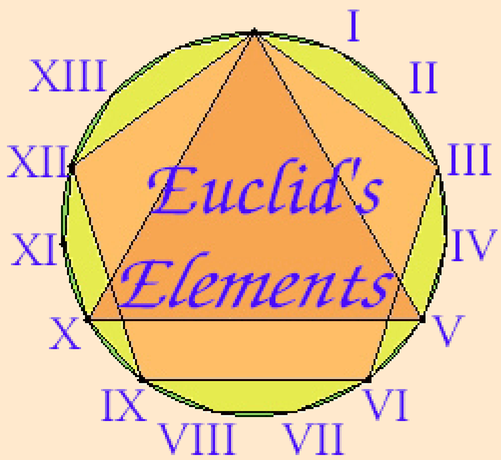
Book I



Book I

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Definitions

Definition 1.
A *point* is that which has no part.

Definition 2.
A *line* is breadthless length.

Definition 3.
The ends of a line are points.

Definition 4.
A *straight line* is a line which lies evenly with the points on itself.

Definition 5.
A *surface* is that which has length and breadth only.

Definition 6.
The edges of a surface are lines.

Definition 7.
A *plane surface* is a surface which lies evenly with the straight lines on itself.

Definition 8.

A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Definition 9.
And when the lines containing the angle are straight, the angle is called *rectilinear*.

Definition 10.
When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.

Definition 11.
An *obtuse angle* is an angle greater than a right angle.

Definition 12.
An *acute angle* is an angle less than a right angle.

Definition 13.
A *boundary* is that which is an extremity of anything.

Definition 14.
A *figure* is that which is contained by any boundary or boundaries.

Definition 15.
A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16.
And the point is called the *center* of the circle.

Definition 17.
A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Definition 18.
A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Definition 19.
Rectilinear figures are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.

Definition 20.
Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

Definition 21.
Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.

Definition 22.
Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not equilateral; a *rhombus* that which is equilateral but not right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia*.

Definition 23

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

Let the following be postulated:

Postulate 1.

To draw a straight line from any point to any point.

Postulate 2.

To produce a finite straight line continuously in a straight line.

Postulate 3.

To describe a circle with any center and radius.

Postulate 4.

That all right angles equal one another.

Postulate 5.

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

Common notion 1.

Things which equal the same thing also equal one another.

Common notion 2.

If equals are added to equals, then the wholes are equal.

Common notion 3.

If equals are subtracted from equals, then the remainders are equal.

Common notion 4.

Things which coincide with one another equal one another.

Common notion 5.

The whole is greater than the part.

Propositions

Proposition 1.

To construct an equilateral triangle on a given finite straight line.

Proposition 2.

To place a straight line equal to a given straight line with one end at a given point.

Proposition 3.

To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition 4.

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proposition 5.
In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition 6.
If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition 7.
Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition 8.
If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition 9.
To bisect a given rectilinear angle.

Proposition 10.
To bisect a given finite straight line.

Proposition 11.
To draw a straight line at right angles to a given straight line from a given point on it.

Proposition 12.
To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition 13.
If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition 14.
If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition 15.
If two straight lines cut one another, then they make the vertical angles equal to one another.

Corollary. If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles.

Proposition 16.
In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Proposition 17.
In any triangle the sum of any two angles is less than two right angles.

Proposition 18.
In any triangle the angle opposite the greater side is greater.

Proposition 19.
In any triangle the side opposite the greater angle is greater.

Proposition 20.
In any triangle the sum of any two sides is greater than the remaining one.

Proposition 21.
If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition 22.
To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition 23.
To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition 24.
If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition 25.
If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

Proposition 26.
If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Proposition 27.
If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition 28.
If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Proposition 29.
A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Proposition 30.
Straight lines parallel to the same straight line are also parallel to one another.

Proposition 31.
To draw a straight line through a given point parallel to a given straight line.

Proposition 32.
In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Proposition 33.
Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Proposition 34.
In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

Proposition 35.
Parallelograms which are on the same base and in the same parallels equal one another.

Proposition 36.
Parallelograms which are on equal bases and in the same parallels equal one another.

Proposition 37.
Triangles which are on the same base and in the same parallels equal one another.

Proposition 38.
Triangles which are on equal bases and in the same parallels equal one another.

Proposition 39.
Equal triangles which are on the same base and on the same side are also in the same parallels.

Proposition 40.
Equal triangles which are on equal bases and on the same side are also in the same parallels.

Proposition 41.
If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

Proposition 42.
To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Proposition 43.
In any parallelogram the complements of the parallelograms about the diameter equal one another.

Proposition 44.
To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Proposition 45.
To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Proposition 46.
To describe a square on a given straight line.

Proposition 47.
In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Proposition 48.
If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

Guide

About the Definitions

The *Elements* begins with a list of definitions. Some of these indicate little more than certain concepts will be discussed, such as Def.I.1, Def.I.2, and Def.I.5, which introduce the terms point, line, and surface. (Note that for Euclid, the concept of line includes curved lines.) Others are substantial definitions which actually describe new concepts in terms of old ones. For example, Def.I.10 defines a *right angle* as one of two equal adjacent angles made when one straight line meets another. Other definitions look like they're substantial, but actually are not. For instance, Def.I.4 says a *straight line* "is a line which lies evenly with the points on itself." No where in the *Elements* is the defining phrase "which lies evenly with the points on itself" applicable. Thus, this definition indicates, at most, that some lines under discussion will be straight lines.

It has been suggested that the definitions were added to the *Elements* sometime after Euclid wrote them. Another possibility is that they are actually from a different work, perhaps older. In Def.I.22 special kinds of quadrilaterals are defined including square, oblong (a rectangle that are not squares), rhombus (equilateral but not a square), and rhomboid (parallelogram but not a rhombus). Except for squares, these other shapes are not mentioned in the *Elements*. Euclid does use parallelograms, but they're not defined in this definition. Also, the exclusive nature of some of these terms—the part that indicates not a square—is contrary to Euclid's practice of accepting squares and rectangles as kinds of parallelograms.

About the Postulates

Following the list of definitions is a list of postulates. Each postulate is an axiom—which means a statement which is accepted without proof— specific to the subject matter, in this case, plane geometry. Most of them are constructions. For instance, Post.I.1 says a straight line can be drawn between two points, and Post.I.3 says a circle can be drawn given a specified point to be the center and another point to be on the circumference. The fourth postulate, Post.I.4, is not a constuction, but says that all right angles are equal.

About magnitudes and the Common Notions

The Common Notions are also axioms, but they refer to magnitudes of various kinds. The kind of magnitude that appears most frequently is that of straight line. Other important kinds are rectilinear angles and areas (plane figures). Later books include other kinds.

In proposition III.16 (but nowhere else) angles with curved sides are compared with rectilinear angles which shows that rectilinear angles are to be considered as a special kind of plane angle. That agrees with Euclid's definition of them in I.Def.9 and I.Def.8.

Also in Book III, parts of circumferences of circles, that is, arcs, appear as magnitudes. Only arcs of equal circles can be compared or added, so arcs of equal circles comprise a kind of magnitude, while arcs of unequal circles are magnitudes of different kinds. These kinds are all different from straight lines. Whereas areas of figures are comparable, different kinds of curves are not.

Book V includes the general theory of ratios. No particular kind of magnitude is specified in that book. It may come as a surprise that ratios do not themselves form a kind of magnitude since they can be compared, but they cannot be added. See the guide on Book V for more information.

Number theory is treated in Books VII through IX. It could be considered that numbers form a kind of magnitude as pointed out by Aristotle.

Beginning in Book XI, solids are considered, and they form the last kind of magnitude discussed in the *Elements*.

The propositions

Following the definitions, postulates, and common notions, there are 48 propositions. Each of these propositions includes a statement followed by a proof of the statement. Each statement of the proof is logically justified by a definition, postulate, common notion, or an earlier proposition that has already been proven. There are gaps in the logic of some of the proofs, and these are mentioned in the commenaries after the propositions. Also included in the proof is a diagram illustrating the proof.

Some of the propositions are constructions. A construction depends, ultimately, on the constructive postulates about drawing lines and circles. The first part of a proof for a constuctive proposition is how to perform the construction. The rest of the proof (usually the longer part), shows that the proposed construction actually satisfies the goal of the proposition. In the list of propositions in each book, the constructions are displayed in red.

Most of the propositions, however, are not constructions. Their statements say that under certain conditions, certain other conditions logically follow. For example, Prop.I.5 says that if a triangle has the property that two of its sides are equal, then it follows that the angles opposite these sides (called the "base angles") are also equal. Even the propositions that are not constructions may have constructions included in their proofs since auxillary lines or circles may be needed in the explanation. But the bulk of the proof is, as for the constructive propositions, a sequence of statements that are logically justified and which culminates in the statement of the proposition.

Logical structure of Book I

The various postulates and common notions are frequently used in Book I. Only two of the propositions rely solely on the postulates and axioms, namely, I.1 and I.4. The logical chains of propositions in Book I are longer than in the other books; there are long sequences of propositions each relying on the previous.

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Euclid's Elements

Book I

Common Notions

C.N.1. Things which equal the same thing also equal one another.

C.N.2. If equals are added to equals, then the wholes are equal.

C.N.3. If equals are subtracted from equals, then the remainders are equal.

C.N.4. Things which coincide with one another equal one another.

C.N.5. The whole is greater than the part.

Guide

These common notions, sometimes called axioms, refer to magnitudes of one kind. The various kinds of magnitudes that occur in the *Elements* include lines, angles, plane figures, and solid figures. The first Common Notion could be applied to plane figures to say, for instance, that if a triangle equals a rectangle, and the rectangle equals a square, then the triangle also equals the square. Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot. For instance, a line cannot be added to a rectangle, nor can an angle be compared to a pentagon.

C.N.4 requires interpretation. On the face of it, it seems to say that if two things are identical (that is, they are the same one), then they are equal, in other words, anything equals itself. But the way it traditionally is interpreted is as a justification of a principle of superposition, which is used, for instance, in proposition I.4. Using this principle, if one thing can be moved to coincide with another, then they are equal. See the notes on I.4 for more discussion on this point.

C.N.5, the whole is greater than the part, could be interpreted as a definition of "greater than." To say one magnitude B is a part of another A could be taken as saying that A is the sum of B and C for some third magnitude C, the remainder. Symbolically, $A > B$ means that there is some C such that $A = B + C$. At any rate, Euclid frequently treats these two conditions as being equivalent.

There are a number of properties of magnitudes used in Book I besides the listed Common Notions. Here are a few of them and locations where they are used.

1.	If not $x = y$, then $x > y$ or $x < y$.	I.6
2.	Not both $x < y$ and $x = y$.	I.6
3.	If not not $x = y$, then $x = y$.	I.6
4.	If $x < y$ and $y = z$, then $x < z$.	I.7
5.	If $x < y$ and $y < z$, then $x < z$.	I.7
6.	If $x = y$ and $y < z$, then $x < z$.	I.16
7.	If $x < y$, then $x + z < y + z$.	I.17

8. If not $x > y$, then $x = y$ or $x < y$. I.19
9. If not $x < y$ and not $x = y$, then $x > y$. I.19
10. If $2x = 2y$, then $x = y$. I.37
11. If $x = y$, then $2x = 2y$. I.42

Number 3 is an instance of the logical principle of double negation, rather than a common notion. Number 11 is a special case of C.N.2 since doubling is a special case of addition, that is, $2x$ is just $x + x$. Some of the others are logical variants of each other, for instance, numbers 1, 8, and 9 are all equivalent to the statement that at least one of the three cases $x < y$, $x = y$, or $x > y$ holds. Statement 2 says that two of those cases cannot simultaneously hold. The statement that

> for two magnitudes x and y of the same kind, exactly one of the three cases $x < y$, $x = y$, or $x > y$ holds

is called the *law of trichotomy* for magnitudes. This law, in particular, really ought to have been made an explicit common notion.

A modern presentation

In modern mathematics, axioms such as these would form the basis of an abstract algebra. Typically a presentation is given symbolically and in terms of set theory, although the set theory isn't necessary. Here is an outline for a presentation for magnitudes. This outline doesn't have many of the details that would normally be included.

First, assume there is a binary relation on a set of magnitudes of the same kind called *equality*, denoted as usual with an equal sign as in $x = y$. (This equality is not identity as we want different magnitudes, such as two different triangles, to be equal. Alternatively, we could identify equal magnitudes so that equality is identity.) Assume that equality is what is called an *equivalence relation*, that is, it satisfies three axioms:

> Reflexivity: For each x, $x = x$.
>
> Symmetry: If $x = y$, then $y = x$.
>
> Transitivity: If $x = y$ and $y = z$, then $x = z$.

Next, assume a binary operation called *addition* and written the usual way, $x + y$. Furthermore, assume addition satisfies the axioms

> Substitution of equals: If $x = y$, then $x + z = y + z$, and $z + x = z + y$.
>
> Associativity: For each x, y, and z, $(x + y) + z = x + (y + z)$.
>
> Commutativity: For each x and y, $x + y = y + x$.

Associativity and commutativity together imply that the order that addition is performed is irrelevant. An algebra satisfying only associativity is called a *semigroup*, while a semigroup that also satisfies commutativity is called a *commutative semigroup* or an *Abelian semigroup*. When other axioms are added for zero and negation, then the algebra is called a *group*, and when commutative, an Abelian group. Groups are some of the most important algebraic structures in modern mathematics.

We can now define order in terms of addition. Define a binary relation *less than* by taking $x < y$ to mean that there is some z such that $x + z = y$. And let *greater than* just have the opposite order, that is, $x > y$ means $y < x$. A number of properties of order can be easily proved.

> If $x < y$ and $y = z$, then $x < z$.

If $x = y$ and $y < z$, then $x < z$.

If $x < y$ and $y < z$, then $x < z$.

If $x < y$, then $x + z < y + z$, and $z + x < z + y$.

Next, assume an axiom for cancellation:

If $x + z = y + z$, then $x = y$.

With this axiom, subtraction can be defined, at least up to equality. If $x < y$, that is to say, there is some z such that $x + z = y$, then we may define $y - x$ as that z, since, under the axiom of cancelation, any other magnitude w such that $x + w = y$ would equal z. Subtraction is characterized by the property that

$x + z = y$ if and only if $z = y - x$.

The expected properties of subtraction, listed below, can be easily proved. Whenever a difference is indicated, such as $x - y$, it is implicitly assumed that $x > y$. Only a few of these properties are used in Book I.

If $x = y$, then $x - z = y - z$, and $w - x = w - y$

If $x = y$ and $w = z$, then $x - w = y - z$.

$(x + y) - y = x$.

