}} = c(1+\cos\alpha),$$

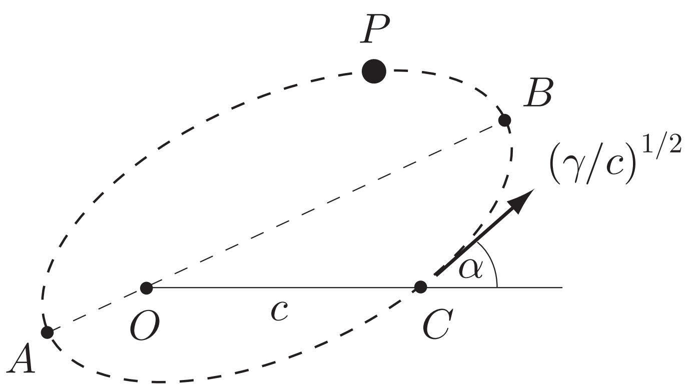

FIGURE 7.2 The orbit in Problem 7.2. The points A and B are the apses of the orbit.

Since we are given that the orbit is an ellipse with O at a focus, we know that

$$r^{\text{min}} = OA = a(1-e),$$
$$r^{\text{max}} = OB = a(1+e),$$

where a, e are the **semi-major axis**, and the **eccentricity**, of the orbit. Hence

$$c(1-\cos\alpha) = a(1-e),$$
$$c(1+\cos\alpha) = a(1+e),$$

from which it follows that $a = c$ and $e = \cos\alpha$. The **semi-minor** axis b is then given by

$$b^2 = a^2(1-e^2) = c^2(1-\cos^2\alpha) = c^2\sin^2\alpha.$$

Hence $b = c\sin\alpha$. ■

Problem 7.3

A particle of mass m moves under the attractive inverse square field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$. Show that the equation satisfied by the apsidal distances is

$$2Er^2 + 2\gamma r - L^2 = 0,$$

where E and L are the specific total energy and angular momentum of the particle. When $E < 0$, the orbit is known to be an ellipse with O as a focus. By considering the sum and product of the roots of the above equation, establish the elliptic orbit formulae

$$L^2 = \gamma b^2/a, \qquad E = -\gamma/2a.$$

Solution

The (specific) **potential energy** corresponding to the force field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$ is

$$V = -\frac{\gamma}{r}.$$

The energy and angular momentum **conservation equations** therefore have the form

$$\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\gamma}{r} = E,$$
$$r^2\dot{\theta} = L,$$

where E, L are the energy and angular momentum constants of the orbit. On eliminating $\dot{\theta}$ between these two equations, we obtain the **radial motion** equation in the form

$$\dot{r}^2 = 2E + \frac{2\gamma}{r} - \frac{L^2}{r^2}.$$

Since $\dot{r} = 0$ at an apse, it follows that the apsidal distances satisfy the equation

$$Q(r) = 0, \qquad (1)$$

where $Q(r) = 2(-E)r^2 - 2\gamma r + L^2$. When the energy $E < 0$, equation (1) generally has two distinct roots. (The special case in which the roots are coincident corresponds to a circular orbit.) Hence there are **two possible apsidal distances**. Since $\dot{r}^2$ cannot be negative, r is restricted to those values that satisfy the inequality

$$2E + \frac{2\gamma}{r} - \frac{L^2}{r^2} \ge 0,$$

which is equivalent to

$$Q(r) \leq 0.$$

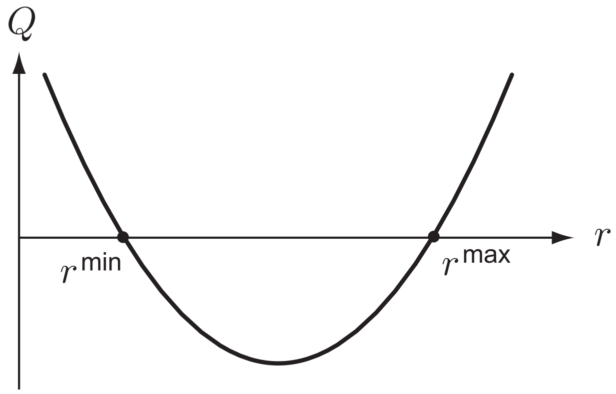

FIGURE 7.3 The function $Q(r)$ when the energy $E < 0$.

It is evident from Figure 7.3 that, when $E < 0$, the permitted range of r lies *between* the roots of the equation $Q(r) = 0$. It follows that r must oscillate in the range $r^{\text{min}} \leq r \leq r^{\text{max}}$, where r^{min} is the smaller of We note that the sum and product of these distances are given by

$$r^{\text{min}} + r^{\text{max}} = -\frac{\gamma}{E},$$
$$r^{\text{min}} \times r^{\text{max}} = -\frac{L^2}{2E}.$$

Since we are given that the orbit is an ellipse with O at a focus, we know that

$$r^{\text{min}} = a(1 - e),$$
$$r^{\text{max}} = a(1 + e),$$

where a, e are the **semi-major axis** and the **eccentricity** of the orbit. Hence the sum and product of r^{min} and r^{max} can also be expressed as

$$r^{\text{min}} + r^{\text{max}} = 2a,$$
$$r^{\text{min}} \times r^{\text{max}} = a^2 \left(1 - e^2\right) = b^2.$$

On equating these different expressions for the sum and product of r^{min} and r^{max}, we obtain

$$E = -\frac{\gamma}{2a},$$
$$L^2 = -2Eb^2 = \frac{\gamma b^2}{a},$$

which are the **E- and L-formulae** for the attractive inverse square elliptic orbit. ■

Problem 7.4

A particle P of mass m moves under the simple harmonic field $\boldsymbol{F} = -(m\Omega^2 r)\,\widehat{\boldsymbol{r}}$, where Ω is a positive constant. Obtain the radial motion equation and show that all orbits of P are bounded.

Initially P is at a point C, a distance c from O, when it is projected with speed Ωc in a direction making an acute angle α with OC. Find the equation satisfied by the apsidal distances. Given that the orbit of P is an ellipse with centre O, find the semi-major and semi-minor axes of this ellipse.

Solution

The (specific) **potential energy** corresponding to the force field $\boldsymbol{F} = -m\Omega^2 r\,\widehat{\boldsymbol{r}}$ is

$$V = \tfrac{1}{2}\Omega^2 r^2.$$

The energy and angular momentum **conservation equations** therefore have the form

$$\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \tfrac{1}{2}\Omega^2 r^2 = E,$$
$$r^2\dot{\theta} = L,$$

where E, L are the energy and angular momentum constants of the orbit. On eliminating $\dot{\theta}$ between these two equations, we obtain the **radial motion** equation in the form

$$\tfrac{1}{2}\dot{r}^2 + V^* = E,$$

where the **effective potential** V^* is given by

$$V^* = \tfrac{1}{2}\Omega^2 r^2 + \frac{L^2}{2r^2}.$$

Since $\dot{r}^2$ cannot be negative, r is restricted to those values that satisfy the inequality $V^* \le E$. It is evident from the graph of V^* shown in Figure 7.4, that, whatever the values of E and L, r must oscillate between two apsidal distances r^{min} and r^{max}. Thus **all orbits are bounded**.

With the special initial conditions given,

$$E = \tfrac{1}{2}\Omega^2 c^2 + \tfrac{1}{2}\Omega^2 c^2 = \Omega^2 c^2$$
$$L = \Omega c^2 \sin\alpha,$$

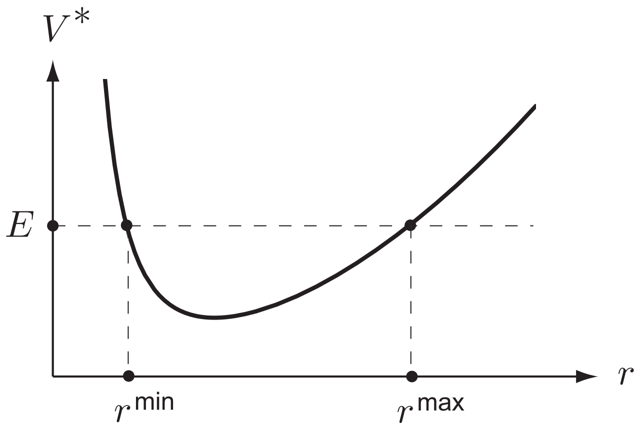

FIGURE 7.4 The effective potential V^* in Problem 7.4.

and V^* becomes

$$V^* = \tfrac{1}{2}\Omega^2 r^2 + \frac{\Omega^2 c^4 \sin^2 \alpha}{2r^2}.$$

Since $\dot{r} = 0$ at an apse, it follows that the apsidal distances satisfy the equation $V^*(r) = E$, that is,

$$r^4 - 2c^2 r^2 + c^4 \sin^2 \alpha = 0.$$

The **apsidal distances** of the orbit are the *positive* roots of this equation. Hence

$$r^{\text{max}} = \sqrt{2}c \cos \tfrac{1}{2}\alpha,$$
$$r^{\text{min}} = \sqrt{2}c \sin \tfrac{1}{2}\alpha.$$

Since we are given that the orbit is an ellipse with its *centre* at O, we know that r^{max} and r^{min} are the **major** and **minor axes** of this ellipse. Hence $a = \sqrt{2}c \cos \frac{1}{2}\alpha$ and $b = \sqrt{2}c \sin \frac{1}{2}\alpha$. ■

Problem 7.5

A particle P moves under the attractive inverse square field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$. Initially P is at the point C, a distance c from O, and is projected with speed $(3\gamma/c)^{1/2}$ perpendicular to OC. Find the polar equation of the path make a sketch of it. Deduce the angle between OC and the final direction of departure of P.

Solution

In the force field $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$, the outward force per unit mass is $f(r) = -\gamma/r^2$ and so $f(1/u) = -\gamma u^2$. Also, from the initial conditions, the angular momentum constant of the orbit is $L = c(3\gamma/c)^{1/2} = (3\gamma c)^{1/2}$. The **path equation** for the orbit is therefore

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{3c},$$

which is a second order linear ODE with constant coefficients. Its general solution is

$$u = \frac{1}{3c} + A\cos\theta + B\sin\theta,$$

where A and B are arbitrary constants.

The values of the constants A, B can be determined from the **initial conditions.** Take the line $\theta = 0$ of the polar coordinate system to pass through the initial position C of the particle (see Figure 7.5). Then

(i) the initial condition $r = c$ when $t = 0$ gives $u = 1/c$ when $\theta = 0$, and

(ii) the initial condition $\dot{r} = 0$ when $t = 0$ gives

$$\frac{du}{d\theta} = -\frac{\dot{r}}{L} = 0,$$

when $\theta = 0$.

The condition $u = 1/c$ when $\theta = 0$ gives $A = 2/3c$ and the condition $du/d\theta = 0$ when $\theta = 0$ gives $B = 0$. The **polar equation** of the orbit is therefore

$$u = \frac{1}{3c} + \frac{2}{3c}\cos\theta,$$

that is,

$$r = \frac{3c}{1 + 2\cos\theta}.$$

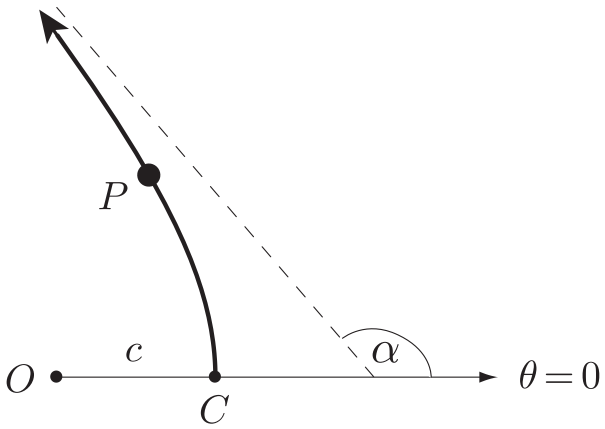

FIGURE 7.5 The orbit in Problem 7.5 (not to scale).

The graph of the orbit is shown in Figure 7.5. The particle **departs to infinity** when

$$1 + 2\cos\theta = 0,$$

that is, when $\theta = 120°$. This is the angle α shown in Figure 7.5. ■

Problem 7.6

A comet moves under the gravitational attraction of the Sun. Initially the comet is at a great distance from the Sun and is moving towards it with speed V along a straight line whose perpendicular distance from the Sun is p. By using the path equation, find the angle through which the comet is deflected and the distance of closest approach.

Solution

In this problem, the force field is $\boldsymbol{F} = -(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$, where $\gamma = M_\odot G$ and $M_\odot$ is the mass of the Sun. The outward force per unit mass is $f(r) = -\gamma/r^2$ and so $f(1/u) = -\gamma u^2$. Also, from the initial conditions, the angular momentum constant of the orbit is $L = pV$. The **path equation** for the orbit is therefore

$$\frac{d^2u}{d\theta^2} + u = \frac{\gamma}{p^2V^2},$$

which is a second order linear ODE with constant coefficients. Its general solution is

$$u = \frac{\gamma}{p^2V^2} + A\cos\theta + B\sin\theta,$$

where A and B are arbitrary constants.

The values of the constants A, B can be determined from the **initial conditions.** Take the line $\theta = 0$ of the polar coordinate system to be parallel to the direction of approach of the comet (see Figure 7.6). Then

(i) the condition that $r \to \infty$ as $t \to -\infty$ gives $u = 0$ when $\theta = 0$, and

(ii) the condition that $\dot{r} \to -V$ as $t \to -\infty$ gives

$$\frac{du}{d\theta} = -\frac{\dot{r}}{L} = -\frac{(-V)}{pV} = \frac{1}{p}$$

when $\theta = 0$.

The initial condition $u = 0$ when $\theta = 0$ gives $A = -\gamma/p^2V^2$ and the initial condition $du/d\theta = 1/p$ when $\theta = 0$ gives $B = 1/p$. The **polar equation** of the orbit is therefore

$$\frac{p}{r} = \frac{\gamma}{pV^2}(1 - \cos\theta) + \sin\theta.$$

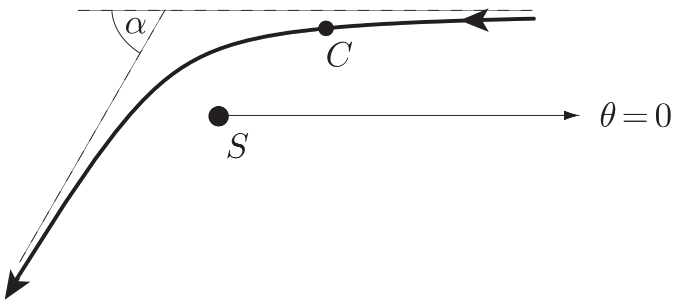

FIGURE 7.6 The orbit in Problem 7.6.

The graph of the orbit is shown in Figure 7.6. The comet **departs to infinity** when

$$\frac{\gamma}{pV^2}(1 - \cos\theta) + \sin\theta = 0.$$

This equation is best solved by expressing it in terms of the angle $\frac{1}{2}\theta$ in which case it becomes

$$\tan\tfrac{1}{2}\theta = -\frac{pV^2}{\gamma}.$$

The **deflection angle** α $(= \pi - \theta)$ shown in Figure 7.6 is therefore given by

$$\tan\tfrac{1}{2}\alpha = \tan\left(\tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right) = -\cot\tfrac{1}{2}\theta = \frac{\gamma}{pV^2}.$$

Hence

$$\alpha = 2\tan^{-1}\frac{\gamma}{pV^2}.$$

To find the **distance of closest approach** of the comet, consider the function

$$q(1 - \cos\theta) + \sin\theta.$$

where $q = \gamma/pV^2$. The *largest* value attained by this function in the range $0 \le \theta \le \alpha + \pi$ is

$$q + \left(q^2 + 1\right)^{1/2}$$

and hence

$$r^{\text{min}} = \frac{p}{q + \left(q^2+1\right)^{1/2}} = p\left(\left(q^2+1\right)^{1/2} - q\right) = \left(\frac{\gamma^2}{V^4} + p^2\right)^{1/2} - \frac{\gamma}{V^2}. \blacksquare$$

Problem 7.7

A particle P of mass m moves under the attractive inverse cube field $\boldsymbol{F} = -(m\gamma^2/r^3)\,\widehat{\boldsymbol{r}}$, where γ is a positive constant. Initially P is at a great distance from O and is projected towards O with speed V along a line whose perpendicular distance from O is p. Obtain the path equation for P.

For the case in which

$$V = \frac{15\gamma}{\sqrt{209}\,p},$$

find the polar equation of the path of P and make a sketch of it. Deduce the distance of closest approach to O, and the final direction of departure.

Solution

In the force field $\boldsymbol{F} = -(m\gamma^2/r^3)\,\widehat{\boldsymbol{r}}$, the outward force per unit mass is $f(r) = -\gamma^2/r^3$ and so $f(1/u) = -\gamma^2 u^3$. Also, from the initial conditions, the angular momentum constant of the orbit is $L = pV$. The **path equation** for the orbit is therefore

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\gamma^2}{p^2V^2}\right)u = 0.$$

For the special case in which

$$V = \frac{15\gamma}{\sqrt{209}\,p},$$

the path equation reduces to

$$\frac{d^2u}{d\theta^2} + \frac{16}{225}\,u = 0,$$

which is the SHM equation with $\Omega = \frac{4}{15}$. Its general solution is

$$u = A\cos\Omega\theta + B\sin\Omega\theta,$$

where A and B are arbitrary constants.

The values of the constants A, B can be determined from the **initial conditions.** Take the line $\theta = 0$ of the polar coordinate system to be parallel to the direction of approach of the particle (see Figure 7.6). Then

(i) the condition that $r \to \infty$ as $t \to -\infty$ gives $u = 0$ when $\theta = 0$, and

(ii) the condition that $\dot{r} \to -V$ as $t \to -\infty$ gives

$$\frac{du}{d\theta} = -\frac{\dot{r}}{L} = -\frac{(-V)}{pV} = \frac{1}{p}$$

when $\theta = 0$.

The initial condition $u = 0$ when $\theta = 0$ gives $A = 0$ and the initial condition $du/d\theta = 1/p$ when $\theta = 0$ gives $B = 1/\Omega p$. The **polar equation** of the orbit is therefore

$$r = \frac{4p}{15 \sin \frac{4}{15}\theta}.$$

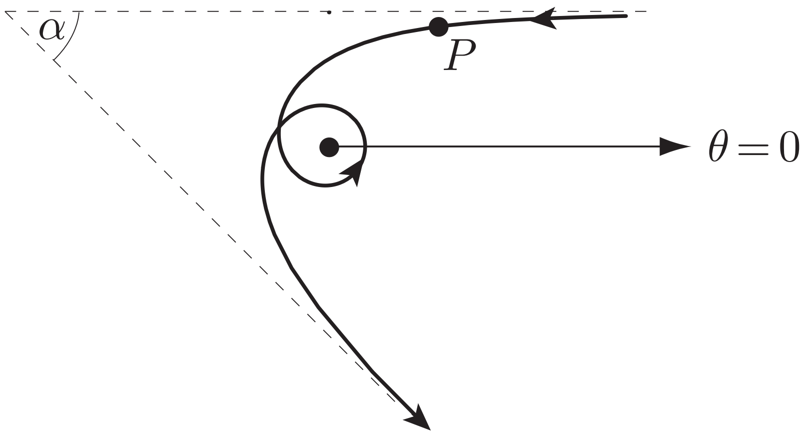

FIGURE 7.7 The orbit in Problem 7.7.

The graph of the orbit is shown in Figure 7.7. The particle **departs to infinity** when

$$\sin \tfrac{4}{15}\theta = 0,$$

that is, when $\theta = \frac{15}{4}\pi$. The angle α $(= 2\pi - \theta)$ shown in Figure 7.7 is therefore 45°.

The **distance of closest approach** is achieved when $\sin \frac{4}{15}\theta$ takes its *maximum* value for θ in the range $0 \le \theta \le \frac{15}{4}\pi$. This maximum is $+1$ and hence

$$r^{\min} = \tfrac{4}{15}p. \blacksquare$$

Problem 7.8 ✻

A particle P of mass m moves under the central field $\boldsymbol{F} = -(m\gamma^2/r^5)\,\widehat{\boldsymbol{r}}$, where γ is a positive constant. Initially P is at a great distance from O and is projected towards O with speed $\sqrt{2}\gamma/p^2$ along a line whose perpendicular distance from O is p. Show that the polar equation of the path of P is given by

$$r = \frac{p}{\sqrt{2}}\coth\left(\frac{\theta}{\sqrt{2}}\right).$$

Make a sketch of the path.

Solution

In the force field $\boldsymbol{F} = -(m\gamma^2/r^5)\,\widehat{\boldsymbol{r}}$, the outward force per unit mass is $f(r) = -\gamma^2/r^5$ and so $f(1/u) = -\gamma^2u^5$. Also, from the initial conditions, the angular momentum constant of the orbit is $L = p\left(\sqrt{2}\gamma/p^2\right) = \sqrt{2}\gamma/p$. The **path equation** for the orbit is therefore

$$\frac{d^2u}{d\theta^2} + u = \tfrac{1}{2}p^2u^3,$$

which is a *non-linear* second order ODE. Such equations cannot usually be solved, but, when the independent variable does not appear explicitly, the equation can always be reduced to first order. Let $v = du/d\theta$. Then

$$\frac{d^2u}{d\theta^2} = \frac{dv}{d\theta} = \frac{dv}{du} \times \frac{du}{d\theta} = v\frac{dv}{du},$$

and the path equation can be written

$$v\frac{dv}{du} + u = \tfrac{1}{2}p^2u^3.$$

This is a separable first order ODE for v as a function of u. On separating, we obtain

$$\tfrac{1}{2}v^2 = \tfrac{1}{8}p^2u^4 - \tfrac{1}{2}u^2 + C,$$

where C is the integration constant.

Take the line $\theta = 0$ of the polar coordinate system to be parallel to the direction of approach of the particle (see Figure 7.8). Then

(i) the condition that $r \to \infty$ as $t \to -\infty$ gives $u = 0$ when $\theta = 0$, and

(ii) the condition that $\dot{r} \to -V$ as $t \to -\infty$ gives

$$v = \frac{du}{d\theta} = -\frac{\dot{r}}{L} = -\frac{(-V)}{pV} = \frac{1}{p}$$

when $\theta = 0$.

The condition $v = 1/p$ when $u = 0$ gives $C = 1/2p^2$ and hence

$$v^2 = \frac{1}{4p^2}\left(2 - p^2u^2\right)^2.$$

The initial condition on $\dot{r}$ implies that r initially *decreases* so that u initially *increases*. Hence u satisfies the equation

$$\frac{du}{d\theta} = +\frac{1}{2p}\left(2 - p^2u^2\right).$$

We have thus reduced the path equation to a first order separable ODE for u as a function of θ. On separating, we obtain

$$\begin{aligned}\theta &= 2p\int \frac{du}{2 - p^2u^2}\\ &= \sqrt{2}\,\psi + D,\end{aligned}$$

on making the substitution $pu = \sqrt{2}\tanh\psi$. The initial condition $u = 0$ when $\theta = 0$ gives $D = 0$ and the solution is

$$\tanh\left(\frac{\theta}{\sqrt{2}}\right) = \frac{pu}{\sqrt{2}},$$

that is,

$$r = \frac{p}{\sqrt{2}}\coth\left(\frac{\theta}{\sqrt{2}}\right).$$

This is the **polar equation** of the orbit.

The graph of the orbit is shown in Figure 7.8. The path spirals inwards and is asymptotic to the circle $r = p/\sqrt{2}$. ■

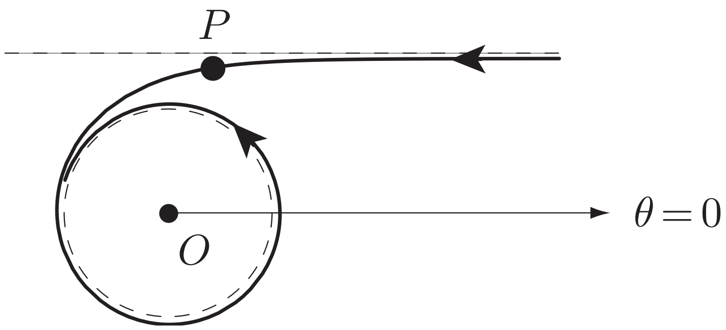

FIGURE 7.8 The orbit in Problem 7.8.

Problem 7.9 ✻

A particle of mass m moves under the central field

$$\boldsymbol{F} = -m\gamma^2\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)\widehat{\boldsymbol{r}},$$

where γ and a are positive constants. Initially the particle is at a distance a from the centre of force and is projected at right angles to the radius vector with speed $3\gamma/\sqrt{2}a$. Find the polar equation of the resulting path and make a sketch of it.

Find the time taken for the particle to reach the centre of force.

Solution

In the given force field, the outward force per unit mass is

$$f(r) = -\gamma^2\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

and so $f(1/u) = -\gamma^2\left(4u^3 + a^2u^5\right)$. Also, from the initial conditions, the angular momentum constant of the orbit is $L = a\big(3\gamma/\sqrt{2}\,a\big) = 3\gamma/\sqrt{2}$. The **path equation** for the orbit is therefore

$$\frac{d^2u}{d\theta^2} + u = \tfrac{2}{9}\left(4u + a^2u^3\right),$$

that is

$$\frac{d^2u}{d\theta^2} = \tfrac{2}{9}a^2u^3 - \tfrac{1}{9}u.$$

This is a *non-linear* second order ODE. Such equations cannot usually be solved, but, when the independent variable does not appear explicitly, the equation can always be reduced to first order. Let $v = du/d\theta$. Then

$$\frac{d^2u}{d\theta^2} = \frac{dv}{d\theta} = \frac{dv}{du}\times\frac{du}{d\theta} = v\frac{dv}{du},$$

and the path equation can be written

$$v\frac{dv}{du} = \tfrac{2}{9}a^2u^3 - \tfrac{1}{9}u.$$

This is a separable first order ODE for v as a function of u. On separating, we obtain

$$\tfrac{1}{2}v^2 = \tfrac{1}{18}a^2u^4 - \tfrac{1}{18}u^2 + C,$$

where C is the integration constant.

Take the line $\theta = 0$ of the polar coordinate system to pass through the point A where the motion begins(see Figure 7.9). Then

(i) the condition $r = a$ when $t = 0$ gives $u = 1/a$ when $\theta = 0$, and

(ii) the condition that $\dot{r} = 0$ when $t = 0$ gives

$$v = \frac{du}{d\theta} = -\frac{\dot{r}}{L} = 0$$

when $\theta = 0$.

The condition $v = 0$ when $u = 1/a$ gives $C = 0$ and hence

$$\frac{du}{d\theta} = \pm\tfrac{1}{3}u\left(a^2u^2 - 1\right)^{1/2}.$$

It is not immediately clear which sign to take since $\dot{r} = 0$ initially. However, since the initial value of $d^2u/d\theta^2$ is *positive*, u must *increase* initially. Hence u satisfies the equation

$$\frac{du}{d\theta} = +\tfrac{1}{3}u\left(a^2u^2 - 1\right)^{1/2}.$$

We have thus reduced the path equation to a first order separable ODE for u as a function of θ. On separating, we obtain

$$\begin{aligned}\theta &= 3\int \frac{du}{u\left(a^2u^2 - 1\right)^{1/2}}\\ &= 3\psi + D,\end{aligned}$$

on making the substitution $au = \sec\psi$. The initial condition $u = 1/a$ when $\theta = 0$ gives $D = 0$ and the solution is

$$\sec\tfrac{1}{3}\theta = au,$$

that is,

$$r = a\cos\tfrac{1}{3}\theta.$$

This is the **polar equation** of the orbit. The graph of the orbit is shown in Figure 7.8. The path spirals inwards and reaches the centre when $\theta = \tfrac{3}{2}\pi$.

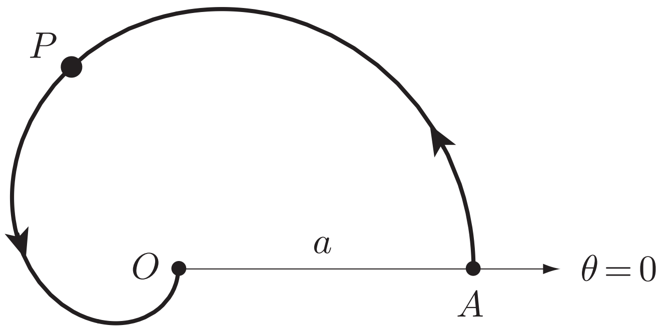

FIGURE 7.9 The orbit in Problem 7.9.

To find the **time taken** to reach the centre, consider the **angular momentum conservation** equation

$$r^2\dot{\theta} = \frac{3\gamma}{\sqrt{2}}.$$

Since we now know that the path of the particle is $r = a\cos\frac{1}{3}\theta$, it follows that θ satisfies the equation

$$\left(a^2\cos^2\tfrac{1}{3}\theta\right)\dot{\theta} = \frac{3\gamma}{\sqrt{2}}.$$

This is a separable first order ODE for θ as a function of t. On separating, we obtain

$$a^2\int_0^{3\pi/2}\cos^2\tfrac{1}{3}\theta\,d\theta = \frac{3\gamma}{\sqrt{2}}\int_0^{\tau}dt,$$

where τ is the required time. Hence

$$\tau = \frac{\sqrt{2}\,a^2}{3\gamma}\int_0^{3\pi/2}\cos^2\tfrac{1}{3}\theta\,d\theta$$
$$= \frac{\pi a^2}{2\sqrt{2}\,\gamma}. \blacksquare$$

Problem 7.10

A particle of mass m moves under the central field

$$\boldsymbol{F} = -m\left(\frac{\gamma e^{-\epsilon r/a}}{r^2}\right)\widehat{\boldsymbol{r}},$$

where γ, a and ϵ are positive constants. Find the apsidal angle for a nearly circular orbit of radius a. When ϵ is small, show that the perihelion of the orbit advances by approximately $\pi\epsilon$ on each revolution.

Solution

Let the nearly circular orbit of radius a be

$$u = \frac{1}{a} + \xi,$$

where $u = 1/r$ and $\xi = \xi(\theta)$. Then, in the linear approximation, ξ satisfies the equation

$$\frac{d^2\xi}{d\theta^2} + \Omega^2\xi = 0,$$

where

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)}$$

and $f(r)$ is the *inward* force per unit mass (see section 7.4 of the book).

In the present problem,

$$f(r) = \frac{\gamma e^{-\epsilon r/a}}{r^2},$$
$$f'(r) = -\frac{2\gamma e^{-\epsilon r/a}}{r^3} - \frac{\gamma\epsilon e^{-\epsilon r/a}}{ar^2},$$

so that

$$f(a) = \frac{\gamma e^{-\epsilon}}{a^2},$$
$$f'(a) = -\frac{(2+\epsilon)\gamma e^{-\epsilon}}{a^3},$$

and

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)} = 1 - \epsilon.$$

The general solution for ξ has the form

$$\xi = C\cos(\Omega\theta + \delta),$$

where C and δ are arbitrary constants. The **apsidal angle** of the orbit is therefore

$$\begin{aligned}\alpha &= \frac{\pi}{\Omega}\\ &= \pi(1-\epsilon)^{-1/2}\\ &= \pi\left(1 + \tfrac{1}{2}\epsilon\right)\end{aligned}$$

in the linear approximation when ϵ is small. Hence the **perihelion advances** by approximately $\pi\epsilon$ on each revolution. ■

Problem 7.11 *Solar oblateness*

A planet of mass m moves in the equatorial plane of a star that is a uniform oblate spheroid. The planet experiences a force field of the form

$$\boldsymbol{F} = -\frac{m\gamma}{r^2}\left(1 + \frac{\epsilon\, a^2}{r^2}\right)\widehat{\boldsymbol{r}},$$

approximately, where γ, a and ϵ are positive constants and ϵ is small. If the planet moves in a nearly circular orbit of radius a, find an approximation to the 'annual' advance of the perihelion. [It has been suggested that oblateness of the Sun might contribute significantly to the precession of the planets, thus undermining the success of general relativity. This point has yet to be resolved conclusively.]

Solution

Let the nearly circular orbit of radius a be

$$u = \frac{1}{a} + \xi,$$

where $u = 1/r$ and $\xi = \xi(\theta)$. Then, in the linear approximation, ξ satisfies the equation

$$\frac{d^2\xi}{d\theta^2} + \Omega^2\xi = 0,$$

where

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)}$$

and $f(r)$ is the *inward* force per unit mass (see section 7.4 of the book).

In the present problem,

$$f(r) = \frac{\gamma}{r^2}\left(1 + \frac{\epsilon a^2}{r^2}\right),$$

$$f'(r) = -\frac{\gamma}{r^3}\left(2 + \frac{4\epsilon a^2}{r^2}\right),$$

so that

$$f(a) = \frac{\gamma}{a^2}(1+\epsilon),$$

$$f'(a) = -\frac{\gamma}{a^3}(2+4\epsilon),$$

and

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)} = \frac{1-\epsilon}{1+\epsilon}$$

The general solution for ξ has the form

$$\xi = C\cos(\Omega\theta + \delta),$$

where C and δ are arbitrary constants. The **apsidal angle** of the orbit is therefore

$$\begin{aligned}\alpha &= \frac{\pi}{\Omega} \\ &= \pi\left(\frac{1-\epsilon}{1+\epsilon}\right)^{-1/2} \\ &= \pi(1+\epsilon)\end{aligned}$$

in the linear approximation when ϵ is small. Hence the 'annual' **advance of the perihelion** of the orbit is approximately $2\pi\epsilon$. ■

Problem 7.12

Suppose the solar system is embedded in a dust cloud of uniform density ρ. Find an approximation to the 'annual' advance of the perihelion of a planet moving in a nearly circular orbit of radius a. (For convenience, let $\rho = \epsilon M/a^3$, where M is the solar mass and ϵ is small.)

Solution

Suppose that the dust cloud is spherically symmetric about the Sun. Then the gravitational force that it exerts on a planet of mass m acts towards the Sun and has magnitude

$$\frac{mG}{r^2}\left(\tfrac{4}{3}\pi r^3 \rho\right) = mMG\left(\frac{4\pi\epsilon}{3a^3}\right) r,$$

where r is the distance of the planet from the Sun, $\epsilon = \rho a^3/M$, and M is the mass of the Sun. The total inward force per unit mass acting on the planet is therefore

$$f(r) = \frac{\gamma}{r^2} + \gamma\left(\frac{4\pi\epsilon}{3a^3}\right) r,$$

where $\gamma = MG$.

Suppose the planet has the nearly circular orbit

$$u = \frac{1}{a} + \xi,$$

where $u = 1/r$ and $\xi = \xi(\theta)$. Then, in the linear approximation, ξ satisfies the equation

$$\frac{d^2\xi}{d\theta^2} + \Omega^2\xi = 0,$$

where

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)}$$

and $f(r)$ is the *inward* force per unit mass (see section 7.4 of the book).

In the present problem,

$$f(r) = \frac{\gamma}{r^2} + \gamma\left(\frac{4\pi\epsilon}{3a^3}\right) r,$$

$$f'(r) = -\frac{2\gamma}{r^3} + \gamma\left(\frac{4\pi\epsilon}{3a^3}\right),$$

so that

$$f(a) = \frac{\gamma}{a^2}\left(1 + \tfrac{4}{3}\pi\epsilon\right),$$
$$f'(a) = -\frac{\gamma}{a^3}\left(2 - \tfrac{4}{3}\pi\epsilon\right),$$

and

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)} = \frac{1 + \frac{16}{3}\pi\epsilon}{1 + \frac{4}{3}\pi\epsilon}$$

The general solution for ξ has the form

$$\xi = C\cos(\Omega\theta + \delta),$$

where C and δ are arbitrary constants. The **apsidal angle** of the orbit is therefore

$$\begin{aligned}\alpha &= \frac{\pi}{\Omega}\\ &= \pi\left(\frac{1 + \frac{16}{3}\pi\epsilon}{1 + \frac{4}{3}\pi\epsilon}\right)^{-1/2}\\ &= \pi(1 - 2\pi\epsilon)\end{aligned}$$

in the linear approximation when ϵ is small. Hence the 'annual' **advance of the perihelion** of the planetary orbit is approximately $-4\pi^2\epsilon$. ■

Problem 7.13 *Orbits in general relativity*

In the theory of general relativity, the path equation for a planet moving in the gravitational field of the Sun is, in the standard notation,

$$\frac{d^2u}{d\theta^2} + u = \frac{MG}{L^2} + \left(\frac{3MG}{c^2}\right)u^2,$$

where c is the speed of light. Find an approximation to the 'annual' advance of the perihelion of a planet moving in a nearly circular orbit of radius a.

Solution

The general relativistic path equation for a planet is the same as that in Newtonian mechanics with a slightly modified law of force. The modified inward force $f(r)$ per unit mass is chosen so that

$$\frac{f(1/u)}{L^2u^2} = \frac{MG}{L^2} + \left(\frac{3MG}{c^2}\right)u^2,$$

that is,

$$f(r) = \frac{MG}{r^2}\left(\frac{3MGL^2}{c^2}\right)\frac{1}{r^4}.$$

We will take the value of the constant L to be that in a non-relativistic circular orbit of radius a, that is, $L^2 = MGa$. The **modified law of force** then has the form

$$f(r) = \frac{MG}{r^2}\left(1 + \frac{3\epsilon a^2}{r^2}\right),$$

where the dimensionless constant ϵ is defined by $\epsilon = MG/ac^2$. In the context of the solar system, the parameter ϵ is very small, being about 10^{-7} for the planet Mercury.

Suppose the planet has the nearly circular orbit

$$u = \frac{1}{a} + \xi,$$

where $u = 1/r$ and $\xi = \xi(\theta)$. Then, in the linear approximation, ξ satisfies the equation

$$\frac{d^2\xi}{d\theta^2} + \Omega^2\xi = 0,$$

where

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)}$$

and $f(r)$ is the *inward* force per unit mass (see section 7.4 of the book).

In the present problem,

$$f(r) = \frac{MG}{r^2}\left(1 + \frac{3\epsilon a^2}{r^2}\right),$$
$$f'(r) = -\frac{MG}{r^3}\left(2 + \frac{12\epsilon a^2}{r^2}\right),$$

so that

$$f(a) = \frac{\gamma}{a^2}(1 + 3\epsilon),$$
$$f'(a) = -\frac{\gamma}{a^3}(2 + 12\epsilon),$$

and

$$\Omega^2 = 3 + \frac{af'(a)}{f(a)} = \frac{1 - 3\epsilon}{1 + 3\epsilon}.$$

The general solution for ξ has the form

$$\xi = C\cos(\Omega\theta + \delta),$$

where C and δ are arbitrary constants. The **apsidal angle** of the orbit is therefore

$$\begin{aligned}\alpha &= \frac{\pi}{\Omega} \\ &= \pi\left(\frac{1 - 3\epsilon}{1 + 3\epsilon}\right)^{-1/2} \\ &= \pi(1 + 3\epsilon)\end{aligned}$$

in the linear approximation when ϵ is small. Hence the ‘annual’ **advance of the perihelion** of the planetary orbit is approximately $6\pi\epsilon$, where $\epsilon = MG/ac^2$. This is Einstein’s famous formula (specialised to the case of a nearly circular orbit). ■

Problem 7.14

A uniform flux of particles is incident upon a fixed hard sphere of radius a. The particles that strike the sphere are reflected elastically. Find the differential scattering cross section.

Solution

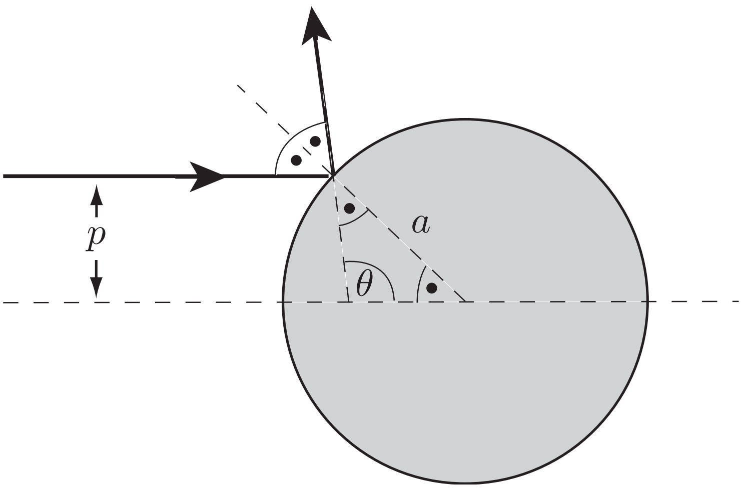

FIGURE 7.10 An incident particle with impact parameter p is elastically scattered through an angle θ. The angles marked with a bullet ($\bullet$) are all equal.

Consider an incident particle with impact parameter p as shown in Figure 7.10. Let the 'angle of incidence' of the particle be ψ; then, since the collision is elastic, all the angles marked with a bullet ($\bullet$) are equal to ψ. Then

$$p = a \sin \psi$$

and the scattering angle θ is

$$\theta = \pi - 2\psi.$$

On eliminating ψ between these two formulae, we obtain

$$p = a \cos \tfrac{1}{2}\theta,$$

which expresses the impact parameter p as a function of the scattering angle θ.

The **differential scattering cross section** σ is now given by

$$\begin{aligned}\sigma &= -\frac{p}{\sin\theta}\frac{dp}{d\theta}\\ &= -\frac{a\cos\frac{1}{2}\theta}{\sin\theta}\left(-\tfrac{1}{2}a\sin\tfrac{1}{2}\theta\right)\\ &= \tfrac{1}{4}a^2.\end{aligned}$$

Thus (somewhat surprisingly) the particles are *scattered equally in all directions*.

The **total scattering cross section** S is given by

$$\begin{aligned}S &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\sigma\sin\theta\,d\theta\,d\phi\\ &= 2\pi\left(\tfrac{1}{4}a^2\right)\int_{\theta=0}^{\theta=\pi}\sin\theta\,d\theta\\ &= \pi a^2.\end{aligned}$$

This is the answer expected since the particles that are scattered are those with impact parameters $p \le a$. ■

Problem 7.15

A uniform flux of particles, each of mass m and speed V, is incident upon a fixed scatterer that exerts the repulsive radial force $\boldsymbol{F} = (m\gamma^2/r^3)\,\widehat{\boldsymbol{r}}$. Find the impact parameter p as a function of the scattering angle θ, and deduce the differential scattering cross section. Find the total back-scattering cross-section.

Solution

In the force field $\boldsymbol{F} = (m\gamma^2/r^3)\,\widehat{\boldsymbol{r}}$, the outward force per unit mass is $f(r) = \gamma^2/r^3$ and so $f(1/u) = \gamma^2 u^3$. Consider a particle with impact parameter p. Then the angular momentum constant of its orbit is $L = pV$. The **path equation** for this particle is therefore

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{\gamma^2}{p^2V^2}\right)u = 0.$$

which is the SHM equation with

$$\Omega^2 = 1 + \frac{\gamma^2}{p^2V^2}.$$

The general solution is

$$u = A\cos\Omega\theta + B\sin\Omega\theta,$$

where A and B are arbitrary constants.

The values of the constants A, B can be determined from the **initial conditions.** Take the line $\theta = 0$ of the polar coordinate system to be parallel to the direction of approach of the particle. Then

(i) the condition that $r \to \infty$ as $t \to -\infty$ gives $u = 0$ when $\theta = 0$, and

(ii) the condition that $\dot{r} \to -V$ as $t \to -\infty$ gives

$$\frac{du}{d\theta} = -\frac{\dot{r}}{L} = -\frac{(-V)}{pV} = \frac{1}{p}$$

when $\theta = 0$.

The initial condition $u = 0$ when $\theta = 0$ gives $A = 0$ and the initial condition $du/d\theta = 1/p$ when $\theta = 0$ gives $B = 1/\Omega p$. The **polar equation** of the orbit is therefore

$$r = \frac{\Omega p}{\sin\Omega\theta}.$$

The particle **departs to infinity** when

$$\sin \Omega\theta = 0,$$

that is, when $\theta = \pi/\Omega$. The **scattering angle** Θ $(= \pi - (\pi/\Omega))$ is therefore

$$\Theta = \pi - \pi \left(1 + \frac{\gamma^2}{p^2V^2}\right)^{-1/2}.$$

On making p the subject of this formula, we obtain

$$p^2 = \frac{\gamma^2(\pi - \Theta)^2}{V^2\Theta(2\pi - \Theta)},$$

which is the required expression for the **impact parameter** p as a function of the scattering angle Θ.

The **differential scattering cross section** σ is now given by

$$\begin{aligned}\sigma &= -\frac{p}{\sin\Theta}\frac{dp}{d\Theta}\\ &= -\frac{1}{2\sin\Theta}\frac{dp^2}{d\Theta}\\ &= -\left(\frac{\gamma^2}{2V^2\sin\Theta}\right)\frac{d}{d\Theta}\left(\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)}\right)\\ &= \frac{\pi^2\gamma^2(\pi-\Theta)}{V^2\Theta^2(2\pi-\Theta)^2\sin\Theta}.\end{aligned}$$

The total **back scattering cross section** S^B is then given by

$$\begin{aligned}S^B &= \int_{\Theta=\pi/2}^{\Theta=\pi}\int_{\phi=0}^{\phi=2\pi} \sigma \sin\Theta \, d\Theta \, d\phi\\ &= \frac{2\pi^3\gamma^2}{V^2}\int_{\Theta=\pi/2}^{\Theta=\pi}\frac{\pi-\Theta}{\Theta^2(2\pi-\Theta)^2}\,d\theta\\ &= -\frac{\pi^3\gamma^2}{V^2}\left[\frac{1}{\Theta(2\pi-\Theta)}\right]_{\pi/2}^{\pi}\\ &= \frac{\pi\gamma^2}{3V^2}. \blacksquare\end{aligned}$$

Problem 7.16

In Yuri Gagarin's first manned space flight in 1961, the perigee and apogee were 181 km and 327 km above the Earth. Find the period of his orbit and his maximum speed in the orbit.

Solution

Suppose that the perigee and apogee of a satellite orbit are at height h and H above the Earth. The corresponding apsidal distances are therefore $R + h$ and $R + H$ respectively, where R is the Earth's radius. Then

$$(R + h) + (R + H) = 2a,$$

where a is the semi-major axis of the orbit. The parameter a is therefore given by

$$a = R + \tfrac{1}{2}(h + H).$$

The **period** τ of the orbit can now be found from the period formula

$$\tau^2 = \frac{4\pi^2 a^3}{MG},$$

where M is the mass of the Earth.

Let V be the speed of the satellite at the perigee. Then, from the energy conservation equation and the E-formula,

$$\tfrac{1}{2}V^2 - \frac{MG}{R + h} = -\frac{MG}{2a},$$

so that the **speed** at the **perigee** is given by

$$V^2 = MG\left(\frac{2}{R + h} - \frac{1}{a}\right).$$

On using the given data , we find that $\tau = 89.6$ min and $V = 7.84$ km s^{-1}. ■

Problem 7.17

An Earth satellite has a speed of 8.60 km per second at its perigee 200 km above the Earth's surface. Find the apogee distance above the Earth, its speed at the apogee, and the period of its orbit.

Solution

Suppose a satellite has speed V at its perigee, which is height h above the Earth. The corresponding apsidal distance is therefore $R+h$, where R is the Earth's radius. Then, from the energy conservation equation and the E-formula,

$$\tfrac{1}{2}V^2 - \frac{MG}{R+h} = -\frac{MG}{2a},$$

where M is the mass of the Earth and a is the semi-major axis of the orbit. Hence

$$a = \frac{MG(R+h)}{2MG - V^2(R+h)},$$

and the **period** τ of the orbit can now be found from the period formula

$$\tau^2 = \frac{4\pi^2 a^3}{MG}.$$

Let H be the height of the satellite above the Earth at the apogee. Then

$$(R+h) + (R+H) = 2a$$

so that H is given by

$$H = 2a - 2R - h.$$

The **speed** v at the **apogee** can now be found from the angular momentum conservation formula

$$(R+h)V = (R+H)v,$$

which gives

$$v = \left(\frac{R+h}{R+H}\right)V.$$

On using the given data , we find that $\tau = 128$ min, $H = 3910$ km, and $v = 5.50\ \mathrm{km\,s^{-1}}$. ■

Problem 7.18

A spacecraft is orbiting the Earth in a circular orbit of radius c when the motors are fired so as to multiply the speed of the spacecraft by a factor k ($k > 1$), its direction of motion being unaffected. [You may neglect the time taken for this operation.] Find the range of k for which the spacecraft will escape from the Earth, and the eccentricity of the escape orbit.

Solution

In a circular orbit of radius c the spacecraft has speed $(\gamma/c)^{1/2}$, where $\gamma = MG$, M being the mass of the Earth. Firing the motors causes the speed to suddenly increase to $k(\gamma/c)^{1/2}$. The energy E of the new orbit is therefore

$$\begin{aligned} E &= \tfrac{1}{2}k^2\left(\frac{\gamma}{c}\right) - \frac{\gamma}{c} \\ &= \frac{\gamma}{2c}\left(k^2 - 2\right). \end{aligned}$$

The spacecraft will **escape** if $E \geq 0$, that is, if $k \geq \sqrt{2}$.

Suppose then that $k \geq \sqrt{2}$ so that the new orbit is a hyperbola. The E-formula then gives

$$\frac{\gamma}{2c}\left(k^2 - 2\right) = +\frac{\gamma}{2a},$$

where a is the standard hyperbola parameter. Hence

$$a = \frac{c}{k^2 - 2}.$$

The angular momentum of the new orbit is

$$L = ck\left(\frac{\gamma}{c}\right)^{1/2} = k(\gamma c)^{1/2}.$$

The L-formula then gives

$$\frac{\gamma b^2}{a} = k^2\gamma c,$$

where b is the other standard hyperbola parameter. Hence

$$\frac{b^2}{a} = k^2 c.$$

The **eccentricity** e of the new orbit is given by

$$\begin{aligned} e^2 &= 1 + \frac{b^2}{a^2} \\ &= 1 + k^2\left(k^2 - 2\right) \\ &= \left(k^2 - 1\right)^2. \end{aligned}$$

Hence $e = k^2 - 1$. ■

Problem 7.19

A spacecraft travelling with speed V approaches a planet of mass M along a straight line whose perpendicular distance from the centre of the planet is p. When the spacecraft is at a distance c from the planet, it fires its engines so as to multiply its current speed by a factor k $(0 < k < 1)$, its direction of motion being unaffected. [You may neglect the time taken for this operation.] Find the condition that the spacecraft should go into orbit around the planet.

Solution

In the initial orbit, the total energy is $\frac{1}{2}V^2$. Suppose that, when the spacecraft is distance c from the planet, its speed is v. Then, by energy conservation,

$$\tfrac{1}{2}v^2 - \frac{\gamma}{c} = \tfrac{1}{2}V^2,$$

where $\gamma = MG$, M being the mass of the planet. Hence the speed of the spacecraft just before the motors are fired is

$$v = \left(V^2 + \frac{2\gamma}{c}\right)^{1/2}.$$

Firing the motors causes the speed to suddenly increase to

$$k\left(V^2 + \frac{2\gamma}{c}\right)^{1/2}.$$

The total energy E of the new orbit is therefore

$$E = \tfrac{1}{2}k^2\left(V^2 + \frac{2\gamma}{c}\right) - \frac{\gamma}{c}.$$

The spacecraft will **go into orbit** around the planet if $E < 0$, that is, if

$$k^2 < \frac{2}{2+\alpha},$$

where $\alpha = cV^2/MG$. ■

Problem 7.20

A body moving in an inverse square attractive field traverses an elliptical orbit with major axis $2a$. Show that the time average of the potential energy $V = -\gamma/r$ is $-\gamma/a$. [Transform the time integral to an integral with repect to the eccentric angle ψ.]

Deduce the time average of the kinetic energy in the same orbit.

Solution

The time average of the potential energy V over a period τ of the motion is

$$\begin{aligned}
\overline{V} &= \frac{1}{\tau}\int_0^\tau V\,dt \\
&= \frac{1}{\tau}\int_0^\tau \left(-\frac{\gamma}{r}\right)\,dt \\
&= -\frac{\gamma}{\tau}\int_0^{2\pi} \frac{1}{r}\left(\frac{dt}{d\theta}\right)\,d\theta \\
&= -\frac{\gamma}{\tau}\int_0^{2\pi} \left(\frac{r}{r^2\dot{\theta}}\right)\,d\theta \\
&= -\frac{\gamma}{\tau L}\int_0^{2\pi} r\,d\theta \\
&= \frac{\gamma b^2}{\tau L a}\int_0^{2\pi} \frac{d\theta}{1+e\cos\theta}.
\end{aligned}$$

This integral can be evaluated by standard methods (for example, by making the substitution $t = \tan\frac{1}{2}\theta$), but the easiest way is to transform the integration variable to the eccentric angle ψ by means of the formulae

$$(1-e\cos\psi)(1+e\cos\theta) = \frac{b^2}{a^2},$$

$$\frac{d\theta}{d\psi} = \frac{b}{a(1-e\cos\psi)},$$

(see the book p.175). This gives

$$\overline{V} = -\frac{\gamma b}{\tau L}\int_0^{2\pi} d\psi$$
$$= -\frac{2\pi\gamma b}{\tau L}$$
$$= -\frac{\gamma}{a},$$

on making use of the L-formula $L^2 = \gamma b^2/a$ and the period formula $\tau^2 = 4\pi^2 a^3/\gamma$. This is the required time average of the **potential energy** over a period of the motion. It is *exactly* the same as if the orbit were a circle of radius a.

Since $T + V = -\gamma/2a$, it follows that

$$\overline{T} + \overline{V} = -\frac{\gamma}{2a}$$

and hence that

$$\overline{T} = \frac{\gamma}{2a}.$$

This is the required time average of the **kinetic energy** over a period of the motion. It is also *exactly* the same as if the orbit were a circle of radius a. ■

Problem 7.21

A body moving in an inverse square attractive field traverses an elliptical orbit with eccentricity e and major axis $2a$. Show that the time average of the distance r of the body from the centre of force is $a(1+\frac{1}{2}e^2)$. [Transform the time integral to an integral with respect to the eccentric angle ψ.]

Solution

The time average of the radial distance r over a period τ of the motion is

$$\begin{aligned}\overline{r} &= \frac{1}{\tau}\int_0^\tau r\,dt\\ &= \frac{1}{\tau}\int_0^{2\pi} r\left(\frac{dt}{d\theta}\right)d\theta\\ &= \frac{1}{\tau}\int_0^{2\pi}\left(\frac{r^3}{r^2\dot{\theta}}\right)d\theta\\ &= \frac{1}{\tau L}\int_0^{2\pi} r^3\,d\theta\\ &= \frac{b^6}{\tau L a^3}\int_0^{2\pi}\frac{d\theta}{(1+e\cos\theta)^3}.\end{aligned}$$

This integral can be evaluated by standard methods (for example, by making the substitution $t = \tan\frac{1}{2}\theta$), but the easiest way is to transform the integration variable to the eccentric angle ψ by means of the formulae

$$(1-e\cos\psi)(1+e\cos\theta) = \frac{b^2}{a^2},$$

$$\frac{d\theta}{d\psi} = \frac{b}{a(1-e\cos\psi)},$$

(see the book p.175). This gives

$$\begin{aligned}\overline{r} &= \frac{a^2 b}{\tau L}\int_0^{2\pi}(1-e\cos\psi)^2\,d\psi\\ &= \frac{a^2 b}{\tau L}\left[2\pi+\pi e^2\right]\\ &= a\left(1+\tfrac{1}{2}e^2\right),\end{aligned}$$

on making use of the L-formula $L^2 = \gamma b^2/a$ and the period formula $\tau^2 = 4\pi^2 a^3/\gamma$. This is the required time average of the **radial distance** over a period of the motion. Note that it is always *greater* than a. ■

Problem 7.22

A spacecraft is 'parked' in a circular orbit 200 km above the Earth's surface. The spacecraft is to be sent to the Moon's orbit by Hohmann transfer. Find the velocity changes Δv^E and Δv^M that are required at the Earth and Moon respectively. How long does the journey take? [The radius of the Moon's orbit is 384,000 km. Neglect the gravitation of the Moon.]

Solution

Let A be the radius of the initial orbit of the spacecraft and B be the radius of the Moon's orbit. Then

$$A + B = 2a,$$

where a is the semi-major axis of the connecting Hohmann orbit. Hence

$$a = \tfrac{1}{2}(A + B).$$

The **journey time** T is then given by the period formula to be

$$T = \tfrac{1}{2}\tau = \left(\frac{\pi^2 a^3}{MG}\right)^{1/2} = \left(\frac{\pi^2 (B + A)^3}{8MG}\right)^{1/2},$$

where M is the mass of the Earth.

Let v^E be the speed of the spacecraft at the perigee of the connecting orbit. Then, by energy conservation and the E-formula,

$$\tfrac{1}{2}\left(v^E\right)^2 - \frac{MG}{A} = -\frac{MG}{2a},$$

from which it follows that

$$v^E = \left(\frac{2MGB}{A(B + A)}\right)^{1/2}.$$

By a similar argument, the speed v^M of the spacecraft at the apogee of the connecting orbit is

$$v^M = \left(\frac{2MGA}{B(B + A)}\right)^{1/2}.$$

The required **speed boosts** at the Earth and Moon are therefore

$$\Delta v^E = \left(\frac{2MGB}{A(B+A)}\right)^{1/2} - \left(\frac{MG}{A}\right)^{1/2},$$

$$\Delta v^M = \left(\frac{MG}{B}\right)^{1/2} - \left(\frac{2MGA}{B(B+A)}\right)^{1/2},$$

respectively.

On using the given data , we find that the **journey time** is 119 hours, and the **speed boosts** required are 3.13 km s^{-1} at the Earth and 0.83 km s^{-1} at the Moon. ■

Problem 7.23 ✻

A spacecraft is 'parked' in an *elliptic* orbit around the Earth. What is the most fuel efficient method of escaping from the Earth by using a single impulse?

Solution

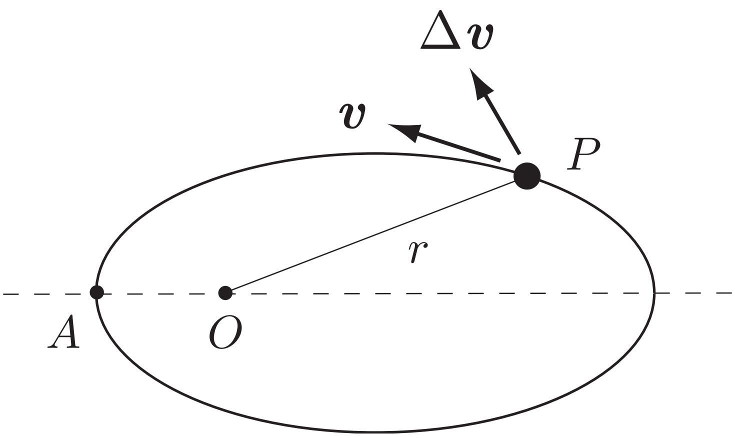

FIGURE 7.11 The initial orbit in Problem 7.23.

Suppose that the spacecraft is at a general point P of its orbit when it receives a velocity boost $\Delta\boldsymbol{v}$ as shown in Figure 7.11. Then the total energy E of the new orbit is

$$\begin{aligned} E &= \tfrac{1}{2}|\boldsymbol{v} + \Delta\boldsymbol{v}|^2 - \frac{\gamma}{r} \\ &= \tfrac{1}{2}|\boldsymbol{v}|^2 + \boldsymbol{v}\cdot\Delta\boldsymbol{v} + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 - \frac{\gamma}{r} \\ &= \boldsymbol{v}\cdot\Delta\boldsymbol{v} + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 + E_0, \end{aligned}$$

where E_0 $(= -\gamma/2a)$ is the total energy of the original orbit. The **spacecraft will escape** if $E \geq 0$, that is, if

$$\boldsymbol{v}\cdot\Delta\boldsymbol{v} + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 \geq -E_0.$$

We wish to choose the point P, and the *direction* of $\Delta\boldsymbol{v}$, so that this is achieved with the least value of $|\Delta\boldsymbol{v}|$.

The **optimum direction** of $\Delta\boldsymbol{v}$ must be parallel to $\boldsymbol{v}$. Suppose the least possible value of $|\Delta\boldsymbol{v}|$ is achieved with $|\Delta\boldsymbol{v}|$ *not* parallel to $\boldsymbol{v}$. Then it would be possible to increase the left side of the above inequality by changing only the *direction* of $\Delta\boldsymbol{v}$, after which $|\Delta\boldsymbol{v}|$ could be reduced without violating the inequality. With $\Delta\boldsymbol{v}$ made parallel to $\boldsymbol{v}$, the condition for escape becomes

$$|\boldsymbol{v}|\,|\Delta\boldsymbol{v}| + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 \geq -E_0.$$

The **optimum position** of the point P must be such that $|\boldsymbol{v}|$ has its maximum value there. If $|\boldsymbol{v}|$ were *not* a maximum at the optimum point, then it would be possible to increase the left side of the above inequality by changing only the position of P, after which $|\Delta\boldsymbol{v}|$ could be reduced without violating the inequality. Since the maximum value of $|\boldsymbol{v}|$ is achieved at the **perigee** of the orbit, this is where the boost should be applied. ■

Problem 7.24

A satellite already in the Earth's heliocentric orbit can fire its engines only once. What is the most fuel efficient method of sending the satellite on a 'flyby' visit to another planet? The satellite can visit either Mars or Venus. Which trip would use less fuel? Which trip would take the shorter time? [The orbits of Mars and Venus have radii 1.524 AU and 0.723 AU respectively.]

Solution

Although this single-impulse problem is not the same as the two-impulse problem discussed in section 7.6 of the book, The **Hohmann orbit** still provides the optimum fuel conserving strategy. This can be shown by modifying the optimality proof given in Appendix B to Chapter 7 so as to include only the velocity boost $\Delta \boldsymbol{v}^A$.

Let A be the radius of the Earth's orbit and B be the radius of the orbit of the other planet. Then

$$A + B = 2a,$$

where a is the semi-major axis of the connecting Hohmann orbit. Hence

$$a = \tfrac{1}{2}(A + B).$$

The (one way) **journey time** T is then given by the period formula to be

$$T = \tfrac{1}{2}\tau = \left(\frac{\pi^2 a^3}{M_\odot G}\right)^{1/2} = \left(\frac{\pi^2 (B + A)^3}{8 M_\odot G}\right)^{1/2},$$

where $M_\odot$ is the mass of the Sun.

Let v^A be the speed of the spacecraft at the perigee of the connecting orbit. Then, by energy conservation and the E-formula,

$$\tfrac{1}{2}\left(v^A\right)^2 - \frac{M_\odot G}{A} = -\frac{M_\odot G}{2a},$$

from which it follows that

$$v^A = \left(\frac{2 M_\odot G B}{A(B + A)}\right)^{1/2}.$$

The required **speed boost** is therefore

$$\Delta v^A = \left(\frac{2 M_\odot G B}{A(B + A)}\right)^{1/2} - \left(\frac{M_\odot G}{A}\right)^{1/2}.$$

On using the given data , we find that:

Mars: The **journey time** is 259 days, and the **speed boost** required is $2.95\ \mathrm{km\,s^{-1}}$.
Venus: The **journey time** is 146 days, and the **speed 'boost'** required is $-2.50\ \mathrm{km\,s^{-1}}$.

Thus the Venus flyby uses less fuel and takes a shorter time. ■

Problem 7.25

A satellite is 'parked' in a circular orbit 250 km above the Earth's surface. What is the most fuel efficient method of transferring the satellite to an (elliptical) synchronous orbit by using a single impulse? [A synchronous orbit has a period of 23 hr 56 m.] Find the value of $\Delta \boldsymbol{v}$ and apogee distance.

Solution

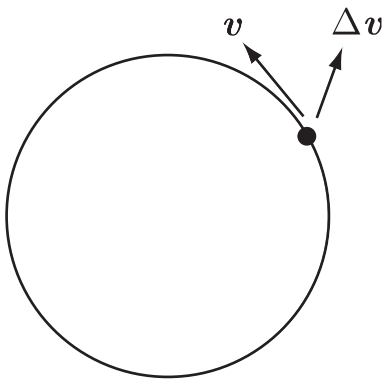

FIGURE 7.12 The initial orbit in Problem 7.25.

Suppose that the spacecraft is moving in a circular orbit of radius A. Its speed is then $(MG/A)^{1/2}$, where M is the mass of the Earth. At some point of the orbit, the spacecraft receives a velocity boost $\Delta \boldsymbol{v}$ shown in Figure 7.11. Then the total energy E of the new orbit is

$$
\begin{aligned}
E &= \tfrac{1}{2}|\boldsymbol{v} + \Delta\boldsymbol{v}|^2 - \frac{MG}{A} \\
&= \tfrac{1}{2}|\boldsymbol{v}|^2 + \boldsymbol{v} \cdot \Delta\boldsymbol{v} + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 - \frac{MG}{A} \\
&= \boldsymbol{v} \cdot \Delta\boldsymbol{v} + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 - \frac{MG}{2A}.
\end{aligned}
$$

The new orbit is required to have a specified period τ. From the period formula

$$\tau^2 = \frac{4\pi^2 a^3}{MG},$$

this is equivalent to specifying the parameter a and, by the E-formula

$$E = -\frac{MG}{2a},$$

this is in turn equivalent to specifying the total energy E of the new orbit. The new orbit will therefore have the correct period if $\Delta\boldsymbol{v}$ is such that

$$\boldsymbol{v} \cdot \Delta\boldsymbol{v} + \tfrac{1}{2}|\Delta\boldsymbol{v}|^2 = E + \frac{MG}{2A},$$

where E is the value of the total energy that corresponds to the required period τ. We wish to choose the *direction* of $\Delta\boldsymbol{v}$ so that this is achieved with the least possible value of $|\Delta\boldsymbol{v}|$.

Suppose that the required period is *longer* than the period of the origional circular orbit; then the right side of the above period condition *positive*. The **optimum direction** of $\Delta\boldsymbol{v}$ must be parallel to $\boldsymbol{v}$. Suppose the least possible value of $|\Delta\boldsymbol{v}|$ is achieved with $|\Delta\boldsymbol{v}|$ *not* parallel to $\boldsymbol{v}$. Then it would be possible to increase the left side of the above equality by changing only the *direction* of $\Delta\boldsymbol{v}$, after which the period condition could be satisfied again by *reducing* $|\Delta\boldsymbol{v}|$. Hence the optimum strategy is to **apply the impulse in the direction of motion**. It does not matter at what point of the orbit this impulse is applied.

For a synchronous orbit of 1436 mins, the semi-major axis of the new orbit is required to be 42,170 km. Since the perigee distance is 6630 km, the **apogee distance** must be 77,720 km. The speed that the spacecraft must have at the perigee of this orbit is $10.53\ \text{km s}^{-1}$ in contrast with the constant speed of $7.76\ \text{km s}^{-1}$ that the spacecraft has in its circular orbit. The **velocity boost** needed is therefore $+2.77\ \text{km s}^{-1}$.
■

Problem 7.26

A satellite of mass m moves under the attractive inverse square field $-(m\gamma/r^2)\,\widehat{\boldsymbol{r}}$ and is also subject to the linear resistance force $-mK\boldsymbol{v}$, where K is a positive constant. Show that the governing equations of motion can be reduced to the form

$$\ddot{r} + K\dot{r} + \frac{\gamma}{r^2} - \frac{L_0^2\, e^{-2Kt}}{r^3} = 0, \qquad r^2\dot{\theta} = L_0\, e^{-Kt},$$

where L_0 is a constant which will be assumed to be positive.

Suppose now that the effect of resistance is slight and that the satellite is executing a 'circular' orbit of slowly changing radius. By neglecting the terms in $\dot{r}$ and $\ddot{r}$, find an approximate solution for the time variation of r and θ in such an orbit. Deduce that small resistance causes the circular orbit to contract slowly, but that the satellite speeds up!

Solution

Newton's equations of motion for the satellite are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\gamma}{r^2} - K\dot{r},$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -Kr\dot{\theta}.$$

On mutiplying through by r, the second equation can be written

$$\frac{dL}{dt} = -KL,$$

where $L = r^2\dot{\theta}$. Note that L is the angular momentum at time t; in this problem, L is *not a constant*. On solving this ODE, the time dependence of L is found to be

$$L = L_0\, e^{-Kt},$$

where L_0 is a constant determined by the initial conditions. On making the substitution

$$\dot{\theta} = \frac{L_0 e^{-Kt}}{r^2}$$

into the first Newton equation, we obtain

$$\ddot{r} + K\dot{r} + \frac{\gamma}{r^2} - \frac{L_0^2\, e^{-2Kt}}{r^3} = 0.$$

This is the **radial motion** equation.

If $\dot{r}$ and $\ddot{r}$ are negligible compared to the other terms in this equation, then the time variation of r is given approximately by

$$r = \left(\frac{L_0^2}{\gamma}\right) e^{-2Kt},$$

so that the **orbit slowly contracts**. From the angular momentum equation, the corresponding time variation of $\dot{\theta}$ is

$$\dot{\theta} = \left(\frac{\gamma^2}{L_0^3}\right) e^{+3Kt}.$$

The time variation of the **circumferential velocity** $v\ (= r\dot{\theta})$ is therefore

$$v = \left(\frac{\gamma}{L_0}\right) e^{+Kt},$$

which is an *increasing* function of t. Thus, contrary to most expectations, the effect of small resistance is that the **satellite speeds up**. This is not contrary to energy conservation however since potential energy is lost as the orbit contracts. ∎

Problem 7.27

Repeat the last problem for the case in which the particle moves under the simple harmonic attractive field $-(m\Omega^2 r)\,\widehat{\boldsymbol{r}}$ with the same law of resistance. Show that, in this case, the body slows down as the orbit contracts. [This problem can be solved exactly in Cartesian coordinates, but do not do it this way.]

Solution

Newton's equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = -\Omega^2 r - K\dot{r},$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -Kr\dot{\theta}.$$

On mutiplying through by r, the second equation can be written

$$\frac{dL}{dt} = -KL,$$

where $L = r^2\dot{\theta}$. Note that L is the angular momentum at time t; in this problem, L is *not a constant*. On solving this ODE, the time dependence of L is found to be

$$L = L_0\, e^{-Kt},$$

where L_0 is a constant determined by the initial conditions. On making the substitution

$$\dot{\theta} = \frac{L_0 e^{-Kt}}{r^2}$$

into the first Newton equation, we obtain

$$\ddot{r} + K\dot{r} + \Omega^2 r - \frac{L_0^2\, e^{-2Kt}}{r^3} = 0.$$

This is the **radial motion** equation.

If $\dot{r}$ and $\ddot{r}$ are negligible compared to the other terms in this equation, then the time variation of r is given approximately by

$$r = \left(\frac{L_0}{\Omega}\right)^{1/2} e^{-\frac{1}{2}Kt},$$

so that the **orbit slowly contracts**. From the angular momentum equation, the corresponding time variation of $\dot{\theta}$ is

$$\dot{\theta} = \Omega$$

so that $\dot{\theta}$ is constant in this approximation. The time variation of the **circumferential velocity** v $(= r\dot{\theta})$ is therefore

$$v = (\Omega L_0)^{1/2}\, e^{-\frac{1}{2}Kt},$$

which is a *decreasing* function of t. Hence the **body slows down** as the orbit contracts. ■

Chapter Eight

Non-linear oscillations and phase space

Problem 8.1

A non-linear oscillator satisfies the equation

$$\left(1+\epsilon x^2\right)\ddot{x}+x=0,$$

where ϵ is a small parameter. Use Linstedt's method to obtain a two-term approximation to the oscillation frequency when the oscillation has unit amplitude. Find also the corresponding two-term approximation to $x(t)$. [You will need the identity $4\cos^3 s = 3\cos s + \cos 3s$.]

Solution

The problem is solved using **Lindstedt's method**. Define the new independent variable s (the dimensionless time) by $s=\omega(\epsilon)t$, where $\omega(\epsilon)$ is the angular frequency of the required solution. Then $x(s,\epsilon)$ satisfies the equation

$$\big(\omega(\epsilon)\big)^2\left(1+\epsilon x^2\right)x''+x=0,$$

with the initial conditions $x=1$ and $x'=0$ when $s=0$. (Here $'$ means d/ds.) These initial conditions correspond to an oscillation of unit amplitude. We now expand x and ω in the **perturbation series**

$$x(s,\epsilon)=x_0(s)+\epsilon x_1(s)+\epsilon^2 x_2(s)+\cdots,$$
$$\omega(\epsilon)=1+\epsilon\omega_1+\epsilon^2\omega_2+\cdots,$$

which is possible when ϵ is small. By construction, this solution must have period 2π *for all* ϵ from which it follows that each of the functions $x_0(s)$, $x_1(s)$, $x_2(s)$, ... must also have period 2π. On substituting these expansions into the governing equation and its initial conditions, we obtain:

$$\big(1+\omega_1\epsilon+\cdots\big)^2\left(1+\epsilon\big(x_0+\epsilon x_1+\cdots\big)^2\right)\big(x_0''+\epsilon x_1''+\cdots\big)$$
$$+\,(x_0+\epsilon x_1+\cdots)=0,$$

with

$$x_0+\epsilon x_1+\epsilon^2 x_2+\cdots=1,$$
$$x_0'+\epsilon x_1'+\epsilon^2 x_2'+\cdots=0,$$

when $s=0$. If we now equate coefficients of powers of ϵ in these equalities, we obtain a succession of ODEs and initial conditions, the first two of which are as follows:

- From coefficients of ϵ^0, we obtain the **zero order** equation

$$x_0'' + x_0 = 0,$$

with $x_0 = 1$ and $x_0' = 0$ when $s = 0$.

- From coefficients of ϵ^1, we obtain the **first order** equation

$$x_1'' + x_1 = -2\omega_1 x_0'' - x_0^2 x_0'',$$

with $x_1 = 0$ and $x_1' = 0$ when $s = 0$.

The solution of the **zero order** equation and initial conditions is

$$x_0 = \cos s$$

and this can now be substituted into the first order equation to give

$$\begin{aligned} x_1'' + x_1 &= 2\omega_1 \cos s + \cos^3 s \\ &= \tfrac{1}{4}(8\omega_1 + 3)\cos s + \tfrac{1}{4}\cos 3s, \end{aligned} \qquad (1)$$

on using the trigonometric identity $4\cos^3 s = 3\cos s + \cos 3s$. The coefficient of $\cos s$ on the right side of this equation must be zero, for otherwise $x_1(s)$ would not be periodic. Hence

$$\omega_1 = -\frac{3}{8}.$$

The general solution of the **first order** equation is then

$$x_1 = -\frac{1}{32}\cos 3s + A_1 \cos s + B_1 \sin s,$$

where A_1, B_1 are arbitrary constants. The initial conditions $x_1 = x_1' = 0$ when $s = 0$ give $A_1 = \frac{1}{32}$ and $B_1 = 0$ so that

$$x_1 = \tfrac{1}{32}(\cos s - \cos 3s).$$

Hence, when ϵ is small, **approximate frequency** of the oscillation of unit amplitude is given by

$$\omega = 1 - \tfrac{3}{8}\epsilon + O\left(\epsilon^2\right),$$

and the **approximate displacement** at time t is given by

$$x = \cos s + \tfrac{1}{32}\big(\cos s - \cos 3s\big)\epsilon + O\left(\epsilon^2\right),$$

where $s = \left(1 - \frac{3}{8}\epsilon + O\left(\epsilon^2\right)\right)t$. ■

Problem 8.2

A non-linear oscillator satisfies the equation

$$\ddot{x} + x + \epsilon x^5 = 0,$$

where ϵ is a small parameter. Use Linstedt's method to obtain a two-term approximation to the oscillation frequency when the oscillation has unit amplitude. [You will need the identity $16\cos^5 s = 10\cos s + 5\cos 3s + \cos 5s$.]

Solution

The problem is solved using **Lindstedt's method**. Define the new independent variable s (the dimensionless time) by $s = \omega(\epsilon)t$, where $\omega(\epsilon)$ is the angular frequency of the required solution. Then $x(s, \epsilon)$ satisfies the equation

$$\big(\omega(\epsilon)\big)^2 x'' + x + \epsilon x^5 = 0,$$

with the initial conditions $x = 1$ and $x' = 0$ when $s = 0$. (Here $'$ means d/ds.) These initial conditions correspond to an oscillation of unit amplitude. We now expand x and ω in the **perturbation series**

$$\begin{aligned} x(s,\epsilon) &= x_0(s) + \epsilon x_1(s) + \epsilon^2 x_2(s) + \cdots, \\ \omega(\epsilon) &= 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots, \end{aligned}$$

which is possible when ϵ is small. By construction, this solution must have period 2π *for all* ϵ from which it follows that each of the functions $x_0(s)$, $x_1(s)$, $x_2(s)$, ... must also have period 2π. On substituting these expansions into the governing equation and its initial conditions, we obtain:

$$\big(1 + \omega_1\epsilon + \cdots\big)^2\big(x_0'' + \epsilon x_1'' + \cdots\big) + (x_0 + \epsilon x_1 + \cdots)$$

$$+\, \epsilon(x_0 + \epsilon x_1 + \cdots)^5 = 0,$$

with

$$\begin{aligned} x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots &= 1, \\ x_0' + \epsilon x_1' + \epsilon^2 x_2' + \cdots &= 0, \end{aligned}$$

when $s = 0$. If we now equate coefficients of powers of ϵ in these equalities, we obtain a succession of ODEs and initial conditions, the first two of which are as follows:

- From coefficients of ϵ^0, we obtain the **zero order** equation

$$x_0'' + x_0 = 0,$$

with $x_0 = 1$ and $x_0' = 0$ when $s = 0$.

- From coefficients of ϵ^1, we obtain the **first order** equation

$$x_1'' + x_1 = -2\omega_1 x_0'' - x_0^5,$$

with $x_1 = 0$ and $x_1' = 0$ when $s = 0$.

The solution of the **zero order** equation and initial conditions is

$$x_0 = \cos s$$

and this can now be substituted into the first order equation to give

$$\begin{aligned} x_1'' + x_1 &= 2\omega_1 \cos s - \cos^5 s \\ &= \tfrac{1}{8}(16\omega_1 - 5)\cos s - \tfrac{1}{16}\big(\cos 5s + 5\cos 3s\big), \end{aligned} \tag{1}$$

on using the trigonometric identity $16\cos^5 s = 10\cos s + 5\cos 3s + \cos 5s$. The coefficient of $\cos s$ on the right side of this equation must be zero, for otherwise $x_1(s)$ would not be periodic. Hence

$$\omega_1 = \frac{5}{16}.$$

The general solution of the **first order** equation is then

$$x_1 = \tfrac{1}{384}\big(\cos 5s + 15\cos 3s\big) + A_1 \cos s + B_1 \sin s,$$

where A_1, B_1 are arbitrary constants. The initial conditions $x_1 = x_1' = 0$ when $s = 0$ give $A_1 = -\frac{1}{24}$ and $B_1 = 0$ so that

$$x_1 = \tfrac{1}{384}\big(\cos 5s + 15\cos 3s - 16\cos s\big).$$

Hence, when ϵ is small, **approximate frequency** of the oscillation of unit amplitude is given by

$$\omega = 1 + \tfrac{5}{16}\epsilon + O\left(\epsilon^2\right),$$

and the **approximate displacement** at time t is given by

$$x = \cos s + \tfrac{1}{384}\big(\cos 5s + 15\cos 3s - 16\cos s\big)\epsilon + O\left(\epsilon^2\right),$$

where $s = \left(1 + \frac{5}{16}\epsilon + O\left(\epsilon^2\right)\right)t$. ■

Problem 8.3 *Unsymmetrical oscillations*

A non-linear oscillator satisfies the equation

$$\ddot{x} + x + \epsilon x^2 = 0,$$

where ϵ is a small parameter. Explain why the oscillations are unsymmetrical about $x = 0$ in this problem.

Use Linstedt's method to obtain a two-term approximation to $x(t)$ for the oscillation in which the *maximum* value of x is unity. Deduce a two-term approximation to the *minimum* value achieved by $x(t)$ in this oscillation.

Solution

In this problem, the effective spring stiffness depends on the sign of x. For $\epsilon > 0$, we have a *hardening* spring when x is positive, and a *softening* spring when x is negative. Hence the **oscillations are unsymmetrical** about $x = 0$.

The problem is solved using **Lindstedt's method**. Define the new independent variable s (the dimensionless time) by $s = \omega(\epsilon)t$, where $\omega(\epsilon)$ is the angular frequency of the required solution. Then $x(s, \epsilon)$ satisfies the equation

$$\big(\omega(\epsilon)\big)^2 x'' + x + \epsilon x^2 = 0,$$

with the initial conditions $x = 1$ and $x' = 0$ when $s = 0$. (Here $'$ means d/ds.) These initial conditions correspond to an oscillation in which the *right* amplitude is unity. We now expand x and ω in the **perturbation series**

$$x(s, \epsilon) = x_0(s) + \epsilon x_1(s) + \epsilon^2 x_2(s) + \cdots ,$$
$$\omega(\epsilon) = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots ,$$

which is possible when ϵ is small. By construction, this solution must have period 2π *for all* ϵ from which it follows that each of the functions $x_0(s)$, $x_1(s)$, $x_2(s)$, ... must also have period 2π. On substituting these expansions into the governing equation and its initial conditions, we obtain:

$$\big(1 + \omega_1\epsilon + \cdots\big)^2\big(x_0'' + \epsilon x_1'' + \cdots\big) + (x_0 + \epsilon x_1 + \cdots)$$
$$+ \epsilon(x_0 + \epsilon x_1 + \cdots)^2 = 0,$$

with

$$x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots = 1,$$
$$x_0' + \epsilon x_1' + \epsilon^2 x_2' + \cdots = 0,$$

when $s = 0$. If we now equate coefficients of powers of ϵ in these equalities, we obtain a succession of ODEs and initial conditions, the first two of which are as follows:

- From coefficients of ϵ^0, we obtain the **zero order** equation

$$x_0'' + x_0 = 0,$$

  with $x_0 = 1$ and $x_0' = 0$ when $s = 0$.
- From coefficients of ϵ^1, we obtain the **first order** equation

$$x_1'' + x_1 = -2\omega_1 x_0'' - x_0^2,$$

  with $x_1 = 0$ and $x_1' = 0$ when $s = 0$.

The solution of the **zero order** equation and initial conditions is

$$x_0 = \cos s$$

and this can now be substituted into the first order equation to give

$$\begin{aligned} x_1'' + x_1 &= 2\omega_1 \cos s - \cos^2 s \\ &= 2\omega_1 \cos s - \tfrac{1}{2}\big(\cos 2s + 1\big). \end{aligned}$$

The coefficient of $\cos s$ on the right side of this equation must be zero, for otherwise $x_1(s)$ would not be periodic. Hence

$$\omega_1 = 0,$$

which means that there is no correction to the oscillation frequency at first order. The general solution of the **first order** equation is then

$$x_1 = -\tfrac{1}{2} + \tfrac{1}{6}\cos 2s + A_1 \cos s + B_1 \sin s,$$

where A_1, B_1 are arbitrary constants. The initial conditions $x_1 = x_1' = 0$ when $s = 0$ give $A_1 = \frac{1}{3}$ and $B_1 = 0$ so that

$$x_1 = \tfrac{1}{6}\big(\cos 2s + 2\cos s - 3\big).$$

Hence, when ϵ is small, the **approximate frequency** of the oscillation of unit right amplitude is given by

$$\omega = 1 + O\left(\epsilon^2\right),$$

and the **approximate displacement** at time t is given by

$$x = \cos s + \tfrac{1}{6}\big(\cos 2s + 2\cos s - 3\big)\epsilon + O\left(\epsilon^2\right),$$

where $s = \left(1 + O\left(\epsilon^2\right)\right) t$.

To find the *left* amplitude of the oscillation, consider $\dot{x}$, which, correct to order ϵ, is given by

$$\begin{aligned}\dot{x} &= -\sin t - \tfrac{1}{3}\big(\sin 2t + \sin t\big)\epsilon \\ &= -\sin t\left[1 + \tfrac{1}{3}\big(2\cos t + 1\big)\epsilon\right].\end{aligned}$$

Since ϵ is small, the factor in the square brackets is close to unity and is therefore never zero. Hence the stationary points of $x(t)$ occur when $\sin t = 0$, that is, when $t = 0, \pm\pi, \pm 2\pi, \ldots$. The values $t = 0, \pm 2\pi, \pm 4\pi \ldots$ correspond to x achieving its right amplitude while the values $t = \pm\pi, \pm 3\pi \ldots$ correspond to x achieving its left amplitude. In the latter case,

$$x = -1 - \tfrac{2}{3}\epsilon$$

and hence the **approximate left amplitude** of the oscillation is $1 + \frac{2}{3}\epsilon$. As expected, this is bigger than the right amplitude when ϵ is positive. ■

Problem 8.4 ✻ *A limit cycle by perturbation theory*

Use perturbation theory to investigate the limit cycle of **Rayleigh's equation**, taken here in the form

$$\ddot{x} + \epsilon\left(\tfrac{1}{3}\dot{x}^2 - 1\right)\dot{x} + x = 0,$$

where ϵ is a small positive parameter. Show that the zero order approximation to the limit cycle is a circle and determine its centre and radius. Find the frequency of the limit cycle correct to order ϵ^2, and find the function $x(t)$ correct to order ϵ.

Solution

The difference between this problem and problems 8.1–8.3 is that the *amplitude of the limit cycle cannot be prescribed*. It must be determined along with the rest of the solution. This is because the limit cycle is an *isolated* periodic solution rather than a member of a family of such solutions. With this modification, the problem is solved using **Lindstedt's method**.

Define the new independent variable s (the dimensionless time) by $s = \omega(\epsilon)t$, where $\omega(\epsilon)$ is the angular frequency of the limit cycle. Then $x(s, \epsilon)$ satisfies the equation

$$\big(\omega(\epsilon)\big)^2 x'' + \epsilon\left(\tfrac{1}{3}x'^2 - 1\right)x' + x = 0,$$

with the initial conditions $x = a$ and $x' = 0$ when $s = 0$, where a $(= a(\epsilon))$ is the unknown amplitude of the limit cycle. (Here $'$ means d/ds.) We now expand x, ω and a in the **perturbation series**

$$\begin{aligned} x(s,\epsilon) &= x_0(s) + \epsilon x_1(s) + \epsilon^2 x_2(s) + \cdots\,, \\ \omega(\epsilon) &= 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\,, \\ a(\epsilon) &= a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots\,, \end{aligned}$$

which we assume to be possible when ϵ is small. By construction, this solution must have period 2π *for all* ϵ from which it follows that each of the functions $x_0(s)$, $x_1(s)$, $x_2(s)$, …must also have period 2π. On substituting these expansions into the governing equation and its initial conditions, we obtain:

$$\big(1 + \omega_1\epsilon + \cdots\big)^2\big(x_0'' + \epsilon x_1'' + \cdots\big) +$$

$$\epsilon\left(\tfrac{1}{3}\big(x_0' + \epsilon x_1' + \cdots\big)^2 - 1\right)\big(x_0' + \epsilon x_1' + \cdots\big) + \big(x_0 + \epsilon x_1 + \cdots\big) = 0,$$

with

$$x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots ,$$
$$x_0' + \epsilon x_1' + \epsilon^2 x_2' + \cdots = 0,$$

when $s = 0$. If we now equate coefficients of powers of ϵ in these equalities, we obtain a succession of ODEs and initial conditions, the first three of which are as follows:

- From coefficients of ϵ^0, we obtain the **zero order** equation

$$x_0'' + x_0 = 0,$$

with $x_0 = a_0$ and $x_0' = 0$ when $s = 0$.

- From coefficients of ϵ^1, we obtain the **first order** equation

$$x_1'' + x_1 = -2\omega_1 x_0'' - \left(\tfrac{1}{3}{x_0'}^2 - 1\right) x_0'$$

with $x_1 = a_1$ and $x_1' = 0$ when $s = 0$.

- From coefficients of ϵ^2, we obtain the **second order** equation

$$x_2'' + x_2 = -2\omega_1 x_1'' - (\omega_1^2 + 2\omega_2) x_0'' - {x_0'}^2 x_1' + x_1'$$

with $x_2 = a_2$ and $x_2' = 0$ when $s = 0$.