$(x - y) + y = x$.

$(x - y) - (w - z) = (x - w) - (y - z)$.

If $x < y$, then $z - x > z - y$.

If $x < y$ and $w = z$, then $x - w < y - z$.

If $x = y$ and $w < z$, then $x - w > y - z$.

If $x < y$ and $w > z$, then $x - w < y - z$.

The law of trichotomy still isn't covered. It can be split into two parts: at most one of the three cases can occur, and at least one of the three cases occurs. The first can be stated as an axiom of addition as

It is not the case that $x = x + y$.

And that says it is not the case that $x > x$. The other half requires the axiom

For each x and y, either $x = y$ or there is some z such that $x + z = y$, or there is some z such that $x = y + z$.

With these axioms, all the properties of magnitudes needed in the first few books of the *Elements* can be proved. For instance, we can prove

If $2x = 2y$, then $x = y$.

using the same outline that Euclid used to prove proposition I.6.

Let twice x equal twice y. I say that x equals y.

If x does not equal y, then one of them is greater. Let x be greater. Then $x + x > y + y$, that is, twice x is greater than twice y. But twice x was assumed to equal twice y, the less equals the greater, which is absurd. Therefore x and y are not unequal. Therefore they are equal. Q.E.D.

Book V will require more properties of magnitudes, and in that book, pairs of magnitudes of different kinds will be compared by using ratios.

Next proposition: I.1

Previous: Postulate 5

Book I introduction

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Euclid's Elements

Book I

Definition 1

A *point* is that which has no part.

Guide

The *Elements* is the prime example of an axiomatic system from the ancient world. Its form has shaped centuries of mathematics. An axiomatic system should begin with a list of the terms that it will use. This definition says that one term that will be used is that of *point*. The next few definitions give some more terms that will be used. Although there is some description to go along with the terms, that description is actually never used in the exposition of the axiomatic system. It can, at most, be used to orient the reader.

The description of a point, "that which has no part," indicates that Euclid will be treating a point as having no width, length, or breadth, but as an indivisible location.

Later definitions will define terms by means of terms defined before them, but the first few terms in the *Elements* are not defined by means of other terms; they're "primitive" terms. Their meaning comes from properties about them that are assumed later in axioms. In the *Elements*, the axioms come in two kinds: postulates and common notions. The first postulate, I.Post.1, for instance, gives some meaning to the term "point." It states that a straight line may be drawn between any two points. Other postulates add more meaning to the term "point."

Actually, Euclid failed to notice that he made a number of conclusions without complete justification at a number of places in the *Elements*. This usually means that a postulate, that is, a explicit assumption, is missing.

Next definition: I.Def.2

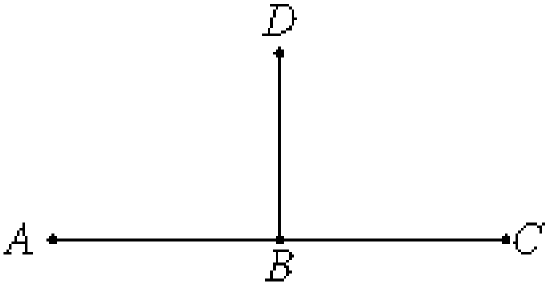
Book I introduction

Select from Book I

Select book

Select topic

D.E.Joyce
Clark University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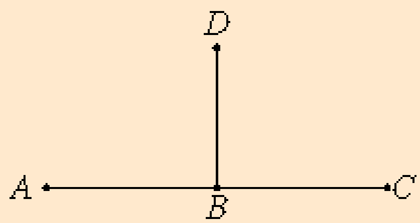
Euclid's Elements

Book I

Definition 10

When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.

Guide



In the figure, the two angles *DBA* and *DBC* are equal, so they are right angles by definition, and so the line *BD* set up on the line *AC* is perpendicular to it. Later there will be a postulate (Post.4) which states that all right angles are equal, and after a few propositions, it can be shown that *AC* is also perpendicular to *BD.* There are no postulates that explicitly state perpendiculars exist. Instead a construction for them is given and proved in proposition I.11.

The word "orthogonal" is frequently used in mathematics as a synonym for "perpendicular."

This is the first mention in the *Elements* of magnitudes being equal. There are several different kinds of magnitudes in the *Elements* besides angles. Lines, plane figures, and solids are also kinds of magnitudes. Some of the assumptions about magnitudes are stated later as "common notions" C.N., which are often called "axioms." One thing that magnitudes of the same kind can be is "equal," as the angles in this definition can be. Nowhere does Euclid explicitly state what it means for angles to be equal, or for that matter, for lines, plane figures, or solids to be equal, although much can be determined by the way he uses equality.

Next definition: I.Def.11-12

Previous: I.Def.9

Book I introduction

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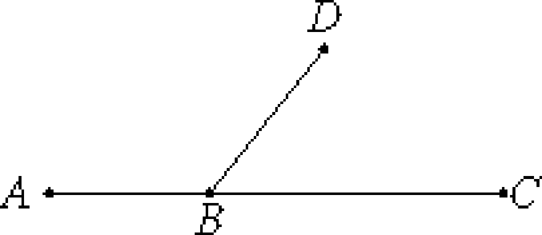
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Euclid's Elements

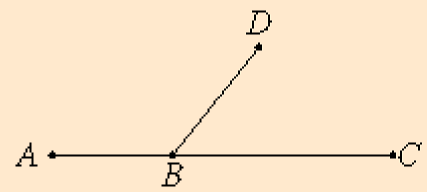
Book I

Definitions 11 and 12

Def. 11. An *obtuse angle* is an angle greater than a right angle.

Def. 12. An *acute angle* is an angle less than a right angle.

Guide



The angle *ABD* in the figure is an obtuse angle. It is greater than a right angle, but less than two right angles. Recall that Euclid required that any angle be less than two right angles.

The angle *CBD* is an acute angle. It is less than a right angle.

Note that there is no requirement that the angle be rectilinear, indeed, the horn angles mentioned before are not rectilinear, but they are less than right angles, and so are acute (notwithstanding Proclus' remarks to the contrary).

With these definition, we see another aspect of magnitudes, namely, two magnitudes of the same kind, such as two angles, can be compared for size. Euclid frequently uses what is known as the *law of trichotomy*: given two magnitudes *F* and *G* of the same kind, exactly one of the following three situations hold, *F* is less than *G*, *F* equals *G*, or *F* is greater than *G*. See the comments after the Common Notions for more discussion on magnitudes and the law of trichotomy.

Next definition: I.Def.13-14

Previous: I.Def.10

Book I introduction

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Euclid's Elements
Book I
Definitions 13 and 14

Def. 13. A *boundary* is that which is an extremity of anything.

Def. 14. A *figure* is that which is contained by any boundary or boundaries.

Guide

These are rather nebulous definitions since they are based on the undefined terms "extremity" and "contained by." Euclid deals with two kinds of figures in the *Elements*: plane figures and solid figures. Plane figures are defined in the upcoming definitions: circles and semicircles in I.Def.15 and I.Def.18, rectilinear figures in I.Def.19 and particular kinds of rectilinear figures such as triangles and quadrilaterals following that. Specific solid figures such as spheres, cones, pyramids, and various polyhedra are defined in Book XI. Plane figures are not solid figures since they are not contained by any boundaries in space. Thus, implicit to the concept of figure is the ambient plane or space of the figure.

Extremities, boundaries, and topology

Euclid deals with three kinds of extremities, or boundaries. There are the ends of lines (I.Def.3), the edges of surfaces (I.Def.3), and the surfaces of solids (XI.Def.2). A finite line has two points as its boundaries. A circle is defined in I.Def.15 as is a plane figure and has its circumference as its boundary. A sphere is defined in XI.Def.14 as a solid figure and has a spherical surface as its boundary.

The modern subject of topology studies space in a different way than geometry does. The geometric concepts of straightness, distance, and angle are excluded from topology, but the concept of boundary is central to topology. In topology, a sphere remains a sphere even when it's squeezed or stretched.

Not everything has a boundary. For instance, the circumference of a circle has no boundary. Also a spherical surface has no boundary. In topology, a finite region with no boundary is called a *cycle*. Circles and spherical surfaces are cycles. In general, if something is a boundary, it has no boundary itself. So boundaries are cycles. But not all cycles are boundaries.

Topology uses observation to distinguish various spaces. For instance, on a spherical surface, every circle is the boundary of a region on that surface. But on a toroidal surface (rotate a circle around a line in the plane of the circle that doesn't meet the circle), there are circles (for instance, that circle mentioned parenthetically) that don't bound any region on the surface. Thus, spherical surfaces are topologically different from toroidal surfaces.

Figures and their boundaries

The definition of figure needs to be fleshed out. In order to be a figure, a region must be bounded, that is, held in by a boundary. For instance, an infinite plane is unbounded, so it is not intended to be a figure. Neither is the region between two parallel lines even though that region has the two parallel lines as its extremities.

Other figures may be considered if other ambient spaces are allowed, although Euclid only uses plane and solid figures. For a one-dimensional example, a line segment could be considered to be a figure in an infinite line with its endpoints as its boundary. Also, a hemisphere could be considered to be a figure on the surface of a sphere with the equator as its boundary.

Next definition: I.Def.15-18

Previous: I.Def.11-12

Book I introduction

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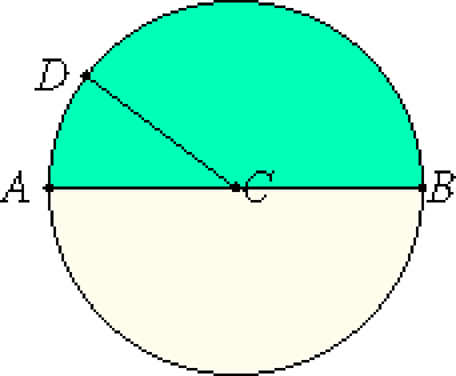
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Euclid's Elements
Book I
Definitions 15, 16, 17, and 18

Def. 15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Def. 16. And the point is called the *center* of the circle.

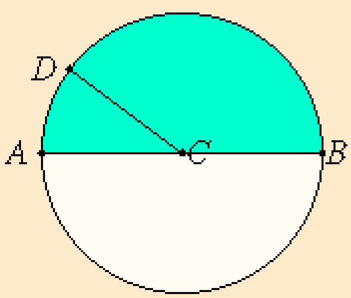
Def. 17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Def. 18. A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Guide

A definition such as this describes what circles are. Definitions do not guarantee the existence of the things they define. The existence of circles follows from a postulate, namely, Post.3.

Note that a circle for Euclid is a two-dimensional figure. But in modern mathematics, usually the word "circle" refers to what Euclid calls the circumference of a circle.



The center of the circle in the diagram is the point *C*. It's interesting that the English word "center" derives from the Greek word which also means a prod or a poker, and it refers to the fixed leg of a compass used to draw a circle.

The (curved) line *ABD* that contains the circle is its circumference. Euclid typically names a circle by three points on its circumference. Perhaps a better translation than "circumference" would be "periphery" since that is the Greek word while "circumference" derives from the Latin.

Euclid doesn't have a term for "radius" other than "that from the center," but "radius" is such a useful word that it is used here to translate "that from the center," such as the radius *CD.*

An example diameter is the line *AB* which passes through the center. Of course, a diameter is twice a radius, and since the radii are all equal to each other by definition, the diameters also all equal to each other.

That diameters "also bisects the circle" should not be part of the definition, but either assumed as a postulate or proved as a proposition. It depends on the fact that circles are drawn on planes, and planes have constant curvature. The analogous figure on a surface of nonconstant curvature does not have this property. For such figures the two "semicircles" on either side of a "diameter" need not be equal.

Although circles are used throughout Book I, the proper theory of circles doesn't begin until Book III. That book begins with more definitions relating to circles including the equality of circles, when circles touch (are tangent to) lines and other circles, and so forth.

Next definition: I.Def.19

Previous: I.Def.13-14

Book I introduction

Select from Book I

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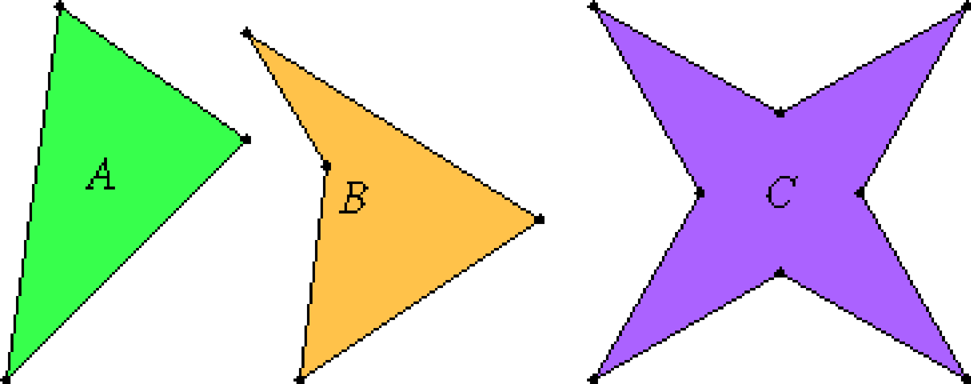
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Euclid's Elements

Book I

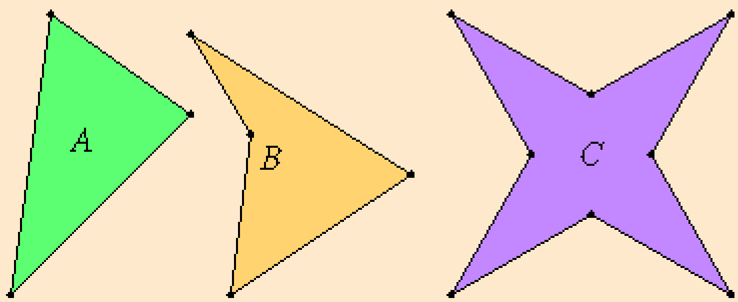
Definition 19

Rectilinear figures are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.

Guide

Euclid classifies rectilinear figures by their number of sides in this definition. Classifying them by their number of angles could lead to complications since an angle has to be less than two right angles, and a non-convex figure would have an internal angle greater than two right angles.

The modern English names, however, are based an the number of angles (except quadrilateral): triangle, pentagon, hexagon, heptagon, octagon, etc. From pentagon on up these names derive from the Greek, but they're rarely used past octagon.



Next proposition: I.Def.20-21

Previous: I.Def.15-18

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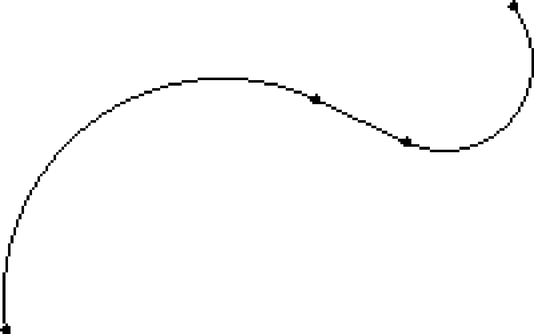


Euclid's Elements

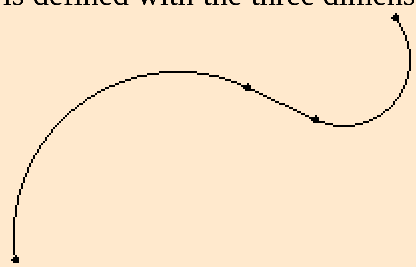
Book I

Definition 2

A *line* is breadthless length.

Guide

"Line" is the second primitive term in the *Elements*. The description, "breadthless length," says that a line will have one dimension, length, but it won't have breadth or depth. In I.Def.5 a surface is defined with the two dimensions length and breadth, and in XI.Def.1 a solid is defined with the three dimensions length, breadth, and depth.



One cannot tell from this definition what kind of line is meant by "line," but later a "straight" line defined to be a special kind of line. One can conclude, then, that "lines" need not be straight. Perhaps "curve" would be a better translation than "line" since Euclid meant what is commonly called a curve in modern English, where a curve may or may not be straight.

Also, from the next definition, it is apparent that Euclid's lines may have ends, so they are "line segments" or "curve segments." But they need not have ends in all cases since the entire circumference of a circle is an example of a line. Indeed, lines need not be finite in all cases; there are a few instances in the *Elements* where a line is not bounded, and that is usually indicated by the language. See, for example, proposition I.12.

One piece of terminology that Euclid did not mention explicitly in a definition is a phrase to indicate when a line passes through a point. That would be a "primitive" relation that could hold between a line and a point. Postulates would be included as well to give meaning to the phrase as they are in modern treatments of elementary geometry.

Next definition: I.Def.3

Previous: I.Def.1

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Select from Book I

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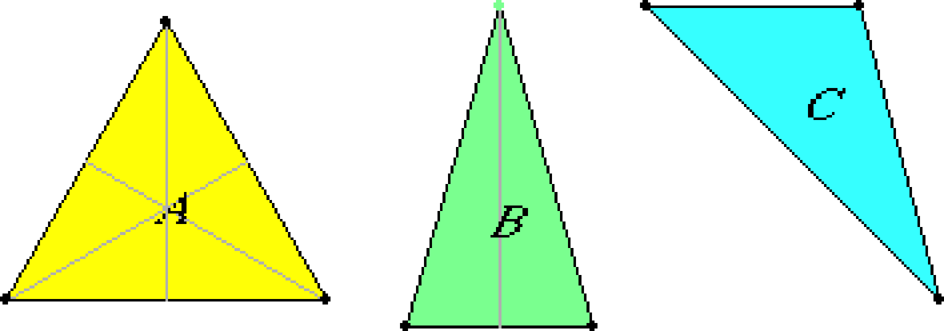
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Euclid's Elements

Book I

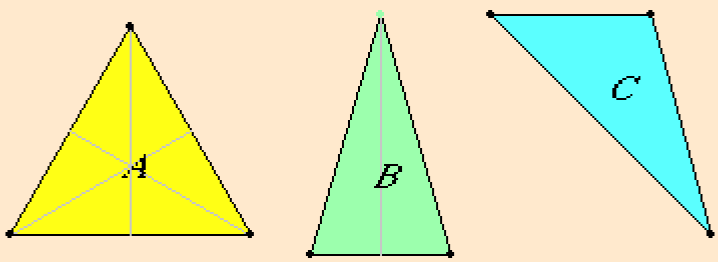
Definitions 20 and 21

Def. 20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

Def. 21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.

Guide

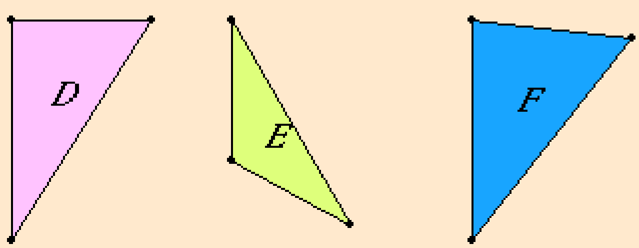
Definition 20 classifies triangles by their symmetries, while definition 21 classifies them by the kinds of angles they contain.



The scalene triangle *C* has no symmetries, but the isosceles triangle *B* has a bilateral symmetry. The equilateral triangle *A* not only has three bilateral symmetries, but also has 120°-rotational symmetries.

As defined by Euclid, an equilateral triangle is not to be considered as an isosceles triangle, but in modern terminology, it is usually the case that equilateral triangles are included among the isosceles triangles, that is, it is only required that at least two sides be equal in order for a triangle to be isosceles. Generally speaking, modern definitions are inclusive whereas Euclid's definitions are usually exclusive.

Equilateral triangles are constructed in the very first proposition of the Elements, I.1. An alternate characterization of isosceles triangles, namely that their base angles are equal, is demonstrated in propositions I.5 and I.6.



Since triangle *D* has a right angle, it is a right triangle. Proposition I.17 states that the sum of any two angles in a triangle is less than two right angles, therefore, no triangle can contain more than one right angle. Furthermore, there can be at most one obtuse angle, and a right angle and an obtuse angle cannot occur in the same triangle.

Triangle *E* is an obtuse triangle since it has an obtuse angle, while triangle *F* is an acute triangle since all its angles are acute.

Next definition: I.Def.22

Previous: I.Def.19

Book I introduction

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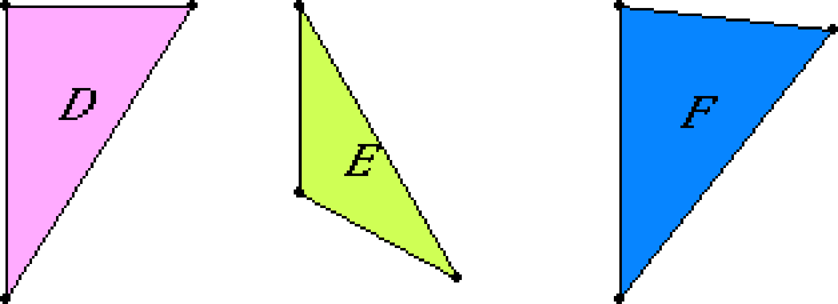
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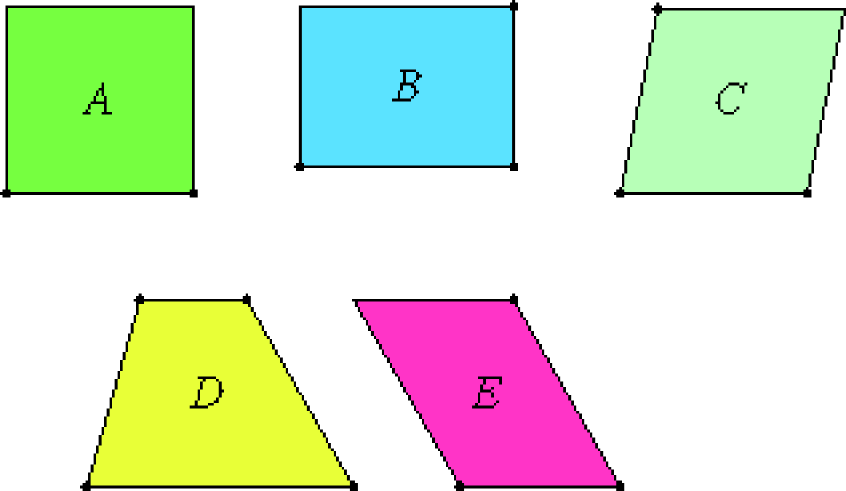
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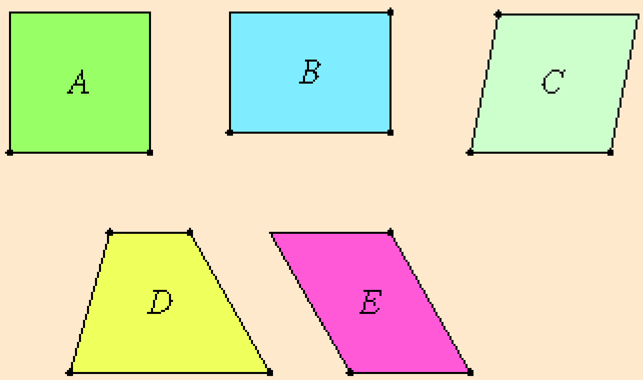
Euclid's Elements

Book I

Definition 22

Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not equilateral; a *rhombus* that which is equilateral but not right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia*.

Guide



The figure *A* is, of course, a square. Figure *B* is an oblong, or a rectangle. Figure *C* is a rhombus. Figure *D* is a trapezium (sometimes called a trapeze or trapezoid). And figure *E* is a parallelogram.

The only figure defined here that Euclid actually uses is the square. The other names of figures may have been common at the time of Euclid's writing, or they may have been left over from earlier authors' versions of the *Elements*. Euclid makes much use of parallelogram, or parallelogrammic area, which he does not define, but clearly means quadrilateral with parallel opposite sides. Parallelograms include rhombi and rhomboids as special cases. And rather than oblong, he uses rectangle, or rectangular parallelogram, which includes both squares and oblongs.

Squares and oblongs are defined to be "right-angled." Of course, that is intended to mean that all four angles are right angles. Sometimes Euclid's definitions are too brief, but the intended meaning can easily be determined from the way the definitions are used. In particular, proposition I.46 constructs a square, and all four angles are constructed to be right, not just one of them.

Next definition: I.Def.23

Previous: I.Def.20-21

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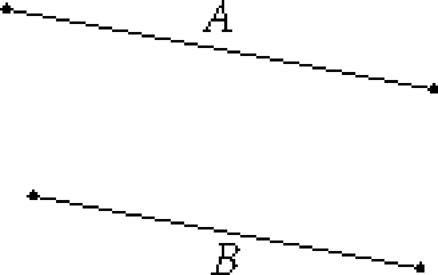
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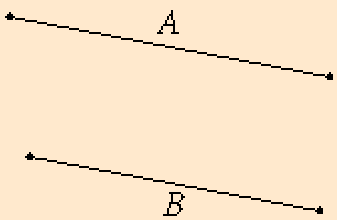
Euclid's Elements

Book I

Definition 23

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Guide



This definition only defines what it means for straight lines to be parallel; it does not say that there are any parallel lines. Proposition I.31 gives a construction for a line parallel to a given line through a given point.

Next proposition: I.Post.1

Previous: I.Def.22

Book I introduction

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Euclid's Elements

Book I

Definition 3

The ends of a line are points.

Guide

This statement can be taken as indicating that between certain lines and points a relation holds, that a point can be an end of a line. It doesn't say what ends are. It also doesn't indicate how many ends a line can have. For instance, the circumference of a circle has no ends, but a finite line has its two end points.

Next definition: I.Def.4

Previous: I.Def.2

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Euclid's Elements

Book I

Definition 4

A *straight line* is a line which lies evenly with the points on itself.

Guide

This statement indicates, at least, that the term "straight line" refers to a kind of line. It is hard to tell what else it means, if anything. Various commentators have interpreted in a variety of ways. The definition of plane surface in I.Def.7 uses a similar language that is equally opaque.

There are a some postulates that come a little later in Book I and give meaning to straight lines. I.Post.1 says that a straight line can be drawn between any two points, I.Post.2 says that a straight line can be extended, and the remaining postulates use the concept of straight line in one way or another.

Next definition: I.Def.5

Previous: I.Def.3

Book I introduction

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Euclid's Elements
Book I
Definition 5

A *surface* is that which has length and breadth only.

Guide

This statement suggests that a surface has two dimensions, but says very little, if anything, since neither length nor breadth have been defined yet, nor will they be. From the next definition, it is clear that a surface does not have to be a plane. Other examples of surfaces that appear in the *Elements* are surfaces of cones, cylinders, and spheres.

Next proposition: I.Def.6

Previous: I.Def.4

Book I introduction

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Euclid's Elements

Book I

Definition 6

The edges of a surface are lines.

Guide

As in I.Def.3, this statement describes a certain relationship, but this time between surfaces and lines. For example, a hemisphere is a surface, and its edge is the circumference of a circle, a kind of line.

This definition cannot actually be used since there are no postulates to go along with it to connect the edges of a surface in any way to the surface.

Euclid uses the same term for the end of a line in I.Def.3, the edge of a surface in this definition, and the surface of a solid in XI.Def.2. That term could be translated as "that which is around," "the limits of," or "the extremities of," but in English the terms "the ends of" a line, "the edges of" a surface, and either "the surfaces of" or "the faces of" a solid are fairly standard for different dimensions.

Next definition: I.Def.7

Previous: I.Def.5

Book I introduction

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Euclid's Elements

Book I

Definition 7

A *plane surface* is a surface which lies evenly with the straight lines on itself.

Guide

We see now that a plane surface, usually abbreviated to the single word "plane," is a kind of surface. Perhaps the remainder of the statement is a definition of content, but, if so, some words are missing.

One interpretation often given is that if a plane surface contains two points, then it contains the line connecting the two points. If that were the meaning, then it would be just as well to make that the explicit definition or to make it a postulate. But that does not seem to be Euclid's intent. His proposition XI.7 has a detailed proof that the line joining two points on two parallel lines lies in the plane of the two parallel lines. No proof at all would be necessary if that line were by definition or by postulate contained in a plane that contained its ends.

Note that a plane surface may be infinite, but needn't be infinite. It can be a square, a circle, or any other plane figure (Def.I.19).

There are no postulates in the *Elements* for the existance of plane surfaces, either finite or infinite. Post.3 says circles can be drawn, but a ambient plane is implicitly required there. Rectilinear figures are assumed to exist once the bounding lines have been constructed, but again, a plane is presumed to exist first. Throughout Books I through IV and Book VI, the books on plane geometry, there is the implicit assumption of one plane in which all the points, lines, and circles lie. In the books on solid geometry, Books XI through XIII, there is sometimes mentioned a "plane of reference," and proposition XI.2 claims that two intersecting lines determine a plane as does any triangle (but its proof fails completely).