The solution of the **zero order** equation and initial conditions is

$$x_0 = a_0 \cos s,$$

where the positive constant a_0 cannot be determined at the zero order stage. On substituting this expression for x_0 into the first order equation, we obtain

$$\begin{aligned} x_1'' + x_1 &= 2a_0\omega_1 \cos s + \left(\tfrac{1}{3}a_0^2 \sin^2 s - 1\right) a_0 \sin s \\ &= 2a_0\omega_1 \cos s + \tfrac{1}{12}a_0 \left[\left(3a_0^2 - 12\right) \sin s - a_0^2 \sin 3s\right], \end{aligned}$$

on using the trigonometric identity $4\sin^3 s = 3\sin s - \sin 3s$. The coefficients of $\cos s$ and $\sin s$ on the right side of this equation must both be zero, for otherwise $x_1(s)$ would not be periodic. Since a_0 is positive, this implies that $\omega_1 = 0$ and $a_0 = 2$. Hence the **zero order approximation** to the limit cycle is

$$x = 2\cos s,$$

where $s = \big(1 + O(\epsilon)\big)\, t$.

The **first order** equation now reduces to

$$x_1'' + x_1 = -\tfrac{2}{3}\sin 3s,$$

the general solution of which is

$$x_1 = \frac{1}{12}\sin 3s + A_1 \cos s + B_1 \sin s,$$

where A_1, B_1 are arbitrary constants. The initial conditions $x_1 = a_1$ and $x_1' = 0$ when $s = 0$ give $A_1 = a_1$ and $B_1 = -\frac{1}{4}$ so that

$$x_1 = \tfrac{1}{12}(\sin 3s - 3\sin s) + a_1 \cos s,$$

where the constant a_1 cannot be determined at the first order stage.

To determine a_1, and to find the leading correction to the frequency, we must proceed to the second order. On substituting the expressions for x_0 and x_1 into the second order equation, we obtain

$$\begin{aligned} x_2'' + x_2 &= -2\omega_1 x_1'' - \big(\omega_1^2 + 2\omega_2\big)x_0'' - {x_0'}^2 x_1' + x_1' \\ &= 4\omega_2 \cos s - \sin^2 s(\cos 3s - \cos s) + 4a_1 \sin^3 s + \tfrac{1}{4}(\cos 3s - \cos s) - a_1 \sin s \\ &= \left(4\omega_2 + \tfrac{1}{4}\right)\cos s + a_1 \sin s - \tfrac{1}{2}\cos 3s - a_1 \sin 3s + \tfrac{1}{4}\cos 5s, \end{aligned}$$

after some trigonometric simplification. The coefficients of $\cos s$ and $\sin s$ on the right side of this equation must both be zero, for otherwise $x_2(s)$ would not be periodic. Hence $a_1 = 0$ and that $\omega_2 = -\frac{1}{16}$. Mercifully, this is as far as we need to go.

Hence, when ϵ is small, the **approximate frequency** of the limit cycle is

$$\omega = 1 - \tfrac{1}{16}\epsilon^2 + O\left(\epsilon^3\right),$$

and the **approximate displacement** at time t is given by

$$x = 2\cos s + \tfrac{1}{12}(\sin 3s - 3\sin s)\epsilon + O\left(\epsilon^2\right),$$

where $s = \left(1 + O\left(\epsilon^2\right)\right) t$. ■

Problem 8.5 *Phase paths in polar form*

Show that the system of equations

$$\dot{x}_1 = F_1(x_1, x_2, t), \qquad \dot{x}_2 = F_2(x_1, x_2, t)$$

can be written in polar coordinates in the form

$$\dot{r} = \frac{x_1 F_1 + x_2 F_2}{r}, \qquad \dot{\theta} = \frac{x_1 F_2 - x_2 F_1}{r^2},$$

where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$.

A dynamical system satisfies the equations

$$\dot{x} = -x + y,$$
$$\dot{y} = -x - y.$$

Convert this system into polar form and find the polar equations of the phase paths. Show that every phase path encircles the origin infinitely many times in the clockwise direction. Show further that every phase path terminates at the origin. Sketch the phase diagram.

Solution

From the relations $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, it follows that

$$r = \left(x_1^2 + x_2^2\right)^{1/2},$$
$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right),$$

and on differentiating these formulae with respect to t, we obtain

$$\dot{r} = \left(x_1^2 + x_2^2\right)^{-1/2} x_1\dot{x}_1 + \left(x_1^2 + x_2^2\right)^{-1/2} x_2\dot{x}_2 = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r},$$
$$\dot{\theta} = \frac{1}{1 + \left(x_2/x_1\right)^2}\left(\frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2}\right) = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}.$$

Hence, if x_1, x_2 satisfy the equations $\dot{x}_1 = F_1$, $\dot{x}_2 = F_2$, then r, θ must satisfy the equations

$$\dot{r} = \frac{x_1 F_1 + x_2 F_2}{r},$$
$$\dot{\theta} = \frac{x_1 F_2 - x_2 F_1}{r^2},$$

where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$.

The system of equations

$$\dot{x} = -x + y,$$
$$\dot{y} = -x - y$$

can be expressed in the **polar form**

$$\dot{r} = \frac{x(-x+y) + y(-x-y)}{r} = -r,$$
$$\dot{\theta} = \frac{x(-x-y) - y(-x+y)}{r^2} = -1.$$

The general solution of this pair of (now) uncoupled ODEs is

$$r = a e^{-t},$$
$$\theta = -t + \alpha,$$

where a and α are integration constants. We see that, whatever the initial conditions, θ tends to negative infinity and r tends to zero as t tends to infinity. In other words, *every phase path encircles the origin infinitely many times in the clockwise direction*, and *every phase path terminates at the origin*. Figure 8.1 shows some typical phase paths. ■

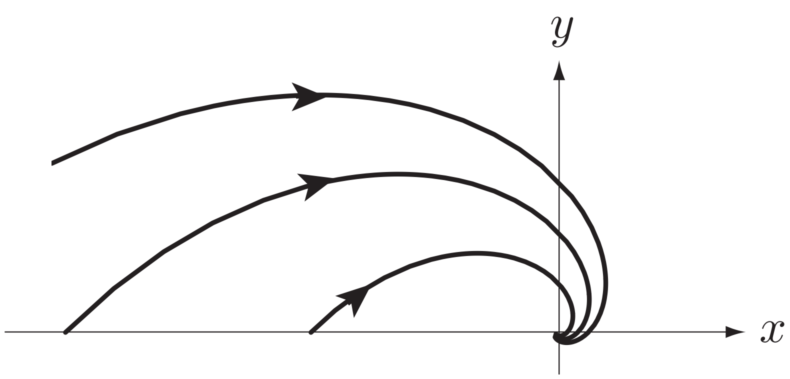

FIGURE 8.1 Three typical phase paths in problem 8.5.

Problem 8 . 6

A dynamical system satisfies the equations

$$\dot{x} = x - y - (x^2 + y^2)x,$$
$$\dot{y} = x + y - (x^2 + y^2)y.$$

Convert this system into polar form and find the polar equations of the phase paths that begin in the domain $0 < r < 1$. Show that all these phase paths spiral anti-clockwise and tend to the limit cycle $r = 1$. Show also that the same is true for phase paths that begin in the domain $r > 1$. Sketch the phase diagram.

Solution

The system of equations

$$\dot{x} = x - y - (x^2 + y^2)x,$$
$$\dot{y} = x + y - (x^2 + y^2)y$$

can be expressed in the **polar form**

$$\dot{r} = \frac{x(x - y - r^2x) + y(x + y - r^2y)}{r},$$
$$\dot{\theta} = \frac{x(x + y - r^2y) - y(x - y - r^2x)}{r^2},$$

that is,

$$\dot{r} = r\left(1 - r^2\right),$$
$$\dot{\theta} = 1.$$

The general solution of the second equation is $\theta = t + \alpha$, where α is the integration constant. It follows that, whatever the initial conditions, *every phase path encircles the origin infinitely many times in the anti-clockwise direction*. The first equation is a separable first order ODE whose general solution is

$$t = \int \frac{dr}{r\left(1 - r^2\right)}.$$

The integral on the right can be evaluated by first writing the integrand in partial fractions and the result is

$$t + \tau = \begin{cases} \frac{1}{2}\ln\left(\dfrac{r^2}{1-r^2}\right) & (r < 1), \\ \frac{1}{2}\ln\left(\dfrac{r^2}{r^2-1}\right) & (r > 1), \end{cases}$$

where τ is the integration constant. Hence the time variation of r is given by

$$r^2 = \begin{cases} \dfrac{1}{1+e^{-2(t+\tau)}} & (r < 1), \\ \dfrac{1}{1-e^{-2(t+\tau)}} & (r > 1). \end{cases}$$

We see that if $r < 1$ initially, then r *increases* with time and tends to unity from below. Conversely, if $r > 1$ initially, then r *decreases* with time and tends to unity from above. Thus, whatever the initial conditions, *every phase path spirals anti-clockwise and tends to the limit cycle* $r = 1$. Figure 8.2 shows some typical phase paths.

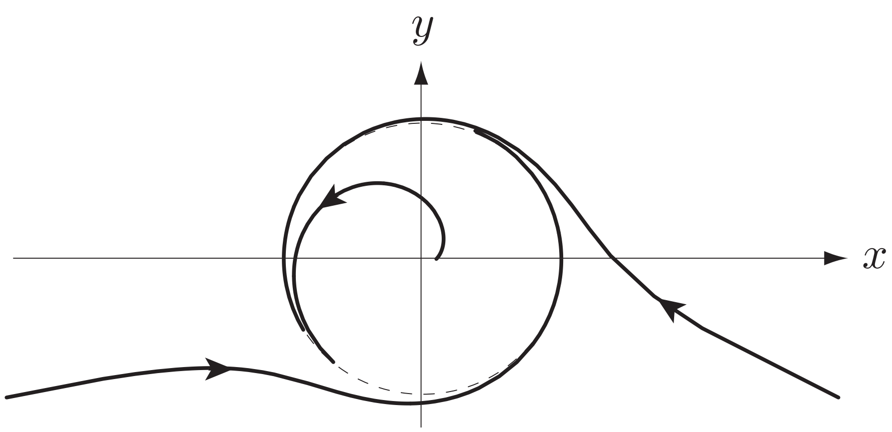

FIGURE 8.2 Three typical phase paths tending to the limit cycle in problem 8.6.

Problem 8.7

A damped linear oscillator satisfies the equation

$$\ddot{x} + \dot{x} + x = 0.$$

Show that the polar equations for the motion of the phase points are

$$\dot{r} = -r\sin^2\theta, \qquad \dot{\theta} = -\left(1 + \tfrac{1}{2}\sin 2\theta\right).$$

Show that every phase path encircles the origin infinitely many times in the clockwise direction. Show further that these phase paths terminate at the origin.

Solution

The second order ODE

$$\ddot{x} + \dot{x} + x = 0$$

is equivalent to the system of first order ODEs

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= -x - v.\end{aligned}$$

These equations can be expressed in the **polar form**

$$\begin{aligned}\dot{r} &= \frac{xv + v(-x - v)}{r},\\ \dot{\theta} &= \frac{x(-x - v) - v^2}{r^2},\end{aligned}$$

that is,

$$\begin{aligned}\dot{r} &= -r\sin^2\theta,\\ \dot{\theta} &= -\left(1 + \tfrac{1}{2}\sin 2\theta\right).\end{aligned}$$

We first wish to show that θ tends to negative infinity as t tends to infinity. Although the second equation can be integrated explicitly, it is a *very* messy job. One can however argue that since $\dot{\theta} \le -\frac{1}{2}$ for all t, then

$$\theta - \alpha \le -\tfrac{1}{2}t,$$

where α is the initial value of θ. It follows immediately that, whatever the initial conditions, *θ tends to negative infinity as t tends to infinity*. This simple argument does not provide the *value* of θ at time t, but it does yield the required result.

We proceed in a similar way to show that r tends to zero as t tends to infinity. In order not to get confused by the negative signs, we introduce the new variable $\theta' = -\theta$ so that $\frac{1}{2} \le \dot{\theta}' \le \frac{3}{2}$ and θ' tends to *positive* infinity as t tends to infinity. The first ODE then becomes

$$\dot{r} = -r \sin^2 \theta'$$

and hence

$$\begin{aligned}
-\ln r &= \int \sin^2 \theta' \, dt \\
&= \int \frac{\sin^2 \theta'}{\dot{\theta}'} \, d\theta' \\
&\ge \tfrac{2}{3} \int \sin^2 \theta' \, d\theta' \\
&= \tfrac{1}{3}\left(\theta' - \tfrac{1}{2} \sin 2\theta'\right) + C,
\end{aligned}$$

where C is the integration constant. Whatever the value of C, this tends to infinity as θ' tends to infinity. Thus $-\ln r$ tends to infinity and hence r tends to zero. Thus, whatever the initial conditions, *r tends to zero as t tends to infinity*. As before, this simple argument does not provide the *value* of r at time t, but it does yield the required result. ■

Problem 8.8

A non-linear oscillator satisfies the equation

$$\ddot{x} + \dot{x}^3 + x = 0.$$

Find the polar equations for the motion of the phase points. Show that phase paths that begin within the circle $r < 1$ encircle the origin infinitely many times in the clockwise direction. Show further that these phase paths terminate at the origin.

Solution

The second order ODE

$$\ddot{x} + \dot{x}^3 + x = 0.$$

is equivalent to the system of first order ODEs

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x - v^3. \end{aligned}$$

These equations can be expressed in the **polar form**

$$\begin{aligned} \dot{r} &= \frac{xv + v\big(-x - v^3\big)}{r}, \\ \dot{\theta} &= \frac{x\big(-x - v^3\big) - v^2}{r^2}, \end{aligned}$$

that is,

$$\begin{aligned} \dot{r} &= -r^3 \sin^4 \theta, \\ \dot{\theta} &= -1 - r^2 \cos\theta \sin^3 \theta. \end{aligned}$$

Since this pair of coupled equations cannot be solved explicitly, we must use inequalities in the remainder of the problem. It is clear from the first equation that r is a decreasing function of t. It follows that any phase path that begins in the region $r \le R$ remains there. Suppose that $R < 1$. On phase paths that begin in this region,

$$\begin{aligned} \dot{\theta} &= -1 - r^2 \cos\theta \sin^3 \theta \\ &\le -1 + R^2 \end{aligned}$$

and hence

$$\theta - \alpha \le -\big(1 - R^2\big)t,$$

where α is the initial value of θ. It follows that, whatever the initial conditions, *θ tends to negative infinity as t tends to infinity*. This simple argument does not provide the *value* of θ at time t, but it does yield the required result.

We proceed in a similar way to show that r tends to zero as t tends to infinity. In order not to get confused by the negative signs, we introduce the new variable $\theta' = -\theta$ so that $1 - R^2 \le \dot{\theta}' \le 1 + R^2$ and θ' tends to *positive* infinity as t tends to infinity. The first ODE then becomes

$$\dot{r} = -r^3 \sin^4 \theta'$$

and hence

$$\begin{aligned}
\frac{1}{r^2} &= 2 \int \sin^4 \theta' \, dt \\
&= 2 \int \frac{\sin^4 \theta'}{\dot{\theta}'} \, d\theta' \\
&\ge \frac{2}{1 + R^2} \int \sin^4 \theta' \, d\theta'.
\end{aligned}$$

We could evaluate this integral but it obviously tends to infinity as θ' tends to infinity since the integrand is positive and periodic. Thus $1/r^2$ tends to infinity and hence r tends to zero. Thus, whatever the initial conditions, *r tends to zero as t tends to infinity*. As before, this simple argument does not provide the *value* of r at time t, but it does yield the required result. ■

Problem 8.9

A non-linear oscillator satisfies the equation

$$\ddot{x} + \left(x^2 + \dot{x}^2 - 1\right)\dot{x} + x = 0.$$

Find the polar equations for the motion of the phase points. Show that any phase path that starts in the domain $1 < r < \sqrt{3}$ spirals clockwise and tends to the limit cycle $r = 1$. [The same is true of phase paths that start in the domain $0 < r < 1$.] What is the period of the limit cycle?

Solution

The second order ODE

$$\ddot{x} + \left(x^2 + \dot{x}^2 - 1\right) + x = 0.$$

is equivalent to the system of first order ODEs

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x - \left(x^2 + v^2 - 1\right)v. \end{aligned}$$

These equations can be expressed in the **polar form**

$$\begin{aligned} \dot{r} &= \frac{xv + v\left(-x - (x^2 + v^2 - 1)v\right)}{r}, \\ \dot{\theta} &= \frac{x\left(-x - (x^2 + v^2 - 1)v\right) - v^2}{r^2}, \end{aligned}$$

that is,

$$\begin{aligned} \dot{r} &= -r\left(r^2 - 1\right)\sin^2\theta, \\ \dot{\theta} &= -1 - \tfrac{1}{2}\left(r^2 - 1\right)\sin 2\theta. \end{aligned}$$

Since this pair of coupled equations cannot be solved explicitly, we use inequalities in the remainder of the problem. Consider those phase paths that begin in the domain $1 < r < R$. Such a phase path cannot cross the circle $r = 1$. This is because the circle $r = 1$ is itself a phase path (by virtue of the fact that $r = 1$, $\dot{\theta} = -1$ satisfies the above equations) and phase paths of an autonomous system cannot cross each other. The phase path is therefore restricted to the domain $r > 1$. But it then follows from the first equation that r must be a *decreasing* function of t, that is, the phase point must move *inwards*. The second equation then implies that, on such a path,

$$\begin{aligned}\dot{\theta} &= -1 - \tfrac{1}{2}\left(r^2 - 1\right)\sin 2\theta\\ &\le -1 + \tfrac{1}{2}\left(R^2 - 1\right)\\ &= -\tfrac{1}{2}\left(3 - R^2\right)\end{aligned}$$

and hence

$$\theta - \alpha \le -\tfrac{1}{2}\left(3 - R^2\right)t,$$

where α is the initial value of θ. Suppose now that $R < \sqrt{3}$. Then, whatever the initial conditions, *θ tends to negative infinity as t tends to infinity*.

We proceed in a similar way to show that r tends to unity as t tends to infinity. In order not to get confused by the negative signs, we introduce the new variable $\theta' = -\theta$ so that

$$\tfrac{1}{2}\left(3 - R^2\right) \le \dot{\theta}' \le \tfrac{1}{2}\left(R^2 + 1\right)$$

and θ' tends to *positive* infinity as t tends to infinity. The first ODE then becomes

$$\dot{r} = -r\left(r^2 - 1\right)\sin^2\theta'$$

and hence

$$-\int \frac{dr}{r\left(r^2 - 1\right)} = \int \sin^2\theta'\, dt.$$

The integral on the left can be evaluated by first putting the integrand into partial fractions. This gives

$$\begin{aligned}\ln\left(\frac{r^2}{r^2 - 1}\right) &= 2\int \sin^2\theta'\, dt\\ &= 2\int \frac{\sin^2\theta'}{\dot{\theta}'}\, d\theta'\\ &\ge \frac{4}{R^2 + 1}\int \sin^2\theta'\, d\theta'\\ &= \frac{2}{R^2 + 1}\left(\theta' - \tfrac{1}{2}\sin 2\theta'\right) + C,\end{aligned}$$

where C is the integration constant. Whatever the value of C, this tends to infinity as θ' tends to infinity. Thus $\ln\left(r^2/(r^2-1)\right)$ tends to infinity and hence r tends to unity. Thus, whatever the initial conditions, *r tends to unity as t tends to infinity*.

The above analysis, together with the corresponding result for phase paths that start in the domain $0 < r < 1$, shows that the periodic solution $r = 1$, $\dot{\theta} = -1$ is a **limit cycle**. (It also shows that there are no other limit cycles lying wholly or partly in the domain $0 < r < \sqrt{3}$.) This limit cycle is executed in the clockwise sense and its **period** is 2π. ■

Problem 8.10 *Predator–prey*

Consider the symmetrical predator–prey equations

$$\dot{x} = x - xy, \qquad \dot{y} = xy - y,$$

where $x(t)$ and $y(t)$ are positive functions. Show that the phase paths satisfy the equation

$$(xe^{-x})(ye^{-y}) = A,$$

where A is a constant whose value determines the particular phase path. By considering the shape of the surface

$$z = (xe^{-x})(ye^{-y}),$$

deduce that each phase path is a simple closed curve that encircles the equilibrium point at $(1, 1)$. Hence *every solution* of the equations is periodic! [This prediction can be confirmed by solving the original equations numerically.]

Solution

The phase paths of the predator prey system satisfy the equation

$$\frac{dy}{dx} = \frac{xy - y}{x - xy},$$

which is a separable first order ODE. On separating and solving, we find that

$$\ln y - y = x - \ln x + \text{constant},$$

which can be written in the form

$$xye^{-x-y} = A,$$

where A is a constant whose value determines the particular phase path. The **phase paths** are therefore the curves in which the surface

$$z = xye^{-x-y}$$

meets the family of planes $z = A$.

The surface $z = xye^{-x-y}$ is shown in Figure 8.3 (left). It has a single maximum at the equilibrium point $(1, 1)$, it is zero on the axes $x = 0$ and $y = 0$ and it tends

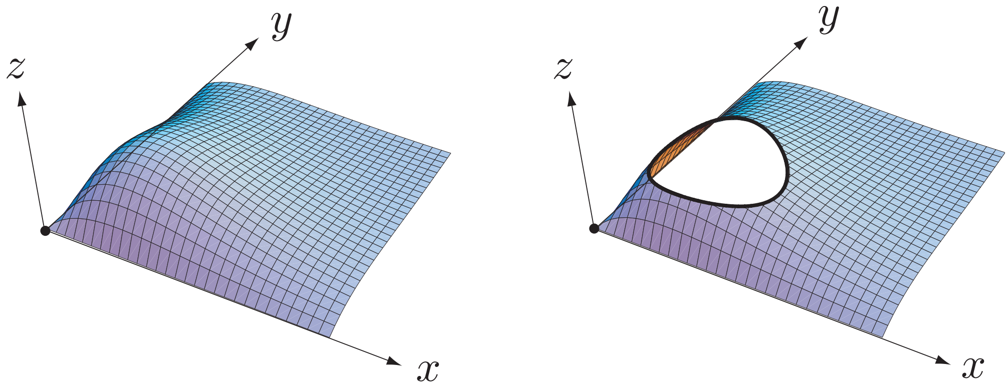

FIGURE 8.3 **Left**: The surface $z = \left(xe^{-x}\right)\left(ye^{-y}\right)$. **Right**: The intersection of the surface with the plane $z =$ constant is a *closed* curve.

to zero as $\left(x^2 + y^2\right)^{1/2}$ tends to infinity. The intersection of this surface with a typical plane $z = A$ is shown in Figure 8.3 (right). It is evident from the shape of the surface that the intersection must be a simple closed curve that (when projected down on to the (x, y)-plane) encircles the equilibrium point at $(1, 1)$. Hence *every solution of the predator prey system must be periodic*. Some typical phase paths are shown in Figure 8.4.

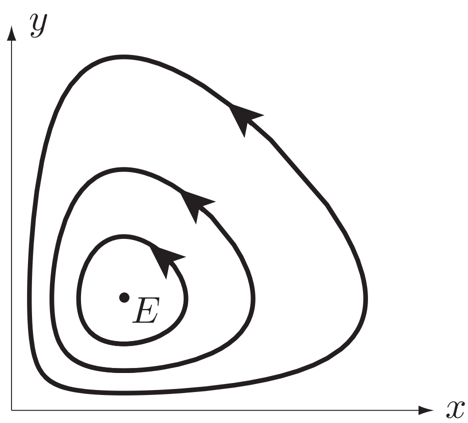

FIGURE 8.4 Three typical phase paths of the predator-prey equations. E is the equilibrium point.

Problem 8.11

Use Poincaré-Bendixson to show that the system

$$\dot{x} = x - y - (x^2 + 4y^2)x,$$
$$\dot{y} = x + y - (x^2 + 4y^2)y,$$

has a limit cycle lying in the annulus $\frac{1}{2} < r < 1$.

Solution

The equilibrium points of the system satisfy the equations

$$x - y - (x^2 + 4y^2)x = 0,$$
$$x + y - (x^2 + 4y^2)y = 0.$$

On multiplying the first equation by y and the second equation by x and then subtracting, we find that $x^2 + y^2 = 0$. Hence the only **equilibrium point** of the system is at the origin.

The system of equations

$$\dot{x} = x - y - (x^2 + 4y^2)x,$$
$$\dot{y} = x + y - (x^2 + 4y^2)y$$

can be expressed in the **polar form**

$$\dot{r} = \frac{x\big(x - y - (x^2 + 4y^2)x\big) + y\big(x + y - (x^2 + 4y^2)y\big)}{r},$$
$$\dot{\theta} = \frac{x\big(x + y - (x^2 + 4y^2)y\big) - y\big(x - y - (x^2 + 4y^2)x\big)}{r^2},$$

that is,

$$\dot{r} = r\left(1 - r^2\cos^2\theta - 4r^2\sin^2\theta\right),$$
$$\dot{\theta} = 1.$$

At points on the circle $r = 1$,

$$\begin{aligned}\dot{r} &= 1 - \cos^2\theta - 4\sin^2\theta \\ &= -3\sin^2\theta \\ &\le 0.\end{aligned}$$

Hence, except *possibly* for the two points $(\pm 1, 0)$, *phase points that start on the circle* $r = 1$ *move towards the origin*.

Similarly. at points on the circle $r = \frac{1}{2}$,

$$\begin{aligned}\dot{r} &= \tfrac{1}{2}\big(1 - \tfrac{1}{4}\cos^2\theta - \sin^2\theta\big)\\ &= \tfrac{3}{8}\cos^2\theta\\ &\geq 0.\end{aligned}$$

Hence, except *possibly* for the two points $(0, \pm\frac{1}{2})$, *phase points that start on the circle* $r = \frac{1}{2}$ *move away from the origin*.

We are now in a position to apply the **Poincaré-Bendixson** theorem. Let $\mathcal{D}$ be the annular domain $\frac{1}{2} < r < 1$. Then, with perhaps four exceptions, phase points that start anywhere on the boundaries $r = \frac{1}{2}$ and $r = 1$ *enter* the domain $\mathcal{D}$. It follows that *infinitely many phase paths enter the domain* $\mathcal{D}$ *and never leave*. Since $\mathcal{D}$ is a *bounded* domain with *no equilibrium points* within it or on its boundaries, it follows from Poincaré-Bendixson that any such path must either be a simple closed loop or tend to a limit cycle. In fact these phase paths cannot close themselves (that would mean leaving $\mathcal{D}$) and so can only tend to a limit cycle. It follows that the system must have (at least one) **limit cycle** lying in the domain $\mathcal{D}$. Some typical phase paths tending to the limit cycle are shown in Figure 8.5. ■

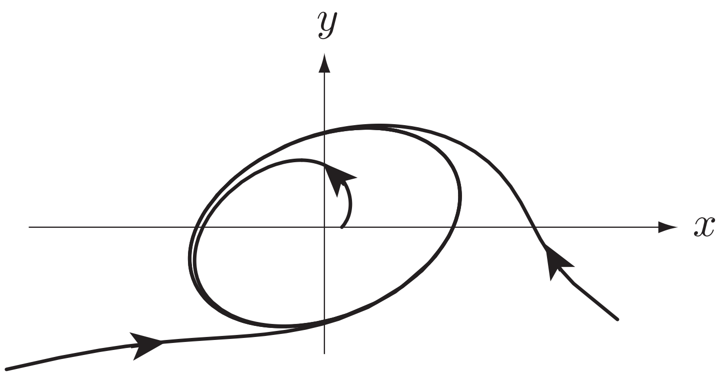

FIGURE 8.5 Three typical phase paths tending to the limit cycle in problem 8.11.

Problem 8.12 *Van der Pol's equation*

Show that Van der Pol's equation

$$\ddot{x} + \epsilon\,\dot{x}\left(x^2 - 1\right) + x = 0$$

is equivalent to the system of first order equations

$$\begin{aligned} \dot{u} &= -x, \\ \dot{x} &= u - \epsilon\,x\left(\tfrac{1}{3}x^2 - 1\right), \end{aligned}$$

and, by making appropriate changes of variable, that this system is in turn equivalent to the system

$$\begin{aligned} \dot{X} &= V, \\ \dot{V} &= -X - \epsilon\,V\left(V^2 - 1\right). \end{aligned}$$

By comparing this last system with the system (8.20) discussed in Example 8.4, deduce that Van der Pol's equation has a limit cycle for any positive value of the parameter ϵ.

Solution

Van der Pol's equation can be written in the form

$$\frac{d}{dt}\left(\dot{x} + \epsilon\left(\tfrac{1}{3}x^3 - x\right)\right) + x = 0,$$

that is,

$$\dot{u} + x = 0,$$

where $u = \dot{x} + \epsilon\,x\left(\tfrac{1}{3}x^2 - 1\right)$. Thus Van der Pol's equation is equivalent to the system of first order equations

$$\begin{aligned} \dot{u} &= -x, \\ \dot{x} &= u - \epsilon\,x\left(\tfrac{1}{3}x^2 - 1\right). \end{aligned}$$

If we now make the changes of varaible $x = \sqrt{3}\,V$, $u = -\sqrt{3}\,X$, then X, V satisfy the system of equations

$$\begin{aligned} \dot{X} &= V, \\ \dot{V} &= -X - \epsilon\,V\left(V^2 - 1\right), \end{aligned}$$

as required. These equations are identical to equations (8.20), which were obtained from Rayleigh's equation (see Example 8.4). Since we have already proved that these equations have a limit cycle for all positive values of ϵ, it follows that the same must be true of Van der Pol's equation. ■

Problem 8.13

A driven non-linear oscillator satisfies the equation

$$\ddot{x} + \epsilon \dot{x}^3 + x = \cos pt,$$

where ϵ, p are positive constants. Use perturbation theory to find a two-term approximation to the driven response when ϵ is small. Are there any restrictions on the value of p?

Solution

The driven response satisfies the equation

$$\ddot{x} + \epsilon \dot{x}^3 + x = \cos pt$$

and has period $2\pi/p$. We expand the required solution in the **perturbation series**

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots,$$

where the expansion functions $x_0(t)$, $x_1(t)$, $x_2(t)$, $\ldots$ each have period $2\pi/p$. If we now substitute this series into the equation and equate coefficients of powers of ϵ, we obtain a succession of ODEs the first two of which are as follows:

- From coefficients of ϵ^0:

$$\ddot{x}_0 + x_0 = \cos pt.$$

- From coefficients of ϵ^1:

$$\ddot{x}_1 + x_1 = -\dot{x}_0^3. \tag{1}$$

For $p \neq 1$, the general solution of the zero order equation is

$$x_0 = \frac{\cos pt}{1 - p^2} + A_0 \cos t + B_0 \sin t,$$

where A_0 and B_0 are arbitrary constants. Since x_0 is known to have period $2\pi/p$, it follows that A_0 and B_0 must be zero unless $1/p$ is an integer; we will assume this is *not* the case. Then the required solution of the **zero order** equation is

$$x_0 = -\frac{\cos pt}{p^2 - 1}.$$

The **first order** equation can now be written

$$\begin{aligned}\ddot{x}_1 + x_1 &= -\left(\frac{p}{p^2-1}\right)^3 \sin^3 pt\\ &= -\frac{p^3}{4(p^2-1)^3}\big(3\sin pt - \sin 3pt\big),\end{aligned}$$

on using the trigonometric identity $4\sin^3\theta = 3\sin\theta - \sin 3\theta$. Since $1/p$ is not an integer, the only solution of this equation that has period $2\pi/p$ is

$$x_1 = \frac{p^3}{4(p^2-1)^3}\left(\frac{3\sin pt}{p^2-1} - \frac{\sin 3pt}{9p^2-1}\right).$$

Hence the **driven response** of the oscillator is given by

$$x(t) = -\frac{\cos pt}{p^2-1} + \left(\frac{3p^3\sin pt}{4(p^2-1)^4} - \frac{p^3\sin 3pt}{4(p^2-1)^3(9p^2-1)}\right)\epsilon + O\left(\epsilon^2\right)$$

This is the approximate solution correct to the first order in the small parameter ϵ. In the course of the derivation we have assumed that $1/p$ is not an integer. When $1/p = 1, 3, \ldots$, this expression is clearly invalid. When $1/p = 2, 4, \ldots$, it does provide a solution (although possibly not the only one). ■

Chapter Nine

The energy principle

and energy conservation

Problem 9.1

Book Figure 9.12 shows two particles P and Q, of masses M and m, that can move on the smooth outer surface of a fixed horizontal cylinder. The particles are connected by a light inextensible string of length $\pi a/2$. Find the equilibrium configuration and show that it is unstable.

Solution

In the configuration shown, the potential energy of the system is

$$V = Mga\cos\theta + mga\sin\theta.$$

In equilibrium, it is necessary that $V'(\theta) = 0$, which implies that θ must satisfy

$$m\cos\theta - M\sin\theta = 0.$$

The **equilibrium positions** are therefore

$$\theta = \tan^{-1}\left(\frac{m}{M}\right) \quad \text{and} \quad \theta = \pi + \tan^{-1}\left(\frac{m}{M}\right).$$

In the present problem, only the first value is permissible. (The second value would also be permissible if the particles were sliding on a circular wire.)

To investigate **stability**, we examine the value of V'' at the equilibrium position.

$$\begin{aligned} V'' &= -ga\big(M\cos\theta + m\sin\theta\big) \\ &= -ga\left(M^2 + m^2\right)^{1/2} \end{aligned}$$

when $\theta = \tan^{-1}(m/M)$. This value of V'' is *negative* and so V has a local *maximum* at $\theta = \tan^{-1}(m/M)$. This equilibrium position is therefore **unstable**. ■

Problem 9.2

A uniform rod of length $2a$ has one end smoothly pivoted at a fixed point O. The other end is connected to a fixed point A, which is a distance $2a$ vertically above O, by a light elastic spring of natural length a and modulus $\frac{1}{2}mg$. The rod moves in a vertical plane through O. Show that there are two equilibrium positions for the rod, and determine their stability. [The vertically upwards position for the rod would compress the spring to zero length and is excluded.]

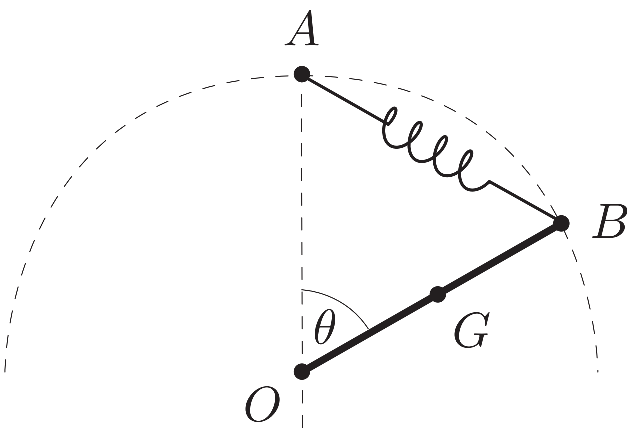

FIGURE 9.1 The rod and the spring in Problem 9.2.

Solution

The system of the rod and the spring is shown in Figure 9.1. In this configuration, the spring has length $AB = 4a \sin \frac{1}{2}\theta$ and the extension Δ is therefore

$$\Delta = a\left(4 \sin \tfrac{1}{2}\theta - 1\right).$$

The modulus of the spring is given to be $\frac{1}{2}mg$ so that the strength (as defined in Section 5.1, p. 106) is $\frac{1}{2}mg/a$. The potential energy of the spring is therefore

$$\begin{aligned} V^S &= \tfrac{1}{2}\left(\frac{mg}{2a}\right)\Delta^2 \\ &= \tfrac{1}{4}mga\left(4 \sin \tfrac{1}{2}\theta - 1\right)^2. \end{aligned}$$

The gravitational potential energy of the rod is simply $V^G = mga \cos\theta$ and so the total **potenetial energy** of the system is

$$\begin{aligned} V &= V^S + V^G \\ &= \tfrac{1}{4}mga\left(4\sin\tfrac{1}{2}\theta - 1\right)^2 + mga\cos\theta \\ &= \tfrac{1}{4}mga\left(8\sin^2\tfrac{1}{2}\theta - 8\sin\tfrac{1}{2}\theta + 5\right), \end{aligned}$$

on using the trigonometric identity $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$.

In equilibrium, it is necessary that $V'(\theta) = 0$, which implies that θ must satisfy

$$\cos\tfrac{1}{2}\theta\left(2\sin\tfrac{1}{2}\theta - 1\right) = 0.$$

The **equilibrium positions** are therefore

$$\theta = \pi \quad \text{and} \quad \theta = \tfrac{1}{3}\pi.$$

To investigate **stability**, we examine the value of V'' at each of the equilibrium positions. It is easily shown that $V''(\pi) < 0$ and $V''\left(\frac{1}{3}\pi\right) > 0$ so that

(i) the vertically downwards position of the rod is **unstable**, and

(ii) the position with $\theta = \frac{1}{3}\pi$ is **stable**. ■

Problem 9.3

The internal potential energy function for a diatomic molecule is approximated by the **Morse potential**

$$V(r) = V_0\left(1 - e^{-(r-a)/b}\right)^2 - V_0,$$

where r is the distance of separation of the two atoms, and V_0, a, b are positive constants. Make a sketch of the Morse potential.

Suppose the molecule is restricted to *vibrational* motion in which the centre of mass G of the molecule is fixed, and the atoms move on a fixed straight line through G. Show that there is a single equilibrium configuration for the molecule and that it is stable. If the atoms each have mass m, find the angular frequency of small vibrational oscillations of the molecule.

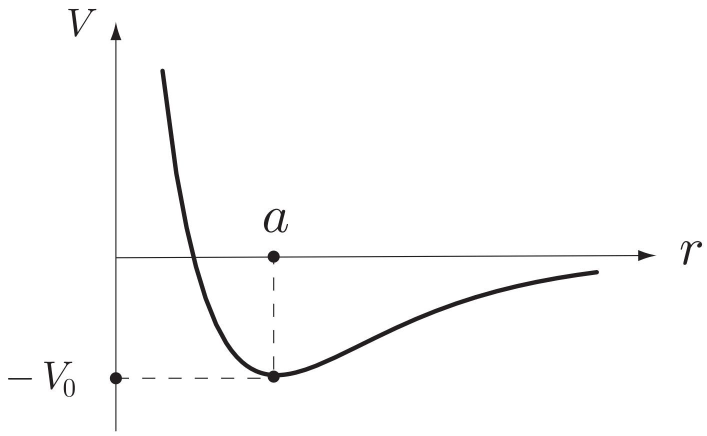

FIGURE 9.2 The Morse potential.

Solution

If V is the **Morse potential**

$$V(r) = V_0\left(1 - e^{-(r-a)/b}\right)^2 - V_0,$$

then V' is given by

$$\begin{aligned} V' &= 2V_0\left(1 - e^{-(r-a)/b}\right)\left(\frac{1}{b}e^{-(r-a)/b}\right) \\ &= \frac{2V_0}{b}\left(e^{-(r-a)/b} - e^{-2(r-a)/b}\right). \end{aligned}$$

Hence, at the stationary points of V, r must satisfy the equation

$$e^{-(r-a)/b} - e^{-2(r-a)/b} = 0,$$

which has the single solution $r = a$. To determine the nature of this stationary point, we examine the value of V''.

$$\begin{aligned} V'' &= \frac{2V_0}{b^2}\left(2e^{-2(r-a)/b} - e^{-(r-a)/b}\right) \\ &= +\frac{2V_0}{b^2} \end{aligned}$$

when $r = a$. The point $r = a$ is therefore a local **minimum** point of the function $V(r)$. The graph of the Morse potential is shown in Figure 9.2.

In rectilinear vibrational motion, the molecule has **kinetic energy**

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\tfrac{1}{2}\dot{r}\right)^2 + \tfrac{1}{2}m\left(\tfrac{1}{2}\dot{r}\right)^2 \\ &= \tfrac{1}{4}m\dot{r}^2 \end{aligned}$$

and the **energy conservation** equation is

$$\tfrac{1}{4}m\dot{r}^2 + V(r) = E,$$

where $V(r)$ is the Morse potential. On differentiating this equation with respect to t (and cancelling by $\dot{r}$), we obtain the vibrational **equation of motion**

$$\tfrac{1}{2}m\ddot{r} + V'(r) = 0.$$

In *small* vibrational motions near $r = a$, we can approximate $V'(r)$ by the first two terms of its Taylor series, namely,

$$\begin{aligned} V'(r) &= V'(a) + (r-a)V''(a) + \cdots \\ &= \frac{2V_0}{b^2}(r-a) + \cdots . \end{aligned}$$

The **linearised equation** for small vibrations is therefore

$$\tfrac{1}{2}m\ddot{r} + \frac{2V_0}{b^2}(r-a) = 0,$$

which can be written in the form

$$\frac{d^2}{dt^2}\left(r-a\right) + \frac{4V_0}{mb^2}\left(r-a\right) = 0.$$

This is the SHM equation with **angular frequency** Ω given by

$$\Omega^2 = \frac{4V_0}{mb^2}.$$

The **period** τ of small extensional vibrations of the molecule is therefore

$$\tau = \frac{2\pi}{\Omega} = \pi \left(\frac{mb^2}{V_0}\right)^{1/2}. \blacksquare$$

Problem 9.4 ✻

The internal gravitational potential energy of a system of masses is sometimes called the **self energy** of the system. (The reference configuration is taken to be one in which the particles are all a great distance from each other.) Show that the self energy of a uniform sphere of mass M and radius R is $-3M^2G/5R$. [Imagine that the sphere is built up by the addition of successive thin layers of matter brought in from infinity.]

Solution

When the sphere has been built up to radius r, its mass is $M(r/R)^3$. Suppose that a new layer of thickness dr is now added. The volume of this layer is $4\pi r^2 dr$ and its mass is

$$M\left(\frac{4\pi r^2 dr}{\frac{4}{3}\pi R^3}\right) = \frac{3Mr^2 dr}{R^3}.$$

Since the gravitation of the sphere is the same as that of a particle of equal mass at the centre, the potential energy of the new layer is

$$dV = \left(\frac{Mr^3}{R^3}\right)\left(\frac{3Mr^2 dr}{R^3}\right)\frac{G}{r} = \left(\frac{3M^2G}{R^6}\right) r^4\, dr,$$

where G is the constant of gravitation. The **potential energy** of the complete sphere of radius R is therefore given by

$$\begin{aligned} V &= -\left(\frac{3M^2G}{R^6}\right)\int_0^R r^4\, dr \\ &= -\left(\frac{3M^2G}{R^6}\right)\left(\frac{R^5}{5}\right) \\ &= -\frac{3M^2G}{5R}. \blacksquare \end{aligned}$$

Problem 9.5

Book Figure 9.13 shows two blocks of masses M and m that slide on smooth planes inclined at angles α and β to the horizontal. The blocks are connected by a light inextensible string that passes over a light frictionless pulley. Find the acceleration of the block of mass m up the plane, and deduce the tension in the string.

Solution

Let x be the displacement of the mass m up the plane, measured from some reference configuration, and let $v\,(=\dot{x})$ be the velocity of m. Then the **kinetic energy** of the system is

$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}Mv^2,$$

the **potential energy** is

$$V = mgx\sin\beta - Mgx\sin\alpha,$$

and the **energy conservation** equation is

$$\tfrac{1}{2}(m+M)v^2 + g\big(m\sin\beta - M\sin\alpha\big)x = E.$$

If we now differentiate this equation with respect to t (and cancel by v), we obtain the **equation of motion**

$$(m+M)\frac{dv}{dt} + g\big(m\sin\beta - M\sin\alpha\big) = 0.$$

The **acceleration** of the mass m up the plane is therefore

$$\frac{dv}{dt} = \left(\frac{M\sin\alpha - m\sin\beta}{M+m}\right)g,$$

which is a constant.

To find the **tension** S the string, consider the Newton equation for the mass m resolved parallel to the plane. This gives

$$m\frac{dv}{dt} = S - mg\sin\beta,$$

which, on using the calculated value for dv/dt, gives

$$S = \frac{Mmg\big(\sin\alpha + \sin\beta\big)}{M+m}. \blacksquare$$

Problem 9.6

Consider the system shown in book Figure 9.12 for the special case in which the particles P, Q have masses $2m$, m respectively. The system is released from rest in a symmetrical position with θ, the angle between OP and the upward vertical, equal to $\pi/4$. Find the energy conservation equation for the subsequent motion in terms of the coordinate θ.

✱ Find the normal reactions of the cylinder on each of the particles. Show that P is first to leave the cylinder and that this happens when $\theta = 70°$ approximately.

Solution

In terms of the coordinate θ, the **kinetic energy** of the system is

$$T = \tfrac{1}{2}M\left(a\dot{\theta}\right)^2 + \tfrac{1}{2}m\left(a\dot{\theta}\right)^2 = \tfrac{1}{2}(M+m)a^2\dot{\theta}^2,$$

the **potential energy** is

$$V = Mga\cos\theta + mga\sin\theta,$$

and the **energy conservation** equation has the form

$$\tfrac{1}{2}\left(m+M\right)a^2\dot{\theta}^2 + ga\left(M\cos\theta + m\sin\theta\right) = E.$$

For the special case in which $M = 2m$, this becomes

$$\tfrac{3}{2}ma^2\dot{\theta}^2 + mga\left(2\cos\theta + \sin\theta\right) = E.$$

The initial conditions $\theta = \pi/4$ and $\dot{\theta} = 0$ when $t = 0$ imply that $E = 3mga/\sqrt{2}$ and the final form of the energy conservation equation is therefore

$$\dot{\theta}^2 = \frac{g}{3a}\left(3\sqrt{2} - 4\cos\theta - 2\sin\theta\right).$$

To find the **normal reaction** R^P exerted on the particle P, consider the Newton equation for P resolved in the radial direction. This is

$$2mg\cos\theta - R^P = \frac{(2m)(a\dot{\theta})^2}{a}$$

which gives

$$R^P = \frac{2mg}{3}\left(7\cos\theta + 2\sin\theta - 3\sqrt{2}\right).$$

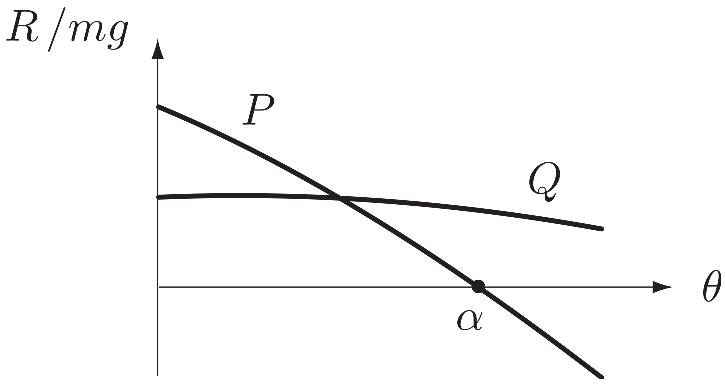

FIGURE 9.3 The (dimensionless) normal reactions R^P/mg and R^Q/mg for $\pi/4 \le \theta \le \pi/2$.

In the same way, the **normal reaction** R^Q exerted on the particle Q is found to be

$$R^Q = \frac{mg}{3}\left(4\cos\theta + 5\sin\theta - 3\sqrt{2}\right).$$

Figure 9.3 shows the dimensionless normal reactions R^P/mg and R^Q/mg for $\pi/4 \le \theta \le \pi/2$. Although R^P is initially larger than R^Q, it is the first to become zero as θ increases. A numerical evaluation gives $\alpha = 70°$ approximately.

This value can be obtained analytically by solving the trigonometric equation

$$7\cos\theta + 2\sin\theta - 3\sqrt{2} = 0.$$

On writing

$$7\cos\theta + 2\sin\theta = \left(7^2 + 2^2\right)^{1/2}\cos(\theta - \beta),$$

where $\beta = \tan^{-1}\frac{2}{7}$, the equation becomes

$$\cos(\theta - \beta) = \frac{3\sqrt{2}}{\sqrt{53}}$$

and the solution is

$$\theta = \tan^{-1}\left(\frac{2}{7}\right) + \cos^{-1}\left(\frac{3\sqrt{2}}{\sqrt{53}}\right) \approx 70°. \blacksquare$$

Problem 9.7

A heavy uniform rope of length $2a$ is draped symmetrically over a *thin* smooth horizontal peg. The rope is then disturbed slightly and begins to slide off the peg. Find the speed of the rope when it finally leaves the peg.

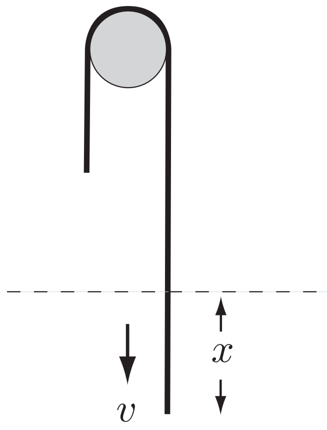

FIGURE 9.4 Rope sliding off a smooth fixed peg.

Solution

Let x be the downward displacement of the rope (see Figure 9.4) and let $v = \dot{x}$. Then, since every particle of the rope has the same speed, the **kinetic energy** of the rope is

$$T = \tfrac{1}{2}Mv^2$$

where M is the total mass.

The displaced configuration is the same as if the rope were held still and a length x were cut from the left side and hung from the end of the right side. The mass of this segment is $Mx/2a$ and its centre of mass is lowered by a distance x. The **potential energy** of the rope in the displaced configuration is therefore

$$V = -\left(\frac{Mx}{2a}\right)gx = -\frac{Mgx^2}{2a}.$$

The **energy conservation** equation for the rope then has the form

$$\tfrac{1}{2}Mv^2 - \frac{Mgx^2}{2a} = E.$$

The initial conditions $x = 0$ and $v = 0$ when $t = 0$ imply that $E = 0$ and the final form of the energy conservation equation is

$$v^2 = \frac{gx^2}{a}.$$

This gives the **speed of the rope** when its displacement is x. The rope leaves the peg when $x = a$ at which time its speed is $(ag)^{1/2}$. ■

Problem 9.8

A uniform heavy rope of length a is held at rest with its two ends close together and the rope hanging symmetrically below. (In this position, the rope has two long vertical segments connected by a small curved segment at the bottom.) One of the ends is then released. Find the velocity of the free end when it has descended by a distance x.

Deduce a similar formula for the acceleration of the free end and show that it always *exceeds* g. Find how far the free end has fallen when its acceleration has risen to $5g$.

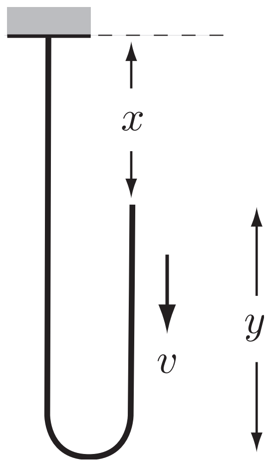

FIGURE 9.5 Rope falling with one end supported and the other free.

Solution

Let x be the downward displacement of the free end of the rope (see Figure 9.5) and let v $(= \dot{x})$ be its velocity. In the displaced configuration, the length y of the right side of the rope is given by $x + 2y = a$, that is, $y = \frac{1}{2}(a - x)$.

Since the left side of the rope is at rest and every particle of the right side has the same velocity v, the **kinetic energy** of the rope is

$$T = 0 + \tfrac{1}{2}\left(\frac{My}{a}\right)v^2 = \frac{M}{4a}(a - x)v^2.$$

The **potential energy** of the rope in the displaced configuration is

$$\begin{aligned} V &= -\left(\frac{M(x+y)}{a}\right) g \left(\tfrac{1}{2}(x+y)\right) - \left(\frac{My}{a}\right) g \left(x + \tfrac{1}{2}y\right) \\ &= -\frac{Mg}{4a}\left(a^2 + 2ax - x^2\right), \end{aligned}$$

after some simplification.

The **energy conservation** equation for the rope then has the form

$$\frac{M}{4a}(a-x)v^2 - \frac{Mg}{4a}\left(a^2 + 2ax - x^2\right) = E.$$

The initial conditions $x = 0$ and $v = 0$ when $t = 0$ imply that $E = -\frac{1}{4}Mga$ and the final form of the energy conservation equation is

$$v^2 = \frac{x(2a-x)g}{a-x}.$$

This gives the **speed of the rope** when its displacement is x.

On differentiating this formula with respect to t (and cancelling by v) we find that the downward **acceleration** of the free end is given by

$$\frac{dv}{dt} = \left(\frac{2a^2 - 2ax + x^2}{2(a-x)^2}\right) g,$$

after more simplification.

It follows that

$$\frac{dv}{dt} - g = \left(\frac{x(2a-x)}{2(a-x)^2}\right) g,$$

which is positive for x in the physical range $0 < x < a$. Hence the downward acceleration of the free end always exceeds g!

If dv/dt is to have the value $5g$, then the displacement x must satisfy

$$\frac{2a^2 - 2ax + x^2}{2(a-x)^2} = 5,$$

which yields the quadratic equation

$$9x^2 - 18ax + 8a^2 = 0,$$

whose solutions are $x = \frac{2}{3}a$ and $x = \frac{4}{3}a$. The solution $x = \frac{4}{3}a$ lies outside the physical range. Hence, the downward acceleration of the free end becomes equal to $5g$ when it has fallen a distance $\frac{2}{3}a$. ■

Problem 9.9

A heavy uniform rope of mass M and length $4a$ has one end connected to a fixed point on a smooth horizontal table by light elastic spring of natural length a and modulus $\frac{1}{2}Mg$, while the other end hangs down over the edge of the table. When the spring has its natural length, the free end of the rope hangs a distance a vertically below the level of the table top. The system is released from rest in this position. Show that the free end of the rope executes simple harmonic motion, and find its period and amplitude.

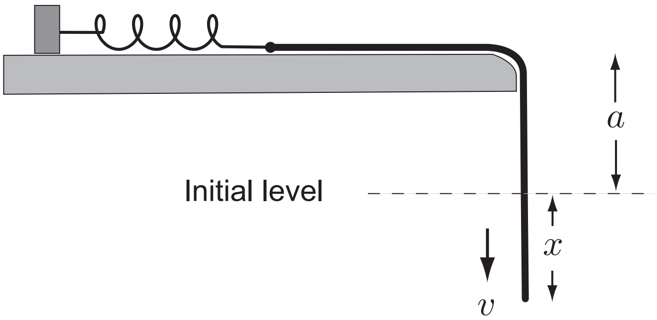

FIGURE 9.6 The rope and the spring.

Solution

Let x be the downward displacement of the free end of the rope from its initial position (see Figure 9.6) and let v $(= \dot{x})$ be its velocity.

Since every particle of the rope has the same speed, the **kinetic energy** of the rope is simply

$$T = \tfrac{1}{2}Mv^2.$$

In the displaced configuration, the potential energy of the spring is

$$V^S = \tfrac{1}{2}\left(\frac{\frac{1}{2}Mg}{a}\right)x^2 = \frac{Mgx^2}{4a},$$

while the gravitational potential energy of the rope (relative to the initial configuration) is

$$V^G = -\left(\frac{Mx}{4a}\right)g\left(a + \tfrac{1}{2}x\right) = -\frac{Mgx}{8a}(2a + x).$$

The total **potential energy** of the system is therefore

$$\begin{aligned} V &= V^S + V^G \\ &= \frac{Mgx}{8a}(x - 2a). \end{aligned}$$

The **energy conservation** equation for the rope then has the form

$$\tfrac{1}{2}Mv^2 + \left(\frac{Mg}{8a}\right) x(x - 2a) = E.$$

The initial conditions $x = 0$ and $v = 0$ when $t = 0$ imply that $E = 0$ and so the final form of the energy conservation equation is

$$v^2 = \left(\frac{g}{4a}\right) x(2a - x).$$

This gives the **speed** of the rope when its displacement is x.

On differentiating this formula with respect to t (and cancelling by v) we find that

$$\frac{dv}{dt} = \frac{g}{4a}(a - x).$$

Hence the **equation of motion** for the displacement x can be written in the form

$$\frac{d^2}{dt^2}(x - a) + \frac{g}{4a}(x - a) = 0.$$

Thus the free end of the rope performs simple harmonic oscillations about the point $x = a$. The **period** τ of these oscillations is

$$\tau = 4\pi \left(\frac{a}{g}\right)^{1/2}$$

and, since $v = 0$ when $x = 0$, the **amplitude** of the oscillations is a. ■

Problem 9.10

A circular hoop is rolling with speed v along level ground when it encounters a slope leading to more level ground, as shown in book Figure 9.14. If the hoop loses altitude h in the process, find its final speed.

Solution

In the **initial state**, the **kinetic energy** of the hoop is

$$T^I = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(Ma^2\right)\left(\frac{v}{a}\right)^2 = Mv^2,$$

where M is the mass of the hoop. The gravitational **potential energy** (relative to the *lower* level ground) is

$$V^I = Mg(h+a).$$

The corresponding values in the **final state** are

$$T^F = MV^2 \qquad V^F = Mga,$$

where V is the final speed of the hoop.

Energy conservation then requires that

$$MV^2 + Mga = Mv^2 + Mg(h+a).$$

Hence, the **final speed** of the hoop is

$$V = \left(v^2 + gh\right)^{1/2}. \blacksquare$$

Problem 9.11

A uniform ball is rolling in a straight line down a *rough* plane inclined at an angle α to the horizontal. Assuming the ball to be in planar motion, find the energy conservation equation for the ball. Deduce the acceleration of the ball.

Solution

Let x be the displacement of the ball down the plane (measured from some reference configuration) and let v $(= \dot{x})$ be its velocity.

Then the **kinetic energy** of the ball is

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}Ma^2\right)\left(\frac{v}{a}\right)^2 = \tfrac{7}{10}Mv^2,$$

where M is the mass of the ball. The gravitational **potential energy** of the ball (relative to its initial configuration) is

$$V = -Mgx\sin\alpha.$$

The **energy conservation** equation then has the form

$$\tfrac{7}{10}Mv^2 - Mgx\sin\alpha = E,$$

where E is the constant total energy.

On differentiating this formula with respect to t (and cancelling by v) we find that the **acceleration** of the ball down the plane is given by

$$\frac{dv}{dt} = \tfrac{5}{7}g\sin\alpha. \blacksquare$$

Problem 9.12

A uniform circular cylinder (a yo-yo) has a light inextensible string wrapped around it so that it does not slip. The free end of the string is secured to a fixed point and the yo-yo descends in a vertical straight line with the straight part of the string also vertical. Explain why the string does no work on the yo-yo. Find the energy conservation equation for the yo-yo and deduce its acceleration.

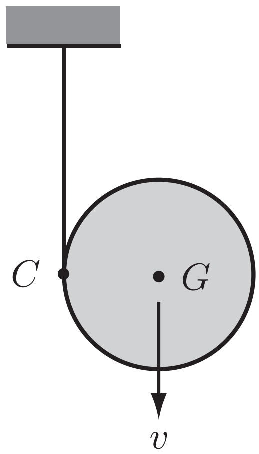

FIGURE 9.7 The yo-yo in vertical motion.

Solution

The string does no work on the yo-yo because (i) the support is fixed, (ii) there is no slippage between the string and the yo-yo.

Let z be the *downward* displacement of the yo-yo (measured from the support) and let v $(= \dot{z})$ be its velocity.

Then the **kinetic energy** of the yo-yo is

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}Ma^2\right)\left(\frac{v}{a}\right)^2 = \tfrac{3}{4}Mv^2,$$

where M is the mass of the yo-yo. The gravitational **potential energy** of the yo-yo (relative to the support) is

$$V = -Mgz.$$

The **energy equation** for the yo-yo then has the form

$$\tfrac{3}{4}Mv^2 - Mgz = E,$$

where E is the constant total energy.

On differentiating this formula with respect to t (and cancelling by v) we find that the downward **acceleration** of the yo-yo is given by

$$\frac{dv}{dt} = \tfrac{2}{3}g. \blacksquare$$

Problem 9.13

Book Figure 9.15 shows a partially unrolled roll of paper on a horizontal floor. Initially the paper on the roll has radius a and the free paper is laid out in a straight line on the floor. The roll is then projected horizontally with speed V in such a way that the free paper is gathered up on to the roll. Find the speed of the roll when its radius has increased to b. [Neglect the bending stiffness of the paper.] Deduce that the radius of the roll when it comes to rest is

$$a\left(\frac{3V^2}{4ga}+1\right)^{1/3}.$$

Solution

In the **initial state**, the **kinetic energy** of the paper is

$$T^I = \tfrac{1}{2}mV^2 + \tfrac{1}{2}\left(\tfrac{1}{2}ma^2\right)\left(\frac{V}{a}\right)^2 = \tfrac{3}{4}mV^2,$$

where m is the mass of the roll when it has radius a. The gravitational **potential energy** of the paper (relative to the ground) is

$$V^I = mga.$$

The corresponding values in the **final state** are

$$T^F = \tfrac{3}{4}Mv^2 \qquad V^F = Mgb,$$

where M is the mass and v is the speed of the roll when its radius is b.

Energy conservation then requires that

$$\tfrac{3}{4}mV^2 + mga = \tfrac{3}{4}Mv^2 + Mgb.$$

Hence, the **speed** of the roll when its radius has increased to b is given by

$$\begin{aligned} v^2 &= \frac{m}{M}\left(V^2 + \tfrac{4}{3}ga\right) - \tfrac{4}{3}gb \\ &= \frac{a^2}{b^2}\left(V^2 + \tfrac{4}{3}ga\right) - \tfrac{4}{3}gb \\ &= \frac{a^2V^2}{b^2} - \frac{4}{3}\left(\frac{b^3-a^3}{b^2}\right)g, \end{aligned}$$

on making use of the fact that $m/M = a^2/b^2$.

When the roll comes to rest, $v = 0$ and so the final radius R must satisfy the equation

$$\frac{a^2V^2}{R^2} - \frac{4}{3}\left(\frac{R^3 - a^3}{R^2}\right)g = 0.$$

On solving, we find that the **final radius** of the roll is

$$R = a\left(1 + \frac{3V^2}{4ag}\right)^{1/3}. \blacksquare$$

Problem 9.14

A rigid body of general shape has mass M and can rotate freely about a fixed horizontal axis. The centre of mass of the body is distance h from the rotation axis, and the moment of inertia of the body about the rotation axis is I. Show that the period of small oscillations of the body about the downward equilibrium position is

$$2\pi \left(\frac{I}{Mgh}\right)^{1/2}.$$

Deduce the period of small oscillations of a uniform rod of length $2a$, pivoted about a horizontal axis perpendicular to the rod and distance b from its centre.

Solution

Let θ be the angular displacement of the body from the downward equilibrium position and let ω $(= \dot{\theta})$ be its angular velocity.

Then the **kinetic energy** of the body is

$$T = \tfrac{1}{2} I \omega^2$$

where I is the moment of inertia of the body about its rotation axis. The gravitational **potential energy** of the body (relative to the axis level) is

$$V = -Mgh \cos\theta,$$

where M is the mass of the the body.

The **energy equation** for the body then has the form

$$\tfrac{1}{2} I \omega^2 - Mgh \cos\theta = E,$$

where E is the constant total energy.

On differentiating this formula with respect to t (and cancelling by ω) we find that the **equation of motion** for θ is

$$\ddot{\theta} + \left(\frac{Mgh}{I}\right) \sin\theta = 0.$$

This is the exact equation of motion for large oscillations. The **linearised equation** for small oscillations is

$$\ddot{\theta} + \left(\frac{Mgh}{I}\right) \theta = 0,$$

which is the SHM equation. The **period** τ of small oacillations is therefore

$$\tau = 2\pi \left(\frac{I}{Mgh} \right)^{1/2}.$$

For the special case of the rod, $h = b$ and $I = \frac{1}{3}Ma^2 + Mb^2$. In this case, the period of small oscillations is

$$2\pi \left(\frac{a^2 + 3b^2}{3gb} \right)^{1/2}. \blacksquare$$

Problem 9.15

A uniform ball of radius a can roll without slipping on the *outside* surface of a fixed sphere of (outer) radius b and centre O. Initially the ball is at rest at the highest point of the sphere when it is slightly disturbed. Find the speed of the centre G of the ball in terms of the variable θ, the angle between the line OG and the upward vertical. [Assume planar motion.]

Solution

Let θ be the angle between OG and the *upward* vertical and let v $(= (a+b)\dot{\theta})$ be the velocity of G .

Then the **kinetic energy** of the ball is

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}Ma^2\right)\left(\frac{v}{a}\right)^2 = \tfrac{7}{10}Mv^2,$$

where M is the mass of the ball.

The gravitational **potential energy** of the ball (relative to the level of O) is

$$V = Mg(a+b)\cos\theta.$$

The **energy conservation** equation for the ball then has the form

$$\tfrac{7}{10}Mv^2 + Mg(a+b)\cos\theta = E,$$

where E is the constant total energy. The initial conditions $\theta = 0$ and $v = 0$ when $t = 0$ imply that $E = Mg(a+b)$ and the final form of the energy conservation equation is

$$v^2 = \frac{10g(a+b)}{7}(1-\cos\theta).$$

This gives the **speed** of the ball when its angular displacement is θ. ■

Problem 9.16

A uniform ball of radius a and centre G can roll without slipping on the *inside* surface of a fixed hollow sphere of (inner) radius b and centre O. The ball undergoes planar motion in a vertical plane through O. Find the energy conservation equation for the ball in terms of the variable θ, the angle between the line OG and the downward vertical. Deduce the period of small oscillations of the ball about the equilibrium position.

Solution

Let θ be the angle between OG and the *downward* vertical and let $v\ (= (b-a)\dot{\theta})$ be the velocity of G.

Then the **kinetic energy** of the ball is

$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}Ma^2\right)\left(\frac{v}{a}\right)^2 = \tfrac{7}{10}Mv^2,$$

where M is the mass of the ball.

The gravitational **potential energy** of the ball (relative to the level of O) is

$$V = -Mg(b-a)\cos\theta.$$

The **energy conservation** equation for the ball then has the form

$$\tfrac{7}{10}Mv^2 - Mg(b-a)\cos\theta = E,$$

where E is the constant total energy.

If we now differentiate this formula with respect to t (and cancel by $\dot{\theta}$) we find that the **equation of motion** for θ is

$$\ddot{\theta} + \left(\frac{5g}{7(b-a)}\right)\sin\theta = 0.$$

This is the exact equation for large motions of the ball. The **linearised equation** for small motions is

$$\ddot{\theta} + \left(\frac{5g}{7(b-a)}\right)\theta = 0,$$

which is the SHM equation. The **period** τ of small oscillations of the ball is therefore

$$\tau = 2\pi\left(\frac{7(b-a)}{5g}\right)^{1/2}. \blacksquare$$

Problem 9.17 ✻

Figure 9.6 shows a uniform thin rigid plank of length $2b$ which can roll without slipping on top of a rough circular log of radius a. The plank is initially in equilibrium, resting symmetrically on top of the log, when it is slightly disturbed. Find the period of small oscillations of the plank.

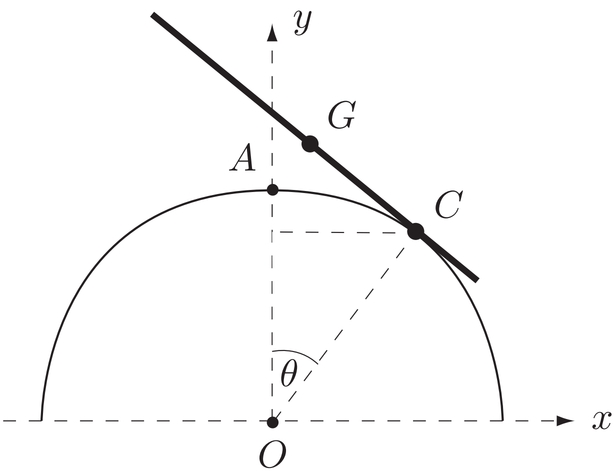

FIGURE 9.8 Plank rolling on a log.

Solution

Let C be the point of contact between the plank and the log (see Figure 9.8) and let θ be the angle between OC and the upward vertical OA; then θ is also the inclination of the plank to the horizontal. Note also that, since the plank rolls on the log, the length GC is equal to the length of the circular arc AC.

Let G have coordinates (X, Z) in the Cartesian coordinate system Oxy shown in Figure 9.8. Then

$$\begin{aligned} X &= a\sin\theta - (a\theta)\cos\theta, \\ &= a\big(\sin\theta - \theta\cos\theta\big) \end{aligned}$$

and

$$\begin{aligned} Z &= a\cos\theta + (a\theta)\sin\theta \\ &= a\big(\cos\theta + \theta\sin\theta\big). \end{aligned}$$

Hence

$$\dot{X} = a(\theta \sin\theta)\dot{\theta}$$
$$\dot{Z} = a(\theta \cos\theta)\dot{\theta}.$$

We can now calculate T and V for the plank in terms of the coordinate θ. The **kinetic energy** of the plank is

$$\begin{aligned} T &= \tfrac{1}{2}M\left(\dot{X}^2 + \dot{Z}^2\right) + \tfrac{1}{2}\left(\tfrac{1}{3}Mb^2\right)\dot{\theta}^2 \\ &= \tfrac{1}{2}Ma^2\theta^2\dot{\theta}^2 + \tfrac{1}{6}Mb^2\dot{\theta}^2, \end{aligned}$$

and the gravitational **potential energy** (relative to the level of O) is

$$\begin{aligned} V &= MgZ \\ &= Mga\big(\cos\theta + \theta\sin\theta\big). \end{aligned}$$

The **energy conservation** equation for the plank thus has the form

$$\tfrac{1}{2}Ma^2\theta^2\dot{\theta}^2 + \tfrac{1}{6}Mb^2\dot{\theta}^2 + Mga\big(\cos\theta + \theta\sin\theta\big) = E,$$

where E is the constant total energy.

If we now differentiate this equation with respect to t (and cancel by $\dot{\theta}$), we find that the **equation of motion** for θ is

$$\left(a^2\theta^2 + \tfrac{1}{3}b^2\right)\ddot{\theta} + \left(a^2\theta\right)\dot{\theta}^2 + ga\theta\cos\theta = 0.$$

This is the exact equation for large oscillations of the plank. The **linearised equation** for small oscillations is

$$\ddot{\theta} + \left(\frac{3ga}{b^2}\right)\theta = 0,$$

which is the SHM equation. The **period** τ of small oscillations of the plank is therefore

$$\tau = 2\pi\left(\frac{b^2}{3ga}\right)^{1/2}. \blacksquare$$

Chapter Ten

The linear momentum principle

and linear momentum conservation

Problem 10 . 1

Show that, if a system moves from one state of rest to another over a certain time interval, then the average of the total external force over this time interval must be zero.

An hourglass of mass M stands on a fixed platform which also measures the apparent weight of the hourglass. The sand is at rest in the upper chamber when, at time $t = 0$, a tiny disturbance causes the sand to start running through. The sand comes to rest in the lower chamber after a time $t = \tau$. Find the time average of the apparent weight of the hourglass over the time interval $[0, \tau]$. [The apparent weight of the hourglass is however *not constant* in time. One can advance an argument that, when the sand is steadily running through, the apparent weight of the hourglass *exceeds* the real weight!]

Solution

Let $\boldsymbol{P}$ be the linear momentum of the system and $\boldsymbol{F}$ the total external force acting on it. Then, by the **linear momentum principle**,

$$\dot{\boldsymbol{P}} = \boldsymbol{F}$$

and, for any time interval $0 \le t \le \tau$,

$$\int_0^\tau \boldsymbol{F}\, dt = \boldsymbol{P}(\tau) - \boldsymbol{P}(0).$$

In particular, if the system moves from one state of rest to another in the time interval $0 \le t \le \tau$, then $\boldsymbol{P}(0) = \boldsymbol{P}(\tau) = \boldsymbol{0}$ and

$$\frac{1}{\tau}\int_0^\tau \boldsymbol{F}\, dt = \boldsymbol{0},$$

that is, the *mean value of* $\boldsymbol{F}$ *over the time interval* $0 \le t \le \tau$ *must be zero.*

Suppose the hourglass is supported by a fixed platform that measures the upthrust $\boldsymbol{X}(t)$ that it applies to the hourglass. Then $-\boldsymbol{X}$ is the apparent weight of the hourglass. In this problem, $\boldsymbol{F} = \boldsymbol{X} - Mg\boldsymbol{k}$, where M is the total mass of the hourglass and its contents, and $\boldsymbol{k}$ is the unit vector pointing vertically upwards. Since this system moves between two states of rest, it follows that

$$\frac{1}{\tau}\int_0^\tau (\boldsymbol{X} - Mg\boldsymbol{k})\, dt = \boldsymbol{0},$$

that is,

$$\frac{1}{\tau}\int_0^\tau (-\boldsymbol{X})\, dt = -Mg\boldsymbol{k}.$$

Hence the *mean value* of the **apparent weight** of the hourglass is the same as its static weight. ■

Problem 10.2

Show that, if a system moves periodically, then the average of the total external force over a period of the motion must be zero.

A juggler juggles four balls of masses M, $2M$,$3M$ and $4M$ in a periodic manner. Find the time average (over a period) of the total force he applies to the balls. The juggler wishes to cross a shaky bridge that cannot support the combined weight of the juggler and his balls. Would it help if he juggles his balls while he crosses?

Solution

Let $\boldsymbol{P}$ be the linear momentum of the system and $\boldsymbol{F}$ the total external force acting on it. Then, by the **linear momentum principle**,

$$\dot{\boldsymbol{P}} = \boldsymbol{F}$$

and, for any time interval $0 \leq t \leq \tau$,

$$\int_0^\tau \boldsymbol{F}\, dt = \boldsymbol{P}(\tau) - \boldsymbol{P}(0).$$

In particular, if the system moves periodically with period τ, then $\boldsymbol{P}(0) = \boldsymbol{P}(\tau)$ and

$$\frac{1}{\tau}\int_0^\tau \boldsymbol{F}\, dt = \boldsymbol{0},$$

that is, the *mean value of* $\boldsymbol{F}$ *over a period of the motion must be zero*.

For simplicity, suppose that the juggler walks over the bridge with constant velocity and that we observe the motion from an inertial reference frame moving with this velocity. Then

$$\boldsymbol{F} = \boldsymbol{X} - (m + 10M)g\boldsymbol{k},$$

where $\boldsymbol{X}(t)$ is the upthrust that the bridge applies to the juggler at time t, m is the mass of the juggler, and $\boldsymbol{k}$ is the unit vector pointing vertically upwards.

In the new reference frame, the system moves periodically (this is what jugglers do) and it follows that

$$\frac{1}{\tau}\int_0^\tau \big(\boldsymbol{X} - (m + 10M)g\boldsymbol{k}\big)\, dt = \boldsymbol{0},$$

that is,

$$\frac{1}{\tau}\int_0^\tau \boldsymbol{X}\, dt = (m + 10M)g\boldsymbol{k}.$$

Hence, by the Third Law, the *mean value* of the **total force** that the juggler applies to the bridge is simply equal to his own weight plus the combined weight of the balls. Hence, *averaged over a juggling period, there is nothing to be gained by juggling the balls*. In fact, since $\boldsymbol{X}(t)$ is not a constant, there must be times when its instantaneous value is *greater* than its mean value, which makes juggling worse than simply carrying the balls across. [He could have carried the balls across one at a time, but he never thought of that!] ■

Problem 10.3 ✻

A boat of mass M is at rest in still water and a man of mass m is sitting at the bow. The man stands up, walks to the stern of the boat and then sits down again. If the water offers a resistance to the motion of the boat proportional to the velocity of the boat, show that the boat will *eventually* come to rest at its orginal position. [This remarkable result is independent of the resistance constant and the details of the man's motion.]

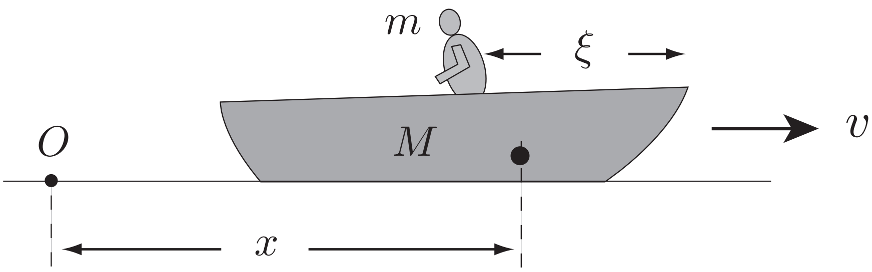

FIGURE 10.1 Man walking on a boat.

Solution

Let x be the displacement of the boat at time t, and v $(= \dot{x})$ be its velocity. Let ξ be the diplacement of the man *relative to the boat* at time t, measured in the negative x direction (see Figure 10.1). Then the true velocity of the man (in the positive x direction) is $v - \dot{\xi}$.

The **linear momentum** of the system in the positive x-direction is therefore

$$P = Mv + m(v - \dot{\xi}) = (M + m)v - m\dot{\xi}.$$

The only horizontal **force** acting on the system is the resistance force R exerted by the water. Since the resistance is known to be linear, R has the form

$$R = -(M + m)Kv,$$

where K is a constant; the factor $M + m$ has been included purely for convenience. Then, by the linear momentum principle, $\dot{P} = R$, which can be written in the form

$$\dot{v} + Kv = \left(\frac{m}{M + m}\right)\ddot{\xi}.$$

On integrating this equation with respect to t, we obtain

$$\dot{x} + Kx = \left(\frac{m}{M+m}\right)\dot{\xi} + C,$$

where C is a constant of integration. Since the whole system starts from rest with $x = 0$, it follows that $C = 0$ and we obtain

$$\dot{x} + Kx = \left(\frac{m}{M+m}\right)\dot{\xi}.$$

This is the **equation of motion** satisfied by x. The function $\xi(t)$ (the motion of the man) is supposed to be known.

This is a first order linear ODE with integrating factor e^{Kt}. On solving, we find that

$$x = \left(\frac{m}{M+m}\right)e^{-Kt}\int_0^t \dot{\xi}(t')e^{Kt'}\,dt' + De^{-kt},$$

where D is a second constant of integration. The initial condition $x = 0$ when $t = 0$ implies that $D = 0$ and so the **displacement** x of the boat at time t is

$$x = \left(\frac{m}{M+m}\right)e^{-Kt}\int_0^t \dot{\xi}(t')e^{Kt'}\,dt'.$$

We now wish to show that the boat eventually regains its original position. Let us suppose that the man has taken his seat at the back of the boat by time τ. Then, for $t \geq \tau$, $\dot{\xi} = 0$ and the integral can be restricted to the range $0 \leq t' \leq \tau$. The solution for x for $t > \tau$ can therefore be written

$$\begin{aligned} x &= \left(\frac{m}{M+m}\right)e^{-Kt}\int_0^\tau \dot{\xi}(t')e^{Kt'}\,dt' \\ &= \left(\frac{m}{M+m}\int_0^\tau \dot{\xi}(t')e^{Kt'}\,dt'\right)e^{-Kt}. \end{aligned}$$

The expression in the brackets looks complicated but, since the limits of integration are now constants, it is simply a *constant* x_0, say. Hence, for $t \geq \tau$, the solution for x has the form

$$x = x_0\,e^{-Kt}.$$

This tends to zero as t tends to infinity, which is the required result. ∎

Problem 10.4

A uniform rope of mass M and length a is held at rest with its two ends close together and the rope hanging symmetrically below. (In this position, the rope has two long vertical segments connected by a small curved segment at the bottom.) One of the ends is then released. It can be shown by energy conservation (see Problem 9.8) that the velocity of the free end when it has descended by a distance x is given by

$$v^2 = \left(\frac{x(2a-x)}{a-x}\right) g.$$

Find the reaction R exerted by the support at the *fixed* end when the free end has descended a distance x. The support will collapse if R exceeds $\frac{3}{2}Mg$. Find how far the free end will fall before this happens.

Solution

The motion of the rope in this problem was found in Solution 9.8 by energy methods. We will make use of the notation and results from this solution.

The downwards **linear momentum** P of the rope is

$$P = 0 + \left(\frac{My}{a}\right) v = \frac{M}{2a}(a-x)v,$$

and total downwards force F is

$$F = Mg - R,$$

where R is the reaction of the support. The **linear momentum principle** $\dot{P} = F$ then implies that

$$\frac{M}{2a}\frac{d}{dt}\big((a-x)v\big) = Mg - R$$

which gives

$$R = Mg - \frac{M}{2a}\left((a-x)\dot{v} - v^2\right).$$

If we now make use of the formulae

$$v^2 = \frac{x(2a-x)g}{a-x}, \qquad \dot{v} = \left(\frac{2a^2 - 2ax + x^2}{2(a-x)^2}\right) g$$

that were obtained in Solution 9.8, we find that

$$R = \frac{Mg}{4a}\left(\frac{2a^2 + 2ax - 3x^2}{a - x}\right),$$

after some simplification. This is the **reaction** exerted by the support at the fixed end of the rope when the free end has descended a distance x.

This reaction will be equal to $\frac{3}{2}Mg$ when

$$\frac{Mg}{4a}\left(\frac{2a^2 + 2ax - 3x^2}{a - x}\right) = \tfrac{3}{2}Mg,$$

a condition which reduces to the quadratic equation

$$3x^2 - 8ax + 4a^2 = 0.$$

The solutions are $x = \frac{2}{3}a$ and $x = 2a$. Since the solution $x = 2a$ lies outside the physical range $0 \le x \le a$, the **support will collapse** when $x = \frac{2}{3}a$. ■

Problem 10.5

A fine uniform chain of mass M and length a is held at rest hanging vertically downwards with its lower end just touching a fixed horizontal table. The chain is then released. Show that, while the chain is falling, the force that the chain exerts on the table is always *three times* the weight of chain actually lying on the table. [Assume that, before hitting the table, the chain falls freely under gravity.]

✱ When all the chain has landed on the table, the loose end is pulled upwards with the constant force $\frac{1}{3}Mg$. Find the height to which the chain will first rise. [This time, assume that the force exerted on the chain by the table is *equal* to the weight of chain lying on the table.]

Solution

Let x be the downward displacement of the top end of the chain and $v\ (= \dot{x})$ its velocity. The mass of the vertical part of the chain is $M(a - x)/a$.

Then the downward **linear momentum** P of the chain is

$$P = \frac{M}{a}(a - x)v$$

and the total downwards **force** F acting on the chain is

$$F = Mg - R,$$

where R is the reaction of the table. The **linear momentum principle** $\dot{P} = F$ then implies that

$$\frac{M}{a}\frac{d}{dt}\big((a - x)v\big) = Mg - R$$

which gives

$$R = Mg - \frac{M}{a}\left((a - x)\dot{v} - v^2\right).$$

Since the chain is assumed to be falling freely under gravity, $\dot{v} = g$ and $v^2 = 2gx$, from which it follows that

$$R = 3\left(\frac{Mx}{a}\right)g.$$

This is the **reaction** of the table when the chain has fallen by a distance x. By the Third Law, it is equal to the force that the chain exerts on the table. Thus, the *force* *that the chain exerts on the table is always three times the weight of chain lying on the table*.

When the chain is being pulled up, let x be the height of the top end of the chain above the table and v $(= \dot{x})$ its upwards velocity. The mass of the vertical part of the chain is Mx/a.

Then the upward **linear momentum** P of the chain is

$$P = \frac{M}{a} x v$$

and the total upwards **force** F acting on the chain is

$$F = \tfrac{1}{3}Mg + R - Mg,$$

where R is the reaction of the table. The **linear momentum principle** $\dot{P} = F$ then implies that

$$\frac{M}{a}\frac{d}{dt}\big(x v\big) = R - \tfrac{2}{3}Mg,$$

that is,

$$\frac{M}{a}\left(v^2 + x\dot{v}\right) = R - \tfrac{2}{3}Mg.$$

In the *upwards* motion, we assume that the force exerted on the chain by the table is *equal* to the weight of chain lying on the table, that is,

$$R = \left(\frac{M(a-x)}{a}\right) g.$$

We then obtain

$$x\dot{v} + v^2 = \tfrac{1}{3}g(a - 3x),$$

which is the **equation of motion** for the chain.

To solve this equation, write $\dot{v} = v\,dv/dx$ and introduce the new dependent variable $w = v^2$. The equation for w is then

$$x\frac{dw}{dx} + 2w = \tfrac{2}{3}g(a - 3x).$$

This is a first order linear ODE with integrating factor x. On solving, we find that

$$v^2 = \tfrac{1}{3}g(a - 2x) + \frac{C}{x^2},$$

where C is a constant of integration.

A curious feature of this problem is that the initial conditions $x = 0$ and $v = 0$ when $t = 0$ cannot be satisfied. The easiest way to make sense of this is to suppose that the motion starts from rest with $x = b$ (instead of $x = 0$), find the solution, and then let $b \to 0$. The solution obtained turns out to be the same as putting $C = 0$ in the above expression. Hence the **velocity** of the chain when its upward displacement is x is given by

$$v^2 = \tfrac{1}{3} g(a - 2x).$$

The **chain** first **comes to rest** when $v = 0$, that is when $x = \frac{1}{2}a$. ■

Problem 10.6

A uniform ball of mass M and radius a can roll without slipping on the rough outer surface of a fixed sphere of radius b and centre O. Initially the ball is at rest at the highest point of the sphere when it is slightly disturbed. Find the speed of the centre G of the ball in terms of the variable θ, the angle between the line OG and the upward vertical. [Assume planar motion.] Show that the ball will leave the sphere when $\cos\theta = \frac{10}{17}$.

Solution

The motion of the ball in this problem was found in Solution 9.15 by energy methods. We will make use of the notation and results from this solution.

On making use of the centre of mass form of the **linear momentum principle** (resolved in the direction $\overrightarrow{GO}$), we obtain

$$Mg\cos\theta - R = M\left(\frac{v^2}{a+b}\right),$$

where R is the *normal* component of the reaction exerted on the ball by the fixed sphere. If we now make use of the formula

$$v^2 = \frac{10g(a+b)}{7}$$

that was obtained in Solution 9.15, we find that

$$R = \frac{Mg}{7}\left(17\cos\theta - 10\right).$$

The ball will leave the sphere when $R = 0$, that is, when

$$\cos\theta = \tfrac{10}{17},$$

that is, when $\theta = 54°$ approximately. ■

Problem 10.7

A rocket of initial mass M, of which $M - m$ is fuel, burns its fuel at a constant rate in time τ and ejects the exhaust gases with constant speed u. The rocket starts from rest and moves vertically under uniform gravity. Show that the maximum speed acheived by the rocket is $u \ln \gamma - g\tau$ and that its height at burnout is

$$u\tau\left(1 - \frac{\ln\gamma}{\gamma - 1}\right) - \tfrac{1}{2}g\tau^2,$$

where $\gamma = M/m$. [Assume that the thrust is such that the rocket takes off immediately.]

Solution

Let v be the upward velocity of the rocket at time t. Then, from the text p. 255, the solution for v is given by

$$v = u\ln\left(\frac{m(0)}{m(t)}\right) - gt,$$

where $m(t)$ is the mass of the rocket and the remaining fuel at time t. In the present problem, $m(0) = M$ and

$$m(t) = M - \frac{t}{\tau}\big(M - m(\tau)\big)$$

so that

$$\frac{m(t)}{m(0)} = 1 - \left(\frac{\gamma - 1}{\gamma\tau}\right)t,$$

where $\gamma = M/m(\tau)$. [Note that the question uses the symbol m for $m(\tau)$, but, in order to avoid confusion with the previous usage of $m(t)$, we will not use this in the solution.]

Hence the **velocity** of the rocket at time t into the burn is

$$v = -u\ln(1 - kt) - gt,$$

where

$$k = \frac{\gamma - 1}{\gamma\tau}.$$

In particular, $v_{\max}$, the **velocity at burnout**, is given by

$$v_{\max} = u \ln \gamma - g\tau.$$

The height z achieved by the rocket at time t satisfies the equation

$$\frac{dz}{dt} = -u \ln(1 - kt) - gt,$$

with the initial condition $z = 0$ when $t = 0$. The solution is messy but straightforward and gives

$$z = -ut \ln(1 - kt) + \frac{u}{k} \ln(1 - kt) + ut - \tfrac{1}{2}gt^2.$$

This is the **height** of the rocket at time t into the burn. In particular, h, the **height at burnout** is given by

$$h = u\tau \left(1 - \frac{\ln \gamma}{\gamma - 1}\right) - \tfrac{1}{2}g\tau^2. \blacksquare$$

Problem 10.8 *Saturn V rocket*

In first stage of the Saturn V rocket, the initial mass was 2.8×10^6 kg, of which 2.1×10^6 kg was fuel. The fuel was burned at a constant rate over 150 s and the exhaust speed was $2,600 \text{ m s}^{-1}$. Use the results of the last problem to find the speed and height of the Saturn V at first stage burnout. [Take g to be constant at 9.8 m s^{-2} and neglect air resistance.]