Next definition: I.Def.8

Previous: I.Def.6

Book I introduction

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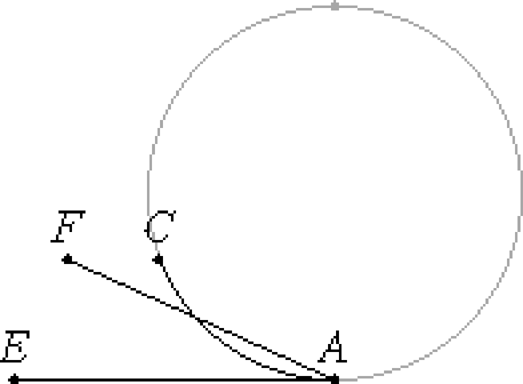
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Euclid's Elements

Book I

Definition 8

A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

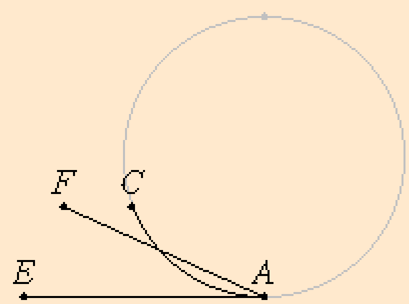
Guide

The concept of angle is a very important concept for all of Greek geometry. Many of the propositions require angles even for their statements.

The two lines are meant to emanate from the same point; two intersecting lines will actually make four angles.

The concept is also a difficult one, and, surprisingly, broader than our modern concept of angle.

As can be seen from the next definition of rectilinear angle, angles do not have to have straight sides; they can have curves as sides. The size of the angle does not depend on the length of the sides, but is determined only by how the two sides meet. In the *Elements* nearly all the angles are rectilinear, but angles with curved sides appear in proposition III.16. In that proposition, a so-called horn angle *CAE* is described as the angle between a circle and a straight tangent line and is shown to be smaller than any rectilinear angle *FAE*. Even though the curved side of the horn angle extends beyond any rectilinear angle, it is considered to be smaller since near the vertex *A* of the angle, the curvilinear angle *CAE* is entirely contained in the rectilinear angle *FAE*. Thus, an angle doesn't have an extent.



Next definition: I.Def.9

Previous: I.Def.7

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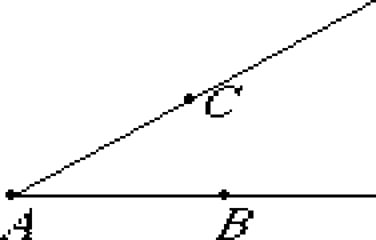
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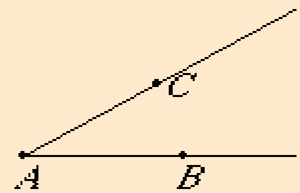
Euclid's Elements

Book I

Definition 9

And when the lines containing the angle are straight, the angle is called *rectilinear*.

Guide



This continues the previous definition of angle. Nearly all the angles that appear in the *Elements* are rectilinear as is the illustrated angle *BAC*. Angles usually are named by three points, the middle point the vertex of the angle. When there is no ambiguity it is sufficient to name the angle by its vertex, in this example, *A*.

Angles as magnitudes

As treated by Euclid, rectilinear angles are magnitudes that can be added together. When the sum of angles happens greater than two right angles, it is continued to be treated as a sum of angles rather than an individual angle. For instance, in proposition I.32 it is proved that the sum of the interior angles of a triangle equals two right angles.

Treating angles as magnitudes should not be confused with measuring angles. The angles themselves are the magnitudes. The only measurement of angles in the *Elements* is in terms of right angles (defined in the next definition). Degree measurement and radian measurement were not used until later. In terms of degrees a right angle is 90°, while in terms of radians a right angle is pi/2 radians.

Throughout ancient Greek mathematics, only positive magnitudes were considered. Zero and negative magnitudes were not conceived. For the most part, a lack of zero and negative magnitudes complicates mathematics, but occasionally simplifies it. In any case, the power of a mathematics without zero and negative magnitudes is no less in the sense that any statement made using the language of zero or negative magnitudes can be translated into a statement that doesn't use them, although the translated statement may be longer and less understandable.

Although in modern mathematics, angles can be positive, negative, or zero, and can be greater than a full circle (360° or 2 pi radians), in the *Elements* angles are always greater than zero and less than two right angles (180° or pi radians), except perhaps in one interpretation of proposition III.20 where the central angle of a circle could be greater than two right angles.

Next definition: I.Def.10

Previous: I.Def.8

Book I introduction

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Euclid's Elements
Book I Guide
A gentle introduction to Book I

For someone unfamiliar with the *Elements* it may take a while to see what's going on. The place to start is here, in Book I, the first of 13 books. It includes basic plane geometry. In the nineteenth century it was common for students to take a year to study the content of Books I and II, usually in slightly simplified form. The to take is that there is a lot of material in the book, and it's not especially easy.

Book I contains 23 definitions, five postulates, five common notions, and 47 propositions. Each proposition includes a proof based on the defintions, postulates, common notions, and earlier propositions. Although one could study book I sequentially, perhaps a better program is to start with the propositions and refer back to the definitions, postulates, and common notions as needed, and that is what's done here. We'll begin with the first proposition, I.1, and continue through the third, I.3, and refer back to the earlier material only as needed.

Proposition I.1

> *To construct an equilateral triangle on a given finite straight line.*

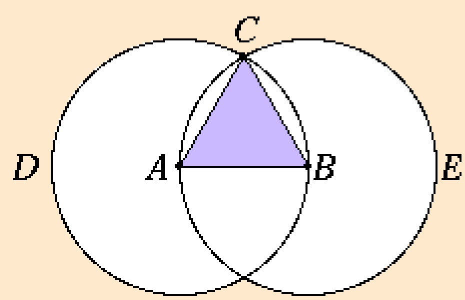
The very first proposition is a construction of an equilateral triangle given one of its sides. The page for this proposition has what Euclid wrote as well as a more detailed guide for the proposition. Click here to get the page to appear in a separate window.

At the top you see the statement:

> *To construct an equilateral triangle on a given finite straight line.*

Below the statement is a proof which (1) gives a construction, and (2) shows that the resulting figure includes an equilateral triangle. There is also a figure which illustrates the proposition. (Typically, each proposition in the *Elements* includes one figure.)

What does Euclid mean by a "construction"? His meaning is very limited. You are allowed (1) to draw a straight line between two points and extend that line indefinitely in either direction, and (2) draw a circle given its center and a point on its circumference. A construction consists of a sequence of these two operations.



When you look at the figure and read the first few sentences, it is apparent what the construction is. You start with a line *AB*, draw a couple circles and a couple lines, and you're done.

There are, however, many things that need clarification before beginning. Let's start with the concept of **equilateral triangle. In his 20th definition in the book, Def.I.20, Euclid defines and *equilateral triangle* as a triangle with three equal sides. Note that he doesn't inlude in his definition that the three angles must be equal, too; that would be called an *equiangular triangle.* Now, we all know that equilateral triangles are equiangular, but at this stage that has not been proved. If you skip forward to I.5, you'll see a proposition that implies that equilateral triangles are equiangular, and the next proposition, I.6, implies that equangular triangles are equilateral. But they come later. Here in I.1 we only need to construct a triangle with three equal sides.**

The next thing to clarify is the concept of straight line. What Euclid usually meant by this is a line segment, that

is, a piece of a straight line with two endpoints. His actual definition, which is found in Def.I.2 through Def.I.4, has no more useful information than the intuitive idea.

A remaining concept to clarify is that of triangle. Paraphrasing Euclid's definition in Def.I.19, (which relies on Def.I.13 and Def.I.14) a *triangle* is a two-dimensional figure bounded by three straight lines as sides. There are actually a lot of subtleties in these defintions, but in a gentle introduction like this, it is best to pass over them.

Let's return to the construction in I.1. We begin with a straight line *AB* which is to be one side of the constructed equilateral triangle. That is the given datum. We are then instructed to draw a circle with center *A* and radius *AB*. Over to the right of this statement is a reference to Postulate 3. This brings us to the concept of "postulate". For Euclid, a "postulate" was a statement about the particular subject of geometry that is to be *assumed* true. Postulates do not have proofs; they're literally taken for granted. This particular one, Post.3, says that given a point, such as *A*, which is to be the center of a circle, and another point *B*, which is to be on the circumference of a circle, you can construct the required circle. This postulate is an assumed construction, something that says a particular thing, in this case a circle, exists. So Post.3 is the *justification* required for the first construction in I.1. Every statement in a proof requires some kind of justification. Post.3 is also the justification for the next circle, the one with center *B* and radius *BA*.

Now, the two circles meet, so let *C* be one of the points where they meet. You'll notice that this is not justified, even though it's obvious. But obvious isn't enough. Sometimes statements seem obvious, but they're actually wrong. It's important in mathematics to know why you accept a statement, not just that you accept a statement. In this case Euclid should have included some postulate about when circles intersect.

The next construction is to draw straight lines *CA* and *CB*. These are justified by Post.1, which states that given two points, there is a line that has them as endpoints.

That concludes the construction part of the proof. The rest of it shows what has been constructed is an equilateral triangle. First, using the definition of a circle, Def.I.15 that all radii in a circle are equal, we conclude that line *AC* equals line *AB*, also that line *BC* equals line *BA*.

Next, we get to use a common notion as justification: C.N.1. For Euclid, a "common notion" was a statement about magnitudes in general that is to be *assumed* true. They are like postulates, but instead of being about geometry in particular, they're about magnitudes. Logically postulates and common notions are both axioms--explicitly assumed statements. The magnitudes that appear in I.1 are the three straight lines *AB*, *AC*, and *BC*. Although Euclid does not explicitly state how, straight lines can be added, compared, and subtracted. All we need here is the case when straight lines are equal, and C.N.1. says

> *Things which equal the same thing also equal one another.*

Since *AC* and *BC* both equal *AB*, C.N.1 justifies the conclusion that *AC* and *BC* are equal.

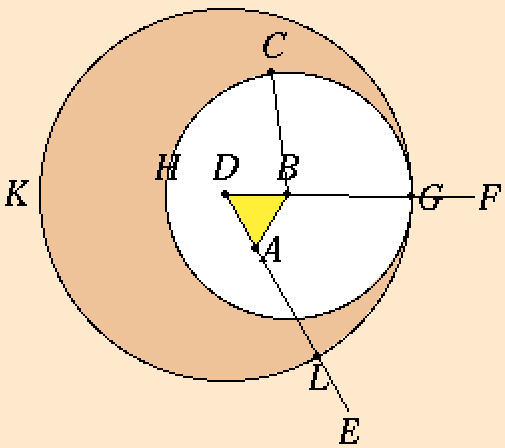
We now know that the three sides of the figure are equal straight lines. Therefore, we've constructed an equilateral triangle on the given straight line *AB*, as required. The Guide below I.1 goes into further detail about critiques of the proof, but we will continue on with

Proposition I.2

> *To place a straight line equal to a given straight line with one end at a given point.*

This second proposition, I.2 is also a construction. This time, the data include a point and a straight line (finite as usual), and the goal is to move the line over to the point; more precisely, to draw a line with one endpoint at the given point and equal to the given line.

This is the basic construction associated with a compass. You usually open the compass to a certain size, the length of a given line, and then draw a circle with a given center and that radius. But Euclid did not take this basic compass construction as a postulate because it is enough to assume a more limited postulate, Post.3, the one we used in the first proposition. That corresponds to a compass that doesn't transfer distances. Such a limited compass, which probably never actually existed, is called a "collapsible" compass. The reason that a more limited postulate is enough is in the proof of this second proposition, I.2.



Note that the figure in I.2 varies quite a bit in its appearance depending on the relative positions and distances of the data.

As you study the construction, you'll note that the construction we just made for an equilateral triangle is needed. Many of the propositions in the *Elements*, especially those in Book I, directly depend on the preceding proposition. Postulates 1 and three are used again, and for the first time Post.2, which allows a finite straight line to be extended indefinitely. Also I.Def.15, about equal radii in a circle, is used again, as is C.N.1.

C.N.3 is used for the first time:

If equals are subtracted from equals, then the remainders are equal.

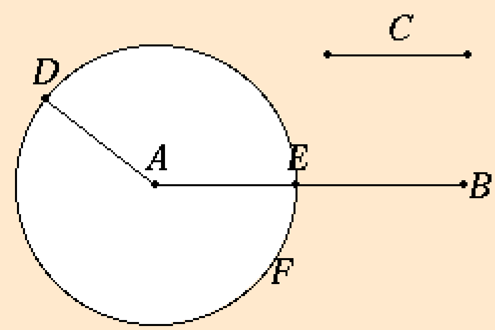
This common notion is used in the last step of the proof.

The construction and the proof in this proposition take time to understand well enough to reproduce. Nonetheless, it's rewarding to completely comprehend them, and valuable to understand that an ordinary compass can be simulated by a collapsible compass. Mathematicians often search for the simplest axioms and definitions for any field.

Proposition I.3

> *To cut off from the greater of two given unequal straight lines a straight line equal to the less.*

The third proposition, I.3, is also a construction. It performs a subtraction of straight lines, that is, it is a construction to cut a smaller line off from a greater line. The construction is quite easy, given the previous construction in I.2, as is the proof that the construction works. Simply move the shorter line so that one of its endpoints coincides with an endpoint of the longer line, and draw a circle with that point as center and the moved shorter line as radius. That circle will cut off the correct length from the longer line.



No new definitions, postulates, or common notions are needed in the proof of I.3.

As mentioned in the guide to I.3, this proposition is used more than any other proposition in the *Elements*.

Guide to Book I, continued

Book I Introduction

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Euclid's Elements
Book I Guide

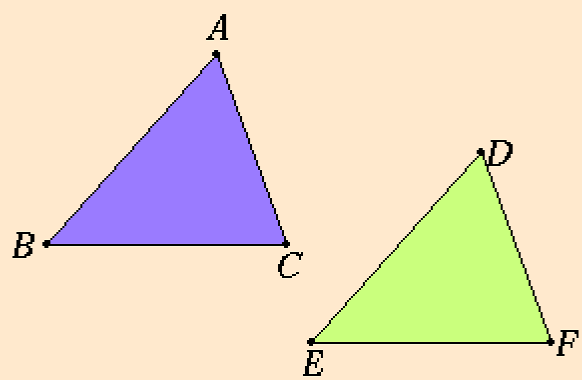
On Propositions I.4 through I.8

This next group of propositions includes two congruence propositions for triangles, I.4 and I.8, and two propositions about isosceles triangles, I.5 and I.6. Proposition I.7 is only used in I.8 and could have been made part of I.8, but was probably separated in order to reduce the length of the proof in I.8.

Proposition I.4

> *If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

This is the familiar side-angle-side congruence proposition for triangles. When two triangles have two sides and the included angle equal, then the remaining sides, angle, and area are also equal, that is to say, they're congruent. Symbolically, given triangles *ABC* and *DEF* with *AB* = *DE*, *AC* = *DF*, and angle *BAC* = angle *EDF*, then the rest of the parts of the triangles are the same.



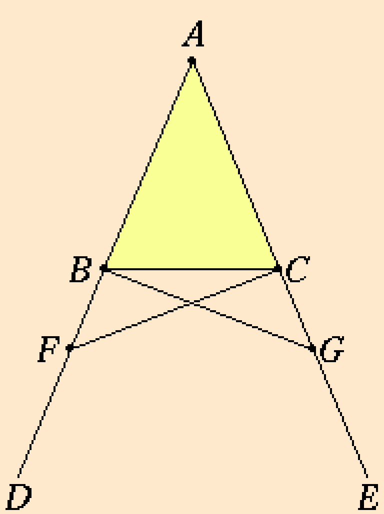
Euclid's proof of this proposition relies on the "principle of superposition". This principle is not supported by his postulates, and so it would be more appropriate to take I.4 as a postulate than pretend that it is adequately justified by its proof. For more discussion on this point, see the Guide for I.4.

There will be two other congruence propositions: I.8, side-side-side, and I.26, side and two angles.

Proposition I.5

> *In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.*

An *isosceles* triangle is defined in I.Def.20 as a triangle with two equal sides.



In the figure *AB* = *AC*, so triangle *ABC* is isosceles. One of the properties of such triangles is that they also have two equal angles, the base angles of the triangle, angles *ABC* and *ACB*. That's the first part of the statement of Proposition I.5. The other conclusion is that the angles supplementary to the base angles are also equal, that is, angles *DBC* and *ECB* are equal.

Guide to Book I, continued

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Book I Introduction

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Euclid's Elements
Book I Guide

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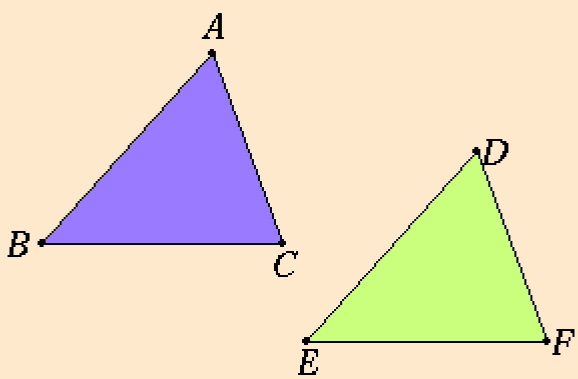
Proposition I.4



This is the familiar side-angle-side congruence proposition. When two triangles have two sides and the included angle equal, then the remaining sides, angle, and area are also equal, that is to say, they're congruent. Symbolically, given triangles *ABC* and *DEF* with *AB* = *DE*, *AC* = *DF*, and angle *BAC* = angle *EDF*, then the rest of the parts of the triangles are the same.

Guide to Book I, continued

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Book I Introduction

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The requested URL /~djoyce/java/elements/bookI/guide3.html was not found on this server.

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	propI27.gif	15-May-1997 21:43	1k
	propI26b.gif	15-May-1997 21:43	1k
	propI26a.gif	15-May-1997 21:43	1k
	propI26.gif	15-May-1997 21:43	1k
	propI25.gif	15-May-1997 21:43	1k
	propI24.gif	15-May-1997 21:43	2k
	propI23.gif	15-May-1997 21:42	1k
	propI22.gif	15-May-1997 21:42	2k
	propI21.gif	15-May-1997 21:42	1k
	propI20.gif	15-May-1997 21:42	1k
	propI18.gif	15-May-1997 21:41	1k
	propI17.gif	15-May-1997 21:41	1k
	propI16.gif	15-May-1997 21:41	1k
	propI15.gif	15-May-1997 21:41	1k
	propI13.gif	15-May-1997 21:41	1k
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	post4.gif	15-May-1997 21:40	1k
	post3.gif	15-May-1997 21:40	1k
	post2.gif	15-May-1997 21:40	1k
	post1.gif	15-May-1997 21:40	1k
	defI8.gif	15-May-1997 21:40	1k
	defI23.gif	15-May-1997 21:39	1k
	defI22.gif	15-May-1997 21:39	2k
	defI2.gif	15-May-1997 21:39	1k
	defI19.gif	15-May-1997 21:39	2k
	defI15.gif	15-May-1997 21:39	1k
	defI11.gif	15-May-1997 21:39	1k
	defI10.gif	15-May-1997 21:39	1k
	bkI.gif	15-May-1997 21:38	1k

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Index of /~djoyce/java/elements/bookI

Name	Last modified	Size	Description
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bkI.gif	15-May-1997 21:38	1k	
bookI.html	21-Oct-2002 08:55	34k	
cn.html	21-Oct-2002 08:56	17k	
cn.html%	06-Oct-2000 10:44	11k	
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defI10.gif	15-May-1997 21:39	1k	
defI10.html	21-Oct-2002 08:55	7k	
defI11.gif	15-May-1997 21:39	1k	
defI11.html	21-Oct-2002 08:55	7k	
defI13.html	21-Oct-2002 08:55	9k	
defI15.gif	15-May-1997 21:39	1k	
defI15.html	21-Oct-2002 08:55	9k	
defI19.gif	15-May-1997 21:39	2k	
defI19.html	21-Oct-2002 08:56	7k	
defI2.gif	15-May-1997 21:39	1k	
defI2.html	21-Oct-2002 08:55	8k	
defI20.gif	11-Jul-1997 22:47	2k	
defI20.html	21-Oct-2002 08:56	10k	
defI21.gif	01-Jul-1997 21:18	1k	
defI22.gif	15-May-1997 21:39	2k	
defI22.html	21-Oct-2002 08:56	9k	
defI23.gif	15-May-1997 21:39	1k	
defI23.html	21-Oct-2002 08:56	6k	
defI3.html	21-Oct-2002 08:55	6k	
defI4.html	21-Oct-2002 08:55	6k	
defI5.html	21-Oct-2002 08:55	6k	
defI6.html	21-Oct-2002 08:55	6k	
defI7.html	21-Oct-2002 08:55	7k	
defI8.gif	15-May-1997 21:40	1k	
defI8.html	21-Oct-2002 08:55	8k	
defI9.gif	01-Jul-1997 21:18	1k	
defI9.html	21-Oct-2002 08:55	8k	
favicon.ico	07-May-2001 14:54	1k	
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guide1.html%	12-May-1999 12:40	12k	
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guide2.html%	12-May-1999 13:16	3k	

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guide2.html`%	12-May-1999	12:56	2k
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post1.html	21-Oct-2002	08:56	7k
post2.gif	15-May-1997	21:40	1k
post2.html	21-Oct-2002	08:56	10k
post2a.gif	11-Jul-1997	22:48	1k
post3.gif	15-May-1997	21:40	1k
post3.html	21-Oct-2002	08:56	7k
post4.gif	15-May-1997	21:40	1k
post4.html	21-Oct-2002	08:56	7k
post5.gif	15-May-1997	21:40	1k
post5.html	21-Oct-2002	08:56	8k
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propI1.gif	12-May-1999	10:05	1k
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propI11.html	18-Oct-2002	13:28	10k
propI11a.gif	17-Jun-1997	15:01	2k
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propI12.html	18-Oct-2002	13:28	10k
propI13.gif	15-May-1997	21:41	1k
propI13.html	18-Oct-2002	13:28	9k
propI14.gif	27-Jan-1998	14:02	1k
propI14.html	18-Oct-2002	13:28	9k
propI15.gif	15-May-1997	21:41	1k
propI15.html	18-Oct-2002	13:28	9k
propI16.gif	15-May-1997	21:41	1k
propI16.html	18-Oct-2002	13:28	14k
propI16a.gif	21-May-1997	13:18	2k
propI17.gif	15-May-1997	21:41	1k
propI17.html	18-Oct-2002	13:28	8k
propI18.gif	15-May-1997	21:41	1k
propI18.html	18-Oct-2002	13:28	9k
propI19.gif	03-Mar-1998	14:07	1k
propI19.html	18-Oct-2002	13:28	9k
propI19a.gif	21-May-1997	13:18	1k
propI1a.gif	11-Jul-1997	22:49	1k
propI2.gif	12-May-1999	10:05	2k
propI2.html	18-Oct-2002	13:27	12k
propI20.gif	15-May-1997	21:42	1k

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propI20.html	18-Oct-2002 13:28	12k
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propI21.gif	15-May-1997 21:42	1k
propI21.html	18-Oct-2002 13:28	10k
propI22.gif	15-May-1997 21:42	2k
propI22.html	18-Oct-2002 13:28	11k
propI23.gif	15-May-1997 21:42	1k
propI23.html	18-Oct-2002 13:28	11k
propI23a.gif	19-May-1997 20:44	2k
propI24.gif	15-May-1997 21:43	2k
propI24.html	18-Oct-2002 13:28	9k
propI25.gif	15-May-1997 21:43	1k
propI25.html	18-Oct-2002 13:28	8k
propI26.gif	15-May-1997 21:43	1k
propI26.html	18-Oct-2002 13:28	15k
propI26a.gif	15-May-1997 21:43	1k
propI26b.gif	15-May-1997 21:43	1k
propI27.gif	15-May-1997 21:43	1k
propI27.html	18-Oct-2002 13:28	11k
propI28.gif	15-May-1997 21:43	1k
propI28.html	18-Oct-2002 13:28	9k
propI29.gif	15-May-1997 21:43	1k
propI29.html	18-Oct-2002 13:28	14k
propI29a.gif	21-May-1997 13:18	1k
propI2a.gif	19-May-1997 20:44	2k
propI3.gif	12-May-1999 10:05	1k
propI3.html	18-Oct-2002 13:27	10k
propI30.gif	15-May-1997 21:43	1k
propI30.html	18-Oct-2002 13:28	10k
propI31.gif	15-May-1997 21:44	1k
propI31.html	18-Oct-2002 13:29	8k
propI32.gif	15-May-1997 21:44	1k
propI32.html	18-Oct-2002 13:29	9k
propI33.gif	15-May-1997 21:44	1k
propI33.html	18-Oct-2002 13:29	9k
propI34.gif	15-May-1997 21:44	1k
propI34.html	18-Oct-2002 13:29	9k
propI35.gif	15-May-1997 21:44	1k
propI35.html	18-Oct-2002 13:29	11k
propI35a.gif	12-Jun-1997 21:01	1k
propI36.gif	15-May-1997 21:45	1k
propI36.html	18-Oct-2002 13:29	9k
propI37.gif	15-May-1997 21:45	1k
propI37.html	18-Oct-2002 13:29	9k

	propI38.gif	15-May-1997 21:45	1k
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	propI44.html	18-Oct-2002 13:29	11k
	propI45.gif	15-May-1997 21:46	2k
	propI45.html	18-Oct-2002 13:29	11k
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	propI47x.gif	12-Jun-1997 21:05	1k
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	propI5.html	18-Oct-2002 13:27	12k
	propI6.gif	12-May-1999 10:59	1k
	propI6.html	18-Oct-2002 13:27	12k
	propI7.gif	12-May-1999 10:59	1k
	propI7.html	18-Oct-2002 13:27	10k
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	propI9.html	18-Oct-2002 13:27	11k
	propI9a.gif	01-Jul-1997 21:18	2k

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Name	Last modified	Size	Description
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propI8.gif	12-May-1999 10:59	2k	
propI7.html	18-Oct-2002 13:27	10k	
propI7.gif	12-May-1999 10:59	1k	
propI6.html	18-Oct-2002 13:27	12k	
propI6.gif	12-May-1999 10:59	1k	
propI5.html	18-Oct-2002 13:27	12k	
propI5.gif	12-May-1999 10:59	1k	
propI48.html	18-Oct-2002 13:29	8k	
propI48.gif	15-May-1997 21:47	1k	
propI47x.gif	12-Jun-1997 21:05	1k	
propI47b.gif	03-Mar-1998 14:07	1k	
propI47a.gif	11-Jul-1997 22:47	3k	
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propI46a.gif	12-Jun-1997 21:01	2k	
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propI46.gif	03-Mar-1998 14:07	1k	
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propI45.gif	15-May-1997 21:46	2k	
propI44.html	18-Oct-2002 13:29	11k	
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propI43.gif	15-May-1997 21:46	1k	
propI42.html	18-Oct-2002 13:29	10k	
propI42.gif	03-Mar-1998 14:07	2k	
propI41.html	18-Oct-2002 13:29	8k	
propI41.gif	15-May-1997 21:46	1k	
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	propI36.html	18-Oct-2002 13:29	9k
	propI36.gif	15-May-1997 21:45	1k
	propI35a.gif	12-Jun-1997 21:01	1k
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	propI32.html	18-Oct-2002 13:29	9k
	propI32.gif	15-May-1997 21:44	1k
	propI31.html	18-Oct-2002 13:29	8k
	propI31.gif	15-May-1997 21:44	1k
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	propI30.gif	15-May-1997 21:43	1k
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	propI3.gif	12-May-1999 10:05	1k
	propI2a.gif	19-May-1997 20:44	2k
	propI29a.gif	21-May-1997 13:18	1k
	propI29.html	18-Oct-2002 13:28	14k
	propI29.gif	15-May-1997 21:43	1k
	propI28.html	18-Oct-2002 13:28	9k
	propI28.gif	15-May-1997 21:43	1k
	propI27.html	18-Oct-2002 13:28	11k
	propI27.gif	15-May-1997 21:43	1k
	propI26b.gif	15-May-1997 21:43	1k
	propI26a.gif	15-May-1997 21:43	1k
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	propI26.gif	15-May-1997 21:43	1k
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	propI25.gif	15-May-1997 21:43	1k
	propI24.html	18-Oct-2002 13:28	9k
	propI24.gif	15-May-1997 21:43	2k
	propI23a.gif	19-May-1997 20:44	2k
	propI23.html	18-Oct-2002 13:28	11k
	propI23.gif	15-May-1997 21:42	1k
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	propI22.gif	15-May-1997 21:42	2k
	propI21.html	18-Oct-2002 13:28	10k
	propI21.gif	15-May-1997 21:42	1k

propI20a.gif	07-Oct-2002 09:52	2k
propI20.html	18-Oct-2002 13:28	12k
propI20.gif	15-May-1997 21:42	1k
propI2.html	18-Oct-2002 13:27	12k
propI2.gif	12-May-1999 10:05	2k
propI1a.gif	11-Jul-1997 22:49	1k
propI19a.gif	21-May-1997 13:18	1k
propI19.html	18-Oct-2002 13:28	9k
propI19.gif	03-Mar-1998 14:07	1k
propI18.html	18-Oct-2002 13:28	9k
propI18.gif	15-May-1997 21:41	1k
propI17.html	18-Oct-2002 13:28	8k
propI17.gif	15-May-1997 21:41	1k
propI16a.gif	21-May-1997 13:18	2k
propI16.html	18-Oct-2002 13:28	14k
propI16.gif	15-May-1997 21:41	1k
propI15.html	18-Oct-2002 13:28	9k
propI15.gif	15-May-1997 21:41	1k
propI14.html	18-Oct-2002 13:28	9k
propI14.gif	27-Jan-1998 14:02	1k
propI13.html	18-Oct-2002 13:28	9k
propI13.gif	15-May-1997 21:41	1k
propI12.html	18-Oct-2002 13:28	10k
propI12.gif	03-Mar-1998 14:07	2k
propI11a.gif	17-Jun-1997 15:01	2k
propI11.html	18-Oct-2002 13:28	10k
propI11.gif	27-Jan-1998 14:02	1k
propI10a.gif	01-Jul-1997 21:18	2k
propI10.html	18-Oct-2002 13:27	10k
propI10.gif	11-Jul-1997 22:49	1k
propI1.html	18-Oct-2002 13:27	15k
propI1.gif	12-May-1999 10:05	1k
prop11a.gif	19-May-1997 20:44	2k
post5.html	21-Oct-2002 08:56	8k
post5.gif	15-May-1997 21:40	1k
post4.html	21-Oct-2002 08:56	7k
post4.gif	15-May-1997 21:40	1k
post3.html	21-Oct-2002 08:56	7k
post3.gif	15-May-1997 21:40	1k
post2a.gif	11-Jul-1997 22:48	1k
post2.html	21-Oct-2002 08:56	10k
post2.gif	15-May-1997 21:40	1k
post1.html	21-Oct-2002 08:56	7k
post1.gif	15-May-1997 21:40	1k