Solution

This is a numerical application of the results of Problem 10.7. For the Saturn V rocket, $\gamma = 4$, $\tau = 150$ s, $u = 2,600 \text{ m s}^{-1}$ and $g = 9.8 \text{ m s}^{-2}$. The calculated values of v_{max} and h are

$$v_{\text{max}} = 2,100 \text{ m s}^{-1}, \qquad h = 100 \text{ km},$$

approximately. ∎

Problem 10.9 *Rocket in resisting medium*

A rocket of initial mass M, of which $M - m$ is fuel, burns its fuel at a constant rate k and ejects the exhaust gases with constant speed u. The rocket starts from rest and moves through a medium that exerts the resistance force $-\epsilon k v$, where v is the forward velocity of the rocket, and ϵ is a small positive constant. Gravity is absent. Find the maximum speed V achieved by the rocket. Deduce a two term approximation for V, valid when ϵ is small.

Solution

Let v be the velocity of the rocket at time t. Then, on incorporating the resistance force $-\epsilon k v$ into the rocket equation on p. 253 of the text, the **equation of motion** for v is

$$m\frac{dv}{dt} = -\dot{m}u - \epsilon k v,$$

where m $(= m(t))$ is the mass of the rocket and its fuel at time t. In the present problem, $m(0) = M$ and $\dot{m} = -k$ so that

$$m(t) = M - kt.$$

The equation of motion therefore becomes

$$\frac{dv}{dt} + \left(\frac{\epsilon k}{M - kt}\right) v = \frac{ku}{M - kt}.$$

This is a first order linear ODE with integrating factor $(M - kt)^{-\epsilon}$. The general solution is

$$v = \frac{u}{\epsilon} + C(M - kt)^{\epsilon},$$

where C is a constant of integration. On applying the initial condition $v = 0$ when $t = 0$, we find that

$$C = -\frac{u}{\epsilon M^{\epsilon}}.$$

The **velocity** of the rocket at time t into the burn is therefore

$$v = \frac{u}{\epsilon}\left[1 - \left(\frac{M - kt}{M}\right)^{\epsilon}\right].$$

In particular, at **burnout**, $M - kt = m$ and the rocket velocity is

$$V = \frac{u}{\epsilon}\big(1 - \gamma^{-\epsilon}\big),$$

where $\gamma = M/m$. [Note that the symbol m used here *is* the same as the m used in the question. There is now little chance of confusion with the previous usage of $m(t)$.]

When ϵ is small,

$$\begin{aligned} \gamma^{-\epsilon} &= \exp\big(-\epsilon \ln \gamma\big) \\ &= 1 - \epsilon \ln \gamma + \tfrac{1}{2}(\epsilon \ln \gamma)^2 + O\left(\epsilon^3\right) \end{aligned}$$

and so the required **approximation** to V when ϵ is small is

$$V = u \ln \gamma \left[1 - \tfrac{1}{2} \ln \gamma\, \epsilon + O\left(\epsilon^2\right)\right]. \blacksquare$$

Problem 10 . 10 *Two-stage rocket*

A two-stage rocket has a first stage of initial mass M_1, of which $(1-\eta)M_1$ is fuel, a second stage of initial mass M_2, of which $(1-\eta)M_2$ is fuel, and an inert payload of mass m_0. In each stage, the exhaust gases are ejected with the same speed u. The rocket is initially at rest in free space. The first stage is fired and, on completion, the first stage carcass (of mass ηM_1) is discarded. The second stage is then fired. Find an expression for the final speed V of the rocket and deduce that V will be maximised when the mass ratio $\alpha = M_2/(M_1 + M_2)$ satisfies the equation

$$\alpha^2 + 2\beta\alpha - \beta = 0,$$

where $\beta = m_0/(M_1 + M_2)$. [Messy algebra.]

Show that, when β is small, the optimum value of α is approximatelely $\beta^{1/2}$ and the maximum velocity reached is approximately $2u \ln \gamma$, where $\gamma = 1/\eta$.

Solution

It follows from the formula (10.9) on p. 254 of the text that the rocket velocity when the **first stage** is completed is

$$v_1 = u \ln\left(\frac{M_1 + M_2 + m_0}{\eta M_1 + M_2 + m_0}\right),$$

and that the rocket velocity when the **second stage** is completed is

$$\begin{aligned} V &= v_1 + u \ln\left(\frac{M_2 + m_0}{\eta M_2 + m_0}\right) \\ &= u \ln\left(\frac{(M_1 + M_2 + m_0)(M_2 + m_0)}{(\eta M_1 + M_2 + m_0)(\eta M_2 + m_0)}\right) \\ &= u \ln\left(\frac{(1+\beta)(\alpha+\beta)}{(\eta + (1-\eta)\alpha + \beta)(\eta\alpha + \beta)}\right), \end{aligned}$$

where

$$\alpha = \frac{M_2}{M_1 + M_2}, \qquad \beta = \frac{m_0}{M_1 + M_2}.$$

We must now choose the mass ratio α so as to maximise V. The equation $dV/d\alpha = 0$ gives

$$-\frac{1-\eta}{\eta + (1-\eta)\alpha + \beta} + \frac{1}{\alpha + \beta} - \frac{\eta}{\eta\alpha + \beta} = 0,$$

which reduces to

$$\alpha^2 + 2\beta\alpha - \beta = 0$$

after much labour. On selecting the positive root, the **optimium value** of α is found to be

$$\alpha = -\beta + \left(\beta^2 + \beta\right)^{1/2}.$$

When β is small, the optimum value of α is approximately

$$\begin{aligned}
\alpha &= -\beta + \left(\beta^2 + \beta\right)^{1/2} \\
&= -\beta + \beta^{1/2}(1+\beta)^{1/2} \\
&= -\beta + \beta^{1/2}(1 + O(\beta)) \\
&= \beta^{1/2} + O(\beta).
\end{aligned}$$

Hence, when the mass ratio β is small, the optimum value of the mass ratio α is approximately $\beta^{1/2}$. In this limit, the final velocity achieved by the rocket is

$$\begin{aligned}
V &= u \ln\left(\frac{(1+\beta)\left(\beta^{1/2} + O(\beta)\right)}{\left((\eta + (1-\eta)\beta^{1/2} + O(\beta)\right)\left((\eta\beta^{1/2} + O(\beta)\right)}\right) \\
&= u \ln\left(\frac{\beta^{1/2} + O(\beta)}{\eta^2\beta^{1/2} + O(\beta)}\right) \\
&= u \ln\left(\frac{1}{\eta^2} + O(\beta^{1/2})\right) \\
&= u \ln\left(\gamma^2 + O(\beta^{1/2})\right), \\
&= 2u \ln\gamma\left(1 + O(\beta^{1/2})\right)
\end{aligned}$$

where $\gamma = 1/\eta$. Hence, when the mass ratio β is small and the mass ratio α takes its optimum value, the **maximum velocity** achieved by the rocket is approximately $2u \ln\gamma$.

Problem 10.11 ✻

A raindrop falls vertically through stationary mist, collecting mass as it falls. The raindrop remains spherical and the rate of mass accretion is proportional to its speed and the square of its radius. Show that, if the drop starts from rest with a negligible radius, then it has constant acceleration $g/7$. [Tricky ODE.]

Solution

Suppose that the drop has mass m and downward velocity v at time t. Then m and v satisfy the following two conditions:

(i) Since the mass gained by the drop is at rest, the **linear momentum** equation becomes

$$\frac{d}{dt}(mv) = mg.$$

(ii) The rate of **mass increase** is given to be

$$\frac{dm}{dt} = kr^2v,$$

where k is a constant and r is the radius of the drop at time t.

It is convenient to work with the radius of the drop rather than its mass. Since the mass is proportional to the cube of the radius, the above equations become

$$\frac{d}{dt}(r^3v) = r^3g, \qquad (1)$$

$$\frac{dr}{dt} = Kv, \qquad (2)$$

where K is a new constant. These are a pair of simultaneous first order ODEs for the unknown functions v and r.

The trick is to eliminate the time and to obtain a single ODE for v as a function of r. [In the language of dynamical systems, we are finding the phase paths of an autonomous system.] Now

$$\begin{aligned}\frac{d}{dt}(r^3v) &= \frac{d}{dr}(r^3v) \times \frac{dr}{dt}\\ &= \left(3r^2v + r^3\frac{dv}{dr}\right) \times \frac{dr}{dt}\\ &= Kv\left(3r^2v + r^3\frac{dv}{dr}\right),\end{aligned}$$

on making use of equation (2). On substituting this result into equation (1), we obtain

$$r v \frac{dv}{dr} + 3v^2 = \left(\frac{g}{K}\right) r.$$

This is the required ODE for v as a function of r. To solve, we introduce the new independent variable $w = v^2$. The equation for w is then

$$\frac{dw}{dr} + \left(\frac{6}{r}\right) w = \frac{2g}{K},$$

which is a first order linear ODE with integrating factor r^6. The general solution is

$$w = \left(\frac{2g}{7K}\right) r + \frac{C}{r^6},$$

where C is a constant of integration. The initial conditions $v = 0$ and $r = 0$ when $t = 0$ then imply that $C = 0$ so that the solution for v is

$$v^2 = \left(\frac{2g}{7K}\right) r.$$

This gives the **velocity** of the drop when its radius is r.

To find the **acceleration** of the drop, we differentiate this last equation with respect to t. This gives

$$\begin{aligned} 2v\frac{dv}{dt} &= \left(\frac{2g}{7K}\right) \frac{dr}{dt} \\ &= \left(\frac{2g}{7K}\right) K v, \end{aligned}$$

on using equation (2) again. Hence

$$\frac{dv}{dt} = \frac{2g}{7},$$

as is required. ■

Problem 10.12

A body of mass $4m$ is at rest when it explodes into *three* fragments of masses $2m$, m and m. After the explosion the two fragments of mass m are observed to be moving with the same speed in directions making 120° with each other. Find the proportion of the total kinetic energy carried by each fragment.

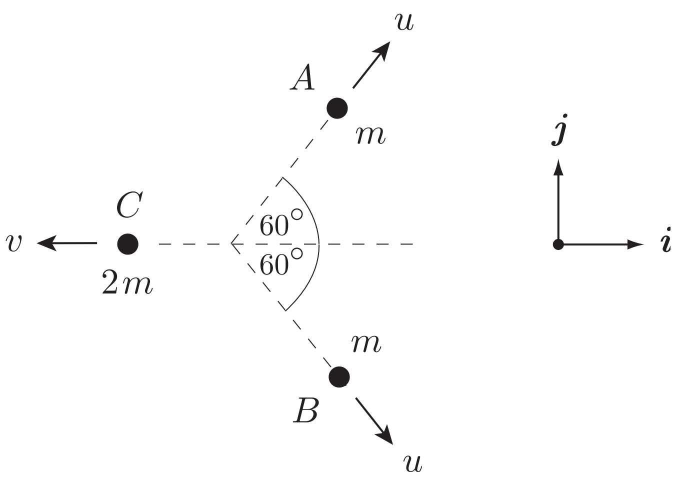

FIGURE 10.2 The three emerging fragments.

Solution

Since the explosion conserves linear momentum, and the body is initially at rest, the total **linear momentum** of the fragments must be zero. The three velocities must therefore be coplanar. Also, since the two fragments of mass m have equal speeds, the third velocity must lie along the bisector of the angle between their paths (see Figure 10.2).

Let the speed of the fragments A and B be u and the speed of fragment C be v. Then, since the total linear momentum in the $\boldsymbol{i}$-direction must be zero, we have

$$mu\cos 60^\circ + mu\cos 60^\circ - (2m)v = 0.$$

Hence, the **speed** of C is $v = \frac{1}{2}u$.

It follows that the three **kinetic energies** are

$$T^A = T^B = \tfrac{1}{2}mu^2,$$
$$T^C = \tfrac{1}{2}(2m)\left(\tfrac{1}{2}u\right)^2 = \tfrac{1}{4}mu^2.$$

The total kinetic energy is therefore $T = \frac{5}{4}mu^2$. Hence, the **proportions** of the total kinetic energy carried by each fragment are

$$\frac{T^A}{T} = \frac{2}{5}, \qquad \frac{T^B}{T} = \frac{2}{5}, \qquad \frac{T^C}{T} = \frac{1}{5}. \blacksquare$$

Problem 10.13

Show that, in an elastic head-on collision between two spheres, the relative velocity of the spheres after impact is the negative of the relative velocity before impact.

A tube is fixed in the vertical position with its lower end on a horizontal floor. A ball of mass M is released from rest at the top of the tube followed closely by a second ball of mass m. The first ball bounces off the floor and immediately collides with the second ball coming down. Assuming that both collisions are elastic, show that, when m/M is small, the second ball will be projected upwards to a height nearly nine times the length of the tube.

Solution

In a head-on collision between spheres, the motion must be entirely rectilinear. Suppose that the spheres have masses m_1, m_2, that their initial velocities are u_1, u_2, and that their final velocities are v_1, v_2. These velocities are all measured in the same direction along the line of motion.

Then conservation of **linear momentum** requires that

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2,$$

and conservation of **energy** requires that

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2.$$

We wish to show that $v_2 - v_1 = u_1 - u_2$. It is possible to grind this out directly, but the following argument is neater.

Let the collision be observed from a reference frame in which the velocity of m_1 is *reversed* by the collision, that is, $v_1 = -u_1$. Such a choice is always possible. [This reference frame is actually the ZM frame.] Then, in this reference frame, the energy equation becomes

$$u_2^2 = v_2^2$$

so that $v_2 = \pm u_2$. The linear momentum equation shows that the sign must be negative so that $v_2 = -u_2$. Then $v_2 - v_1 = (-u_2) - (-u_1) = u_1 - u_2$, as required.

Suppose the ball B_1 of mass M has speed v when it hits the floor. Since the collision with the floor is elastic, the ball will be reflected with initial upward speed v. The ball is then immediately in collision with the second ball B_2 (of mass m) that has downwards speed v. Suppose that, after this second collision, B_2 has upward speed V. Since the collision between the balls is elastic, the result obtained above applies and so the upward speed of B_1 must be $V - 2v$. Conservation of **linear**

momentum then implies that

$$Mv + m(-v) = M(V - 2v) + mV$$

from which it follows that

$$V = \left(\frac{3M - m}{M + m}\right) v.$$

This is the upward **velocity** of B_2 after its collision with B_1. B_2 will then rise to the **height**

$$H = \frac{V^2}{2g} = \left(\frac{3M - m}{M + m}\right)^2 \frac{v^2}{2g}$$
$$= \left(\frac{3M - m}{M + m}\right)^2 h,$$

where h is the height from which the balls were dropped. When the mass ratio m/M is small, this height is nearly $9h$. [This experiment makes quite a spectacular demonstration.] ■

Problem 10.14

Two particles with masses m_1, m_2 and velocities $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ collide and stick together. Find the velocity of this composite particle and show that the loss in kinetic energy due to the collision is

$$\frac{m_1 m_2}{2(m_1 + m_2)} \left|\boldsymbol{v}_1 - \boldsymbol{v}_2\right|^2 .$$

Solution

Let $\boldsymbol{V}$ be the velocity of the composite particle. Then, since **linear momentum** is conserved in the collision,

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = (m_1 + m_2)\boldsymbol{V}.$$

Hence the **velocity** of the composite particle is

$$\boldsymbol{V} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2}.$$

The loss in **kinetic energy** in the collision is therefore

$$\begin{aligned}
\Delta T &= \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 - \tfrac{1}{2}(m_1 + m_2)V^2 \\
&= \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 - \frac{\left|m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2\right|^2}{2(m_1 + m_2)} \\
&= \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 - \frac{m_1^2 v_1^2 + m_2^2 v_2^2 + 2m_1 m_2 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{2(m_1 + m_2)} \\
&= \frac{m_1 m_2}{2(m_1 + m_2)} \left(v_1^2 + v_2^2 - 2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2\right) \\
&= \frac{m_1 m_2}{2(m_1 + m_2)} \left|\boldsymbol{v}_1 - \boldsymbol{v}_2\right|^2 \quad \blacksquare
\end{aligned}$$

Problem 10 . 15

In an elastic collision between a proton moving with speed u and a helium nucleus at rest, the proton was scattered through an angle of 45°. What proportion of its initial energy did it lose? What was the recoil angle of the helium nucleus?

Solution

In the standard notation of the **elastic scattering formulae**, we are given that $m_1 = 1$, $m_2 = 4$ and $\theta_1 = 45°$. Then $\gamma = \frac{1}{4}$ and formula **A** gives

$$\frac{\sin\psi}{\cos\psi + \frac{1}{4}} = 1,$$

where ψ is the scattering angle in the ZM-frame. This equation for ψ can be written in the form

$$4\sqrt{2}\,\sin\left(\psi - \tfrac{\pi}{4}\right) = 1$$

and the solution is

$$\psi = \frac{\pi}{4} + \sin^{-1}\left(\frac{1}{4\sqrt{2}}\right) \approx 55°.$$

The proportion of **energy lost** by the proton is given by formula **D**:

$$\frac{E_2}{E_0} = \frac{4\gamma}{(\gamma+1)^2}\sin^2\left(\tfrac{1}{2}\psi\right) = \tfrac{16}{25}\sin^2\left(\tfrac{1}{2}\psi\right) \approx 14\%.$$

The **recoil angle** of the helium nucleus is given by formula **B**:

$$\theta_2 = \tfrac{1}{2}(\pi - \psi) \approx 62°. \blacksquare$$

Problem 10.16

In an elastic collision between an alpha particle and an unknown nucleus at rest, the alpha particle was deflected through a right angle and lost 40% of its energy. Identify the mystery nucleus.

Solution

In the standard notation of the **elastic scattering formulae**, we are given that

$$\theta_1 = 90^\circ, \qquad \frac{E_2}{E_0} = \frac{2}{5}, \qquad \gamma = \frac{4}{M},$$

where M is the mass of the mystery nucleus.

Formula **A** then gives

$$\cos\psi + \gamma = 0,$$

while formula **D** gives

$$\frac{4\gamma}{(\gamma+1)^2}\sin^2\left(\tfrac{1}{2}\psi\right) = \frac{2}{5},$$

which can be written in the form

$$\frac{2\gamma}{(\gamma+1)^2}(1-\cos\psi) = \frac{2}{5}.$$

The mass ratio γ therefore satisfies the equation

$$\frac{2\gamma}{(\gamma+1)^2}(1+\gamma) = \frac{2}{5},$$

which solves to give $\gamma = \frac{1}{4}$. Hence the **mystery nucleus** has mass 16 and must therefore be **oxygen**. ■

Problem 10.17 *Some inequalities in elastic collisions*

Use the elastic scattering formulae to show the following inequalities:

(i) When $m_1 > m_2$, the scattering angle θ_1 is restricted to the range $0 \le \theta_1 \le \sin^{-1}(m_2/m_1)$.

(ii) If $m_1 < m_2$, the opening angle is obtuse, while, if $m_1 > m_2$, the opening angle is acute.

(iii)

$$\frac{E_1}{E_0} \ge \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2, \qquad \frac{E_2}{E_0} \le \frac{4m_1 m_2}{(m_1 + m_2)^2}.$$

Solution

(i) From formula **A**, the **scattering angle** θ_1 is given by

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + \gamma},$$

where $\gamma = m_1/m_2$ and ψ is the scattering angle in the ZM-frame.
Let $F(\psi)$ be the function

$$F(\psi) = \frac{\sin\psi}{\cos\psi + \gamma}.$$

Then $\tan\theta_1$ lies between zero and the maximum value achieved by $F(\psi)$ for ψ in the interval $0 \le \psi \le \pi$. When the constant $\gamma > 1$, F is a continuous function of ψ and so the maximum certainly exists. The stationary points of F satisfy the equation $F'(\psi) = 0$, that is,

$$\frac{1 + \gamma\cos\psi}{(\cos\psi + \gamma)^2} = 0.$$

It follows that there is just **one stationary point** of F in the range $0 \le \psi \le \pi$ at

$$\psi = \cos^{-1}\left(-\frac{1}{\gamma}\right).$$

One could show that this sationary point is a *local* maximum of F by finding F'', but there is no need. The function F is zero at the end points $\psi = 0, \pi$

and is positive within the interval. It follows that the stationary point $\psi = \cos^{-1}(-1/\gamma)$ must give rise to $F^{\max}$, the **global maximum** of F. Hence

$$F^{\max} = \frac{\left(1-\gamma^{-2}\right)^{1/2}}{-\gamma^{-1}+\gamma} = (\gamma^2-1)^{-1/2}.$$

The maximum value of θ_1 is therefore

$$\theta_1^{\max} = \tan^{-1}\left(\gamma^2-1\right)^{-1/2}.$$

This is the answer, but it can be written more simply since

$$\operatorname{cosec}^2\theta_1^{\max} = 1+\cot^2\theta_1^{\max} = 1+\left(\gamma^2-1\right) = \gamma^2$$

and so $\sin\theta_1^{\max} = 1/\gamma = m_2/m_1$. We thus have the simpler formula

$$\theta_1^{\max} = \sin^{-1}\left(\frac{m_2}{m_1}\right).$$

The scattering angle therefore lies in the range $0 \le \theta_1 \le \sin^{-1}(m_2/m_1)$. ■

(ii) From formula **C**, the **opening angle** θ is given by

$$\tan\theta = \left(\frac{\gamma+1}{\gamma-1}\right)\cot\left(\tfrac{1}{2}\psi\right) = \begin{cases} >0 & \text{for} \quad \gamma > 1, \\ <0 & \text{for} \quad \gamma < 1. \end{cases}$$

Hence, the opening angle is acute for $m_1 > m_2$ and obtuse for $m_1 < m_2$. ■

(iii) From formula **D**,

$$\begin{aligned} \frac{E_1}{E_0} &= 1-\frac{E_2}{E_0} \\ &= 1-\frac{4\gamma}{(\gamma+1)^2}\sin^2\left(\tfrac{1}{2}\psi\right) \\ &\ge 1-\frac{4\gamma}{(\gamma+1)^2} = \frac{(\gamma-1)^2}{(\gamma+1)^2} \\ &= \left(\frac{m_1-m_2}{m_1+m_2}\right)^2. \ ■ \end{aligned}$$

Also from formula **D**,

$$\begin{aligned}\frac{E_2}{E_0} &= \frac{4\gamma}{(\gamma+1)^2}\sin^2\left(\tfrac{1}{2}\psi\right)\\ &\le \frac{4\gamma}{(\gamma+1)^2}\\ &= \frac{4m_1m_2}{(m_1+m_2)^2}. \blacksquare\end{aligned}$$

Problem 10.18 *Equal masses*

Show that, when the particles are of equal mass, the elastic scattering formulae take the simple form

$$\theta_1 = \tfrac{1}{2}\psi \qquad \theta_2 = \tfrac{1}{2}\pi - \tfrac{1}{2}\psi \qquad \theta = \tfrac{1}{2}\pi \qquad \frac{E_1}{E_0} = \cos^2 \tfrac{1}{2}\psi \qquad \frac{E_2}{E_0} = \sin^2 \tfrac{1}{2}\psi$$

where ψ is the scattering angle in the ZM frame.

In the scattering of neutrons of energy E by neutrons at rest, in what directions should the experimenter look to find neutrons of energy $\tfrac{1}{4}E$? What other energies would be observed in these directions?

Solution

When $m_1 = m_2$, the mass ratio $\gamma = 1$.

(i) Formula **A** then becomes

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + 1} = \tan\left(\tfrac{1}{2}\psi\right).$$

Hence $\theta_1 = \tfrac{1}{2}\psi$. ■

(ii) Formula **B** is unchanged. ■

(iii) Formula **C** becomes $\tan\theta = \infty$ so that $\theta = \pi/2$. If you do not like the infinity, simply use formulae **A** and **B** to give

$$\theta = \theta_1 + \theta_2 = \tfrac{1}{2}\psi + \left(\tfrac{1}{2}\pi - \tfrac{1}{2}\psi\right) = \tfrac{1}{2}\pi. \blacksquare$$

(iv) Formula **D** becomes

$$\frac{E_2}{E_0} = \sin^2 \tfrac{1}{2}\psi \ \blacksquare$$

from which we deduce that

$$\frac{E_1}{E_0} = 1 - \frac{E_2}{E_0} = 1 - \sin^2 \tfrac{1}{2}\psi = \cos^2 \tfrac{1}{2}\psi. \blacksquare$$

If it is **scattered neutrons** that are being observed, then

$$\frac{E_1}{E} = \frac{1}{4}.$$

Hence, from formula **D**,

$$\cos^2\left(\tfrac{1}{2}\psi\right) = \tfrac{1}{4}$$

and $\psi = 120°$. Formula **A** now tells us that $\theta_1 = 60°$. Hence, in order to see *scattered* neutrons with energy $\frac{1}{4}E$, we must look at an angle of 60° to the direction of the incident beam.

However, if it is **recoiling neutrons** that are being observed, then

$$\frac{E_2}{E} = \frac{1}{4}.$$

Hence, from formula **D**,

$$\sin^2\left(\tfrac{1}{2}\psi\right) = \tfrac{1}{4}$$

and $\psi = 60°$. Formula **A** now tells us that $\theta_1 = 30°$. Hence, in order to see *recoiling* neutrons with energy $\frac{1}{4}E$, we must look at an angle of 30° to the direction of the incident beam. Thus neutrons with energy $\frac{1}{4}E$ will be seen at angles of 30° and 60° to the direction of the incident beam. At the 30° angle, we see recoiling neutrons of energy $\frac{1}{4}E$ and scattered neutrons of energy

$$E_1 = \cos^2 30°\,E = \tfrac{3}{4}E,$$

while, at the 60° angle, we see scattered neutrons of energy $\frac{1}{4}E$ and recoiling neutrons of energy

$$E_2 = \sin^2 60°\,E = \tfrac{3}{4}E.$$

Hence neutrons of energy $\frac{3}{4}E$ are also seen at the 30° and 60° angles. ■

Problem 10.19

Use the elastic scattering formulae to express the energy of the scattered particle as a function of the scattering angle, and the energy of the recoiling particle as a function of the recoil angle, as follows:

$$\frac{E_1}{E_0} = \frac{1 + \gamma^2 \cos 2\theta_1 + 2\gamma \cos\theta_1 \left(1 - \gamma^2 \sin^2\theta_1\right)^{1/2}}{(\gamma + 1)^2}, \qquad \frac{E_2}{E_0} = \frac{4\gamma}{(\gamma + 1)^2} \cos^2\theta_2.$$

Make polar plots of E_1/E_0 as a function of θ_1 for the case of neutrons scattered by the nuclei of hydrogen, deuterium, helium and carbon.

Solution

(i) From formula **D**, the energy of the **scattered particle** is given by

$$\frac{E_1}{E_0} = 1 - \frac{4\gamma}{(\gamma + 1)^2} \sin^2\left(\tfrac{1}{2}\psi\right),$$
$$= \frac{1 + \gamma^2 + 2\gamma\cos\psi}{(\gamma + 1)^2},$$

where the ZM scattering angle ψ is related to the actual scattering angle θ_1 by formula **A**, namely,

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + \gamma}.$$

The object is to eliminate ψ and express E_1/E_0 in terms of θ_1. To do this, we need to invert formula **A**. On clearing the fractions, we obtain

$$\sin(\psi - \theta_1) = \gamma\sin\theta_1$$

so that

$$\psi = \theta_1 + \sin^{-1}(\gamma\sin\theta_1).$$

Consequently,

$$\cos\psi = \cos\theta_1 \left(1 - \gamma^2 \sin^2\theta_1\right)^{1/2} - \gamma\sin^2\theta_1,$$

and, on substituting this expression into the formula for E_1/E_0, we obtain

$$\frac{E_1}{E_0} = \frac{1 + \gamma^2 \cos 2\theta_1 + 2\gamma \cos \theta_1 \left(1 - \gamma^2 \sin^2 \theta_1\right)^{1/2}}{(\gamma + 1)^2}. \blacksquare$$

(ii) From formula **D**, the energy of the **recoiling particle** is given by

$$\frac{E_2}{E_0} = \frac{4\gamma}{(\gamma + 1)^2} \sin^2\left(\tfrac{1}{2}\psi\right),$$

where the ZM scattering angle ψ is related to the recoil angle θ_2 by formula **B**, namely,

$$\theta_2 = \tfrac{1}{2}(\pi - \psi).$$

The object is to eliminate ψ and express E_2/E_0 in terms of θ_2. In this case, $\psi = \pi - 2\theta_2$ and consequently $\sin\left(\frac{1}{2}\psi\right) = \cos\theta_2$. Hence

$$\frac{E_2}{E_0} = \frac{4\gamma}{(\gamma + 1)^2} \cos^2 \theta_2. \blacksquare$$

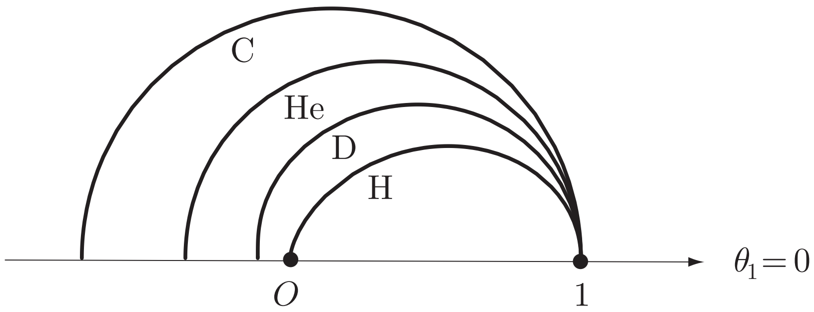

FIGURE 10.3 Polar plots of E_1/E_0 against θ_1 for neutrons scattered by nuclei of hydrogen, deuterium, helium and carbon.

The polar plots of E_1/E_0 against θ_1 are shown in Figure 10.3. $\blacksquare$

Problem 10.20 *Binary star*

The observed period of the binary star Cygnus X-1 (of which only one component is visible) is 5.6 days, and the semi-major axis of the orbit of the visible component is about 0.09 AU. The mass of the visible component is believed to be about $20M_\odot$. Estimate the mass of its dark companion. [Requires the numerical solution of a cubic equation.]

Solution

Let m_1, m_2 be the masses of the bright and dark components of Cygnus X-1 and let a_1, a_2 be the semi-major axes of their respective orbits. Then $a_2 = (m_1/m_2)a_1$ and a, the semi-major axis of the orbit of *relative motion*, is

$$\begin{aligned} a &= a_1 + a_2 \\ &= \left(1 + \frac{1}{\gamma}\right) a_1, \end{aligned}$$

where $\gamma = m_2/m_1$. The period τ of the orbit is then given by

$$\begin{aligned} \tau^2 &= \frac{a^3}{m_1 + m_2} \\ &= \frac{(1+\gamma)^2 a_1^3}{\gamma^3 m_1}, \end{aligned}$$

in astronomical units. The mass ratio γ therefore satisfies the cubic equation

$$\left(\frac{\tau^2 m_1}{a_1^3}\right)\gamma^3 - \gamma^2 - 2\gamma - 1 = 0.$$

On inserting the given values for τ, a_1 and m_1 and performing a numerical solution, we find that the cubic has *one* real root whose value is approximately 0.79. Hence the mass of the **dark component** of Cygnus X-1 is about $16M_\odot$. [This dark component is thought to be a black hole.] ■

Problem 10.21

In two-body elastic scattering, show that the angular distribution of the *recoiling* particles is given by

$$4\cos\theta_2\,\sigma^{ZM}(\pi - 2\theta_2),$$

where $\sigma^{ZM}(\psi)$ is defined by text equation (10.32).

In a Rutherford scattering experiment, alpha particles of energy E were scattered by a target of ionised helium. Find the angular distribution of the emerging particles.

Solution

Let p be the impact parameter of an incoming particle. Then σ^R, the recoil cross section of the helium nuclei, is given by

$$\begin{aligned}
\sigma^R &= \left(\frac{p}{\sin\theta_2}\right)\frac{dp}{d\theta_2}\\
&= \left(\frac{p}{\sin\theta_2}\right)\left(\frac{dp}{d\psi}\times\frac{d\psi}{d\theta_2}\right)\\
&= -\left(-\frac{p}{\sin\psi}\frac{dp}{d\psi}\right)\left(\frac{\sin\psi}{\sin\theta_2}\right)\frac{d\psi}{d\theta_2}\\
&= -\sigma^{ZM}(\psi)\left(\frac{\sin\psi}{\sin\theta_2}\right)\frac{d\psi}{d\theta_2}.
\end{aligned}$$

Now, from formula **B**,

$$\psi = \pi - 2\theta_2 \quad\text{and}\quad \frac{d\psi}{d\theta_2} = -2$$

and so

$$\begin{aligned}
\sigma^R &= 2\,\sigma^{ZM}(\pi - 2\theta_2)\left(\frac{\sin(\pi - 2\theta_2)}{\sin\theta_2}\right)\\
&= 4\cos\theta_2\,\sigma^{ZM}(\pi - 2\theta_2),
\end{aligned}$$

as required.

For **Rutherford scattering**, we know from the text that

$$\sigma^{ZM}(\psi) = \frac{q^4}{4E^2}\left(\frac{1}{\sin^4\left(\frac{1}{2}\psi\right)}\right),$$

in the standard notation. Hence, the **recoil cross section** of the helium nuclei is

$$\sigma^R = \frac{q^4}{4E^2}\left(\frac{4\cos\theta_2}{\sin^4\left(\frac{1}{2}\pi - \theta_2\right)}\right)$$
$$= \frac{q^4}{E^2}\left(\frac{1}{\cos^3\theta_2}\right).$$

Since alpha particles and helium nuclei are the same thing, the **angular distribution** of the emerging particles is the sum of the recoil cross section σ^R and the two-body Rutherford **scattering cross section** for equal masses, namely,

$$\sigma^S(\theta_1) = \frac{q^4}{E^2}\left(\frac{\cos\theta_1}{\sin^4\theta_1}\right).$$

This gives the **angular distribution**

$$\frac{q^4}{E^2}\left(\frac{\cos\theta}{\sin^4\theta} + \frac{1}{\cos^3\theta}\right),$$

where θ is measured from the direction of the incident beam.

Problem 10.22 ✱

Consider two-body elastic scattering in which the incident particles have energy E_0. Show that the energies of the *recoiling* particles lie in the interval $0 \le E \le E_{\text{max}}$, where $E_{\text{max}} = 4\gamma E_0/(1+\gamma)^2$. Show further that the energies of the recoiling particles are distributed over the interval $0 \le E \le E_{\text{max}}$ by the frequency distribution

$$f(E) = \left(\frac{4\pi}{E_{\text{max}}}\right) \sigma^{ZM}(\psi),$$

where σ^{ZM} is defined by text equation (10.32), and

$$\psi = 2\sin^{-1}\left(\frac{E}{E_{\text{max}}}\right)^{1/2}.$$

In the elastic scattering of neutrons of energy E_0 by protons at rest, the energies of the recoiling protons were found to be uniformly distributed over the interval $0 \le E \le E_0$, the total cross section being A. Find the *angular* distribution of the recoiling protons and the scattering cross section of the incident neutrons.

Solution

From Problem 10.17 (iii), we know that

$$\begin{aligned}\frac{E}{E_0} &\le \frac{4m_1 m_2}{(m_1+m_2)^2}\\ &= \frac{4\gamma}{(1+\gamma)^2}.\end{aligned}$$

Hence, the energies of the recoiling particles are bounded by

$$0 \le E \le \frac{4\gamma E_0}{(1+\gamma)^2} = E_{\text{max}}.$$

Let the recoil cross section be $\sigma^R(\theta_2)$. Then dF, the flux of recoiling particles that have recoil angles between θ_2 and $\theta_2 + d\theta_2$ is

$$dF = 2\pi N \sin\theta_2\, \sigma^R(\theta_2)\, d\theta_2,$$

where N is the incident flux per unit area. On using the result of Problem 10.21, this can be written

$$\begin{aligned}dF &= 8\pi N \sin\theta_2 \cos\theta_2\, \sigma^{ZM}(\pi - 2\theta_2)\, d\theta_2\\ &= 4\pi N \sin 2\theta_2\, \sigma^{ZM}(\pi - 2\theta_2)\, d\theta_2\\ &= -2\pi N \sin\psi\, \sigma^{ZM}(\psi)\, d\psi,\end{aligned}$$

where ψ $(= \pi - 2\theta_2)$ is the ZM scattering angle.

Now, from formula **D**, the recoil energy E is given in terms of ψ by

$$E = E_{\max} \sin^2 \tfrac{1}{2}\psi,$$

and hence dE and $d\psi$ are related by

$$dE = \tfrac{1}{2} E_{\max} \sin\psi \, d\psi.$$

Hence

$$\begin{aligned} dF &= -\left(\frac{4\pi N}{E_{\max}}\right) \sigma^{ZM}(\psi)\, dE \\ &= +\left(\frac{4\pi N}{E_{\max}}\right) \sigma^{ZM}(\psi)\, |dE|. \end{aligned}$$

Now the **frequency distribution** $f(E)$ is *defined* by the relation

$$dF = N f(E)\, dE,$$

where dE is now positive, and is hence given by

$$f(E) = \left(\frac{4\pi}{E_{\max}}\right) \sigma^{ZM}(\psi),$$

where the ZM scattering angle ψ is expressed in terms of E by the formula

$$\psi = 2 \sin^{-1} \left(\frac{E}{E_{\max}}\right)^{1/2}.$$

In the given example, the function $f(E)$ is constant and hence so is the function $\sigma^{ZM}(\psi)$. It follows that the recoil cross section has the form

$$\sigma^{R} = k \cos\theta_2,$$

where k is a constant. This constant can be determined from the fact that the total cross section is A. This implies that

$$\begin{aligned} A &= \int_0^{\pi/2} k \cos\theta_2 \,(2\pi \sin\theta_2)\, d\theta_2 \\ &= \pi k \int_0^{\pi/2} \sin 2\theta_2 \, d\theta_2 \\ &= \pi k. \end{aligned}$$

Hence $k = A/\pi$. The **recoil cross section** of the protons is therefore given by

$$\sigma^R = \frac{A}{\pi}\cos\theta_2 \qquad\qquad (0 \le \theta_2 \le \tfrac{1}{2}\pi).$$

The **scattering cross section** of the incident neutrons is given by the standard two-body formula for equal masses (see the text page 272), namely,

$$\begin{aligned}\sigma^{TB}(\theta_1) &= 4\cos\theta_1\,\sigma^{ZM}(2\theta_1)\\ &= \frac{A}{\pi}\cos\theta_1 \qquad\qquad (0 \le \theta_1 \le \tfrac{1}{2}\pi).\end{aligned}$$

Although the functions σ^R and σ^{TB} happen to be the same in this example, this is not true in general, even for particles of equal mass. ■

Problem 10.23

A particle Q has mass $2m$ and two other particles P, R, each of mass m, are connected to Q by light inextensible strings of length a. The system is free to move on a smooth horizontal table. Initially P, Q R are at the points $(0, a)$, $(0, 0)$, $(0, -a)$ respectively so that they lie in a straight line with the strings taut. Q is then projected in the positive x-direction with speed u. Express the conservation of linear momentum and energy for this system in terms of the coordinates x (the displacement of Q) and θ (the angle turned by each of the strings).

Show that θ satisfies the equation

$$\dot{\theta}^2 = \frac{u^2}{a^2}\left(\frac{1}{2 - \cos^2\theta}\right)$$

and deduce that P and R will collide after a time

$$\frac{a}{u}\int_0^{\pi/2}\left[2 - \cos^2\theta\right]^{\frac{1}{2}}\, d\theta.$$

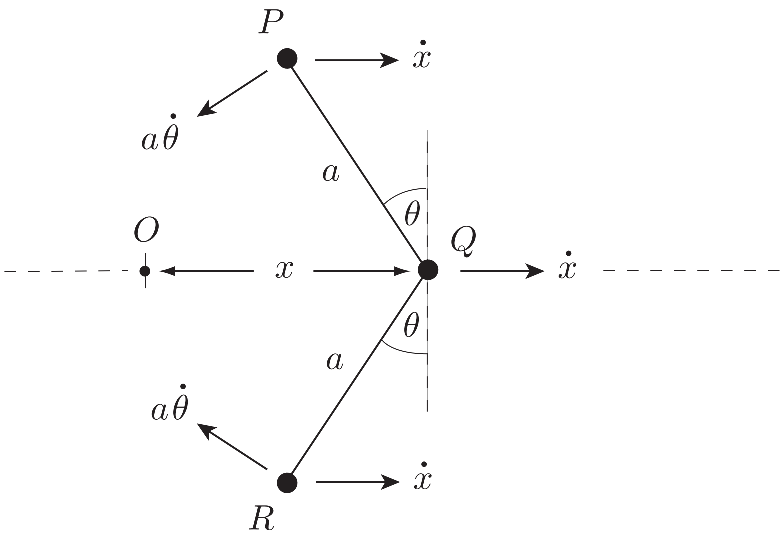

FIGURE 10.4 Generalised coordinates and velocity diagram for the system in Problem 10.23.

Solution

Take x, θ as generalised coordinates. The corresponding velocity diagram is shown in Figure 10.4.

Since the x-component of the total **linear momentum** is conserved,

$$(2m)\dot{x} + 2 \times m\left(\dot{x} - a\dot{\theta}\cos\theta\right) = C,$$

where C is a constant. From the initial conditions, we find that $C = (2m)u$ so that the first conservation relation becomes

$$2\dot{x} - a\dot{\theta}\cos\theta = u. \qquad (1)$$

The **total energy** is also conserved. Since there is no potential energy, this gives

$$\tfrac{1}{2}(2m)\dot{x}^2 + 2 \times \tfrac{1}{2}m\left(\dot{x}^2 + (a\dot{\theta})^2 - 2\dot{x}(a\dot{\theta})\cos\theta\right) = E,$$

where E is the constant total energy. From the initial conditions, we find that $E = \frac{1}{2}(2m)u^2$ so that the second conservation relation becomes

$$2\dot{x}^2 + a^2\dot{\theta}^2 - 2a\dot{x}\dot{\theta}\cos\theta = u^2. \qquad (2)$$

On eliminating $\dot{x}$ between equations (1), (2), we obtain

$$\dot{\theta}^2 = \frac{u^2}{a^2}\left(\frac{1}{2 - \cos^2\theta}\right),$$

after some simplification. This is the required equation for the coordinate θ.

Since θ is an increasing function of t in the motion,

$$\frac{d\theta}{dt} = +\frac{u}{a}\left(\frac{1}{2 - \cos^2\theta}\right)^{1/2},$$

which is a first order separable ODE. On separating and integrating, we find that τ, the time at which P and R collide is given by

$$\tau = \frac{a}{u}\int_0^{\pi/2}\left(2 - \cos^2\theta\right)^{1/2} d\theta \approx 1.91\left(\frac{a}{u}\right). \blacksquare$$

Problem 10.24

A uniform rod of length $2a$ has its lower end in contact with a smooth horizontal table. Initially the rod is released from rest in a position making an angle of 60° with the upward vertical. Express the conservation of linear momentum and energy for this system in terms of the coordinates x (the horizontal displacement of the centre of mass of the rod) and θ (the angle between the rod and the upward vertical). Deduce that the centre of mass of the rod moves in a vertical straight line, and that θ satisifies the equation

$$\dot{\theta}^2 = \frac{3g}{a}\left(\frac{1 - 2\cos\theta}{4 - 3\cos^2\theta}\right).$$

Find how long it takes for the rod to hit the table.

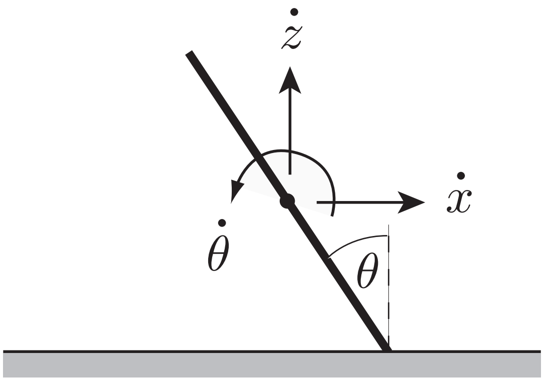

FIGURE 10.5 The velocity diagram for the falling rod.

Solution

Suppose first that the rod can move freely in a vertical plane. Then $\{x, z, \theta\}$ is a possible set of generalised coordinates, where x and z are the horizontal and vertical displacements of the centre of mass G (relative to a fixed origin on the table), and θ is the angle between the rod and the upward vertical. The corresponding velocity diagram is shown in Figure 10.5. If one end of the rod is now constrained to slide on the table, these coordinates are no longer independent since $z = a\cos\theta$. The system is thus reduced to two degrees of freedom and we will take $\{x, \theta\}$ as its generalised coordinates.

Since the horizontal component of the total **linear momentum** is conserved,

$$M\dot{x} = C,$$

where C is a constant. From the initial conditions, we find that $C = 0$ so that the first conservation relation becomes

$$\dot{x} = 0.$$

Hence, G moves in a vertical straight line.

The **total energy** is also conserved. The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\left(\dot{x}^2 + \dot{z}^2\right) + \tfrac{1}{2}\left(\tfrac{1}{3}Ma^2\right)\dot{\theta}^2 \\ &= \tfrac{1}{2}Ma^2 \sin^2\theta\, \dot{\theta}^2 + \tfrac{1}{6}Ma^2\dot{\theta}^2, \end{aligned}$$

on using the fact that $\dot{x} = 0$ and $\dot{z} = -a\sin\theta\,\dot{\theta}$. The gravitational potential energy is

$$V = Mgz = Mga\cos\theta.$$

The energy conservation equation thus has the form

$$\tfrac{1}{6}Ma^2\left(1 + 3\sin^2\theta\right)\dot{\theta}^2 + Mga\cos\theta = E,$$

where E is the constant total energy. From the initial conditions, we find that $E = \frac{1}{2}Mga$ so that the second conservation relation becomes

$$\dot{\theta}^2 = \frac{3g}{a}\left(\frac{1 - 2\cos\theta}{4 - 3\cos^2\theta}\right).$$

This is the required equation for θ.

Since θ is an increasing function of t in the motion,

$$\frac{d\theta}{dt} = +\left(\frac{3g}{a}\right)^{1/2}\left(\frac{1 - 2\cos\theta}{4 - 3\cos^2\theta}\right)^{1/2},$$

which is a first order separable ODE. On separating and integrating, we find that τ, the time at which the rod hits the table, is given by

$$\tau = \left(\frac{a}{3g}\right)^{1/2}\int_{\pi/3}^{\pi/2}\left(\frac{4 - 3\cos^2\theta}{1 - 2\cos\theta}\right)^{1/2} d\theta \approx 1.18\left(\frac{a}{g}\right)^{1/2}. \blacksquare$$

Chapter Eleven

The angular momentum principle

and angular momentum conservation

Problem 11.1 *Non-standard angular momentum principle*

If A is a generally moving point of space and $\boldsymbol{L}_A$ is the angular momentum of a system $\mathcal{S}$ about A *in its motion relative to* A, show that the angular momentum principle for $\mathcal{S}$ about A takes the non-standard form

$$\frac{d\boldsymbol{L}_A}{dt} = \boldsymbol{K}_A - M(\boldsymbol{R} - \boldsymbol{a})\times\frac{d^2\boldsymbol{a}}{dt^2}.$$

[Begin by expanding the expression for $\boldsymbol{L}_A$.]

When does this formula reduce to the standard form? [This non-standard version of the angular momentum principle is rarely needed. However, see Problem 11.9.]

Solution

By definition, the angular momentum of $\mathcal{S}$ about A *in its motion relative to* A is

$$\boldsymbol{L}_A = \sum m(\boldsymbol{r} - \boldsymbol{a})\times(\dot{\boldsymbol{r}} - \dot{\boldsymbol{a}}),$$

where the summing is taken over all the particles of $\mathcal{S}$. On differentiating with respect to t, this becomes

$$\begin{aligned}
\dot{\boldsymbol{L}}_A &= \sum m(\boldsymbol{r} - \boldsymbol{a})\times(\ddot{\boldsymbol{r}} - \ddot{\boldsymbol{a}})\\
&= \sum \Big(m\boldsymbol{r}\times\dot{\boldsymbol{r}} - \boldsymbol{a}\times\big(m\ddot{\boldsymbol{r}}\big) - \big(m\boldsymbol{r}\big)\times\ddot{\boldsymbol{a}} + m\boldsymbol{a}\times\ddot{\boldsymbol{a}}\Big)\\
&= \dot{\boldsymbol{L}}_O - \boldsymbol{a}\times\big(M\ddot{\boldsymbol{R}}\big) - \big(M\boldsymbol{R}\big)\times\ddot{\boldsymbol{a}} + M\boldsymbol{a}\times\ddot{\boldsymbol{a}}\\
&= \boldsymbol{K}_O - \boldsymbol{a}\times\boldsymbol{F} - M(\boldsymbol{R} - \boldsymbol{a})\times\ddot{\boldsymbol{a}}\\
&= \boldsymbol{K}_A - M(\boldsymbol{R} - \boldsymbol{a})\times\ddot{\boldsymbol{a}},
\end{aligned}$$

as required.

This non-standard form of the angular momentum principle reduces to the **standard form** if

$$(\boldsymbol{R} - \boldsymbol{a})\times\ddot{\boldsymbol{a}} = \boldsymbol{0}$$

at all times. This will be true if

(i) $\boldsymbol{a} = \boldsymbol{R}$, that is, the point A is the centre of mass of $\mathcal{S}$, or

(ii) $\ddot{\boldsymbol{a}} = \boldsymbol{0}$, that is, A moves with constant velocity, or

(iii) $\boldsymbol{R} - \boldsymbol{a}$ happens to be parallel to $\ddot{\boldsymbol{a}}$ *at all times*.

Condition (i) leads to the standard angular momentum principle $\dot{\boldsymbol{L}}_G = \boldsymbol{K}_G$. Condition (ii) is equivalent to the result that the standard angular momentum principle applies in the inertial frame moving with the constant velocity $\dot{\boldsymbol{a}}$. The author is not aware of any practical problem in which condition (iii) holds. ■

Problem 11.2

A fairground target consists of a uniform circular disk of mass M and radius a that can turn freely about a diameter which is fixed in a vertical position. Initially the target is at rest. A bullet of mass m is moving with speed u along a horizontal straight line at right angles to the target. The bullet embeds itself in the target at a point distance b from the rotation axis. Find the final angular speed of the target. [The moment of inertia of the disk about its rotation axis is $Ma^2/4$.]

Show also that the energy lost in the impact is

$$\frac{1}{2}mu^2\left(\frac{Ma^2}{Ma^2+4mb^2}\right).$$

Solution

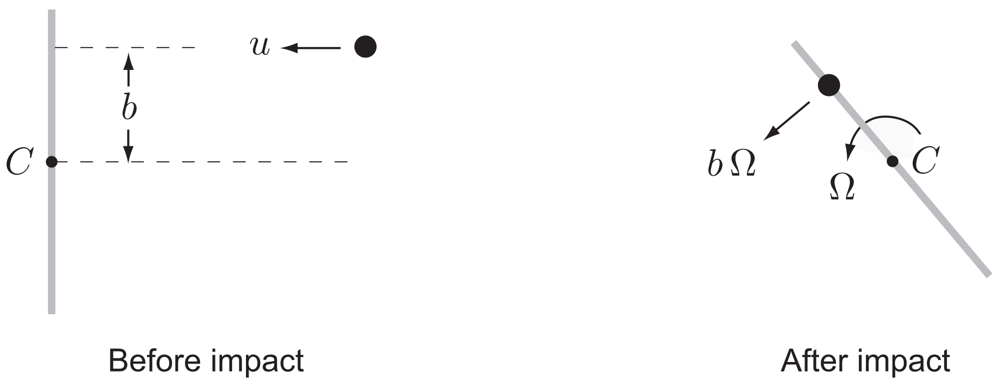

FIGURE 11.1 The system in problem 11.2 (viewed from above).

Since the target is *smoothly* pivoted about a *vertical* axis, the angular momentum of the system about this axis is conserved. (The proof is similar to that given in Example 11.8.) Thus $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is conserved, where C is the centre of the disk and $\boldsymbol{k}$ is the unit vector pointing vertically upwards.

Figure 11.1 shows the system before and after the impact.

(i) Before the impact, the bullet has angular momentum mbu and the target is at rest. Hence the initial value of $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is mbu.

(ii) After the impact, the target has the unknown angular velocity Ω and the embedded bullet has speed $b\Omega$, as shown in Figure 11.1 (right). The final value of $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is therefore $mb(b\Omega) + \left(\frac{1}{4}Ma^2\right)\Omega$.

Since the angular momentum $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is conserved, we have

$$\left(mb^2 + \tfrac{1}{4}Ma^2\right)\Omega = mbu,$$

from which it follows that, after the impact, the **angular velocity** of the target is

$$\Omega = \frac{4mbu}{Ma^2 + 4mb^2}.$$

The **kinetic energy** of the system after the impact is then given by

$$\begin{aligned}
T_2 &= \tfrac{1}{2}m(b\Omega)^2 + \tfrac{1}{2}\left(\tfrac{1}{4}Ma^2\right)\Omega^2 \\
&= \tfrac{1}{8}\left(Ma^2 + 4mb^2\right)\Omega^2 \\
&= \frac{2m^2b^2u^2}{Ma^2 + 4mb^2},
\end{aligned}$$

on using the calculated value of Ω. The **loss of energy** in the impact is therefore

$$\begin{aligned}
T_1 - T_2 &= \tfrac{1}{2}mu^2 - \frac{2m^2b^2u^2}{Ma^2 + 4mb^2} \\
&= \frac{mMa^2u^2}{2\left(Ma^2 + 4mb^2\right)}. \blacksquare
\end{aligned}$$

Problem 11.3

A uniform circular cylinder of mass M and radius a can rotate freely about its axis of symmetry which is fixed in a vertical position. A light string is wound around the cylinder so that it does not slip and a particle of mass m is attached to the free end. Initially the system is at rest with the free string taut, horizontal and of length b. The particle is then projected horizontally with speed u at right angles to the string. The string winds itself around the cylinder and eventually the particle strikes the cylinder and sticks to it. Find the final angular speed of the cylinder.

Solution

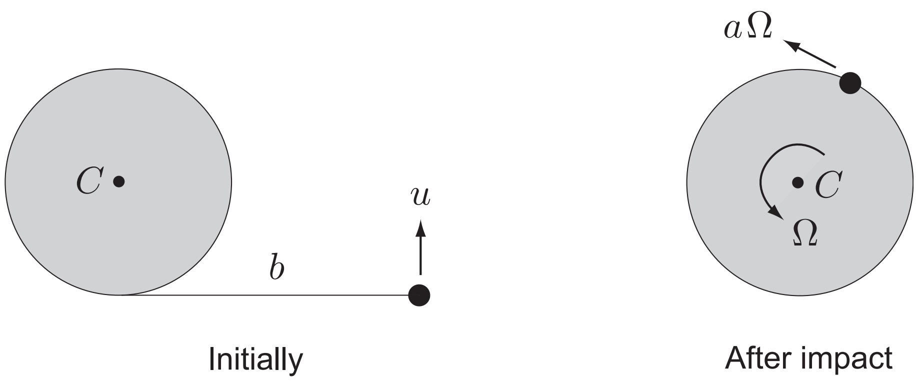

FIGURE 11.2 The system in problem 11.3 (viewed from above).

Since the cylinder is *smoothly* pivoted about a *vertical* axis, the angular momentum of the system about this axis is conserved. (The proof is similar to that given in Example 11.8.) Thus $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is conserved, where C is the centre of the cylinder and $\boldsymbol{k}$ is the unit vector pointing vertically upwards.

Figure 11.2 shows the system initially and after the impact.

(i) Before the impact, the particle has angular momentum mbu and the cylinder is at rest. Hence the initial value of $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is mbu.

(ii) After the impact, the cylinder has the unknown angular velocity Ω and the attached particle has speed $a\Omega$, as shown in Figure 11.2 (right). The final value of $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is therefore $ma(a\Omega) + \left(\frac{1}{2}Ma^2\right)\Omega$.

Since the angular momentum $\boldsymbol{L}_C \cdot \boldsymbol{k}$ is conserved, we have

$$\left(ma^2 + \tfrac{1}{2}Ma^2\right)\Omega = mbu,$$

from which it follows that, after the impact, the **angular velocity** of the cylinder is

$$\Omega = \frac{2mbu}{(M + 2m)a^2}. \blacksquare$$

Problem 11.4 *Rotating gas cloud*

A cloud of interstellar gas of total mass M can move freely in space. Initially the cloud has the form of a uniform sphere of radius a rotating with angular speed Ω about an axis through its centre. Later, the cloud is observed to have changed its form to that of a thin uniform circular disk of radius b which is rotating about an axis through its centre and perpendicular to its plane. Find the angular speed of the disk and the increase in the kinetic energy of the cloud.

Solution

Since the gas cloud is an *isolated* system, the angular momentum $\boldsymbol{L}_G$ is conserved, where G is its centre of mass. For simplicity, we will suppose that the motion of the cloud is viewed from an inertial frame in which G is at rest.

Initially, the cloud is a uniform sphere of radius a rotating with angular speed Ω about a fixed axis through G. Its angular momentum is therefore

$$\boldsymbol{L}_G = \left(\tfrac{2}{5}Ma^2\right)\Omega\boldsymbol{k},$$

where $\boldsymbol{k}$ is a unit vector pointing along the rotation axis. Later, the cloud is a uniform disk of radius b rotating with unknown angular speed Ω' about a fixed axis through G perpendicular to the plane of the disk. The angular momentum of the cloud in this configuration is

$$\boldsymbol{L}_G = \left(\tfrac{1}{2}Mb^2\right)\Omega'\boldsymbol{k}',$$

where $\boldsymbol{k}'$ is a unit vector pointing along the new rotation axis.

Since the angular momentum $\boldsymbol{L}_G$ is conserved, we therefore have

$$\tfrac{2}{5}Ma^2\Omega\boldsymbol{k} = \tfrac{1}{2}Mb^2\Omega'\boldsymbol{k}'.$$

Hence $\boldsymbol{k}' = \boldsymbol{k}$ (that is, the two rotation axes must be the same) and the new **angular speed** of the cloud is

$$\Omega' = \left(\frac{4a^2}{5b^2}\right)\Omega.$$

The increase in the **kinetic energy** of the cloud is then given by

$$
\begin{aligned}
\Delta T &= \tfrac{1}{2}\left(\tfrac{1}{2}Mb^2\right)\Omega'^2 - \tfrac{1}{2}\left(\tfrac{2}{5}Ma^2\right)\Omega^2 \\
&= \tfrac{1}{4}Mb^2\left(\frac{4a^2}{5b^2}\right)^2\Omega^2 - \tfrac{1}{5}Ma^2\Omega^2 \\
&= \frac{Ma^2\left(4a^2 - 5b^2\right)\Omega^2}{25b^2}. \blacksquare
\end{aligned}
$$

Problem 11.5 *Conical pendulum with shortening string*

A particle is suspended from a support by a light inextensible string which passes through a small fixed ring vertically below the support. Initially the particle is performing a conical motion of angle 60°, with the moving part of the string of a. The support is now made to move slowly upwards so that the motion remains nearly conical. Find the angle of this conical motion when the support has been raised by a distance $a/2$. [Requires the numerical solution of a trigonometric equation.]

Solution

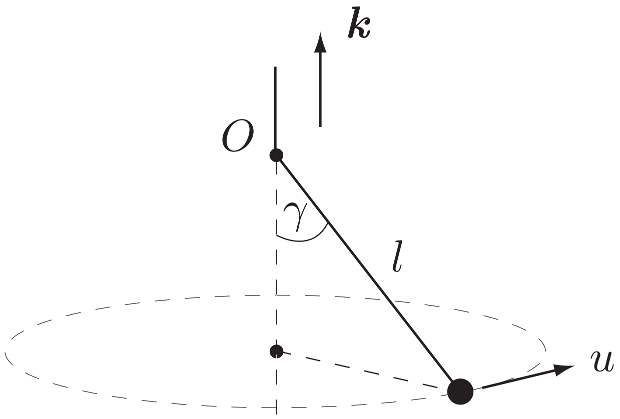

FIGURE 11.3 A pendulum in conical motion .

Consider first a true conical pendulum with a string of *fixed* length l inclined at a *constant* angle γ to the downward vertical. What is its angular momentum about the axis $\{O, \boldsymbol{k}\}$ (see Figure 11.3)? Newton's equations of motion for the bob are

$$0 = T\cos\gamma - mg,$$
$$\frac{mu^2}{l\sin\gamma} = T\sin\gamma,$$

where T is the tension in the string and u is the speed of the bob. It follows that

$$u^2 = \frac{lg\sin^2\gamma}{\cos\gamma}.$$

The axial angular momentum of the pendulum is therefore

$$\begin{aligned}\boldsymbol{L}_O \cdot \boldsymbol{k} &= m(l \sin\gamma)u \\ &= \frac{m\left(l^3 g\right)^{1/2} \sin^2\gamma}{(\cos\gamma)^{1/2}}.\end{aligned}$$

Suppose now that the string is pulled upwards. Since the external forces are either vertical or act at O, the axial angular momentum $\boldsymbol{L}_O \cdot \boldsymbol{k}$ is conserved. The motion of the pendulum is not now conical but, if the string is pulled up *slowly*, it remains approximately conical. (Numerical solution of the full equations of motion confirms this.) Hence, if the pendulum passes slowly from a conical motion with a string of length a and inclination α to a nearly conical motion with a string of length b and inclination β, conservation of angular momentum requires that

$$\frac{m\left(a^3 g\right)^{1/2} \sin^2\alpha}{(\cos\alpha)^{1/2}} = \frac{m\left(b^3 g\right)^{1/2} \sin^2\beta}{(\cos\beta)^{1/2}},$$

that is,

$$\frac{a^3 \sin^4\alpha}{\cos\alpha} = \frac{b^3 \sin^4\beta}{\cos\beta}.$$

In particular, if the initial inclination $\alpha = 60°$ and the final length $b = \frac{1}{2}a$, the **final inclination** β must satisfy the equation

$$\frac{\sin^4\beta}{\cos\beta} = 9.$$

Numerical solution of this trigonometric equation shows that $\beta = 84°$ approximately. ■

Problem 11.6 *Baseball bat*

A baseball bat has mass M and moment of inertia Mk^2 about any axis through its centre of mass G that is perpendicular to the axis of symmetry. The bat is at rest when a ball of mass m, moving with speed u, is normally incident along a straight line through the axis of symmetry at a distance b from G. Show that, whether the impact is elastic or not, there is a point on the axis of symmetry of the bat that is instantaneously at rest after the impact and that the distance c of this point from G is given by $bc = k^2$. In the elastic case, find the speed of the ball after the impact. [Gravity (and the batter!) should be ignored throughout this question.]

Solution

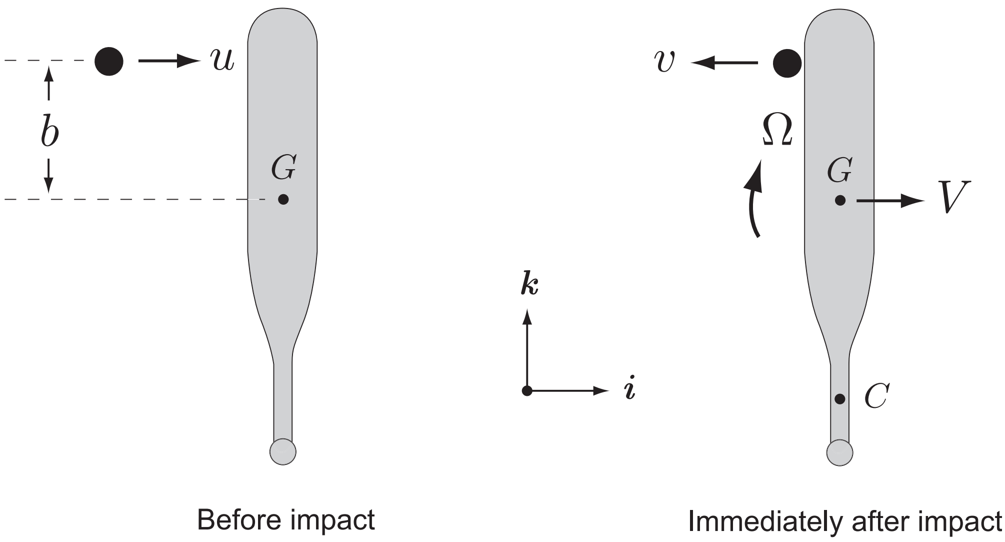

FIGURE 11.4 The ball and the bat in problem 11.6 .

Since the system is assumed to be *isolated*, its **linear momentum** is conserved. Initially, the ball has linear momentum $mu\boldsymbol{i}$ and the bat is at rest. Hence the initial value of $\boldsymbol{P}$ is $mu\boldsymbol{i}$. Immediately after the impact, the motion is assumed to have the form shown in Figure 11.4 (right). The ball has linear momentum $-mv\boldsymbol{i}$ and the bat has linear momentum $MV\boldsymbol{i}$. The final value of $\boldsymbol{P}$ is therefore $(-mv + MV)\boldsymbol{i}$. Since $\boldsymbol{P}$ is conserved, we therefore have

$$-mv + MV = mu,$$

that is,

$$m(u + v) = MV.$$

The fact that the system is *isolated* also has the consequence that its **angular momentum** about any fixed point is conserved. We will apply this principle about G, the centre of mass of the bat. Since G is *not* a fixed point (nor is it the centre of mass of the whole system) this needs some justification. What we are actually doing is using angular momentum conservation about G_0, the *fixed point of space occupied by G before the impact*. In the subsequent motion, G will move away from G_0, but the two points are still coincident immediately after the impact. The value of L_G *at this instant* is therefore the same as that before the impact.

Before the impact, the angular momentum of the ball about G is $mbu\,\boldsymbol{j}$ and the bat is at rest. Hence the initial value of $\boldsymbol{L}_G$ is $mbu\,\boldsymbol{j}$. Immediately after the impact, the angular momentum of the ball about G is $-mbv\,\boldsymbol{j}$ and that of the bat is $\left(Mk^2\right)\Omega\,\boldsymbol{j}$, where Mk^2 is the moment of inertia of the bat about G. The 'final' value of $\boldsymbol{L}_G$ is therefore $\left(-mbv + Mk^2\Omega\right)\boldsymbol{j}$. Since $\boldsymbol{L}_G$ is conserved, we therefore have

$$-mbv + \left(Mk^2\right)\Omega = mbu.$$

It follows from these two conservation relations that

$$\Omega = \frac{bV}{k^2}.$$

Let C be some point on the axis of the bat (see Figure 11.4) and let c be the distance GC. Then v_C, the forward velocity of C immediately after the impact, is given by

$$\begin{aligned} v_C &= V - \Omega c \\ &= V\left(1 - \frac{bc}{k^2}\right). \end{aligned}$$

The point C will be **instantaneously at rest** after the impact if $v_C = 0$, that is, if $bc = k^2$. [Note that the point C that satisfies this equation may not lie within the bat!]

In the special case in which the impact is **elastic**, the total kinetic energy is also conserved so that

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2 + \tfrac{1}{2}\left(Mk^2\right)\Omega^2 = \tfrac{1}{2}mu^2.$$

On using the relation $\Omega = bV/k^2$, this can be written in the form

$$m\left(u^2 - v^2\right) = M\left(1 + \frac{b^2}{k^2}\right)V^2.$$

Since we also have the linear momentum conservation relation

$$m(u + v) = MV,$$

it follows that

$$u - v = \left(1 + \frac{b^2}{k^2}\right)V.$$

The last two equations can now be solved for the unknowns v and V. In particular, the **velocity of the ball** after the impact is

$$v = \left(\frac{1-\beta}{1+\beta}\right)u,$$

where

$$\beta = \frac{m}{M}\left(1 + \frac{b^2}{k^2}\right). \blacksquare$$

Problem 11.7 *Hoop mounting a step*

A uniform hoop of mass M and radius a is rolling with speed V along level ground when it meets a step of height h ($h < a$). The particle C of the hoop that makes contact with the step is suddenly brought to rest. Find the instantaneous speed of the centre of mass, and the instantaneous angular velocity of the hoop, immediately after the impact. Deduce that the particle C cannot remain at rest on the edge of the step if

$$V^2 > (a-h)g\left(1-\frac{h}{2a}\right)^{-2}.$$

Suppose that the particle C *does* remain on the edge of the step. Show that the hoop will go on to mount the step if

$$V^2 > hg\left(1-\frac{h}{2a}\right)^{-2}.$$

Deduce that the hoop cannot mount the step in the manner described if $h > a/2$.

Solution

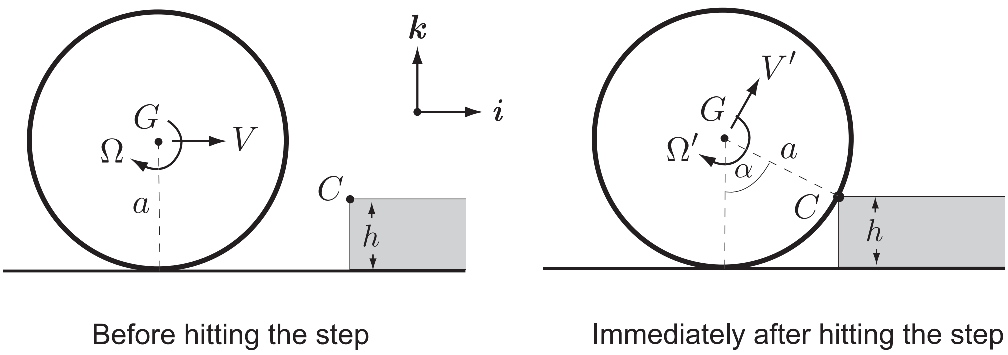

FIGURE 11.5 The hoop and the step in problem 11.7 .

Consider the angular momentum of the hoop about C, the corner of the step. While the hoop is rolling,

$$\begin{aligned}\boldsymbol{L}_C &= M(a-h)V\,\boldsymbol{j} + \left(Ma^2\right)\Omega\,\boldsymbol{j}\\ &= M(2a-h)V\,\boldsymbol{j},\end{aligned}$$

on using the rolling condition. When the hoop hits the step, the contact particle is suddenly brought to rest by an impulsive reaction supplied by the step. Since this reaction acts through C, its moment about C is zero. It follows that the value of $\boldsymbol{L}_C$ is not changed *by the impact*. Immediately after the impact, the angular momentum of the hoop about C is

$$\begin{aligned} \boldsymbol{L}_C &= MaV'\boldsymbol{j} + \left(Ma^2\right)\Omega' \\ &= 2MaV', \end{aligned}$$

since the contact particle is instantaneously at rest. The velocity V' and angular velocity Ω' are those shown in Figure 11.5 (right). Since $\boldsymbol{L}_C$ is unchanged by the impact, it follows that

$$V' = \left(1 - \frac{h}{2a}\right) V$$

and hence that

$$\Omega' = \left(1 - \frac{h}{2a}\right) \frac{V}{a}.$$

What happens next is not clear. The hoop *must* leave the floor, but it may or may not maintain contact with the step. Suppose that, at least for a short time, the hoop maintains contact with the step without slipping so that G moves on an arc of a circle. The centre of mass equation for G then implies that

$$Mg\cos\alpha - R = \frac{MV'^2}{a},$$

where R is the initial reaction of the step (resolved in the direction $\overrightarrow{CG}$), and α is the angle between GC and the downward vertical (see Figure 11.5 (right)). It follows that

$$R = \frac{Mg(a-h)}{a} - \frac{MV^2}{a}\left(1 - \frac{h}{2a}\right)^2.$$

Since R must be *positive*, it follows that the hoop will **leave the step immediately** if

$$V^2 \ > \ (a-h)g\left(1 - \frac{h}{2a}\right)^{-2}.$$

Suppose from now on that

$$V^2 < (a-h)g\left(1-\frac{h}{2a}\right)^{-2}$$

and that the hoop maintains contact with the step (without slipping) until it either mounts the step or falls back. Now $\boldsymbol{L}_C$ is *not* conserved in this motion because of the moment of the gravity force about C. However, the **total energy** is conserved. The hoop will **mount the step** if, and only if,

$$\tfrac{1}{2}MV'^2 + \tfrac{1}{2}\left(Ma^2\right)\left(\frac{V'}{a}\right)^2 + Mga > Mg(a+h),$$

that is

$$V^2 > gh\left(1-\frac{h}{2a}\right)^{-2}.$$

This condition, together with the condition that the hoop should maintain contact without slipping, means that the hoop **cannot mount the step** (by rolling up it) if

$$gh\left(1-\frac{h}{2a}\right)^{-2} > (a-h)g\left(1-\frac{h}{2a}\right)^{-2},$$

that is, if $h > \frac{1}{2}a$. ■

Problem 11.8 *Particle sliding on a cone*

A particle P slides on the smooth inner surface of a circular cone of semi-angle α. The axis of symmetry of the cone is vertical with the vertex O pointing downwards. Show that the vertical component of angular momentum about O is conserved in the motion. State a second dynamical quantity that is conserved.

Initially P is a distance a from O when it is projected horizontally along the inside surface of the cone with speed u. Show that, in the subsequent motion, the distance r of P from O satisfies the equation

$$\dot{r}^2 = (r-a)\left[\frac{u^2(r+a)}{r^2} - 2g\cos\alpha\right].$$

Case A For the case in which gravity is absent, find r and the azimuthal angle ϕ explicitly as functions of t. Make a sketch of the path of P (as seen from 'above') when $\alpha = \pi/6$.

Case B For the case in which $\alpha = \pi/3$, find the value of u such that r oscillates between a and $2a$ in the subsequent motion. With this value of u, show that r will first return to the value $r = a$ after a time

$$2\sqrt{3}\left(\frac{a}{g}\right)^{1/2}\int_1^2 \frac{\xi\, d\xi}{[(\xi-1)(2-\xi)(2+3\xi)]^{1/2}}.$$

Solution

The forces acting on P are shown in Figure 11.6 (left). Since the cone is *smooth*, the reaction $\boldsymbol{N}$ is always normal to its surface. The total moment of forces about O is

$$\boldsymbol{K}_O = \boldsymbol{r}\times(-mg\boldsymbol{k}) + \boldsymbol{r}\times\boldsymbol{N}$$

and hence

$$\boldsymbol{K}_O\cdot\boldsymbol{k} = -mg(\boldsymbol{r}\times\boldsymbol{k})\cdot\boldsymbol{k} + (\boldsymbol{r}\times\boldsymbol{N})\cdot\boldsymbol{k}.$$

Now the triple scalar product $(\boldsymbol{r}\times\boldsymbol{k})\cdot\boldsymbol{k}$ is zero since two of its vectors are the same. Also the triple scalar product $(\boldsymbol{r}\times\boldsymbol{N})\cdot\boldsymbol{k}$ is zero since its three vectors are coplanar. Thus $\boldsymbol{K}_O\cdot\boldsymbol{k} = 0$ and hence the **axial angular momentum $\boldsymbol{L}_O\cdot\boldsymbol{k}$ is conserved**.

The fact that the cone is smooth also has the consequence that the **total energy** of the particle is **conserved**.

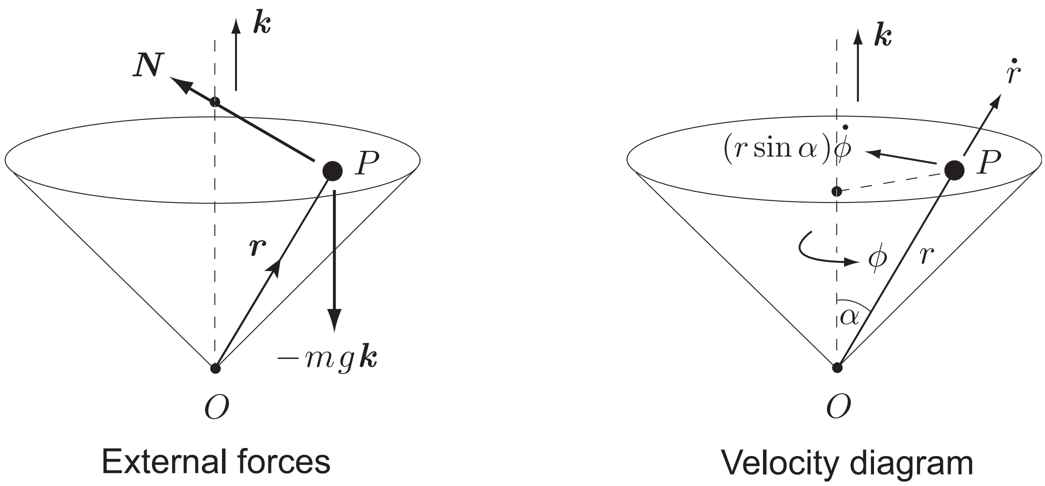

FIGURE 11.6 The cone and the particle in problem 11.8 .

The coordinates r, ϕ and the corresponding velocity diagram are shown in Figure 11.6 (right). In terms of these coordinates,

$$\begin{aligned} \boldsymbol{L}_O \cdot \boldsymbol{k} &= m\big(r\sin\alpha\big)\big(r\sin\alpha\,\dot{\phi}\big) \\ &= m\sin^2\alpha\, r^2\dot{\phi}. \end{aligned}$$

Since the initial value of $\boldsymbol{L}_O \cdot \boldsymbol{k}$ is $m(a\sin\alpha)u$, the **angular momentum conservation equation** is

$$\sin\alpha\, r^2\dot{\phi} = au.$$

Similarly, the **energy conservation equation** is

$$\tfrac{1}{2}m\left(\dot{r}^2 + \big(r\sin\alpha\,\dot{\phi}\big)^2\right) + mg(r\cos\alpha) = \tfrac{1}{2}mu^2 + mg(a\cos\alpha),$$

that is,

$$\dot{r}^2 + \sin^2\alpha\, r^2\dot{\phi}^2 = u^2 + 2g\cos\alpha(a - r).$$

On eliminating $\dot{\phi}$ between these two conservation equations, we find that r satisfies the equation

$$\dot{r}^2 = (a - r)\left(\frac{u^2(r + a)}{r^2} - 2g\cos\alpha\right),$$

as required. ■

Case A In the special case in which gravity is absent, the equation for r becomes

$$\dot{r}^2 = \frac{u^2(r^2 - a^2)}{r^2}$$

so that the motion takes place with r in the range $a \le r < \infty$. On taking square roots, we obtain

$$\dot{r} = +\frac{u(r^2 - a^2)^{1/2}}{r},$$

which is a separable first order ODE for r as a function of t. On separating and integrating, we obtain

$$(r^2 - a^2)^{1/2} = ut + C,$$

where C is the integration constant. Since $r = a$ when $t = 0$, it follows that $C = 0$ and on solving for r we find that the **time variation** of r is given by

$$r = \left(a^2 + u^2t^2\right)^{1/2}.$$

The corresponding time variation of ϕ can now be found by substituting this formula into the angular momentum conservation equation. This gives

$$\sin\alpha \left(a^2 + u^2t^2\right)\dot{\phi} = au,$$

which is a separable first order ODE for ϕ as a function of t. On separating and integrating, we obtain

$$\phi = \frac{1}{\sin\alpha}\tan^{-1}\left(\frac{ut}{a}\right) + D,$$

where D is the integration constant. If we suppose that $\phi = 0$ when $t = 0$, then $D = 0$ and the **time variation** of ϕ is given by

$$\phi = \frac{1}{\sin\alpha}\tan^{-1}\left(\frac{ut}{a}\right).$$

The path of the particle when $\alpha = \pi/6$ is shown in Figure 11.7. Since ϕ tends to π as t tends to infinity, the path is asymptotically parallel to the line $\phi = \pi$. ■

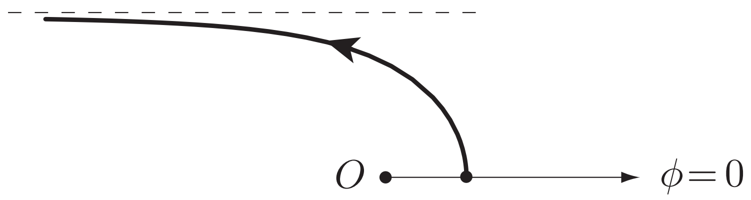

FIGURE 11.7 The path of P in the absence of gravity when the cone angle $\alpha = \pi/6$ (viewed from above).

Case B The stationary values of r are achieved when $\dot{r} = 0$, that is, when

$$(a - r)\left(\frac{u^2(r + a)}{r^2} - 2g\cos\alpha\right) = 0,$$

which becomes

$$(a - r)\left(\frac{u^2(r + a)}{r^2} - g\right) = 0$$

when the cone angle $\alpha = \pi/3$. Hence $r = a$ is one stationary value (a consequence of the initial conditions) and any other stationary values must satisfy the equation

$$r^2 = \frac{u^2}{g}(r + a).$$

If $r = 2a$ is to be the maximum value achieved by r, then it must be a root of the above equation which in turn implies that $u^2 = \frac{4}{3}ag$. With this value of u (and with $\alpha = \pi/3$), the equation satisfied by r then becomes

$$\dot{r}^2 = \frac{g(r - a)(2a - r)(2a + 3r)}{3r^2},$$

after some simplification. On examining the *sign* of the right side of this equation, we see that r must oscillate periodically between a and $2a$, which is the required result.

It remains to find the period of this motion. In the first half period, r is increasing so that

$$\dot{r} = +\left(\frac{g}{3}\right)^{1/2}\frac{\big((r - a)(2a - r)(2a + 3r)\big)^{1/2}}{r},$$

which is a separable first order ODE for r as a function of t. On separating and integrating, we obtain

$$\int_a^{2a} \frac{r\,dr}{\big((r-a)(2a-r)(2a+3r)\big)^{1/2}} = \left(\frac{g}{3}\right)^{1/2} \int_0^{\tau/2} dt,$$

where τ is the period of the oscillations of r. Hence

$$\begin{aligned}\tau &= 2\left(\frac{3}{g}\right)^{1/2} \int_a^{2a} \frac{r\,dr}{\big((r-a)(2a-r)(2a+3r)\big)^{1/2}}\\ &= 2\sqrt{3}\left(\frac{a}{g}\right)^{1/2} \int_1^{2} \frac{\xi\,d\xi}{\big((\xi-1)(2-\xi)(2+3\xi)\big)^{1/2}},\end{aligned}$$

on making the substitution $r = a\xi$. This is the **time taken** for r to first return to the value $r = a$. A numerical integration shows that $\tau \approx 5.19(a/g)^{1/2}$. ■

Problem 11.9 ✱ *Bug running on a hoop*

A uniform circular hoop of mass M can slide freely on a smooth horizontal table, and a bug of mass m can run on the hoop. The system is at rest when the bug starts to run. What is the angle turned through by the hoop when the bug has completed one lap of the hoop? [This is a classic problem, but difficult. Apply the angular momentum principle about the centre of the hoop, using the *non-standard* version given in Problem 11.1]

Solution

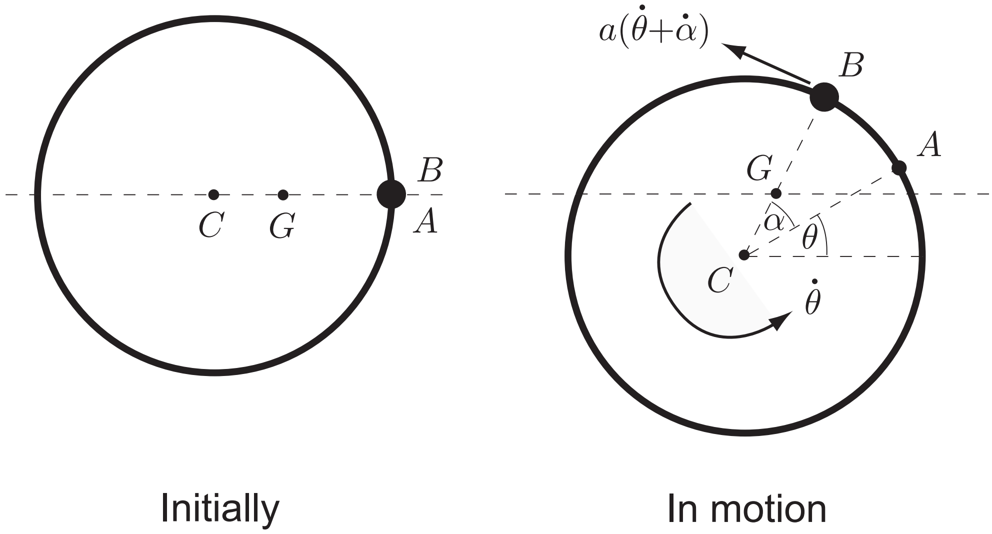

FIGURE 11.8 The bug B runs around the hoop with centre C. Note that the velocity of B shown is not its absolute velocity but that relative to C.

This problem can be solved by using linear and angular momentum conservation principles. Since there are no horizontal forces, and the vertical forces cancel, the total external force is zero. Hence the **linear momentum** is **conserved** which implies that the centre of mass G of the system moves with constant velocity. Moreover, since the system starts from rest, this constant velocity must be zero. Hence G **remains at rest** during the motion.

We will apply the angular momentum principle about C, the centre of the hoop. Since C is not fixed (nor is it the centre of mass of the *whole system*), this requires the *non-standard* form of the **angular momentum principle** given in problem 11.1.

In the present case, $\boldsymbol{K}_C = \boldsymbol{0}$ and, if we take the fixed point G as origin, $\boldsymbol{R} = \boldsymbol{0}$. The principle then reduces to

$$\dot{\boldsymbol{L}}_C = (M + m)\boldsymbol{c}\times\ddot{\boldsymbol{c}},$$

where $\boldsymbol{c}$ is the position vector of C relative to the fixed origin G. Since

$$\boldsymbol{c}\times\ddot{\boldsymbol{c}} = \frac{d}{dt}\left(\boldsymbol{c}\times\dot{\boldsymbol{c}}\right),$$

this equation can be integrated with respect to t to give

$$\boldsymbol{L}_C = (M + m)\boldsymbol{c}\times\dot{\boldsymbol{c}} + \boldsymbol{D},$$

where $\boldsymbol{D}$ is the (vector) integration constant. Moreover, since the motion starts from rest, $\boldsymbol{L}_C$ and $\dot{\boldsymbol{c}}$ are initially zero so that $\boldsymbol{D} = \boldsymbol{0}$. We thus obtain

$$\boldsymbol{L}_C = (M + m)\boldsymbol{c}\times\dot{\boldsymbol{c}}$$

as our (non-standard) **angular momentum conservation equation**.

Suppose that initially the bug B is at a marked point A of the hoop, as shown in Figure 11.8 (left). Suppose also that, after time t, the angle turned through by the hoop is θ while the angular displacement of the bug *relative to the hoop* is α, as shown in Figure 11.8 (right). (In this figure. the angles θ and α are shown with the same sign. This is simply to assist the drawing. If α is positive then θ will turn out to be negative!) Then $\boldsymbol{L}_C$, the angular momentum of the system about C *in its motion relative to C* is

$$\begin{aligned}\boldsymbol{L}_C &= ma\left(a\dot{\theta} + a\dot{\alpha}\right)\boldsymbol{k} + \left(Ma^2\right)\dot{\theta}\boldsymbol{k}\\ &= a^2\left((M + m)\dot{\theta} + m\dot{\alpha}\right)\boldsymbol{k},\end{aligned}$$

where $\boldsymbol{k}$ is the unit vector pointing vertically upwards. Also, since G divides the line CB in the ratio m/M,

$$\boldsymbol{c} = -\left(\frac{m}{M + m}\right)\boldsymbol{b}_C,$$

where $\boldsymbol{b}_C$ is the position vector of the bug *relative to C*. Hence

$$
\begin{aligned}
(M+m)\boldsymbol{c}\times\dot{\boldsymbol{c}} &= \left(\frac{m^2}{M+m}\right)\boldsymbol{b}_C\times\dot{\boldsymbol{b}}_C \\
&= \left(\frac{m^2}{M+m}\right) a\left(a\dot{\theta}+a\dot{\alpha}\right)\boldsymbol{k} \\
&= \left(\frac{m^2a^2}{M+m}\right)\left(\dot{\theta}+\dot{\alpha}\right)\boldsymbol{k}.
\end{aligned}
$$

The angular momentum conservation equation is therefore

$$a^2\left((M+m)\dot{\theta}+m\dot{\alpha}\right)\boldsymbol{k} = \left(\frac{m^2a^2}{M+m}\right)\left(\dot{\theta}+\dot{\alpha}\right)\boldsymbol{k},$$

that is,

$$(M+2m)\dot{\theta} = -m\dot{\alpha},$$

after a little simplification. On integrating with respect to t, we obtain

$$(M+2m)\theta = -m\alpha + D,$$

where D is the integration constant. Since $\theta = 0$ and $\alpha = 0$ when $t = 0$, $D = 0$ and the solution for θ is

$$\theta = -\left(\frac{m}{M+2m}\right)\alpha.$$

This is the angle turned through by the hoop when the bug has advanced to angular displacement α. In particular, when the bug has completed one lap of the hoop, $\alpha = 2\pi$ and the **angle turned through by the hoop** is

$$\frac{2\pi m}{M+2m}$$

in the *opposite* direction to the bug. Note that this result is independent of the details of the bug's motion. ■

Problem 11.10 *General rigid pendulum*

A rigid body of general shape has mass M and can rotate freely about a fixed horizontal axis. The centre of mass of the body is distance h from the rotation axis, and the moment of inertia of the body about the rotation axis is I. Show that the period of small oscillations of the body about the downward equilibrium position is

$$2\pi\left(\frac{I}{Mgh}\right)^{1/2}.$$

Deduce the period of small oscillations of a uniform rod of length $2a$, pivoted about a horizontal axis perpendicular to the rod and distance b from its centre.

Solution

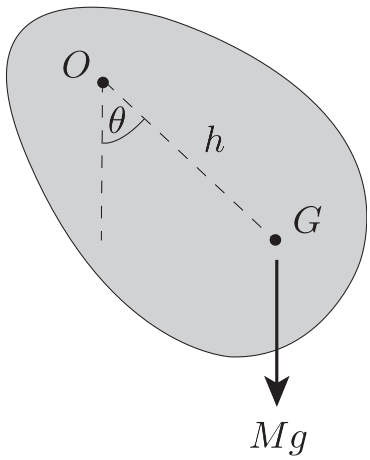

FIGURE 11.9 A rigid body of general shape rotates freely about a fixed horizontal axis through O.

Since the body is constrained to rotate about a fixed axis through O, its equation of motion is the planar angular momentum principle

$$\frac{dL_O}{dt} = K_O.$$

Since the body is *smoothly* pivoted at O, the only contribution to the **planar moment** K_O is from the gravity force so that

$$\begin{aligned} K_O &= (h\sin\theta)Mg \\ &= Mgh\sin\theta \end{aligned}$$

(see Figure 11.9). The **planar angular momentum** L_O is

$$L_O = -I\dot{\theta},$$

where I is the moment of inertia of the body about the rotation axis. Here we are using the sign convention that *clockwise* moments, angular velocities and angular momenta are *positive*.

The **equation of motion** is therefore

$$\frac{d}{dt}\left(-I\dot{\theta}\right) = Mgh\sin\theta,$$

that is,

$$\ddot{\theta} + \left(\frac{Mgh}{I}\right)\sin\theta = 0.$$

For oscillations of small amplitude, this can be approximated by the **linearised equation**

$$\ddot{\theta} + \left(\frac{Mgh}{I}\right)\theta = 0,$$

which is the SHM equation with $\Omega^2 = Mgh/I$. The **period** τ of small oscillations of the body is therefore

$$\tau = 2\pi\left(\frac{I}{Mgh}\right)^{1/2},$$

as required.

For the particular case of the rod, $h = b$ and

$$I = \tfrac{1}{3}Ma^2 + Mb^2 = \tfrac{1}{3}M\left(a^2 + 3b^2\right)$$

so that

$$\tau = 2\pi\left(\frac{a^2 + 3b^2}{3gb}\right)^{1/2}. \blacksquare$$

Problem 11.11 *From sliding to rolling*

A snooker ball is at rest on the table when it is projected forward with speed V and no angular velocity. Find the speed of the ball when it eventually begins to roll. What proportion of the original kinetic energy is lost in the process?

Solution

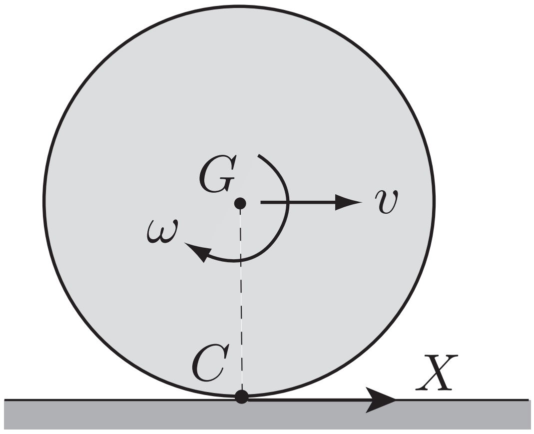

FIGURE 11.10 The snooker ball moves in contact with the table but is not necessarily rolling. Vertical forces are omitted for clarity.

Since the ball moves horizontally, its **planar equations of motion** reduce to

$$M\frac{dV_x}{dt} = F_x,$$

$$I_G\frac{d\omega}{dt} = K_G,$$

that is,

$$M\dot{v} = X,$$

$$\tfrac{2}{5}Ma^2\dot{\omega} = -aX,$$

where v, ω and X are shown in Figure 11.10. Here we are using the sign convention that *clockwise* moments, angular velocities and angular momenta are *positive*. On eliminating the unknown frictional force X, we find that

$$\dot{v} + \tfrac{2}{5}a\dot{\omega} = 0$$

and, on integrating with respect to t, we obtain

$$v + \tfrac{2}{5}a\omega = C,$$

where C is the integration constant. Initially, $v = V$ and $\omega = 0$ so that C = V. We have thus established the non-standard **conservation principle**

$$v + \tfrac{2}{5}a\omega = V$$

which holds in the subsequent motion whether the ball slides or rolls.

Suppose that the **ball eventually rolls** with speed V'. By the rolling condition, its angular velocity will then be V'/a. Then, from the conservation principle,

$$V' + \tfrac{2}{5}a\left(\frac{V'}{a}\right) = V,$$

so that the **speed of the ball when rolling** must be

$$V' = \tfrac{5}{7}V.$$

The final kinetic energy of the ball is therefore

$$\begin{aligned} T' &= \tfrac{1}{2}MV'^2 + \tfrac{1}{2}\left(\tfrac{2}{5}Ma^2\right)\left(\frac{V'}{a}\right)^2 \\ &= \tfrac{7}{10}MV'^2 \\ &= \tfrac{7}{10}M\left(\tfrac{5}{7}V\right)^2 \\ &= \tfrac{5}{14}MV^2 \\ &= \tfrac{5}{7}T, \end{aligned}$$

where T is the *initial* kinetic energy. Hence the **ball loses $\frac{2}{7}$ of its kinetic energy** in the transition from sliding to rolling. ■

Problem 11.12 *Rolling or sliding?*

A uniform ball is released from rest on a rough plane inclined at angle α to the horizontal. The coefficient of friction between the ball and the plane is μ. Will the ball roll or slide down the plane? Find the acceleration of the ball in each case.

Solution

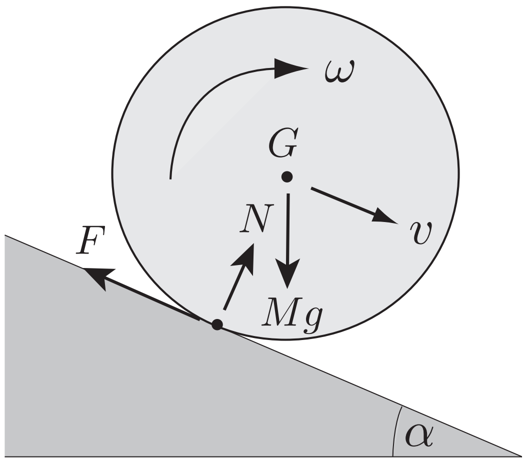

FIGURE 11.11 The ball and the inclined plane in problem 11.12.

The **planar equations of motion** are

$$M\frac{dV_x}{dt} = F_x,$$
$$M\frac{dV_z}{dt} = F_z,$$
$$I_G\frac{d\omega}{dt} = K_G,$$

where the x-axis points *down* the plane. In the present problem, these equations give

$$M\dot{v} = Mg\sin\alpha - F,$$
$$0 = N - Mg\cos\alpha,$$
$$\tfrac{2}{5}Ma^2\dot{\omega} = aF,$$

where v, ω, F and N are shown in Figure 11.11. Here we are using the sign convention that *clockwise* moments, angular velocities and angular momenta are *positive*. The second equation shows that, in all cases, the normal reaction $N = Mg\cos\alpha$.

Rolling Suppose that the ball rolls down the plane. Then, by the rolling condition, when the velocity of the ball is v, its angular velocity must be v/a. In this case, the third equation of motion gives $F = \frac{2}{5}M\dot{v}$ and the first equation then gives the **acceleration** of the ball to be

$$\dot{v} = \tfrac{5}{7}g\sin\alpha.$$

The required **frictional force** F is therefore

$$F = \tfrac{2}{7}Mg\sin\alpha.$$

Thus, in any period of rolling,

$$\frac{F}{N} = \tfrac{2}{7}\tan\alpha.$$

Hence **rolling is impossible** at *any* stage of the motion if the coefficient of friction $\mu < \frac{2}{7}\tan\alpha$. Conversely, if $\mu > \frac{2}{7}\tan\alpha$ *and the motion starts from rest*, then the ball will roll. ■

Sliding Suppose now that $\mu < \frac{2}{7}\tan\alpha$ so that the ball always slides. In this case, F has its maximum value, that is

$$\begin{aligned} F &= \mu N \\ &= \mu Mg\cos\alpha. \end{aligned}$$

The first equation of motion then gives the **acceleration** of the ball to be

$$\dot{v} = (\sin\alpha - \mu\cos\alpha)g. \blacksquare$$

Problem 11.13

A circular disk of mass M and radius a is smoothly pivoted about its axis of symmetry which is fixed in a horizontal position. A bug of mass m runs with constant speed u around the rim of the disk. Initially the disk is held at rest and is released when the bug reaches its lowest point. What is the condition that the bug will reach the highest point of the disk?

Solution

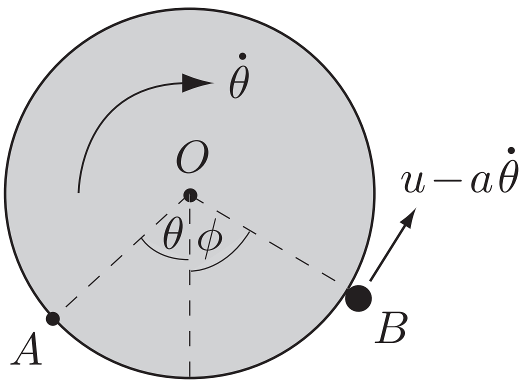

FIGURE 11.12 The bug and the disk in problem 11.13.

We solve this problem by using the planar angular momentum principle

$$\frac{dL_O}{dt} = K_O$$

applied to the *whole system* of the disk *and* the bug.

Let ϕ be the angular displacement of the bug at time t, *measured from the downward vertical*, and let θ be the angle turned through by the disk at this instant, measured in the *opposite* direction to ϕ (see Figure 11.12). Since the bug moves with speed u *relative to the disk*, its velocity relative to a fixed reference frame is $u - a\dot{\theta}$. The planar angular momentum L_O is then

$$L_O = -ma\left(u - a\dot{\theta}\right) + \left(\tfrac{1}{2}Ma^2\right)\dot{\theta}.$$

Here we are using the sign convention that *clockwise* moments, angular velocities and angular momenta are *positive*. Since the disk is smoothly pivoted at O, the only contribution to the **planar moment** K_O is from the gravity force so that

$$K_O = (a \sin\phi) mg$$
$$= mga \sin\phi.$$

The **equation of motion** is therefore

$$\frac{d}{dt}\Big(-ma\big(u - a\dot{\theta}\big) + \big(\tfrac{1}{2}Ma^2\big)\dot{\theta}\Big) = mga \sin\phi,$$

which, since u is constant, simplifies to give

$$(M + 2m)a\ddot{\theta} = 2mg \sin\phi.$$

Since the bug runs with constant speed u, and $\theta = \phi = 0$ when $t = 0$, it follows that

$$a(\theta + \phi) = ut$$

and hence that $\ddot{\theta} = -\ddot{\phi}$. Hence ϕ satisfies the equation

$$(M + 2m)a\ddot{\phi} = -2mg \sin\phi,$$

which is the equation for large amplitude pendulum motion. The initial conditions are $\phi = 0$ and $\dot{\phi} = u/a$ when $t = 0$.

In order to find if the bug reaches the top we need to integrate this equation. On multiplying through by $\dot{\phi}$ and integrating, we obtain the 'energy' equation

$$\tfrac{1}{2}(M + 2m)a\dot{\phi}^2 = 2mg \cos\phi + C,$$

where C is the integration constant. Since $\dot{\phi} = u/a$ when $\phi = 0$,

$$C = \tfrac{1}{2}(M + 2m)\frac{u^2}{a} - 2mg$$

so that ϕ satisfies the first order ODE

$$(M + 2m)a^2\dot{\phi}^2 = (M + 2m)u^2 - 4mga(1 - \cos\phi).$$

If the bug is to reach the top of the disk, $\dot{\phi}$ must remain *positive* for $0 \le \phi \le \pi$. This requires that

$$(M + 2m)u^2 - 4mga(1 - \cos\phi) > 0 \quad \text{for} \quad 0 \le \phi \le \pi$$

which is satisfied if, and only if,

$$(M + 2m)u^2 > 8mga.$$

Hence, the bug **will reach the top** of the disk if, and only if,

$$u^2 > \frac{8mga}{M + 2m}. \blacksquare$$

Problem 11.14 *Yo-yo with moving support*

A uniform circular cylinder (a yo-yo) has a light inextensible string wrapped around it so that it does not slip. The free end of the string is fastened to a support and the yo-yo moves in a vertical straight line with the straight part of the string also vertical. At the same time the support is made to move vertically having upward displacement $Z(t)$ at time t. Find the acceleration of the yo-yo. What happens if the system starts from rest and the support moves upwards with acceleration $2g$?

Solution

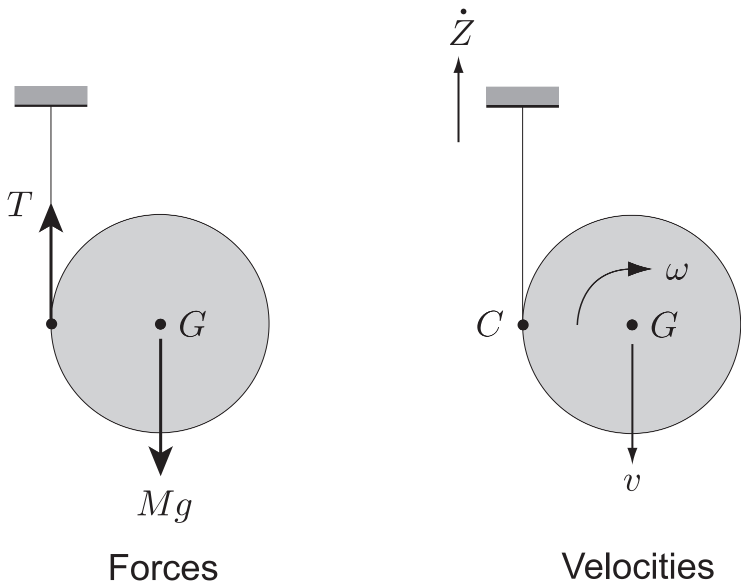

FIGURE 11.13 The yo-yo with a moving support.

Since the yo-yo moves vertically, its **planar equations of motion** reduce to

$$M\frac{dV_z}{dt} = F_z,$$
$$I_G\frac{d\omega}{dt} = K_G,$$

that is,

$$M\dot{v} = Mg - T,$$
$$\tfrac{1}{2}Ma^2\dot{\omega} = aT,$$

where v, ω and T are shown in Figure 11.13. Here we are using the sign convention that *clockwise* moments, angular velocities and angular momenta are *positive*. On eliminating the unknown string tension T, we find that v and ω are related by

$$\dot{v} + \tfrac{1}{2}a\dot{\omega} = g. \qquad (1)$$

Since the string does not slip on the yo-yo, the velocity of the point C of the string must be equal to the velocity of the particle of the yo-yo with which it is in contact. This implies that

$$\dot{Z} = a\omega - v$$

and hence

$$\ddot{Z} = a\dot{\omega} - \dot{v}. \qquad (2)$$

Equations (1) and (2) can now be solved for $\dot{v}$ and $\dot{\omega}$ which gives

$$\dot{v} = \tfrac{2}{3}g - \tfrac{1}{3}\ddot{Z},$$
$$a\dot{\omega} = \tfrac{2}{3}g + \tfrac{2}{3}\ddot{Z}.$$

Thus the downwards **acceleration of the yo-yo** is $\tfrac{2}{3}g - \tfrac{1}{3}\ddot{Z}$.

In particular, when $\ddot{Z} = 2g$, $\dot{v} = 0$ so that the yo-yo moves with **constant velocity**. ■

Problem 11.15 *Supermarket belt*

A circular cylinder, which is axially symmetric but not uniform, has mass M and moment of inertia Mk^2 about its axis of symmetry. The cylinder is placed on a rough horizontal belt at right angles to the direction in which the belt can move. Initially the cylinder and the belt are both at rest when the belt begins to move with velocity $V(t)$. Given that there is no slipping, find the velocity of the cylinder at time t.

Explain why drinks bottles tend to spin on a supermarket belt (instead of moving forwards) if they are placed at right-angles to the belt.

Solution

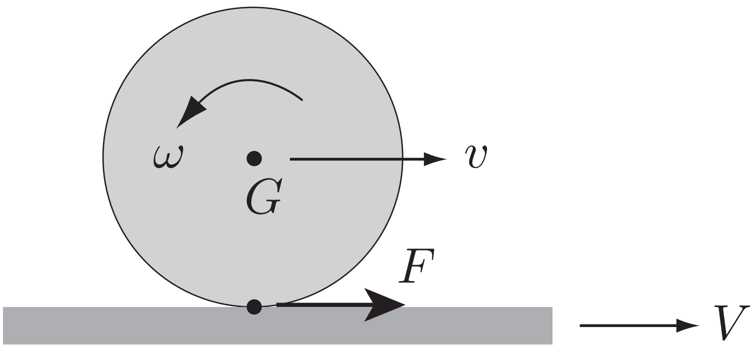

FIGURE 11.14 The cylinder and the belt in problem 11.15. Vertical forces have been omitted for clarity.

Since the cylinder moves horizontally, its **planar equations of motion** reduce to

$$M\frac{dV_x}{dt} = F_x,$$
$$I_G\frac{d\omega}{dt} = K_G,$$

that is,

$$M\dot{v} = F,$$
$$Mk^2(-\dot{\omega}) = -aF,$$

where v, ω and F are shown in Figure 11.14. Here we are using the sign convention that *clockwise* moments, angular velocities and angular momenta are *positive*. On eliminating the unknown frictional force F, we find that

$$a\dot{v} - k^2\dot{\omega} = 0$$

and, on integrating with respect to t, we obtain

$$av - k^2\omega = C,$$

where C is the integration constant. Since the cylinder is initially at rest, $C = 0$ and hence v and ω are related by

$$av - k^2\omega = 0.$$

Since the cylinder does not slip on the belt, the velocity of the belt must be equal to the velocity of the particles of the cylinder with which it is in contact. This implies that

$$V = v + a\omega.$$

These two equations can now be solved for v and ω which gives

$$v = \left(\frac{k^2}{a^2 + k^2}\right) V,$$
$$\omega = \left(\frac{a}{a^2 + k^2}\right) V.$$

This is the **velocity** (and angular velocity) **of the cylinder** at time t.

The drinks bottle is not a rigid body since it is filled with an inviscid fluid (expensive water!). When the bottle begins to rotate, the water hardly moves for several revolutions. The effect is that the bottle and its contents have a *small moment of inertia*, that is, $k \ll a$. In this case, the above formulae imply that $v \ll V$ while $\omega \approx V/a$. Thus the **bottle spins on the belt** instead of moving forwards. ■

Problem 11.16 ✻ *Falling chimney*

A uniform rod of length $2a$ has one end on a rough table and is balanced in the vertically upwards position. The rod is then slightly disturbed. Given that its lower end does not slip, show that, in the subsequent motion, the angle θ that the rod makes with the upward vertical satisfies the equation

$$2a\dot{\theta}^2 = 3g(1 - \cos\theta).$$

Consider now the the *upper part* of the rod of length $2\gamma a$, as shown in book Figure 11.15. Let T, S and K be the tension force, the shear force and the couple exerted on the upper part of the rod by the lower part. By considering the upper part of the rod to be a rigid body in planar motion, find expressions for S and K in terms of θ.

If a tall thin chimney begins to fall, at what point along its length would you expect it to break first?

Solution

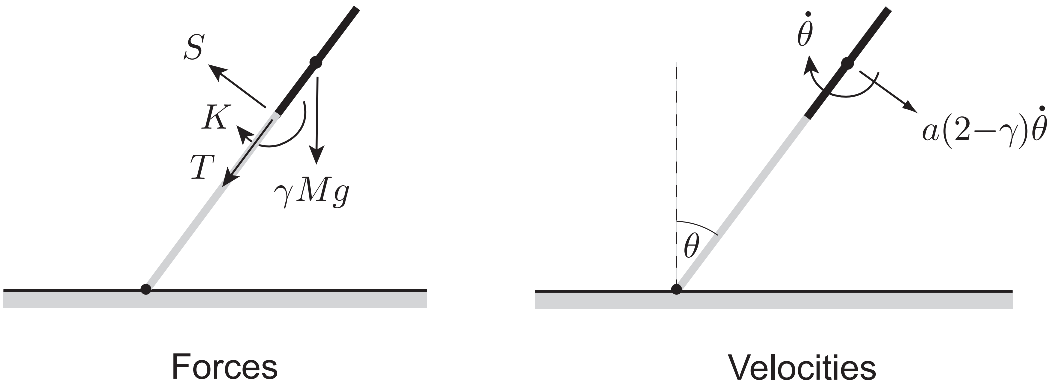

FIGURE 11.15 The falling rod in problem 11.16.

The first part of the problem is a straightforward application of **energy conservation** applied to the *whole rod*. This gives

$$\tfrac{1}{2}M\left(a\dot{\theta}\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{3}Ma^2\right)\dot{\theta}^2 + Mga\cos\theta = E,$$

where E is the constant total energy. Since the rod starts from rest in the vertically upright position, $E = Mga$ and the **energy conservation equation** becomes

$$2a\dot{\theta}^2 = 3g(1 - \cos\theta), \qquad (1)$$

as required.

Condider now the planar motion of the *upper segment* of the rod shown in Figure 11.15. The planar equations for the centre of mass of the segment in the radial and transverse directions are

$$\frac{\big(\gamma M\big)\big((2-\gamma)a\dot{\theta}\big)^2}{(2-\gamma)a} = T + \big(\gamma M\big)g\cos\theta,$$
$$\big(\gamma M\big)\big(a(2-\gamma)\big)\ddot{\theta} = \big(\gamma M\big)g\sin\theta - S,$$

and the planar angular momentum equation about the centre of mass is

$$\left(\tfrac{1}{3}\big(\gamma M\big)\big(\gamma a\big)^2\right)\ddot{\theta} = K + \big(\gamma a\big)S.$$

Here, γ (as defined in the problem) is the ratio of the length of the segment to the length of the whole rod. These three equations can be solved to find the stress and couple resultants T, S and K. This gives

$$T = \gamma(2-\gamma)Ma\dot{\theta}^2 - \gamma Mg\cos\theta,$$
$$S = \gamma Mg\sin\theta - \gamma(2-\gamma)Ma\ddot{\theta},$$
$$K = \tfrac{2}{3}\gamma^2(3-\gamma)Ma^2\ddot{\theta} - \gamma^2 Mga\sin\theta.$$

We now wish to express T, S and K in terms of the angle θ alone. Now $\dot{\theta}^2$ is already given as a function of θ by the energy equation

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\cos\theta).$$

Moreover, if we differentiate this equation with respect to t, we find that $\ddot{\theta}$ is given by

$$\ddot{\theta} = \frac{3g}{4a}\sin\theta.$$

On making use of these relations, we find that the **stress** and **couple resultants** exerted on the segment are given by

$$T = \tfrac{1}{2}\gamma\big(3(2-\gamma) - (8-3\gamma)\cos\theta\big)Mg,$$
$$S = \tfrac{1}{4}\gamma(3\gamma-2)Mg\sin\theta,$$
$$K = \tfrac{1}{2}\gamma^2(1-\gamma)Mga\sin\theta.$$

We model the chimney as a long thin rod whose base does not slip. If the rod is weak (as brick-built chimneys are), it will fracture at the point where the **couple resultant** K is largest. This is because the internal *pointwise* stresses in the rod due to resultants T, S and K are of orders $O\,(T/h)$, $O\,(S/h)$ and $O\left(K/h^2\right)$ respectively, where h is the thickness of the rod. Since h is small, the pointwise stresses due to K predominate. The variation of K along the rod is determined by the function

$$f = \gamma^2(1-\gamma),$$

which is positive for γ in the range $0 < \gamma < 1$. Elementary calculus shows that f achieves its maximum value when $\gamma = \frac{2}{3}$ which is one third way up the rod from the base. *We therefore expect the chimney to break by bending one third way up from the base*. Qualitatively, this is what is observed when tall brick-built chimneys are demolished. Real chimneys are tapered however and the actual bending point is a little higher than that predicted our simple theory (see the photograph below). ■

FIGURE 11.16 The 120 ft chimney of the former Co-op brush factory at Wymondham, Norfolk being demolished in 1988.

Problem 11.17 *Leaning triangular panel*

A rough floor lies in the horizontal plane $z = 0$ and the planes $x = 0$, $y = 0$ are occupied by smooth vertical walls. A rigid uniform triangular panel ABC has mass m. The vertex A of the panel is placed on the floor at the point $(2, 2, 0)$ and the vertices B, C rest in contact with the walls at the points $(0, 1, 6)$, $(1, 0, 6)$ respectively. Given that the vertex A does not slip, find the reactions exerted by the walls. Deduce the reaction exerted by the floor.

Solution

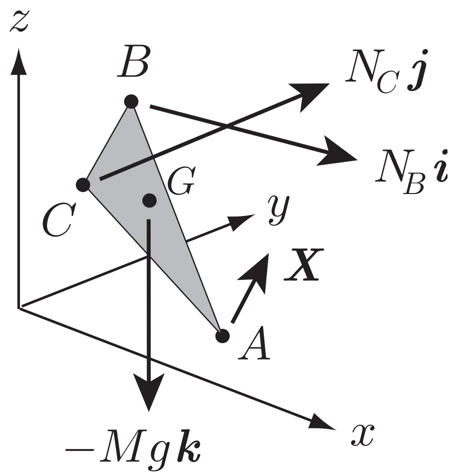

FIGURE 11.17 The triangular panel in problem 11.17.

Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be the position vectors of the vertices A, B, C relative to the origin O. Then

$$\begin{aligned} \boldsymbol{a} &= 2\boldsymbol{i} + 2\boldsymbol{j}, \\ \boldsymbol{b} &= \boldsymbol{j} + 6\boldsymbol{k}, \\ \boldsymbol{c} &= \boldsymbol{i} + 6\boldsymbol{k}, \end{aligned}$$

and $\boldsymbol{R}$, the position vector of the centre of mass G, is given by

$$\begin{aligned} \boldsymbol{R} &= \tfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) \\ &= \boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}. \end{aligned}$$

Since the walls are *smooth*, the reactions they exert are perpendicular to their surfaces. Hence the reactions at B and C are in the $\boldsymbol{i}$- and $\boldsymbol{j}$-directions respectively.

Let the reaction at B be $N_B\boldsymbol{i}$ and the reaction at C be $N_C\,\boldsymbol{j}$. We now apply the **equilibrium conditions**.

(i) The equilibrium condition $\boldsymbol{F} = \boldsymbol{0}$ gives

$$\boldsymbol{X} + N_B\boldsymbol{i} + N_C\,\boldsymbol{j} - Mg\boldsymbol{k} = \boldsymbol{0},$$

where $\boldsymbol{X}$ is the reaction of the floor. (The reaction $\boldsymbol{X}$ need not be vertical because the floor is rough.) This equation merely serves to determine $\boldsymbol{X}$ once N_B and N_C are known.

(ii) The equilibrium condition $\boldsymbol{K}_A = \boldsymbol{0}$ gives

$$\boldsymbol{0}\times\boldsymbol{X} + (\boldsymbol{b}-\boldsymbol{a})\times(N_B\boldsymbol{i}) + (\boldsymbol{c}-\boldsymbol{a})\times(N_C\,\boldsymbol{j}) + (\boldsymbol{R}-\boldsymbol{a})\times(-Mg\boldsymbol{k}) = \boldsymbol{0}.$$

On substituting in the values of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{R}$, this condition reduces to

$$(Mg - 6N_C)\,\boldsymbol{i} + (6N_B - Mg)\,\boldsymbol{i} + (N_B - N_C)\,\boldsymbol{k} = \boldsymbol{0}.$$

Since $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a linearly independent set of vectors, this relation can hold only when all the coefficients are zero, that is, when

$$\begin{aligned} Mg - 6N_C &= 0,\\ 6N_B - Mg &= 0,\\ N_B - N_C &= 0. \end{aligned}$$

Thus we must satisfy three linear equations in only two unknowns. However, the equations *are* consistent and the solution is

$$\begin{aligned} N_B &= \tfrac{1}{6}Mg,\\ N_C &= \tfrac{1}{6}Mg. \end{aligned}$$

These are the required **reactions at the walls**.

Now that N_B and N_C are known, the condition $\boldsymbol{F} = \boldsymbol{0}$ reduces to

$$\boldsymbol{X} + \tfrac{1}{6}Mg\boldsymbol{i} + \tfrac{1}{6}Mg\,\boldsymbol{j} - Mg\boldsymbol{k} = \boldsymbol{0}$$

so that the **reaction at the floor** is

$$\boldsymbol{X} = -\tfrac{1}{6}Mg\boldsymbol{i} - \tfrac{1}{6}Mg\,\boldsymbol{j} + Mg\boldsymbol{k}. \blacksquare$$

Problem 11.18 *Triangular coffee table*

A trendy swedish coffee table has an unsymmetrical triangular glass top supported by a leg at each vertex. Show that, whatever the shape of the triangular top, each leg bears one third of its weight.

Solution

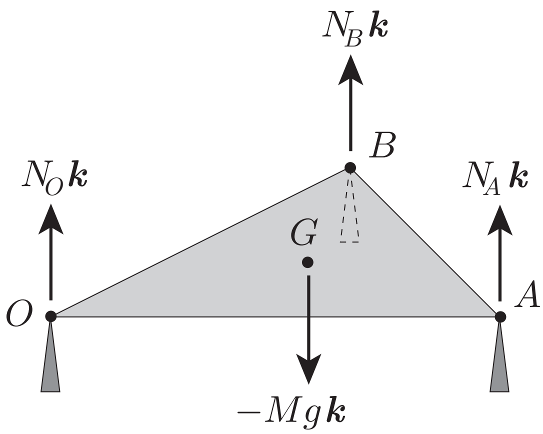

FIGURE 11.18 The table in problem 11.18.

Let the triangle have vertices O, A, B and let $\boldsymbol{a}$, $\boldsymbol{b}$ be the position vectors of the vertices A, B relative to the origin O. Then $\boldsymbol{R}$, the position vector of the centre of mass G, is given by

$$\begin{aligned}\boldsymbol{R} &= \tfrac{1}{3}(\boldsymbol{0} + \boldsymbol{a} + \boldsymbol{b}) \\ &= \tfrac{1}{3}(\boldsymbol{a} + \boldsymbol{b}).\end{aligned}$$

Let the reactions at O, A, B be $N_O\boldsymbol{k}$, $N_A\boldsymbol{k}$, $N_B\boldsymbol{k}$ respectively. We now apply the **equilibrium conditions**.

(i) The equilibrium condition $\boldsymbol{F} = \boldsymbol{0}$ gives

$$N_O\boldsymbol{k} + N_A\boldsymbol{k} + N_B\boldsymbol{k} - Mg\boldsymbol{k} = \boldsymbol{0},$$

that is

$$N_O + N_A + N_B = Mg.$$

This simply means that the sum of the reactions must balance the weight force.

(ii) The equilibrium condition $\boldsymbol{K}_O = \boldsymbol{0}$ gives

$$\boldsymbol{0}\times(N_O\boldsymbol{k}) + \boldsymbol{a}\times(N_A\boldsymbol{k}) + \boldsymbol{b}\times(N_B\boldsymbol{k}) + \boldsymbol{R}\times(-Mg\boldsymbol{k}) = \boldsymbol{0},$$

that is,

$$\left[\left(N_A - \tfrac{1}{3}Mg\right)\boldsymbol{a} + \left(N_B - \tfrac{1}{3}Mg\right)\boldsymbol{b}\right]\times\boldsymbol{k} = \boldsymbol{0},$$

on using that fact that $\boldsymbol{R} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$. This equation is satisfied only when the expression in the square brackets is zero, that is, when

$$\left(N_A - \tfrac{1}{3}Mg\right)\boldsymbol{a} + \left(N_B - \tfrac{1}{3}Mg\right)\boldsymbol{b} = \boldsymbol{0}.$$

Furthermore, since $\boldsymbol{a}$, $\boldsymbol{b}$ are linearly independent vectors, this last relation is satisfied only when both the coefficients are zero, that is, when

$$N_A - \tfrac{1}{3}Mg = 0,$$
$$N_B - \tfrac{1}{3}Mg = 0.$$

Hence $N_A = N_B = \frac{1}{3}Mg$.

Thus the legs at A and B each bear one third of the weight and the first equilibrium condition then implies that $N_O = \frac{1}{3}Mg$. Hence **each leg bears one third of the weight**. ■

Problem 11.19 *Pile of balls*

Three identical balls are placed in contact with each other on a horizontal table and a fourth identical ball is placed on top of the first three. Show that the four balls cannot be in equilibrium unless (i) the coefficient of friction between the balls is at least $\sqrt{3}-\sqrt{2}$, and (ii) the coefficient of friction between each ball and the table is at least $\frac{1}{4}(\sqrt{3}-\sqrt{2})$.

Solution

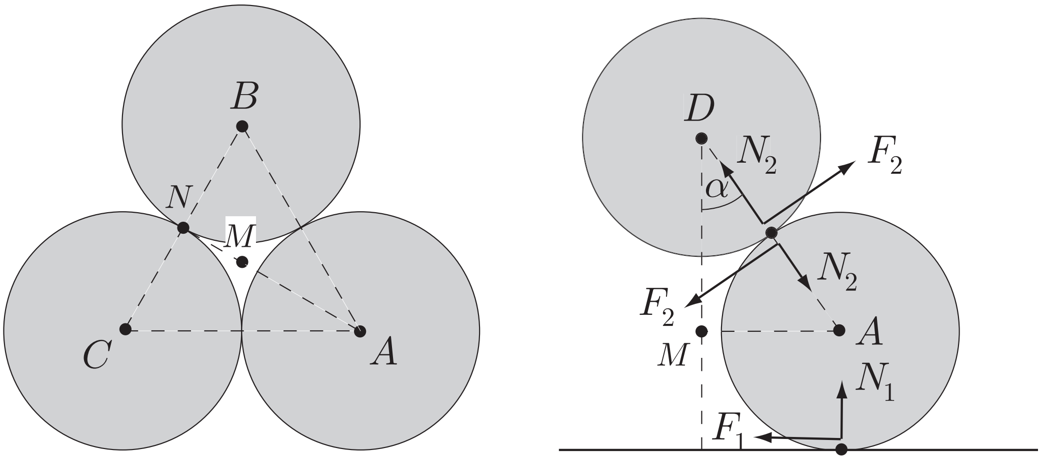

FIGURE 11.19 The balls in problem 11.19. **Left** The lower three balls seen from above. **Right** The upper ball and *one* of the lower balls seen from the side. Gravity forces are omitted for clarity.

Consider first the equilibrium of one of the lower balls. Rotational equilibrium implies that $F_1 = F_2$ (see Figure 11.19 (right)) and we will denote the common value of these forces by F. Horizontal and vertical equilibrium then imply that

$$N_2 \sin\alpha - F - F\cos\alpha = 0,$$
$$N_1 - N_2\cos\alpha - Mg = 0,$$

where α is the angle shown in Figure 11.19 (right). We will show later that $\alpha = \sin^{-1}\sqrt{\frac{1}{3}}$, but it is easier not to use this fact at the moment.

Now consider the equilibrium of the upper ball. Vertical equilibrium implies that

$$3N_2\cos\alpha + 3F\sin\alpha - Mg = 0,$$

while horizontal and rotational equilibrium are automatically satisfied by symmetry.

We thus have three equations for the unknown forces N_1, N_2 and F. On solving, we find that

$$F = \frac{Mg \sin\alpha}{3(1+\cos\alpha)},$$
$$N_1 = \tfrac{4}{3}Mg,$$
$$N_2 = \tfrac{1}{3}Mg.$$

(The formula $N_1 = \frac{4}{3}Mg$ follows immediately from the vertical equilibrium of the whole system of four balls. The formula $N_2 = \frac{1}{3}Mg$ does not seem to have a simple explanation.) Hence

$$\frac{F}{N_1} = \frac{\sin\alpha}{4(1+\cos\alpha)},$$
$$\frac{F}{N_2} = \frac{\sin\alpha}{1+\cos\alpha}.$$

It follows that if μ_T is the coefficient of friction between each ball and the table, and μ_B is the coefficient of friction between any two balls, then, for the balls to be in equilibrium, the inequalities

$$\mu_T > \frac{\sin\alpha}{4(1+\cos\alpha)},$$
$$\mu_B > \frac{\sin\alpha}{1+\cos\alpha},$$

must both be satisfied.

It remains to evaluate the angle α. In Figure 11.19 (left), ABC is an equilateral triangle of side $2a$ and M is its median centre. Then

$$AM = \tfrac{2}{3}AN = \tfrac{2}{3}\big(\sqrt{3}a\big) = \frac{2a}{\sqrt{3}}.$$

This is the same distance AM shown in Figure 11.19 (right). Hence

$$\sin\alpha = \frac{AM}{AD} = \frac{1}{\sqrt{3}}$$

and hence

$$\cos\alpha = \sqrt{\frac{2}{3}}.$$

On substituting in these numerical values, we find that, **for the balls to be equilibrium**, the inequalities

$$\mu_T > \tfrac{1}{4}\left(\sqrt{3} - \sqrt{2}\right) \approx 0.08,$$
$$\mu_B > \sqrt{3} - \sqrt{2} \approx 0.32$$

must both be satisfied. Tennis balls can be stacked in this way, but snooker balls cannot. ■

Chapter Twelve

Lagrange's equations

and conservation principles

Problem 12.1

A bicycle chain consists of N freely jointed links forming a closed loop. The chain can slide freely on a smooth horizontal table. How many degrees of freedom has the chain? How many conserved quantities are there in the motion? What is the maximum number of links the chain can have for its motion to be determined by conservation principles alone?

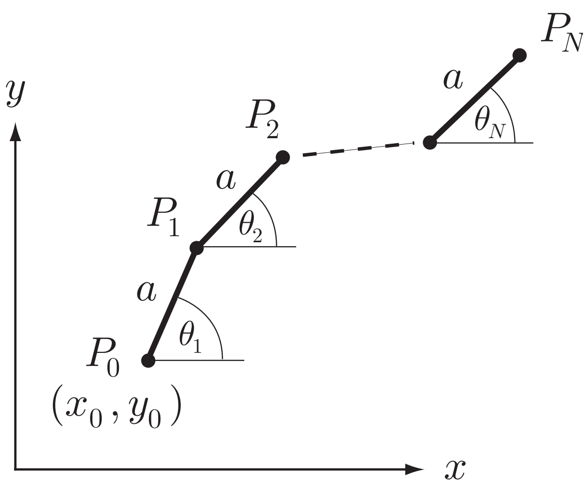

FIGURE 12.1 Generalised coordinates for an *unclosed* chain with N links.

Solution

Suppose first that the chain is *unclosed* as shown in Figure 12.1. Then the Cartesian coordinates x_0, y_0 together with the angles $\theta_1, \theta_2, \ldots \theta_N$ are sufficient to determine its position on the table. Since these variables are also independent, they are therefore a set of generalised coordinates for the unclosed chain. The **unclosed chain** with N links therefore has $N + 2$ degrees of freedom.

Now suppose that the chain is *closed*. The previous coordinates still specify the position of the chain, but now they are not independent since the point P_N must coincide with the point P_0. The Cartesian coordinates of P_N are given by

$$x_N = x_0 + a\cos\theta_1 + a\cos\theta_2 + \cdots + a\cos\theta_N,$$
$$y_N = y_0 + a\sin\theta_1 + a\sin\theta_2 + \cdots + a\sin\theta_N,$$

and so the old coordinates must satisfy the two functional relations

$$a\cos\theta_1 + a\cos\theta_2 + \cdots + a\cos\theta_N = 0,$$
$$a\sin\theta_1 + a\sin\theta_2 + \cdots + a\sin\theta_N = 0.$$

These functional relations reduce the number of degrees of freedom by two so that the **closed chain** has N degrees of freedom.

There are four conserved quantities, namely, the linear momentum components P_x, P_y, the angular momentum component L_z (about any fixed point on the table) and the kinetic energy T. These are sufficient to determine the motion of an **unclosed chain** with **two** links, or a **closed chain** with **four** links. ■

Problem 12.2 *Attwood's machine*

A uniform circular pulley of mass $2m$ can rotate freely about its axis of symmetry which is fixed in a horizontal position. Two masses m, $3m$ are connected by a light inextensible string which passes over the pulley without slipping. The whole system undergoes planar motion with the masses moving vertically. Take the rotation angle of the pulley as generalised coordinate and obtain Lagrange's equation for the motion. Deduce the upward acceleration of the mass m.

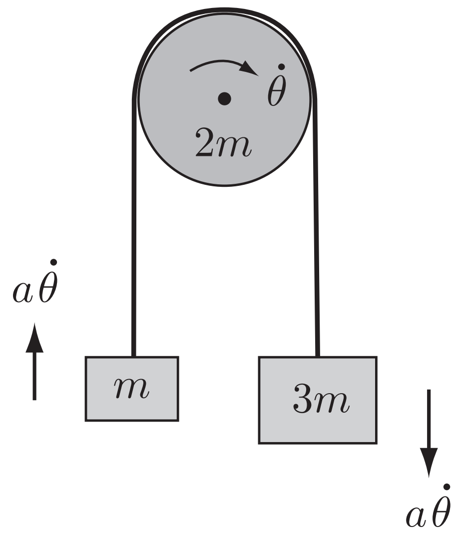

FIGURE 12.2 The velocity diagram for the single Attwood machine.

Solution

Let θ be the rotation angle of the pulley measured from some reference configuration. Then the velocity diagram is shown in Figure 12.2. The **kinetic energy** of the system is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(a\dot{\theta}\right)^2 + \tfrac{1}{2}(3m)\left(a\dot{\theta}\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{2}(2m)a^2\right)\dot{\theta}^2 \\ &= \tfrac{5}{2}ma^2\dot{\theta}^2 \end{aligned}$$

and the **potential energy** relative to the reference configuration is

$$\begin{aligned} V &= mg\,(a\theta) + (3m)g\,(-a\theta) \\ &= -2mga\theta. \end{aligned}$$

Lagrange's equation for the system is therefore

$$\frac{d}{dt}\left(5ma^2\dot{\theta}\right) - 0 = 2mga,$$

that is,

$$a\ddot{\theta} = \tfrac{2}{5}g.$$

The **upward acceleration** of the mass m is therefore $\frac{2}{5}g$. ■

Problem 12.3 *Double Attwood machine*

A light pulley can rotate freely about its axis of symmetry which is fixed in a horizontal position. A light inextensible string passes over the pulley. At one end the string carries a mass $4m$, while the other end supports a second light pulley. A second string passes over this pulley and carries masses m and $4m$ at its ends. The whole system undergoes planar motion with the masses moving vertically. Find Lagrange's equations and deduce the acceleration of each of the masses.

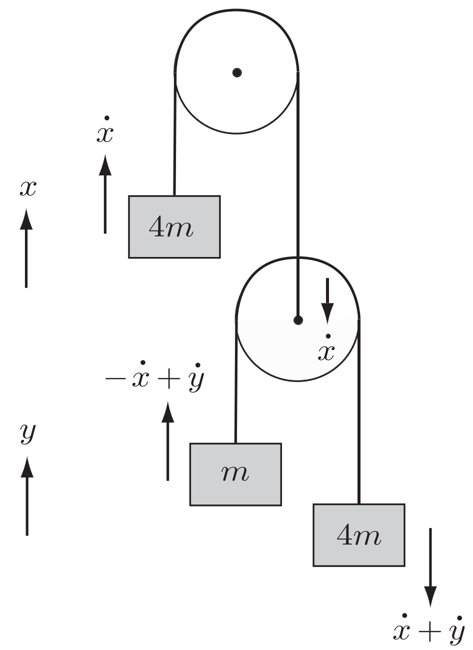

FIGURE 12.3 The coordinates and velocity diagram for the double Attwood machine. Note that the displacement y is measured relative to the centre of the lower pulley.

Solution

Let x be the upward displacement of the first mass $4m$, and let y be the upward displacement of the mass m *measured relative to the centre of the lower pulley*. Then the velocity diagram is shown in Figure 12.3. The **kinetic energy** of the system is

$$\begin{aligned} T &= \tfrac{1}{2}(4m)\dot{x}^2 + \tfrac{1}{2}m\left(-\dot{x} + \dot{y}\right)^2 + \tfrac{1}{2}(4m)\left(\dot{x} + \dot{y}\right)^2 \\ &= \tfrac{1}{2}m\left(9\dot{x}^2 + 6\dot{x}\dot{y} + 5\dot{y}^2\right) \end{aligned}$$

and the **potential energy** is

$$\begin{aligned} V &= (4m)gx + mg(-x + y) - 4mg(x + y) \\ &= -mgx - 3mgy. \end{aligned}$$

Lagrange equations for the system are therefore

$$\frac{d}{dt}(9\dot{x} + 3\dot{y}) = g,$$
$$\frac{d}{dt}(3\dot{x} + 5\dot{y}) = 3g,$$

that is,

$$9\ddot{x} + 3\ddot{y} = g,$$
$$3\ddot{x} + 5\ddot{y} = 3g.$$

These simultaneous linear equations have the solution

$$\ddot{x} = -\tfrac{1}{9}\,g, \qquad \ddot{y} = \tfrac{2}{3}\,g.$$

The **accelerations** of the three masses are therefore $\frac{1}{9}\,g$ downwards, $\frac{7}{9}\,g$ upwards, and $\frac{5}{9}\,g$ downwards. ■

Problem 12.4 *The swinging door*

A uniform rectangular door of width $2a$ can swing freely on its hinges. The door is misaligned and the line of the hinges makes an angle α with the upward vertical. Take the rotation angle of the door from its equilibrium position as generalised coordinate and obtain Lagrange's equation for the motion. Deduce the period of small oscillations of the door about the equilibrium position.

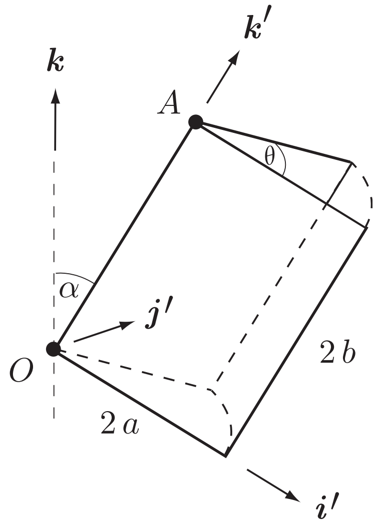

FIGURE 12.4 The door is pivoted about the fixed axis OA which makes an angle α with the upward vertical. The angle θ is the opening angle of the door from its equilibrium position.

Solution

Let $\{\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'\}$ be a standard set of basis vectors with $\boldsymbol{k}'$ along the line of the hinges and $\boldsymbol{i}'$ along the equilibrium position of the bottom edge of the door, as shown in Figure 12.4; the unit vector $\boldsymbol{k}$ points vertically upwards. Let θ be the opening angle of the door. Then the **kinetic energy** of the door is

$$\begin{aligned} T &= \tfrac{1}{2} I_{OA}\dot{\theta}^2 \\ &= \tfrac{1}{2}\left(\tfrac{1}{3}Ma^2 + Ma^2\right)\dot{\theta}^2 \\ &= \tfrac{2}{3}Ma^2\dot{\theta}^2. \end{aligned}$$

To find the **potential energy** of the door we need to find the vertical displacement of the centre of mass G when the door is opened. Relative to O, the position vector of G is $a\boldsymbol{i}' + b\boldsymbol{k}'$ in the equilibrium position and $a\cos\theta\,\boldsymbol{i}' + a\sin\theta\,\boldsymbol{j}' + b\boldsymbol{k}'$ in the open position. The displacement of G when the door is opened through an angle θ is therefore

$$a(\cos\theta - 1)\boldsymbol{i'} + a\sin\theta\,\boldsymbol{j'}.$$

The *vertical component* of this displacement is

$$\begin{aligned}\left(a(\cos\theta - 1)\boldsymbol{i'} + a\sin\theta\,\boldsymbol{j'}\right)\cdot\boldsymbol{k} &= a(\cos\theta - 1)(\boldsymbol{i'}\cdot\boldsymbol{k}) + a\sin\theta(\boldsymbol{j'}\cdot\boldsymbol{k})\\ &= a(\cos\theta - 1)(-\sin\alpha)\ +\ 0\\ &= a\sin\alpha(1 - \cos\theta).\end{aligned}$$

The potential energy of the door is therefore

$$V = Mga\sin\alpha(1 - \cos\theta).$$

Lagrange's equation for the door is therefore

$$\tfrac{4}{3}Ma^2\ddot{\theta} = -Mga\sin\alpha\sin\theta,$$

that is

$$\ddot{\theta} + \left(\frac{3g\sin\alpha}{4a}\right)\sin\theta = 0.$$

This is the equation for large oscillations of the door. The linearised equation for small oscillations is

$$\ddot{\theta} + \left(\frac{3g\sin\alpha}{4a}\right)\theta = 0$$

and the **period** of small oscillations is therefore

$$4\pi\left(\frac{a}{3g\sin\alpha}\right)^{1/2}. \blacksquare$$

Problem 12.5

A uniform solid cylinder $\mathcal{C}$ with mass m and radius a rolls on the rough outer surface of a fixed horizontal cylinder of radius b. In the motion, the axes of the two cylinders remain parallel to each other. Let θ be the angle between the plane containing the cylinder axes and the upward vertical. Taking θ as generalised coordinate, obtain Lagrange's equation and verify that it is equivalent to the energy conservation equation.

Initially the cylinder $\mathcal{C}$ is at rest on top of the fixed cylinder when it is given a very small disturbance. Find, as a function of θ, the normal component of the reaction force exerted on $\mathcal{C}$. Deduce that $\mathcal{C}$ will leave the fixed cylinder when $\theta = \cos^{-1}(4/7)$. Is the assumption that rolling persists up to this moment realistic?

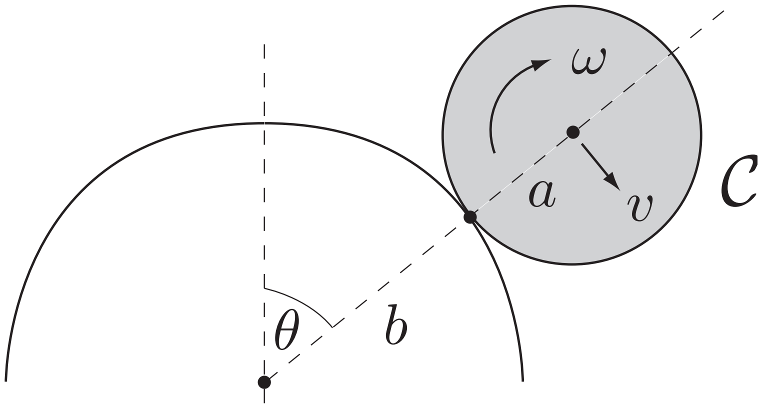

FIGURE 12.5 A uniform solid cylinder $\mathcal{C}$ of radius a rolls on the rough outer surface of a fixed horizontal cylinder of radius b.

Solution

Let v the velocity of the centre of mass of the cylinder $\mathcal{C}$, and ω its angular velocity. Then $v = (a + b)\dot{\theta}$ and, by the rolling condition,

$$\omega = \left(\frac{a+b}{a}\right)\dot{\theta}.$$

The **kinetic energy** of $\mathcal{C}$ is therefore

$$\begin{aligned} T &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_G\,\omega^2 \\ &= \tfrac{1}{2}m(a+b)^2\dot{\theta}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}ma^2\right)\left(\frac{a+b}{a}\right)^2\dot{\theta}^2 \\ &= \tfrac{3}{4}m(a+b)^2\dot{\theta}^2. \end{aligned}$$

The **potential energy** of $\mathcal{C}$ (relative to the centre of the fixed cylinder) is

$$V = mg(a+b)\cos\theta.$$

Lagrange's equation for the cylinder is therefore

$$\tfrac{3}{2}m(a+b)^2\ddot{\theta} = mg(a+b)\sin\theta,$$

that is,

$$\ddot{\theta} = \frac{2g}{3(a+b)}\sin\theta.$$

The **energy equation** $T + V = E$ is

$$\tfrac{3}{4}m(a+b)^2\dot{\theta}^2 + mg(a+b)\cos\theta = E,$$

which, on differentiation with respect to t, gives Lagrange's equation. The two equations are therefore equivalent.

On using the initial conditions $\theta = 0$ and $\dot{\theta} = 0$ when $t = 0$ we find that the total energy E is given by

$$E = mg(a+b),$$

and the energy equation becomes

$$\dot{\theta}^2 = \frac{4g}{3(a+b)}(1-\cos\theta).$$

To find when $\mathcal{C}$ leaves the fixed cylinder, we need to find the normal reaction N that it exerts on $\mathcal{C}$. To do this we apply the centre of mass form of the linear momentum principle. Its *normal* component gives

$$\begin{aligned} mg\cos\theta - N &= \frac{mv^2}{a+b} \\ &= \frac{m(a+b)^2\dot{\theta}^2}{a+b} \\ &= \tfrac{4}{3}mg(1-\cos\theta), \end{aligned}$$

on using the energy equation. It follows that

$$N = \tfrac{1}{3}mg(7\cos\theta - 4).$$

The cylinder $\mathcal{C}$ will leave the fixed cylinder when $N = 0$, that is, when $\theta = \cos^{-1}\tfrac{4}{7}$, which is approximately $55°$.

The assumption that rolling persists up to this moment is not realistic. For any finite coefficient of friction, slipping will occur first. ■

Problem 12.6

A uniform disk of mass M and radius a can roll along a rough horizontal rail. A particle of mass m is suspended from the centre C of the disk by a light inextensible string of length b. The whole system moves in the vertical plane through the rail. Take as generalised coordinates x, the horizontal displacement of C, and θ, the angle between the string and the downward vertical. Obtain Lagrange's equations. Show that x is a cyclic coordinate and find the corresponding conserved momentum p_x. Is p_x the horizontal linear momentum of the system?

Given that θ remains small in the motion, find the period of small oscillations of the particle.

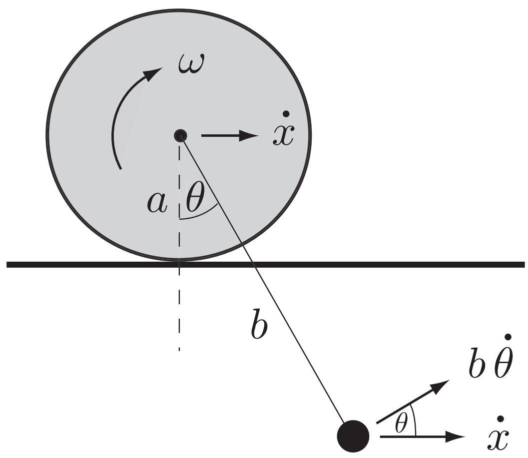

FIGURE 12.6 The velocity diagram for the system in Problem 12.6.

Solution

The velocity diagram for the system is shown in Figure 12.6. On applying the rolling condition, the angular velocity ω of the disk is given by $\omega = \dot{x}/a$. The kinetic energy of the disk is

$$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}Ma^2\right)\left(\frac{\dot{x}}{a}\right)^2 = \tfrac{3}{4}M\dot{x}^2.$$

and the kinetic energy of the particle is

$$\tfrac{1}{2}m\left(\dot{x}^2 + (b\dot{\theta})^2 + 2\dot{x}(b\dot{\theta})\cos\theta\right).$$

The total **kinetic energy** of the system is therefore

$$T = \tfrac{3}{4}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}\cos\theta\right).$$

The **potential energy** of the system (relative to the centre of the disk) is

$$V = -mgb\cos\theta.$$

Since $\partial T/\partial x$ and $\partial V/\partial x$ are both zero, the coordinate x is **cyclic**. The conserved momentum p_x is

$$\begin{aligned} p_x &= \frac{\partial T}{\partial \dot{x}} \\ &= \tfrac{3}{2}M\dot{x} + m(\dot{x} + b\dot{\theta}\cos\theta). \end{aligned}$$

This is *not* the same as the horizontal component of linear momentum, which is

$$M\dot{x} + m(\dot{x} + b\dot{\theta}\cos\theta).$$

Lagrange's equations for the system are therefore

$$\frac{d}{dt}\left(\tfrac{3}{2}M\dot{x} + m(\dot{x} + b\dot{\theta}\cos\theta)\right) - 0 = 0,$$
$$\frac{d}{dt}\left(mb^2\dot{\theta} + mb\dot{x}\cos\theta\right) - \left(-mb\dot{x}\dot{\theta}\sin\theta\right) = -mgb\sin\theta.$$

On expanding these equations and eliminating $\ddot{x}$, we find that θ satisfies the equation

$$\left(3M + 2m\sin^2\theta\right)\ddot{\theta} + 2m\sin\theta\cos\theta\,\dot{\theta}^2 + \left(\frac{(3M+2m)g}{b}\right)\sin\theta = 0.$$

This is the equation for large oscillations of the particle. The linearised equation for small oscillations is

$$\ddot{\theta} + \left(\frac{(3M+2m)g}{3Mb}\right)\theta = 0$$

and the **period** of small oscillations is therefore

$$2\pi\left(\frac{3Mb}{(3M+2m)g}\right)^{1/2}. \blacksquare$$

Problem 12.7

A uniform ball of mass m rolls down a rough wedge of mass M and angle α, which itself can slide on a smooth horizontal table. The whole system undergoes planar motion. How many degrees of freedom has this system? Obtain Lagrange's equations. For the special case in which $M = 3m/2$, find (i) the acceleration of the wedge, and (ii) the acceleration of the ball relative to the wedge.

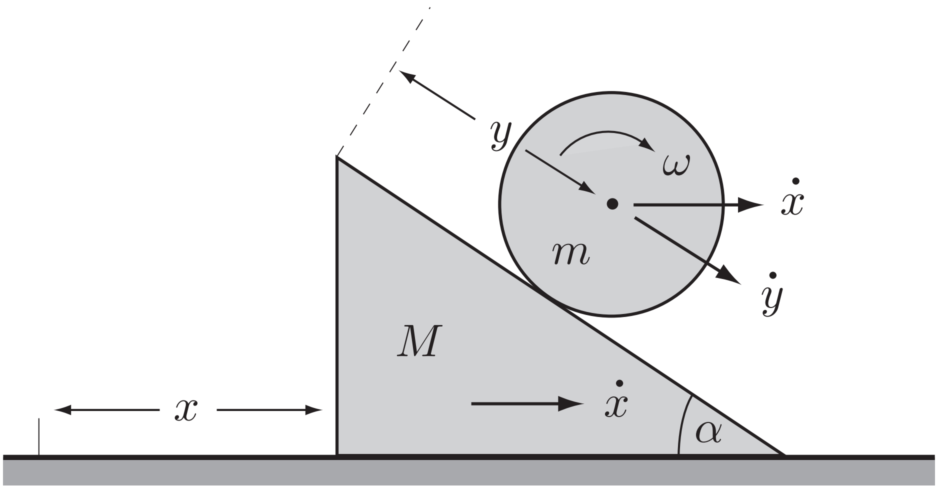

FIGURE 12.7 The coordinates and the velocity diagram from the system in Problem 12.7

Solution

The velocity diagram for the system is shown in Figure 12.7. On applying the rolling condition, the angular velocity ω of the ball is given by $\omega = \dot{y}/a$. The total **kinetic energy** of the system is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha\right) + \tfrac{1}{2}\left(\tfrac{2}{5}ma^2\right)\left(\frac{\dot{y}}{a}\right)^2 \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \tfrac{7}{5}\dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha\right) \end{aligned}$$

and the **potential energy** of the system (relative to the reference position) is

$$V = -mgy\sin\alpha.$$

Lagrange's equations for the system are therefore

$$\frac{d}{dt}\Big(M\dot{x} + m(\dot{x} + \cos\alpha\,\dot{y})\Big) - 0 = 0,$$
$$\frac{d}{dt}\left(\tfrac{7}{5}m\dot{y} + m\cos\alpha\,\dot{x}\right) - 0 = mg\sin\alpha.$$

For the particular case in which $M = \frac{3}{2}m$, these equations become

$$5\ddot{x} + 2\cos\alpha\,\ddot{y} = 0,$$
$$5\cos\alpha\,\ddot{x} + 7\ddot{y} = 5g\sin\alpha,$$

which give the required **accelerations** to be

$$\ddot{x} = -\frac{2g\sin\alpha\cos\alpha}{7 - 2\cos^2\alpha}, \qquad \ddot{y} = \frac{5g\sin\alpha}{7 - 2\cos^2\alpha}. \blacksquare$$

Problem 12.8

A rigid rod of length $2a$ has its lower end in contact with a smooth horizontal floor. Initially the rod is at an angle α to the upward vertical when it is released from rest. The subsequent motion takes place in a vertical plane. Take as generalised coordinates x, the horizontal displacement of the centre of the rod, and θ, the angle between the rod and the upward vertical. Obtain Lagrange's equations. Show that x remains constant in the motion and verify that the θ-equation is equivalent to the energy conservation equation.

✱ Find, in terms of the angle θ, the reaction exerted on the rod by the floor.

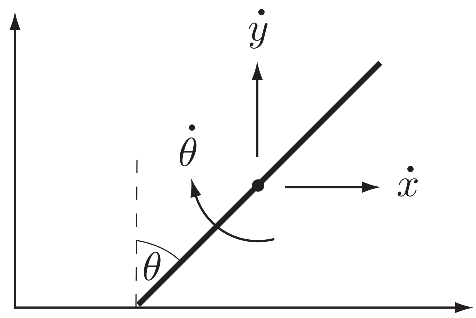

FIGURE 12.8 The velocity diagram for the rod in terms of the *non-independent* coordinates x, y, θ.

Solution

Let x, y be the Cartesian coordinates of the centre of mass of the rod, and let θ be the angle between the rod and the upward vertical. These three coordinates are *not independent* since $y = a\cos\theta$. In order to express T and V in terms of the generalised coordinates x and θ, we will need to eliminate y and $\dot{y}$. However, since $y = a\cos\theta$ it follows that $\dot{y} = -a\sin\theta\,\dot{\theta}$.

The **kinetic energy** of the rod is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2}I_G\dot{\theta}^2 \\ &= \tfrac{1}{2}m\left(\dot{x}^2 + \left(-a\sin\theta\,\dot{\theta}\right)^2\right) + \tfrac{1}{2}\left(\tfrac{1}{3}ma^2\right)\dot{\theta}^2 \\ &= \tfrac{1}{2}m\left(\dot{x}^2 + a^2\left(\tfrac{1}{3} + \sin^2\theta\right)\dot{\theta}^2\right). \end{aligned}$$

The **potential energy** of the rod (relative to the ground) is

$$V = mgy = mga\cos\theta.$$

Since $\partial T/\partial x$ and $\partial V/\partial x$ are both zero, the coordinate x is **cyclic**. The conserved momentum p_x is

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x},$$

which is the horizontal component of linear momentum of the rod. Thus $\dot{x}$ is a constant of the motion. But since the rod is initially at rest, $\dot{x} = 0$ initially and so must *always* be zero. Hence x is also **constant** in this motion.

The Lagrange equation for θ is

$$\frac{d}{dt}\left(ma^2\left(\tfrac{1}{3} + \sin^2\theta\right)\dot{\theta}\right) - \left(ma^2 \sin\theta\cos\theta\,\dot{\theta}^2\right) = mga\sin\theta,$$

which, after simplification, becomes

$$a\left(\tfrac{1}{3} + \sin^2\theta\right)\ddot{\theta} + a\sin\theta\cos\theta\,\dot{\theta}^2 = g\sin\theta.$$

The **energy equation** $T + V = E$ is

$$\tfrac{1}{2}m\left(\dot{x}^2 + a^2\left(\tfrac{1}{3} + \sin^2\theta\right)\dot{\theta}^2\right) + mga\cos\theta = E,$$

which becomes

$$\tfrac{1}{2}ma^2\left(\tfrac{1}{3} + \sin^2\theta\right)\dot{\theta}^2 + mga\cos\theta = E,$$

on using the linear momentum conservation equation $m\dot{x} = 0$. On differentiation with respect to t, this gives Lagrange's equation for θ. The two equations are therefore equivalent.

On using the initial conditions $\theta = \alpha$ and $\dot{\theta} = 0$ when $t = 0$, we find that the total energy E is given by $E = mga\cos\alpha$, and the energy equation becomes

$$a\left(\tfrac{1}{3} + \sin^2\theta\right)\dot{\theta}^2 = 2g(\cos\alpha - \cos\theta).$$

To find the normal reaction N exerted by the floor, we apply the **angular momentum** principle about the centre of mass G of the rod. This gives

$$\frac{d}{dt}\left(I_G\dot{\theta}\right) = N(a\sin\theta) + 0,$$

since gravity has no moment about G. Hence, since $I_G = \frac{1}{3}ma^2$, N is given by

$$N = \frac{ma\,\ddot{\theta}}{3\sin\theta} \quad .$$

Now $\ddot{\theta}$ can be expressed in terms of θ and $\dot{\theta}^2$ by using the Lagrange equation for θ, and $\dot{\theta}^2$ can, in turn, be expressed in terms of θ by using the energy equation. After some labour, this gives the **normal reaction** of the floor to be

$$N = \frac{mg}{\left(1 + 3\sin^2\theta\right)^2}\left(4 + 3\cos^2\theta - 6\cos\alpha\cos\theta\right). \blacksquare$$

Problem 12.9

A particle P is connected to one end of a light inextensible string which passes through a small hole O in a smooth horizontal table and extends below the table in a vertical straight line. P slides on the upper surface of the table while the string is pulled downwards from below in a prescribed manner. (Suppose that the length of the horizontal part of the string is $R(t)$ at time t.) Take θ, the angle between OP and some fixed reference line in the table, as generalised coordinate and obtain Lagrange's equation. Show that θ is a cyclic coordinate and find (and identify) the corresponding conserved momentum p_θ. Why is the kinetic energy not conserved?

If the constant value of p_θ is mL, find the tension in the string at time t.

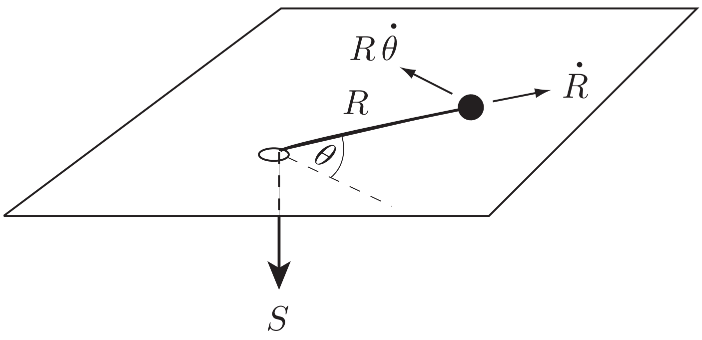

FIGURE 12.9 The velocity diagram for the system in Problem 12.9.

Solution

The velocity diagram is shown in Figure 12.9. Remember that θ is the generalised coordinate while R is a prescribed function of the time t. The **kinetic energy** of the particle is

$$T = \tfrac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2\right)$$

and the **potential energy** (relative to the table top) is zero.

Since $\partial T/\partial\theta$ and $\partial V/\partial\theta$ are both zero, θ is a **cyclic** coordinate. The corresponding conserved momentum p_θ is given by

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = mR^2\dot{\theta},$$

which is the vertical component of the **angular momentum** of the particle about O. The kinetic energy of the particle is not conserved because the tension S does work (but no virtual work!).

To find the tension S, apply the second law to the particle, resolved in the radially outwards direction. Then

$$\begin{aligned} -S &= m\left(\ddot{R} - R\dot{\theta}^2\right) \\ &= m\left(\ddot{R} - \frac{L^2}{R^3}\right), \end{aligned}$$

on using the fact that $p_\theta = mL$. Hence the **tension** in the string is

$$S = m\left(\frac{L^2}{R^3} - \ddot{R}\right).$$

Thus (unless $L = 0$) it is impossible to pull the particle through the hole!

Problem 12.10

A particle P of mass m can slide along a smooth rigid straight wire. The wire has one of its points fixed at the origin O, and is made to rotate in the (x, y)-plane with angular speed Ω. Take r, the distance of P from O, as generalised coordinate and obtain Lagrange's equation.

Initially the particle is a distance a from O and is at rest relative to the wire. Find its position at time t. Find also the energy function h and show that it is conserved even though there is a time dependent constraint.

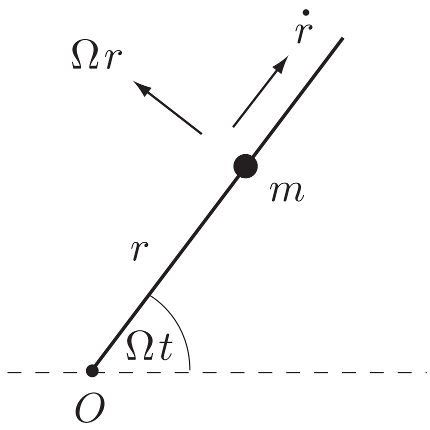

FIGURE 12.10 The velocity diagram for the particle sliding on a rotating wire.

Solution

The velocity diagram is shown in Figure 12.10. Remember that r is the generalised coordinate while Ω is a prescribed constant. The **kinetic energy** of the particle is

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + \Omega^2 r^2\right)$$

and, since there is no gravity, the **potential energy** is zero.

The **Lagrange equation** for the coordinate r is therefore

$$\frac{d}{dt}(m\dot{r}) - \left(m\Omega^2 r\right) = 0,$$

that is,

$$\ddot{r} - \Omega^2 r = 0.$$

The general solution of this equation can be written in the form

$$r = A\cosh\Omega t + B\sinh\Omega t,$$

and, on applying the initial conditions $r = a$ and $\dot{r} = 0$ when $t = 0$, we find that $A = a$ and $B = 0$. The **position** of the particle at time t is therefore given by

$$r = a \cosh \Omega t.$$

The **energy function** h is given by

$$\begin{aligned} h &= \dot{r}\frac{\partial L}{\partial \dot{r}} - L = \dot{r}\frac{\partial T}{\partial \dot{r}} - T \\ &= m\dot{r}^2 - \tfrac{1}{2}m\left(\dot{r}^2 + \Omega^2 r^2\right) \\ &= \tfrac{1}{2}m\left(\dot{r}^2 - \Omega^2 r^2\right) \\ &= \tfrac{1}{2}ma^2\Omega^2\left(\sinh^2 \Omega t - \cosh^2 \Omega t\right) \\ &= -\tfrac{1}{2}ma^2\Omega^2, \end{aligned}$$

which is a **constant**. The fact that h is constant (though not its value) can be obtained more quickly by remembering that

$$\begin{aligned} \frac{dh}{dt} &= -\frac{\partial L}{\partial t} \\ &= -\frac{\partial T}{\partial t} = 0. \blacksquare \end{aligned}$$

Problem 12.11 *Yo-yo with moving support*

A uniform circular cylinder (a yo-yo) has a light inextensible string wrapped around it so that it does not slip. The free end of the string is fastened to a support and the yo-yo moves in a vertical straight line with the straight part of the string also vertical. At the same time the support is made to move vertically having upward displacement $Z(t)$ at time t. Take the rotation angle of the yo-yo as generalised coordinate and obtain Lagrange's equation. Find the acceleration of the yo-yo. What upwards acceleration must the support have so that the centre of the yo-yo can remain at rest?

Suppose the whole system starts from rest. Find an expression for the total energy $E = T + V$ at time t.

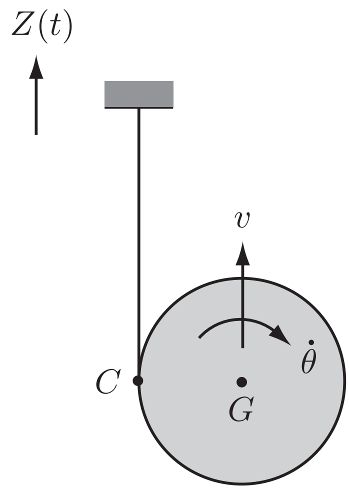

FIGURE 12.11 The velocity diagram for the yo-yo with a moving support.

Solution

The velocity diagram for the yo-yo is shown in Figure 12.11. Remember that θ is the generalised coordinate while $Z(t)$ is a prescribed function of the time t. Since the string does not slip on the yo-yo, the velocity of the string and the yo-yo at the point C must be equal. Hence $v + a\dot{\theta} = \dot{Z}$ and so

$$v = \dot{Z} - a\dot{\theta}.$$

The **kinetic energy** of the particle is therefore

$$\begin{aligned} T &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_G\,\omega^2 \\ &= \tfrac{1}{2}m\left(\dot{Z} - a\dot{\theta}\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{2}ma^2\right)\dot{\theta}^2 \\ &= \tfrac{1}{2}m\left(\dot{Z}^2 - 2a\dot{Z}\dot{\theta} + \tfrac{3}{2}a^2\dot{\theta}^2\right). \end{aligned}$$

The **potential energy** of the yo-yo (relative to the reference position in which $Z = \theta = 0$) is

$$V = mg\left(Z - a\theta\right).$$

Lagrange's equation for the yo-yo is therefore

$$m\frac{d}{dt}\left(\tfrac{3}{2}a^2\dot{\theta} - a\dot{Z}\right) - 0 = mga,$$

so that

$$a\ddot{\theta} = \tfrac{2}{3}\left(g + \ddot{Z}\right).$$

The upward **acceleration** of the yo-yo is then given by

$$\begin{aligned} \dot{v} &= \ddot{Z} - a\ddot{\theta} \\ &= \ddot{Z} - \tfrac{2}{3}\left(g + \ddot{Z}\right) \\ &= \tfrac{1}{3}\left(\ddot{Z} - 2g\right). \end{aligned}$$

If the centre of the yo-yo remains at rest, then $\dot{v} = 0$ and $\ddot{Z} = 2g$. This is the required upwards acceleration of the support.

If the motion begins from rest in the reference position, then the Lagrange equation integrates to give

$$a\dot{\theta} = \tfrac{2}{3}\left(gt + \dot{Z}\right) \qquad \text{and} \qquad a\theta = \tfrac{2}{3}\left(\tfrac{1}{2}gt^2 + Z\right).$$

Then

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{Z}^2 - \tfrac{4}{3}\dot{Z}\left(gt + \dot{Z}\right) + \tfrac{2}{3}\left(gt + \dot{Z}\right)^2\right) \\ &= \tfrac{1}{6}m\left(\dot{Z}^2 + 2g^2t^2\right) \end{aligned}$$

and

$$V = mg\left(Z - a\theta\right) = \tfrac{1}{3}mg\left(Z - gt^2\right).$$

Hence the total energy E is

$$\begin{aligned} E &= T + V \\ &= \tfrac{1}{6}m\left(\dot{Z}^2 + 2g^2t^2\right) + \tfrac{1}{3}mg\left(Z - gt^2\right) \\ &= \tfrac{1}{6}m\left(\dot{Z}^2 + 2gZ\right). \blacksquare \end{aligned}$$

Problem 12.12 *Pendulum with a shortening string*

A particle is suspended from a support by a light inextensible string which passes through a small fixed ring vertically below the support. The particle moves in a vertical plane with the string taut. At the same time the support is made to move vertically having an upward displacement $Z(t)$ at time t. The effect is that the particle oscillates like a simple pendulum whose string length at time t is $a - Z(t)$, where a is a positive constant. Take the angle between the string and the downward vertical as generalised coordinate and obtain Lagrange's equation. Find the energy function h and the total energy E and show that $h = E - m\dot{Z}^2$. Is either quantity conserved?

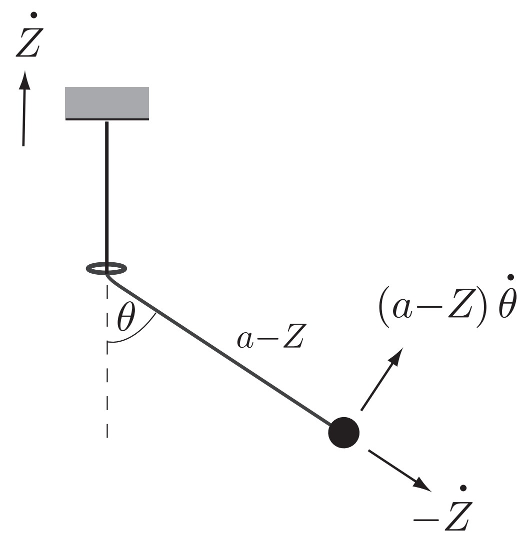

FIGURE 12.12 The velocity diagram for the pendulum with a shortening string.

Solution

The velocity diagram for the pendulum is shown in Figure 12.12. Remember that θ is the generalised coordinate while $Z(t)$ is a prescribed function of the time t.

The **kinetic energy** of the particle is

$$T = \tfrac{1}{2}m\left(\dot{Z}^2 + (a - Z)^2\dot{\theta}^2\right)$$

and the **potential energy** of the particle (relative to the restraining ring) is

$$V = -mg(a - Z)\cos\theta.$$

Lagrange's equation for the pendulum is therefore

$$m\frac{d}{dt}\left((a-Z)^2\dot{\theta}\right) - 0 = -mg(a-Z)\sin\theta,$$

that is,

$$(a-Z)\ddot{\theta} - 2\dot{Z}\dot{\theta} + g\sin\theta = 0,$$

which is the **equation of motion** of the pendulum.

The **total energy** E is given by

$$E = T + V = \tfrac{1}{2}m\left(\dot{Z}^2 + (a-Z)^2\dot{\theta}^2 - 2g(a-Z)\cos\theta\right),$$

while the **energy function** h is given by

$$\begin{aligned} h &= \dot{\theta}\,p_\theta - L = \dot{\theta}\frac{\partial T}{\partial\dot{\theta}} - T + V \\ &= m(a-Z)^2\dot{\theta}^2 - \tfrac{1}{2}m\left(\dot{Z}^2 + (a-Z)^2\dot{\theta}^2\right) - mg(a-Z)\cos\theta \\ &= \tfrac{1}{2}m\left(-\dot{Z}^2 + (a-Z)^2\dot{\theta}^2 - 2g(a-Z)\cos\theta\right). \end{aligned}$$

Thus $h - E - m\dot{Z}^2$, as required.

Neither h nor E is conserved in general. Consider, for example, the special case in which the string is pulled *upwards* with *constant speed* V. Then, since E and h differ by a constant, dE/dt and dh/dt are equal. Also

$$\begin{aligned} \frac{dh}{dt} &= -\frac{\partial L}{\partial t} \\ &= mV\left((a-Z)\dot{\theta}^2 + g\cos\theta\right), \end{aligned}$$

which is certainly *positive* while θ is an acute angle. Thus h is not constant and so neither is E. We would not expect E to be conserved since the tension in the string does work (but no virtual work!). ■

Problem 12.13 ✻ *Bug on a hoop*

A uniform circular hoop of mass M can slide freely on a smooth horizontal table, and a bug of mass m can run on the hoop. The system is at rest when the bug starts to run. What is the angle turned through by the hoop when the bug has completed one lap of the hoop?

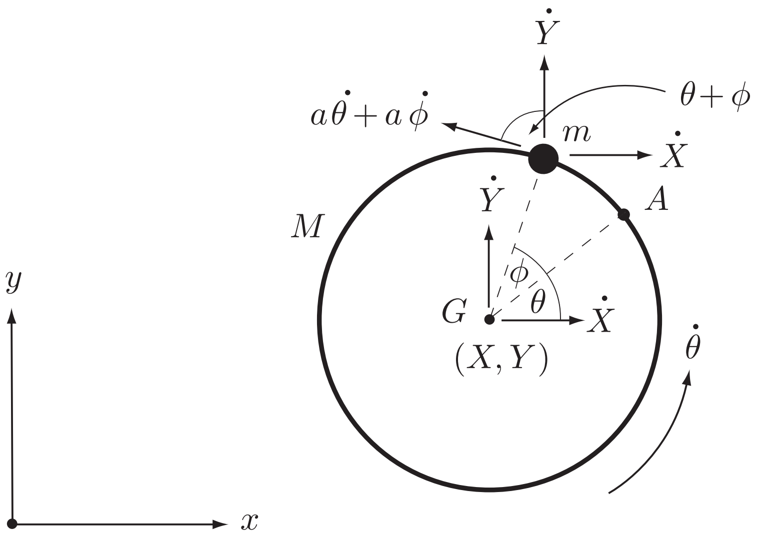

FIGURE 12.13 The coordinates and velocity diagram for the hoop and the bug.

Solution

Take the generalised coordinates to be X, Y and θ, as shown in Figure 12.13. X and Y are the Cartesian coordinates of the centre of mass G of the hoop, and θ is the rotation angle of the hoop from its initial position. A is a fixed point of the hoop which, in the initial position, is such that GA is parallel to the positive x-axis. The angle ϕ is the angular displacement of the bug *relative to the hoop*. Remember that ϕ is not a generalised coordinate but is regarded as a known function of the time t. The velocity diagram corresponding to this choice of coordinates is also shown in Figure 12.13.

The total **kinetic energy** of the hoop and the bug is

$$
\begin{aligned}
T &= \tfrac{1}{2} M V^2 + \tfrac{1}{2} I_G \, \omega^2 + \tfrac{1}{2} m v^2 \\
&= \tfrac{1}{2} M \left(\dot{X}^2 + \dot{Y}^2 \right) + \tfrac{1}{2} \left(M a^2 \right) \dot{\theta}^2 +
\end{aligned}
$$

$$\tfrac{1}{2}m\left(\dot{X}^2+\dot{Y}^2+a^2\big(\dot\theta+\dot\phi\big)^2-2a\dot{X}\big(\dot\theta+\dot\phi\big)\sin(\theta+\phi)+2a\dot{Y}\big(\dot\theta+\dot\phi\big)\cos(\theta+\phi)\right),$$

and the **potential energy** (relative to the level of the table) is zero.

Since $\partial L/\partial X$ and $\partial L/\partial Y$ are both zero, it follows that the coordinates X and Y are cyclic. The corresponding momenta p_X and p_Y, given by

$$p_X=\frac{\partial L}{\partial \dot{X}}=M\dot{X}+m\left(\dot{X}-a(\dot\theta+\dot\phi)\sin(\theta+\phi)\right),$$
$$p_Y=\frac{\partial L}{\partial \dot{Y}}=M\dot{Y}+m\left(\dot{X}+a(\dot\theta+\dot\phi)\cos(\theta+\phi)\right),$$

are therefore constants of the motion. The whole system (including the bug) starts from rest, so that $\dot{X}$, $\dot{Y}$, $\dot\theta$ and $\dot\phi$ are all zero initially. We therefore obtain the two **conservation relations**

$$M\dot{X}+m\left(\dot{X}-a(\dot\theta+\dot\phi)\sin(\theta+\phi)\right)=0,$$
$$M\dot{Y}+m\left(\dot{Y}+a(\dot\theta+\dot\phi)\cos(\theta+\phi)\right)=0.$$

Our third equation is the **Lagrange equation** for θ. Now

$$\begin{aligned}\frac{\partial L}{\partial\dot\theta}&=\frac{\partial T}{\partial\dot\theta}\\&=Ma^2\dot\theta+ma\left(a\big(\dot\theta+\dot\phi\big)-\dot{X}\sin(\theta+\phi)+\dot{Y}\cos(\theta+\phi)\right)\\&=(M+m)a^2\dot\theta+ma^2\dot\phi-\frac{m^2a^2}{M+m}\big(\dot\theta+\dot\phi\big)\end{aligned}$$

on eliminating $\dot{X}$ and $\dot{Y}$ by using the conservation relations. Also,

$$\begin{aligned}\frac{\partial L}{\partial\theta}&=\frac{\partial T}{\partial\theta}\\&=-ma\big(\dot\theta+\dot\phi\big)\big(\dot{X}\cos(\theta+\phi)+\dot{Y}\sin(\theta+\phi)\big)\\&=0,\end{aligned}$$

on using the conservation relations again. The Lagrange equation for θ is therefore

$$\frac{d}{dt}\left((M+m)a^2\dot\theta+ma^2\dot\phi-\frac{m^2a^2}{M+m}\big(\dot\theta+\dot\phi\big)\right)-0=0,$$

which simplifies to give

$$\ddot\theta=-\left(\frac{m}{M+2m}\right)\ddot\phi.$$

On integrating and using the initial conditions, the solution for $\theta(t)$ is

$$\theta = -\left(\frac{m}{M + 2m}\right)\phi,$$

where $\phi(t)$ is the known angular displacement of the bug. The bug completes one lap of the hoop when $\phi = 2\pi$. The **angle turned** by the ring at this instant is therefore

$$\frac{2\pi m}{M + 2m}$$

in the opposite sense to the motion of the bug. ■

Problem 12.14

Suppose a particle is subjected to a time dependent force of the form $\boldsymbol{F} = f(t)\,\text{grad}\,W(\boldsymbol{r})$. Show that this force can be represented by the time dependent potential $U = -f(t)W(\boldsymbol{r})$. What is the value of U when $\boldsymbol{F} = f(t)\boldsymbol{i}$?

Solution

If $U = -f(t)W(\boldsymbol{r})$, then

$$\frac{\partial U}{\partial \dot{x}} = 0, \qquad \frac{\partial U}{\partial x} = -f(t)\frac{\partial W}{\partial x},$$

and

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) - \frac{\partial U}{\partial x} &= f(t)\frac{\partial W}{\partial x} = F_x \\ &= Q_x,\end{aligned}$$

since, in Cartesian coordinates, the generalised force Q_x corresponding to x is just the x-component of the actual force $\boldsymbol{F}$. A similar argument applies to the y and z components.

In particular, if $\boldsymbol{F} = f(t)\boldsymbol{i}$, then $U = -f(t)x$. ∎

Problem 12.15 *Charged particle in an electrodynamic field*

Show that the velocity dependent potential

$$U = e\,\phi(\boldsymbol{r}, t) - e\,\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}, t)$$

represents the Lorentz force $\boldsymbol{F} = e\,\boldsymbol{E} + e\,\boldsymbol{v}\times\boldsymbol{B}$ that acts on a charge e moving with velocity $\boldsymbol{v}$ in the general *electrodynamic* field $\{\boldsymbol{E}(\boldsymbol{r}, t), \boldsymbol{B}(\boldsymbol{r}, t)\}$. [Here $\{\phi, \boldsymbol{A}\}$ are the *electrodynamic potentials* that generate the field $\{\boldsymbol{E}, \boldsymbol{B}\}$ by the formulae

$$\boldsymbol{E} = -\operatorname{grad}\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \operatorname{curl}\boldsymbol{A}.]$$

Show that the potentials $\phi = 0$, $\boldsymbol{A} = tz\,\boldsymbol{i}$ generate a field $\{\boldsymbol{E}, \boldsymbol{B}\}$ that satisfies all four Maxwell equations in free space. A particle of mass m and charge e moves in this field. Find the Lagrangian of the particle in terms of Cartesian coordinates. Show that x and y are cyclic coordinates and find the conserved momenta p_x, p_y.