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guide1.html	12-May-1999	12:40	12k
guide.html%	12-May-1999	09:40	12k
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defI7.html	21-Oct-2002	08:55	7k
defI6.html	21-Oct-2002	08:55	6k
defI5.html	21-Oct-2002	08:55	6k
defI4.html	21-Oct-2002	08:55	6k
defI3.html	21-Oct-2002	08:55	6k
defI23.html	21-Oct-2002	08:56	6k
defI23.gif	15-May-1997	21:39	1k
defI22.html	21-Oct-2002	08:56	9k
defI22.gif	15-May-1997	21:39	2k
defI21.gif	01-Jul-1997	21:18	1k
defI20.html	21-Oct-2002	08:56	10k
defI20.gif	11-Jul-1997	22:47	2k
defI2.html	21-Oct-2002	08:55	8k
defI2.gif	15-May-1997	21:39	1k
defI19.html	21-Oct-2002	08:56	7k
defI19.gif	15-May-1997	21:39	2k
defI15.html	21-Oct-2002	08:55	9k
defI15.gif	15-May-1997	21:39	1k
defI13.html	21-Oct-2002	08:55	9k
defI11.html	21-Oct-2002	08:55	7k
defI11.gif	15-May-1997	21:39	1k
defI10.html	21-Oct-2002	08:55	7k
defI10.gif	15-May-1997	21:39	1k
defI1.html	21-Oct-2002	08:55	7k
cn.html%	06-Oct-2000	10:44	11k
cn.html	21-Oct-2002	08:56	17k
bookI.html	21-Oct-2002	08:55	34k
bkI.gif	15-May-1997	21:38	1k

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A
B

Euclid's Elements

Book I

Postulate 1

To draw a straight line from any point to any point.

Guide

This first postulate says that given any two points such as *A* and *B*, there is a line *AB* which has them as endpoints. This is one of the constructions that may be done with a straightedge (the other being described in the next postulate).

Although it doesn't explicitly say so, there is a unique line between the two points. Since Euclid uses this postulate as if it includes the uniqueness as part of it, he really ought to have stated the uniqueness explicitly.

The last three books of the *Elements* cover solid geometry, and for those, the two points mentioned in the postulate may be any two points in space. Proposition XI.1 claims that if part of a line is contained in a plane, then the whole line is. In the books on plane geometry, it is implicitly assumed that the line *AB* joining *A* to *B* lies in the plane of discussion.

Next postulate: I.Post.2

Previous: I.Def.23

Book I introduction

Select from Book I

Select book

Select topic


C
A
B
D

Euclid's Elements

Book I

Postulate 2

To produce a finite straight line continuously in a straight line.

Guide

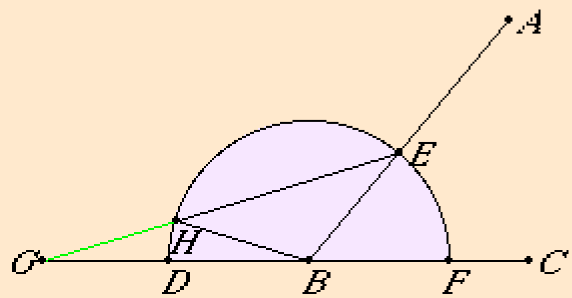
Here we have the second ability of a straightedge, namely, to extend a given line *AB* to *CD.* This postulate does not say how far a line can be extended. Sometimes it is used so that the extension equals some other line. Other times it is extended arbitrarily far.

As with the first postulate, it is implicitly assumed in the books on plane geometry that when a line is extended, it remains in the plane of discussion. The first proposition on solid geometry, proposition XI.1, claims that line can't be only partly in a plane. The central step in the proof of that proposition is to show that a line cannot be extended in two ways, that is, there is only one continuation of a line. The proof is hardly convincing. Rather, this postulate should include a clause to that effect.

Neusis: fitting a line into a diagram

Other uses of a straightedge can be imagined. For instance, it might be marked at two points on it, then fit into a diagram so that the two points fall on two lines, perhaps curved. This operation is an example of "neusis" or "verging" where lines are adjusted to fit the diagram. For instance, Archimedes, who lived in the century after Euclid, used neusis in several constructions in his work *On Spirals.* In the *Book of Lemmas,* attributed by Thabit ibn-Qurra to Archimedes, neusis is used to trisect an angle.

Suppose the angle *ABC* is to be trisected. Draw a circle *DEF* with center *B* and any radius. Extend *CB* through *D* and beyond. Fit in a line *GHE* passing through *E* and a point *G* on the line *CB* extended so that a segment (colored green) equal to the radius *BD* of the circle starts at *G* and ends at a point *H* on the circle. (You'll have to move *G* around until *H* lands on the circle.) Draw *BH.*



With the help of Euclid's propositions here in Book I, we can show that angle *EGC* is one-third of angle *ABC.* Since the lines *GH, HB,* and *BE* are equal, therefore the triangles *GHB* and *HBE* are isosceles. Therefore, by I.5, angle *HGB* equals *HBG,* and angle *BHE* equals angle *BEH.*

By I.32, the exterior angle *BHE* of triangle *GHB* equals the sum of the equal angles *HGB* and *HBG,* therefore angle *BHE* is double angle *HGB.* And angle *BEH* equals *BHE,* so it is also double angle *HGB.*

Again by I.32, the exterior angle *ABC* of triangle *BEG* is the sum of angles *HGB* and *BEH.* But angle *BEH* is double angle *HGB,* therefore angle *ABC* is triple angle *GHB.*

Therefore, angle *GHB* is one-third of angle *ABC.* Q.E.D.

The ancient Greek geometers believed that angle trisection required tools beyond those given in Euclid's postulates.

They were right, but it wasn't proved until Wantzel in 1837 proved that a 60°-angle cannot be so trisected using only Euclidean tools.

Euclid has no postulate for neusis constructions, and since neusis constructions can trisect angles, we conclude from Wantzel's theorem that another postulate is required to justify neusis constructions.

Next postulate: I.Post.3

Previous: I.Post.1

Book I introduction

Select from Book I

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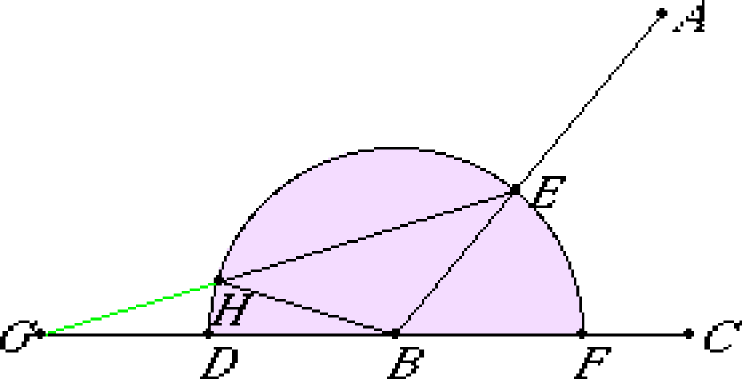
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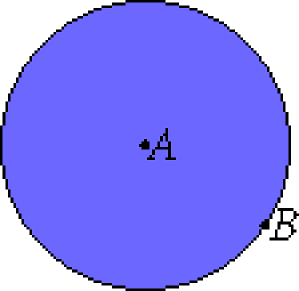
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Euclid's Elements

Book I

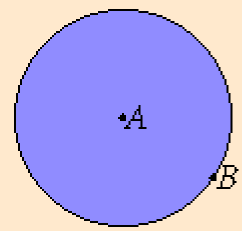
Postulate 3

To describe a circle with any center and radius.

Guide

This is the third and final assumed construction in the *Elements.* It corresponds to drawing a circle with a compass.

The given data are (1) a point *A* to be the center of the circle, (2) another point *B* to be on the circumference of the circle, and (3) a plane in which the two points lie. In the first few books of the *Elements,* there is but one plane under consideration and needn't be mentioned, but in the last three books which develop solid geometry, the plane has to be specified.



Note that this postulate does not allow for the compass to be moved. The usual way that a compass is used is that is is opened to a given width, then the pivot is placed on the drawing surface, then a circle is drawn as the compass is rotated around the pivot. But this postulate does not allow for transferring distances. It is as if the compass collapses as soon as it's removed from the plane. Proposition I.3, however, gives a construction for transferring distances. Therefore, the same constructions that can be made with a regular compass can also be made with Euclid's collapsing compass.

Next postulate: I.Post.4

Previous: I.Post.2

Book I introduction

Select from Book I

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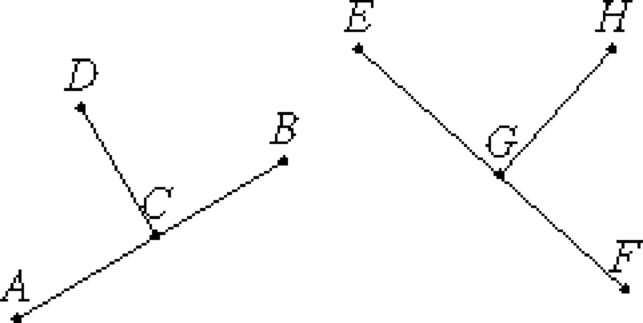
D.E.Joyce
Clark University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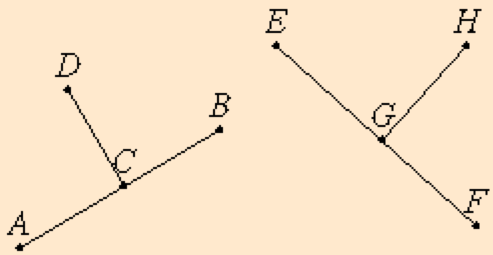
Euclid's Elements

Book I

Postulate 4

That all right angles equal one another.

Guide



In the definition of right angle, it is clear that the two angles at the foot of a perpendicular, such as angles *ACD* and *BCD*, are equal. This postulate says that an angle at the foot of one perpendicular, such as angle *ACD*, equals an angle at the foot of any other perpendicular, such as angle *EGH*.

Next postulate: I.Post.5

Previous: I.Post.3

Book I introduction

Select from Book I

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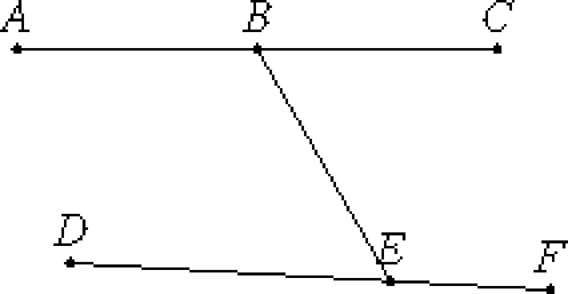
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Euclid's Elements

Book I

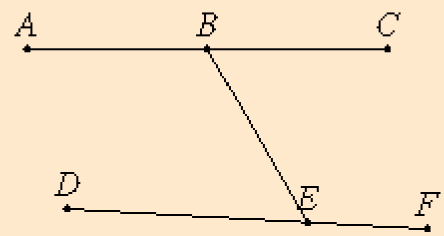
Postulate 5

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Guide

Of course, this is a postulate for plane geometry. It should include the condition that the two straight lines lie in a plane, otherwise, skew lines in space would satisfy the hypotheses. Also, without an ambient plane, the term "that side [of the straight line]" has no meaning.

In the diagram, if angle *ABE* plus angle *BED* is less than two right angles (180°), then lines *AC* and *DF* will meet when extended in the direction of *A* and *D*.



This postulate is usually called the "parallel postulate" since it can be used to prove properties of parallel lines. Euclid develops the theory of parallel lines in propositions through I.31.

The parallel postulate is historically the most interesting postulate. Geometers throughout the ages have tried to show that it could be proved from the remaining postulates so that it wasn't necessary to assume it. The process tried was to assume its falsehood, then derive a contradiction. Many strange conclusions follow from denying the parallel postulate, and several geometers found such great absurdities that they concluded that the parallel postulate did follow from the rest.

Nevertheless, these apparent absurdities are not contradictions. In the early nineteenth century, Bolyai, Lobachevsky, and Gauss found ways of dealing with this "non-Euclidean" geometry by means of analysis and accepted it as a valid kind of geometry, although very different from Euclidean geometry. This hyperbolic geometry, as it is called, is just as consistent as Euclidean geometry and has many uses.

Thus, we know now that we must include the parallel postulate to derive Euclidean geometry. For more on noneuclidean geometries, see the notes on hyperbolic geometry after I.29 and elliptic geometry after I.16.