Solution

We are given that

$$\begin{aligned} U &= e\,\phi - e\,\dot{\boldsymbol{r}} \cdot \boldsymbol{A} \\ &= e\,\phi(\boldsymbol{r}, t) - e\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right). \end{aligned}$$

Then (dropping the e for the moment)

$$\frac{\partial U}{\partial \dot{x}} = -A_x, \qquad \frac{\partial U}{\partial x} = \frac{\partial \phi}{\partial x} - \dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_y}{\partial x} - \dot{z}\frac{\partial A_z}{\partial x}$$

and, by the chain rule,

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) = -\frac{\partial A_x}{\partial x}\dot{x} - \frac{\partial A_x}{\partial y}\dot{y} - \frac{\partial A_x}{\partial z}\dot{z} - \frac{\partial A_x}{\partial t}.$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) - \frac{\partial U}{\partial x} &= -\frac{\partial A_x}{\partial x}\dot{x} - \frac{\partial A_x}{\partial y}\dot{y} - \frac{\partial A_x}{\partial z}\dot{z} - \frac{\partial A_x}{\partial t} \\
&\quad -\frac{\partial \phi}{\partial x} + \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x} \\
&= -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\
&= -\big[\operatorname{grad}\phi\big]_x - \left[\frac{\partial \boldsymbol{A}}{\partial t}\right]_x + \dot{y}\big[\operatorname{curl}\boldsymbol{A}\big]_z - \dot{z}\big[\operatorname{curl}\boldsymbol{A}\big]_y \\
&= -\big[\operatorname{grad}\phi\big]_x - \left[\frac{\partial \boldsymbol{A}}{\partial t}\right]_x + \big[\dot{\boldsymbol{r}}\times\operatorname{curl}\boldsymbol{A}\big]_x \\
&= E_x + \big[\dot{\boldsymbol{r}}\times\boldsymbol{B}\big]_x,
\end{aligned}
$$

which, on restoring the e, is the x component of the Lorentz force. A similar argument applies to the y and z components.

If $\phi = 0$ and $\boldsymbol{A} = tz\,\boldsymbol{i}$, then $\boldsymbol{E} = -z\,\boldsymbol{i}$ and

$$
\begin{aligned}
\boldsymbol{B} &= \operatorname{curl}\boldsymbol{A} = \operatorname{grad}(tz)\times\boldsymbol{i} = t\,\boldsymbol{k}\times\boldsymbol{i} \\
&= t\,\boldsymbol{j}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{div}\boldsymbol{E} &= 0, \\
\operatorname{curl}\boldsymbol{E} &= -\operatorname{grad}z\times\boldsymbol{i} = -\boldsymbol{k}\times\boldsymbol{i} = -\boldsymbol{j} = -\frac{\partial \boldsymbol{B}}{\partial t}, \\
\operatorname{div}\boldsymbol{B} &= 0, \\
\operatorname{curl}\boldsymbol{B} &= \boldsymbol{0} = \frac{\partial \boldsymbol{E}}{\partial t},
\end{aligned}
$$

so that **Maxwell's equations** for free space are satisfied.

When a particle of mass m and charge e moves in this field, its Lagrangian is

$$
\begin{aligned}
L &= T - U \\
&= \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - e\phi + e\dot{\boldsymbol{r}}\cdot\boldsymbol{A} \\
&= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + etz\dot{x}.
\end{aligned}
$$

The coordinates x and y are **cyclic** and the corresponding conserved momenta are

$$
\begin{aligned}
p_x &= m\dot{x} + etz, \\
p_y &= m\dot{y}.
\end{aligned}
$$

The conserved momentum p_y is the linear momentum of the particle in the y-direction, but p_x is *not* the corresponding x-component. ■

Problem 12.16 ✻ *Relativistic Lagrangian*

The relativistic Lagrangian for a particle of rest mass m_0 moving along the x-axis under the simple harmonic potential field $V = \frac{1}{2}m_0\Omega^2x^2$ is given by

$$L = m_0c^2\left(1 - \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2}\right) - \tfrac{1}{2}m_0\Omega^2x^2.$$

Obtain the energy integral for this system and show that the period of oscillations of amplitude a is given by

$$\tau = \frac{4}{\Omega}\int_0^{\pi/2}\frac{1 + \frac{1}{2}\epsilon^2\cos^2\theta}{\left(1 + \frac{1}{4}\epsilon^2\cos^2\theta\right)^{1/2}}\,d\theta,$$

where the dimensionless parameter $\epsilon = \Omega a/c$.

Deduce that

$$\tau = \frac{2\pi}{\Omega}\left[1 + \tfrac{3}{16}\epsilon^2 + O\left(\epsilon^4\right)\right],$$

when ϵ is small.

Solution

Since L is given by

$$L = m_0c^2\left(1 - \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2}\right) - \tfrac{1}{2}m_0\Omega^2x^2,$$

it follows that

$$\frac{\partial L}{\partial \dot{x}} = m_0\dot{x}\left(1 - \frac{\dot{x}^2}{c^2}\right)^{-1/2}$$

and so the **energy function** h is given by

$$\begin{aligned}
h &= \dot{x}\frac{\partial L}{\partial \dot{x}} - L \\
&= m_0\dot{x}^2\left(1 - \frac{\dot{x}^2}{c^2}\right)^{-1/2} - m_0c^2\left(1 - \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2}\right) + \tfrac{1}{2}m_0\Omega^2x^2 \\
&= m_0c^2\left(1 - \frac{\dot{x}^2}{c^2}\right)^{-1/2} + \tfrac{1}{2}m_0\Omega^2x^2 - m_0c^2.
\end{aligned}$$

Since $\partial L/\partial t = 0$, h is a constant of the motion; the condition $\dot{x} = 0$ when $x = a$ shows that the value of this constant is $\frac{1}{2}m_0\Omega^2 a^2$. Hence, the relativistic **energy equation** can be written

$$\left(1 - \frac{\dot{x}^2}{c^2}\right)^{-1/2} = 1 + \tfrac{1}{2}\epsilon^2\left(1 - \frac{x^2}{a^2}\right),$$

where the dimensionless constant ϵ is defined by $\epsilon = \Omega a/c$.

On solving for $\dot{x}$, this gives

$$\dot{x} = \pm\,\Omega a\,\frac{\left(1 + \frac{1}{4}\epsilon^2\left(1 - \frac{x^2}{a^2}\right)\right)^{1/2}}{1 + \frac{1}{2}\epsilon^2\left(1 - \frac{x^2}{a^2}\right)}\left(1 - \frac{x^2}{a^2}\right)^{1/2},$$

where the $\pm$ sign depends on whether the particle is moving in positive or negative x-direction. Consider the particle moving in the positive x-direction from $x = 0$ to $x = a$, a motion that takes a quarter of the period τ. On separating variables and integrating, we obtain

$$\int_0^{\tau/4} dt = \frac{1}{\Omega a}\int_0^a \frac{1 + \frac{1}{2}\epsilon^2\left(1 - \frac{x^2}{a^2}\right)}{\left(1 - \frac{x^2}{a^2}\right)^{1/2}\left(1 + \frac{1}{4}\epsilon^2\left(1 - \frac{x^2}{a^2}\right)\right)^{1/2}}\,dx.$$

The **period** τ of the motion is therefore given by

$$\tau = \frac{4}{\Omega a}\int_0^a \frac{1 + \frac{1}{2}\epsilon^2\left(1 - \frac{x^2}{a^2}\right)}{\left(1 - \frac{x^2}{a^2}\right)^{1/2}\left(1 + \frac{1}{4}\epsilon^2\left(1 - \frac{x^2}{a^2}\right)\right)^{1/2}}\,dx.$$

This formula can be written in a simpler form by making the change of variable $x = a\sin\theta$ $(0 \le \theta \le \pi/2)$. This gives

$$\tau = \frac{4}{\Omega}\int_0^{\pi/2} \frac{1 + \frac{1}{2}\epsilon^2\cos^2\theta}{\left(1 + \frac{1}{4}\epsilon^2\cos^2\theta\right)^{1/2}}\,d\theta,$$

as required.

This is the formula for the exact period, but when the parameter ϵ is small we can find a simple approximation. In this case, the integrand can be approximated by

$$
\begin{aligned}
\frac{1+\frac{1}{2}\epsilon^2\cos^2\theta}{\left(1+\frac{1}{4}\epsilon^2\cos^2\theta\right)^{1/2}} &= \left(1+\tfrac{1}{2}\epsilon^2\cos^2\theta\right)\left(1+\tfrac{1}{4}\epsilon^2\cos^2\theta\right)^{-1/2} \\
&= \left(1+\tfrac{1}{2}\epsilon^2\cos^2\theta\right)\left(1-\tfrac{1}{8}\epsilon^2\cos^2\theta+O\left(\epsilon^4\right)\right) \\
&= \left(1+\tfrac{3}{8}\epsilon^2\cos^2\theta+O\left(\epsilon^4\right)\right)
\end{aligned}
$$

and, when this is substituted into the integral, we obtain

$$
\tau = \frac{2\pi}{\Omega}\left(1+\tfrac{3}{16}\epsilon^2+O\left(\epsilon^4\right)\right),
$$

as required. The period of the motion is therefore *lengthened* by the inclusion of relativistic effects. ■

Problem 12.17

A particle of mass m moves under the gravitational attraction of a fixed mass M situated at the origin. Take polar coordinates r, θ as generalised coordinates and obtain Lagrange's equations. Show that θ is a cyclic coordinate and find (and identify) the conserved momentum p_θ.

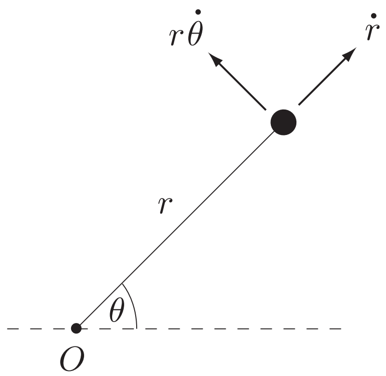

FIGURE 12.14 The coordinates and velocity diagram for the particle in Problem 12.17.

Solution

The coordinates and velocity diagram for the particle are shown in Figure 12.14.

The **kinetic energy** of the particle is

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and the **potential energy** of the particle (relative to infinity) is

$$V = -\frac{MG}{r}.$$

The **Lagrangian** of the particle is therefore

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{MG}{r}.$$

Since $\partial L/\partial\theta = 0$, the coordinate θ is **cyclic**. The corresponding conserved momentum p_θ is given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta},$$

which is the **angular momentum** of the particle about the axis through O perpendicular to the plane of motion. ■

Problem 12 . 18

A particle P of mass m slides on the smooth inner surface of a circular cone of semi-angle α. The axis of symmetry of the cone is vertical with the vertex O pointing downwards. Take as generalised coordinates r, the distance OP, and ϕ, the azimuthal angle about the vertical through O. Obtain Lagrange's equations. Show that ϕ is a cyclic coordinate and find (and identify) the conserved momentum p_ϕ.

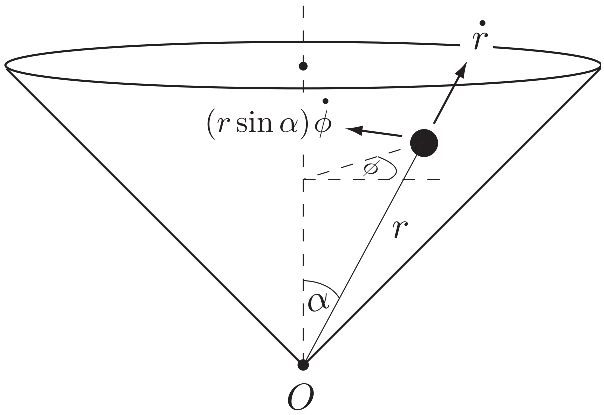

FIGURE 12.15 The coordinates and velocity diagram for a particle sliding on a cone.

Solution

The coordinates and velocity diagram for the particle sliding on the cone are shown in Figure 12.15.

The **kinetic energy** of the particle is

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2\right)$$

and the **potential energy** of the particle (relative to O) is

$$V = mgr\cos\alpha.$$

The **Lagrangian** of the particle is therefore

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2\right) - mgr\cos\alpha.$$

Lagrange's equations are therefore

$$\frac{d}{dt}(m\dot{r}) - \left(mr\sin^2\alpha\,\dot{\phi}^2\right) = -mg\cos\alpha,$$
$$\frac{d}{dt}\left(mr^2\sin^2\alpha\,\dot{\phi}\right) - 0 = 0.$$

The second equation has the form $dp_\phi/dt = 0$, where $p_\phi = \partial L/\partial\dot{\phi}$, the momentum corresponding to ϕ. This is because $\partial L/\partial\phi = 0$, that is, ϕ is a **cyclic** coordinate. The momentum p_ϕ is the **angular momentum** of the particle about the vertical axis through O. ■

Problem 12.19

A particle of mass m and charge e moves in the magnetic field produced by a current I flowing in an infinite straight wire that lies along the z-axis. The vector potential $\boldsymbol{A}$ of the induced magnetic field is given by

$$A_r = A_\theta = 0, \qquad A_z = -\left(\frac{\mu_0 I}{2\pi}\right) \ln r,$$

where r, θ, z are cylindrical polar coordinates. Find the Lagrangian of the particle. Show that θ and z are cyclic coordinates and find the corresponding conserved momenta.

Solution

The **kinetic energy** of the particle is

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

and the (velocity dependent) **potential energy** of the particle is given by

$$\begin{aligned} U &= -e\,\dot{\boldsymbol{r}} \cdot \boldsymbol{A} \\ &= -e\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right) = +\left(\frac{e\mu_0 I}{2\pi}\right)\dot{z}\ln r. \end{aligned}$$

The **Lagrangian** of the particle is therefore

$$\begin{aligned} L &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(\frac{e\mu_0 I}{2\pi}\right)\dot{z}\ln r \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - \left(\frac{e\mu_0 I}{2\pi}\right)\dot{z}\ln r \end{aligned}$$

in cylindrical polar coordinates.

Since $\partial L/\theta$ and $\partial L/\partial z$ are both zero, the coordinates θ and z are **cyclic**. The corresponding conserved momenta p_θ and p_z are given by

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} - \left(\frac{e\mu_0 I}{2\pi}\right)\ln r. \end{aligned}$$

The momentum p_θ is the **angular momentum** of the particle about the z-axis, but p_z is *not* the linear momentum of the particle in the z-direction. ■

Problem 12.20

A particle moves freely in the gravitational field of a fixed mass distribution. Find the conservation principles that correspond to the symmetries of the following fixed mass distributions: (i) a uniform sphere, (ii) a uniform half plane, (iii) two particles, (iv) a uniform right circular cone, (v) an infinite uniform circular cylinder.

Solution

In every case, the particle is free to move in any direction. It remains to find those motions (translations or rotations) that preserve the gravitational potential energy V.

- (i) V is preserved if the particle is *rotated* about any fixed axis through the centre G of the sphere. The full vector **angular momentum** $\boldsymbol{L}_G$ is therefore conserved.
- (ii) Suppose the half-plane is $x \geq 0$, $z = 0$. Then V is preserved if the particle is *translated* in the y-direction. The **linear momentum** component P_y is therefore conserved.
- (iii) V is preserved if the particle is *rotated* about the fixed axis passing through the two particles. The **angular momentum** about this axis is therefore conserved.
- (iv) V is preserved if the particle is *rotated* about the axis of symmetry of the cone. The **angular momentum** about the symmetry axis is therefore conserved.
- (v) Since the cylinder is infinite, V is preserved if the particle is *translated* parallel to the symmetry axis of the cylinder. The **linear momentum** component in this direction is therefore conserved. Also, V is preserved if the particle is *rotated* about the symmetry axis. The **angular momentum** about the symmetry axis is therefore also conserved.

Problem 12.21 ✻ *Helical symmetry*

A particle moves in a conservative field whose potential energy V has *helical symmetry*. This means that V is invariant under the *simultaneous* operations (i) a rotation through any angle α about the axis Oz, and (ii) a translation $c\alpha$ in the z-direction. What conservation principle corresponds to this symmetry?

Solution

This is essentially a linear combination of Theorems 12.1 and 12.2. In the present case, there is a family of mappings $\{\mathfrak{M}^{\lambda}\}$ with the effect $\boldsymbol{r}_i \longrightarrow \boldsymbol{r}_i^{\lambda}$, where

$$\frac{\partial \boldsymbol{r}_i^{\lambda}}{\partial \lambda} = c\,\boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{r}_i^{\lambda},$$

that preserve the potential energy V. The corresponding **conserved quantity** is therefore $cP_z + L_z$.

Chapter Thirteen

The calculus of variations

and Hamilton's principle

Problem 13.1

Find the extremal of the functional

$$J[x] = \int_1^2 \frac{\dot{x}^2}{t^3}\, dt$$

that satisfies $x(1) = 3$ and $x(2) = 18$. Show that this extremal provides the global minimum of J.

Solution

For this functional, the integrand $F(x, \dot{x}, t)$ is

$$F = \frac{\dot{x}^2}{t^3}$$

and the corresponding **Euler-Lagrange equation** is

$$\frac{d}{dt}\left(\frac{2\dot{x}}{t^3}\right) - 0 = 0.$$

This equation integrates immediately to give

$$\frac{2\dot{x}}{t^3} = 8a,$$

where a is an integration constant. (The factor 8 is introduced simply to avoid fractions.) Hence

$$\dot{x} = 4at^3,$$

and so

$$x = at^4 + b,$$

where b is a second integration constant. This is the **family of extremals** of the functional J.

We must now find the extremals that satisfy the given **end conditions**. The condition $x = 3$ when $t = 1$ gives $a+b = 3$, and the condition $x = 18$ when $t = 2$ gives $16a + b = 18$. These simultaneous equations have the unique solution $a = 1$, $b = 2$. Hence there is exactly **one admissible extremal**, namely

$$x = t^4 + 2.$$

To investigate the nature of this extremal, consider the function $x = t^4 + 2 + h$, where $h(t)$ is any admissible variation. Then

$$
\begin{aligned}
J[t^4 + 2 + h] &= \int_1^2 \frac{(4t^3 + \dot{h})^2}{t^3} dt \\
&= \int_1^2 \left(16t^3 + 8\dot{h} + \frac{\dot{h}^2}{t^3}\right) dt \\
&= \Big[4t^4\Big]_{t=1}^{t=2} + \Big[8h\Big]_{t=1}^{t=2} + \int_1^2 \frac{\dot{h}^2}{t^3}\, dt \\
&= 60 + 0 + \int_1^2 \frac{\dot{h}^2}{t^3}\, dt,
\end{aligned}
$$

since $h(1) = h(2) = 0$ in an *admissible* variation. In particular, by taking $h \equiv 0$, $J[t^4 + 2] = 60$. Hence

$$
\begin{aligned}
J[t^4 + 2 + h] &= J[t^4 + 2] + \int_1^2 \frac{\dot{h}^2}{t^3}\, dt \\
&\geq J[t^4 + 2],
\end{aligned}
$$

since the integrand $\dot{h}^2/t^3$ is *positive* in the range $t > 0$. Since h is a general admissible variation, it follows that the extremal $x = t^4 + 2$ provides the **global minimum** for the functional J.

Problem 13.2

Find the extremal of the functional

$$J[x] = \int_0^{\pi} \left(2x \sin t - \dot{x}^2\right) \, dt$$

that satisfies $x(0) = x(\pi) = 0$. Show that this extremal provides the global maximum of J.

Solution

For this functional, the integrand $F(x, \dot{x}, t)$ is

$$F = 2x \sin t - \dot{x}^2$$

and the corresponding **Euler-Lagrange equation** is

$$\frac{d}{dt}\left(-2\dot{x}\right) - 2\sin t = 0,$$

that is

$$\ddot{x} = -\sin t.$$

The general solution of this equation is

$$x = \sin t + at + b,$$

where a and b are integration constants. This is the **family of extremals** of the functional J.

We must now find the extremals that satisfy the given **end conditions**. The condition $x = 0$ when $t = 0$ gives $b = 0$, and the condition $x = 0$ when $t = \pi$ then gives $a = 0$. Hence there is exactly **one admissible extremal**, namely

$$x = \sin t.$$

To investigate the nature of this extremal, consider the function

$$x = \sin t + h,$$

where $h(t)$ is any admissible variation. Then

$$\begin{aligned} J[\sin t + h] &= \int_0^{\pi} 2(\sin t + h)\sin t - \left(\cos t + \dot{h}\right)^2 \, dt \\ &= \int_0^{\pi} 2\sin^2 t - \cos^2 t + 2h\sin t - 2\dot{h}\cos t - \dot{h}^2 \, dt \\ &= \int_0^{\pi} \left(2\sin^2 t - \cos^2 t\right) - \frac{d}{dt}\left(2h\cos t\right) - \dot{h}^2 \, dt \\ &= \tfrac{1}{2}\pi - \Big[2h\cos t\Big]_{t=0}^{t=\pi} - \int_0^{\pi} \dot{h}^2 \, dt, \\ &= \tfrac{1}{2}\pi + 0 - \int_0^{\pi} \dot{h}^2 \, dt, \end{aligned}$$

since $h(0) = h(\pi) = 0$ in an *admissible* variation. In particular, by taking $h \equiv 0$, $J[\sin t] = \frac{1}{2}\pi$. Hence

$$\begin{aligned} J[\sin t + h] &= J[\sin t] - \int_1^2 \dot{h}^2 dt \\ &\leq J[\sin t]. \end{aligned}$$

since the integrand $\dot{h}^2$ is *positive*. Since h is a general admissible variation, it follows that the extremal $x = \sin t$ provides the **global maximum** for the functional J.

Problem 13.3

Find the extremal of the path length functional

$$L[y] = \int_0^1 \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

that satisfies $y(0) = y(1) = 0$ and show that it does provide the global minimum for L.

Solution

For this functional, the integrand $F(y, \dot{y}, x)$ is

$$F = \left(1 + \dot{y}^2\right)^{1/2}$$

which has no explicit x dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{y}\frac{\partial F}{\partial \dot{y}} - F = \text{ constant.}$$

In the present case, this simplifies to give

$$\frac{1}{\left(1 + \dot{y}^2\right)^{1/2}} = \text{constant}$$

that is

$$\dot{y} = a,$$

where a is a constant. The general solution of this equation is

$$y = ax + b,$$

where b is an integration constant. Except possibly for constant solutions, this family of straight lines is the **family of extremals** of the length functional J.

We must now find the extremals that satisfy the given **end conditions**. The condition $y = 0$ when $x = 0$ gives $b = 0$, and the condition $y = 0$ when $x = 1$ then gives $a = 0$. However, since the function $y \equiv 0$ is a *constant* solution of the integrated equation, it may not actually be an extremal. We must check whether or not it satisfies the original Euler-Lagrange equation, namely

$$\frac{d}{dx}\left(\frac{\dot{y}}{\left(1+\dot{y}^2\right)^{1/2}}\right) - 0 = 0.$$

The function $y \equiv 0$ clearly satisfies this equation and hence is the **only admissible extremal**. It represents the straight line joining the points $(0,0)$ and $(1,0)$.

To investigate the nature of this extremal, consider the path

$$y = 0 + h,$$

where $h(x)$ is any admissible variation. Then

$$\begin{aligned} L[y] &= \int_0^1 \left[1 + \dot{h}^2\right]^{1/2} dx \\ &\geq 1, \end{aligned}$$

since the integrand is always *greater than unity*. Thus the **straight line** $y = 0$ really is the **path of shortest length** joining the points (0,0) and (1,0).

Problem 13.4

An aircraft flies in the (x, z)-plane from the point $(-a, 0)$ to the point $(a, 0)$. ($z = 0$ is ground level and the z-axis points vertically upwards.) The cost of flying the aircraft at height z is $\exp(-kz)$ per unit *distance* of flight, where k is a positive constant. Find the extremal for the problem of minimising the total cost of the journey. [Assume that $ka < \pi/2$.]

Solution

The cost functional $C[z]$ for the flight path $z(x)$ is given by

$$C[z] = \int_{-a}^{a} e^{-kz}\left(1 + \dot{z}^2\right)^{1/2} dx.$$

For this functional, the integrand $F(z, \dot{z}, x)$ is

$$F = e^{-kz}\left(1 + \dot{z}^2\right)^{1/2}$$

which has no explicit x dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{z}\frac{\partial F}{\partial \dot{z}} - F = \text{constant.}$$

In the present case, this simplifies to give

$$\frac{e^{-kz}}{\left(1 + \dot{z}^2\right)^{1/2}} = \text{constant}$$

that is

$$\dot{z}^2 = b^2 e^{-2kz} - 1,$$

where b is a positive constant. This equation evidently has the family of solutions $z =$ constant but these solutions do not satisfy the Euler-Lagrange equation and are therefore *not* extremals. Other solutions can be found by taking square roots and separating, as usual. This gives

$$x = \pm \int \frac{dz}{\left(b^2 e^{-2kz} - 1\right)^{1/2}}$$

$$= \pm \int \frac{e^{kz}\, dx}{\left(b^2 - e^{2kz}\right)^{1/2}}.$$

On making the substitution $bu = e^{kz}$, this becomes

$$x = \pm\frac{1}{k}\int \frac{du}{\left(1-u^2\right)^{1/2}} = \pm\frac{1}{k}\cos^{-1} u + c,$$

$$= \pm\frac{1}{k}\cos^{-1}\left(\frac{e^{kz}}{b}\right) + c,$$

where c is an integration constant. Hence the **family of extremals** of the functional C is given by

$$e^{kz} = b\cos k(x-c).$$

We must now find the extremals that satisfy the given **end conditions**. The conditions $z = 0$ when $x = \pm a$ give

$$b\cos k(a+c) = b\cos k(a-c) = 1$$

from which it follows that $c = 0$ and $b = 1/\cos ka$. (We know that $ka < \pi/2$ so that $\cos ka$ is positive.) Hence there is exactly **one admissible extremal**, namely

$$z = \frac{1}{k}\ln\left(\frac{\cos kx}{\cos ka}\right).$$

Figure 13.1 shows the optimal flight path in Problem 13.4 for three different values of the parameter ka. ■

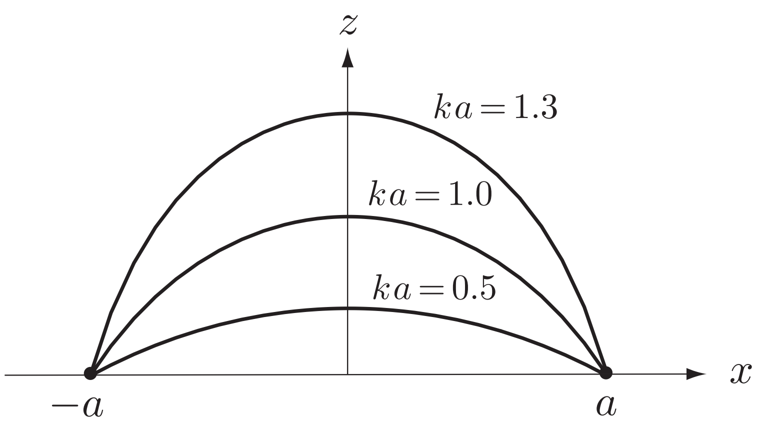

FIGURE 13.1 The optimal flight path in Problem 13.4 for three different values of the parameter ka.

Problem 13.5 ✻ *Geodesics on a cone*

Solve the problem of finding a shortest path over the surface of a cone of semi-angle α by the calculus of variations. Take the equation of the path in the form $\rho = \rho(\theta)$, where ρ is distance from the vertex O and θ is the cylindrical polar angle measured around the axis of the cone. Obtain the general expression for the path length and find the extremal that satisfies the end conditions $\rho(-\pi/2) = \rho(\pi/2) = a$.

Verify that this extremal is the same as the shortest path that would be obtained by developing the cone on to a plane.

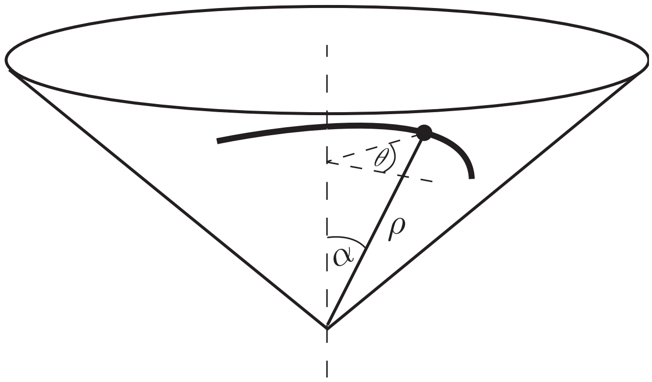

FIGURE 13.2 The coordinates ρ and θ used in Problem 13.5.

Solution

The coordinates ρ and θ are shown in Figure 13.2. In terms of these coordinates, the length element ds is given by

$$(ds)^2 = (d\rho)^2 + (\rho \sin\alpha \, d\theta)^2,$$

where α is the semi-angle of the cone. Hence

$$ds = \left[\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \sin^2\alpha \right]^{1/2} d\theta$$

and the length functional for paths over the surface of the cone is

$$L[\rho] = \int_{\theta_1}^{\theta_2} \left[\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \sin^2\alpha \right]^{1/2} d\theta,$$

where θ_1 and θ_2 are the initial and final values of θ on the path. In the present case, these values are $-\pi/2$ and $\pi/2$ respectively.

For this functional, the integrand $F(\rho, \dot{\rho}, \theta)$ is

$$F = \left(\dot{\rho}^2 + \rho^2 \sin^2 \alpha\right)^{1/2}$$

which has no explicit θ dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{\rho}\frac{\partial F}{\partial \dot{\rho}} - F = \text{constant.}$$

In the present case, this simplifies to give

$$\frac{\rho^2}{\left(\rho^2 \sin^2 \alpha + \dot{\rho}^2\right)^{1/2}} = \text{constant}$$

that is

$$\dot{\rho}^2 = b^2 \rho^4 - \rho^2 \sin^2 \alpha,$$

where b is a positive constant. This equation evidently has the family of solutions $\rho =$ constant but these solutions do not satisfy the Euler-Lagrange equation and are therefore *not* extremals.

Other solutions can be found by taking square roots and separating, as usual. This gives

$$\theta = \pm \int \frac{d\rho}{\rho\left(b^2\rho^2 - \sin^2 \alpha\right)^{1/2}}.$$

On making the substitution $b\rho = \sin\alpha \sec\psi$, this becomes

$$\theta = \pm\frac{\psi}{\sin\alpha} + c,$$

where c is an integration constant. In reintroducing the variable ρ instead of ψ we find that the **family of extremals** of the length functional L is given by

$$\rho = \left(\frac{\sin\alpha}{b}\right) \sec\left((\theta - c)\sin\alpha\right).$$

We must now find the extremals that satisfy the prescribed **end conditions**. The conditions $\rho = a$ when $\theta = \pm\pi/2$ give

$$a = \left(\frac{\sin\alpha}{b}\right)\sec\left(\left(\tfrac{1}{2}\pi + c\right)\sin\alpha\right) = \left(\frac{\sin\alpha}{b}\right)\sec\left(\left(\tfrac{1}{2}\pi - c\right)\sin\alpha\right)$$

from which it follows that $c = 0$ and

$$b = \left(\frac{\sin\alpha}{a}\right)\sec\left(\tfrac{1}{2}\pi\sin\alpha\right).$$

Hence there is exactly **one admissible extremal**, namely

$$\rho = \frac{\cos\left(\tfrac{1}{2}\pi\sin\alpha\right)}{\cos\left(\theta\sin\alpha\right)}.$$

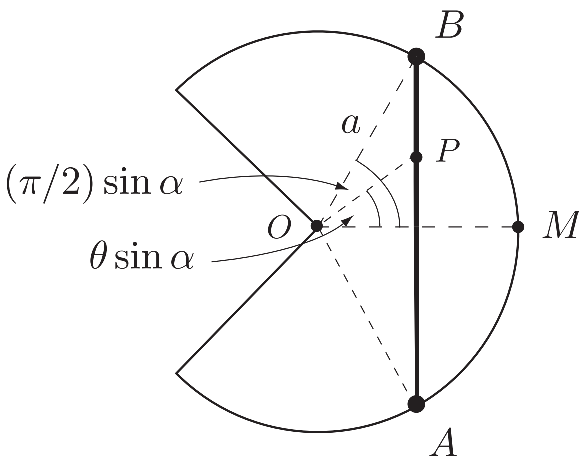

FIGURE 13.3 A cone of semi-angle α developed on to a plane. The shortest path from $A(\rho = a, \theta = -\pi/2)$ to $B(\rho = a, \theta = \pi/2)$ is the line segment AB. P is a general point on this path.

It remains to identify this extremal with the minimum length path obtained by developing the cone on to a plane. Figure 13.3 shows what the cone would look like if it were slit along the generator $\theta = \pi$ and then rolled out on a flat table. A and B are the starting and end points of the path and the straight line AB is the shortest path. Let P be a general point on this path with coordinates ρ, θ. Then $OP = \rho$ and the angle between OP and OM is $\theta \sin\alpha$. The equation of the straight line AB is therefore

$$\rho \cos\left(\theta \sin\alpha\right) = a \cos\left(\tfrac{1}{2}\pi \sin\alpha\right),$$

which is the same as obtained from the Euler-Lagrange equation. Figure 13.4 shows a path of shortest length on a cone of semi-angle $\pi/6$.

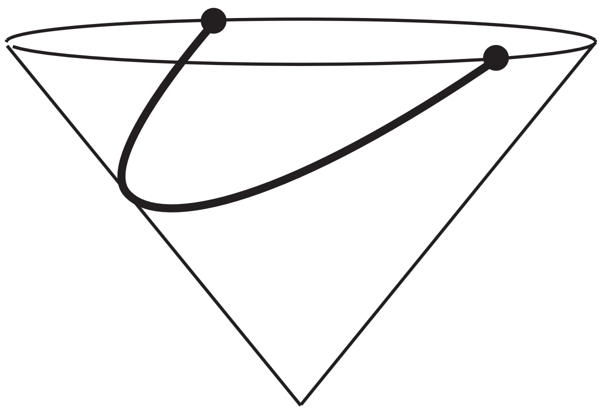

FIGURE 13.4 A path of shortest length on a cone of semi-angle $\pi/6$.

Problem 13.6 *Cost functional*

A manufacturer wishes to minimise the cost functional

$$C[x] = \int_0^4 \Big((3 + \dot{x})\dot{x} + 2x \Big)\, dt$$

subject to the conditions $x(0) = 0$ and $x(4) = X$, where X is volume of goods to be produced. Find the extremal of C that satisfies the given conditions and prove that this function provides the global minimum of C.

Why is this solution not applicable when $X < 8$?

Solution

For this functional, the integrand $F(x, \dot{x}, t)$ is

$$F = (3 + \dot{x})\dot{x} + 2x$$

which has no explicit t dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{x}\frac{\partial F}{\partial \dot{x}} - F = \text{constant.}$$

In the present case, this simplifies to give

$$\dot{x}^2 = 2x + a,$$

where a is a constant. This equation evidently has the family of solutions $x =$ constant but these solutions do not satisfy the Euler-Lagrange equation and are therefore *not* extremals.

Other solutions can be found by taking square roots and separating, as usual. This gives

$$\begin{aligned} t &= \pm \int \frac{dx}{(2x + a)^{1/2}} \\ &= (2x + a)^{1/2} + b, \end{aligned}$$

where b is an integration constant. On solving for x, we find that the **family of extremals** of the cost functional C is

$$x = \tfrac{1}{2}(t - b)^2 - \tfrac{1}{2}a,$$

which is a family of parabolas in the (t, x)-plane.

We must now find the extremals that satisfy the prescribed **end conditions**. The condition $x = 0$ when $t = 0$ gives

$$0 = b^2 - a,$$

and the condition $x = X$ when $t = 4$ gives

$$2X = (4 - b)^2 - a.$$

These equations have the solution

$$a = \tfrac{1}{16}(8 - X)^2, \qquad b = \tfrac{1}{4}(8 - X),$$

so that there is exactly **one admissible extremal**, namely

$$\begin{aligned} x &= \tfrac{1}{32}(4t + X - 8)^2 - \tfrac{1}{32}(8 - X)^2, \\ &= \tfrac{1}{4}t(2t + X - 8). \end{aligned}$$

To investigate the nature of this extremal, let $x^* = \frac{1}{4}t(2t + X - 8)$ and consider the function $x^* + h$, where $h(t)$ is any admissible variation. Then

$$\begin{aligned} C\big[x^* + h\big] &= C\big[x^*\big] + \int_0^4 \Big[\big(3 + 2t + \tfrac{1}{2}(X - 8)\big)\dot{h} + \dot{h}^2 + 2h\Big]\,dt \\ &= C\big[x^*\big] + \Big[\big(3 + 2t + \tfrac{1}{2}(X - 8)\big)h\Big]_{t=0}^{t=4} + \int_0^4 \dot{h}^2\,dt \\ &= C\big[x^*\big] + 0 + \int_0^4 \dot{h}^2\,dt, \end{aligned}$$

since $h(0) = h(4) = 0$ in an *admissible* variation. Hence

$$C\big[x^* + h\big] \geq C\big[x^*\big]$$

since the integrand $\dot{h}^2$ is *positive*. Since h is a general admissible variation, it follows that x^* provides the **global minimum** for the cost functional C.

This minimising function is not necessarily appropriate since solutions of the actual problem must also satisfy a condition not previously mentioned, namely, that the *rate of production of goods must always be positive*. Since

$$\dot{x}^* = t + \tfrac{1}{4}(X - 8),$$

this condition will be satisfied if $X \geq 8$ but not otherwise. [Can you guess what the optimum solution is when $X < 8$?] ■

Problem 13.7 *Soap film problem*

Consider the soap film problem for which it is required to minimise

$$J[y] = \int_{-a}^{a} y\left(1 + \dot{y}^2\right)^{\frac{1}{2}}\, dx$$

with $y(-a) = y(a) = b$. Show that the extremals of J have the form

$$y = c\cosh\left(\frac{x}{c} + d\right),$$

where c, d are constants, and that the end conditions are satisfied if (and only if) $d = 0$ and

$$\cosh\lambda = \left(\frac{b}{a}\right)\lambda,$$

where $\lambda = a/c$. Show that there are *two* admissible extremals provided that the aspect ratio b/a exceeds a certain critical value and *none* if b/a is less than this crirical value. Sketch a graph showing how this critical value is determined.

The remainder of this question requires computer assistance. Show that the critical value of the aspect ratio b/a is about 1.51. Choose a value of b/a larger than the critical value ($b/a = 2$ is suitable) and find the two values of λ. Plot the two admissible extremals on the same graph. Which one looks like the actual shape of the soap film? Check your guess by perturbing each extremal by small admissible variations and finding the change in the value of the functional $J[y]$.

Solution

For this functional, the integrand $F(y, \dot{y}, x)$ is

$$F = y\left(1 + \dot{y}^2\right)^{1/2}$$

which has no explicit x dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{y}\frac{\partial F}{\partial \dot{y}} - F = \text{ constant.}$$

In the present case, this simplifies to give

$$\frac{y}{\left(1 + \dot{y}^2\right)^{1/2}} = \text{constant},$$

that is ,

$$\dot{y}^2 = \frac{y^2}{c^2} - 1,$$

where c is a positive constant. This equation evidently has the family of solutions $y =$ constant but these solutions do not satisfy the Euler-Lagrange equation and are therefore *not* extremals. Other solutions can be found by taking square roots and separating, as usual. This gives

$$\begin{aligned} x &= \pm \int \frac{dy}{\left((y^2/c^2) - 1\right)^{1/2}} \\ &= \pm c \cosh^{-1}\left(\frac{y}{c}\right) + d, \end{aligned}$$

where d is an integration constant. On solving for y, we obtain

$$y = c \cosh\left(\frac{x - d}{c}\right).$$

This family of catenaries (hanging chains) is the **family of extremals** of the functional J.

We must now find the extremals that satisfy the given **end conditions**. The conditions $y = b$ when $x = \pm a$ give

$$b = c \cosh\left(\frac{a + d}{c}\right) = c \cosh\left(\frac{a - d}{c}\right)$$

from which it follows that $d = 0$ and that c must satisfy the equation

$$\frac{b}{c} = \cosh\left(\frac{a}{c}\right).$$

This equation determines the value of the constant c. If we introduce the dimensionless unknown $\lambda = a/c$, then λ satisfies the equation

$$\cosh \lambda = \left(\frac{b}{a}\right)\lambda.$$

This equation cannot be solved explicitly, but the nature of its solutions can be investigated graphically. Figure 13.5 shows that graphs of $\cosh \lambda$ and $k\lambda$ for various

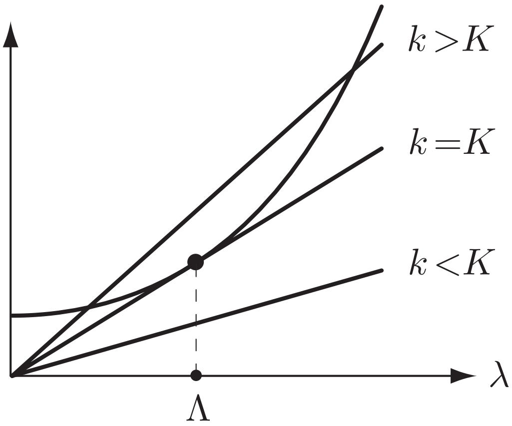

FIGURE 13.5 Graphs of $\cosh\lambda$ and $k\lambda$ for various values of the gradient k.

values of the gradient k. There is a **critical value** K of the gradient such that the straight line $K\lambda$ *touches* the catenary $\cosh\lambda$. When k $(= b/a)$ is less than K, there are no intersections, and when k is greater than K there are *two*. Each intersection corresponds to an **admissible extremal** of the functional J.

The critical gradient K can be found by observing that, at the touching point $\lambda = \Lambda$, the functions $\cosh\lambda$ and $k\lambda$ are equal, and so are their gradients. This gives the simultaneous equations

$$\cosh\lambda = k\lambda,$$
$$\sinh\lambda = k$$

for the critical values of k and λ. On eliminating k, we find that the critical value Λ satisfies the equation

$$\lambda \tanh\lambda = 1.$$

This equation cannot be solved explicitly, but it is easy to solve numerically and the value of Λ is found to be about 1.20. The corresponding value of K $(= \sinh\Lambda)$ is about 1.51.

When the aspect ratio b/a is *less* than K there are no extremals and hence no soap film can exist. However, when the aspect ratio b/a is *greater* than K there are two extremals and apparently *two different shapes that the film can have*. Figure 13.6 shows the two extremals for the case in which $b/a = 2$. It seems baffling that two different solutions can exist since experience indicates that there is only one configuration for the soap film (or none at all). Although we cannot prove this here, it can be shown that *the upper extremal provides a* ***minimum*** *for the functional J while the lower extremal does not*. Hence it is the **upper extremal** that provides the **unique stable configuration** of the film. ■

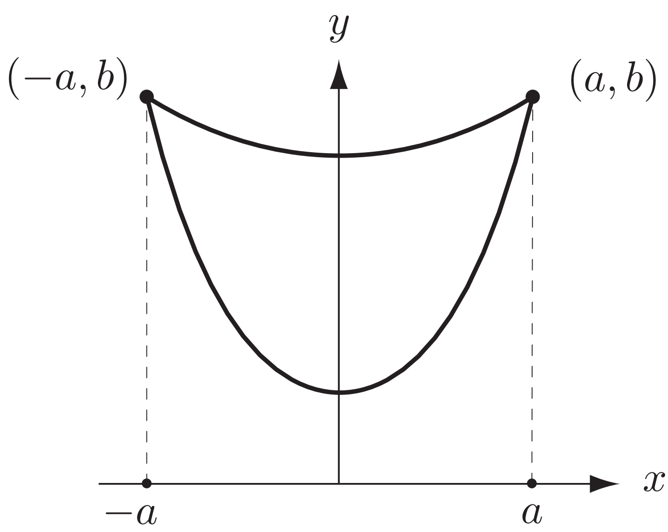

FIGURE 13.6 The two extremals in the soap film problem for the case in which the aspect ratio $b/a = 2$.

Problem 13.8

A sugar solution has a refractive index n that increases with the depth z according to the formula

$$n = n_0 \left(1 + \frac{z}{a}\right)^{1/2},$$

where n_0 and a are positive constants. A particular ray is horizontal when it passes through the origin of coordinates. Show that the path of the ray is not the straight line $z = 0$ but the parabola $z = x^2/4a$.

Solution

Rays in this medium make the **Fermat time functional**

$$T[z] = \int n \left(1 + \dot{z}^2\right)^{1/2} dx$$

stationary, where n $(= n(z))$ is the refractive index of the medium. For this functional, the integrand $F(z, \dot{z}, x)$ is

$$F = n(z) \left(1 + \dot{z}^2\right)^{1/2}$$

which has no explicit x dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{z}\frac{\partial F}{\partial \dot{z}} - F = \text{constant}.$$

In the present case, this simplifies to give

$$\frac{n_0 \left(1 + \dfrac{z}{a}\right)^{1/2}}{\left(1 + \dot{z}^2\right)^{1/2}} = \text{constant},$$

that is,

$$\dot{z}^2 = A^2 n_0^2 \left(1 + \frac{z}{a}\right) - 1,$$

where A is a positive constant. We are interested in a particular ray that is horizontal when it passes through the origin, that is, $z = 0$ and $\dot{z} = 0$ when $x = 0$. For this ray, $A = 1/n_0$ and we have the simple ODE

$$\dot{z}^2 = \frac{z}{a}.$$

This equation evidently has the constant solution $z = 0$ but this is not an extremal unless it satisfies the original Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{n\dot{z}}{\left(1+\dot{z}^2\right)^{1/2}}\right) - \left(n'\left(1+\dot{z}^2\right)^{1/2}\right) = 0,$$

where $n' = dn/dz$. This equation admits $z = 0$ as a solution if $n'(0) = 0$, but not otherwise. Since this condition is *not* satisfied by the medium in the present problem, the **straight line solution is excluded**.

Other solutions can be found by taking square roots and separating in the usual way. This gives

$$\begin{aligned} x &= \pm a^{1/2} \int \frac{dz}{z^{1/2}} \\ &= \pm 2a^{1/2} z^{1/2} + B, \end{aligned}$$

where B is an integration constant. The initial condition $z = 0$ when $x = 0$ gives $B = 0$ and hence the only **admissible extremal** is the parabola

$$z = \frac{x^2}{4a}.$$

This is the **path of the ray**. ■

Problem 13.9

Consider the propagation of light rays in an axially symmetric medium, where, in a system of cylindrical polar co-ordinates (r, θ, z), the refractive index $n = n(r)$ and the rays lie in the plane $z = 0$. Show that Fermat's time functional has the form

$$T[r] = c^{-1} \int_{\theta_0}^{\theta_1} n \left(r^2 + \dot{r}^2 \right)^{1/2} d\theta,$$

where $r = r(\theta)$ is the equation of the path, and $\dot{r}$ means $dr/d\theta$.

(i) Show that the extremals of T satisfy the ODE

$$\frac{n\, r^2}{(r^2 + \dot{r}^2)^{1/2}} = \text{constant}.$$

Show further that, if we write $\dot{r} = r \tan \psi$, where ψ is the angle between the tangent to the ray and the local cylindrical surface $r =$ constant, this equation becomes

$$r\, n \cos \psi = \text{constant},$$

which is the form of Snell's law for this case. Deduce that circular rays with centre at the origin exist only when the refractive index $n = a/r$, where a is a positive constant.

Solution

In plane polar coordinates, the element of length ds is given by

$$(ds)^2 = (dr)^2 + (r\, d\theta)^2,$$

that is,

$$ds = \left(r^2 + \left(\frac{dr}{d\theta} \right)^2 \right)^{1/2} d\theta.$$

Hence, rays in this medium make the **Fermat time functional**

$$T[r] = \int n \left(r^2 + \left(\frac{dr}{d\theta} \right)^2 \right)^{1/2} d\theta$$

stationary, where n $(= n(r))$ is the refractive index of the medium. For this functional, the integrand $F(r, \dot{r}, \theta)$ is

$$F = n\left(r^2 + \dot{r}^2\right)^{1/2}$$

which has no explicit θ dependence. We may therefore replace the Euler-Lagrange equation by the **integrated form**

$$\dot{r}\frac{\partial F}{\partial \dot{r}} - F = \text{constant}.$$

In the present case, this simplifies to give

$$\frac{r^2 n}{\left(r^2 + \dot{r}^2\right)^{1/2}} = \text{constant},$$

that is,

$$\dot{r}^2 = r^2\left(\frac{n^2 r^2}{b^2} - 1\right), \tag{1}$$

where b is a positive constant. This equation evidently has the family of solutions $r =$ constant but these solutions do not *generally* satisfy the Euler-Lagrange equation

$$\frac{d}{d\theta}\left(\frac{\dot{r}n}{\left(1 + \dot{r}^2\right)^{1/2}}\right) - \left(n'\left(1 + \dot{r}^2\right)^{1/2} - \frac{rn}{\left(1 + \dot{r}^2\right)^{1/2}}\right) = 0,$$

where $n' = dn/dr$. Thus, if the circle $r = c$ is an extremal, then

$$n'(c) + cn(c) = 0,$$

and if circles of *all radii* are extremals, then the function $n(r)$ must satisfy the ODE

$$n'(r) + rn(r) = 0$$

for $r > 0$. Hence n must have the **special radial dependence**

$$n = \frac{a}{r}, \tag{2}$$

where a is a positive constant.

Other solutions may be investigated by introducing the angle ψ, which is the angle between the tangent to the ray and the local cylindrical surface $r =$ constant. The relationship is

$$\tan\psi = \frac{1}{r}\frac{dr}{d\theta}.$$

On replacing $\dot{r}$ in equation (1) by $r\tan\psi$, we find that, along each ray, r and ψ must related by the formula

$$rn\cos\psi = \text{constant}.$$

This is the form of **Snell's law** for this geometry. [It is tempting to try to establish the formula (1) by putting $\psi = 0$ in Snell's law. However, although this gives the right answer, this step is not justified. Why not?] ■

Problem 13.10

A particle of mass 2 kg moves under uniform gravity along the z-axis, which points verically downwards. Show that (in S.I. units) the action functional for the time interval $[0, 2]$ is

$$S[z] = \int_0^2 \left(\dot{z}^2 + 20z\right) dt,$$

where g has been taken to be $10\ \mathrm{m\,s^{-2}}$.

Show directly that, of all the functions $z(t)$ that satisfy the end conditions $z(0) = 0$ and $z(2) = 20$, the actual motion $z = 5t^2$ provides the *least* value of S.

Solution

For this mechanical system, the **Lagrangian** is

$$\begin{aligned} L = T - V &= \tfrac{1}{2}(2)\dot{z}^2 - \big(-(2)(10)z\big) \\ &= \dot{z}^2 + 20z, \end{aligned}$$

and the **action functional** for the time interval $[0, 2]$ is

$$S[z] = \int_0^2 \left(\dot{z}^2 + 20z\right) dt.$$

By **Hamilton's principle**, the motion $z = 5t^2$ makes the action functional stationary. To investigate the nature of this extremal, consider the function

$$x = 5t^2 + h,$$

where h is an admissible variation. Then

$$\begin{aligned} S[5t^2 + h] &= \int_0^2 \left(10t + \dot{h}\right)^2 + 20\left(5t^2 + h\right)\, dt \\ &= \int_0^2 \left(200t^2 + 20t\dot{h} + 20h + \dot{h}^2\right) dt \\ &= \left[\frac{200t^3}{3}\right]_{t=0}^{t=2} + \Big[20th\Big]_{t=0}^{t=2} + \int_0^2 \dot{h}^2 dt \\ &= \frac{1600}{3} + 0 + \int_0^2 \dot{h}^2 dt, \end{aligned}$$

since $h(0) = h(2) = 0$ in an *admissible* variation. In particular, by taking $h \equiv 0$, $S[5t^2] = 1600/3$. Hence

$$\begin{aligned} S[5t^2 + h] &= S[5t^2] + \int_0^2 \dot{h}^2 dt \\ &\geq S[5t^2], \end{aligned}$$

since the integrand $\dot{h}^2$ is *positive*. Since h is a general admissible variation, it follows that the extremal $x = 5t^2$ provides the **global minimum** for the action functional S. ■

Problem 13.11

A certain oscillator with generalised coordinate q has Lagrangian

$$L = \dot{q}^2 - 4q^2.$$

Verify that $q^* = \sin 2t$ is a motion of the oscillator, and show directly that it makes the action functional $S[q]$ satationary in any time interval $[0, \tau]$.

For the time interval $0 \le t \le \pi$, find the variation in the action functional corresponding to the variations (i) $h = \epsilon \sin 4t$, (ii) $h = \epsilon \sin t$, where ϵ is a small parameter. Deduce that the motion $q^* = \sin 2t$ does not make S a minimum or a maximum.

Solution

The **equation of motion** corresponding to the Lagrangian $L = \dot{q}^2 - 4q^2$ is

$$\ddot{q} + 4q = 0,$$

which is the classical SHM equation with $\omega = 2$. Hence $q^* = \sin 2t$ is a possible motion of the system.

The **action functional** for the time interval $[0, \tau]$ is

$$S[q] = \int_0^\tau \left(\dot{q}^2 - 4q^2\right) dt$$

and, by **Hamilton's principle**, the motion $q^* = \sin 2t$ makes S stationary. To prove this from first principles, consider the function

$$q = q^* + h,$$

where h is an admissible variation. Then

$$\begin{aligned} S[q^* + h] &= \int_0^\tau \left(2\cos 2t + \dot{h}\right)^2 - 4\left(\sin 2t + h\right)^2 dt \\ &= \int_0^\tau \left(4\cos 4t + 4\dot{h}\cos 2t - 8h\sin 2t + \dot{h}^2 - 4h^2\right) dt \\ &= \sin 4\tau + \Big[4h\cos 2t\Big]_{t=0}^{t=\tau} + \int_0^\tau \left(\dot{h}^2 - 4h^2\right) dt \\ &= \sin 4\tau + 0 + \int_0^\tau \left(\dot{h}^2 - 4h^2\right) dt, \end{aligned}$$

since $h(0) = h(\tau) = 0$ in an *admissible* variation. In particular, by taking $h \equiv 0$,

$S[q^*] = \sin 4\tau$. Hence

$$S[q^* + h] = S[q^*] + \int_0^\tau \left(\dot{h}^2 - 4h^2\right) dt$$
$$= S[q^*] + O\left(||h||^2\right).$$

Thus, in accordance with Hamilton's principle, the motion $q^* = \sin 2t$ makes the action functional **stationary**.

To investigate the nature of this extremal for the particular time interval $[0, \pi]$, let q^* first be perturbed by the variation $h_1 = \epsilon \sin 4t$, where ϵ is a positive constant. Then

$$S[q^* + h] = S[q^*] + \int_0^\pi \left(\dot{h}^2 - 4h^2\right) dt$$
$$= S[q^*] + \int_0^\pi \left(16\epsilon^2 \cos^2 4t - 4\epsilon^2 \sin^2 4t\right) dt$$
$$= S[q^*] + 6\pi\epsilon^2.$$

On the other hand, when q^* is perturbed by the variation $h_2 = \epsilon \sin t$,

$$S[q^* + h] = S[q^*] + \int_0^\pi \left(\dot{h}^2 - 4h^2\right) dt$$
$$= S[q^*] + \int_0^\pi \left(\epsilon^2 \cos^2 t - 4\epsilon^2 \sin^2 t\right) dt$$
$$= S[q^*] - \tfrac{3}{2}\pi\epsilon^2.$$

Thus S is *increased* by the variation h_1 and *decreased* by the variation h_2. It follows that the motion q^* provides **neither a minimum nor a maximum** for the action functional S over the time interval $[0, \pi]$. ■

Problem 13.12

A particle is constrained to move over a smooth fixed surface under *no forces* other than the force of constraint. By using Hamilton's principle and energy conservation, show that the path of the particle must be a geodesic of the surface. (The term geodesic has been extended here to mean those paths that make the length functional *stationary*).

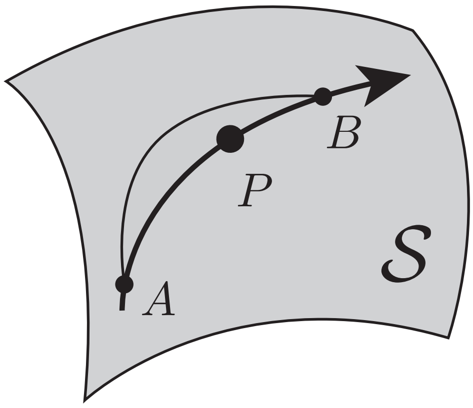

FIGURE 13.7 The particle P slides over the smooth surface $\mathcal{S}$.

Solution

Let A and B be two points on an actual path traced out by the particle P (see Figure 13.7) and suppose that the motion between A and B takes place in the time interval $0 \le t \le \tau$.

Since the surface is smooth and there are no forces other than the constraint force, the **Lagrangian** $L = \frac{1}{2}m|\boldsymbol{v}|^2$ and the **action functional** for the interval $[0, \tau]$ is

$$S = \tfrac{1}{2}m \int_0^\tau |\boldsymbol{v}|^2\, dt.$$

Let $\boldsymbol{q}$ $(= (q_1, q_2))$ be a set of generalised coordinates for the particle and let $\boldsymbol{q}^*(t)$ be the actual motion shown in Figure 13.7. Then, by **Hamilton's principle**,

$$S[\boldsymbol{q}^* + \boldsymbol{h}] = S[\boldsymbol{q}^*] + O\left(||\boldsymbol{h}||^2\right),$$

where $\boldsymbol{h}(t)$ is any admissible variation. In the present problem, this states that

$$\int_0^\tau |\boldsymbol{v}|^2\, dt = \int_0^\tau |\boldsymbol{v}^*|^2\, dt + O\left(||\boldsymbol{h}||^2\right),$$

where $\boldsymbol{v}$ is the velocity corresponding to the geometrically possible trajectory $\boldsymbol{q} = \boldsymbol{q}^* + \boldsymbol{h}$ and $\boldsymbol{v}^*$ is the velocity corresponding to the actual motion $\boldsymbol{q}^*$.

Since the surface is smooth and there are no prescribed forces, **energy conservation** applies in the form $T =$ constant. Hence, in the actual motion, P moves with *constant speed*. Since the motion takes place over the time interval $[0, \tau]$, this speed must be L^*/τ, where L^* is the path length of the actual motion connecting A and B. Hence

$$\int_0^\tau |\boldsymbol{v}|^2\, dt = \frac{(L^*)^2}{\tau} + O\left(||\boldsymbol{h}||^2\right).$$

Now comes the clever bit. Let $\mathcal{C}$ be some path on $\mathcal{S}$ connecting the points A and B, and let $\boldsymbol{q}(t)$ be the trajectory in which P traverses $\mathcal{C}$ at *constant speed*. Since all trajectories are supposed to take place over the time interval $[0, \tau]$, the constant speed required is L/τ, where L is the length of $\mathcal{C}$. Then Hamilton's principle implies that

$$\frac{L^2}{\tau} = \frac{(L^*)^2}{\tau} + O\left(||\boldsymbol{h}||^2\right),$$

which in turn implies that

$$L = L^* + O\left(||\boldsymbol{h}||^2\right).$$

Since $\mathcal{C}$ can be *any* path connecting A and B, this formula states that the motion $\boldsymbol{q}^*$ makes the path length functional stationary. In other words, the path of the particle is a **geodesic of the surface** $\mathcal{S}$. ■

Problem 13.13

By using Hamilton's principle, show that, if the Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is modified to L' by any transformation of the form

$$L' = L + \frac{d}{dt}\, g(\boldsymbol{q}, t),$$

then the equations of motion are unchanged.

Solution

The action functional S'corresponding to the Lagrangian L' over the time interval $[t_A, t_B]$ is

$$\begin{aligned} S'[\boldsymbol{q}] &= \int_{t_A}^{t_B} L'\, dt \\ &= \int_{t_A}^{t_B} \left(L + \frac{d}{dt}\, g(\boldsymbol{q}, t) \right) dt \\ &= \int_{t_1}^{t_2} L\, dt + \Big[g(\boldsymbol{q}, t) \Big]_{t=t_A}^{t=t_B} \\ &= S[\boldsymbol{q}] + \Big(g(\boldsymbol{q}_B, t_B) - g(\boldsymbol{q}_A, t_A) \Big). \end{aligned}$$

Thus S and S' differ by a constant and hence have the same family of extremals extremals. The Lagrange equations for L and L' therefore have **the same family of solutions**. ■

Chapter Fourteen

Hamilton’s equations and phase space

Problem 14.1

Find the Legendre transform $G(v_1, v_2, w)$ of the function

$$F(u_1, u_2, w) = 2u_1^2 - 3u_1u_2 + u_2^2 + 3wu_1,$$

where w is a passive variable. Verify that $\partial F/\partial w = -\partial G/\partial w$.

Solution

In a specific example such as this it is *always* easier to work from first principles rather than from the relation $F + G = u_1v_1 + u_2v_2$.

The new variables v_1, v_2 are expressed in terms of the old variables u_1, u_2 by

$$v_1 = \frac{\partial F}{\partial u_1} = 4u_1 - 3u_2 + 3w,$$
$$v_2 = \frac{\partial F}{\partial u_2} = -3u_1 + 2u_2.$$

The first step is to invert these relations by solving the simultaneous equations

$$4u_1 - 3u_2 = v_1 - 3w,$$
$$3u_1 - 2u_2 = -v_2,$$

which gives

$$u_1 = -2v_1 - 3v_2 + 6w,$$
$$u_2 = -3v_1 - 4v_2 + 9w.$$

The Legendre transform $G(v_1, v_2, w)$ then satisfies the equations

$$\frac{\partial G}{\partial v_1} = -2v_1 - 3v_2 + 6w,$$
$$\frac{\partial G}{\partial v_2} = -3v_1 - 4v_2 + 9w,$$

from which it follows that

$$G = -v_1^2 - 3v_1v_2 - 2v_2^2 + 6wv_1 + 9wv_2 + f(w).$$

In this method of solution, we must *make* G satisfy the equation $\partial G/\partial w = -\partial F/\partial w$ by an appropriate choice of the function $f(w)$. Now

$$\frac{\partial G}{\partial w} = 6v_1 + 9v_2 + f'(w),$$
$$-\frac{\partial F}{\partial w} = -3u_1 = 6v_1 + 9v_2 - 18w,$$

and these expressions are equal if $f'(w) = -18w$. Hence $f(w) = -9w^2$ to within an added constant. The **Legendre transform** of F is therefore

$$G = -v_1^2 - 3v_1 v_2 - 2v_2^2 + 6wv_1 + 9wv_2 - 9w^2. \blacksquare$$

Problem 14.2

A smooth wire has the form of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$, where θ is a real parameter, and a, b are positive constants. The wire is fixed with the axis Oz pointing vertically upwards. A particle P of mass m can slide freely on the wire. Taking θ as generalised coordinate, find the Hamiltonian and obtain Hamilton's equations for this system.

Solution

In terms of the coordinate θ, the particle has **kinetic energy**

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ &= \tfrac{1}{2}m\left((-a\sin\theta)^2 + (a\cos\theta)^2 + b^2\right)\dot{\theta}^2 \\ &= \tfrac{1}{2}m\left(a^2 + b^2\right)\dot{\theta}^2. \end{aligned}$$

The **potential energy** is $V = mgz = mgb\theta$. Hence the **Lagrangian** of the system is

$$L(\theta, \dot{\theta}) = T - V = \tfrac{1}{2}m\left(a^2 + b^2\right)\dot{\theta}^2 - mgb\theta.$$

The **conjugate momentum** p_θ is then given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\left(a^2 + b^2\right)\dot{\theta}$$

and the corresponding inverse relation is

$$\dot{\theta} = \frac{p_\theta}{m(a^2 + b^2)}.$$

Since this system is conservative, the **Hamiltonian** is given by

$$\begin{aligned} H(\theta, p_\theta) = T + V &= \tfrac{1}{2}m\left(a^2 + b^2\right)\left(\frac{p_\theta}{m(a^2 + b^2)}\right)^2 + mgb\theta \\ &= \frac{p_\theta^2}{2m(a^2 + b^2)} + mgb\theta. \end{aligned}$$

Hamilton's equations for the system are then given by

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \end{aligned}$$

that is,

$$\dot{\theta} = \frac{p_\theta}{m(a^2 + b^2)}$$
$$\dot{p}_\theta = -mgb \qquad \blacksquare$$

Problem 14.3 *Projectile*

Using Cartesian coordinates, find the Hamiltonian for a projectile of mass m moving under uniform gravity. Obtain Hamilton's equations and identify any cyclic coordinates.

Solution

In terms of Cartesian coordinates x, z, the particle has **kinetic energy**

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$$

and **potential energy** $V = mgz$. Hence the **Lagrangian** of the system is

$$L(x, z, \dot{x}, \dot{z}) = T - V = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) - mgz.$$

The **conjugate momenta** p_x, p_z are then given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x},$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

and the corresponding inverse relations are

$$\dot{x} = \frac{p_x}{m},$$
$$\dot{z} = \frac{p_z}{m}.$$

Since this system is conservative, the **Hamiltonian** is given by

$$H(x, y, p_x, p_z) = T + V = \tfrac{1}{2}m\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + mgz$$
$$= \frac{p_x^2 + p_z^2}{2m} + mgz.$$

Hamilton's equations for the system are then given by

$$\dot{x} = \frac{\partial H}{\partial p_x}, \qquad \dot{p}_x = -\frac{\partial H}{\partial x},$$
$$\dot{z} = \frac{\partial H}{\partial p_z}, \qquad \dot{p}_z = -\frac{\partial H}{\partial z},$$

that is,

$$\dot{x} = \frac{p_x}{m}, \qquad \dot{p}_x = 0,$$
$$\dot{z} = \frac{p_z}{m}, \qquad \dot{p}_z = -mg.$$

The coordinate x does not appear in H and is therefore **cyclic**. As a result, the conjugate momentum p_x is conserved. ■

Problem 14.4 *Spherical pendulum*

The spherical pendulum is a particle of mass m attached to a fixed point by a light inextensible string of length a and moving under uniform gravity. It differs from the simple pendulum in that the motion is not restricted to lie in a vertical plane. Show that the Lagrangian is

$$L = \tfrac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + mga\cos\theta,$$

where the polar angles θ, ϕ are shown in Figure 11.7.

Find the Hamiltonian and obtain Hamilton's equations. Identify any cyclic coordinates.

Solution

In terms of the polar angles θ, ϕ, the system has **kinetic energy**

$$T = \tfrac{1}{2}m\left((a\dot{\theta})^2 + (a\sin\theta\,\dot{\phi})^2\right)$$

and **potential energy** $V = -mga\cos\theta$. Hence the **Lagrangian** of the system is

$$L(\theta,\phi,\dot{\theta},\dot{\phi}) = T - V = \tfrac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + mga\cos\theta.$$

The **conjugate momenta** p_θ, p_ϕ are then given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta},$$
$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ma^2\sin^2\theta\,\dot{\phi}.$$

Since this system is conservative, the **Hamiltonian** is given by

$$\begin{aligned} H(\theta,\phi,p_\theta,p_\phi) &= T + V \\ &= \tfrac{1}{2}ma^2\left(\left(\frac{p_\theta}{ma^2}\right)^2 + \sin^2\theta\left(\frac{p_\phi}{ma^2\sin^2\theta}\right)^2\right) - mga\cos\theta \\ &= \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mga\cos\theta. \end{aligned}$$

Hamilton's equations for the system are then given by

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial\theta},$$
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi}, \qquad \dot{p}_\phi = -\frac{\partial H}{\partial\phi},$$

that is,

$$\dot{\theta} = \frac{p_\theta}{ma^2}, \qquad \dot{p_\theta} = \frac{p_\phi^2 \cos\theta}{ma^2 \sin^3\theta} - mga\cos\theta,$$
$$\dot{\phi} = \frac{p_\phi}{ma^2 \sin^2\theta}, \qquad \dot{p_\phi} = 0.$$

The coordinate ϕ does not appear in H and is therefore **cyclic**. As a result, the conjugate momentum p_ϕ is conserved. ■

Problem 14.5

The system shown in Figure 10.9 consists of two particles P_1 and P_2 connected by a light inextensible string of length a. The particle P_1 is constrained to move along a fixed smooth horizontal rail, and the whole system moves under uniform gravity in the vertical plane through the rail. For the case in which the particles are of equal mass m, show that the Lagrangian is

$$L = \tfrac{1}{2}m\left(2\dot{x}^2 + 2a\dot{x}\dot{\theta} + a^2\dot{\theta}^2\right) + mga\cos\theta,$$

where x and θ are the coordinates shown in Figure 10.9.

Find the Hamiltonian and verify that it satisfies the equations $\dot{x} = \partial H/\partial p_x$ and $\dot{\theta} = \partial H/\partial p_\theta$. [Messy algebra.]

Solution

In terms of coordinates x, θ, the system has **kinetic energy**

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \left(a\dot{\theta}\right)^2 + 2\dot{x}\left(a\dot{\theta}\right)\cos\theta\right) \\ &= \tfrac{1}{2}m\left(2\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right). \end{aligned}$$

The **potential energy** is $V = -mga\cos\theta$. Hence the **Lagrangian** of the system is

$$L(x, \theta, \dot{x}, \dot{\theta}) = T - V = \tfrac{1}{2}m\left(2\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta\right) + mga\cos\theta.$$

The **conjugate momenta** p_x, p_θ are then given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\left(2\dot{x} + a\dot{\theta}\cos\theta\right), \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = ma\left(a\dot{\theta} + \dot{x}\cos\theta\right). \end{aligned}$$

This is more typical of the general case in that we must solve a pair of *coupled* equations to obtain $\dot{x}$ and $\dot{\theta}$ in terms of p_x and p_θ. This gives

$$\begin{aligned} \dot{x} &= \frac{p_x - \cos\theta\,(p_\theta/a)}{m\left(2 - \cos^2\theta\right)}, \\ a\dot{\theta} &= \frac{2\,(p_\theta/a) - \cos\theta\, p_x}{m\left(2 - \cos^2\theta\right)}. \end{aligned}$$

Since this system is conservative, the **Hamiltonian** is given by

$$H(x, \theta, p_x, p_\theta) = T + V = \frac{p_x^2 + 2\,(p_\theta/a)^2 - 2\cos\theta\, p_x\,(p_\theta/a)}{2m\left(2 - \cos^2\theta\right)} - mga\cos\theta,$$

after much algebra.

It may now be verified that H satisfies

$$\frac{\partial H}{\partial p_x} = \frac{p_x - \cos\theta\,(p_\theta/a)}{m\left(2 - \cos^2\theta\right)} = \dot{x},$$

$$\frac{\partial H}{\partial p_\theta} = \frac{2\,(p_\theta/a) - \cos\theta\, p_x}{m\left(2 - \cos^2\theta\right)} = \dot{\theta}. \blacksquare$$

Problem 14.6 *Pendulum with a shortening string*

A particle is suspended from a support by a light inextensible string which passes through a small fixed ring vertically below the support. The particle moves in a vertical plane with the string taut. At the same time, the support is made to move vertically having an upward displacement $Z(t)$ at time t. The effect is that the particle oscillates like a simple pendulum whose string length at time t is $a - Z(t)$, where a is a positive constant. Show that the Lagrangian is

$$L = \tfrac{1}{2}m\left((a-Z)^2\dot{\theta}^2 + \dot{Z}^2\right) + mg(a-Z)\cos\theta,$$

where θ is the angle between the string and the downward vertical.

Find the Hamiltonian and obtain Hamilton's equations. Is H conserved?

Solution

In terms of coordinate θ, the system has **kinetic energy**

$$T = \tfrac{1}{2}m\left(\dot{Z}^2 + (a-Z)^2\dot{\theta}^2\right)$$

and **potential energy** is $V = -mg(a-Z)\cos\theta$. (Remember that Z is not a coordinate but a specified function of t.) Hence the **Lagrangian** of the system is

$$L(\theta,\dot{\theta}) = T - V = \tfrac{1}{2}m\left(\dot{Z}^2 + (a-Z)^2\dot{\theta}^2\right) + mg(a-Z)\cos\theta.$$

The **conjugate momentum** p_θ is then given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m(a-Z)^2\dot{\theta}$$

and the corresponding inverse relation is

$$\dot{\theta} = \frac{p_\theta}{m(a-Z)^2}.$$

Since this system is *not* conservative, the **Hamiltonian** must be found from the general expression

$$\begin{aligned}
H(\theta, p_\theta) &= \dot{\theta}\,p_\theta - L \\
&= \left(\frac{p_\theta}{m(a-Z)^2}\right)p_\theta - \tfrac{1}{2}m\left(\dot{Z}^2 + \frac{p_\theta^2}{m^2(a-Z)^2}\right) - mg(a-Z)\cos\theta \\
&= \frac{p_\theta^2}{2m(a-Z)^2} - \tfrac{1}{2}m\dot{Z}^2 - mg(a-Z)\cos\theta.
\end{aligned}$$

Hamilton's equations for the system are then given by

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta},$$

that is,

$$\dot{\theta} = \frac{p_\theta}{m(a - Z)^2}, \qquad \dot{p}_\theta = -mg(a - Z)\sin\theta.$$

Since H has an explicit time dependence through $Z(t)$, H will not *generally* be conserved. It *will* be conserved however if $\dot{Z}(t)$ is constant. [Why is this?] ■

Problem 14.7 *Charged particle in an electrodynamic field*

The Lagrangian for a particle with mass m and charge e moving in the general electrodynamic field $\{\boldsymbol{E}(\boldsymbol{r},t), \boldsymbol{B}(\boldsymbol{r},t)\}$ is given in Cartesian coordinates by

$$L(\boldsymbol{r},\dot{\boldsymbol{r}},t) = \tfrac{1}{2}m\,\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - e\,\phi(\boldsymbol{r},t) + e\,\dot{\boldsymbol{r}}\cdot\boldsymbol{A}(\boldsymbol{r},t),$$

where $\boldsymbol{r} = (x, y, z)$ and $\{\phi, \boldsymbol{A}\}$ are the electrodynamic potentials of field $\{\boldsymbol{E}, \boldsymbol{B}\}$. Show that the corresponding Hamiltonian is given by

$$H(\boldsymbol{r}, \boldsymbol{p}, t) = \frac{(\boldsymbol{p} - e\boldsymbol{A})\cdot(\boldsymbol{p} - e\boldsymbol{A})}{2m} + e\,\phi,$$

where $\boldsymbol{p} = (p_x, p_y, p_x)$ are the generalised momenta conjugate to the coordinates (x, y, z). [Note that $\boldsymbol{p}$ is *not* the ordinary linear momentum of the particle.] Under what circumstances is H conserved?

Solution

In terms of Cartesian coordinates x, y, z, the system has **Lagrangian**

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - e\phi + e\dot{x}A_x + e\dot{y}A_y + e\dot{z}A_z.$$

The **conjugate momenta** p_x, p_y, p_z are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + eA_x,$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + eA_y,$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + eA_z.$$

and the corresponding inverse relations are

$$\dot{x} = \frac{1}{m}\left(p_x - eA_x\right),$$
$$\dot{y} = \frac{1}{m}\left(p_y - eA_y\right),$$
$$\dot{z} = \frac{1}{m}\left(p_z - eA_z\right).$$

Since this system is *not* conservative, the **Hamiltonian** must be found from the general expression

$$
\begin{aligned}
H(\boldsymbol{r}, \boldsymbol{p}) &= \dot{x} p_x + \dot{y} p_y + \dot{z} p_z - L \\
&= \frac{1}{m}\big(p_x(p_x - eA_x) + p_y(p_y - eA_y) + p_z(p_z - eA_z)\big) \\
&\quad -\frac{1}{2m}\Big((p_x - eA_x)^2 + (p_y - eA_y)^2 + (p_z - eA_z)^2\Big) \\
&\quad + e\phi - \frac{e}{m}\big(A_x(p_x - eA_x) + A_y(p_y - eA_y) + A_z(p_z - eA_z)\big) \\
&= \frac{1}{2m}\Big((p_x - eA_x)^2 + (p_y - eA_y)^2 + (p_z - eA_z)^2\Big) + e\phi \\
&= \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}) \cdot (\boldsymbol{p} - e\boldsymbol{A}) + e\phi.
\end{aligned}
$$

This is the required **Hamiltonian**. Since H has an explicit time dependence through $\{\phi(t), \boldsymbol{A}(t)\}$, H will not *generally* be conserved. It *will* be conserved however if the potentials $\{\phi, \boldsymbol{A}\}$ are independent of t, that is, if the fields are static. ■

Problem 14.8 *Relativistic Hamiltonian*

The relativistic Lagrangian for a particle of rest mass m_0 moving along the x-axis under the potential field $V(x)$ is given by

$$L = m_0\, c^2 \left(1 - \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2}\right) - V(x).$$

Show that the corresponding Hamiltonian is given by

$$H = m_0\, c^2 \left(1 + \left(\frac{p_x}{m_0\, c}\right)^2\right)^{1/2} - m_0\, c^2 + V(x),$$

where p_x is the generalised momentum conjugate to x.

Solution

Since the particle has **Lagrangian**

$$L = m_0\, c^2 \left(1 - \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2}\right) - V(x),$$

the **conjugate momentum** p_x is given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} \\ &= m_0 \dot{x} \left(1 - \frac{\dot{x}^2}{c^2}\right)^{-1/2} \end{aligned}$$

and the corresponding inverse relation is

$$\dot{x} = c p_x \left(m_0^2 c^2 + p_x^2\right)^{-1/2}.$$

This system is conservative, but the non-standard form of the 'kinetic energy' part of L means that the Hamiltonian must be found from the general expression

$$\begin{aligned} H(x, p_x) &= \dot{x}\, p_x - L \\ &= c p_x^2 \left(m_0^2 c^2 + p_x^2\right)^{-1/2} \\ &\quad - m_0 c^2 + m_0 c^2 \left(m_0^2 c^2 + p_x^2\right)^{-1/2} + V(x) \\ &= m_0\, c^2 \left(1 + \left(\frac{p_x}{m_0\, c}\right)^2\right)^{1/2} - m_0\, c^2 + V(x). \end{aligned}$$

This is the required relativistic **Hamiltonian**. ■

Problem 14.9 *A variational principle for Hamilton's equations*

Consider the functional

$$J[\boldsymbol{q}(t), \boldsymbol{p}(t)] = \int_{t_0}^{t_1} \Big(H(\boldsymbol{q}, \boldsymbol{p}, t) - \dot{\boldsymbol{q}} \cdot \boldsymbol{p} \Big)\, dt$$

of the $2n$ independent functions $q_1(t), \dots, q_n(t),\ p_1(t), \dots, p_n(t)$. Show that the extremals of J satisfy Hamilton's equations with Hamiltonian H.

Solution

In expanded form, the integrand is

$$F = H(\boldsymbol{q}, \boldsymbol{p}, t) \ - \ (\dot{q}_1 p_1 + \dot{q}_2 p_2 + \cdots + \dot{q}_n p_n)$$

and there is one Euler Lagrange equation corresponding to each of the $\{q_j\}$ and one corresponding to each of the $\{p_j\}$, making $2n$ equations in all.

The Euler Lagrange equation corresponding to the variable q_j is

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_j}\right) - \frac{\partial F}{\partial q_j} = 0,$$

that is,

$$\frac{d}{dt}(-p_j) - \frac{\partial H}{\partial q_j} = 0,$$

which becomes

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad\qquad (1 \le j \le n). \tag{1}$$

Similarly, the Euler Lagrange equation corresponding to the variable p_j is

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{p}_j}\right) - \frac{\partial F}{\partial p_j} = 0,$$

that is,

$$\frac{d}{dt}(0) - \left(\frac{\partial H}{\partial p_j} - \dot{q}_j\right) = 0,$$

which becomes

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad\qquad (1 \le j \le n). \tag{2}$$

Equations (1) and (2) are exactly Hamilton's equations for a system with Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p}, t)$. ■

Problem 14.10

In the theory of dynamical systems, a point is said to be an *asymptotically stable equilibrium point* if it 'attracts' points in a nearby volume of the phase space. Show that such points cannot occur in Hamiltonian dynamics.

Solution

FIGURE 14.1 The motion of phase points towards the asymptotically stable equilibrium point $\boldsymbol{x}_0$.

Suppose that there is an asymptotically stable equilibrium point $\boldsymbol{x}_0$ and that the sphere $\mathcal{R}_0$ is sufficiently small so that *all* of its phase points are attracted to $\boldsymbol{x}_0$ (see Figure 14.1, left). Then, with increasing time, the region $\mathcal{R}$ occupied by these points will shrink in size as its points are drawn towards $\boldsymbol{x}_0$ (see Figure 14.1, right). Thus the volume of this region is *not* conserved. However, by Liouville's theorem, volumes in phase space *are* conserved for any Hamiltonian system. The conclusion is that asymptotically stable equilibrium points cannot be a feature of Hamiltonian systems. ∎

Problem 14.11

A one dimensional damped oscillator with coordinate q satisfies the equation $\ddot{q} + 4\dot{q} + 3q = 0$, which is equivalent to the first order system

$$\dot{q} = v, \qquad \dot{v} = -3q - 4v.$$

Show that the area $a(t)$ of any region of points moving in (q, v)-space has the time variation

$$a(t) = a(0)\, e^{-4t}.$$

Does this result contradict Liouville's theorem?

Solution

Let $a(t)$ be the area of a region $\mathcal{A}_t$ of phase points moving in the phase plane (q, v) of the first order system of equations

$$\begin{aligned} \dot{q} &= v, \\ \dot{v} &= -3q - 4v. \end{aligned}$$

Then, as in the proof of Liouville's theorem,

$$\frac{da}{dt} = \int_{\mathcal{A}_t} \operatorname{div} \boldsymbol{F}\; dq dv,$$

where

$$\begin{aligned} \operatorname{div} \boldsymbol{F} &= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \\ &= \frac{\partial}{\partial q}\big(v\big) + \frac{\partial}{\partial v}\big(- 3q - 4v\big) \\ &= -4. \end{aligned}$$

Hence

$$\begin{aligned} \frac{da}{dt} &= \int_{\mathcal{A}_t} \operatorname{div} \boldsymbol{F}\; dq dv \\ &= \int_{\mathcal{A}_t} \big(- 4\big)\, dq dv \\ &= -4a. \end{aligned}$$

The area $a(t)$ therefore satisfies the equation

$$\frac{da}{dt} = -4a$$

the **general solution** of which is

$$a(t) = a(0)\, e^{-4t}.$$

This example does not contradict Liouville's theorem since the original oscillator equation $\ddot{q} + 4\dot{q} + 3q = 0$ contains the 'damping term' $4\dot{q}$ and is therefore *not* derivable from a Lagrangian. ■

Problem 14.12 *Ensembles in statistical mechanics*

In statistical mechanics, a macroscopic property of a system $\mathcal{S}$ is calculated by averaging that property over a set, or *ensemble*, of points moving in the phase space of $\mathcal{S}$. The number of ensemble points in any volume of phase space is represented by a *density function* $\rho(\boldsymbol{q}, \boldsymbol{p}, t)$. If the system is autonomous and in *statistical equilibrium*, it is required that, even though the ensemble points are moving (in accordance with Hamilton's equations), their density function should remain the same, that is, $\rho = \rho(\boldsymbol{q}, \boldsymbol{p})$. This places a restriction on possible choices for $\rho(\boldsymbol{q}, \boldsymbol{p})$. Let $\mathcal{R}_0$ be any region of the phase space and suppose that, after time t, the points of $\mathcal{R}_0$ occupy the region $\mathcal{R}_t$. Explain why statistical equilibrium requires that

$$\int_{\mathcal{R}_0} \rho(\boldsymbol{q}, \boldsymbol{p})\, dv = \int_{\mathcal{R}_t} \rho(\boldsymbol{q}, \boldsymbol{p})\, dv$$

and show that the *uniform* density function $\rho(\boldsymbol{q}, \boldsymbol{p}) = \rho_0$ satisfies this condition. [It can be proved that the above condition is also satisfied by any density function that is constant along the streamlines of the phase flow.]

Solution

The equation

$$\int_{\mathcal{R}_0} \rho(\boldsymbol{q}, \boldsymbol{p})\, dv = \int_{\mathcal{R}_t} \rho(\boldsymbol{q}, \boldsymbol{p})\, dv$$

merely expresses the condition that the *number* of ensemble points lying in the moving region $\mathcal{R}_t$ remains constant.

If $\rho(\boldsymbol{q}, \boldsymbol{p}) = \rho_0$, then

$$\int_{\mathcal{R}_t} \rho(\boldsymbol{q}, \boldsymbol{p})\, dv = \int_{\mathcal{R}_t} \rho_0\, dv = \rho_0 v(t),$$

where $v(t)$ is the volume of the region $\mathcal{R}_t$. Since $v(t)$ is known to be constant by Liouville's theorem, it follows that the uniform density function $\rho(\boldsymbol{q}, \boldsymbol{p}) = \rho_0$ satisfies the condition for statistical equilibrium. ■

Problem 14.13

Decide if the energy surfaces in phase space are bounded in the following cases:

(i) The two-body gravitation problem with $E < 0$.

(ii) The two-body gravitation problem viewed from the zero momentum frame and with $E < 0$.

(iii) The three-body gravitation problem viewed from the zero momentum frame and with $E < 0$. Does the solar system have the recurrence property?

Solution

(i) Let us take generalised coordinates $\{\boldsymbol{R}, \boldsymbol{r}\}$, where $\boldsymbol{R}$ is the position vector of G and $\boldsymbol{r}$ is the position vector of *one* of the particles relative to G. Then the conjugate momenta $\{\boldsymbol{P}, \boldsymbol{p}\}$ are bounded in *any* motion, and the coordinate $\boldsymbol{r}$ is bounded in a motion with negative energy. However, the coordinate $\boldsymbol{R}$ is not bounded. The energy surfaces in phase space are therefore **unbounded**.

(ii) The difference with (i) is that, in the zero momentum frame, G is at rest and we are left with the coordinate $\boldsymbol{r}$. Then the conjugate momuntum $\boldsymbol{p}$ is bounded in *any* motion and the coordinate $\boldsymbol{r}$ is bounded in a motion with negative energy. Hence, surfaces in phase space with constant negative energy are **bounded**. The recurrence theorem therefore applies, but this does not yield an interesting result since motions with negative energy were already known to be periodic.

(iii) In the three body problem, take generalised coordinates $\{\boldsymbol{R}, \boldsymbol{r}_1, \boldsymbol{r}_2\}$, where $\boldsymbol{R}$ is the position vector of G, and $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ are the position vectors of *two* of the particles relative to G. In the zero momentum frame, G is at rest and we are left with the coordinates $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$. Then the conjugate momenta $\{\boldsymbol{p}_1, \boldsymbol{p}_2\}$ are bounded in *any* motion and we need to decide whether $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are bounded in a motion in which the total energy is negative. In general, the answer is no. In the three body problem, it is known to be possible for one body to *escape*, even though the total energy is negative. In such a case, one of the position vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ must be unbounded. Hence, surfaces in phase space with constant negative energy are generally **unbounded**. The recurrence theorem does not apply and so we have no right to expect that the three body problem has the recurrence property. Similar remarks apply (even more so!) to the solar system. ■

Problem 14.14 *Poisson brackets*

Suppose that $u(\boldsymbol{q}, \boldsymbol{p})$ and $v(\boldsymbol{q}, \boldsymbol{p})$ are any two functions of position in the phase space $(\boldsymbol{q}, \boldsymbol{p})$ of a mechanical system $\mathcal{S}$. Then the **Poisson bracket** $[u, v]$ of u and v is defined by

$$[u, v] = \text{grad}_{\boldsymbol{q}}\, u \cdot \text{grad}_{\boldsymbol{p}}\, v - \text{grad}_{\boldsymbol{p}}\, u \cdot \text{grad}_{\boldsymbol{q}}\, v = \sum_{j=1}^{n} \left(\frac{\partial u}{\partial q_j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q_j} \right).$$

The *algebraic* behaviour of the Poisson bracket of two functions resembles that of the cross product $\boldsymbol{U} \times \boldsymbol{V}$ of two vectors or the commutator $\boldsymbol{U}\boldsymbol{V} - \boldsymbol{V}\boldsymbol{U}$ of two matrices. The Poisson bracket of two functions is closely related to the commutator of the corresponding operators in quantum mechanics.*

Prove the following properties of Poisson brackets.

Algebraic properties

$$[u, u] = 0, \qquad [v, u] = -[u, v], \qquad [\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 [u_1, v] + \lambda_2 [u_2, v]$$

$$[[u, v], w] + [[w, u], v] + [[v, w], u] = 0.$$

This last formula is called *Jacobi's identity*. It is quite important, but there seems to be no way of proving it apart from crashing it out, which is very tedious. Unless you can invent a smart method, leave this one alone.

Fundamental Poisson brackets

$$[q_j, q_k] = 0, \qquad [p_j, p_k] = 0, \qquad [q_j, p_k] = \delta_{jk},$$

where δ_{jk} is the Kroneker delta.

Hamilton's equations

Show that Hamilton's equations for $\mathcal{S}$ can be written in the form

$$\dot{q}_j = [q_j, H], \qquad \dot{p}_j = [p_j, H], \qquad (1 \le j \le n).$$

Constants of the motion

* The commutator $[\mathbf{U}, \mathbf{V}]$ of two quantum mechanical operators $\mathbf{U}$, $\mathbf{V}$ corresponds to $i\hbar[u, v]$, where $\hbar$ is the modified Planck constant, and $[u, v]$ is the Poisson bracket of the corresponding classical variables u, v.

(i) Show that the *total* time derivative of $u(\boldsymbol{q}, \boldsymbol{p})$ is given by

$$\frac{du}{dt} = [u, H]$$

and deduce that u is a constant of the motion of $\mathcal{S}$ if, and only if, $[u, H] = 0$.

(ii) If u and v are constants of the motion of $\mathcal{S}$, show that the Poisson bracket $[u, v]$ is another constant of the motion. [Use Jacobi's identity.] Does this mean that you can keep on finding more and more constants of the motion ?

Solution

Algebraic properties

(i)

$$[u, u] = \text{grad}_{\boldsymbol{q}}\, u \cdot \text{grad}_{\boldsymbol{p}}\, u - \text{grad}_{\boldsymbol{p}}\, u \cdot \text{grad}_{\boldsymbol{q}}\, u = 0.$$

(ii)

$$\begin{aligned}[v, u] &= \text{grad}_{\boldsymbol{q}}\, v \cdot \text{grad}_{\boldsymbol{p}}\, u - \text{grad}_{\boldsymbol{p}}\, v \cdot \text{grad}_{\boldsymbol{q}}\, u \\ &= -\Big(\text{grad}_{\boldsymbol{q}}\, u \cdot \text{grad}_{\boldsymbol{p}}\, v - \text{grad}_{\boldsymbol{p}}\, u \cdot \text{grad}_{\boldsymbol{q}}\, v\Big) \\ &= -[u, v].\end{aligned}$$

(iii)

$$\begin{aligned}[\lambda_1 u_1 + \lambda_2 u_2, v] &= \text{grad}_{\boldsymbol{q}}\, (\lambda_1 u_1 + \lambda_2 u_2) \cdot \text{grad}_{\boldsymbol{p}}\, v - \text{grad}_{\boldsymbol{p}}\, (\lambda_1 u_1 + \lambda_2 u_2) \cdot \text{grad}_{\boldsymbol{q}}\, v \\ &= \lambda_1 \Big(\text{grad}_{\boldsymbol{q}}\, u_1 \cdot \text{grad}_{\boldsymbol{p}}\, v - \text{grad}_{\boldsymbol{p}}\, u_1 \cdot \text{grad}_{\boldsymbol{q}}\, v\Big) \\ &\qquad + \lambda_2 \Big(\text{grad}_{\boldsymbol{q}}\, u_2 \cdot \text{grad}_{\boldsymbol{p}}\, v - \text{grad}_{\boldsymbol{p}}\, u_2 \cdot \text{grad}_{\boldsymbol{q}}\, v\Big) \\ &= \lambda_1 [u_1, v] + \lambda_2 [u_2, v].\end{aligned}$$

(iv) You must be joking!

Fundamental Poisson brackets

(i)

$$\begin{aligned}[q_j, q_k] &= \text{grad}_{\boldsymbol{q}}\, q_j \cdot \text{grad}_{\boldsymbol{p}}\, q_k - \text{grad}_{\boldsymbol{p}}\, q_j \cdot \text{grad}_{\boldsymbol{q}}\, q_k \\ &= \boldsymbol{e}_j \cdot \boldsymbol{0} - \boldsymbol{0} \cdot \boldsymbol{e}_k \\ &= 0.\end{aligned}$$

Here $\boldsymbol{e}_j$ is the n-dimensional basis vector with zeros everywhere except in the j-th position where there is a one. For example, $\boldsymbol{e}_1 = (1, 0, 0, \ldots, 0)$ and $\boldsymbol{e}_2 = (0, 1, 0, 0, \ldots, 0)$.

(ii)

$$\begin{aligned}[p_j, p_k] &= \text{grad}_{\boldsymbol{q}}\, p_j \cdot \text{grad}_{\boldsymbol{p}}\, p_k - \text{grad}_{\boldsymbol{p}}\, p_j \cdot \text{grad}_{\boldsymbol{q}}\, p_k \\ &= \boldsymbol{0} \cdot \boldsymbol{e}_k - \boldsymbol{e}_j \cdot \boldsymbol{0} \\ &= 0.\end{aligned}$$

(iii)

$$\begin{aligned}[q_j, p_k] &= \text{grad}_{\boldsymbol{q}}\, q_j \cdot \text{grad}_{\boldsymbol{p}}\, p_k - \text{grad}_{\boldsymbol{p}}\, q_j \cdot \text{grad}_{\boldsymbol{q}}\, p_k \\ &= \boldsymbol{e}_j \cdot \boldsymbol{e}_k - \boldsymbol{0} \cdot \boldsymbol{0} \\ &= \delta_{ij}.\end{aligned}$$

Hamilton's equations

(i)

$$\begin{aligned}[q_j, H] &= \text{grad}_{\boldsymbol{q}}\, q_j \cdot \text{grad}_{\boldsymbol{p}}\, H - \text{grad}_{\boldsymbol{p}}\, q_j \cdot \text{grad}_{\boldsymbol{q}}\, H \\ &= \boldsymbol{e}_j \cdot \text{grad}_{\boldsymbol{p}}\, H - \boldsymbol{0} \cdot \text{grad}_{\boldsymbol{q}}\, H \\ &= \frac{\partial H}{\partial p_j} \\ &= \dot{q}_j.\end{aligned}$$

(ii)

$$\begin{aligned}[p_j, H] &= \text{grad}_{\boldsymbol{q}}\, p_j \cdot \text{grad}_{\boldsymbol{p}}\, H - \text{grad}_{\boldsymbol{p}}\, p_j \cdot \text{grad}_{\boldsymbol{q}}\, H \\ &= \boldsymbol{0} \cdot \text{grad}_{\boldsymbol{p}}\, H - \boldsymbol{e}_j \cdot \text{grad}_{\boldsymbol{q}}\, H \\ &= -\frac{\partial H}{\partial q_j} \\ &= \dot{p}_j.\end{aligned}$$

Constants of the motion

(i)

$$\begin{aligned}[u, H] &= \text{grad}_{\boldsymbol{q}}\, u \cdot \text{grad}_{\boldsymbol{p}}\, H - \text{grad}_{\boldsymbol{p}}\, u \cdot \text{grad}_{\boldsymbol{q}}\, H \\ &= \text{grad}_{\boldsymbol{q}}\, u \cdot \dot{\boldsymbol{q}} - \text{grad}_{\boldsymbol{p}}\, u \cdot (-\dot{\boldsymbol{p}}) \\ &= \text{grad}_{\boldsymbol{q}}\, u \cdot \dot{\boldsymbol{q}} + \text{grad}_{\boldsymbol{p}}\, u \cdot \dot{\boldsymbol{p}} \\ &= \frac{du}{dt}.\end{aligned}$$

(ii) Fom Jacobi's identity,

$$[[u, v], H] + [[H, u], v] + [[v, H], u] = 0.$$

However, since u and v are known to be constants of the motion,

$$[u, H] = [v, H] = 0,$$

and so

$$[[u, v], H] = 0.$$

Hence $[u, v]$ must be another constant of the motion.

Obviously one cannot keep on finding more and more constants of the motion! The reason is that $[u, v]$ may be simply some combination of u and v and therefore not an *independent* constant. ■

Problem 14.15 *Integrable systems and chaos*

A mechanical system is said to be **integrable** if its equations of motion are soluble in the sense that they can be reduced to integrations. (You do not need to be able to evaluate the integrals in terms of standard functions.) A theorem due to Liouville states that *any Hamiltonian system with n degrees of freedom is integrable if it has n independent constants of the motion, and all these quantities commute in the sense that all their mutual Poisson brackets are zero*.* The qualitative behaviour of integrable Hamiltonian systems is well investigated (see Goldstein [**?**]). In particular, *no integrable Hamiltonian system can exhibit chaos*.

Use Liouville's theorem to show that any autonomous system with n degrees of freedom and $n-1$ cyclic coordinates must be integrable.

Solution

For any autonomous system, H is a constant of the motion. Also this system has $n-1$ cyclic coordinates $q_1, q_2, \ldots, q_{n-1}$ and therefore $n-1$ conserved momenta $p_1, p_2, \ldots, p_{n-1}$. Hence there are a total of n constants of the motion and Liuoville's theorem will be satisfied if all these variables commute. We already know that $[p_j, p_k] = 0$, and

$$[p_j, H] = \dot{p}_j = 0,$$

since each p_j is conserved. Hence the conditions of Liuoville's theorem are satisfied and so the system must be **integrable**. Most integrable systems are like this. ■

* This result is really very surprising. A *general* system of first order ODEs in $2n$ variables needs $2n$ integrals in order to be integrable in the Liouville sense. Hamiltonian systems need only half that number. The theorem does not rule out the possibility that that there could be other classes of integrable systems. However, according to Arnold [**?**], every system that has ever been integrated is of the Liouville kind!

Chapter Fifteen

The general theory of small oscillations

Problem 15.1

A particle P of mass $3m$ is connected to a particle Q of mass $8m$ by a light elastic spring of natural length a and strength α. Two similar springs are used to connect P and Q to the fixed points A and B respectively, which are a distance $3a$ apart on a smooth horizontal table. The particles can perform longitudinal oscillations along the straight line AB. Find the normal frequencies and the forms of the normal modes.

The system is in equlilibrium when the particle P receives a blow that gives it a speed u in the direction $\overrightarrow{AB}$. Find the displacement of each particle at time t in the subsequent motion.

Solution

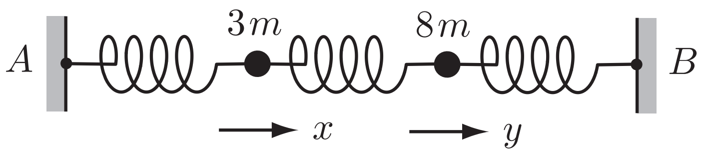

FIGURE 15.1 The system in problem 15.1.

Let the displacements of the two particles from their equilibrium positions be x, y, as shown in Figure 15.1. Then the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2}(3m)\dot{x}^2 + \tfrac{1}{2}(8m)\dot{y}^2,$$

and the T-matrix is

$$\mathbf{T} = \tfrac{1}{2}m\begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}.$$

Likewise, since the springs are linear, the exact and approximate **potential energies** are the same, namely

$$\begin{aligned} V = V^{\text{app}} &= \tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\alpha(y-x)^2 + \tfrac{1}{2}\alpha y^2, \\ &= \tfrac{1}{2}\alpha\left(2x^2 - 2xy + 2y^2\right), \end{aligned}$$

and the V-matrix is

$$\mathbf{V} = \tfrac{1}{2}\alpha\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathbf{V} - \omega^2\mathbf{T}) = 0$ can be written

$$\begin{vmatrix} 2-3\mu & -1 \\ -1 & 2-8\mu \end{vmatrix} = 0,$$

where $\mu = m\omega^2/\alpha$. When expanded, this gives the quadratic equation

$$24\mu^2 - 22\mu + 3 = 0$$

whose roots are $\mu = \frac{1}{6}$ and $\mu = \frac{3}{4}$. Since $\mu = m\omega^2/\alpha$, the **normal frequencies** are therefore given by

$$\omega_1^2 = \frac{\alpha}{6m}, \qquad \omega_2^2 = \frac{3\alpha}{4m}.$$

Since the normal frequencies are non-degenerate, the corresponding amplitude vectors are unique to within multiplied constants. In the **slow mode**, $\mu = \frac{1}{6}$ and the equations $(\mathbf{V} - \omega^2\mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

on clearing fractions. It is evident that $X = 2$, $Y = 3$ is a solution so that the amplitude vector for the ω_1-mode is $\boldsymbol{a}_1 = (2,\ 3)$. The other mode is treated in a similar way and its amplitude vector is found to be $\boldsymbol{a}_2 = (4, -1)$. In column vector form, the **amplitude vectors** of the normal modes are therefore

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \qquad \mathbf{a}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

These are the **forms** of the normal modes.

It follows that the **general solution** of the small motion equations is

$$\begin{aligned} x &= 2C_1 \cos(\omega_1 t - \gamma_1) + 4C_2 \cos(\omega_2 t - \gamma_2) \\ y &= 3C_1 \cos(\omega_1 t - \gamma_1) - C_2 \cos(\omega_2 t - \gamma_2), \end{aligned}$$

where C_1, C_2, γ_1, γ_2 are arbitrary constants. This can be written in the alternative form

$$\begin{aligned} x &= 2\big(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t\big) + 4\big(A_2 \cos\omega_2 t + B_2 \sin\omega_2 t\big), \\ y &= 3\big(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t\big) - \big(A_2 \cos\omega_2 t + B_2 \sin\omega_2 t\big), \end{aligned}$$

where A_1, B_1, A_2, B_2 are arbitrary constants.

It remains to determine the constants A_1, B_1, A_2, B_2 from the **initial conditions** $x = 0$, $y = 0$, $\dot{x} = u$, $\dot{y} = 0$ when $t = 0$. These conditions require that

$$\begin{aligned} 2A_1 + 4A_2 &= 0, \\ 3A_1 - A_2 &= 0, \\ 2\omega_1 B_1 + 4\omega_2 B_2 &= u, \\ 3\omega_1 B_1 - \omega_2 B_2 &= 0, \end{aligned}$$

from which it follows that $A_1 = A_2 = 0$ and

$$B_1 = \frac{u}{14\omega_1}, \qquad B_2 = \frac{3u}{14\omega_2}.$$

The **motion** resulting from the given initial conditions is therefore

$$x = \frac{2u}{14\omega_1\omega_2}\big(\omega_2 \sin\omega_1 t + 6\omega_1 \sin\omega_2 t\big),$$

$$y = \frac{3u}{14\omega_1\omega_2}\big(\omega_2 \sin\omega_1 t - \omega_1 \sin\omega_2 t\big). \blacksquare$$

Problem 15.2

A particle A of mass $3m$ is suspended from a fixed point O by a spring of strength α and a second particle B of mass $2m$ is suspended from A by a second identical spring. The system performs small oscillations in the vertical straight line through O. Find the normal frequencies, the forms of the normal modes, and a set of normal coordinates.

Solution

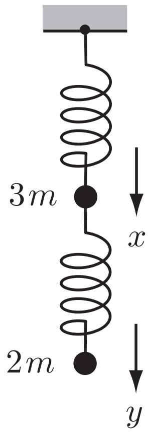

FIGURE 15.2 The system in problem 15.2. The displacements of the particles are measured from the *equilibrium configuration* of the system.

Let the displacements of the two particles from their equilibrium positions be x, y, as shown in Figure 15.2. In the equilibrium configuration, the tension in the upper spring is $5mg$ while the tension in the lower spring is $2mg$. Hence, in the displaced configuration, the total potential energy of the springs, relative to the equilibrium configuration, is given by

$$\begin{aligned} V^S &= \int_0^x (5mg + \alpha\xi)\, d\xi + \int_0^{y-x} (2mg + \alpha\xi)\, d\xi \\ &= 3mgx + 2mgy + \tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\alpha(y-x)^2 \\ &= 3mgx + 2mgy + \tfrac{1}{2}\alpha\left(2x^2 - 2xy + y^2\right). \end{aligned}$$

The total gravitational potential energy, relative to the equilibrium configuration, is

$$\begin{aligned} V^G &= -(3m)gx - (2m)gy \\ &= -3mgx - 2mgy. \end{aligned}$$

Hence, the total **potential energy** of the system is

$$\begin{aligned} V &= V^S + V^G \\ &= \tfrac{1}{2}\alpha\left(2x^2 - 2xy + y^2\right). \end{aligned}$$

The exact and approximate **potential energies** are the same so that

$$V^{\text{app}} = \tfrac{1}{2}\alpha\left(2x^2 - 2xy + y^2\right)$$

and the V-matrix is

$$\mathbf{V} = \tfrac{1}{2}\alpha\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Likewise, the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2}(3m)\dot{x}^2 + \tfrac{1}{2}(2m)\dot{y}^2,$$

and the T-matrix is

$$\mathbf{T} = \tfrac{1}{2}m\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathbf{V} - \omega^2\mathbf{T}) = 0$ can be written

$$\begin{vmatrix} 2 - 3\mu & -1 \\ -1 & 1 - 2\mu \end{vmatrix} = 0,$$

where $\mu = m\omega^2/\alpha$. When expanded, this gives the quadratic equation

$$6\mu^2 - 7\mu + 1 = 0$$

whose roots are $\mu = \frac{1}{6}$ and $\mu = 1$. Since $\mu = m\omega^2/\alpha$, the **normal frequencies** are therefore given by

$$\omega_1^2 = \frac{\alpha}{6m}, \qquad \omega_2^2 = \frac{\alpha}{m}.$$

Since the normal frequencies are non-degenerate, the corresponding amplitude vectors are unique to within multiplied constants. In the **slow mode**, $\mu = \frac{1}{6}$ and the equations $(\mathbf{V} - \omega^2\mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

on clearing fractions. It is evident that $X = 2$, $Y = 3$ is a solution so that the amplitude vector for the ω_1-mode is $\boldsymbol{a}_1 = (2,\ 3)$. The other mode is treated in a similar way and its amplitude vector is found to be $\boldsymbol{a}_2 = (1, -1)$. In column vector form, the **amplitude vectors** of the normal modes are therefore

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These are the **forms** of the normal modes.

The matrix $\mathbf{P}$, whose columns are the (un-normalised) amplitude vectors, is therefore

$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$$

and a set of normal coordinates is given by

$$\begin{aligned} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \mathbf{P}' \cdot \mathbf{T} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \tfrac{1}{2} m \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \tfrac{1}{2} m \begin{pmatrix} 6 & 6 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

These are a set of **normal coordinates**, but we may remove inessential scaling factors and take the set

$$\begin{aligned} \eta_1 &= x + y, \\ \eta_2 &= 3x - 2y \end{aligned}$$

instead. ■

Problem 15.3 *Rod pendulum*

A uniform rod of length $2a$ is suspended from a fixed point O by a light inextensible string of length b attached to one of its ends. The system moves in a vertical plane through O. Take as coordinates the angles θ, ϕ between the string and the rod respectively and the downward vertical. Show that the equations governing small oscillations of the system about $\theta = \phi = 0$ are

$$b\ddot{\theta} + a\ddot{\phi} = -g\theta,$$
$$b\ddot{\theta} + \tfrac{4}{3}a\ddot{\phi} = -g\phi.$$

For the special case in which $b = 4a/5$, find the normal frequencies and the forms of the normal modes. Is the general motion periodic?

Solution

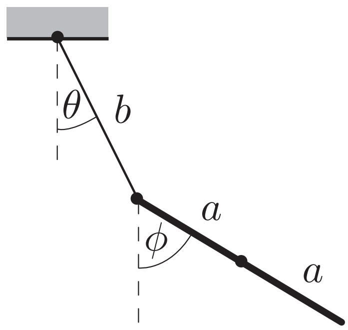

FIGURE 15.3 The rod pendulum in problem 15.3.

The **kinetic energy** of the rod can be expressed as the sum of its translational and rotational contributions in the form

$$T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I_G\omega^2,$$

where M is the mass of the rod, V is the speed of its centre of mass G, I_G is its moment of inertia about G, and ω is its angular speed. The value of T^{app} can be found by evaluating T when the system is *passing through its equilibrium configuration*. In terms of the coordinates θ, ϕ shown in Figure 15.3, this is

$$T^{\text{app}} = \tfrac{1}{2}M\left(b\dot{\theta} + a\dot{\phi}\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{3}Ma^2\right)\dot{\phi}^2,$$
$$= \tfrac{1}{2}M\left(b^2\dot{\theta}^2 + 2ba\dot{\theta}\dot{\phi} + \tfrac{4}{3}a^2\dot{\phi}^2\right).$$

The T-matrix is therefore

$$\mathsf{T} = \tfrac{1}{2}M \begin{pmatrix} b^2 & ba \\ ba & \tfrac{4}{3}a^2 \end{pmatrix}.$$

The gravitational **potential energy** of the rod relative to the equilibrium configuration is

$$V = Mg\big(b(1-\cos\theta) + a(1-\cos\phi)\big),$$

from which it follows that

$$V^{\text{app}} = \tfrac{1}{2}Mg\left(b\theta^2 + a\phi^2\right).$$

The V-matrix is therefore

$$\mathsf{V} = \tfrac{1}{2}Mg \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

The **small oscillation equations** are then

$$\mathsf{T}\cdot\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} + \mathsf{V}\cdot\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is,

$$\begin{aligned} b\ddot{\theta} + a\ddot{\phi} + g\theta &= 0, \\ b\ddot{\theta} + \tfrac{4}{3}a\ddot{\phi} + g\phi &= 0, \end{aligned}$$

as required.

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2\mathsf{T}) = 0$ is

$$\begin{vmatrix} gb - b^2\omega^2 & -ba\omega^2 \\ -ba\omega^2 & ga - \tfrac{4}{3}a^2\omega^2 \end{vmatrix} = 0.$$

For the special case in which $b = \tfrac{4}{5}a$, this reduces to

$$\begin{vmatrix} 5-4\mu & -5\mu \\ -12\mu & 15-20\mu \end{vmatrix} = 0,$$

where $\mu = a\omega^2/g$. When expanded, this gives the quadratic equation

$$4\mu^2 - 32\mu + 15 = 0$$

whose roots are $\mu = \frac{1}{2}$ and $\mu = \frac{15}{2}$. Since $\mu = a\omega^2/g$, the **normal frequencies** are therefore given by

$$\omega_1^2 = \frac{g}{2a}, \qquad \omega_2^2 = \frac{15g}{2a}.$$

Since the normal frequencies are non-degenerate, the corresponding amplitude vectors are unique to within multiplied constants. In the **slow mode**, $\mu = \frac{1}{2}$ and the equations $(\mathbf{V} - \omega^2\mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} 6 & -5 \\ -6 & 5 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

on clearing fractions. It is evident that $A = 5$, $B = 6$ is a solution so that the amplitude vector for the ω_1-mode is $\boldsymbol{a}_1 = (5,\ 6)$. The other mode is treated in a similar way and its amplitude vector is found to be $\boldsymbol{a}_2 = (3, -2)$. In column vector form, the **amplitude vectors** of the normal modes are therefore

$$\mathbf{a}_1 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}. \qquad \mathbf{a}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

These are the **forms** of the normal modes.

For this system, the ratio of the normal frequencies is $\omega_2/\omega_1 = \sqrt{15}$ which is an *irrational* number. It follows that τ_2/τ_1 is also irrational and that the general motion of the pendulum is **not periodic**. ■

Problem 15 . 4 *Triple pendulum*

A triple pendulum has three strings of equal length a and the three particles (starting from the top) have masses $6m$, $2m$, m respectively. The pendulum performs small oscillations in a vertical plane. Show that the normal frequencies satisfy the equation

$$12\mu^3 - 60\mu^2 + 81\mu - 27 = 0,$$

where $\mu = a\omega^2/g$. Find the normal frequencies, the forms of the normal modes, and a set of normal coordinates. [$\mu = 3$ is a root of the equation.]

Solution

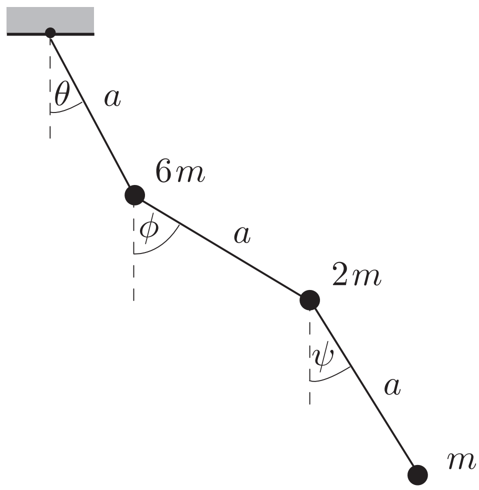

FIGURE 15.4 The system in problem 15.4.

The value of T^{app} can be found by evaluating T when the system is *passing through its equilibrium configuration*. In terms of the coordinates θ, ϕ, ψ shown in Figure 15.4, this is

$$\begin{aligned} T^{\text{app}} &= \tfrac{1}{2}(6m)\left(a\dot{\theta}\right)^2 + \tfrac{1}{2}(2m)\left(a\dot{\theta} + a\dot{\phi}\right)^2 + \tfrac{1}{2}m\left(a\dot{\theta} + a\dot{\phi} + a\dot{\psi}\right)^2 \\ &= \tfrac{1}{2}ma^2\left(9\dot{\theta}^2 + 3\dot{\phi}^2 + \dot{\psi}^2 + 6\dot{\theta}\dot{\phi} + 2\dot{\theta}\dot{\psi} + 2\dot{\phi}\dot{\psi}\right). \end{aligned}$$

The T-matrix is therefore

$$\mathsf{T} = \tfrac{1}{2}ma^2\begin{pmatrix} 9 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The gravitational **potential energy** of the system relative to the equilibrium configuration is

$$V = 6mga(1-\cos\theta) + 2mg\big(a(1-\cos\theta) + a(1-\cos\phi)\big) + \\ mga\big((1-\cos\theta) + a(1-\cos\phi) + a(1-\cos\psi)\big),$$

from which it follows that

$$V^{\text{app}} = \tfrac{1}{2}mga\left(9\theta^2 + 3\phi^2 + \psi^2\right).$$

The V-matrix is therefore

$$\mathbf{V} = \tfrac{1}{2}mga\begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathbf{V} - \omega^2\mathbf{T}) = 0$ can be written

$$\begin{vmatrix} 9-9\mu & -3\mu & -\mu \\ -3\mu & 3-3\mu & -\mu \\ -\mu & -\mu & 1-\mu \end{vmatrix} = 0.$$

where $\mu = a\omega^2/g$. When expanded, this gives the cubic equation

$$4\mu^3 - 20\mu^2 + 27\mu - 9 = 0,$$

as required. We are given that $\mu = 3$ is a root, and, on extracting the factor $\mu - 3$, we are left with the quadratic equation

$$4\mu^2 - 8\mu + 3 = 0,$$

whose roots are $\mu = \frac{1}{2}$ and $\mu = \frac{3}{2}$. Since $\mu = a\omega^2/g$, the **normal frequencies** are therefore given by

$$\omega_1^2 = \frac{g}{2a}, \qquad \omega_2^2 = \frac{3g}{2a}, \quad \omega_3^2 = \frac{3g}{a},$$

Since the normal frequencies are non-degenerate, the corresponding amplitude vectors are unique to within multiplied constants. In the **slow mode**, $\mu = \frac{1}{2}$ and the equations $(\mathbf{V} - \omega^2\mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} 9 & -3 & -1 \\ -3 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

on clearing fractions. On discarding the first equation and solving the remaining two, we find that $B = 2A$ and $C = 3A$. The amplitude vector for the ω_1-mode is therefore $\boldsymbol{a}_1 = (1,\ 2,\ 3)$. The other modes are treated in a similar way and the amplitude vectors are found to be $\boldsymbol{a}_2 = (1,\ 0, -3)$ and $\boldsymbol{a}_3 = (1, -3,\ 3)$. In column vector form, the **amplitude vectors** of the normal modes are therefore

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}.$$

These are the **forms** of the normal modes.

The matrix $\mathbf{P}$, whose columns are the (un-normalised) amplitude vectors, is therefore

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -3 \\ 3 & -3 & 3 \end{pmatrix}$$

and a set of normal coordinates is given by

$$\begin{aligned} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} &= \mathbf{P}' \cdot \mathbf{T} \cdot \begin{pmatrix} \theta \\ \phi \\ \psi \end{pmatrix} \\ &= \tfrac{1}{2} ma^2 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -3 \\ 1 & -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 9 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \theta \\ \phi \\ \psi \end{pmatrix} \\ &= \tfrac{1}{2} ma^2 \begin{pmatrix} 18 & 12 & 6 \\ 6 & 0 & -2 \\ 3 & -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} \theta \\ \phi \\ \psi \end{pmatrix}. \end{aligned}$$

These are a set of **normal coordinates**, but we may remove inessential scaling factors and take the set

$$\begin{aligned} \eta_1 &= 3\theta + 2\phi + \psi, \\ \eta_2 &= 3\theta \qquad\ \ - \psi, \\ \eta_3 &= 3\theta - 3\phi + \psi \end{aligned}$$

instead. ■

Problem 15.5

A light *elastic* string is stretched to tension T_0 between two fixed points A and B a distance $3a$ apart, and two particles of mass m are attached to the string at equally spaced intervals. The strength of *each* of the three sections of the string is α. The system performs small oscillations in a plane through AB. Without making any prior assumptions, prove that the particles oscillate longitudinally in two of the normal modes and transversely in the other two. Find the four normal frequencies.

Solution

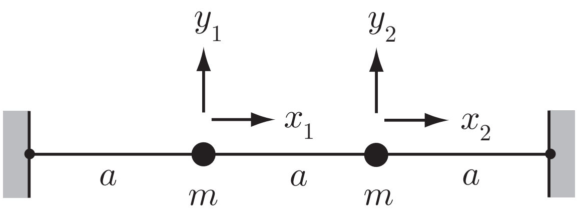

FIGURE 15.5 The system in problem 15.5.

Let the displacements of the two particles from their equilibrium positions be x_1, y_1 and x_2, y_2, as shown in Figure 15.5. Then the exact and approximate **kinetic energies** are the same, namely

$$
\begin{aligned}
T = T^{\text{app}} &= \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right)\\
&= \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right),
\end{aligned}
$$

so that the T-matrix is

$$
\mathbf{T} = \tfrac{1}{2}m\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Let Δ_2 be the extension of the middle segment of the string, relative to its state in the equilibrium configuration of the system. Then

$$\begin{aligned}
\Delta_2 &= \Big((a + x_2 - x_1)^2 + (y_2 - y_1)^2\Big)^{1/2} - a \\
&= a\left(1 + \frac{2(x_2 - x_1)}{a} + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{a^2}\right)^{1/2} - a \\
&= a\left(1 + \frac{x_2 - x_1}{a} + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2a^2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{2(x_2 - x_1)}{a}\right)^2 + \cdots\right) - a \\
&= (x_2 - x_1) + \frac{(y_2 - y_1)^2}{2a},
\end{aligned}$$

correct to quadratic terms. The exact potential energy of the segment relative to the equilibrium configuration is

$$\begin{aligned}
V_2 &= \int_0^{\Delta_2} \big(T_0 + \alpha\,\xi\big)\, d\xi, \\
&= T_0 \Delta_2 + \tfrac{1}{2}\alpha \Delta_2^2,
\end{aligned}$$

and so the corresponding approximate potential energy is

$$V_2^{\text{app}} = T_0(x_2 - x_1) + \frac{T_0}{2a}(y_2 - y_1)^2 + \tfrac{1}{2}\alpha(x_2 - x_1)^2.$$

The approximate potential energies of the other segments of the string are calculated similarly and are given by

$$\begin{aligned}
V_1^{\text{app}} &= T_0 x_1 + \frac{T_0}{2a} y_1^2 + \tfrac{1}{2}\alpha x_1^2, \\
V_3^{\text{app}} &= -T_0 x_2 + \frac{T_0}{2a} y_2^2 + \tfrac{1}{2}\alpha x_2^2.
\end{aligned}$$

The approximate total **potential energy** of the system is therefore

$$V^{\text{app}} = \alpha\left(x_1^2 - x_1 x_2 + x_2^2\right) + \frac{T_0}{a}\left(y_1^2 - y_1 y_2 + y_2^2\right).$$

On taking the coordinates in the order x_1, x_2, y_1, y_2, the V-matrix becomes

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 2\alpha & -\alpha & 0 & 0 \\ -\alpha & 2\alpha & 0 & 0 \\ 0 & 0 & 2T_0/a & -T_0/a \\ 0 & 0 & -T_0/a & 2T_0/a \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2\mathsf{T}) = 0$ is therefore

$$\begin{vmatrix} 2\alpha - m\omega^2 & -\alpha & 0 & 0 \\ -\alpha & 2\alpha - m\omega^2 & 0 & 0 \\ 0 & 0 & 2T_0/a - \alpha & -T_0/a \\ 0 & 0 & -T_0/a & 2T_0/a - m\omega^2 \end{vmatrix} = 0,$$

that is

$$\begin{vmatrix} 2\alpha - m\omega^2 & -m\omega^2 \\ -m\omega^2 & 2\alpha - m\omega^2 \end{vmatrix} \times \begin{vmatrix} 2T_0/a - m\omega^2 & -m\omega^2 \\ -m\omega^2 & 2T_0/a - m\omega^2 \end{vmatrix} = 0.$$

Thus the eigenvalue equation is satisfied if

(a) either

$$\begin{vmatrix} 2-\mu & -1 \\ -1 & 2-\mu \end{vmatrix} = 0,$$

where $\mu = m\omega^2/\alpha$,

(b) or

$$\begin{vmatrix} 2-\nu & -\nu \\ -\nu & 2-\nu \end{vmatrix} = 0.$$

where $\nu = ma\omega^2/T_0$.

In **Case (a)**, the determinant expands to give the quadratic equation

$$\mu^2 - 4\mu + 3 = 0,$$

whose roots are $\mu = 1$ and $\mu = 3$. The corrresponding normal frequencies are

$$\omega_1^L = \left(\frac{\alpha}{m}\right)^{1/2}, \qquad \omega_2^L = \left(\frac{3\alpha}{m}\right)^{1/2}.$$

In order to identify these frequencies with **longitudinal modes**, we determine the corresponding **amplitude vectors.** In the slow mode, $\omega^2 = \alpha/m$ and the equations

$(\mathbf{V} - \omega^2\mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} \alpha & -\alpha & 0 & 0 \\ -\alpha & \alpha & 0 & 0 \\ 0 & 0 & 2T_0/a - \alpha & -T_0/a \\ 0 & 0 & -T_0/a & 2T_0/a - \alpha \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

These equations have the solution $X_1 = X_2$, $Y_1 = Y_2 = 0$ so that the amplitude vector for the ω_1^L-mode is $\boldsymbol{a}_1^L = (1,\ 1,\ 0,\ 0)$. The fast ω_2^L-mode is treated in a similar way and its amplitude vector is found to be $\boldsymbol{a}_2^L = (1, -1,\ 0,\ 0)$. Thus $y_1 = y_2 = 0$ in these two modes, that is, the modes are **purely longitudinal**. The existence of such modes is entirely to be expected since it is clear by the symmetry of the system that purely longitudinal motions do exist.

In **Case (b)**, the determinant expands to give the quadratic equation

$$\nu^2 - 4\nu + 3 = 0,$$

whose roots are $\nu = 1$ and $\nu = 3$. The corrresponding normal frequencies are

$$\omega_1^T = \left(\frac{T_0}{ma}\right)^{1/2}, \qquad \omega_2^T = \left(\frac{3T_0}{ma}\right)^{1/2}.$$

In order to identify these frequencies with **transverse modes**, we determine the corresponding **amplitude vectors.** In the slow mode, $\omega^2 = T_0/ma$ and the equations $(\mathbf{V} - \omega^2\mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} 2\alpha - T_0/a & -\alpha & 0 & 0 \\ -\alpha & 2\alpha - T_0/a & 0 & 0 \\ 0 & 0 & T_0/a & -T_0/a \\ 0 & 0 & -T_0/a & T_0/a \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

These equations have the solution $Y_1 = Y_2$, $X_1 = X_2 = 0$, so that the amplitude vector for the ω_1^T-mode is $\boldsymbol{a}_1^T = (0,\ 0,\ 1,\ 1)$. The fast ω_2^T-mode is treated in a similar way and its amplitude vector is found to be $\boldsymbol{a}_2^L = (0,\ 0,\ 1, -1)$. Thus $x_1 = x_2 = 0$ in these two modes, that is, the modes are **purely transverse**. This was not to be expected since there is no symmetry reason why purely transverse motions should exist. Indeed, in the large displacement theory, they do **not** exist. However, in the linearised *small displacement* theory they do exist and this is what we have found. ■

Problem 15.6

A rod of mass M and length L is suspended from two fixed points at the same horizontal level and a distance L apart by two equal strings of length b attached to its ends. From each end of the rod a particle of mass m is suspended by a string of length a. The system of the rod and two particles performs small oscillations in a vertical plane. Find **V** and **T** for this system. For the special case in which $b = 3a/2$ and $M = 6m/5$, find the normal frequencies. Show that the general small motion is periodic and find the period.

Solution

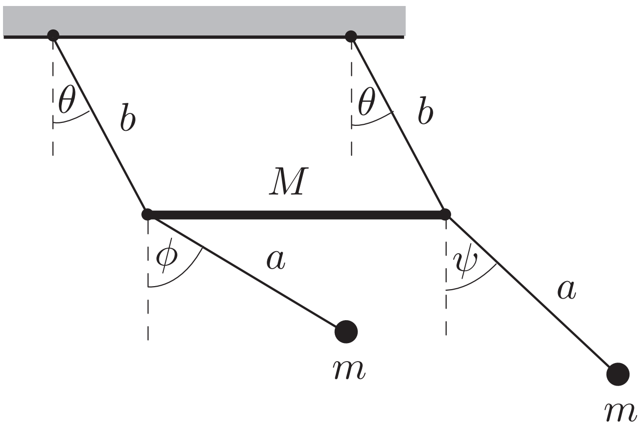

FIGURE 15.6 The system in problem 15.6.

The approximate **kinetic energy** of the system can be found by evaluating T when the system is *passing through its equilibrium configuration*. In terms of the coordinates θ, ϕ, ψ shown in Figure 15.6, this is

$$\begin{aligned} T^{\text{app}} &= \tfrac{1}{2}M\left(b\dot{\theta}\right)^2 + \tfrac{1}{2}m\left(b\dot{\theta} + a\dot{\phi}\right)^2 + \tfrac{1}{2}m\left(b\dot{\theta} + a\dot{\psi}\right)^2 \\ &= \tfrac{1}{2}(M + 2m)b^2\dot{\theta}^2 + \tfrac{1}{2}ma^2\dot{\phi}^2 + \tfrac{1}{2}ma^2\dot{\psi}^2 + +mba\dot{\theta}\dot{\phi} + mba\dot{\theta}\dot{\psi}. \end{aligned}$$

[Note that the *rotational* kinetic energy of the rod is zero.] The T-matrix is therefore

$$\mathbf{T} = \tfrac{1}{2}\begin{pmatrix} (M + 2m)b^2 & mba & mba \\ mba & ma^2 & 0 \\ mba & 0 & ma^2 \end{pmatrix}.$$

The gravitational **potential energy** of the system relative to the equilibrium configuration is

$$V = Mgb(1-\cos\theta)+mg\big(b(1-\cos\theta)+a(1-\cos\phi)\big)+mg\big(b(1-\cos\theta)+a(1-\cos\psi)\big)$$

from which it follows that

$$V^{\text{app}} = \tfrac{1}{2}(M+2m)gb\theta^2 + \tfrac{1}{2}mga\phi^2 + \tfrac{1}{2}mga\psi^2.$$

The V-matrix is therefore

$$\mathsf{V} = \tfrac{1}{2}g\begin{pmatrix}(M+2m)b & 0 & 0\\ 0 & ma & 0\\ 0 & 0 & ma\end{pmatrix}.$$

For the special case in which $b = 3a/2$ and $M = 6m/5$, T and V reduce to

$$\mathsf{T} = \frac{ma^2}{20}\begin{pmatrix}72 & 15 & 15\\ 15 & 10 & 0\\ 15 & 0 & 10\end{pmatrix}, \qquad \mathsf{V} = \frac{mga}{10}\begin{pmatrix}24 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathsf{V}-\omega^2\mathsf{T}) = 0$ is then

$$\begin{vmatrix}48-72\mu & -15\mu & -15\mu\\ -15\mu & 10-10\mu & 0\\ -15\mu & 0 & 10-10\mu\end{vmatrix} = 0.$$

where $\mu = a\omega^2/g$. When expanded, this gives the cubic equation

$$9\mu^3 - 49\mu^2 + 56\mu - 16 = 0$$

whose roots are $\mu = \frac{4}{9}$, $\mu = 1$ and $\mu_3 = 4$. [You were supposed to spot that $\mu = 1$ is a root.] Since $\mu = a\omega^2/g$, the **normal frequencies** are therefore given by

$$\omega_1 = \tfrac{2}{3}n, \qquad \omega_2 = n, \quad \omega_3 = 2n,$$

where $n^2 = g/a$.

The periods of the normal modes are $\tau_1 = 3\pi/n$, $\tau_2 = 2\pi/n$ and $\tau_3 = \pi/n$. The ratios of these periods are *rational numbers* and hence the **general motion is periodic**. The period of the general motion is the lowest common multiple of τ_1, τ_2, τ_3, which is $6\pi/n$ ■

Problem 15.7

A uniform rod is suspended in a horizontal position by *unequal* vertical strings of lengths b, c attached to its ends. Show that the frequency of the in-plane swinging mode is $((b+c)g/2bc)^{1/2}$, and that the frequencies of the other modes satisfy the equation

$$bc\mu^2 - 2a(b+c)\mu + 3a^2 = 0,$$

where $\mu = a\omega^2/g$. Find the normal frequencies for the particular case in which $b = 3a$ and $c = 8a$.

Solution

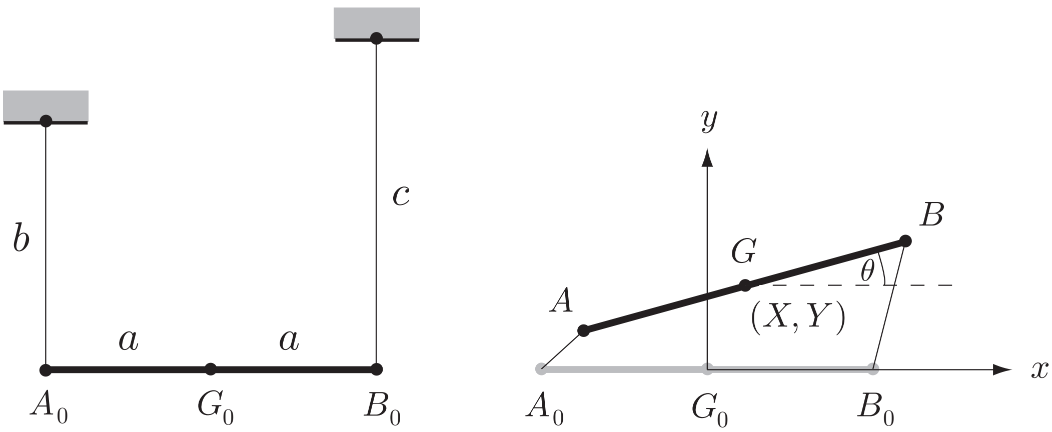

FIGURE 15.7 The system in problem 15.7: **Left**: In the equilibrium position (side view). **Right**: In general position (viewed from above).

This problem is quite similar to that in Example 15.6. Let (X, Y) be the horizontal displacement of the centre of mass G of the rod from its equilibrium position, and let θ be the rotation angle of the rod when viewed from above (see Figure 15.7). The geometry is complicated by the fact that the rod does not remain horizontal in the motion. However, its vertical displacement is *quadratic* in the small quantities X, Y, θ, and this enables us to make approximations. In particular, $\Delta\boldsymbol{a}$, the displacement of the end A, is given by

$$\Delta\boldsymbol{a} = X\boldsymbol{i} + (Y - a\theta)\boldsymbol{j},$$

correct to the *first* order in small quantities. The vertical displacement z^A of the end

A is therefore given by

$$(b - z^A)^2 = b^2 - \Big(X^2 + (Y - a\theta)^2\Big)$$

correct to the *second* order in small quantities. Hence

$$\begin{aligned} z^A &= b - \Big(b^2 - \big(X^2 + (Y - a\theta)^2\big)\Big)^{1/2} \\ &= b - b\left(1 - \frac{X^2 + (Y - a\theta)^2}{b^2}\right)^{1/2} \\ &= b - b\left(1 - \frac{X^2 + (Y - a\theta)^2}{2b^2} + \cdots\right)^{1/2} \\ &= \frac{X^2 + (Y - a\theta)^2}{2b} \end{aligned}$$

correct to the *second* order in small quantities. Similarly z^B, the vertical displacement of the end B, is given by

$$z^B = \frac{X^2 + (Y + a\theta)^2}{2c}$$

correct to the *second* order in small quantities. Hence z^G, the vertical displacement of G is given by

$$\begin{aligned} z^G &= \tfrac{1}{2}\Big(z^A + z^B\Big) \\ &= \frac{X^2 + (Y - a\theta)^2}{4b} + \frac{X^2 + (Y + a\theta)^2}{4c} \end{aligned}$$

correct to the *second* order in small quantities. Since the gravitational **potential energy** of the system is $V = Mgz^G$, the approximate potential energy is

$$V^{\text{app}} = \frac{Mg}{4bc}\Big((b + c)X^2 + (b + c)Y^2 + (b + c)a^2\theta^2 + 2a(b - c)Y\theta\Big).$$

The V-matrix is therefore

$$\mathsf{V} = \frac{Mg}{4bc}\begin{pmatrix} b + c & 0 & 0 \\ 0 & b + c & a(b - c) \\ 0 & a(b - c) & (b + c)a^2 \end{pmatrix}.$$

The **kinetic energy** of the rod can be expressed as the sum of its translational and rotational contributions in the form

$$T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I_G\,\omega^2,$$

where V is the speed of G, I_G is the moment of inertia of the rod about G, and ω is its angular speed. The value of T^{app} can be found by evaluating T when the system is *passing through its equilibrium configuration*. In terms of the chosen coordinates, this is

$$T^{\text{app}} = \tfrac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) + \tfrac{1}{2}\left(\tfrac{1}{3}Ma^2\right)\dot{\theta}^2.$$

The T-matrix is therefore

$$\mathsf{T} = \tfrac{1}{2}M\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tfrac{1}{3}a^2 \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2\mathsf{T}) = 0$ can be written

$$\begin{vmatrix} a(b+c) - 2bc\mu & 0 & 0 \\ 0 & a(b+c) - 2bc\mu & a^2(b-c) \\ 0 & a^2(b-c) & (b+c)a^3 - \tfrac{2}{3}a^2bc\mu \end{vmatrix} = 0.$$

where $\mu = a\omega^2/g$. This equation will be satisfied if

(a) either

$$a(b+c) - 2bc\mu = 0,$$

(b) or

$$\begin{vmatrix} a(b+c) - 2bc\mu & a^2(b-c) \\ a^2(b-c) & a^3(b+c) - \tfrac{2}{3}a^2bc\mu \end{vmatrix} = 0.$$

In **Case (a)**,

$$\mu = \frac{a(b+c)}{2bc}$$

and the corrresponding normal frequency is

$$\omega_0^2 = \frac{(b+c)g}{2bc}.$$

In order to identify this frequency with the longitudinal mode, we determine the corresponding **amplitude vector.** For this normal frequency, the equations $(\mathbf{V} - \omega^2 \mathbf{T}) \cdot \mathbf{a} = \mathbf{0}$ for the amplitude vector $\mathbf{a}$ become

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^2(b-c) \\ 0 & a^2(b-c) & \frac{1}{3}(b+c)a^3 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ \Theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

These equations have the solution $Y = \Theta = 0$ so that the amplitude vector for the ω_0-mode is $\boldsymbol{a}_0 = (1,\ 0,\ 0)$. Thus $y = \theta = 0$ in this mode, that is, the mode is **purely longitudinal**. The existence of such a mode is entirely to be expected since it is clear by the symmetry of the system that purely longitudinal motions do exist.

In **Case (b)**, the determinant expands to give the quadratic equation

$$bc\mu^2 - 2a(b+c)\mu + 3a^2 = 0,$$

as required.

In the special case in which $b = 3a$ and $c = 8a$, the longitudinal frequency becomes $\omega_0^2 = 11g/48a$ and the equation for the other normal frequences becomes

$$24\mu^2 - 22\mu + 3 = 0,$$

the roots of which are $\mu = \frac{1}{6}$ and $\mu = \frac{3}{4}$. Hence, in this special case, the three normal frequencies are

$$\omega_0^2 = \frac{11g}{48a}, \qquad \omega_1^2 = \frac{g}{6a}, \qquad \omega_2^2 = \frac{3g}{4a} \ \blacksquare$$

Problem 15.8 ✻

A uniform rod BC has mass M and length $2a$. The end B of the rod is connected to a fixed point A on a smooth horizontal table by an elastic string of strength α_1, and the end C is connected to a second fixed point D on the table by a second elastic string of strength α_2. In equilibrium, the rod lies along the line AD with the strings having tension T_0 and lengths b, c respectively. Show that the frequency of the longitudinal mode is $((\alpha_1 + \alpha_2)/M)^{1/2}$ and that the frequencies of the transverse modes satisfy the equation

$$b^2c^2\mu^2 - 2bc(2ab + 3bc + 2ac)\mu + 6abc(2a + b + c) = 0,$$

where $\mu = Ma\omega^2/T_0$. [The calculation of V^{app} is very tricky.]

Find the frequencies of the transverse modes for the particular case in which $a = 3c$ and $b = 5c$.

Solution

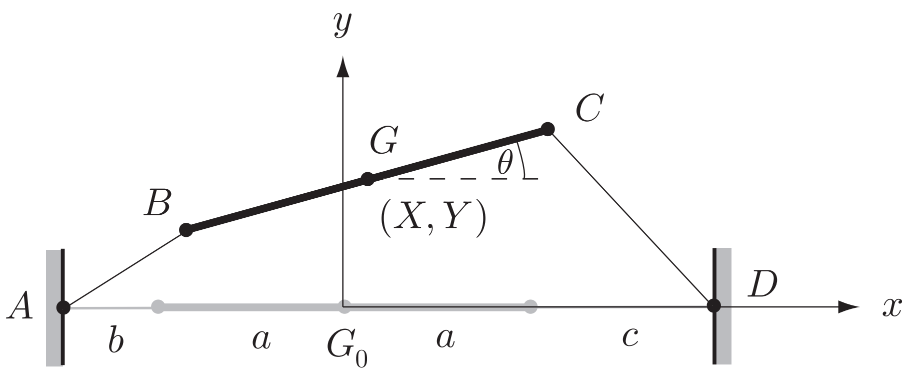

FIGURE 15.8 The system in problem 15.8.

Let (X, Y) be the displacement of the centre of mass G of the rod from its equilibrium position G_0, and let θ be the rotation angle of the rod (see Figure 15.8).

The **kinetic energy** of the rod can be expressed as the sum of its translational and rotational contributions in the form

$$T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I_G\,\omega^2,$$

where M is the mass of the rod, V is the speed of G, I_G is the moment of inertia of the rod about G, and ω is its angular speed. The value of T^{app} can be found by evaluating T when the system is *passing through its equilibrium configuration*. In terms of the chosen coordinates, this is

$$T^{\text{app}} = \tfrac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) + \tfrac{1}{2}\left(\tfrac{1}{3}Ma^2\right)\dot{\theta}^2$$

and the T-matrix is therefore

$$\mathsf{T} = \tfrac{1}{2}M\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tfrac{1}{3}a^2 \end{pmatrix}.$$

The length AB of the left string segment is given by

$$\begin{aligned} AB^2 &= \big(b + X + a(1-\cos\theta)\big)^2 + \big(Y - a\sin\theta\big)^2 \\ &= b^2 + X^2 + a^2(1-\cos\theta)^2 + 2bX + 2ab(1-\cos\theta) + 2aX(1-\cos\theta) \\ &\qquad + Y^2 + a^2\sin^2\theta - 2aY\sin\theta \\ &= b^2 + 2bX + X^2 + Y^2 + a(a+b)\theta^2 - 2aY\theta, \end{aligned}$$

correct to the *second* order in small quantities. The extension Δ_1 of this segment is therefore

$$\begin{aligned} \Delta_1 &= AB - b \\ &= \left(b^2 + 2bX + X^2 + Y^2 + a(a+b)\theta^2 - 2aY\theta + \cdots\right)^{1/2} - b \\ &= b\left(1 + \frac{2X}{b} + \frac{X^2 + Y^2 + a(a+b)\theta^2 - 2aY\theta}{b^2} + \cdots\right)^{1/2} - b \\ &= b\left(1 + \frac{X}{b} + \frac{X^2 + Y^2 + a(a+b)\theta^2 - 2aY\theta}{2b^2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{2X}{b}\right)^2 + \cdots\right) - b \\ &= X + \frac{Y^2 + a(a+b)\theta^2 - 2aY\theta}{2b}, \end{aligned}$$

correct to the *second* order in small quantities.

The potential energy of the segment relative to the equilibrium configuration is

$$\begin{aligned} V_1 &= \int_0^{\Delta_1}\left(T_0 + \alpha_1\xi\right)d\xi, \\ &= T_0\Delta_1 + \tfrac{1}{2}\alpha_1\Delta_1^2, \end{aligned}$$

and so the approximate potential energy of the segment is

$$V_1^{\text{app}} = T_0 X + \frac{T_0}{2b}\left(Y^2 + a(a+b)\theta^2 - 2aY\theta\right) + \tfrac{1}{2}\alpha_1 X^2.$$

The approximate potential energy of the right string segment can be calculated similarly and is given by

$$V_2^{\text{app}} = -T_0 X + \frac{T_0}{2c}\left(Y^2 + a(a+b)\theta^2 + 2aY\theta\right) + \tfrac{1}{2}\alpha_2 X^2.$$

The approximate total **potential energy** of the system is therefore

$$V^{\text{app}} = \tfrac{1}{2}(\alpha_1+\alpha_2)X^2 + \frac{T_0}{2bc}\left((b+c)Y^2 + a(ab+ac+2bc)\theta^2 + 2a(b-c)Y\theta\right)$$

and the V-matrix is

$$\mathsf{V} = \frac{T_0}{2bc}\begin{pmatrix} bc(\alpha_1+\alpha_2)/T_0 & 0 & 0 \\ 0 & b+c & a(b-c) \\ 0 & a(b-c) & a(ab+ac+2bc) \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2\mathsf{T}) = 0$ is therefore

$$\begin{vmatrix} abc(\alpha_1+\alpha_2)/T_0 - bc\mu & 0 & 0 \\ 0 & a(b+c) - bc\mu & a^2(b-c) \\ 0 & a^2(b-c) & a^2(ab+ac+2bc) - \tfrac{1}{3}a^2bc\mu \end{vmatrix} = 0,$$

where $\mu = Ma\omega^2/T_0$. The eigenvalue equation is satisfied if

(a) either

$$abc(\alpha_1+\alpha_2)/T_0 - bc\mu = 0,$$

(b) or

$$\begin{vmatrix} a(b+c) - bc\mu & a^2(b-c) \\ a^2(b-c) & a^2(ab+ac+2bc) - \tfrac{1}{3}a^2bc\mu \end{vmatrix} = 0.$$

In **Case (a)**,

$$\mu = \frac{a(\alpha_1 + \alpha_2)}{T_0}$$

and, since $\mu = Ma\omega^2/T_0$, the corresponding normal frequency is

$$\omega_0^2 = \frac{\alpha_1 + \alpha_2}{M}.$$

It is easily verified that the corresponding **amplitude vector** is $\boldsymbol{a}_0 = (1,\ 0,\ 0)$ so that this mode is **purely longitudinal**. The existence of such a mode is entirely to be expected since it is clear from the symmetry of the system that purely longitudinal motions do exist.

In **Case (b)**, the determinant expands to give the quadratic equation

$$bc\mu^2 - 2(2ab + 2ac + 3bc)\mu + 6a(2a + b + c) = 0,$$

as required.

In the special case in which $a = 3c$ and $b = 5c$, this equation reduces to

$$5\mu^2 - 102\mu + 216 = 0,$$

the roots of which are $\mu = \frac{12}{5}$ and $\mu = 18$. Since $\mu = Ma\omega^2/T_0$, the corrre-sponding normal frequencies are

$$\omega_1^2 = \frac{12T_0}{5Ma}, \qquad \omega_2^2 = \frac{18T_0}{Ma}. \blacksquare$$

Problem 15.9 ✻

A light *elastic* string is stretched between two fixed points A and B a distance $(n+1)a$ apart, and n particles of mass m are attached to the string at equally spaced intervals. The strength of *each* of the $n+1$ sections of the string is α. The system performs small *longitudinal* oscillations along the line AB. Show that the normal frequencies satisfy the determinantal equation

$$\Delta_n \equiv \begin{vmatrix} 2\cos\theta & -1 & 0\cdots & 0 & 0 \\ -1 & 2\cos\theta & -1\cdots & 0 & 0 \\ \vdots & \vdots & \vdots\ddots & \vdots & \vdots \\ 0 & 0 & 0\cdots & 2\cos\theta & -1 \\ 0 & 0 & 0\cdots & -1 & 2\cos\theta \end{vmatrix} = 0,$$

where $\cos\theta = 1 - (m\omega^2/2\alpha)$.

By expanding the determinant by the top row, show that Δ_n satisfies the recurrence relation

$$\Delta_n = 2\cos\theta\Delta_{n-1} - \Delta_{n-2},$$

for $n \geq 3$. Hence, show by induction that

$$\Delta_n = \sin(n+1)\theta/\sin\theta.$$

Deduce the normal frequencies of the system.

Solution

Let the longitudinal displacements of the particles from their equilibrium positions be $x_1, x_2, \ldots, x_n$. Then the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \cdots + \tfrac{1}{2}m\dot{x}_n^2,$$

so that the T-matrix is

$$\mathbf{T} = \tfrac{1}{2}m \begin{pmatrix} 1 & 0\cdots & 0 & 0 \\ 0 & 1\cdots & 0 & 0 \\ \vdots & \vdots\ddots & \vdots & \vdots \\ 0 & 0\cdots & 1 & 0 \\ 0 & 0\cdots & 0 & 1 \end{pmatrix}.$$

Likewise, since the elastic string is linear, the exact and approximate **potential energies** are the same, namely

$$
\begin{aligned}
V = V^{\text{app}} &= \tfrac{1}{2}\alpha x_1^2 + \tfrac{1}{2}\alpha(x_2 - x_1)^2 + \cdots + \tfrac{1}{2}\alpha(x_n - x_{n-1})^2 + \tfrac{1}{2}\alpha x_n^2 \\
&= \alpha\left(x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2 - \cdots + x_{n-1}^2 - x_{n-1}x_n + x_n^2\right),
\end{aligned}
$$

so that the V-matrix is

$$
\mathbf{V} = \tfrac{1}{2}\alpha \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.
$$

The **eigenvalue equation** $\det(\mathbf{V} - \omega^2\mathbf{T}) = 0$ can therefore be written in the form

$$
\Delta_n \equiv \begin{vmatrix} 2\cos\theta & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2\cos\theta & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2\cos\theta \end{vmatrix} = 0,
$$

where $\cos\theta = 1 - (m\omega^2/2\alpha)$.

In order to evaluate Δ_n, we first show that it satisfies the given recurrence relation. On expanding Δ_n by the top row, we obtain

$$
\begin{aligned}
\Delta_n &= 2\cos\theta\,\Delta_{n-1} - (-1)\begin{vmatrix} -1 & -1 & \cdots & 0 & 0 \\ 0 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2\cos\theta & -1 \\ 0 & 0 & \cdots & -1 & 2\cos\theta \end{vmatrix} \\
&= 2\cos\theta\,\Delta_{n-1} + (-1)\Delta_{n-2},
\end{aligned}
$$

on expanding this new determinant by its first column. Hence, for $n \geq 3$,

$$
\Delta_n = 2\cos\theta\,\Delta_{n-1} - \Delta_{n-2},
$$

as required.

We now wish to show that $\Delta_n = D_n$, where

$$D_n = \frac{\sin(n+1)\theta}{\sin\theta}.$$

We prove this by induction.

(i) When $n = 1$,

$$\begin{aligned}\Delta_1 &= 2\cos\theta = \frac{2\sin\theta\cos\theta}{\sin\theta} = \frac{\sin 2\theta}{\sin\theta}\\ &= D_1.\end{aligned}$$

(ii) When $n = 2$,

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 2\cos\theta & -1 \\ -1 & 2\cos\theta \end{vmatrix}\\ &= 4\cos^2\theta - 1 = 3 - 4\sin^2\theta = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} = \frac{\sin 3\theta}{\sin\theta}\\ &= D_2.\end{aligned}$$

(iii) Suppose $\Delta_m = D_m$ for $m = 3, 4, \ldots, n-1$. Then

$$\begin{aligned}\Delta_n &= 2\cos\theta\,\Delta_{n-1} - \Delta_{n-2}\\ &= 2\cos\theta\, D_{n-1} - D_{n-2}\\ &= \frac{2\cos\theta\sin n\theta - \sin(n-1)\theta}{\sin\theta}\\ &= \frac{\sin(n+1)\theta + \sin(n-1)\theta - \sin(n-1)\theta}{\sin\theta}\\ &= \frac{\sin(n+1)\theta}{\sin\theta}\\ &= D_n.\end{aligned}$$

This completes the induction and hence

$$\Delta_n = \frac{\sin(n+1)\theta}{\sin\theta}$$

for all $n \geq 1$.

The normal frequencies are found by solving the equation $\Delta_n = 0$, that is,

$$\sin(n+1)\theta = 0.$$

The roots are

$$\theta = \frac{j\pi}{n+1},$$

where j is any integer. Since $\cos\theta = 1 - (m\omega^2/2\alpha)$, the **normal frequencies** are given by

$$\omega_j^2 = \frac{2\alpha}{m}\left(1 - \cos\left(\frac{j\pi}{n+1}\right)\right)$$
$$= \frac{4\alpha}{m}\sin^2\left(\frac{j\pi}{2(n+1)}\right).$$

Hence the n normal frequencies of the system are all distinct and are given by

$$\omega_j = 2\left(\frac{\alpha}{m}\right)^{1/2}\sin\left(\frac{j\pi}{2(n+1)}\right),$$

where $j = 1, 2, \ldots, n$. Further values of j give nothing new. ■

Problem 15.10

A light string is stretched to a tension T_0 between two fixed points A and B a distance $(n+1)a$ apart, and n particles of mass m are attached to the string at equally spaced intervals. The system performs small plane *transverse* oscillations. Show that the normal frequencies satisfy the same determinantal equation as in the previous question, except that now $\cos\theta = 1 - (ma\omega^2/2T_0)$. Find the normal frequencies of the system.

Solution

Let the transverse displacements of the particles from their equilibrium positions be $y_1, y_2, \dots, y_n$. Then the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2}m\dot{y}_1^2 + \tfrac{1}{2}m\dot{y}_2^2 + \cdots + \tfrac{1}{2}m\dot{y}_n^2,$$

so that the T-matrix is

$$\mathbf{T} = \tfrac{1}{2}m\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The extension Δ_n of the n-th segment of the string is

$$\begin{aligned}\Delta_n &= \left(a^2 + (y_n - y_{n-1})^2\right)^{1/2} - a \\ &= a\left(1 + \frac{(y_n - y_{n-1})^2}{a^2}\right)^{1/2} - a \\ &= a\left(1 + \frac{(y_n - y_{n-1})^2}{2a^2} + \cdots\right) - a \\ &= \frac{(y_n - y_{n-1})^2}{2a},\end{aligned}$$

correct to the second order in small quantities.

The potential energy of the segment relative to the equilibrium configuration is

$$\begin{aligned}V_n &= \int_0^{\Delta_n} \left(T_0 + \alpha\xi\right) d\xi, \\ &= T_0\Delta_n + \tfrac{1}{2}\alpha\Delta_n^2,\end{aligned}$$

and so the approximate potential energy of the n-th segment is

$$\frac{T_0}{2a}(y_n - y_{n-1})^2.$$

The approximate total **potential energy** is therefore

$$V^{\text{app}} = \frac{T_0}{a}\left(y_1^2 - y_1 y_2 + y_2^2 - y_2 y_3 + y_3^2 - \cdots + y_{n-1}^2 - y_{n-1} y_n + y_n^2\right),$$

so that the V-matrix is

$$\mathsf{V} = \frac{T_0}{2a}\begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2 \mathsf{T}) = 0$ can therefore be written in the form

$$\Delta_n \equiv \begin{vmatrix} 2\cos\theta & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2\cos\theta & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2\cos\theta \end{vmatrix} = 0,$$

where $\cos\theta = 1 - (ma\omega^2/2T_0)$.

In order to evaluate Δ_n, we first show that it satisfies the given recurrence relation. On expanding Δ_n by the top row, we obtain

$$\Delta_n = 2\cos\theta\,\Delta_{n-1} - (-1)\begin{vmatrix} -1 & -1 & \cdots & 0 & 0 \\ 0 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2\cos\theta & -1 \\ 0 & 0 & \cdots & -1 & 2\cos\theta \end{vmatrix}$$

$$= 2\cos\theta\,\Delta_{n-1} + (-1)\Delta_{n-2},$$

on expanding this new determinant by its first column. Hence, for $n \geq 3$,

$$\Delta_n = 2\cos\theta\,\Delta_{n-1} - \Delta_{n-2},$$

as required.

We now wish to show that $\Delta_n = D_n$, where

$$D_n = \frac{\sin(n+1)\theta}{\sin\theta}.$$

We prove this by induction.

(i) When $n = 1$,

$$\begin{aligned}\Delta_1 &= 2\cos\theta = \frac{2\sin\theta\cos\theta}{\sin\theta} = \frac{\sin 2\theta}{\sin\theta}\\ &= D_1.\end{aligned}$$

(ii) When $n = 2$,

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 2\cos\theta & -1 \\ -1 & 2\cos\theta \end{vmatrix}\\ &= 4\cos^2\theta - 1 = 3 - 4\sin^2\theta = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} = \frac{\sin 3\theta}{\sin\theta}\\ &= D_2.\end{aligned}$$

(iii) Suppose $\Delta_m = D_m$ for $m = 3, 4, \ldots, n-1$. Then

$$\begin{aligned}\Delta_n &= 2\cos\theta\,\Delta_{n-1} - \Delta_{n-2}\\ &= 2\cos\theta\, D_{n-1} - D_{n-2}\\ &= \frac{2\cos\theta\sin n\theta - \sin(n-1)\theta}{\sin\theta}\\ &= \frac{\sin(n+1)\theta + \sin(n-1)\theta - \sin(n-1)\theta}{\sin\theta}\\ &= \frac{\sin(n+1)\theta}{\sin\theta}\\ &= D_n.\end{aligned}$$

This completes the induction and hence

$$\Delta_n = \frac{\sin(n+1)\theta}{\sin\theta}$$

for all $n \geq 1$.

The normal frequencies are found by solving the equation $\Delta_n = 0$, that is,

$$\sin(n + 1)\theta = 0.$$

The roots are

$$\theta = \frac{j\pi}{n + 1},$$

where j is any integer. Since $\cos\theta = 1 - (ma\omega^2/2T_0)$, the **normal frequencies** are given by

$$\begin{aligned} \omega_j^2 &= \frac{2T_0}{ma}\left(1 - \cos\left(\frac{j\pi}{n + 1}\right)\right) \\ &= \frac{4T_0}{ma}\sin^2\left(\frac{j\pi}{2(n + 1)}\right). \end{aligned}$$

Hence the n normal frequencies of the system are all distinct and are given by

$$\omega_j = 2\left(\frac{T_0}{ma}\right)^{1/2}\sin\left(\frac{j\pi}{2(n + 1)}\right),$$

where $j = 1, 2, \ldots, n$. Further values of j give nothing new. ■

Problem 15.11 *Unsymmetrical linear molecule*

A general linear triatomic molecule has atoms A_1, A_2, A_3 with masses m_1, m_2, m_3. The chemical bond between A_1 and A_2 is represented by a spring of strength α_{12} and the bond between A_2 and A_3 is represented by a spring of strength α_{23}. Show that the vibrational frequences of the molecule satisfy the equation

$$m_1 m_2 m_3\, \omega^4 - \big[\alpha_{12} m_3 (m_1 + m_2) + \alpha_{23} m_1 (m_2 + m_3)\big]\omega^2 \\ + \alpha_{12}\alpha_{23}(m_1 + m_2 + m_3) = 0.$$

Find the vibrational frequencies for the special case in which $m_1 = 3m$, $m_2 = m$, $m_3 = 2m$ and $\alpha_{12} = 3\alpha$, $\alpha_{23} = 2\alpha$.

The molecule O–C–S (carbon oxysulphide) is known to be linear. Use the λ_1^{-1} values given in Table 2 of the book (p. 441) to estimate its vibrational frequencies. [The experimentally measured values are 2174 cm^{-1} and 874 cm^{-1}.]

Solution

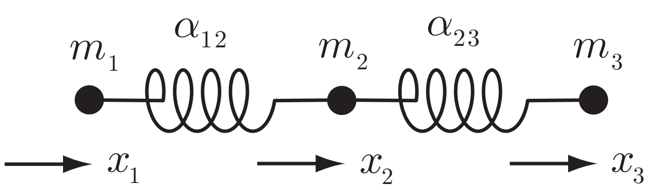

FIGURE 15.9 The system in problem 15.11.

Let the longitudinal displacements of the atoms from their equilibrium positions be x_1, x_2, x_3 as shown in Figure 15.9. Then the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2} m_3 \dot{x}_3^2,$$

so that the T-matrix is

$$\mathbf{T} = \tfrac{1}{2}\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

Likewise, since the springs are linear, the exact and approximate **potential en-**

ergies are the same, namely

$$V = V^{\text{app}} = \tfrac{1}{2}\alpha_{12}(x_2 - x_1)^2 + \tfrac{1}{2}\alpha_{23}(x_3 - x_2)^2$$
$$= \tfrac{1}{2}\left(\alpha_{12}x_1^2 - 2\alpha_{12}x_1x_2 + (\alpha_{12} + \alpha_{23})x_2^2 - 2\alpha_{23}x_2x_3 + \alpha_{23}x_3^2\right),$$

so that the V-matrix is

$$\mathsf{V} = \frac{1}{2}\begin{pmatrix} \alpha_{12} & -\alpha_{12} & 0 \\ -\alpha_{12} & \alpha_{12} + \alpha_{23} & -\alpha_{23} \\ 0 & -\alpha_{23} & \alpha_{23} \end{pmatrix}.$$

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2\mathsf{T}) = 0$ is therefore

$$\begin{vmatrix} \alpha_{12} - m_1\omega^2 & -\alpha_{12} & 0 \\ -\alpha_{12} & \alpha_{12} + \alpha_{23} - m_2\omega^2 & -\alpha_{23} \\ 0 & -\alpha_{23} & \alpha_{23} - m_3\omega^2 \end{vmatrix} = 0.$$

On expanding the determinant, we obtain

$$\omega^2\Big[m_1m_2m_3\,\omega^4 - \big[\alpha_{12}m_3(m_1 + m_2) + \alpha_{23}m_1(m_2 + m_3)\big]\omega^2$$
$$+ \alpha_{12}\alpha_{23}(m_1 + m_2 + m_3)\Big] = 0,$$

a cubic equation in the variable ω^2. The root $\omega = 0$ corresponds to a rigid body translation of the whole molecule. There are therefore only **two vibrational modes**, the frequencies of which satisfy the equation

$$m_1m_2m_3\,\omega^4 - \big[\alpha_{12}m_3(m_1 + m_2) + \alpha_{23}m_1(m_2 + m_3)\big]\omega^2$$
$$+ \alpha_{12}\alpha_{23}(m_1 + m_2 + m_3) = 0,$$

a quadratic equation in the variable ω^2.

In the special case in which $m_1 = 3m$, $m_2 = m$, $m_3 = 2m$ and $\alpha_{12} = 3\alpha$, $\alpha_{23} = 2\alpha$, the equation for the normal frequencies reduces to

$$m^2\omega^4 - 7m\alpha\omega^2 + 6\alpha^2 = 0$$

and the **normal frequencies** are

$$\omega_1^2 = \frac{\alpha}{m}, \qquad \omega_2^2 = \frac{6\alpha}{m}.$$

In order to calculate the vibrational frequencies of **carbon oxysulphide**, we need to know α_{12} and α_{23}, the strengths of the O—C and C—S bonds. These can be found from the vibrational frequencies of CO_2 and CS_2 given in Table 2 of the book. From Example 15.5 , the strength α of the bonds in a *symmetric* triatomic molecule is given by $\alpha = M\omega_1^2$, where M is the mass of each of the outer atoms and ω_1 is the frequency of the symmetric stretching mode.

In order to use the data given in Table 2 more easily, we introduce a non-standard system of units in which the unit of mass is the atomic unit, the unit of length is the centimetre, and the unit of time is taken so that the speed of light is $1/2\pi$. In these units, the mass of an atom is *equal* to its atomic weight, and the angular frequency of a mode is *equal* to its reciprocal wavelength in cm^{-1}. For the carbon oxysulphide molecule, $m_1 = 16$, $m_2 = 12$, $m_3 = 32$ and the bond strengths are given by

$$\alpha_{12} = 16 \times 1337^2, \qquad \alpha_{23} = 32 \times 657^2,$$

on using the values of λ_1^{-1} given in Table 2. On substituting this data into the equation for the normal frequencies, we find that the **vibrational frequencies** of carbon oxysulphide are

$$\lambda_1^{-1} = 2230 \text{ cm}^{-1}, \qquad \lambda_2^{-1} = 880 \text{ cm}^{-1},$$

correct to three significant figures. (Examination of the amplitude vectors reveals that the λ_1-mode is predominantly a C—O stretching mode, while the λ_2-mode is predominantly a C—S stretching mode.) The experimentally measured values are 2174 cm^{-1} and 874 cm^{-1} respectively. ■

Problem 15.12 ✻ *Symmetric V-shaped molecule*

Book Figure 15.7 shows the symmetric V-shaped triatomic molecule XY_2; the X – Y bonds are represented by springs of strength k, while the Y – Y bond is represented by a spring of strength ϵk. Common examples of such molecules include water, hydrogen sulphide, sulphur dioxide and nitrogen dioxide; the apex angle 2α is typically between 90° and 120°. In planar motion, the molecule has six degrees of freedom of which three are rigid body motions; there are therefore *three* vibrational modes. It is best to exploit the reflective symmetry of the molecule and solve separately for the symmetric and antisymmetric modes. Book Figure 15.7 (left) shows a symmetric motion while (right) shows an antisymmetric motion; the displacements X, Y, x, y are measured *from the equilibrium position*. Show that there is one antisymmetric mode whose frequency ω_3 is given by

$$\omega_3^2 = \frac{k}{mM}(M + 2m\sin^2\alpha),$$

and show that the frequencies of the symmetric modes satisfy the equation

$$\mu^2 - \left(1 + 2\gamma\cos^2\alpha + 2\epsilon\right)\mu + 2\epsilon\cos^2\alpha(1 + 2\gamma) = 0,$$

where $\mu = m\omega^2/k$ and $\gamma = m/M$.

Find the three vibrational frequencies for the special case in which $M = 2m$, $\alpha = 60°$ and $\epsilon = 1/2$.

Solution

Anti-symmetric modes

Let the coordinates Y, x, y be those shown in book Figure 15.7 (right). Then the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2}M\dot{Y}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right),$$

so that the T-matrix is

$$\mathbf{T} = m\begin{pmatrix} 1/2\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\gamma = m/M$.

The extension of the upper X—Y bond from its equilibrium configuration is

$$x\cos\alpha + y\sin\alpha - Y\sin\alpha,$$

correct to the *first* order in small quantities. Its approximate potential energy is therefore

$$\tfrac{1}{2}k(x\cos\alpha + y\sin\alpha - Y\sin\alpha)^2$$

and the approximate potential energy of the lower X—Y bond has the same value. The extension of the Y—Y bond from its equilibrium configuration is zero and hence its potential energy is also zero. The approximate total **potential energy** of the molecule is therefore

$$V^{\text{app}} = k(x\cos\alpha + y\sin\alpha - Y\sin\alpha)^2$$

so that the V-matrix is

$$\mathsf{V} = k\begin{pmatrix} s^2 & -sc & -s^2 \\ -sc & c^2 & sc \\ -s^2 & sc & s^2 \end{pmatrix},$$

where $s = \sin\alpha$ and $c = \cos\alpha$.

The **eigenvalue equation** $\det(\mathsf{V} - \omega^2\mathsf{T}) = 0$ can therefore be written

$$\begin{vmatrix} s^2 - \mu/2\gamma & -sc & -s^2 \\ -sc & c^2 - \mu & sc \\ -s^2 & sc & s^2 - \mu \end{vmatrix} = 0,$$

where $\mu = m\omega^2/k$. On expanding the determinant, we obtain

$$\mu^2(1 + 2\gamma\sin^2\alpha - \mu) = 0,$$

a cubic equation in the variable μ. The double root $\mu^2 = 0$ corresponds to rigid body motions of the whole molecule (one translation and one rotation). There is therefore only **one vibrational mode** corresponding to the root $\mu = 1 + 2\gamma\sin^2\alpha$. Since $\mu = m\omega^2/m$ and $\gamma = m/M$, the frequency ω_3 of this mode is given by

$$\omega_3^2 = \frac{k}{mM}(M + 2m\sin^2\alpha).$$

In the special case in which $M = 2m$ and $\alpha = 60°$, the antisymmetric mode has frequency $\omega_3^2 = 7k/4m$.

Symmetric modes

Let the coordinates $X,\ x,\ y$ be those shown in book Figure 15.7 (left). Then the exact and approximate **kinetic energies** are the same, namely

$$T = T^{\text{app}} = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right),$$

so that the T-matrix is

$$\mathbf{T} = m\begin{pmatrix} 1/2\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The extension of the upper X—Y bond from its equilibrium configuration is

$$x\cos\alpha + y\sin\alpha - X\cos\alpha,$$

correct to the *first* order in small quantities. Its approximate potential energy is therefore

$$\tfrac{1}{2}k(x\cos\alpha + y\sin\alpha - X\cos\alpha)^2$$

and the approximate potential energy of the lower X—Y bond has the same value. The extension of the Y—Y bond from its equilibrium configuration is $2y$ and hence its potential energy is $\frac{1}{2}(\epsilon k)(2y)^2 = 2\epsilon k y^2$. The approximate total **potential energy** of the molecule is therefore

$$V^{\text{app}} = k(x\cos\alpha + y\sin\alpha - X\cos\alpha)^2 + 2\epsilon k y^2$$

so that the V-matrix is

$$\mathbf{V} = k\begin{pmatrix} c^2 & -c^2 & -sc \\ -c^2 & c^2 & sc \\ -sc & sc & s^2 + 2\epsilon \end{pmatrix},$$

where $s = \sin\alpha$ and $c = \cos\alpha$.

The **eigenvalue equation** $\det(\mathbf{V} - \omega^2\mathbf{T}) = 0$ can therefore be written

$$\begin{vmatrix} c^2 - \mu/2\gamma & -c^2 & -sc \\ -c^2 & c^2 - \mu & sc \\ -sc & sc & s^2 + 2\epsilon - \mu \end{vmatrix} = 0,$$

where $\mu = m\omega^2/k$. On expanding the determinant, we obtain

$$\mu\Big(\mu^2 - \big(1 + 2\gamma\cos^2\alpha + 2\epsilon\big)\mu + 2\epsilon\cos^2\alpha(1 + 2\gamma)\Big) = 0,$$

a cubic equation in the variable μ. The root $\mu = 0$ corresponds to a rigid body translation of the whole molecule . There are therefore only **two vibrational modes** corresponding to the roots of the quadratic equation

$$\mu^2 - \big(1 + 2\gamma\cos^2\alpha + 2\epsilon\big)\mu + 2\epsilon\cos^2\alpha(1 + 2\gamma) = 0,$$

where $\mu = m\omega^2/m$ and $\gamma = m/M$.

In the special case in which $M = 2m$, $\alpha = 60°$ and $\epsilon = \frac{1}{2}$, this equation reduces to

$$4\mu^2 - 9\mu + 2 = 0,$$

the roots of which are $\mu = \frac{1}{4}$ and $\mu = 2$. Since $\mu = m\omega^2/k$, the symmetric modes therefore have frequencies

$$\omega_1^2 = \frac{k}{4m}, \qquad \omega_2^2 = \frac{2k}{m}. \blacksquare$$

Problem 15.13 *Plane triangular molecule*

The molecule BCl_3 (boron trichloride) is plane and symmetrical. In equlibrium, the Cl atoms are at the vertices of an equilateral triangle with the B atom at the centroid. Show that the molecule has six vibrational modes of which five are in the plane of the molecule; show also that the out-of-plane mode and one of the in-plane modes have axial symmetry; and show finally that the remaining four in-plane modes are in doubly degenerate pairs. Deduce that the BCl_3 molecule has a total of four distinct vibrational frequencies.

Solution

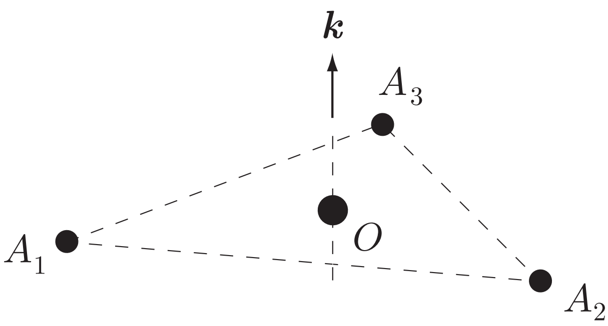

FIGURE 15.10 The boron trichloride molecule.

This problem involves the *classification* of vibrational modes rather than the determination of their frequencies. There is a general method for classifying vibrational modes based on a study of the symmetry group of the molecule. Here however we will find the solution by *ad hoc* symmetry arguments. Such arguments are adequate for small molecules.

(a) **Total number of modes** Since each atom has three degrees of freedom, the molecule has twelve degrees of freedom. However, since any molecule has six possible rigid body motions (three translational and three rotational) there are only **six normal modes**. Note that the number of *distinct* normal frequencies may be less than six.

(b) **In-plane and anti-plane modes** Since the molecule is plane, the particle motions in its normal modes must all lie either (i) in the plane of the molecule (in-plane motion) or (ii) perpendicular to it (anti-plane motion). If we restrict the motion of the molecule to be in-plane, this reduces the number of degrees of freedom to eight and the number of rigid body motions to three (two translational and one rotational). Hence there must be **five in-plane modes** and therefore only **one anti-plane mode**.

(c) Axially symmetric modes Let $\{O, \boldsymbol{k}\}$ be the axis of rotational symmetry of the molecule in its equilibrium position (see Figure 15.10). Then a *motion* is said to have axial symmetry if it is preserved when the molecule is rotated through an angle of 120° about the axis $\{O, \boldsymbol{k}\}$. If we restrict the motion of the molecule to be in-plane *and* to have axial symmetry, this reduces the number of degrees of freedom to two and the number of rigid body motions to one (a rotation). Hence, of the five in-plane modes, only **one is axially symmetric**. Similarly, if we restrict the motion of the molecule to be anti-plane *and* to have axial symmetry, this reduces the number of degrees of freedom to two and the number of rigid body motions to one (a translation). Hence, **the anti-plane mode is axially symmetric**. The forms of these axially symmetric modes are depicted in Figure 15.11.

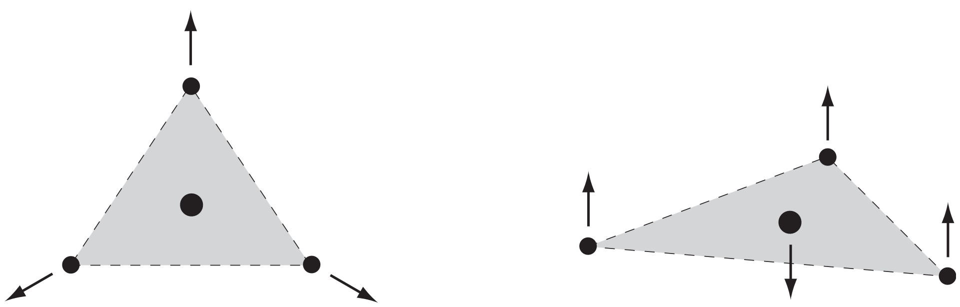

FIGURE 15.11 The axially symmetric modes of boron trichloride. **Left:** the in-plane mode. **Right:** the anti-plane mode.

(d) Degenerate modes It remains to show that the remaining four in-plane modes are in doubly degenerate pairs. This arises from the rotational symmetry of the molecule. Let $\mathcal{M}$ be some normal mode with frequency Ω. Then $\mathcal{M}'$, the motion obtained by rotating $\mathcal{M}$ through 120° about the axis $\{O, \boldsymbol{k}\}$, is also a normal mode with frequency Ω. If $\mathcal{M}$ is axially symmetric, then $\mathcal{M}' = \mathcal{M}$ and we have found nothing new. However, in any other case, we have found a second normal mode with frequency Ω. Hence, except for the axially symmetric modes, all the normal frequencies must be (at least) doubly degenerate. It follows that the remaining four in-plane modes must be in doubly degenerate pairs. [Exceptionally, it might happen that these doubly degenerate frequencies are equal, producing one fourfold degeneracy. Measurement shows that this does *not* happen for boron trichloride.] ■

Chapter Sixteen

Vector angular velocity

and rigid body kinematics

Problem 16.1

A rigid body is rotating in the right-handed sense about the axis Oz with a constant angular speed of 2 radians per second. Write down the angular velocity vector of the body, and find the instantaneous velocity, speed and acceleration of the particle of the body at the point $(4, -3, 7)$, where distances are measured in metres.

Solution

The vector angular velocity of the body is $\boldsymbol{\omega} = 2\boldsymbol{k}$ radians per second. The given particle P has position vector $\boldsymbol{r} = 4\boldsymbol{i} - 3\boldsymbol{j} + 7\boldsymbol{k}$ and its **velocity** $\boldsymbol{v}$ is given by

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{\omega}\times\boldsymbol{r}\\ &= (2\boldsymbol{k})\times(4\boldsymbol{i} - 3\boldsymbol{j} + 7\boldsymbol{k})\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 0 & 2\\ 4 & -3 & 7\end{vmatrix}\\ &= 6\boldsymbol{i} + 8\boldsymbol{j}\ \mathrm{m\,s^{-1}}.\end{aligned}$$

The **speed** of P is $|\boldsymbol{v}| = \left(6^2 + 8^2\right)^{1/2} = 10\ \mathrm{m\,s^{-1}}$.

The **acceleration** of P can then be calculated as follows:

$$\begin{aligned}\boldsymbol{a} = \dot{\boldsymbol{v}} &= \frac{d}{dt}(\boldsymbol{\omega}\times\boldsymbol{r})\\ &= \dot{\boldsymbol{\omega}}\times\boldsymbol{r} + \boldsymbol{\omega}\times\dot{\boldsymbol{r}}\\ &= \boldsymbol{0}\times\boldsymbol{r} + \boldsymbol{\omega}\times\boldsymbol{v}\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 0 & 2\\ 6 & 8 & 0\end{vmatrix}\\ &= -16\boldsymbol{i} + 12\boldsymbol{j}\ \mathrm{m\,s^{-2}}. \blacksquare\end{aligned}$$

Problem 16.2

A rigid body is rotating with constant angular speed 3 radians per second about a fixed axis through the points $A(4,1,1)$, $B(2,-1,0)$, distances being measured in centimetres. The rotation is in the left-handed sense relative to the direction $\overrightarrow{AB}$. Find the instantaneous velocity and acceleration of the particle P of the body at the point $(4,4,4)$.