Next proposition: I.1

Previous: I.Post.4

Book I introduction

Select from Book I

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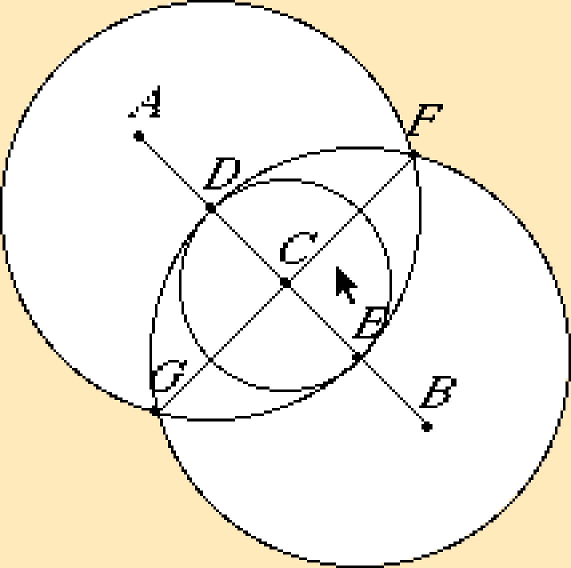
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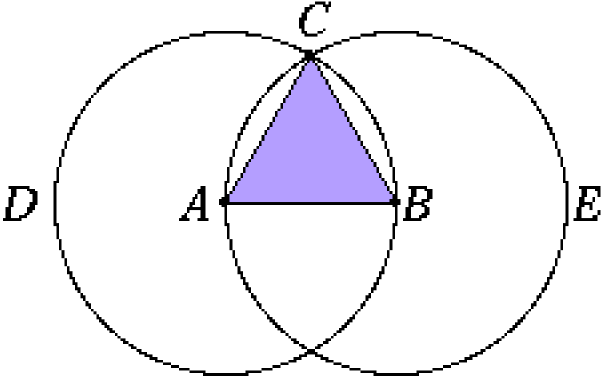
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Euclid's Elements

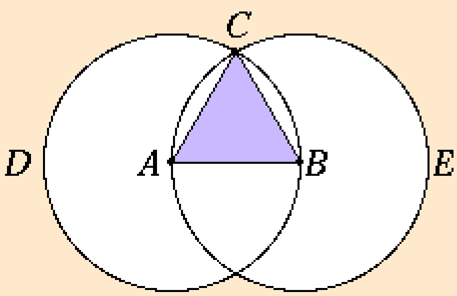
Book I

Proposition 1

To construct an equilateral triangle on a given finite straight line.

Let *AB* be the given finite straight line.

It is required to construct an equilateral triangle on the straight line *AB.*



Describe the circle *BCD* with center *A* and radius *AB.* Again describe the circle *ACE* with center *B* and radius *BA.* Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B.* Post.3 Post.1

Now, since the point *A* is the center of the circle *CDB,* therefore *AC* equals *AB.* Again, since the point *B* is the center of the circle *CAE,* therefore *BC* equals *BA.* I.Def.15

But *AC* was proved equal to *AB,* therefore each of the straight lines *AC* and *BC* equals *AB.*

And things which equal the same thing also equal one another, therefore *AC* also equals *BC.* C.N.1

Therefore the three straight lines *AC, AB,* and *BC* equal one another.

Therefore the triangle *ABC* is equilateral, and it has been constructed on the given finite straight line *AB.* I.Def.20

Q.E.F.

Guide

This proposition is a very pleasant choice for the first proposition in the *Elements.* The construction of the triangle is clear, and the proof that it is an equilateral triangle is evident. Of course, there are two choices for the point *C*, but either one will do.

Euclid could have chosen proposition I.4 to come first, since it doesn't logically depend on the previous three, but there are some good reasons for putting I.1 first. For one thing, the *Elements* ends with constructions of the five regular solids in Book XIII, so it is a nice aesthetic touch to begin with the construction of a regular triangle. More important, though, is I.1 is needed in I.2, and that in I.3. Propositions I.2 and I.3 give constructions for moving lines, and I.4, although not logically dependent on I.2 or I.3, does use the concept of superposition which involves, in some sense, moving points and lines.

Marginal references to postulates, definitions, etc.

The abbreviations in the right column refer to postulates, definitions, common notions, and previously proved propositions. Each indicates a justification of a construction or conclusion in a sentence to its left. They are not part of Euclid's *Elements*, but it is a tradition to include them as a guide to the reader.

Sometimes the justification is quoted as C.N.1 is quoted here, but usually it is left to the reader to determine the justification.

Q.E.F. and Q.E.D. at the ends of proofs

The Q.E.F. at the end of the proof is an abbreviation for the Latin words "quod erat faciendum" which means "which was to be done." A few of the propositions, as this one and the next two, solve problems by constructions. These are the ones that end with Q.E.F. (They're also printed in red here in the listings of propositions for each book.)

The rest of the proofs end with Q.E.D. instead, an abbreviation for "quod erat demonstrandum" which means "which was to be demonstrated." It's convenient to have a standard way to indicate the end of a proof. These Latin abbreviations are a bit of an anachronism. It would be less of an anachronism to use abbreviations for the original Greek phrase, or abbreviations for a modern English phrase since the rest of this version of the *Elements* is in English. But by now, Q.E.F. and Q.E.D. are traditional. In recent decades a small square has become common as a symbol to indicate the end of a proof.

Critiques of the proof

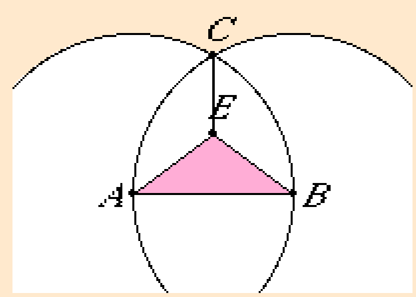
It is surprising that such a short, clear, and understandable proof can be so full of holes. These are logical gaps where statements are made with insufficient justification. Having the first proof in the *Elements* this proposition has probably received more criticism over the centuries than any other.

Why does the point *C* exist? Near the beginning of the proof, the point *C* is mentioned where the circles are supposed to intersect, but there is no justification for its existence. The only one of Euclid's postulate that says a point exists the parallel postulate, and that postulate is not relevant here. Thus, there is no assurance that the point *C* actually exists. Indeed, there are models of geometry in which the circles do not intersect. Thus, other postulates not mentioned by Euclid are required. In Book III, Euclid takes some care in analyzing the possible ways that circles can meet, but even with more care, there are missing postulates.

Why is *ABC* a plane figure? After concluding the three straight lines *AC, AB,* and *BC* are equal, what is the justification that they contain a plane figure *ABC*? Recall that a triangle is a plane figure bounded by contained by three lines. These lines have not been shown to lie in a plane and that the entire figure lies in a plane. It is proposition XI.1 that claims that all parts of a line lie in a plane, and XI.2 that claims that the entire triangle lie in a plane. Logically, they should precede I.1. The reason they don't, of course, is that those propositions belong to solid geometry, and plane geometry is developed first in the *Elements,* also, no doubt, plane geometry developed first historically.

Why does *ABC* contain an equilateral triangle? Proclus relates that early on there were critiques of the proof and describes that of Zeno of Sidon, an Epicurean philosopher of the early first century B.C.E. (not to be confused with Zeno of Elea famous of the paradoxes who lived long before Euclid), and whose criticisms, Proclus says, were refuted in a book by Posidonius. The critique is sound, however, and the refutation faulty.

Zeno of Sidon criticized the proof because it was not shown that the sides do not meet before they reach the vertices. Suppose *AC* and *BC* meet at *E* before they reach *C,* that is, the straight lines *AEC* and *BEC* have a common segment *EC.* Then they would contain a triangle *ABE* which is not equilateral, but isosceles.



Zeno recognized that in order to destroy his counterexample it was necessary to assume that straight lines cannot have a common segment. Proclus relates a supposed proof of that statement, the same one found in proposition XI.1, but it is faulty. Proclus and Posidonius quoted properties of lines and circles that were never proven and never explicitly assumed as postulates.

The possibilities that haven't been excluded are much more numerous than Zeno's example. The sides could meet numerous times and the region they contain could look like a necklace of bubbles. What needs to be shown (or assumed as a postulate) is that two infinitely extended straight lines can meet in at most one point.

Use of Proposition 1

The construction in this proposition is directly used in propositions I.2, I.9, I.10, I.11, XI.11, and XI.22.

Next proposition: I.2

Previous: I.Post.5

Book I introduction

Select from Book I

Select book

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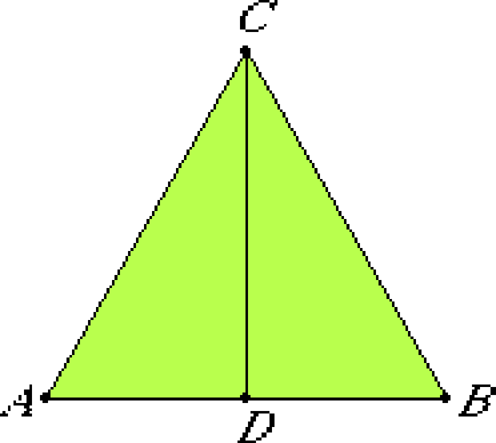
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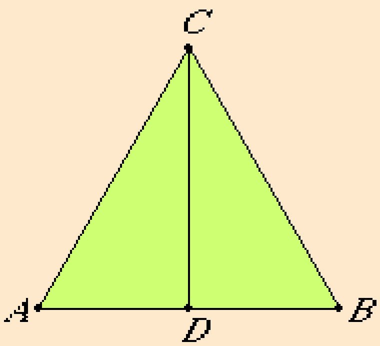
Euclid's Elements

Book I

Proposition 10

To bisect a given finite straight line.

Let *AB* be the given finite straight line.



It is required to bisect the finite straight line *AB*.

Construct the equilateral triangle *ABC* on it, and bisect the angle *ACB* by the straight line *CD*. I.1 I.9

I say that the straight line *AB* is bisected at the point *D*.

Since *CA* equals *CB*, and *CD* is common, therefore the two sides *CA* and *CD* equal the two sides *CB* and *CD* respectively, and the angle *ACD* equals the angle *BCD*, therefore the base *AD* equals the base *BD*. I.Def.20 I.4

Therefore the given straight line *AB* is bisected at *D*.

Q.E.F.

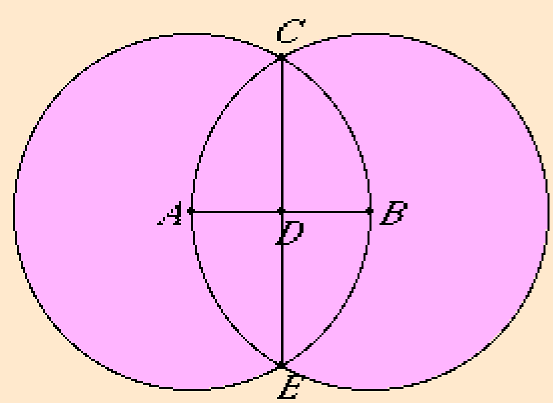
Guide

While this construction divides a line into two equal parts, the construction in proposition VI.9 divides a line into any given number of equal parts.

Construction steps

This method for bisecting lines takes less actual work than it appears to. It is really no more than the double-equilateral-triangle.

First, the equilateral triangle *ABC* needs to be constructed. According to I.1 two circles need to be drawn: one with center *A* and radius *AB*, the other with center *B* and radius *BA*. One of the points of intersection of the two circles is *C*. Then to bisect angle *ACB*, according to I.9, an arbitrary point is chosen on one side of the angle, and it might as well be the point *A* on the side *AC*, and a point equally far from *C* on the side *BC*, which is, of course, *B*. Then an equilateral triangle is constructed on the line *AB*. There are two such equilateral triangles, the one already constructed *ACB*, and another one, call it *AEB*. The point *E* is the other intersection of the two circles already drawn. Then, by I.9, the line *CE* bisects the angle *ACB*, and according to this proposition, the point *D* bisects the line *AB*.



Actually, only two circles and the straight line *CE* need to be drawn. The straight lines *AC*, *CB*, *AE*, and *EB*, aren't necessary for the construction; they are only used to show that the construction is correct.

Use of Proposition 10

The construction of this proposition in Book I is used in propositions I.12, I.16, and I.42. It is also used in several propositions in the Books II, III, IV, X, and XIII.

Next proposition: I.11

Previous: I.9

Book I introduction

Select from Book I

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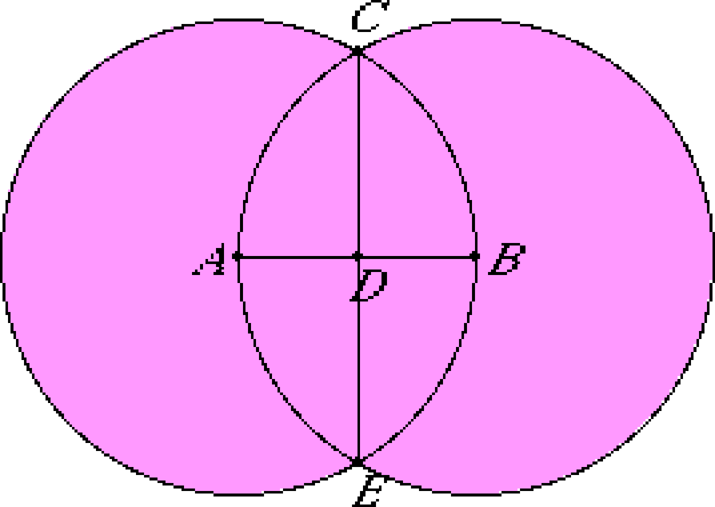
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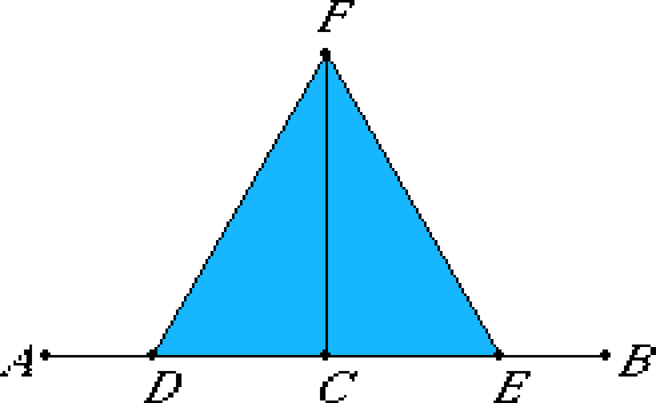
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Euclid's Elements

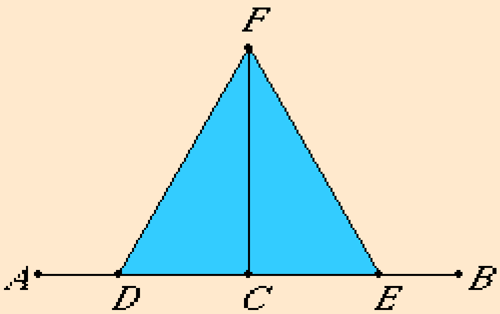
Book I

Proposition 11

To draw a straight line at right angles to a given straight line from a given point on it.