Solution

The points A and B have position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ respectively. The rotation axis $\overrightarrow{AB}$ has direction $\boldsymbol{b} - \boldsymbol{a} = -2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ and so the unit vector $\boldsymbol{n}$ pointing in this direction is given by

$$\begin{aligned}\boldsymbol{n} &= \frac{\boldsymbol{b}-\boldsymbol{a}}{|\boldsymbol{b}-\boldsymbol{a}|}\\ &= -\frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{|2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}|}\\ &= -\tfrac{1}{3}(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}).\end{aligned}$$

The **angular velocity** of the body is therfore

$$\boldsymbol{\omega} = -3\boldsymbol{n} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$$

radians per second.

The instantaneous **velocity** of the particle P that has position vector $4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$ is then given by

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{\omega}\times(\boldsymbol{r}-\boldsymbol{b})\\ &= (2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})\times(2\boldsymbol{i}+5\boldsymbol{j}+4\boldsymbol{k})\\ &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 2 & 1\\ 2 & 5 & 4\end{vmatrix}\\ &= 3\boldsymbol{i} - 6\boldsymbol{j} + 6\boldsymbol{k}\ \mathrm{cm\,s^{-1}}.\end{aligned}$$

The instantaneous **acceleration** of P is can then be calculated as follows:

$$\begin{aligned}
\boldsymbol{a} = \dot{\boldsymbol{v}} &= \frac{d}{dt}(\boldsymbol{\omega}\times(\boldsymbol{r}-\boldsymbol{b})) \\
&= \dot{\boldsymbol{\omega}}\times(\boldsymbol{r}-\boldsymbol{b}) + \boldsymbol{\omega}\times(\dot{\boldsymbol{r}}-\dot{\boldsymbol{b}}) \\
&= \boldsymbol{0}\times\boldsymbol{r} + \boldsymbol{\omega}\times(\boldsymbol{v}-\boldsymbol{0}) \\
&= \boldsymbol{\omega}\times\boldsymbol{v} \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 1 \\ 3 & -6 & 6 \end{vmatrix} \\
&= 18\boldsymbol{i} - 9\boldsymbol{j} - 18\boldsymbol{k}\ \mathrm{cm\,s^{-2}}. \ \blacksquare
\end{aligned}$$

Problem 16.3

A spinning top (a rigid body of revolution) is in general motion with its vertex (a particle on the axis of symmetry) fixed at the origin O. Let $\boldsymbol{a}(t)$ be the unit vector pointing along the axis of symmetry and let $\boldsymbol{\omega}(t)$ be the angular velocity of the top. (In general, $\boldsymbol{\omega}$ does *not* point along the axis of symmetry.) By considering the velocities of particles of the top that lie on the axis of symmetry, show that $\boldsymbol{a}$ satisfies the equation

$$\dot{\boldsymbol{a}} = \boldsymbol{\omega} \times \boldsymbol{a}.$$

Deduce that the most general form $\boldsymbol{\omega}$ can have is

$$\boldsymbol{\omega} = \boldsymbol{a} \times \dot{\boldsymbol{a}} + \lambda \boldsymbol{a},$$

where λ is a scalar function of the time. [This formula is needed in the theory of the spinning top.]

Solution

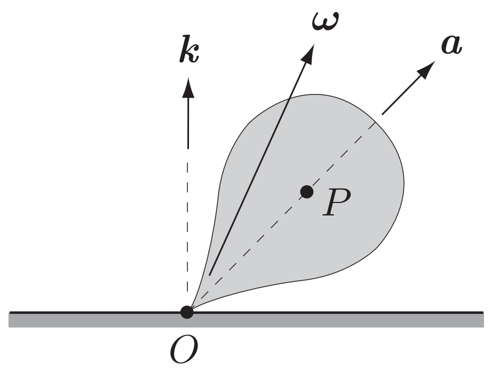

FIGURE 16.1 The symmetrical spinning top with its vertex fixed at O has angular velocity $\boldsymbol{\omega}$. The unit vector $\boldsymbol{a}$ points along the symmetry axis of the top.

Let P be the particle of the top that lies on the symmetry axis and is unit distance from the vertex O. Then the position vector of P (relative to the origin O) is the axial unit vector $\boldsymbol{a}(t)$. It follows that $\boldsymbol{v}$, the velocity of P, is $\dot{\boldsymbol{a}}$. However, $\boldsymbol{v}$ can also be expressed in the form $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{a}$. It follows that the vectors $\boldsymbol{a}$ and $\boldsymbol{\omega}$ must be related by the formula

$$\dot{\boldsymbol{a}} = \boldsymbol{\omega} \times \boldsymbol{a}. \qquad (1)$$

On taking the cross product of this formula with $\boldsymbol{a}$, we obtain

$$\begin{aligned}\boldsymbol{a}\times\dot{\boldsymbol{a}} &= \boldsymbol{a}\times(\boldsymbol{\omega}\times\boldsymbol{a})\\ &= (\boldsymbol{a}\cdot\boldsymbol{a})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{a})\boldsymbol{a}\\ &= \boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{a})\boldsymbol{a},\end{aligned}$$

since $\boldsymbol{a}\cdot\boldsymbol{a} = 1$. Hence $\boldsymbol{\omega}$ must have the form

$$\boldsymbol{\omega} = \boldsymbol{a}\times\dot{\boldsymbol{a}} + \lambda\boldsymbol{a}, \tag{2}$$

where λ is some scalar function of the time. (Actually, $\lambda = \boldsymbol{\omega}\cdot\boldsymbol{a}$, the axial component of $\boldsymbol{\omega}$.) In fact, the expression (2) satisfies equation (1) for *any* choice of the scalar function $\lambda(t)$ and this is therefore the most general form that $\boldsymbol{\omega}$ can have. ■

Problem 16.4

A penny of radius a rolls without slipping on a rough horizontal table. The penny rolls in such a way that its centre G remains fixed (see Figure 16.5). The plane of the penny makes a constant angle α with the table and the point of contact C traces out a circle with centre O and radius $a\cos\alpha$, as shown. At time t, the angle between the radius OC and some fixed radius is θ. Find the angular velocity vector of the penny in terms of the unit vectors $\boldsymbol{a}(t)$, $\boldsymbol{k}$ shown.

Find the velocity of the highest particle of the penny.

Solution

Suppose that the penny is viewed from a frame rotating about the axis $\{O, \boldsymbol{k}\}$ with angular velocity $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k}$. In the rotating frame, G is still fixed and $\boldsymbol{a}$ is now constant. The apparent angular velocity of the penny must therefore have the form $\boldsymbol{\omega}' = \lambda\boldsymbol{a}$, where λ is some scalar function of the time. Hence, by the **addition theorem** for angular velocities, the true angular velocity of the penny is given by

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\omega} + \boldsymbol{\omega}'\\ &= \dot{\theta}\boldsymbol{k} + \lambda\boldsymbol{a}.\end{aligned}$$

It remains to determine the scalar function λ from the **rolling condition**. Since G is permanently at rest and the contact particle C is instantaneously at rest, the instantaneous axis of rotation must lie along the line GC. In particular then, $\boldsymbol{\omega}$ must be perpendicular to $\boldsymbol{a}$. The condition $\boldsymbol{\omega}\cdot\boldsymbol{a} = 0$ gives

$$\dot{\theta}(\boldsymbol{k}\cdot\boldsymbol{a}) + \lambda(\boldsymbol{a}\cdot\boldsymbol{a}) = 0,$$

that is,

$$\dot{\theta}(\boldsymbol{k}\cdot\boldsymbol{a}) + \lambda = 0,$$

since $\boldsymbol{a}\cdot\boldsymbol{a} = 1$. Hence

$$\lambda = -\dot{\theta}(\boldsymbol{k}\cdot\boldsymbol{a}) = -\dot{\theta}\cos\alpha$$

and the **angular velocity** of the penny is

$$\boldsymbol{\omega} = \dot{\theta}(\boldsymbol{k} - \cos\alpha\,\boldsymbol{a}).$$

The velocity of the highest particle of the penny can be found without using the above formula for $\boldsymbol{\omega}$. Since the highest particle of the penny lies on the instantaneous rotation axis, its velocity must be zero. ■

Problem 16.5

A rigid circular cone with altitude h and semi-angle α rolls without slipping on a rough horizontal table. Explain why the vertex O of the cone never moves. Let $\theta(t)$ be the angle between OC, the line of the cone that is in contact with the table, and some fixed horizontal reference line OA. Show that the angular velocity $\boldsymbol{\omega}$ of the cone is given by

$$\boldsymbol{\omega} = -\left(\dot{\theta}\cot\alpha\right)\boldsymbol{i},$$

where $\boldsymbol{i}(t)$ is the unit vector pointing in the direction $\overrightarrow{OC}$. [First identify the *direction* of $\boldsymbol{\omega}$, and then consider the velocities of those particles of the cone that lie on the axis of symmetry.]

Identify the particle of the cone that has the maximum speed and find this speed.

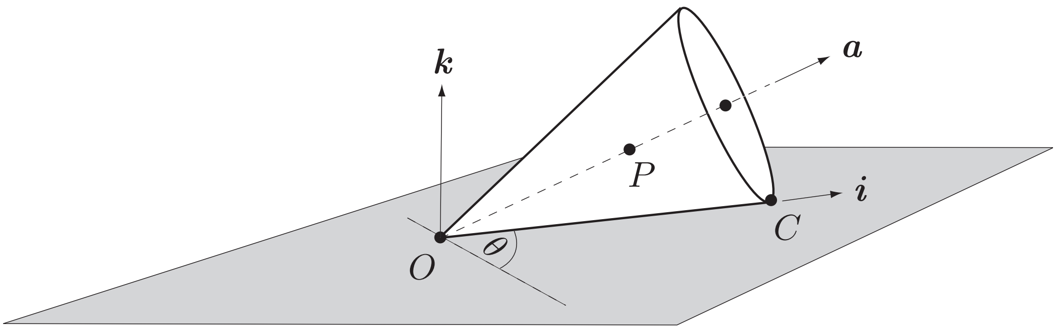

FIGURE 16.2 A cone of semi-angle α rolls on a flat table.

Solution

The cone is shown in Figure 16.2. The unit vector $\boldsymbol{a}$ $(= \boldsymbol{a}(t))$ lies along the symmetry axis, and the unit vector $\boldsymbol{i}$ $(= \boldsymbol{i}(t))$ lies along the generator of the cone that is in instantaneous contact with the table. By the **rolling condition**, every particle of the cone lying on this generator is instantaneously at rest. Hence O must be *permanently* at rest and the angular velocity of the cone must point along the direction OC. Thus O is fixed and $\boldsymbol{\omega}$ has the form

$$\boldsymbol{\omega} = \lambda\boldsymbol{i},$$

where λ is some scalar function of the time.

To determine the scalar function λ, consider the motion of a particle P on the symmetry axis that is distance a from O. Then the position vector of P relative to

O is

$$\boldsymbol{r} = a\cos\alpha\,\boldsymbol{i} + a\sin\alpha\,\boldsymbol{k},$$

and its velocity is given by

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{\omega}\times\boldsymbol{r}\\ &= \lambda\,\boldsymbol{i}\times(a\cos\alpha\,\boldsymbol{i} + a\sin\alpha\,\boldsymbol{k})\\ &= a\lambda\sin\alpha(\boldsymbol{i}\times\boldsymbol{k}).\end{aligned}$$

However, it is evident that P moves on a horizontal circle of radius $a\cos\alpha$ centered on the axis $\{O, \boldsymbol{k}\}$, and that its scalar velocity is $(a\cos\alpha)\dot{\theta}$. On comparing these two formulae, we see that $-a\lambda\sin\alpha = (a\cos\alpha)\dot{\theta}$. Hence

$$\lambda = -\dot{\theta}\cot\alpha$$

and the **angular velocity** of the cone is

$$\boldsymbol{\omega} = -\left(\dot{\theta}\cot\alpha\right)\boldsymbol{i}.$$

The instantaneous speed of *any* particle Q of the cone is ωp, where $\omega = |\boldsymbol{\omega}|$ and p is the perpendicular distance of Q from the instantaneous axis OC. The particle with the highest speed is therefore the particle furthest from OC, namely, the highest particle. This particle has perpendicular distance $(h\sec\alpha)\sin 2\alpha$ from OC and its speed is therefore

$$\begin{aligned}|\boldsymbol{v}| &= \omega p = |-\dot{\theta}\cot\alpha|(h\sec\alpha)\sin 2\alpha\\ &= 2h\cos\alpha\,|\dot{\theta}|. \blacksquare\end{aligned}$$

Problem 16.6 ✻

Two rigid plastic panels lie in the planes $z = -b$ and $z = b$ respectively. A rigid ball of radius b can move in the space between the panels and is gripped by them so that it does not slip. The panels are made to rotate with angular velocities $\omega_1 \boldsymbol{k}$, $\omega_2 \boldsymbol{k}$ about fixed vertical axes that are a distance $2c$ apart. Show that, *with a suitable choice of origin*, the position vector $\boldsymbol{R}$ of the centre of the ball satisfies the equation

$$\dot{\boldsymbol{R}} = \boldsymbol{\omega} \times \boldsymbol{R},$$

where $\boldsymbol{\omega} = \frac{1}{2}(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$. Deduce that the ball must move in a circle and find the position of the centre of this circle.

Solution

Suppose that the panel $z = -b$ rotates with angular velocity $\omega_1 \boldsymbol{k}$ about an axis through the point $(-c, 0, 0)$, and that the panel $z = b$ rotates with angular velocity $\omega_2 \boldsymbol{k}$ about an axis through the point $(c, 0, 0)$. Let $\boldsymbol{r}$ be the position vector of the centre of the ball relative to O and let $\boldsymbol{\omega}$ be its angular velocity.

Then the **rolling conditions** at the points where the ball contacts the panels $z = \pm b$ are

$$\begin{aligned} \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (-b\,\boldsymbol{k}) &= (\omega_1 \boldsymbol{k}) \times (\boldsymbol{r} + c\,\boldsymbol{i}), \\ \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (b\,\boldsymbol{k}) &= (\omega_2 \boldsymbol{k}) \times (\boldsymbol{r} - c\,\boldsymbol{i}). \end{aligned}$$

Adding these equations together gives

$$2\dot{\boldsymbol{r}} = (\omega_1 + \omega_2)(\boldsymbol{k} \times \boldsymbol{r}) + c(\omega_1 - \omega_2)(\boldsymbol{k} \times \boldsymbol{i}),$$

which can be written in the form

$$\dot{\boldsymbol{r}} = \left(\tfrac{1}{2}(\omega_1 + \omega_2)\boldsymbol{k}\right) \times \left(\boldsymbol{r} + c\left(\frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}\right)\boldsymbol{i}\right).$$

Hence, if we define $\boldsymbol{R}$ by

$$\boldsymbol{R} = \boldsymbol{r} + c\left(\frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}\right)\boldsymbol{i},$$

then $\boldsymbol{R}$ satisfies the equation

$$\dot{\boldsymbol{R}} = \boldsymbol{\omega} \times \boldsymbol{R},$$

where $\boldsymbol{\omega} = \frac{1}{2}(\omega_1 + \omega_2)\boldsymbol{k} = \frac{1}{2}(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$. The change from $\boldsymbol{r}$ to $\boldsymbol{R}$ represents a shift in the origin of position vectors from O to the new origin O' whose coordinates are

$$\left(-c\left(\frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}\right),\ 0,\ 0\right).$$

Now the solutions of the equation $\dot{\boldsymbol{R}} = \boldsymbol{\omega} \times \boldsymbol{R}$ (where $\boldsymbol{\omega}$ is a constant) are known to be motions in which the point with position vector $\boldsymbol{R}$ moves on a circle lying in a plane perpendicular to the vector $\boldsymbol{\omega}$, and with centre lying on the axis $\{O, \boldsymbol{\omega}\}$. It follows that, in our case, the centre of the ball moves on a circle in the (x, y)-plane with centre at the point O'.

Problem 16.7

Two hollow spheres have radii a and b ($b > a$), and their common centre O is fixed. A rigid ball of radius $\frac{1}{2}(b-a)$ can move in the annular space between the spheres and is gripped by them so that it does not slip. The spheres are made to rotate with constant angular velocities $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ respectively. Show that the ball must move in a circle whose plane is perpendicular to the vector $a\boldsymbol{\omega}_1 + b\boldsymbol{\omega}_2$.

Solution

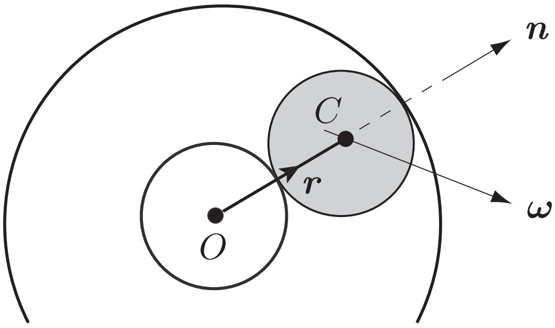

FIGURE 16.3 The ball is gripped between two rotating hollow spheres.

Let $\boldsymbol{r}$ be the position vector of the centre of the ball and let $\boldsymbol{\omega}$ be its angular velocity. Let $\boldsymbol{n}$ be the unit vector in the direction $\overrightarrow{OC}$. Then the **rolling conditions** at the points where the ball contacts the inner and outer spheres are

$$\dot{\boldsymbol{r}} + \boldsymbol{\omega}\times(-\tfrac{1}{2}(b-a)\boldsymbol{n}) = \boldsymbol{\omega}_1\times(a\boldsymbol{n}),$$
$$\dot{\boldsymbol{r}} + \boldsymbol{\omega}\times(\tfrac{1}{2}(b-a)\boldsymbol{n}) = \boldsymbol{\omega}_2\times(b\boldsymbol{n}).$$

Adding these equations together gives

$$2\dot{\boldsymbol{r}} = (a\boldsymbol{\omega}_1 + b\boldsymbol{\omega}_2)\times\boldsymbol{n}.$$

Since $\boldsymbol{r} = \frac{1}{2}(a+b)\boldsymbol{n}$, this equation can be written in the form

$$\dot{\boldsymbol{r}} = \left(\frac{a\boldsymbol{\omega}_1 + b\boldsymbol{\omega}_2}{a+b}\right)\times\boldsymbol{r}.$$

Now the solutions of the equation $\dot{\boldsymbol{r}} = \boldsymbol{\omega}\times\boldsymbol{r}$ (where $\boldsymbol{\omega}$ is a constant) are known to be motions in which the point with position vector $\boldsymbol{r}$ moves on a circle lying in a plane perpendicular to the vector $\boldsymbol{\omega}$, and with centre lying on the axis $\{O, \boldsymbol{\omega}\}$. It follows that, in our case, the centre of the ball moves on a circle whose plane is perpendicular to the vector $a\boldsymbol{\omega}_1 + b\boldsymbol{\omega}_2$. ■

Chapter Seventeen

Rotating reference frames

Problem 17.1

Use the velocity and acceleration transformation formulae to derive the standard expressions for the velocity and acceleration of a particle in plane polar coordinates.

Solution

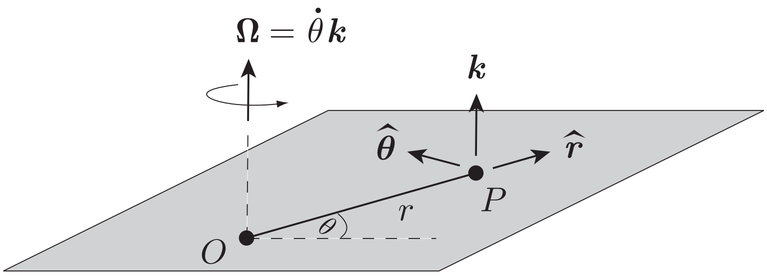

FIGURE 17.1 The frame $\mathcal{F}'$ rotates about the axis $\{O, \boldsymbol{k}\}$ with scalar angular velocity $\dot{\theta}$. In this frame, the unit vectors $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ are constants.

Suppose the polar coordinates r, θ, are measured relative to the frame $\mathcal{F}$ and let $\mathcal{F}'$ be the frame rotating about the axis $\{O, \boldsymbol{k}\}$ with scalar angular velocity $\dot{\theta}$. Then $\boldsymbol{\Omega}$, the vector angular velocity of $\mathcal{F}'$ relative to $\mathcal{F}$ is $\boldsymbol{\Omega} = \dot{\theta}\boldsymbol{k}$, as shown in Figure 17.1.

Suppose that a particle P moves in the plane and is viewed from both frames. Then, in $\mathcal{F}'$, the unit vector $\widehat{\boldsymbol{r}}$ is **constant** so that

$$\boldsymbol{r}' = r\widehat{\boldsymbol{r}}, \qquad \boldsymbol{v}' = \dot{r}\widehat{\boldsymbol{r}}, \qquad \boldsymbol{a}' = \ddot{r}\widehat{\boldsymbol{r}}.$$

The **velocity** of P is given by

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{\Omega}\times\boldsymbol{r}' + \boldsymbol{v}' \\ &= \left(\dot{\theta}\boldsymbol{k}\right)\times\left(r\widehat{\boldsymbol{r}}\right) + \dot{r}\widehat{\boldsymbol{r}} \\ &= \dot{r}\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}, \end{aligned}$$

as required.

In the same way, the **acceleration** of P is given by

$$\begin{aligned} \boldsymbol{a} &= \dot{\boldsymbol{\Omega}}\times\boldsymbol{r}' + 2\boldsymbol{\Omega}\times\boldsymbol{v}' + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{r}'\right) + \boldsymbol{a}' \\ &= \left(\ddot{\theta}\boldsymbol{k}\right)\times\left(r\widehat{\boldsymbol{r}}\right) + 2\left(\dot{\theta}\boldsymbol{k}\right)\times\left(\dot{r}\widehat{\boldsymbol{r}}\right) + \left(\dot{\theta}\boldsymbol{k}\right)\times\Big(\left(\dot{\theta}\boldsymbol{k}\right)\times\left(r\widehat{\boldsymbol{r}}\right)\Big) + \ddot{r}\widehat{\boldsymbol{r}} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \ \blacksquare \end{aligned}$$

Problem 17.2 *Addition of angular velocities*

Prove the 'addition of angular velocities' theorem, Theorem 17.1.

Solution

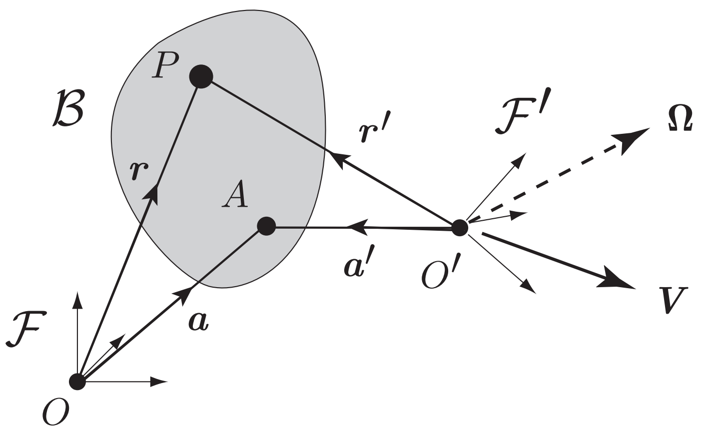

FIGURE 17.2 The angular velocity addition theorem.

Let $\mathcal{B}$ be a rigid body whose motion is observed from the reference frames $\mathcal{F}$ and $\mathcal{F}'$ as shown in Figure 17.2. The frame $\mathcal{F}'$ has velocity $\boldsymbol{V}$ and angular velocity $\boldsymbol{\Omega}$ relative to $\mathcal{F}$. Let A be some reference particle of $\mathcal{B}$ and P a general particle. Then, by the **velocity transformation** formula, $\boldsymbol{v}$, the velocity of P in $\mathcal{F}$ is given by

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{\Omega}\times\boldsymbol{r}' + \boldsymbol{v}',$$

where $\boldsymbol{v}'$, the velocity of P in $\mathcal{F}'$, is given by the **rigid body** formula

$$\boldsymbol{v}' = \boldsymbol{v}'_A + \boldsymbol{\omega}'\times(\boldsymbol{r}' - \boldsymbol{a}'),$$

where $\boldsymbol{\Omega}'$ is the angular velocity of $\mathcal{B}$ in $\mathcal{F}'$. Hence

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{V} + \boldsymbol{\Omega}\times\boldsymbol{r}' + \boldsymbol{v}'_A + \boldsymbol{\omega}'\times(\boldsymbol{r}' - \boldsymbol{a}') \\ &= \big(\boldsymbol{V} + \boldsymbol{\Omega}\times\boldsymbol{a}' + \boldsymbol{v}'_A\big) + \big(\boldsymbol{\Omega} + \boldsymbol{\omega}'\big)\times\big(\boldsymbol{r}' - \boldsymbol{a}'\big) \\ &= \boldsymbol{v}_A + \big(\boldsymbol{\Omega} + \boldsymbol{\omega}'\big)\times(\boldsymbol{r} - \boldsymbol{a})\,. \end{aligned}$$

But $\boldsymbol{v}$ is also given directly by the **rigid body** formula

$$\boldsymbol{v} = \boldsymbol{v}_A + \boldsymbol{\omega}\times(\boldsymbol{r} - \boldsymbol{a}),$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{B}$ in $\mathcal{F}$. By comparing these two formulae for $\boldsymbol{v}$, we see that

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}'.$$

This is exactly the theorem on the **addition of angular velocities**.

Problem 17.3

A circular cone with semi-angle α is fixed with its axis of symmetry vertical and its vertex O upwards. A second circular cone has semi-vertical angle $(\pi/2) - \alpha$ and has its vertex fixed at O. The second cone rolls on the first cone so that its axis of symmetry precesses around the upward vertical with angular speed λ. Find the angular speed of the rolling cone.

Solution

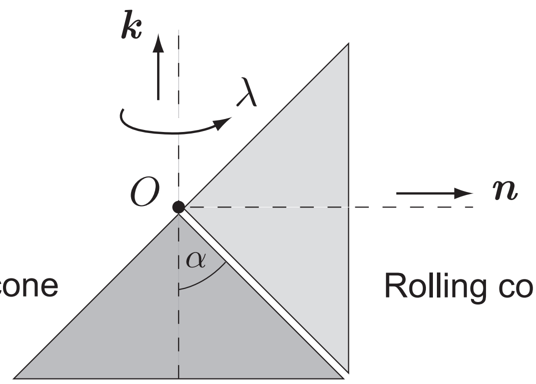

FIGURE 17.3 The upper cone rolls on the fixed lower one.

Suppose the frame $\mathcal{F}$ be fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the axis $\{O, \boldsymbol{k}\}$. Then $\mathcal{F}'$ has vector angular velocity $\lambda\,\boldsymbol{k}$ relative to $\mathcal{F}$.

In the frame $\mathcal{F}'$, the rolling cone has its axis of symmetry fixed and so its angular velocity $\boldsymbol{\omega}'$ must have the form

$$\boldsymbol{\omega}' = \mu\,\boldsymbol{n},$$

where μ is some scalar function of the time. The **addition theorem** for angular velocities then shows that $\boldsymbol{\omega}$, the true angular velocity of the cone, is given by

$$\boldsymbol{\omega} = \lambda\,\boldsymbol{k} + \mu\,\boldsymbol{n}.$$

It remains to determine μ from the **rolling condition**. Since the particles on the contact generator of the rolling cone are instantaneously at rest, it follows that

$$\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{0}$$

for those position vectors $\boldsymbol{r}$ that have the form

$$\boldsymbol{r} = a\,(-\cos\alpha\,\boldsymbol{k} + \sin\alpha\,\boldsymbol{n})\,.$$

Hence

$$(\lambda\,\boldsymbol{k} + \mu\,\boldsymbol{n})\times(-\cos\alpha\,\boldsymbol{k} + \sin\alpha\,\boldsymbol{n}) = \boldsymbol{0},$$

from which it follows that

$$\lambda\sin\alpha + \mu\cos\alpha = 0.$$

Hence $\mu = -\lambda\tan\alpha$ and the **angular velocity** of the rolling cone is therefore

$$\boldsymbol{\omega} = \lambda\,(\boldsymbol{k} - \tan\alpha\,\boldsymbol{n})$$

It follows that the **angular speed** of the rolling cone is

$$\begin{aligned} |\boldsymbol{\omega}| &= |\lambda\,(\boldsymbol{k} - \tan\alpha\,\boldsymbol{n})\,| \\ &= \lambda\left(1 + \tan^2\alpha\right)^{1/2} \\ &= \lambda\sec\alpha \;\blacksquare \end{aligned}$$

Problem 17.4

A particle P of mass m can slide along a smooth rigid straight wire. The wire has one of its points fixed at the origin O, and is made to rotate in a plane through O with constant angular speed Ω. Show that r, the distance of P from O, satisfies the equation

$$\ddot{r} - \Omega^2 r = 0.$$

Initially, P is at rest (relative to the wire) at a distance a from O. Find r as a function of t in the subsequent motion.

Solution

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the axis $\{O, \boldsymbol{k}\}$, where the unit vector $\boldsymbol{k}$ is perpendicular to the plane of motion of the wire. Then the vector **angular velocity** of $\mathcal{F}'$ relative to $\mathcal{F}$ is $\Omega\boldsymbol{k}$. In the frame $\mathcal{F}'$, the unit vectors $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ are constants and so

$$\boldsymbol{r}' = r\,\widehat{\boldsymbol{r}}, \qquad \boldsymbol{v}' = \dot{r}\,\widehat{\boldsymbol{r}}, \qquad \boldsymbol{a}' = \ddot{r}\,\widehat{\boldsymbol{r}}.$$

The equation of motion for the particle P in the rotating frame $\mathcal{F}'$ is therefore

$$m\Big[\,\ddot{r}\,\widehat{\boldsymbol{r}} + \boldsymbol{0} + 2(\Omega\boldsymbol{k})\times(\dot{r}\,\widehat{\boldsymbol{r}}) + (\Omega\boldsymbol{k})\times\big((\Omega\boldsymbol{k})\times(r\,\widehat{\boldsymbol{r}})\big)\,\Big] = N\,\widehat{\boldsymbol{\theta}},$$

where N is the reaction of the wire on the particle. Since the wire is *smooth* this points perpendicular to the wire. This equation simplifies to give

$$m\Big[\,\big(\ddot{r} - \Omega^2 r\big)\widehat{\boldsymbol{r}} + \big(2\Omega\dot{r}\big)\widehat{\boldsymbol{\theta}}\,\Big] = N\,\widehat{\boldsymbol{\theta}}$$

and, on equating components in the $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ directions, we obtain the two scalar equations

$$\ddot{r} - \Omega^2 r = 0,$$
$$2m\Omega\dot{r} = N.$$

The second equation serves to determine the normal reaction N. The general solution of the first equation can be written

$$r = A\cosh\Omega t + B\sinh\Omega t,$$

and, on applying the initial conditions $r = a$ and $\dot{r} = 0$ when $t = 0$, we obtain the solution

$$r = a\cosh\Omega t.$$

This is the **motion** of the particle P along the wire. ■

Problem 17.5 *Larmor precession*

A particle of mass m and charge e moves in the force field $\boldsymbol{F}(\boldsymbol{r})$ and the uniform magnetic field $B\boldsymbol{k}$, where $\boldsymbol{k}$ is a constant unit vector. Its equation of motion is then

$$m\frac{d\boldsymbol{v}}{dt} = \left(\frac{eB}{c}\right)\boldsymbol{v}\times\boldsymbol{k} + \boldsymbol{F}(\boldsymbol{r})$$

in cgs Gaussian units. Show that the term $(eB/c)\boldsymbol{v}\times\boldsymbol{k}$ can be removed from the equation by viewing the motion from an appropriate rotating frame.

For the special case in which $\boldsymbol{F}(\boldsymbol{r}) = -m\omega_0^2\boldsymbol{r}$, show that circular motions with two different frequencies are possible.

Solution

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with vector angular velocity $\boldsymbol{\Omega}$ about some axis through O, the origin of position vectors. Then the equation of motion of the particle in the rotating frame $\mathcal{F}'$ is

$$m\Big[\boldsymbol{a}' + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r}' + 2\boldsymbol{\Omega}\times\boldsymbol{v}' + \boldsymbol{\Omega}\times\big(\boldsymbol{\Omega}\times\boldsymbol{r}'\big)\Big] = \left(\frac{eB}{c}\right)\big(\boldsymbol{v}' + \boldsymbol{\Omega}\times\boldsymbol{r}'\big)\times\boldsymbol{k} + \boldsymbol{F}(\boldsymbol{r}).$$

We see that the terms involving $\boldsymbol{v}'$ can be made to cancel by taking

$$\boldsymbol{\Omega} = -\left(\frac{eB}{2mc}\right)\boldsymbol{k},$$

that is, by taking $\mathcal{F}'$ to be rotating with scalar angular velocity $\Omega = eB/2mc$ about an axis parallel to the uniform magnetic field. The quantity Ω is called the **Larmor frequency**.

On making the substitution $\boldsymbol{\Omega} = -\Omega\,\boldsymbol{k}$ and dropping the dashes, the equation of motion for the particle becomes

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}(\boldsymbol{r}) + m\Omega^2\big(\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r})\big).$$

For the special case in which $\boldsymbol{F} = -m\omega_0^2\boldsymbol{r}$, this reduces to

$$\frac{d\boldsymbol{v}}{dt} = -\omega_0^2\boldsymbol{r} + \Omega^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r}).$$

Let us seek motions in which the particle moves in a plane through O perpendicular to the uniform magnetic field. Then $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r}) = -\boldsymbol{r}$ and the equation of motion becomes

$$\frac{d\boldsymbol{v}}{dt} + \left(\omega_0^2 + \Omega^2\right)\boldsymbol{r} = \boldsymbol{0}.$$

This is the two-dimensional **SHM equation**. The most general motion of the particle is elliptical with centre O and with frequency $(\omega_0^2 + \Omega^2)^{1/2}$. In particular, **circular motions** with centre O and frequency $(\omega_0^2 + \Omega^2)^{1/2}$ are possible.

This is how the motions appear in the rotating frame. On returning to the fixed frame, the circular motions remain circular but have one of **two different frequencies** $(\omega_0^2 + \Omega^2)^{1/2} \pm \Omega$, depending on their *direction* of motion around the axis $\{O, \boldsymbol{k}\}$. When $|\Omega/\omega_0| \ll 1$, as is usually the case, the two frequencies are given approximately by $\omega_0 \pm \Omega$. ■

Problem 17 . 6

A bullet is fired vertically upwards with speed u from a point on the Earth with co-latitude β. Show that it returns to the ground west of the firing point by a distance $4\Omega u^3 \sin\beta/3g^2$.

Solution

In the standard notation, the equation of particle motion relative to the rotating Earth is

$$m\left[\frac{d\boldsymbol{v}}{dt} + 2\boldsymbol{\Omega}\times\boldsymbol{v}\right] = -mg\,\boldsymbol{k},$$

where the Earth's angular velocity $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = \Omega(-\sin\beta\,\boldsymbol{i} + \cos\beta\,\boldsymbol{k})$$

and β is the co-latitude.

One integration with respect to t gives

$$\frac{d\boldsymbol{r}}{dt} + 2\,\boldsymbol{\Omega}\times\boldsymbol{r} = (u - gt)\,\boldsymbol{k},$$

on using the initial conditions $\boldsymbol{v} = u\,\boldsymbol{k}$ and $\boldsymbol{r} = \boldsymbol{0}$ when $t = 0$.

A second integration with respect to t leads to the **integral equation**

$$\boldsymbol{r}(t) = (ut - \tfrac{1}{2}gt^2)\,\boldsymbol{k} - 2\,\boldsymbol{\Omega}\times\int_0^t \boldsymbol{r}(t')\,dt'$$

on using the initial condition $\boldsymbol{r} = \boldsymbol{0}$ when $t = 0$.

Hence the **zero order** approximation to the motion is

$$\boldsymbol{r}^{(0)} = (ut - \tfrac{1}{2}gt^2)\,\boldsymbol{k}$$

and the **first order** approximation is given by

$$\begin{aligned}
\boldsymbol{r}^{(1)} &= \boldsymbol{r}^{(0)} - 2\,\boldsymbol{\Omega}\times\int_0^t \boldsymbol{r}^{(0)}(t')\,dt' \\
&= \boldsymbol{r}^{(0)} - 2\,\boldsymbol{\Omega}\times\int_0^t (ut' - \tfrac{1}{2}gt'^2)\,\boldsymbol{k}\,dt' \\
&= \boldsymbol{r}^{(0)} - 2\,(\boldsymbol{\Omega}\times\boldsymbol{k})\left(\tfrac{1}{2}ut^2 - \tfrac{1}{6}gt^3\right) \\
&= \boldsymbol{r}^{(0)} + \Omega\sin\beta\left(\tfrac{1}{3}gt^3 - ut^2\right)\boldsymbol{j},
\end{aligned}$$

where $\boldsymbol{j}$ is the unit vector pointing east. Hence the **first order correction** to the zero order solution is a deflection of

$$\Omega \sin\beta \left(\tfrac{1}{3}gt^3 - ut^2\right)$$

to the east.

To find the value of this deflection when the bullet returns to the ground, we need an approximation to τ, the time of flight. In this case, the zero order approximation is sufficient, namely

$$\tau^{(0)} = \frac{2u}{g}.$$

On substituting in this value for t, the **deflection** of the bullet is found to be

$$\frac{4\Omega u^3 \sin\beta}{3g^2}$$

to the **west**. ■

Problem 17.7

An artillery shell is fired from a point on the Earth with co-latitude β. The direction of firing is due **south**, the muzzle speed of the shell is u and the angle of elevation of the barrel is α. Show that the effect of the Earth's rotation is to deflect the shell to the west by a distance

$$\frac{4\Omega u^3}{3g^2}\sin^2\alpha\,(3\cos\alpha\cos\beta+\sin\alpha\sin\beta)\,.$$

Solution

In the standard notation, the equation of particle motion relative to the rotating Earth is

$$m\left[\frac{d\boldsymbol{v}}{dt}+2\boldsymbol{\Omega}\times\boldsymbol{v}\right]=-mg\,\boldsymbol{k},$$

where the Earth's angular velocity $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega}=\Omega(-\sin\beta\,\boldsymbol{i}+\cos\beta\,\boldsymbol{k})$$

and β is the co-latitude.

One integration with respect to t gives

$$\frac{d\boldsymbol{r}}{dt}+2\,\boldsymbol{\Omega}\times\boldsymbol{r}=u(\cos\alpha\,\boldsymbol{i}+\sin\alpha\,\boldsymbol{k})-gt\,\boldsymbol{k},$$

on using the initial conditions $\boldsymbol{v}=u(\cos\alpha\,\boldsymbol{i}+\sin\alpha\,\boldsymbol{k})$ and $\boldsymbol{r}=\boldsymbol{0}$ when $t=0$.

A second integration with respect to t leads to the **integral equation**

$$\boldsymbol{r}(t)=u(\cos\alpha\,\boldsymbol{i}+\sin\alpha\,\boldsymbol{k})t-\tfrac{1}{2}gt^2\boldsymbol{k}-2\,\boldsymbol{\Omega}\times\int_0^t\boldsymbol{r}(t')\,dt'$$

on using the initial condition $\boldsymbol{r}=\boldsymbol{0}$ when $t=0$.

Hence the **zero order** approximation to the motion is

$$\boldsymbol{r}^{(0)}=u(\cos\alpha\,\boldsymbol{i}+\sin\alpha\,\boldsymbol{k})t-\tfrac{1}{2}gt^2\boldsymbol{k},$$

and the **first order** approximation is given by

$$\begin{aligned}
\boldsymbol{r}^{(1)} &= \boldsymbol{r}^{(0)} - 2\,\boldsymbol{\Omega}\times\int_0^t \boldsymbol{r}^{(0)}(t')\,dt' \\
&= \boldsymbol{r}^{(0)} - 2\,\boldsymbol{\Omega}\times\int_0^t \Big(u(\cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{k})t' - \tfrac{1}{2}gt'^2\,\boldsymbol{k}\Big)\,dt' \\
&= \boldsymbol{r}^{(0)} - \boldsymbol{\Omega}\times\Big(u(\cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{k})t^2 - \tfrac{1}{3}gt^3\boldsymbol{k}\Big) \\
&= \boldsymbol{r}^{(0)} + \tfrac{1}{3}\Omega g t^3 \sin\beta\,\boldsymbol{j} - \Omega u t^2\big(\cos\alpha\cos\beta + \sin\alpha\sin\beta\big)\boldsymbol{j} \\
&= \boldsymbol{r}^{(0)} + \Omega\left(\tfrac{1}{3}gt^3\sin\beta - ut^2\cos(\alpha-\beta)\right)\boldsymbol{j},
\end{aligned}$$

where $\boldsymbol{j}$ is the unit vector pointing east. Hence the **first order correction** to the zero order solution is a deflection of

$$\Omega\left(\tfrac{1}{3}gt^3\sin\beta - ut^2\cos(\alpha-\beta)\right)$$

to the east.

To find the value of this deflection when the particle returns to the ground, we need an approximation to τ, the time of flight. In this case, the zero order approximation is sufficient, namely

$$\tau^{(0)} = \frac{2u\sin\alpha}{g}.$$

On substituting in this value for t, the required **deflection** of the shell is found to be

$$\frac{4\Omega u^3}{3g^2}\sin^2\alpha\big(3\cos\alpha\cos\beta + 2\sin\alpha\sin\beta\big)$$

to the **west**. ■

Problem 17.8

An artillery shell is fired from a point on the Earth with co-latitude β. The direction of firing is due **east**, the muzzle speed of the shell is u and the angle of elevation of the barrel is α. Show that the effect of the Earth's rotation is to deflect the shell to the south by a distance

$$\frac{4\Omega u^3}{3g^2}\sin^2\alpha\cos\alpha\cos\beta.$$

✱ Show also that the easterly range is increased by

$$\frac{4\Omega u^3}{3g^2}\sin\alpha\sin\beta\left(3-4\sin^2\alpha\right).$$

[*Hint*. The second part requires a corrected value for the flight time.]

Solution

In the standard notation, the equation of particle motion relative to the rotating Earth is

$$m\left[\frac{d\boldsymbol{v}}{dt}+2\boldsymbol{\Omega}\times\boldsymbol{v}\right]=-mg\,\boldsymbol{k},$$

where the Earth's angular velocity $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega}=\Omega(-\sin\beta\,\boldsymbol{i}+\cos\beta\,\boldsymbol{k})$$

and β is the co-latitude.

One integration with respect to t gives

$$\frac{d\boldsymbol{r}}{dt}+2\,\boldsymbol{\Omega}\times\boldsymbol{r}=u(\cos\alpha\,\boldsymbol{j}+\sin\alpha\,\boldsymbol{k})-gt\,\boldsymbol{k},$$

on using the initial conditions $\boldsymbol{v}=u(\cos\alpha\,\boldsymbol{i}+\sin\alpha\,\boldsymbol{k})$ and $\boldsymbol{r}=\boldsymbol{0}$ when $t=0$.

A second integration with respect to t leads to the **integral equation**

$$\boldsymbol{r}(t)=u(\cos\alpha\,\boldsymbol{j}+\sin\alpha\,\boldsymbol{k})t-\tfrac{1}{2}gt^2\,\boldsymbol{k}-2\,\boldsymbol{\Omega}\times\int_0^t\boldsymbol{r}(t')\,dt'$$

on using the initial condition $\boldsymbol{r}=\boldsymbol{0}$ when $t=0$.

Hence the **zero order** approximation to the motion is

$$\boldsymbol{r}^{(0)}=u(\cos\alpha\,\boldsymbol{j}+\sin\alpha\,\boldsymbol{k})t-\tfrac{1}{2}gt^2\,\boldsymbol{k},$$

and the **first order** approximation is given by

$$\begin{aligned} \boldsymbol{r}^{(1)} &= \boldsymbol{r}^{(0)} - 2\,\boldsymbol{\Omega}\times\int_0^t \boldsymbol{r}^{(0)}(t')\,dt' \\ &= \boldsymbol{r}^{(0)} - 2\,\boldsymbol{\Omega}\times\int_0^t \Big(u(\cos\alpha\,\boldsymbol{j} + \sin\alpha\,\boldsymbol{k})t' - \tfrac{1}{2}gt'^2\,\boldsymbol{k}\Big)\,dt' \\ &= \boldsymbol{r}^{(0)} - \boldsymbol{\Omega}\times\Big(u(\cos\alpha\,\boldsymbol{j} + \sin\alpha\,\boldsymbol{k})t^2 - \tfrac{1}{3}gt^3\boldsymbol{k}\Big) \\ &= \boldsymbol{r}^{(0)} + \tfrac{1}{3}\Omega g t^3 \sin\beta\,\boldsymbol{j} + \Omega u t^2\big(\cos\alpha\cos\beta\,\boldsymbol{i} - \sin\alpha\sin\beta\,\boldsymbol{j} + \cos\alpha\sin\beta\,\boldsymbol{k}\big) \end{aligned}$$

Hence the first order **correction** to the zero order solution is

$$\tfrac{1}{3}\Omega g t^3 \sin\beta\,\boldsymbol{j} + \Omega u t^2\big(\cos\alpha\cos\beta\,\boldsymbol{i} - \sin\alpha\sin\beta\,\boldsymbol{j} + \cos\alpha\sin\beta\,\boldsymbol{k}\big).$$

In particular, the shell suffers a deflection of

$$\Omega u t^2 \cos\alpha\cos\beta$$

to the south.

To find the value of this deflection when the particle returns to the ground, we need an approximation to τ, the time of flight. In this case, the zero order approximation is sufficient, namely

$$\tau^{(0)} = \frac{2u\sin\alpha}{g}.$$

On substituting in this value for t, the required **deflection** of the shell is found to be

$$\frac{4\Omega u^3}{g^2}\sin^2\alpha\cos\alpha\cos\beta$$

to the **south**.

Finding the correction to the **easterly range** is very tricky. This part should probably not have been set, but it's too late now!

The total easterly displacement of the shell at time t is given by the first order approximation to be

$$ut\cos\alpha + \tfrac{1}{3}\Omega g t^3\sin\beta - \Omega u t^2 \sin\alpha\sin\beta, \qquad (1)$$

and we now need to replace t in this expression by τ, the time of flight. The second and third terms of this expression are small corrections and it is sufficient to use $\tau^{(0)}$, the zero order approximation to τ. However, the first term is not small and we need to use $\tau^{(1)}$, the **first order** approximation to τ. To find $\tau^{(1)}$, consider the **vertical motion**. The total vertical displacement of the shell at time t is given by the first order approximation to be

$$ut\sin\alpha - \tfrac{1}{2}gt^2 + \Omega u t^2 \cos\alpha\sin\beta$$

and the flight time $\tau^{(1)}$ must make this expression zero. It follows that

$$\tau^{(1)} = \frac{2u\sin\alpha}{g}\left(1 - \frac{2\Omega u}{g}\cos\alpha\sin\beta\right)^{-1}.$$

One last hurdle. Since the first order approximation already neglects squares and higher powers of the small dimensionless paramater $\Omega u/g$, we may replace this expression for $\tau^{(1)}$ by the simpler formula

$$\tau^{(1)} = \frac{2u\sin\alpha}{g}\left(1 + \frac{2\Omega u}{g}\cos\alpha\sin\beta\right).$$

This is the required expression for the **time of flight**, correct to the first order.

It now remains to substitute this value for $\tau^{(1)}$ into the first term of (1) and the value of $\tau^{(0)}$ into the last two terms. After some heavy algebra we find that the **easterly range** of the shell is increased by

$$\frac{4\Omega u^3}{3g^2}\sin\alpha\sin\beta\left(3 - 4\sin^2\alpha\right) \blacksquare$$

Problem 17.9

Consider Problem 17.4 again. This time find the motion of the particle by using the transformed energy equation.

Solution

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the axis $\{O, \boldsymbol{k}\}$, where the unit vector $\boldsymbol{k}$ is perpendicular to the plane of motion of the wire. Then, in the rotating frame $\mathcal{F}'$, the wire is at rest and the system is standard and conservative with **apparent potential energy** zero.

The **apparent kinetic energy** is $T = \frac{1}{2}m\dot{r}^2$ and the **moment of inertia** of the system about the axis $\{O, \boldsymbol{k}\}$ is $I = mr^2$. The **energy conservation principle** for uniformly rotating frames then implies that

$$\tfrac{1}{2}m\dot{r}^2 + 0 - \tfrac{1}{2}m\Omega^2 r^2 = E,$$

where E is a constant. The initial conditions $r = a$ and $\dot{r} = 0$ when $t = 0$ show that $E = -\frac{1}{2}m\Omega^2 a^2$ and so the **energy conservation equation** becomes

$$\dot{r}^2 = \Omega^2\left(r^2 - a^2\right).$$

On taking square roots and separating, we find that

$$\cosh^{-1}\left(\frac{r}{a}\right) = \pm\Omega t + C,$$

where C is a constant of integration. The initial condition $r = a$ when $t = 0$ shows that $C = 0$ and the solution for r is found to be

$$r = a\cosh\Omega t.$$

This is the **motion** of the particle along the wire. ■

Problem 17.10

One end of a straight rod is fixed at a point O on a smooth horizontal table and the rod is made to rotate around O with constant angular speed Ω. A uniform circular disk of radius a lies flat on the table and can slide freely upon it. The disc remains in contact with the rod at all times and is constrained to *roll* along the rod. Initially, the disk is at rest (relative to the rod) with its point of contact at a distance a from O. Find the displacement of the disk as a function of the time.

Solution

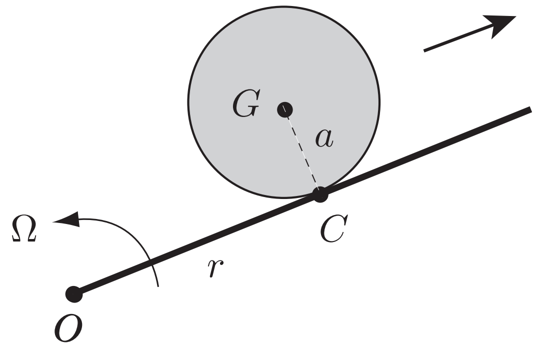

FIGURE 17.4 The disk rolls along the rotating rod.

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the axis $\{O, \boldsymbol{k}\}$, where the unit vector $\boldsymbol{k}$ is perpendicular to the plane of motion of the rod. Then, in the rotating frame $\mathcal{F}'$, the rod is at rest and the system is standard and conservative with **apparent potential energy** zero.

The **apparent kinetic energy** is

$$T = \tfrac{1}{2}M\dot{r}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}Ma^2\right)\left(\frac{\dot{r}}{a}\right)^2$$
$$= \tfrac{3}{4}M\dot{r}^2.$$

The **moment of inertia** of the system about the axis $\{O, \boldsymbol{k}\}$ is

$$I_{\{O,\boldsymbol{k}\}} = I_{\{G,\boldsymbol{k}\}} + M(OG)^2$$
$$= \tfrac{1}{2}Ma^2 + M\left(r^2 + a^2\right)$$
$$= Mr^2 + \tfrac{3}{2}Ma^2.$$

The **energy conservation principle** for uniformly rotating frames then implies that

$$\tfrac{3}{4}M\dot{r}^2 + 0 - \tfrac{1}{2}M\Omega^2\left(r^2 + \tfrac{3}{2}a^2\right) = E,$$

where E is a constant. The initial conditions $r = a$ and $\dot{r} = 0$ when $t = 0$ show that

$$E = -\tfrac{5}{4}M\Omega^2 a^2$$

and so the **energy conservation equation** becomes

$$\dot{r}^2 = \tfrac{2}{3}\Omega^2\left(r^2 - a^2\right).$$

On taking square roots and separating, we find that

$$\cosh^{-1}\left(\frac{r}{a}\right) = \pm\sqrt{\tfrac{2}{3}}\,\Omega t + C,$$

where C is a constant of integration. The initial condition $r = a$ when $t = 0$ shows that $C = 0$ and the solution for r is found to be

$$r = a\cosh\left(\sqrt{\tfrac{2}{3}}\,\Omega t\right).$$

This is the **displacement** of the disk at time t. ■

Problem 17.11

A horizontal turntable is made to rotate about a fixed vertical axis with constant angular speed Ω. A *hollow* uniform circular cylinder of mass M and radius a can *roll* on the turntable. Initially the cylinder is at rest (relative to the turntable), with its centre of mass on the rotation axis, when it is slightly disturbed. Find the speed of the cylinder when it has rolled a distance x on the turntable.

✱ Find also an expression (in terms of x) for the force that the turntable exerts on the cylinder.

Solution

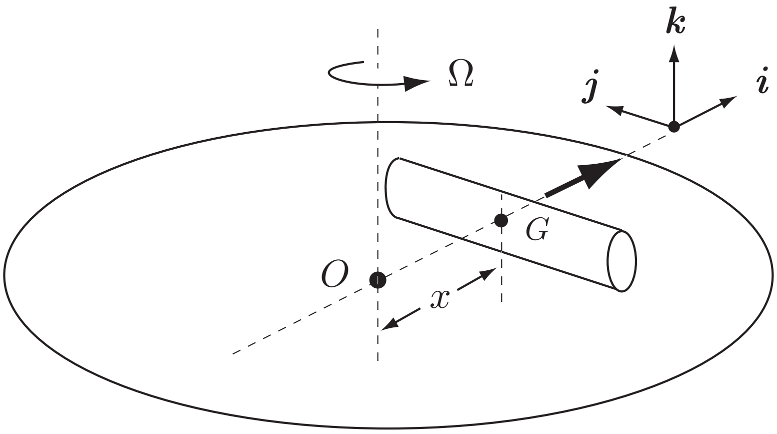

FIGURE 17.5 The hollow circular cylinder rolls on the rotating turntable.

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the axis $\{O, \boldsymbol{k}\}$, where the unit vector $\boldsymbol{k}$ is perpendicular to the plane of the turntable. Then, in the rotating frame $\mathcal{F}'$, the turntable is at rest and the system is standard and conservative with **apparent potential energy** zero.

The **apparent kinetic energy** is

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}\left(Ma^2\right)\left(\frac{\dot{x}}{a}\right)^2$$
$$= M\dot{x}^2.$$

The **moment of inertia** of the system about the axis $\{O, \boldsymbol{k}\}$ is

$$I_{\{O,\boldsymbol{k}\}} = I_{\{G,\boldsymbol{k}\}} + Mx^2.$$

[We could put in the value of $I_{\{G,\boldsymbol{k}\}}$ from the table in the Appendix, but there is no point. It is just a constant and will eventually cancel.]

The **energy conservation principle** for uniformly rotating frames then implies that

$$M\dot{x}^2 + 0 - \tfrac{1}{2}\Omega^2\left(I_{\{G,\boldsymbol{k}\}} + Mx^2\right) = E,$$

where E is a constant. The initial conditions $x = 0$ and $\dot{x} = 0$ when $t = 0$ show that

$$E = -\tfrac{1}{2}\Omega^2 I_{\{G,\boldsymbol{k}\}}$$

and so the **energy conservation equation** becomes

$$\dot{x}^2 = \tfrac{1}{2}\Omega^2 x^2.$$

This equation obviously has the equilibrium solution $x = 0$, but this is not what we are looking for. We are interested in the *non-zero* solution in which

$$\dot{x} = +\left(\frac{\Omega}{\sqrt{2}}\right)x.$$

In this solution, the **velocity** of the cylinder when it has rolled a distance x on the turntable is $\Omega x/\sqrt{2}$.

To find the reaction force $\boldsymbol{X}$ acting on the cylinder, we use the full equation for particle motion in rotating frames. This gives

$$\boldsymbol{X} - Mg\boldsymbol{k} = m\left[\frac{d\boldsymbol{v}}{dt} + \boldsymbol{0} + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})\right],$$

where $\boldsymbol{r} = x\boldsymbol{i}$, $\boldsymbol{v} = \dot{x}\boldsymbol{i}$, $\boldsymbol{\Omega} = \Omega\boldsymbol{k}$ and the unit vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are shown in Figure 17.5. Now, from the energy equation,

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= \ddot{x}\boldsymbol{i} = \left(\frac{\Omega}{\sqrt{2}}\right)\dot{x}\boldsymbol{i} = \left(\frac{\Omega}{\sqrt{2}}\right)\left(\frac{\Omega}{\sqrt{2}}\right)x\boldsymbol{i}\\ &= \tfrac{1}{2}\Omega^2 x\boldsymbol{i},\end{aligned}$$

and, on substituting in all of these values, we find that

$$\begin{aligned}\boldsymbol{X} &= Mg\boldsymbol{k} + M\left[\tfrac{1}{2}\Omega^2 x\boldsymbol{i} + \sqrt{2}\Omega^2\boldsymbol{j} - \Omega^2 x\boldsymbol{i}\right]\\ &= Mg\boldsymbol{k} + M\Omega^2\left[\sqrt{2}\boldsymbol{j} - \tfrac{1}{2}\boldsymbol{i}\right]x.\end{aligned}$$

This is the **force exerted** on the cylinder by the turntable. ■

Problem 17.12 *Newton's bucket*

A bucket half full of water is made to rotate with angular speed Ω about its axis of symmetry, which is vertical. Find, to within a constant, the pressure field in the fluid. By considering the isobars (surfaces of constant pressure) of this pressure field, find the shape of the free surface of the water.

What would the shape of the free surface be if the bucket were replaced by a cubical box?

Solution

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the same vertical axis as the bucket. Then, in the rotating frame $\mathcal{F}'$, the bucket is at rest.

Suppose that the water has come to rest relative to the bucket. The equation of 'hydrostatics' in the rotating frame $\mathcal{F}'$ is

$$\rho\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \boldsymbol{F} - \text{grad}\, p,$$

where ρ is the (constant) water density, p is the pressure field, $\boldsymbol{\Omega}$ $(=\Omega\boldsymbol{k})$ is the angular velocity of the bucket, and $\boldsymbol{F}$ is the body force (per unit volume) acting on the water.

In this problem, the body force is gravity so that

$$\boldsymbol{F} = -\rho g\boldsymbol{k}.$$

It follows that pressure field $p(\boldsymbol{r})$ must satisfy the equation

$$\begin{aligned}\text{grad}\, p &= -\rho g\boldsymbol{k} - \rho\Omega^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r})\\ &= -\rho g\boldsymbol{k} + \rho\Omega^2 R\,\widehat{\boldsymbol{R}},\end{aligned}$$

where R is the distance of the point $\boldsymbol{r}$ from the rotation axis and $\widehat{\boldsymbol{R}}$ is the unit vector pointing in the direction of increasing R. (In other words, R and $\widehat{\boldsymbol{R}}$ relate to the *cylindrical polar* coordinate system whose axis lies along the rotation axis of the bucket.) This equation for p can be integrated to give

$$p = \rho\left(\tfrac{1}{2}\Omega^2 R^2 - gz\right) + \text{constant}$$

This is the **pressure field** in the water, correct to within a constant. The surfaces of constant pressure (of which the free surface must be one) are therefore

$$z = \frac{\Omega^2 R^2}{2g} + \text{constant}$$

Each of these surfaces is a **paraboloid** (a parabola of revolution) whose axis lies along the rotation axis of the bucket. In particular then, the **free surface** of the water must be one of these paraboloids.

If the bucket is replaced by a **cubical box** (or any other container), the above solution still holds. The only difference is that the free surface will now terminate where it meets the sides of the *box* instead of where it met the bucket. ■

Problem 17.13

A sealed circular can of radius a is three-quarters full of water of density ρ, the remainder being air at pressure p_0. The can is taken into gravity free space and then rotated about its axis of symmetry with constant angular speed Ω. Where will the water be when it comes to rest relative to the can? Find the water pressure at the wall of the can.

Solution

Suppose the frame $\mathcal{F}$ is fixed and the frame $\mathcal{F}'$ is rotating with scalar angular velocity Ω about the symmetry axis of the can. Then, in the rotating frame $\mathcal{F}'$, the can is at rest.

Suppose that the water has come to rest relative to the can. The equation of 'hydrostatics' in the rotating frame $\mathcal{F}'$ is

$$\rho\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \boldsymbol{F} - \operatorname{grad} p,$$

where ρ is the (constant) water density, p is the pressure field, $\boldsymbol{\Omega}$ $(=\Omega\boldsymbol{k})$ is the angular velocity of the bucket, and $\boldsymbol{F}$ is the body force (per unit volume) acting on the water.

In this problem, there is no body force so that $\boldsymbol{F} = \boldsymbol{0}$. It follows that pressure field $p(\boldsymbol{r})$ must satisfy the equation

$$\begin{aligned}\operatorname{grad} p &= -\rho\Omega^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{r})\\ &= -\rho\Omega^2 R\,\widehat{\boldsymbol{R}},\end{aligned}$$

where R is the distance of the point $\boldsymbol{r}$ from the rotation axis and $\widehat{\boldsymbol{R}}$ is the unit vector pointing in the direction of increasing R. (In other words, R and $\widehat{\boldsymbol{R}}$ relate to the *cylindrical polar* coordinate system whose axis lies along the symmetry axis of the can.) This equation for p can be integrated to give

$$p = \tfrac{1}{2}\rho\Omega^2R^2 + \text{constant}$$

This is the **pressure field** in the water, correct to within a constant. The **surfaces of constant pressure** (of which the free surface must be one) are therefore

$$R = \text{constant},$$

a family of cylindrical surfaces whose axes lie along the axis of the can. In particular then, the free surface of the water must be one of these surfaces.

The only *stable* configuration of the system is the one in which the water occupies the region between the curved wall of the can and one of the above cylindrical surfaces. The actual one is determined by the fact that the water has *constant volume*, and a little geometry shows that the **free surface** of the water actually lies in the surface $R = \frac{1}{2}a$. At the free surface, the water pressure is known to be p_0, the same as that of the enclosed air. On applying the boundary condition $p = p_0$ when $R = \frac{1}{2}a$, we find that the **pressure field** in the water is

$$p = p_0 + \tfrac{1}{8}\rho\Omega^2 \left(4R^2 - a^2\right).$$

By substituting $R = a$ into this formula, we find that the **pressure at the can wall** is

$$p_0 + \tfrac{3}{8}\rho\Omega^2 a^2. \blacksquare$$

Chapter Eighteen

Tensor algebra and the inertia tensor

Problem 18.1

Show that the matrix

$$\mathbf{A} = \frac{1}{7}\begin{pmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{pmatrix}$$

is orthogonal. If $\mathbf{A}$ is the transformation matrix between the coordinate systems $\mathcal{C}$ and $\mathcal{C}'$, do $\mathcal{C}$ and $\mathcal{C}'$ have the same, or opposite, handedness?

Solution

$$\mathbf{A}\cdot\mathbf{A}^T = \frac{1}{49}\begin{pmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{pmatrix}\begin{pmatrix} 3 & -6 & 2 \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{pmatrix} = \frac{1}{49}\begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix $\mathbf{A}$ is **orthogonal**.

Also, $\det \mathbf{A}$ is given by

$$\det \mathbf{A} = \frac{1}{7^3}\begin{vmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{vmatrix}$$
$$= \frac{1}{7^3}\Big[3(-9-12) - 2(18-4) + 6(-36-6)\Big] = -\frac{343}{7^3}$$
$$= -1.$$

Since $\det \mathbf{A} = -1$, it follows that the transformation represented by $\mathbf{A}$ consists of a rotation followed by a reflection. Hence, the coordinate systems $\mathcal{C}$ and $\mathcal{C}'$ have **opposite handedness**.

Problem 18.2

Find the transformation matrix between the coordinate systems $\mathcal{C}$ and $\mathcal{C}'$ when $\mathcal{C}'$ is obtained

(i) by rotating $\mathcal{C}$ through an angle of 45° about the axis Ox_2,
(ii) by reflecting $\mathcal{C}$ in the plane $x_2 = 0$,
(iii) by rotating $\mathcal{C}$ through a right angle about the axis $\overrightarrow{OB}$, where B is the point with coordinates $(2, 2, 1)$,
(iv) by reflecting $\mathcal{C}$ in the plane $2x_1 - x_2 + 2x_3 = 0$.

In each case, find the new coordinates of the point D whose coordinates in $\mathcal{C}$ are $(3, -3, 0)$.

Solution

(i) The transformation matrix when $\mathcal{C}$ is rotated through an angle ψ about the axis Ox_2 is

$$\mathbf{A} = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Hence, when $\psi = 45°$, the **transformation matrix** is

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The **new coordinates** of the point D are the elements of the column vector

$$\mathbf{A} \cdot \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ -3\sqrt{2} \\ 3 \end{pmatrix}.$$

Hence, in the coordinate system $\mathcal{C}'$, D is the point $(3/\sqrt{2},\ -3,\ 3/\sqrt{2})$. ∎

(ii) The **transformation matrix** when $\mathcal{C}$ is reflected in the plane $x_2 = 0$ is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The **new coordinates** of the point D are the elements of the column vector

$$\mathbf{A}\cdot\begin{pmatrix}3\\-3\\0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}3\\-3\\0\end{pmatrix}$$
$$=\begin{pmatrix}3\\3\\0\end{pmatrix}.$$

Hence, in the coordinate system $\mathcal{C}'$, D is the point $(3, 3, 0)$, which is obvious anyway. ■

(iii) In $\mathcal{C}$, the line segment $\overrightarrow{OB}$ is represented by the vector $2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$. Hence $\boldsymbol{n}$, the unit vector in the direction $\overrightarrow{OB}$ is given by

$$\boldsymbol{n}=\frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{|2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}|}$$
$$=\tfrac{1}{3}(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}).$$

To find the required transformation matrix, we now substitute this value of $\boldsymbol{n}$ (and $\psi=90°$) into the general formula (18.10) on page 497. This gives

$$\mathbf{A}=\frac{1}{9}\begin{pmatrix}4&7&-4\\1&4&8\\8&-4&1\end{pmatrix}.$$

The **new coordinates** of the point D are the elements of the column vector

$$\mathbf{A}\cdot\begin{pmatrix}3\\-3\\0\end{pmatrix}=\frac{1}{9}\begin{pmatrix}4&7&-4\\1&4&8\\8&-4&1\end{pmatrix}\begin{pmatrix}3\\-3\\0\end{pmatrix}$$
$$=\begin{pmatrix}-1\\-1\\4\end{pmatrix}.$$

Hence, in the coordinate system $\mathcal{C}'$, D is the point $(-1, -1, 4)$. ■

(iv) The plane $2x_1 - x_2 + 2x_3 = 0$ can be written in the form $\boldsymbol{n}\cdot\boldsymbol{x}=0$, where the unit vector $\boldsymbol{n}$ is given by

$$\boldsymbol{n}=\frac{2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}}{|2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}|}$$
$$=\tfrac{1}{3}(2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}).$$

To find the required transformation matrix, we now substitute this value of $\boldsymbol{n}$ into the general formula (18.11) on page 498. This gives

$$\mathsf{A} = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}.$$

The **new coordinates** of the point D are the elements of the column vector

$$\mathsf{A}\cdot\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix}.$$

Hence, in the coordinate system $\mathcal{C}'$, D is the point $(-1, -1, -4)$. ■

Problem 18.3

Show that the matrix

$$\mathsf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

is orthogonal and has determinant $+1$. Find the column vectors $\mathbf{v}$ that satisfy the equation $\mathsf{A}\cdot\mathbf{v} = \mathbf{v}$. If A is the transformation matrix between the coordinate systems $\mathcal{C}$ and $\mathcal{C}'$, show that A represents a rotation of $\mathcal{C}$ about the axis $\overrightarrow{OE}$ where E is the point with coordinates $(1,\ 1, -1)$ in $\mathcal{C}$.

✱ Find the rotation angle.

Solution

$$\begin{aligned} \mathsf{A}\cdot\mathsf{A}^T &= \frac{1}{9}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & 1 \\ -1 & 2 & -2 \\ -2 & 1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence the matrix A is **orthogonal**. The **determinant** of A is given by

$$\begin{aligned} \det\mathsf{A} &= \frac{1}{3^3}\begin{vmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} \\ &= \frac{1}{27}\Big[2(4+2) + 1(4-1) - 2(-4-2)\Big] \\ &= +1. \end{aligned}$$

Since $\det\mathsf{A} = +1$, it follows that the transformation represented by A is a **rotation**.

In expanded form, the equation $\mathsf{A}\cdot\mathbf{v} = \mathbf{v}$ is

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

that is,

$$\begin{pmatrix} -1 & -1 & -2 \\ 2 & -1 & 1 \\ 1 & -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The last equation is just the sum of the first two, and the first two can be written in the form

$$\begin{aligned} v_1 + v_2 &= -2v_3, \\ 2v_1 - v_2 &= -v_3, \end{aligned}$$

The general solution of these equations is $v_1 = \lambda$, $v_2 = \lambda$, $v_3 = -\lambda$, where λ may take any value. Hence the **general solution** of the equation $\mathbf{A} \cdot \mathbf{v} = \mathbf{v}$ is

$$\mathbf{v} = \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

where λ may take any value.

It follows that points that have coordinates of the form $(\lambda, \lambda, -\lambda)$ in $\mathcal{C}$ have the *same coordinates* in $\mathcal{C}'$ and so must lie on the rotation axis of the rotation represented by $\mathbf{A}$. In particular, the **rotation axis** must pass through the point $E(1, 1, -1)$.

There are many ways to find the rotation angle. One way is to substitute the value $\boldsymbol{n} = (\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})/\sqrt{3}$ into the general formula (18.10) and pick out the values of $\cos\psi$ and $\sin\psi$. Alternatively, one may work from first principles, using the following homespun method:

Select a point F such that OE and OF are perpendicular. The point $(1, 0, 1)$ will do. Now find the coordinates of F in $\mathcal{C}'$. These are the elements of the column vector

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The rotation angle about $\overrightarrow{OE}$ must therefore be the same as the angle between the vectors $\boldsymbol{i} + \boldsymbol{k}$ and $\boldsymbol{j} + \boldsymbol{k}$, which is $\cos^{-1}\frac{1}{2}$. The angle is therefore $\pm\pi/3$. The correct *sign* can be determined by examining the sign of the triple scalar product $\big((\boldsymbol{i} + \boldsymbol{k})\times(\boldsymbol{j} + \boldsymbol{k})\big)\cdot(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$. It turns out that the **rotation angle** about the axis $\overrightarrow{OE}$ is $+\pi/3$. ■

Problem 18.4

Write out the transformation formula for a fifth order tensor. [The main difficulty is finding enough suffix names!]

Solution

As it says in the text on page 502, the tensor transformation formulae follow a pattern. In the definition of a vector there is only one summation and one appearance of a_{pq}; in the definition of a tensor of the second order, there are two summations and two appearances of a_{pq}, and so on. The suffices of the tensor on the left (in order) must be the same as the first suffix of each of the a_{pq} (in order), and the suffices of the tensor on the right (in order) must be the same as the second suffix of each of the a_{pq} (in order). By observing these rules, one can deduce the transformation formula for a tensor of *any* order. In particular, for a tensor of order five, the **transformation formula** is

$$t'_{ijklm} = \sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{t=1}^{3} a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, a_{mt}\, t_{pqrst} \quad \blacksquare$$

Problem 18.5

In the coordinate system $\mathcal{C}$, a certain second order tensor is represented by the matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the matrix representing the tensor in the coordinate system $\mathcal{C}'$, where $\mathcal{C}'$ is obtained

(i) by rotating $\mathcal{C}$ through an angle of 45° about the axis Ox_1,
(ii) by reflecting $\mathcal{C}$ in the plane $x_3 = 0$.

Solution

(i) The transformation matrix when $\mathcal{C}$ is rotated through an angle ψ about the axis Ox_1 is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}.$$

Hence, when $\psi = 45°$, the **transformation matrix** is

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $\mathbf{T}'$, the matrix representing the tensor in the coordinate system $\mathcal{C}'$, is given by

$$\begin{aligned} \mathbf{T}' &= \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 1 & 1 \\ 1 & \sqrt{2} & 0 \\ 1 & 0 & \sqrt{2} \end{pmatrix} \blacksquare \end{aligned}$$

(ii) The transformation matrix when $\mathcal{C}$ is reflected in the plane $x_3 = 0$ is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $\mathbf{T}'$, the matrix representing the tensor in the coordinate system $\mathcal{C}'$, is given by

$$\begin{aligned}\mathbf{T}' &= \mathbf{A}\cdot\mathbf{T}\cdot\mathbf{A}^T\\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 1\end{pmatrix} \blacksquare\end{aligned}$$

Problem 18.6

The quantities t_{ijk} and u_{ijkl} are third and fourth order tensors respectively. Decide if each of the following quantities is a tensor and, if it is, state its order:

(i) $t_{ijk}u_{lmnp}$ (ii) $t_{ijk}t_{lmn}$ (iii) $\sum_{j=1}^{3} t_{ijj}$

(iv) $\sum_{j=1}^{3} t_{jij}$ (v) $\sum_{i=1}^{3} t_{iii}$ (vi) $\sum_{k=1}^{3} t_{ijk}u_{klmn}$

(vii) $\sum_{i=1}^{3}\sum_{j=1}^{3} u_{iijj}$ (viii) $\sum_{k=1}^{3} u_{klmn}$ (ix) $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} t_{ijk}t_{ijk}$.

Solution

(i) $t_{ijk}u_{lmnp}$ is the *outer product* of the tensors t_{ijk} and u_{ijkl} and is therefore a **seventh order tensor**.

(ii) $t_{ijk}t_{lmn}$ is the *outer product* of the tensor t_{ijk} with itself. It is therefore a **sixth order tensor**.

(iii) $\sum_{j=1}^{3} t_{ijj}$ is the tensor t_{ijk} with the suffix pair $\{j,k\}$ *contracted*. It is therefore a **first order tensor** (a vector).

(iv) $\sum_{j=1}^{3} t_{jij}$ is the tensor t_{ijk} with the suffix pair $\{i,k\}$ *contracted* (and the suffix j renamed as i). It is therefore a **first order tensor** (a vector).

(v) $\sum_{i=1}^{3} t_{iii}$ is *not* a contraction of the tensor t_{ijk} since *three* suffices are set equal and summed. It is therefore **not a tensor**. Mathematicians might like to provide an explicit example of a third order tensor (one is enough) for which $\sum_{i=1}^{3} t_{iii}$ is not preserved under coordinate transformation.

(vi) $\sum_{k=1}^{3} t_{ijk}u_{klmn}$ is the *outer product* of the tensors t_{ijk} and u_{lmnp} with the suffix pair $\{k,l\}$ *contracted* (and the suffices m, n, p renamed as l, m, n). It is therefore a **fifth order tensor**.

(vii) $\sum_{i=1}^{3}\sum_{j=1}^{3} u_{iijj}$ is the tensor u_{ijkl} with the suffix pairs $\{i,j\}$ and $\{k,l\}$ *contracted*. It is therefore a **zero order tensor** (a scalar).

(viii) $\sum_{k=1}^{3} u_{klmn}$ is *not* a contraction of the tensor u_{klmn} since *no* suffices are set equal. It is therefore **not a tensor**.

(ix) $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} t_{ijk}t_{ijk}$ is the tensor $t_{ijk}t_{lmn}$ with the suffix pairs $\{i, l\}$ and $\{j, m\}$ and $\{k, n\}$ all *contracted*. It is therefore a **zero order tensor** (a scalar). Thus the sum of the squares of the elements of the tensor t_{ijk} is an invariant.

Problem 18.7

Show that the sum of the squares of the elements of a tensor is an invariant. [First and second order tensors will suffice.]

Solution

Suppose that the three component quantity v_i is a vector. Then the sum of the squares of its elements is

$$v_1^2 + v_2^2 + v_3^2 = \sum_{i=1}^{3} v_i v_i$$

which is the second order tensor $v_i v_j$ with the suffix pair $\{i, j\}$ contracted. It is therefore an **invariant**.

Similarly, suppose that the nine component quantity t_{ij} is a second order tensor. Then the sum of the squares of its elements is

$$\sum_{i=1}^{3} \sum_{j=1}^{3} t_{ij} t_{ij}$$

which is the fourth order tensor $t_{ij} t_{kl}$ with the suffix pairs $\{i, k\}$ and $\{j, l\}$ contracted. It is therefore an **invariant**.

The corresponding result for third order tensors is part (ix) of question 18.6 and a similar argument applies to tensors of any order. ■

Problem 18.8

If the matrix $\mathbf{T}$ represents a second order tensor, show that det $\mathbf{T}$ is an invariant. [We have now found three invariant functions of a second order tensor: the sum of the diagonal elements, the sum of the squares of all the elements, and the determinant.]

Solution

If $\mathbf{T}$ represents a second order tensor, then it satisfies the transformation formula

$$\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T,$$

where $\mathbf{A}$ is the transformation matrix. Then

$$\begin{aligned}\det \mathbf{T}' &= \det \left(\mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T\right)\\ &= \det \mathbf{A} \times \det \mathbf{T} \times \det \mathbf{A}^T\\ &= \det \mathbf{A} \times \det \mathbf{T} \times \det \mathbf{A}\\ &= \det \mathbf{T},\end{aligned}$$

since $\det \mathbf{A} = 1$ when $\mathbf{A}$ is a rotation matrix. Hence det $\mathbf{T}$ is preserved under coordinate transformation and is therefore an **invariant**. ■

Problem 18.9

In crystalline materials, the ordinary elastic moduli are replaced by c_{ijkl}, a fourth order tensor with eighty one elements. It appears that the most general material has eighty one elastic moduli, but this number is reduced because c_{ijkl} has the following symmetries:

$$\text{(i) } c_{jikl} = c_{ijkl} \qquad \text{(ii) } c_{ijlk} = c_{ijkl} \qquad \text{(iii) } c_{klij} = c_{ijkl}$$

How many elastic moduli does the most general material actually have?

Solution

The symmetry (i) means that, for each choice of the suffix pair $\{k, l\}$, there are *six* independent choices for the suffix pair $\{i, j\}$ instead of nine. Likewise, the symmetry (ii) means that, for each choice of the suffix pair $\{i, j\}$, there are *six* independent choices for the suffix pair $\{k, l\}$ instead of nine. This reduces the number of independent moduli from eighty one to thirty six. These thirty six elements can be set out in a convenient 6×6 array. For example, we can 'number' the rows and columns of this array by using the labels $\{1, 1\}$, $\{2, 2\}$, $\{3, 3\}$, $\{2, 3\}$, $\{3, 1\}$, $\{1, 2\}$. The symmetry (iii) then implies that the elements in this 6×6 array are symmetric about the leading diagonal. On counting up the number of elements on or above this diagonal, we find that the number of independent **elastic moduli** is actually twenty one. [Further symmetries of the crystal lead to further reductions in the number of elastic constants; an isotropic material has only two!]

Triclinic crystals have the full twenty one elastic constants. Such crystals exhibit the least symmetry of all crystal systems. Their axes are unequal and do not intersect at right angles anywhere. **Brazilian axinite** is an example of a triclinic crystal. ■

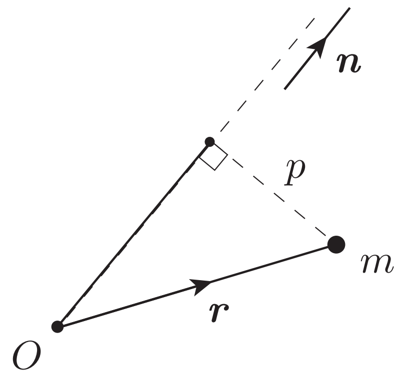

FIGURE 18.1 A typical particle of the body has mass m, position vector $\boldsymbol{r}$, and is distance p from the axis $\{O, \boldsymbol{n}\}$.

Problem 18.10

Show that $I_{\{O,\boldsymbol{n}\}}$, the moment of inertia of a body about an axis through O parallel to the unit vector $\boldsymbol{n}$, is given by

$$I_{\{O,\boldsymbol{n}\}} = \mathsf{n}^T \cdot \mathtt{I}_O \cdot \mathsf{n}$$

where $\mathtt{I}_O$ is the matrix representing the inertia tensor of the body at O (in some coordinate system), and n is the column vector that contains the components of $\boldsymbol{n}$ (in the same coordinate system).

Find the moment of inertia of a uniform rectangular plate with sides $2a$ and $2b$ about a diagonal.

Solution

This formula can be proved by comparing the scalar and tensor expressions for the angular momentum of a rigid body about an axis. However, since the result is entirely geometrical (and has no direct connection with angular momentum), it is perhaps preferable to give a direct geometrical proof.

Figure 18.1 shows a typical particle of the body with mass m, position vector $\boldsymbol{r}$,

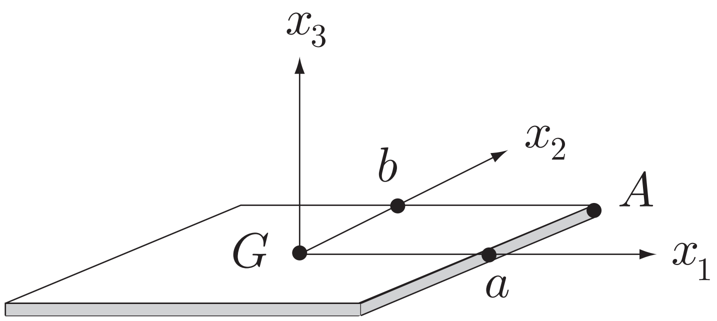

FIGURE 18.2 Principal axes for the reactangular plate at the point G.

and distance p from the axis $\{O, \boldsymbol{n}\}$. Then

$$\begin{aligned}
p^2 &= \boldsymbol{r}\cdot\boldsymbol{r} - (\boldsymbol{r}\cdot\boldsymbol{n})^2\\
&= \left(x_1^2+x_2^2+x_3^2\right)^2 - \left(n_1x_1+n_2x_2+n_3x_3\right)^2\\
&= \left(1-n_1^2\right)x_1^2 + \left(1-n_2^2\right)x_2^2 + \left(1-n_3^2\right)x_3^2\\
&\qquad\qquad -2n_1n_2x_1x_2 - 2n_1n_3x_1x_3 - 2n_2n_3x_2x_3\\
&= \left(n_2^2+n_3^2\right)x_1^2 + \left(n_1^2+n_3^2\right)x_2^2 + \left(n_1^2+n_2^2\right)x_3^2\\
&\qquad\qquad -2n_1n_2x_1x_2 - 2n_1n_3x_1x_3 - 2n_2n_3x_2x_3\\
&= \left(x_2^2+x_3^2\right)n_1^2 + \left(x_1^2+x_3^2\right)n_2^2 + \left(x_1^2+x_2^2\right)n_3^2\\
&\qquad\qquad -2n_1n_2x_1x_2 - 2n_1n_3x_1x_3 - 2n_2n_3x_2x_3\\
&= \left(n_1\ n_2\ n_3\right)\begin{pmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3\\ -x_1x_2 & x_1^2+x_3^2 & -x_2x_3\\ -x_1x_3 & -x_2x_3 & x_1^2+x_2^2\end{pmatrix}\begin{pmatrix} n_1\\ n_2\\ n_3\end{pmatrix}.
\end{aligned}$$

On multiplying this equality by m and summing over all the particles, we obtain

$$\begin{aligned}
I_{\{O,\boldsymbol{n}\}} = \sum mp^2 &= \left(n_1\ n_2\ n_3\right)\begin{pmatrix} I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33}\end{pmatrix}\begin{pmatrix} n_1\\ n_2\\ n_3\end{pmatrix}\\
&= \mathbf{n}^T\cdot \mathbb{I}_O\cdot\mathbf{n}
\end{aligned}$$

which is the required result.

Figure 18.2 shows the rectangular plate and the principal axes at the point G. Relative to these axes, the diagonal GA points in the direction of the unit vector $\boldsymbol{n}$,

where

$$\boldsymbol{n} = \cos\alpha\, \boldsymbol{e}_1 + \sin\alpha\, \boldsymbol{e}_2,$$

and α is the angle between GA and the x_1-axis. Then

$$\mathbf{n} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}$$

and I_{GA}, the moment of inertia of the plate about the axis GA, is given by

$$\begin{aligned} I_{GA} &= \mathbf{n}^T \cdot \mathbb{I}_G \cdot \mathbf{n} \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3}Mb^2 & 0 & 0 \\ 0 & \frac{1}{3}Ma^2 & 0 \\ 0 & 0 & \frac{1}{3}M(a^2+b^2) \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \\ &= \frac{M}{3(a^2+b^2)} \begin{pmatrix} a & b & 0 \end{pmatrix} \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \\ &= \frac{2Ma^2b^2}{3(a^2+b^2)}. \blacksquare \end{aligned}$$

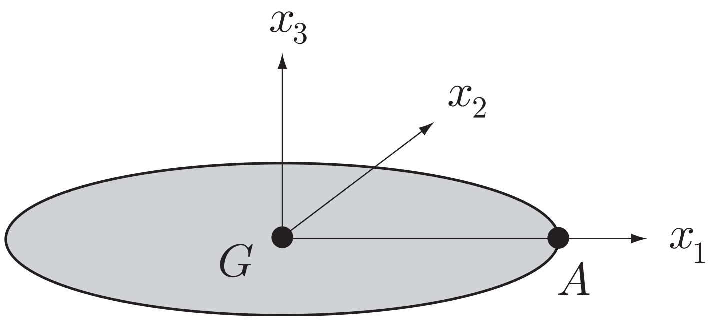

FIGURE 18.3 Principal axes for the circular disk at the point G.

Problem 18.11

Find the principal moments of inertia of a uniform circular disk of mass M and radius a (i) at its centre of mass, and (ii) at a point on the edge of the disk.

Solution

(i) The axes shown in Figure 18.3 are a set of **principal axes** of the disk at G. This follows from the reflective symmetry of the disk in each of the three coordinate planes. Also, since the disk is a lamina lying in the plane $x_3 = 0$, the **perpendicular axes** theorem shows that

$$I_{\{G,\boldsymbol{e}_3\}} = I_{\{G,\boldsymbol{e}_1\}} + I_{\{O,\boldsymbol{e}_2\}},$$

and the rotational symmetry of the disk about the axis $\{G, \boldsymbol{e}_3\}$ implies that $I_{\{G,\boldsymbol{e}_1\}} = I_{\{G,\boldsymbol{e}_2\}}$. From the **table** of moments of inertia on page 570, $I_{\{G,\boldsymbol{e}_3\}} = \frac{1}{2}Ma^2$, and so the **principal moments** of inertia of the disk at G are

$$I_{\{G,\boldsymbol{e}_1\}} = \tfrac{1}{4}Ma^2, \qquad I_{\{G,\boldsymbol{e}_2\}} = \tfrac{1}{4}Ma^2, \qquad I_{\{G,\boldsymbol{e}_3\}} = \tfrac{1}{2}Ma^2 \ \blacksquare$$

(ii) The set of parallel axes $A\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ are **principal axes** of the disk at A. This follows from the reflective symmetry of the disk in each of the *two* coordinate planes $x_2 = 0$ and $x_3 = 0$. [Why is two enough?] The corresponding principal moments can be found by using the **parallel axes** theorem. The **principal moments** of inertia of the disk at A are therefore

$$I_{\{A,\boldsymbol{e}_1\}} = \tfrac{1}{4}Ma^2, \qquad I_{\{A,\boldsymbol{e}_2\}} = \tfrac{5}{4}Ma^2, \qquad I_{\{A,\boldsymbol{e}_3\}} = \tfrac{3}{2}Ma^2 \ \blacksquare$$

Problem 18.12

A uniform circular disk has mass M and radius a. A spinning top is made by fitting the disk with a light spindle AB which passes through the disk and is fixed along its axis of symmetry. The distance of the end A from the disk is equal to the disk radius a. Find the principal moments of inertia of the top at the end A of the spindle.

Solution

Let $Gx_1x_2x_3$ be the set of axes shown in Figure 18.3. Then the set of parallel axes $Ax_1x_2x_3$ are **principal axes** of the top at the tip A. This follows from the rotational symmetry of the top about the axis Ax_3. The corresponding principal moments can be found from those at G by using the **parallel axes** theorem. The **principal moments** of inertia of the top at A are therefore

$$I_{\{A,\boldsymbol{e}_1\}} = \tfrac{5}{4}Ma^2, \qquad I_{\{A,\boldsymbol{e}_2\}} = \tfrac{5}{4}Ma^2, \qquad I_{\{A,\boldsymbol{e}_3\}} = \tfrac{1}{2}Ma^2 \ \blacksquare$$

Problem 18.13

A uniform hemisphere has mass M and radius a. A spinning top is made by fitting the hemisphere with a light spindle AB which passes through the hemisphere and is fixed along its axis of symmetry with the curved surface of the hemisphere facing away from the end A. The distance of A from the point where the spindle enters the flat surface is equal to the radius a of the hemisphere. Find the principal moments of inertia of the top at the end A of the spindle.

Solution

Let $Cx_1x_2x_3$ be a set of axes like those shown in Figure 18.3, where C is the centre of the circular flat face of the hemisphere and Cx_3points along the axis of rotational symmetry of the top. Then the set of parallel axes $Gx_1x_2x_3$ are **principal axes** of the top at the centre of mass G. This follows from the rotational symmetry of the top about the axis of the spindle. The corresponding principal moments can be found from those at C by using the **parallel axes** theorem. The principal moments of inertia of the top at G are therefore

$$I_{\{G,\boldsymbol{e}_1\}} = \tfrac{2}{5}Ma^2 - Md^2, \qquad I_{\{G,\boldsymbol{e}_2\}} = \tfrac{2}{5}Ma^2 - Md^2, \qquad I_{\{G,\boldsymbol{e}_3\}} = \tfrac{2}{5}Ma^2,$$

where d is the distance GC, which was found in Example A.2 to be $\frac{3}{8}a$. It follows that the **principal moments** of the top at G are

$$I_{\{G,\boldsymbol{e}_1\}} = \tfrac{83}{120}Ma^2, \qquad I_{\{G,\boldsymbol{e}_2\}} = \tfrac{83}{120}Ma^2, \qquad I_{\{G,\boldsymbol{e}_3\}} = \tfrac{2}{5}Ma^2.$$

In a similar way, the second set of parallel axes $Ax_1x_2x_3$ are **principal axes** of the top at the tip A. The corresponding principal moments can be found from those at G by a second application of the **parallel axes** theorem. The **principal moments** of inertia of the top at the tip A are therefore

$$I_{\{A,\boldsymbol{e}_1\}} = I_{\{G,\boldsymbol{e}_1\}} + M(d+a)^2 = \tfrac{83}{120}Ma^2 + \left(\tfrac{11}{8}\right)^2 Ma^2 = \tfrac{43}{20}Ma^2,$$
$$I_{\{A,\boldsymbol{e}_2\}} = I_{\{G,\boldsymbol{e}_2\}} + M(d+a)^2 = \tfrac{83}{120}Ma^2 + \left(\tfrac{11}{8}\right)^2 Ma^2 = \tfrac{43}{20}Ma^2,$$
$$I_{\{G,\boldsymbol{e}_2\}} = \tfrac{2}{5}Ma^2 \;\blacksquare$$

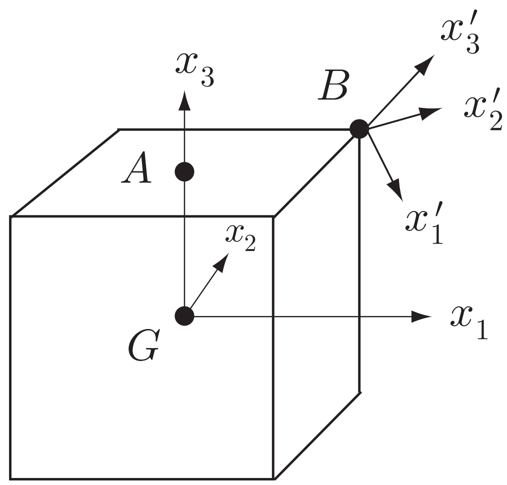

FIGURE 18.4 The cube with principal axes at $Gx_1x_2x_3$ at G and $Bx_1'x_2'x_3'$ at B.

Problem 18.14

Find the principal moments of inertia of a uniform cube of mass M and side $2a$ (i) at its centre of mass, (ii) at the centre of a face, and (iii) at a corner point.

Find the moment of inertia of the cube (i) about a space diagonal, (ii) about a face diagonal, and (iii) about an edge.

Solution

(i) Consider the coordinate system $Gx_1x_2x_3$ shown in Figure 18.4. Since the cube has *reflective* symmetry in each of the three coordinate planes, this is a set of principal axes at G. The moment of inertia of the cube about the axis Gx_1 is the same as that of a uniform plate of mass M occupying the region $x_1 = 0$, $-a \le x_2, x_3 \le a$, which, from Example A.7 is $\frac{2}{3}Ma^2$. The other principal moments have the same value. Hence the **principal moments** of the cube at G are $\frac{2}{3}Ma^2$, $\frac{2}{3}Ma^2$, $\frac{2}{3}Ma^2$. ■

(ii) Now consider a set of parallel axes at the point A. Since the cube has *reflective* symmetry in each of the coordinate planes $x_1 = 0$, $x_2 = 0$, this is a set of principal axes at A. [Why are *two* reflective symmetries enough?] The corresponding principal moments can be found from those at G by using the **parallel axes** theorem. Hence the **principal moments** of the cube at the point A are $\frac{5}{3}Ma^2$, $\frac{5}{3}Ma^2$, $\frac{2}{3}Ma^2$. ■

(iii) Now consider the corner point B. A set of parallel axes at B is *not* a principal set since there are now no reflective symmetries. However, the cube has a *rotational* symmetry (of order three) about the space diagonal GB, which is therefore an axis of dynamical symmetry. Hence GB is a principal axis, and the other principal axes at B can be any axes that form an orthogonal set. In particular, the axes $Bx_1'x_2'x_3'$ shown in Figure 18.4 are a set of principal axes at B. It looks tough to find the corresponding principal moments, but the situation is saved by the fact that the cube has **dynamical spherical symmetry** at G. It follows that the moment of inertia of the cube about *any* axis through G is $\frac{2}{3}Ma^2$. Hence the principal moments at B can be found from those at G by using the **parallel axes** theorem. The **principal moments** of the cube at B are therefore $\frac{11}{3}Ma^2$, $\frac{11}{3}Ma^2$, $\frac{2}{3}Ma^2$. ■

The moment of inertia of the cube about a space diagonal is known to be $\frac{2}{3}Ma^2$ and the others can be found from those at G by using the **parallel axes** theorem. For the face diagonal, the moment is $\frac{5}{3}Ma^2$, and, for the edge, the moment is $\frac{8}{3}Ma^2$. ■

Problem 18 . 15

A uniform rectangular block has mass M and sides $2a$, $2b$ and $2c$. Find the principal moments of inertia of the block (i) at its centre of mass, (ii) at the centre of a face of area $4ab$. Find the moment of inertia of the block (i) about a space diagonal, (ii) about a diagonal of a face of area $4ab$.

Solution

(i) Consider the coordinate system $Gx_1x_2x_3$ shown in Figure 18.4, where the body is now considered to be a rectangular block. Since the block has *reflective* symmetry in each of the three coordinate planes, this is a set of principal axes at G. The moment of inertia of the block about the axis Gx_1 is the same as that of a uniform plate of mass M occupying the region $x_1 = 0$, $-b \le x_2 \le b$, $-c \le x_3 \le c$, which, from Example A.7 is $\frac{1}{3}M(b^2 + c^2)$. The other principal moments are found in a similar way. Hence the **principal moments** of the block at the point G are $\frac{1}{3}M(b^2 + c^2)$, $\frac{1}{3}M(a^2 + c^2)$, $\frac{1}{3}M(a^2 + b^2)$. ■

(ii) Now consider a set of parallel axes $Ax_1x_2x_3$ at the point A. Since the block has *reflective* symmetry in each of the coordinate planes $x_1 = 0$, $x_2 = 0$, this is a set of principal axes at A. [Why are *two* reflective symmetries enough?] The corresponding principal moments can be found from those at G by using the **parallel axes** theorem. Hence the **principal moments** of the block at the point A are $\frac{1}{3}M(b^2 + 4c^2)$, $\frac{1}{3}M(a^2 + 4c^2)$, $\frac{1}{3}M(a^2 + b^2)$. ■

There seems to be no simple way to find the principal axes and moments at a corner point of the block.

(i) To find the moment of inertia of the block about the space diagonal GB, we can use the known principal moments at G, together with the formula

$$I_{\{G,\boldsymbol{n}\}} = \mathsf{n}^T \cdot \mathbb{I}_G \cdot \mathsf{n}.$$

In the present application, the unit vector $\boldsymbol{n}$ is

$$\boldsymbol{n} = \frac{a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}}{(a^2 + b^2 + c^2)^{1/2}}.$$

Hence

$$
\begin{aligned}
I_{GB} &= \mathbf{n}^T \cdot \mathbb{I}_G \cdot \mathbf{n} \\
&= \frac{M}{3(a^2+b^2+c^2)} \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= \frac{2M(a^2b^2+a^2c^2+b^2c^2)}{3(a^2+b^2+c^2)} \blacksquare
\end{aligned}
$$

(ii) To find the moment of inertia of the block about the face diagonal AB, we can use the known principal moments at A, together with the formula

$$I_{\{A,\boldsymbol{n}\}} = \mathbf{n}^T \cdot \mathbb{I}_A \cdot \mathbf{n}.$$

In the present application, the unit vector $\boldsymbol{n}$ is

$$\boldsymbol{n} = \frac{a\boldsymbol{i} + b\boldsymbol{j}}{(a^2+b^2)^{1/2}}.$$

Hence

$$
\begin{aligned}
I_{AB} &= \mathbf{n}^T \cdot \mathbb{I}_A \cdot \mathbf{n} \\
&= \frac{M}{3(a^2+b^2)} \begin{pmatrix} a & b & 0 \end{pmatrix} \begin{pmatrix} b^2+4c^2 & 0 & 0 \\ 0 & a^2+4c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \\
&= \frac{2M(a^2b^2+2a^2c^2+2b^2c^2)}{3(a^2+b^2)} \blacksquare
\end{aligned}
$$

Problem 18.16

Find the principal moments of inertia of a uniform cylinder of mass M, radius a and length $2b$ at its centre of mass G. Is it possible for the cylinder to have dynamical *spherical* symmetry about G?

Solution

Since the cylinder has rotational symmetry (of infinite order) about its axis, this must be an axis of **dynamical axial symmetry**. Hence, any orthogonal coordinate system $Ax_1x_2x_3$ in which A lies on the axis and Ax_3 points along the axis is a set of **principal axes** at A. In particular, this is true at G, the centre of mass. The corresponding **principal moments** at G are given in the table in the Appendix. They are

$$\tfrac{1}{4}Ma^2 + \tfrac{1}{3}Mb^2, \qquad \tfrac{1}{4}Ma^2 + \tfrac{1}{3}Mb^2, \qquad \tfrac{1}{2}Ma^2.$$

The cylinder will have **dynamical spherical symmetry** at G if all the principal moments at G are equal. This will be true if

$$\tfrac{1}{4}Ma^2 + \tfrac{1}{3}Mb^2 = \tfrac{1}{2}Ma^2,$$

that is, if

$$b = \frac{\sqrt{3}}{2}\,a \;\blacksquare$$

Problem 18.17

Determine the dynamical symmetry (if any) of each the following bodies about their centres of mass:

(i) a frisbee,
(ii) a piece of window glass having the shape of an isosceles triangle,
(iii) a two bladed aircraft propellor,
(iv) a three-bladed ship propellor,
(v) an Allen screw (ignore the thread),
(vi) eight particles of equal mass forming a rigid cubical structure,
(vii) a cross-handled wheel nut wrench,
(viii) the great pyramid of Giza,
(ix) a molecule of carbon tetrachloride.

Solution

(i) The frisbee has one rotational symmetry (of infinite order), and so has **dynamical axial symmetry** at G.

(ii) The window glass has a rotational symmetry (of order two), but this is not enough to give rise to any dynamical symmetry.

(iii) The two bladed aircraft propellor has a rotational symmetry of order two, but this is not enough to give rise to any dynamical symmetry.

(iv) The three-bladed ship propellor has one rotational symmetry (of order three) and so has **dynamical axial symmetry** at G.

(v) The Allen screw has one rotational symmetry (of order six) and so has **dynamical axial symmetry** at G.

(vi) The cubical structure has three different rotational symmetries at G; each symmetry is of order four. The structure therefore has **dynamical spherical symmetry** at G.

(vii) The cross-handled wheel nut wrench has one rotational symmetry (of order four) and so has **dynamical axial symmetry** at G.

(viii) The great pyramid of Giza has one rotational symmetry (of order four) and so has **dynamical axial symmetry** at G.

(ix) The molecule of carbon tetrachloride has the form of a regular tetrahedron with the four chlorine atoms at the vertices and the carbon atom at the centre G. [Surely you knew that!] The molecule thus has four different rotational symmetries at G; each symmetry is of order three. The molecule therefore has **dynamical spherical symmetry** at G.

Problem 18.18 ✻

A uniform rectangular plate has mass M and sides $2a$ and $4a$. Find the principal axes and principal moments of inertia at a *corner* point of the plate. [Make use of the formula for $\mathtt{I}_C$ obtained in Example 18.6, with $b = 2a$.]

Solution

In Example 18.6, we found that the inertia tensor of a rectangular plate at a corner point C is

$$\mathtt{I}_C = \frac{1}{3}M \begin{pmatrix} 4b^2 & -3ab & 0 \\ -3ab & 4a^2 & 0 \\ 0 & 0 & 4(a^2+b^2) \end{pmatrix},$$

where the axes $Cx_1x_2x_3$ are those shown in Figure 18.3 (right). When $b = 2a$, this expression becomes

$$\mathtt{I}_C = \tfrac{2}{3}Ma^2 \begin{pmatrix} 8 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix},$$

The object is to move to a new set of coordinates in which the inertia tensor is **diagonal**. The standard method provided by linear algebra is to find the **eigenvalues** and **eigenvectors** of the matrix $\mathtt{I}_C$.

It is convenient to drop the constant multiplier $\frac{2}{3}Ma^2$ for the time being. Let

$$\mathbf{J} = \begin{pmatrix} 8 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

By definition, the eigenvalues λ and eigenvectors $\mathbf{v}$ of $\mathbf{J}$ satisfy the equation

$$\mathbf{J} \cdot \mathbf{v} = \lambda \mathbf{v},$$

which is equivalent to the homogeneous system of linear equations

$$(\mathbf{J} - \lambda \mathbf{1}) \cdot \mathbf{v} = \mathbf{0}.$$

Written out in full, this becomes

$$\begin{pmatrix} 8-\lambda & -3 & 0 \\ -3 & 2-\lambda & 0 \\ 0 & 0 & 10-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For this system of equations to have a *non-trivial* solution for $\mathbf{v}$, the determinant of the matrix must be zero, that is,

$$\begin{vmatrix} 8-\lambda & -3 & 0 \\ -3 & 2-\lambda & 0 \\ 0 & 0 & 10-\lambda \end{vmatrix} = 0.$$

On expanding the determinant, this equation becomes

$$(\lambda - 10)\left(\lambda^2 - 10\lambda + 7\right) = 0,$$

from which it follows that the **eigenvalues** of the matrix $\mathbf{J}$ are

$$\lambda_1 = 5 + 3\sqrt{2}, \qquad \lambda_2 = 5 - 3\sqrt{2}, \qquad \lambda_3 = 10.$$

We must now find the **eigenvector** corresponding to each eigenvalue.

Eigenvalue λ_1 When $\lambda = 5 + 3\sqrt{2}$, the system of equations for $\mathbf{v}$ becomes

$$\begin{pmatrix} \sqrt{2}-1 & 1 & 0 \\ 1 & \sqrt{2}+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which has the general solution

$$v_1 = k, \qquad v_2 = -k\left(\sqrt{2}-1\right), \qquad v_3 = 0,$$

where k can take any value. In particular, the column vector

$$\mathbf{v}_1 = \alpha \begin{pmatrix} 1 \\ -(\sqrt{2}-1) \\ 0 \end{pmatrix},$$

where $\alpha = \left(4 - 2\sqrt{2}\right)^{-1/2}$, is a **normalised eigenvector** of the matrix $\mathbf{J}$ corresponding to the eigenvalue $\lambda = 5 + 3\sqrt{2}$.

Eigenvalue λ_2 By proceeding in a similar way, we find that

$$\mathbf{v}_2 = \alpha \begin{pmatrix} \sqrt{2}-1 \\ 1 \\ 0 \end{pmatrix}$$

is a **normalised eigenvector** of the matrix $\mathbf{J}$ corresponding to the eigenvalue $\lambda = 5 - 3\sqrt{2}$.

Eigenvalue λ_3 This time, we find that

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is a **normalised eigenvector** of the matrix $\mathbf{J}$ corresponding to the eigenvalue $\lambda = 10$.

Let $\mathbf{V}$ be the matrix whose columns are the normalised eigenvectors of $\mathbf{J}$, that is,

$$\mathbf{V} = \big(\mathbf{v}_1\,|\,\mathbf{v}_1\,|\,\mathbf{v}_3\big).$$

Then linear algebra theory tells us that

$$\mathbf{V}^T \cdot \mathbf{J} \cdot \mathbf{V} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

which is a **diagonal matrix**. If we now compare this formula with the tensor transformation formula (18.17) on page 503, we see that we have achieved our object of diagonalising $\mathtt{I}_C$ and that the transformation matrix $\mathbf{A}$ that does the job is $\mathbf{V}^T$. Hence, the required **transformation matrix** is

$$\mathbf{A} = \begin{pmatrix} \alpha & -\alpha(\sqrt{2}-1) & 0 \\ \alpha(\sqrt{2}-1) & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we now compare this transformation matrix with that given by equation (18.8) on page 495, we see that $\mathbf{A}$ represents a **rotation** about the axis Cx_3 through a negative acute angle θ, where

$$\theta = -\tan^{-1}(\sqrt{2}-1) = -\frac{\pi}{8}.$$

This rotated coordinate system is a set of **principal axes** for the plate at C. The corresponding **principal moments** are

$$\tfrac{2}{3}(5+3\sqrt{2})Ma^2, \qquad \tfrac{2}{3}(5-3\sqrt{2})Ma^2, \qquad \tfrac{20}{3}Ma^2 \blacksquare$$

Chapter Nineteen

Problems in rigid body dynamics

Problem 19.1 *Ball rolling on a slope*

A uniform ball can roll or skid on a rough plane inclined at an angle β to the horizontal. Show that, in *any* motion of the ball, the component of w perpendicular to the plane is conserved. If the ball *rolls* on the plane, show that the path of the ball must be a parabola.

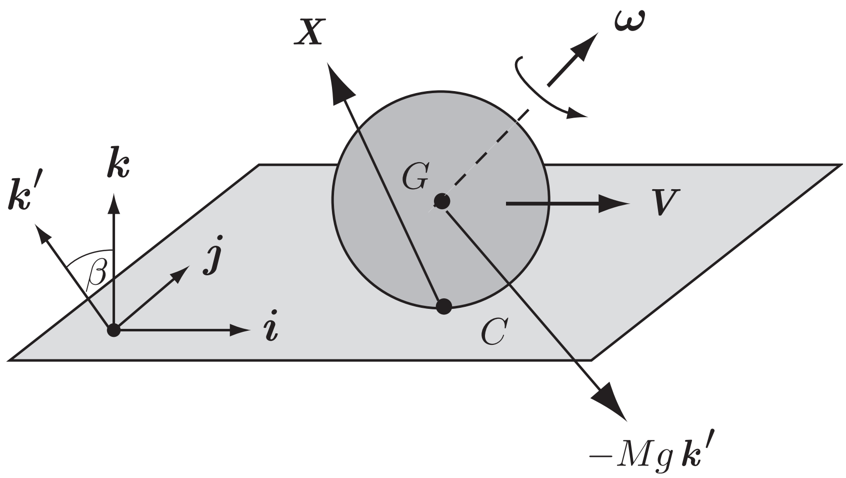

FIGURE 19.1 A ball of mass M and radius a rolls and skids on a rough plane inclined at angle β to the horizontal.

Solution

The plane and the ball are shown in Figure 19.1; the plane appears to be horizontal, but observe the direction of gravity! The vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are a standard basis set with $\boldsymbol{k}$ perpendicular to the plane, $\boldsymbol{j}$ horizontal, and $\boldsymbol{i}$ pointing down the direction of steepest slope. The unit vector $\boldsymbol{k}'$ points vertically upwards.

The equations of motion for the ball are as follows. The equation for the **translational motion** of G is

$$M\dot{\boldsymbol{V}} = \boldsymbol{X} - Mg\boldsymbol{k}', \tag{1}$$

while the equation for the **rotational motion** relative to G is

$$\begin{aligned}\left(\nu Ma^2\right)\dot{\mathrm{w}} &= (-a\boldsymbol{k})\times\boldsymbol{X} + \boldsymbol{0}\times(-Mg\boldsymbol{k}') \\ &= a\boldsymbol{X}\times\boldsymbol{k}.\end{aligned} \tag{2}$$

Here M is the mass and a the radius of the ball, and ν is a constant depending on the moment of inertia of the ball. For a *uniform* ball, $\nu = \frac{2}{5}$ and, for a *hollow* ball, $\nu = \frac{2}{3}$.

On eliminating the reaction $\boldsymbol{X}$ between these two equations, we find that

$$\begin{aligned} \nu a \dot{\boldsymbol{\omega}} &= \big(\dot{\boldsymbol{V}} + g\boldsymbol{k}'\big)\times\boldsymbol{k} \\ &= \dot{\boldsymbol{V}}\times\boldsymbol{k} + g\sin\beta\,\boldsymbol{j}\,. \end{aligned} \tag{3}$$

If we take the scalar product of this equation with $\boldsymbol{k}$, we obtain

$$\begin{aligned} \nu a \dot{\boldsymbol{\omega}}\cdot\boldsymbol{k} &= \big(\dot{\boldsymbol{V}}\times\boldsymbol{k}\big)\cdot\boldsymbol{k} + g\sin\beta\,\boldsymbol{j}\cdot\boldsymbol{k} \\ &= 0. \end{aligned}$$

Since $\boldsymbol{k}$ is a *constant* vector, it follows that

$$\boldsymbol{\omega}\cdot\boldsymbol{k} = n, \tag{4}$$

where n is a constant. Hence the **spin** of the ball perpendicular to the plane is **conserved**.

If we now take the vector product of equation (3) with $\boldsymbol{k}$, we obtain

$$\begin{aligned} \nu a\,\boldsymbol{k}\times\dot{\boldsymbol{\omega}} &= \boldsymbol{k}\times\big(\dot{\boldsymbol{V}}\times\boldsymbol{k}\big) + g\sin\beta\,\boldsymbol{k}\times\boldsymbol{j} \\ &= \Big((\boldsymbol{k}\cdot\boldsymbol{k})\dot{\boldsymbol{V}} - (\boldsymbol{k}\cdot\dot{\boldsymbol{V}})\boldsymbol{k}\Big) - g\sin\beta\,\boldsymbol{i} \\ &= \dot{\boldsymbol{V}} - g\sin\beta\,\boldsymbol{i}\,. \end{aligned}$$

Hence $\boldsymbol{V}$ and $\boldsymbol{\omega}$ must always be related by

$$\dot{\boldsymbol{V}} = \nu a\,\boldsymbol{k}\times\dot{\boldsymbol{\omega}} + g\sin\beta\,\boldsymbol{i}\,. \tag{5}$$

Equations (4), (5) hold for *any motion* of the ball whether skidding or rolling.

Suppose that we now restrict the ball to **rolling motions**. Then by the rolling condition at C,

$$\boldsymbol{v}^C = \boldsymbol{V} + \boldsymbol{\omega}\times(-a\boldsymbol{k}) = \boldsymbol{0},$$

that is,

$$\boldsymbol{V} + a\boldsymbol{k}\times\boldsymbol{\omega} = \boldsymbol{0}. \tag{6}$$

If we now differentiate equation (6) with respect to t, and use this equation to eliminate $\boldsymbol{k}\times\dot{\boldsymbol{\omega}}$ from equation (5), we obtain

$$\dot{\boldsymbol{V}} = \left(\frac{g\sin\beta}{1+\nu}\right)\boldsymbol{i}\,. \tag{7}$$

This is the required **equation of motion** satisfied by G in rolling motions. The most general **rolling motion** therefore consists of

(i) **constant spin** perpendicular to the plane, and

(ii) **constant acceleration** down the plane of magnitude $g \sin\beta/(1+\nu)$. In particular, the path of the point of contact C must be a **parabola**.

For a uniform ball, the acceleration of the ball down the plane is $\frac{5}{7} g \sin\beta$. ■

Problem 19.2 ✻ *Ball rolling on a rotating turntable*

A rough horizontal turntable is made to rotate about a fixed vertical axis through its centre O with *constant* angular velocity $\Omega\boldsymbol{k}$, where the unit vector $\boldsymbol{k}$ points vertically upwards. A uniform ball of radius a can roll or skid on the turntable. Show that, in *any* motion of the ball, the vertical spin w $\cdot\,\boldsymbol{k}$ is conserved. If the ball *rolls* on the turntable, show that

$$\dot{\boldsymbol{V}} = \tfrac{2}{7}\Omega\,\boldsymbol{k}\times\boldsymbol{V},$$

where $\boldsymbol{V}$ is the velocity of the centre of the ball viewed from a *fixed* reference frame. Deduce the amazing result that the path of the rolling ball must be a circle.

Suppose the ball is held at rest (relative to the turntable), with its centre a distance b from the axis $\{O, \boldsymbol{k}\}$, and is then released. Given that the ball rolls, find the radius and the centre of the circular path on which it moves.

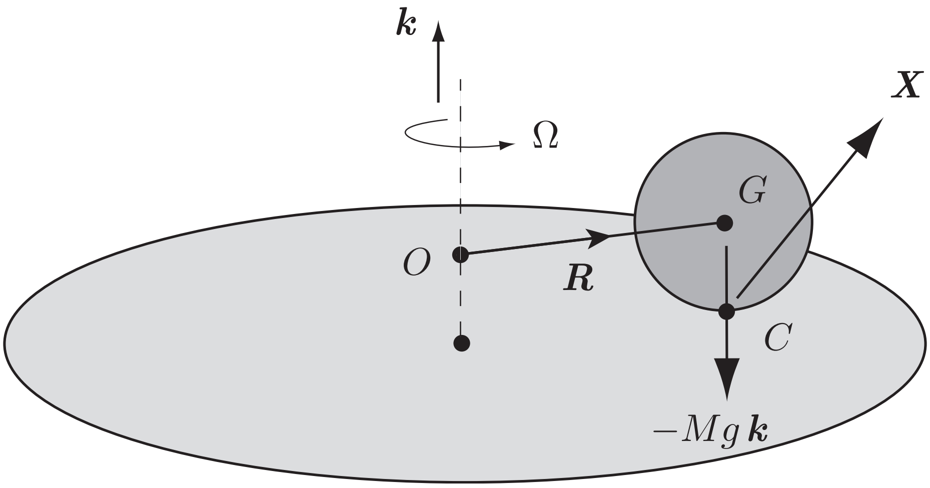

FIGURE 19.2 A ball of mass M and radius a rolls and skids on a rough rotating turntable.

Solution

The turntable and the ball are shown in Figure 19.2. The unit vector $\boldsymbol{k}$ points vertically upwards.

The equations of motion for the ball are as follows. The equation for the **translational motion** of G is

$$M\,\dot{\boldsymbol{V}} = \boldsymbol{X} - Mg\,\boldsymbol{k}, \tag{1}$$

while the equation for the **rotational motion** relative to G is

$$\begin{aligned}\left(\nu M a^2\right)\dot{\mathtt{w}} &= (-a\boldsymbol{k})\times\boldsymbol{X} + \boldsymbol{0}\times(-Mg\boldsymbol{k})\\ &= a\boldsymbol{X}\times\boldsymbol{k}.\end{aligned} \qquad (2)$$

Here M is the mass and a the radius of the ball, and ν is a constant depending on the moment of inertia of the ball. For a *uniform* ball, $\nu = \frac{2}{5}$ and, for a *hollow* ball, $\nu = \frac{2}{3}$.

On eliminating the reaction $\boldsymbol{X}$ between these two equations, we find that

$$\begin{aligned}\nu a\dot{\mathtt{w}} &= \left(\dot{\boldsymbol{V}} + g\boldsymbol{k}\right)\times\boldsymbol{k}\\ &= \dot{\boldsymbol{V}}\times\boldsymbol{k}.\end{aligned} \qquad (3)$$

If we take the scalar product of this equation with $\boldsymbol{k}$, we obtain

$$\begin{aligned}\nu a\dot{\mathtt{w}}\cdot\boldsymbol{k} &= \left(\dot{\boldsymbol{V}}\times\boldsymbol{k}\right)\cdot\boldsymbol{k}\\ &= 0.\end{aligned}$$

Since $\boldsymbol{k}$ is a *constant* vector, it follows that

$$\mathtt{w}\cdot\boldsymbol{k} = n, \qquad (4)$$

where n is a constant. Hence the **vertical spin** of the ball is **conserved**.

If we now take the vector product of equation (4) with $\boldsymbol{k}$, we obtain

$$\begin{aligned}\nu a\boldsymbol{k}\times\dot{\mathtt{w}} &= \boldsymbol{k}\times\left(\dot{\boldsymbol{V}}\times\boldsymbol{k}\right)\\ &= (\boldsymbol{k}\cdot\boldsymbol{k})\dot{\boldsymbol{V}} - (\boldsymbol{k}\cdot\dot{\boldsymbol{V}})\boldsymbol{k}\\ &= \dot{\boldsymbol{V}}.\end{aligned}$$

Hence $\boldsymbol{V}$ and $\mathtt{w}$ must always be related by

$$\dot{\boldsymbol{V}} = \nu a\boldsymbol{k}\times\dot{\mathtt{w}}. \qquad (5)$$

Equations (4), (5) hold for *any motion* of the ball whether skidding or rolling.

Suppose that we now restrict the ball to **rolling motions**. Then by the rolling condition at C,

$$\boldsymbol{v}^C = \boldsymbol{V} + \mathtt{w}\times(-a\boldsymbol{k}) = \left(\Omega\boldsymbol{k}\right)\times\boldsymbol{R},$$

where $\boldsymbol{R}$ is the position vector of G relative to O. Hence

$$\boldsymbol{V} + a\boldsymbol{k}\times\mathtt{w} = \Omega\boldsymbol{k}\times\boldsymbol{R}.$$

On differentiating this equation with respect to t, we obtain

$$\dot{\boldsymbol{V}} + a\,\boldsymbol{k}\times\dot{\mathtt{w}} = \Omega\,\boldsymbol{k}\times\boldsymbol{V},$$

and this equation can now be used to eliminate $\boldsymbol{k}\times\mathtt{w}$ from equation (5). This gives

$$\dot{\boldsymbol{V}} = \left(\frac{\nu\Omega}{1+\nu}\right)\boldsymbol{k}\times\boldsymbol{V},$$

which is the required **equation of motion** satisfied by G in rolling motions. On integrating with respect to t, we obtain

$$\dot{\boldsymbol{R}} = \left(\frac{\nu\Omega}{1+\nu}\right)\boldsymbol{k}\times\boldsymbol{R} + \boldsymbol{C}, \tag{6}$$

where $\boldsymbol{C}$ is a constant of integration.

Suppose that, initially, $\overrightarrow{OG}$ is in the $\boldsymbol{i}$-direction, where the basis vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are *fixed in space* and $\boldsymbol{k}$ points vertically upwards. Then the initial conditions require that $\boldsymbol{R} = b\,\boldsymbol{i}$ and $\dot{\boldsymbol{R}} = \Omega b\,\boldsymbol{j}$ when $t = 0$. It follows that

$$\begin{aligned}\boldsymbol{C} &= \Omega b\,\boldsymbol{j} - \left(\frac{\nu\Omega}{1+\nu}\right)\boldsymbol{k}\times(b\,\boldsymbol{i})\\ &= \left(\frac{\Omega b}{1+\nu}\right)\boldsymbol{j}.\end{aligned}$$

The equation of motion (6) can therefore be written in the form

$$\dot{\boldsymbol{R}} = \left(\frac{\nu\Omega}{1+\nu}\right)\boldsymbol{k}\times\left(\boldsymbol{R} + \left(\frac{b}{\nu}\right)\boldsymbol{i}\right).$$

This solutions of this equation are known to represent **uniform circular motions** with **centre** at the point $-(b/\nu)\,\boldsymbol{i}$. Since the initial value of $\boldsymbol{R}$ is $b\,\boldsymbol{i}$, the **radius** of the circle must be $(1+\nu)b/\nu$. In particular, for a *uniform* ball, the centre of the circle is at the point $-\frac{5}{2}b\,\boldsymbol{i}$ and the radius is $\frac{7}{2}b$. ■

Problem 19.3 ✻ *Ball rolling on a fixed sphere*

A uniform ball with radius a and centre C *rolls* on the rough outer surface of a fixed sphere of radius b and centre O. Show that the radial spin $\mathtt{w} \cdot \boldsymbol{c}$ is conserved, where $\boldsymbol{c}$ $(= \boldsymbol{c}(t))$ is the *unit* vector in the radial direction $\overrightarrow{OC}$. [Take care!] Show also that $\boldsymbol{c}$ satisfies the equation

$$7(a+b)\,\boldsymbol{c}\times\ddot{\boldsymbol{c}} + 2an\,\dot{\boldsymbol{c}} + 5g\,\boldsymbol{c}\times\boldsymbol{k} = \boldsymbol{0},$$

where n is the constant value of $\mathtt{w} \cdot \boldsymbol{c}$ and $\boldsymbol{k}$ is the unit vector pointing vertically upwards.

By comparing this equation with that for the spinning top, deduce the amazing result that the ball can roll on the spherical surface without ever falling off. Find the minimum value of n such that the ball is stable at the highest point of the sphere.

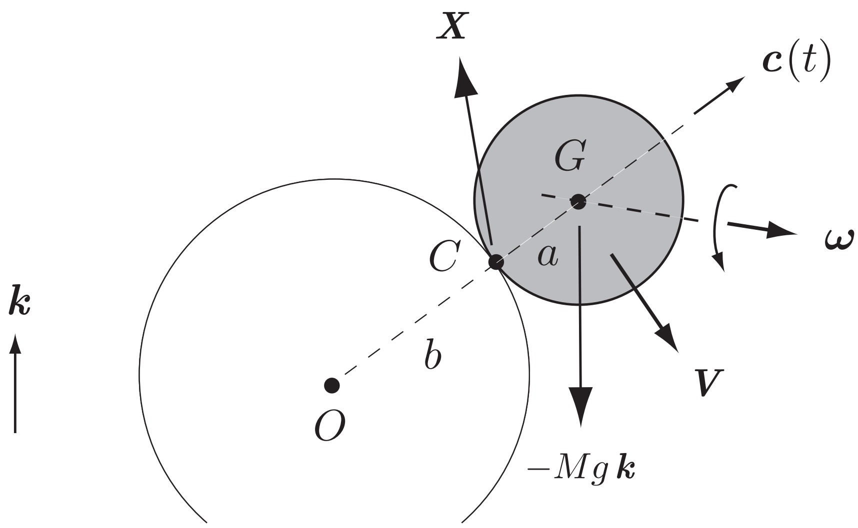

FIGURE 19.3 A ball of mass M and radius a rolls on the rough surface of a fixed ball of radius b.

Solution

The ball and the sphere are shown in Figure 19.3. Note that $\boldsymbol{R}$, the position vector of the centre of the ball, is $\boldsymbol{R} = (a+b)\boldsymbol{c}$, so that $\boldsymbol{V} = (a+b)\dot{\boldsymbol{c}}$.

The equations of motion for the ball are as follows. The equation for the **translational motion** of G is

$$M\dot{\boldsymbol{V}} = \boldsymbol{X} - Mg\boldsymbol{k}, \tag{1}$$

while the equation for the **rotational motion** relative to G is

$$\begin{aligned}\left(\nu M a^2\right)\dot{\mathtt{w}} &= (-a\boldsymbol{c})\times\boldsymbol{X} + \boldsymbol{0}\times(-Mg\,\boldsymbol{k})\\ &= a\,\boldsymbol{X}\times\boldsymbol{c}.\end{aligned} \qquad (2)$$

Here M is the mass and a the radius of the ball, and ν is a constant depending on the moment of inertia of the ball. For a *uniform* ball, $\nu = \frac{2}{5}$ and, for a *hollow* ball, $\nu = \frac{2}{3}$.

On eliminating the reaction $\boldsymbol{X}$ between these two equations, we find that

$$\nu a\,\dot{\mathtt{w}} = \dot{\boldsymbol{V}}\times\boldsymbol{c} + g\boldsymbol{k}\times\boldsymbol{c}, \qquad (3)$$

an equation satisfied in *any motion* of the ball whether skidding or rolling.

If we take the scalar product of this equation with $\boldsymbol{c}$, we obtain

$$\begin{aligned}\nu a\,\dot{\mathtt{w}}\cdot\boldsymbol{c} &= \left(\dot{\boldsymbol{V}}\times\boldsymbol{c}\right)\cdot\boldsymbol{c} + g\left(\boldsymbol{k}\times\boldsymbol{c}\right)\cdot\boldsymbol{c}\\ &= 0.\end{aligned}$$

However, since $\boldsymbol{c}$ is *not* a constant vector, it does not follow (from this) that $\mathtt{w}\cdot\boldsymbol{c}$ is constant. Actually, it *is* constant in rolling motions, but not in general.

Suppose then that we restrict the ball to **rolling motions**. We will proceed in the same manner as in the derivation of the vectorial equation for the top. The rolling condition at the contact point C implies that

$$\boldsymbol{v}^C = \boldsymbol{V} + \mathtt{w}\times(-a\,\boldsymbol{c}) = \boldsymbol{0},$$

and so

$$\boldsymbol{V} + a\,\boldsymbol{c}\times\mathtt{w} = \boldsymbol{0}.$$

If we take the vector product of this equation with $\boldsymbol{c}$, we obtain

$$\boldsymbol{c}\times\boldsymbol{V} + a\,\boldsymbol{c}\times\left(\boldsymbol{c}\times\mathtt{w}\right) = \boldsymbol{0},$$

which simplifies to give

$$\mathtt{w} = \left(\frac{1}{a}\right)\boldsymbol{c}\times\boldsymbol{V} + (\mathtt{w}\cdot\boldsymbol{c})\,\boldsymbol{c}.$$

Hence, $\mathtt{w}$ must have the form

$$\mathtt{w} = \left(\frac{1}{a}\right)\boldsymbol{c}\times\boldsymbol{V} + \lambda\,\boldsymbol{c},$$

where λ is some scalar function of the time. If we now substitute this formula for w into equation (3) and make use of the formula $\boldsymbol{V} = (a+b)\dot{\boldsymbol{c}}$, we obtain

$$(1+\nu)(a+b)\,\boldsymbol{c}\times\ddot{\boldsymbol{c}} + \nu a\big(\dot{\lambda}\boldsymbol{c} + \lambda\dot{\boldsymbol{c}}\big) + g\,\boldsymbol{c}\times\boldsymbol{k} = \boldsymbol{0},$$

after some simplification. This is the equation satisfied by the radial unit vector $\boldsymbol{c}$.

If we take the scalar product of this equation with $\boldsymbol{c}$, the only term on the left that survives is $\nu a\dot{\lambda}\,\boldsymbol{c}\cdot\boldsymbol{c}$ and hence $\dot{\lambda} = 0$. Thus the **radial spin** of the ball is **conserved**.

The **equation of motion** for $\boldsymbol{c}$ then reduces to

$$(1+\nu)(a+b)\,\boldsymbol{c}\times\ddot{\boldsymbol{c}} + \nu an\,\dot{\boldsymbol{c}} + g\,\boldsymbol{c}\times\boldsymbol{k} = \boldsymbol{0},$$

where n is the constant value of the radial spin w $\cdot\,\boldsymbol{c}$. In particular, for a **uniform ball**, the equation for $\boldsymbol{c}$ is

$$7(a+b)\,\boldsymbol{c}\times\ddot{\boldsymbol{c}} + 2an\,\dot{\boldsymbol{c}} + 5g\,\boldsymbol{c}\times\boldsymbol{k} = \boldsymbol{0},$$

as required.

From the vectorial theory of the **top**, the equation for the unit axial vector $\boldsymbol{a}$ is

$$A\,\boldsymbol{a}\times\ddot{\boldsymbol{a}} + Cn\,\dot{\boldsymbol{a}} + Mgh\,\boldsymbol{a}\times\boldsymbol{k} = \boldsymbol{0},$$

in the standard notion. We observe that the equations for the radial vector $\boldsymbol{c}$ of the ball and the axial vector $\boldsymbol{a}$ of the top have the same form. Moreover, they become *exactly* the same if we multiply the equation for $\boldsymbol{c}$ by Ma and give A, C and h the special values

$$\begin{aligned} A &= 7Ma(a+b), \\ C &= 2Ma^2, \\ h &= 5a. \end{aligned}$$

Hence, if we were to construct a top with these parameters and give it spin n, then the motions of its axial vector $\boldsymbol{a}$ would be exactly the same as those of the radial vector $\boldsymbol{c}$ of the ball with spin n. For example, since the *top* can undergo steady precession with $\boldsymbol{a}$ at a fixed angle to the vertical, the *ball* must be able to move so that its point of contact C moves uniformly round a horizontal circle. The ball would never fall off!

Similarly (see Problem 19.5), the *top* is known to be stable in the vertically upright position if

$$C^2n^2 > 4AMgh.$$

On substituting in the above values for A, C and h, it follows that the *ball* will be stable on the top of the sphere if

$$n^2 > \frac{35(a+b)g}{a^2} \blacksquare$$

Problem 19.4

Investigate the steady precession of a top for the case in which the axis of the top moves in the horizontal plane through O. Show that for any $n \neq 0$ there is just *one* rate of steady precession and find its value.

Solution

We give two solutions to this problem, the first based on Lagrangian mechanics and the second on vectorial mechanics.

Lagrangian solution

The **Lagrangian** for the top in terms of Euler's angles is

$$L = \tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}A\left(\dot{\phi}\sin\theta\right)^2 + \tfrac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta.$$

The coordinates ϕ and ψ are *cyclic* and the corresponding conservation relations are

$$\begin{aligned} A\dot{\phi}\sin^2\theta + Cn\cos\theta &= L_z, \\ \dot{\psi} + \dot{\phi}\cos\theta &= n, \end{aligned}$$

where the spin n and angular momentum L_z are constants, determined by the initial conditions.

The coordinate θ is *not cyclic* and the corresponding Lagrange equation is

$$A\ddot{\theta} - \left(A\dot{\phi}^2\cos\theta - Cn\dot{\phi} + Mgh\right)\sin\theta = 0.$$

We now seek solutions in which $\theta = \pi/2$ for all t. This is possible if, and only if,

$$\begin{aligned} A\dot{\phi} &= L_z, \\ \dot{\psi} &= n, \\ Cn\dot{\phi} - Mgh &= 0. \end{aligned}$$

Hence, for any $n \neq 0$, there is just *one* rate of **steady precession**, namely,

$$\dot{\phi} = \frac{Mgh}{Cn} \blacksquare$$

Vectorial solution

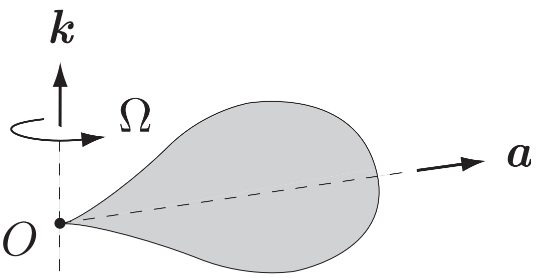

FIGURE 19.4 The top precesses with angular velocity Ω with its axis in the horizontal plane through O.

Figure 19.4 shows the top precessing with angular velocity Ω with its axis in the horizontal plane through O. Then, in the reference frame precessing with the top, the axis vector $\boldsymbol{a}$ is fixed and the *apparent* angular velocity of the top has the form $\lambda \boldsymbol{a}$, where λ is some scalar function of the time. Hence, by the theorem on the addition of angular velocities, the *true* **angular velocity** of the top is

$$\mathtt{w} = \Omega \boldsymbol{k} + \lambda \boldsymbol{a}.$$

Since $\boldsymbol{a}$ and $\boldsymbol{k}$ are principal directions of the top at O, it follows that the corresponding **angular momentum** about O is

$$\boldsymbol{L}_O = A\Omega \boldsymbol{k} + C\lambda \boldsymbol{a},$$

where A, A, C are the principal moments of inertia of the top at O.

The **angular momentum principle** then requires that

$$\frac{d}{dt}\Big(A\Omega \boldsymbol{k} + C\lambda \boldsymbol{a}\Big) = (h\boldsymbol{a})\times(-Mg\boldsymbol{k}),$$

that is,

$$A\Big(\dot{\Omega} \boldsymbol{k}\Big) + C\Big(\dot{\lambda} \boldsymbol{a} + \lambda \dot{\boldsymbol{a}}\Big) = Mgh\, \boldsymbol{k}\times\boldsymbol{a}.$$

If we take the scalar product of this equation with $\boldsymbol{k}$, we find that $\dot{\Omega} = 0$ so that the rate of precession must be constant. Similarly, if we take the scalar product with $\boldsymbol{a}$, we find that $\dot{\lambda} = 0$ so that the axial spin $\mathtt{w} \cdot \boldsymbol{a}$ must be constant. The equation for $\boldsymbol{a}$ then becomes

$$Cn\dot{\boldsymbol{a}} = Mgh\, \boldsymbol{k}\times\boldsymbol{a},$$

where n is the constant value of $\mathtt{w} \cdot \boldsymbol{a}$. However, in this steady precession, $\dot{\boldsymbol{a}} = (\Omega \boldsymbol{k})\times\boldsymbol{a}$ and so there is just *one* **rate of precession** which is given by

$$\Omega = \frac{Mgh}{Cn} \blacksquare$$

Problem 19.5 *The sleeping top*

By performing a perturbation analysis, show that a top will be stable in the vertically upright position if

$$C^2n^2 > 4AMgh,$$

in the standard notation.

Solution

We give two solutions to this problem, the first based on Lagrangian mechanics and the second on vectorial mechanics.

Lagrangian solution

The **Lagrangian** for the top in terms of Euler's angles is

$$L = \tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}A\left(\dot{\phi}\sin\theta\right)^2 + \tfrac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta.$$

The coordinates ϕ and ψ are *cyclic* and the corresponding conservation relations are

$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = L_z,$$
$$\dot{\psi} + \dot{\phi}\cos\theta = n,$$

where the spin n and angular momentum L_z are constants, determined by the initial conditions.

The coordinate θ is *not cyclic* and the corresponding Lagrange equation is

$$A\ddot{\theta} - \left(A\dot{\phi}^2\cos\theta - Cn\dot{\phi} + Mgh\right)\sin\theta = 0.$$

When the top is spinning in the vertically upright position, the constants n and L_z are related by $L_z = Cn$. Suppose now that the top is disturbed from this steady state by being given a horizontal impulse that does not change the *instantaneous* values of $\dot{\psi}$ and $\dot{\phi}$. Then, in the subsequent motion, n and L_z retain their undisturbed values and the angular momentum equation still has the form

$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = Cn.$$

If we now use this equation to eliminate $\dot{\phi}$ from the Lagrange equation for θ, we obtain

$$A\ddot{\theta} + \left(\frac{C^2n^2(1-\cos\theta)^2}{A\sin^4\theta} - Mgh\right)\sin\theta = 0.$$

This is the **equation of motion** for the inclination angle θ. This equation is exact and applies to large disturbances as well as small ones.

To investigate the **stability** of the top when spinning in the upright position, we suppose that θ and its time derivatives are small and approximate the equation for θ by linearising. It is not difficult to show that, when θ is small,

$$\frac{(1-\cos\theta)^2}{\sin^4\theta} \sim \frac{1}{4}$$

so that the **linearised equation** for θ is

$$A\ddot{\theta} + \left(\frac{C^2n^2}{4A} - Mgh\right)\theta = 0.$$

The top will be stable when *small disturbances remain small* and this requires that the bracketed coefficient be *positive*. This in turn requires that

$$C^2n^2 \geq 4AMgh,$$

which is the **condition for stability**. ■

Vectorial solution

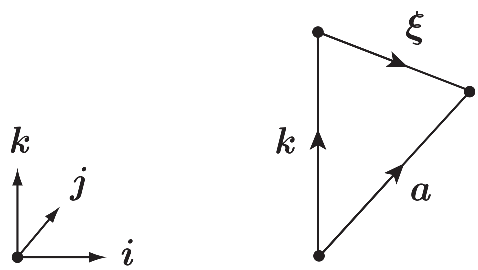

FIGURE 19.5 The axial vector $\boldsymbol{a}$ is expressed in the form $\boldsymbol{a} = \boldsymbol{k} + \boldsymbol{\xi}$.

This time we start from the equation of motion for the axial vector of the top, namely

$$A\,\boldsymbol{a}\times\ddot{\boldsymbol{a}} + Cn\,\dot{\boldsymbol{a}} + Mgh\,\boldsymbol{a}\times\boldsymbol{k} = \boldsymbol{0}.$$

If we write $\boldsymbol{a}$ in the form

$$\boldsymbol{a} = \boldsymbol{k} + \boldsymbol{\xi},$$

then the **equation of motion** for $\boldsymbol{\xi}$ becomes

$$A(\boldsymbol{k}+\boldsymbol{\xi})\times\ddot{\boldsymbol{\xi}} + Cn\,\dot{\boldsymbol{\xi}} + Mgh\,\boldsymbol{\xi}\times\boldsymbol{k} = \boldsymbol{0}.$$

This equation is exact and applies to large disturbances as well as small ones. To investigate the **stability** of the top when spinning in the upright position, we suppose that $\boldsymbol{\xi}$ and its time derivatives are small and approximate the equation by linearising. The **linearised equation** for $\boldsymbol{\xi}$ is

$$A\boldsymbol{k}\times\ddot{\boldsymbol{\xi}} + Cn\dot{\boldsymbol{\xi}} + Mgh\,\boldsymbol{\xi}\times\boldsymbol{k} = \boldsymbol{0}.$$

In order to analyse the solutions of this vector equation, write

$$\boldsymbol{\xi} = \xi_1\boldsymbol{i} + \xi_2\boldsymbol{j} + \xi_3\boldsymbol{k},$$

where the standard basis set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is *fixed* in space.

Then, in the **linear approximation**, ξ_3 is negligible and ξ_1, ξ_2 satisfy the equations

$$A\ddot{\xi}_1 + Cn\dot{\xi}_2 - Mgh\xi_1 = 0,$$
$$A\ddot{\xi}_2 - Cn\dot{\xi}_1 - Mgh\xi_2 = 0.$$

This pair of coupled equations for the components ξ_1, ξ_2 can be combined into the single equation

$$A\ddot{Z} - iCn\dot{Z} - MghZ = 0$$

by introducing the *complex* unknown $Z = \xi_1 + i\xi_2$. This second order ODE for Z is linear, homogeneous and has constant coefficients (like the damped SHO). One of the coefficients is complex, but this does not affect the solution method in any way. The general solution is

$$Z = De^{\lambda_1 t} + Ee^{\lambda_2 t},$$

where

$$\lambda_1 = \frac{iCn + \left(4AMgh - C^2n^2\right)^{1/2}}{2A}, \qquad \lambda_2 = \frac{iCn - \left(4AMgh - C^2n^2\right)^{1/2}}{2A},$$

and D, E are arbitrary constants.

The top will be stable when *small disturbances remain small* and this requires that neither exponent should have a positive real part. This in turn requires that

$$C^2n^2 > 4AMgh,$$

which is the **condition for stability**. ■

Problem 19.6

Estimate how large the spin n of a pencil would have to be for it to be stable in the vertically upright position, spinning on its point. [Take the pencil to be a uniform cylinder 15 cm long and 7 mm in diameter.]

Solution

This is a numerical application of the result in Problem 19.6. If the pencil has mass M, radius a and length $2b$, then (ignoring the fact that one end has been sharpened!)

$$\begin{aligned} A &= \left(\tfrac{1}{4}Ma^2 + \tfrac{1}{3}Mb^2\right) + Mb^2 = \tfrac{1}{4}Ma^2 + \tfrac{4}{3}Mb^2, \\ C &= \tfrac{1}{2}Ma^2, \\ h &= b. \end{aligned}$$

For our pencil, $a = 0.0035$ m and $b = h = 0.075$ m. On taking $g = 9.81\ \mathrm{m\,s^{-2}}$, the **stability condition**

$$C^2n^2 > 4AMgh$$

shows that n must be greater than 24,250 radians per second, which is about 3,860 revolutions per second. ■

Problem 19.7

A juggler is balancing a spinning ball of diameter 20 cm on the end of his finger. Estimate the spin required for stability (i) for a uniform solid ball, (ii) for a uniform thin hollow ball. Which do you suppose the juggler uses?

Solution

This is a numerical application of the result in Problem 19.6. If the ball has mass M and radius a, then

$$A = \nu M a^2 + M a^2,$$
$$C = \nu M a^2,$$
$$h = a,$$

where $\nu = \frac{2}{5}$ for the uniform ball and $\nu = \frac{2}{3}$ for the hollow ball.

For our ball, $a = 0.1$ m and $h = 0.1$ m. On taking $g = 9.81$ m s^{-2}, the **stability condition**

$$C^2 n^2 > 4AMgh$$

shows that n must be greater than about 9.3 revolutions per second for the **uniform ball**, and greater than about 6.2 revolutions per second for the **hollow ball**.

The juggler should therefore use the **hollow ball** since it is stable at lower angular speeds. [It is also has the advantage of being lighter!] ■

Problem 19.8

Solve the problem of the free motion of an axisymmetric body by the Lagrangian method. Compare your results with those in Section 19.4.

Solution

Suppose that G, the centre of mass of the body is at rest. Then, in terms of Euler's angles centred on G, the **Lagrangian** for the body is

$$L = \tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}A\left(\dot{\phi}\sin\theta\right)^2 + \tfrac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta.$$

The coordinates ϕ and ψ are *cyclic* and the corresponding conservation relations are

$$\begin{aligned} A\dot{\phi}\sin^2\theta + Cn\cos\theta &= L_z, \\ \dot{\psi} + \dot{\phi}\cos\theta &= n, \end{aligned}$$

where the angular momentum L_z $(= \boldsymbol{L}_G \cdot \boldsymbol{k})$ and spin n $(= \mathtt{w} \cdot \boldsymbol{a})$ are constants, determined by the initial conditions.

Take the coordinate axis Gz to point in the direction of the angular momentum vector $\boldsymbol{L}_G$. Then $L_z = |\boldsymbol{L}_G| = L$ and the axial angular momentum Cn is related to L by

$$Cn = \boldsymbol{L}_G \cdot \boldsymbol{a} = L\cos\theta.$$

Hence $\theta = \alpha$ (a constant), and the conservation equation for L_z then becomes

$$A\dot{\phi} = L.$$

Hence, the **rate of precession** of the body about the axis $\{G, \boldsymbol{L}_G\}$ is

$$\dot{\phi} = \frac{L}{A}. \tag{1}$$

Furthermore, from the constant spin equation,

$$\begin{aligned} \dot{\psi} &= n - \dot{\phi}\cos\alpha \\ &= \frac{L\cos\alpha}{C} - \left(\frac{L}{A}\right)\cos\alpha \\ &= \left(\frac{A - C}{AC}\right)L\cos\alpha. \end{aligned} \tag{2}$$

This is the **apparent rate of spin**, viewed from the precessing frame. Equations (1) and (2) confirm the results of section 19.4. ■

Problem 19.9 *Frisbee with resistance*

A (wobbling) frisbee moving through air is subject to a frictional couple equal to $K\mathtt{w}$. Find the time variation of the axial spin λ ($= \mathtt{w} \cdot \boldsymbol{a}$), where $\boldsymbol{a}$ is the axial unit vector. Show also that $\boldsymbol{a}$ satisfies the equation

$$A\,\boldsymbol{a}\times\ddot{\boldsymbol{a}} + K\,\boldsymbol{a}\times\dot{\boldsymbol{a}} + C\lambda\,\dot{\boldsymbol{a}} = \boldsymbol{0}.$$

✱ By taking the cross product of this equation with $\dot{\boldsymbol{a}}$, find the time variation of $|\dot{\boldsymbol{a}}|$. Deduce that the angle between $\mathtt{w}$ and $\boldsymbol{a}$ decreases with time if $C > A$ (which it is for a normal frisbee). Thus, in the presence of linear resistance, the wobble dies away.

Solution

By following the same procedure as that in section 19.4, we find that $\mathtt{w}$, the angular velocity of the frisbee can be expressed in the form

$$\mathtt{w} = \boldsymbol{a}\times\dot{\boldsymbol{a}} + \lambda\,\boldsymbol{a},$$

where the unit vector $\boldsymbol{a}$ points along the axis of symmetry, and n is some scalar function of the time t. By the axial symmetry of the frisbee, the corresponding angular momentum about G is

$$\boldsymbol{L}_G = A\,\boldsymbol{a}\times\dot{\boldsymbol{a}} + C\lambda\,\boldsymbol{a},$$

where A, A, C are the principal moments of inertia of the frisbee at G.

The equation of rotational motion is the **angular momentum principle** about G, namely,

$$\frac{d}{dt}\Big(A\,\boldsymbol{a}\times\dot{\boldsymbol{a}} + C\lambda\,\boldsymbol{a}\Big) = \boldsymbol{N}_G,$$

where $\boldsymbol{N}_G$ is the total moment of external forces about G. In the present problem, the resistance forces provide the moment

$$\boldsymbol{N}_G = -K\mathtt{w}$$

and the equation of motion for the axial vector $\boldsymbol{a}$ becomes

$$A\,\boldsymbol{a}\times\ddot{\boldsymbol{a}} + C\dot{\lambda}\,\boldsymbol{a} + C\lambda\,\dot{\boldsymbol{a}} = -K\big(\boldsymbol{a}\times\dot{\boldsymbol{a}} + \lambda\,\boldsymbol{a}\big),$$

that is,

$$A\,\boldsymbol{a}\times\ddot{\boldsymbol{a}} + K\boldsymbol{a}\times\dot{\boldsymbol{a}} + \Big(C\dot{\lambda} + K\lambda\Big)\,\boldsymbol{a} + C\lambda\,\dot{\boldsymbol{a}} = \boldsymbol{0}.$$

If we take the scalar product of this equation with $\boldsymbol{a}$ we find that

$$C\dot{\lambda} + K\lambda = 0,$$

which is an ODE for the unknown axial spin $\lambda(t)$. The general solution of this equation is

$$\lambda = \Omega e^{-Kt/C},$$

where Ω is a constant. Thus, in the presence of resistance, the spin $\boldsymbol{\omega} \cdot \boldsymbol{a}$ is *not constant* but decays exponentially.

On making use of this formula, the **equation of motion** for the axial vector $\boldsymbol{a}$ becomes

$$A\,\boldsymbol{a}\times\ddot{\boldsymbol{a}} + K\boldsymbol{a}\times\dot{\boldsymbol{a}} + C\lambda\,\dot{\boldsymbol{a}} = \boldsymbol{0}.$$

If we now take the vector product of this equation with $\boldsymbol{a}$ we find that

$$A\,(\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}})\,\boldsymbol{a} + K\,(\dot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}})\,\boldsymbol{a} = \boldsymbol{0},$$

which leads to the scalar equation

$$A\,\dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}} + K\,\dot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}} = 0.$$

Now

$$\dot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}} = |\dot{\boldsymbol{a}}|^2 \quad \text{and} \quad \dot{\boldsymbol{a}} \cdot \ddot{\boldsymbol{a}} = \tfrac{1}{2}\frac{d}{dt}\left(|\dot{\boldsymbol{a}}|^2\right)$$

from which it follows that

$$A\frac{d}{dt}\left(|\dot{\boldsymbol{a}}|^2\right) + 2K\left(|\dot{\boldsymbol{a}}|^2\right) = 0.$$

The general solution of this ODE for $|\dot{\boldsymbol{a}}|$ is

$$|\dot{\boldsymbol{a}}| = \epsilon\,\Omega e^{-Kt/A},$$

where ϵ is a dimensionless constant. This is the required **time variation** of $|\dot{\boldsymbol{a}}|$.

✱ The time variation of θ, the angle between $\boldsymbol{\omega}$ and $\boldsymbol{a}$, can now be found explicitly. To do this, we observe that, since $\boldsymbol{\omega} \cdot \boldsymbol{a} = \lambda$ and $\boldsymbol{\omega} \times \boldsymbol{a} = \dot{\boldsymbol{a}}$, the angle θ can be expressed as

$$
\begin{aligned}
\tan\theta &= \frac{|\mathtt{w} \times \boldsymbol{a}|}{\mathtt{w} \cdot \boldsymbol{a}} \\
&= \frac{|\dot{\boldsymbol{a}}|}{\lambda} \\
&= \frac{\epsilon\,\Omega e^{-Kt/A}}{\Omega e^{-Kt/C}} \\
&= \epsilon \exp\left[-K\left(\frac{C-A}{AC}\right)t\right].
\end{aligned}
$$

It follows that the wobble *decays* if $A < C$ but *grows* if $A > C$. ■

Problem 19.10 *Spinning hoop on a smooth floor*

A uniform circular hoop of radius a rolls and slides on a *perfectly smooth* horizontal floor. Find its Lagrangian in terms of the Euler angles, and determine which of the generalised momenta are conserved. [Suppose that G has no *horizontal* motion.]

Investigate the existence of motions in which the angle between the hoop and the floor is a constant α. Show that Ω, the rate of steady precession, must satisfy the equation

$$\cos\alpha\,\Omega^2 - 2n\,\Omega - 2\frac{g}{a}\cot\alpha = 0,$$

where n is the constant axial spin. Deduce that, for $n \neq 0$, there are two possible rates of precession, a faster one going the 'same way' as n, and a slower one in the opposite direction. [These are interesting motions but one would need a *very* smooth floor to observe them.]

Solution

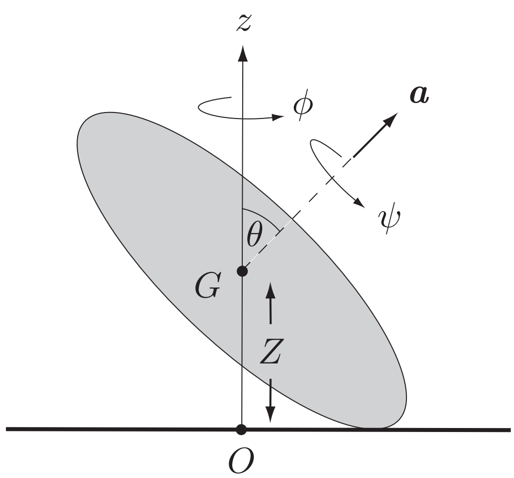

FIGURE 19.6 The hoop (or disk) slides on a perfectly smooth floor.

Since there are no horizontal forces acting on the hoop, the horizontal components of linear momentum are conserved. It follows that G has constant *horizontal* velocity. Any motion of the hoop can therefore be viewed from an inertial frame in which G moves vertically, as shown in Figure 19.6.

The **kinetic energy** of the hoop is the sum of its translational and rotational parts, that is,

$$T = \tfrac{1}{2}M\dot{Z}^2 + \tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}A\left(\dot{\phi}\sin\theta\right)^2 + \tfrac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2,$$

where Z is the vertical displacement of G and $\{\theta, \phi, \psi\}$ are the standard Euler angles based at G. Since $Z = a\sin\theta$,

$$\dot{Z} = a\cos\theta\,\dot{\theta}$$

and the expression for T becomes

$$T = \tfrac{1}{2}\left(A + Ma^2\cos^2\theta\right)\dot{\theta}^2 + \tfrac{1}{2}A\sin^2\theta\,\dot{\phi}^2 + \tfrac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2.$$

The corresponding **potential energy** is simply

$$V = MgZ = Mga\sin\theta.$$

Hence, in terms of the generalised coordinates $\{\theta, \phi, \psi\}$, the hoop has **Lagrangian**

$$L = \tfrac{1}{2}\left(A + Ma^2\cos^2\theta\right)\dot{\theta}^2 + \tfrac{1}{2}A\sin^2\theta\,\dot{\phi}^2 + \tfrac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mga\sin\theta.$$

The coordinates ϕ, ψ are **cyclic** and the corresponding conserved momenta are

$$\begin{aligned} p_\phi &= A\sin^2\theta\,\dot{\phi} + C\cos\theta\left(\dot{\psi} + \dot{\phi}\cos\theta\right), \\ p_\psi &= C\left(\dot{\psi} + \dot{\phi}\cos\theta\right). \end{aligned}$$

This gives the **conservation relations**

$$\begin{aligned} A\dot{\phi}\sin^2\theta + Cn\cos\theta &= L_z, \\ \dot{\psi} + \dot{\phi}\cos\theta &= n, \end{aligned}$$

where the angular momentum L_z $(= \boldsymbol{L}_G \cdot \boldsymbol{k})$ and spin n $(= \mathtt{w} \cdot \boldsymbol{a})$ are constants, determined by the initial conditions.

The coordinate θ is not cyclic and its Lagrange equation is

$$\frac{d}{dt}\left[A\dot{\theta} + Ma^2\cos^2\theta\,\dot{\theta}\right] +$$

$$\left[Ma^2 \sin\theta \cos\theta\, \dot{\theta}^2 - A \sin\theta \cos\theta\, \dot{\phi}^2 + Cn \sin\theta\, \dot{\phi} + Mga \cos\theta\right] = 0,$$

on making use of the spin conservation relation.

We can now investigate **precessional motions** in which $\theta = \alpha$, a constant. In this case, the Lagrange equation for θ is satisfied if

$$A \cos\alpha\, \Omega^2 - Cn\, \Omega - Mga \cot\alpha = 0,$$

where we have now written Ω for the rate of precession $\dot{\phi}$. Hence Ω must be constant and take one of the two possible values

$$\Omega = \frac{Cn \pm \left(C^2n^2 + 4AMga \cos\alpha \cot\alpha\right)^{1/2}}{2A \cos\alpha}.$$

When $C^2n^2 \gg 4AMga$, these values of Ω are given approximately by

$$\Omega^F \sim \frac{Cn}{A \cos\alpha},$$
$$\Omega^S \sim -\frac{Mga \cot\alpha}{Cn},$$

so that the fast precession goes the ‘same way’ as n and the slow precession goes the opposite way. ■

Problem 19.11 *Bicycle wheel*

A bicycle wheel (a hoop) of mass M and radius a is fitted with a smooth spindle lying along its symmetry axis. The wheel is spun with the spindle horizontal, and the spindle is then made to turn with angular speed Ω about a fixed vertical axis through the centre of the wheel. Show that n, the axial spin of the wheel, remains constant and find the moment that must be applied to the spindle to produce this motion.

Solution

Let $Gx_1x_2x_3$ be a set of **embedded principal axes** of the wheel at G with Gx_3 lying along the symmetry axis. Then, since $B = A$, **Euler's equations** for the wheel become

$$\begin{aligned} A\,\dot{\omega}_1 - (A - C)\,\omega_2\omega_3 &= K_1, \\ A\,\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 &= K_2, \\ C\,\dot{\omega}_3 &= K_3, \end{aligned}$$

where $\boldsymbol{K}_G$ $(= K_1\boldsymbol{e}_1 + K_2\boldsymbol{e}_2 + K_3\boldsymbol{e}_3)$ is the applied moment of external forces about G.

Since the wheel is *smoothly* pivoted about its axis, $K_3 = 0$ and the equations become

$$\begin{aligned} A\,\dot{\omega}_1 + (C - A)\,\omega_2\omega_3 &= K_1, \\ A\,\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 &= K_2, \\ \dot{\omega}_3 &= 0. \end{aligned}$$

Hence, in *any* motion, the **axial spin** component $\omega_3 = n$, a **constant**, and the spin components ω_1, ω_2 satisfy the equations

$$A\,\dot{\omega}_1 + (C - A)\,n\omega_2 = K_1, \tag{1}$$

$$A\,\dot{\omega}_2 - (C - A)\,n\omega_1 = K_2. \tag{2}$$

In this problem we are *given* that the spindle is made to turn with constant angular speed Ω about a fixed vertical axis through G. Let $Gxyz$ be a set of Cartesian axes with Gx horizontal, Gy vertical (and fixed), and Gz coincident with the spindle axis Gx_3, as shown in Figure 19.7. Since the axes $Gxyz$ rotate with the wheel around the fixed vertical axis Gy, we will call them the **precessing** axes. Because of the axisymmetry of the wheel, the precessing axes are also a set of principal axes, but they are *not embedded* and Euler's equations do not apply in them.

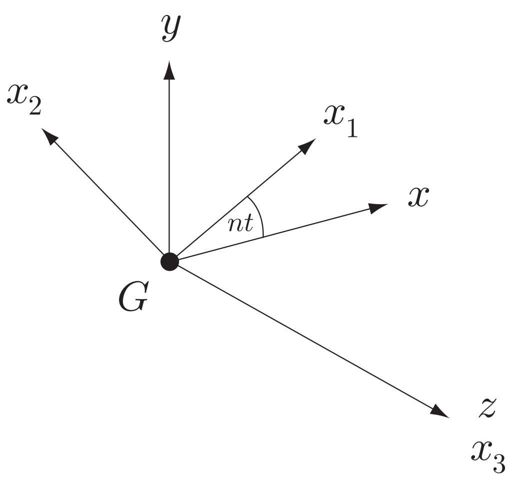

FIGURE 19.7 The **embedded** axes $Gx_1x_2x_3$ and the **precessing** axes $Gxyz$.

Relative to the precessing axes, the spindle axis is at rest and so the *apparent* angular velocity must have the form $\mathtt{w}' = \lambda \boldsymbol{k}$, where λ is some scalar function of the time. Hence, by the theorem on the addition of angular velocities, the *true* angular velocity of the wheel is

$$\mathtt{w} = \Omega \boldsymbol{j} + \lambda \boldsymbol{k},$$

which, since $\mathtt{w} \cdot \boldsymbol{k} = n$, becomes

$$\mathtt{w} = \Omega \boldsymbol{j} + n \boldsymbol{k}.$$

The components of this angular velocity in the embedded axes are therefore

$$\omega_1 = \Omega \sin nt,$$
$$\omega_2 = \Omega \cos nt,$$
$$\omega_3 = n,$$

and, if we now substitute these values for ω_1, ω_2 into the Euler equations (1), (1), we find that

$$K_1 = Cn\Omega \cos nt,$$
$$K_2 = -Cn\Omega \sin nt.$$

The **moment** that must be applied to the wheel is therefore

$$\boldsymbol{K}_G = Cn\Omega \left(\cos nt\, \boldsymbol{e}_1 - \sin nt\, \boldsymbol{e}_2 \right),$$

which, in terms of the unit vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of the precessing axes, becomes

$$\boldsymbol{K}_G = Cn\Omega\,\boldsymbol{i}.$$

This is the formula for the **applied moment** $\boldsymbol{K}_G$. Note that this moment is applied about the *horizontal* axis Gx_1.

If the wheel can be modelled by a hoop of mass M and radius a, then $C = Ma^2$ and the required moment is

$$\boldsymbol{K}_G = Ma^2 n\Omega\,\boldsymbol{i}. \blacksquare$$

Problem 19.12 *Stability of steady rotation*

An unsymmetrical body is in steady rotation about a principal axis through G. By performing a perturbation analysis, investigate the stability of this motion for each of the three principal axes.

Solution

Since the body is subject to no external moments, **Euler's equations** are

$$\begin{aligned} A\,\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 &= 0,\\ B\,\dot{\omega}_2 - (C-A)\,\omega_3\omega_1 &= 0,\\ C\,\dot{\omega}_3 - (A-B)\,\omega_1\omega_2 &= 0, \end{aligned}$$

where $Gx_1x_2x_3$ are a set of **embedded principal axes** of the body at G. Suppose the body is rotating with constant angular velocity Λ about the principal axis Gx_3 when it is *slightly* disturbed. Then, in the subsequent motion,

$$\begin{aligned} \omega_1 &= \xi_1,\\ \omega_2 &= \xi_2,\\ \omega_3 &= \Lambda + \xi_3, \end{aligned}$$

where ξ_1, ξ_2, ξ_3 are *initially* small. We wish to find conditions such that they *remain* small. The motion will then be stable.

On substituting these forms into Euler's equations, we find that, in the **linear appraoximation**, ξ_1, ξ_2, ξ_3 satisfy the equations

$$\begin{aligned} A\,\dot{\xi}_1 - (B-C)\Lambda\,\xi_2 &= 0,\\ B\,\dot{\xi}_2 - (C-A)\Lambda\,\xi_1 &= 0,\\ \dot{\xi}_3 &= 0. \end{aligned}$$

Thus ξ_3 is constant and certainly remains small. It remains to find the time dependencies of ξ_1, ξ_2. If we differentiate the first equation with respect to t and then make use of the second equation, we find that ξ_1 satisfies the equation

$$\ddot{\xi}_1 + \left[\frac{(C-A)(C-B)}{AB}\right]\Lambda^2\,\xi_1 = 0,$$

and it can be shown that ξ_2 satisfies a similar equation. The quantities ξ_1, ξ_2 will therefore remain small if the bracketed coefficient is *positive*, that is, if

$$(C-A)(C-B) > 0.$$

This is true if the principal moment C is the **biggest** or **smallest** of $\{A, B, C\}$, but not otherwise. Hence, *the steady rotation of an unsymmetrical body about a principal axis is stable for the axes with the greatest and least moments of inertia, but unstable for the other axis*. ■

Problem 19.13 *Frisbee with resistance*

Re-solve the problem of the frisbee with resistance (Problem 19.9) by using Euler's equations.

Solution

Since the resistance forces are known to exert the moment $\boldsymbol{N}_G = -K\mathbf{w}$ about G, **Euler's equations** for the frisbee are

$$\begin{aligned} A\,\dot{\omega}_1 - (A-C)\,\omega_2\omega_3 &= -K\omega_1,\\ A\,\dot{\omega}_2 - (C-A)\,\omega_3\omega_1 &= -K\omega_2,\\ C\,\dot{\omega}_3 &= -K\omega_3, \end{aligned}$$

where $Gx_1x_2x_3$ is a set of **embedded principal axes** of the frisbee at G with Gx_3 lying along the symmetry axis.

The third equation is a first order ODE for ω_3 alone and its general solution is

$$\omega_3 = \Lambda e^{-Kt/C},$$

where Λ is a constant of integration. This is the general **time variation** of ω_3.

If we now multiply the first equation by ω_1, the second by ω_2, and add, we obtain

$$A\big(\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2\big) \;+\; 0 \;=\; -K\left(\omega_1^2 + \omega_2^2\right),$$

which can be written in the form

$$A\frac{d}{dt}\left(\omega_1^2 + \omega_2^2\right) + 2K\left(\left(\omega_1^2 + \omega_2^2\right)\right) = 0.$$

This is a first order ODE for the quantity $\left(\omega_1^2 + \omega_2^2\right)$ and its general solution is

$$\omega_1^2 + \omega_2^2 = \Omega^2 e^{-2Kt/A},$$

where Ω is a second constant of integration. Hence

$$\left(\omega_1^2 + \omega_2^2\right)^{1/2} = \Omega e^{-Kt/A}.$$

This is the general **time variation** of $\left(\omega_1^2 + \omega_2^2\right)^{1/2}$.

The **angle** θ between the angular velocity vector $\mathbf{w}$ and the unit axial vector $\boldsymbol{a}$

is then given by

$$\tan\theta = \frac{\left(\omega_1^2 + \omega_2^2\right)^{1/2}}{\omega_3}$$

$$= \frac{\Omega e^{-Kt/A}}{\Lambda e^{-Kt/C}}$$

$$= \left(\frac{\Omega}{\Lambda}\right) \exp\left[-K\left(\frac{C-A}{AC}\right)t\right].$$

It follows that the wobble *decays* if $A < C$ but *grows* if $A > C$.

Problem 19.14 *Wobble on spinning lamina*

An unsymmetrical lamina is in steady rotation about the axis through G perpendicular to its plane. Find an approximation to the wobble of this axis if the body is slightly disturbed. [This is a repeat of Example 19.7 for the special case in which the body is an unsymmetrical *lamina*; in this case $C = A + B$ and there is much simplification.]

Solution

Take **embedded principal axes** at G with Gx_3 perpendicular to the plane of the lamina. Then, by the perpendicular axes theorem, $C = A + B$ and **Euler's equations** reduce to

$$\begin{aligned} \dot{\omega}_1 + \omega_2\omega_3 &= 0, \\ \dot{\omega}_2 - \omega_3\omega_1 &= 0, \\ (A+B)\,\dot{\omega}_3 - (A-B)\,\omega_1\omega_2 &= 0. \end{aligned}$$

Suppose the body is rotating with constant angular velocity Λ about the principal axis Gx_3 when it is *slightly* disturbed. Then, in the subsequent motion,

$$\begin{aligned} \omega_1 &= \xi_1, \\ \omega_2 &= \xi_2, \\ \omega_3 &= \Lambda + \xi_3, \end{aligned}$$

where ξ_1, ξ_2, ξ_3 are small (at least initially). On substituting these forms into Euler's equations, we find that, in the **linear approximation**, ξ_1, ξ_2, ξ_3 satisfy the equations

$$\begin{aligned} \dot{\xi}_1 + \Lambda\,\xi_2 &= 0, \\ \dot{\xi}_2 - \Lambda\,\xi_1 &= 0, \\ \dot{\xi}_3 &= 0. \end{aligned}$$

To find an equation for ξ_1 alone, we differentiate the first equation with respect to t and then make use of the second equation. This gives

$$\ddot{\xi}_1 + \Lambda^2\,\xi_1 = 0.$$

Hence the time variation of ξ_1 has the general form

$$\xi_1 = \epsilon\Lambda\cos(\Lambda t + \gamma),$$

where ϵ and γ are arbitrary constants. The corresponding time variation of ξ_2 is then

$$\xi_2 = \epsilon\Lambda\sin(\Lambda t + \gamma).$$

Now we find the time variaton of the unit vector $\boldsymbol{e}_3$. In general this is obtained by solving the system of *coupled* ODEs

$$\begin{aligned}\dot{\boldsymbol{e}}_1 &= \omega_3\,\boldsymbol{e}_2 - \omega_2\,\boldsymbol{e}_3,\\ \dot{\boldsymbol{e}}_2 &= \omega_1\,\boldsymbol{e}_3 - \omega_3\,\boldsymbol{e}_1,\\ \dot{\boldsymbol{e}}_3 &= \omega_2\,\boldsymbol{e}_1 - \omega_1\,\boldsymbol{e}_2,\end{aligned} \tag{1}$$

for the *three* unknown vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

In the undisturbed motion,

$$\begin{aligned}\boldsymbol{e}_1 &= \cos\Lambda t\,\boldsymbol{i} + \sin\Lambda t\,\boldsymbol{j},\\ \boldsymbol{e}_2 &= -\sin\Lambda t\,\boldsymbol{i} + \cos\Lambda t\,\boldsymbol{j},\\ \boldsymbol{e}_3 &= \boldsymbol{k},\end{aligned}$$

where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a *fixed* orthonormal set. Since ω_1, ω_2 are small in the disturbed motion, the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ that appear in the third equation of (1) can be replaced by their steady (zero order) approximations to give

$$\dot{\boldsymbol{e}}_3 = \xi_2\,(\cos\Lambda t\,\boldsymbol{i} + \sin\Lambda t\,\boldsymbol{j}) - \xi_1\,(-\sin\Lambda t\,\boldsymbol{i} + \cos\Lambda t\,\boldsymbol{j}),$$

correct to the first order. On substituting in the approximate forms for ξ_1 and ξ_2, we find that

$$\begin{aligned}\dot{\boldsymbol{e}}_3 &= \epsilon\Lambda\sin(\Lambda t + \gamma)\,(\cos\Lambda t\,\boldsymbol{i} + \sin\Lambda t\,\boldsymbol{j}) - \\ &\qquad\qquad \epsilon\Lambda\cos(\Lambda t + \gamma)\,(-\sin\Lambda t\,\boldsymbol{i} + \cos\Lambda t\,\boldsymbol{j})\\ &= \epsilon\Lambda\Big[\sin(2\Lambda t + \gamma)\,\boldsymbol{i} - \cos(2\Lambda t + \gamma)\,\boldsymbol{j}\Big].\end{aligned}$$

Hence, correct to the first order, the ODE for $\boldsymbol{e}_3$ is *uncoupled* from the other two and integrates to give

$$\boldsymbol{e}_3 = \boldsymbol{k} - \tfrac{1}{2}\epsilon\Big[\cos(2\Lambda t + \gamma)\,\boldsymbol{i} + \sin(2\Lambda t + \gamma)\,\boldsymbol{j}\Big] + \epsilon\boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant of integration.

Hence, in the first order theory, the principal axis Gx_3 has a **periodic wobble** with **frequency** 2Λ. This result is consistent with the motion of a free *axisymmetric* body, which precesses around the axis $\{G, \boldsymbol{L}_G\}$ with frequency $L/A = Cn/(A\cos\alpha)$, where n is the axial spin and α is the angle between the axial vector $\boldsymbol{a}$ and the angular momentum $\boldsymbol{L}_G$. For an *axisymmetric lamina*, $C = 2A$ and the frequency of the wobble is $2n/\cos\alpha$. ■

Problem 19.15 ✻ *Euler theory for the unsymmetrical lamina*

An unsymmetrical lamina has principal axes $Gx_1x_2x_3$ at G with the corresponding moments of inertia $\{A, B, A+B\}$. Initially the lamina is rotating with angular velocity Ω about an axis through G that lies in the (x_1, x_2)-plane and makes an acute angle α with Gx_1. By using Euler's equations, show that, in the subsequent motion,

$$\omega_1^2 + \omega_2^2 = \Omega^2,$$
$$(B-A)\,\omega_2^2 + (B+A)\,\omega_3^2 = (B-A)\,\Omega^2 \sin^2\alpha.$$

Interpret these results in terms of the motion of the '`w` -point' moving in $(\omega_1, \omega_2, \omega_3)$-space and deduce that `w` is periodic when viewed from the embedded frame.

Find an ODE satisfied by ω_2 alone and deduce that the lamina will once again be rotating about the same axis after a time

$$\frac{4}{\Omega}\left(\frac{B+A}{B-A}\right)^{1/2}\int_0^{\pi/2}\frac{d\theta}{(1-\sin^2\alpha\,\sin^2\theta)^{1/2}}.$$

Solution

Take **embedded principal axes** at G with Gx_3 perpendicular to the plane of the lamina. Then, by the perpendicular axes theorem, $C = A + B$ and **Euler's equations** become

$$\dot\omega_1 + \omega_2\omega_3 = 0,$$
$$\dot\omega_2 - \omega_3\omega_1 = 0,$$
$$(A+B)\,\dot\omega_3 - (A-B)\,\omega_1\omega_2 = 0.$$

Without loss of generality, we will suppose that $A < B$; if not, this can be be made true by rotating the axes through a right angle.

If we multiply the first Euler equation by ω_1, the second by ω_2, and add, we obtain

$$\omega_1\dot\omega_1 + \omega_2\dot\omega_2 = 0,$$

which can be integrated immediately to give

$$\omega_1^2 + \omega_2^2 = C,$$

where C is a constant. It follows from the initial conditions $\omega_1 = \Omega\cos\alpha$ and $\omega_2 = \Omega\sin\alpha$ when $t = 0$ that $C = \Omega^2$. Hence `w` satisfies the first **conservation**

relation

$$\omega_1^2 + \omega_2^2 = \Omega^2.$$

If instead we multiply the second Euler equation by $(B - A)\omega_2$, the third by ω_3, and add, we obtain

$$(B + A)\omega_3\dot{\omega}_3 + (B - A)\omega_2\dot{\omega}_2 = 0,$$

which can also be integrated immediately to give

$$(B - A)\omega_2^2 + (B + A)\omega_3^2 = D,$$

where D is a constant. This time, the initial conditions show that $D = (B - A)\Omega^2 \sin^2 \alpha$. Hence w also satisfies the second **conservation relation**

$$(B - A)\omega_2^2 + (B + A)\omega_3^2 = (B - A)\Omega^2 \sin^2 \alpha.$$

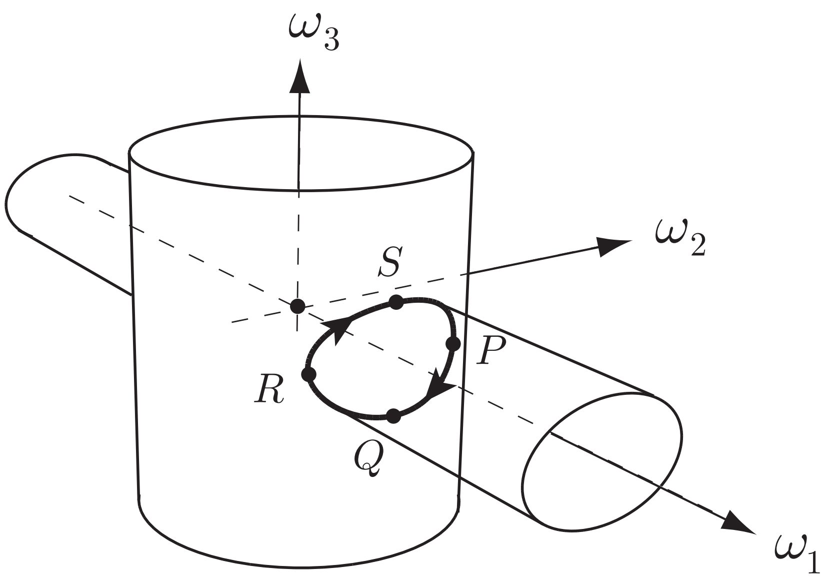

FIGURE 19.8 The 'point' in w -space moves along the curve in which the circular cylinder $\omega_1^2 + \omega_2^2 = \Omega^2$ meets the elliptical cylinder $(B - A)\omega_2^2 + (B + A)\omega_3^2 = (B - A)\Omega^2 \sin^2 \alpha$.

These two conservation relations enable us to find the path of the 'w -point' in $(\omega_1, \omega_2, \omega_3)$ space. The first relation shows that the w -point must move on the 'vertical' circular cylinder shown in Figure 19.8. Similarly, the second relation shows that the w -point must also move on the 'horizontal' elliptic cylinder shown in the same figure. The w -point must therefore move along the **curve of intersection** $PQRS$ of these two surfaces. The w -point begins at $P = (\Omega\cos\alpha, \Omega\sin\alpha, 0)$ at time $t = 0$ and proceeds in the direction shown, eventually returning to P. It follows that w *is **periodic** when viewed from the embedded frame*.

To obtain an ODE satisfied by ω_2 alone, we begin with the second Euler equation and make use of the two conservation relations. This gives

$$\begin{aligned}\dot{\omega}_2 &= \omega_1\omega_3\\ &= \left[\pm\left(\Omega^2-\omega_2^2\right)^{1/2}\right]\times\left[\pm\left(\frac{B-A}{B+A}\right)^{1/2}\left(\Omega^2\sin^2\alpha-\omega_2^2\right)^{1/2}\right]\\ &= \pm\left(\frac{B-A}{B+A}\right)^{1/2}\left(\Omega^2-\omega_2^2\right)^{1/2}\left(\Omega^2\sin^2\alpha-\omega_2^2\right)^{1/2},\end{aligned}$$

where the sign depends on which part of the curve $PQRS$ the w -point is on; on the 'upper' half of the curve, the sign is *positive* and, on the 'lower' half, the sign is *negative*. This first order separable ODE is the **required equation for** ω_2.

Finally we must find the time τ taken for the w -point to return to P. On integrating the ODE for ω_2 over the section SP of the curve, we obtain

$$\int_0^{\Omega\sin\alpha}\frac{d\omega_2}{\left(\Omega^2-\omega_2^2\right)^{1/2}\left(\Omega^2\sin^2\alpha-\omega_2^2\right)^{1/2}} = +\left(\frac{B-A}{B+A}\right)^{1/2}\int_0^{\tau/4}dt$$

and so

$$\begin{aligned}\tau &= 4\left(\frac{B+A}{B-A}\right)^{1/2}\int_0^{\Omega\sin\alpha}\frac{d\omega_2}{\left(\Omega^2-\omega_2^2\right)^{1/2}\left(\Omega^2\sin^2\alpha-\omega_2^2\right)^{1/2}}\\ &= \frac{4}{\Omega}\left(\frac{B+A}{B-A}\right)^{1/2}\int_0^{\pi/2}\frac{d\theta}{(1-\sin^2\alpha\,\sin^2\theta)^{1/2}},\end{aligned}$$

on making the substitution $\omega_2 = \Omega\sin\alpha\sin\theta$. ■