Let *AB* be the given straight line, and *C* the given point on it.

It is required to draw a straight line at right angles to the straight line *AB* from the point *C*.



Take an arbitrary point *D* on *AC*. Make *CE* equal to *CD*. Construct the equilateral triangle *FDE* on *DE*, and join *CF*. I.3 I.1 Post.1

I say that the straight line *CF* has been drawn at right angles to the given straight line *AB* from *C* the given point on it.

Since *CD* equals *CE*, and *CF* is common, therefore the two sides *CD* and *CF* equal the two sides *CE* and *CF* respectively, and the base *DF* equals the base *EF*. Therefore the angle *DCF* equals the angle *ECF*, and they are adjacent angles. I.Def.20 I.8

But, when a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, therefore each of the angles *DCF* and *FCE* is right. I.Def.10

Therefore the straight line *CF* has been drawn at right angles to the given straight line *AB* from the given point *C* on it.

Q.E.F.

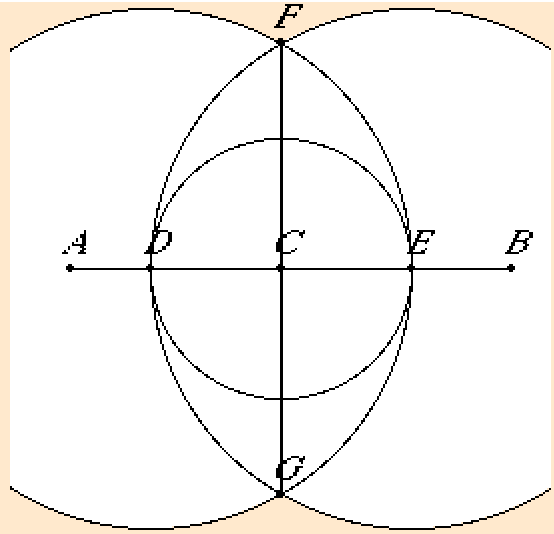
Guide

This and the next proposition both construct a perpendicular to a line through a given point. The difference is that the given point lies on the line in this proposition but doesn't in the next.

Construction steps

The actual construction here is the same double-equilateral-triangle construction of the previous proposition that is used to bisect the line *DE*, except that it is preceded by the selection of points *D* and *E* on *AB* equidistant from *C*.

This construction actually only requires drawing three circles and the one line *FG*.



Use of Proposition 11

Thia construction is used in propositions I.13, I.46, I.48, and numerous propositions in Books II, III, VI, VI, XI, XII, and XIII.

Next proposition: I.12

Previous: I.10

Book I introduction

Select from Book I

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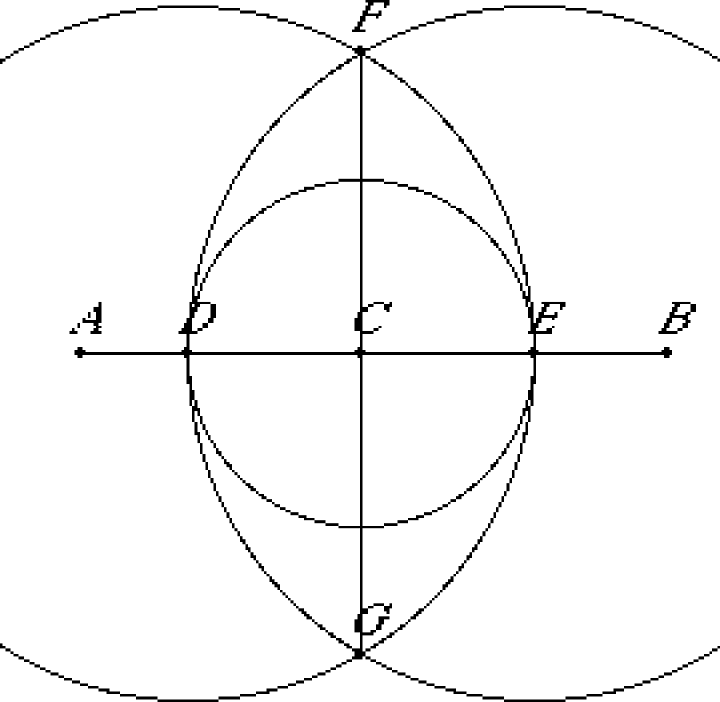
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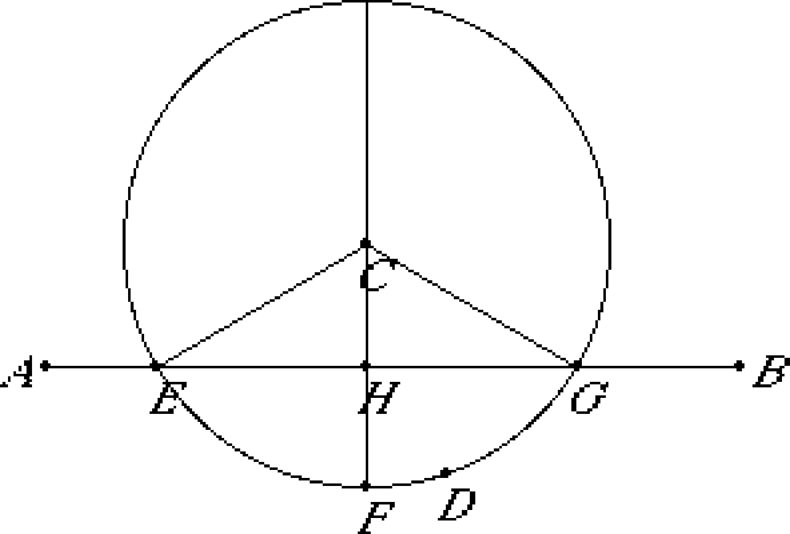
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Euclid's Elements

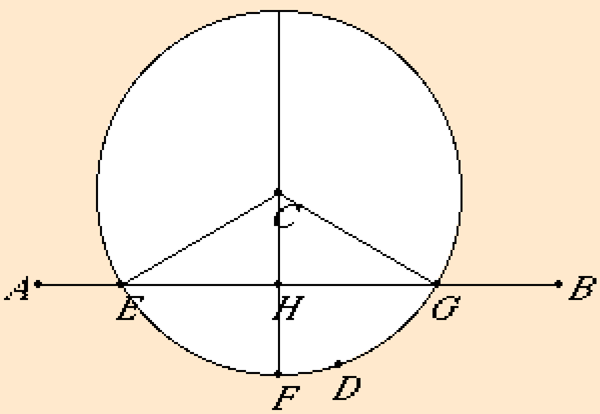
Book I

Proposition 12

To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Let *AB* be the given infinite straight line, and *C* the given point which is not on it.

It is required to draw a straight line perpendicular to the given infinite straight line *AB* from the given point *C* which is not on it.



Take an arbitrary point *D* on the other side of the straight line *AB*, and describe the circle *EFG* with center *C* and radius *CD.* Bisect the straight line *EG* at *H*, and join the straight lines *CG*, *CH*, and *CE.*

Post.3
I.10
Post.1

I say that *CH* has been drawn perpendicular to the given infinite straight line *AB* from the given point *C* which is not on it.

Since *GH* equals *HE,* and *HC* is common, therefore the two sides *GH* and *HC* equal the two sides *EH* and *HC* respectively, and the base *CG* equals the base *CE.* Therefore the angle *CHG* equals the angle *EHC,* and they are adjacent angles.

I.Def.15
I.8

But, when a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

I.Def.10

Therefore *CH* has been drawn perpendicular to the given infinite straight line *AB* from the given point *C* which is not on it.

Q.E.F.

Guide

Again, the double-equilateral-triangle construction is used, but this time the preparation of the starting line *EG* is different. The point *D* is taken on the other side of the line *AB* to insure that circle meets the line *AB* in at least two points, *E* and *G.* If *D* is taken on the line *AB,* it might be taken at *H,* and the resulting circle would touch the line only at *H*; and if *D* is taken on the same side of *AB,* then the circle could miss the line entirely.

Euclid does not precede this proposition with propositions investigating how lines meet circles. He is much more careful in Book III on circles in which the first dozen or so propositions lay foundations. For instance, Proposition III.10 states that a circle does not cut a circle at more than two points. Even so, some propositions are missing. One is needed for this proposition to justify the existence of the two points *C* and *E* where the line *AB* meets circle with center *C* and radius *CD.* Such a proposition would state "A circle whose center is on one side of a line and on whose circumference lies a point on the other side of the line meets the line at two points."

Incidentally, Proclus explains in his commentary on Book I that the problem of constructing the perpendicular was investigated by Oenopides of Chios who lived sometime in the middle of the fifth century B.C.E., a century and a half before Euclid.

Use of Proposition 12

The construction of this proposition is not used in Book I, but it is used on occasion in the remaining geometric books, namely, Books II through IV, VI, and XI through XIII.

Next proposition: I.13

Previous: I.11

Book I introduction

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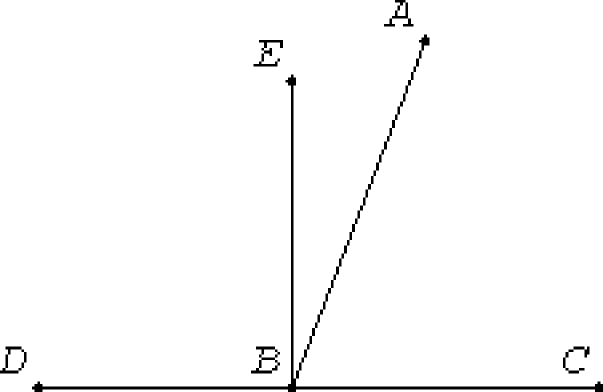
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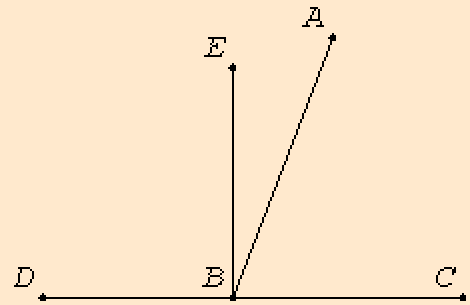
Euclid's Elements

Book I

Proposition 13

If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.



Let any straight line *AB* standing on the straight line *CD* make the angles *CBA* and *ABD.*

I say that either the angles *CBA* and *ABD* are two right angles or their sum equals two right angles.

Now, if the angle *CBA* equals the angle *ABD,* then they are two right angles. I.Def.10

But, if not, draw *BE* from the point *B* at right angles to *CD.* Therefore the angles *CBE* and *EBD* are two right angles. I.11

Since the angle *CBE* equals the sum of the two angles *CBA* and *ABE,* add the angle *EBD* to each, therefore the sum of the angles *CBE* and *EBD* equals the sum of the three angles *CBA*, *ABE*, and *EBD.* C.N.2

Again, since the angle *DBA* equals the sum of the two angles *DBE* and *EBA,* add the angle *ABC* to each, therefore the sum of the angles *DBA* and *ABC* equals the sum of the three angles *DBE, EBA,* and *ABC.* C.N.2

But the sum of the angles *CBE* and *EBD* was also proved equal to the sum of the same three angles, and things which equal the same thing also equal one another, therefore the sum of the angles *CBE* and *EBD* also equals the sum of the angles *DBA* and *ABC.* But the angles *CBE* and *EBD* are two right angles, therefore the sum of the angles *DBA* and *ABC* also equals two right angles. C.N.1

Therefore *if a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.*

Q.E.D.

Guide

With this proposition, we begin to see what the arithmetic of magnitudes means to Euclid, in particular, how to add angles. Euclid says that the angle *CBE* equals the sum of the two angles *CBA* and *ABE.* So, one way a sum of angles occurs is when the two angles have a common vertex (*B* in this case) and a common side (*BA* in this case), and the angles lie on opposite sides of their common side. Thus, addition of angles can be performed by joining adjacent angles.

But that's not the only addition that occurs here. Euclid also says that the sum of the angles *CBE* and *EBD* equals the sum of the three angles *CBA, ABE,* and *EBD.* That sum being mentioned is a straight angle, which is not to be considered as an angle according to Euclid. It is a formal sum equal to two right angles. In other propositions formal sums of four right angles occur. These and larger formal sums are not angles themselves, merely sums of angles. Only if an angle sum is less than two right angles can it be identified with a single angle.

Use of Proposition 13

This proposition is used in the proofs of the next two propositions and several others in this book as well as a few propositions in Books IV and VI.

Next proposition: I.14

Previous: I.12

Book I introduction

Select from Book I

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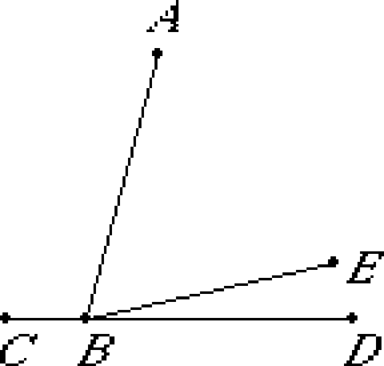
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Euclid's Elements

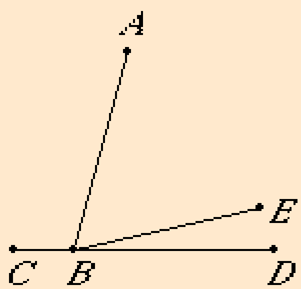
Book I

Proposition 14

If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

With any straight line *AB,* and at the point *B* on it, let the two straight lines *BC* and *BD* not lying on the same side make the sum of the adjacent angles *ABC* and *ABD* equal to two right angles.

I say that *BD* is in a straight line with *CB.*



If *BD* is not in a straight line with *BC,* then produce *BE* in a straight line with *CB.* Post.2

Since the straight line *AB* stands on the straight line *CBE,* therefore the sum of the angles *ABC* and *ABE* equals two right angles. But the sum of the angles *ABC* and *ABD* also equals two right angles, therefore the sum of the angles *CBA* and *ABE* equals the sum of the angles *CBA* and *ABD.* I.13 Post.4 C.N.1

Subtract the angle *CBA* from each. Then the remaining angle *ABE* equals the remaining angle *ABD,* the less equals the greater, which is impossible. Therefore *BE* is not in a straight line with *CB.* C.N.3

Similarly we can prove that neither is any other straight line except *BD.* Therefore *CB* is in a straight line with *BD.*

Therefore *if with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.*

Q.E.D.

Guide

This is a converse of the last proposition.

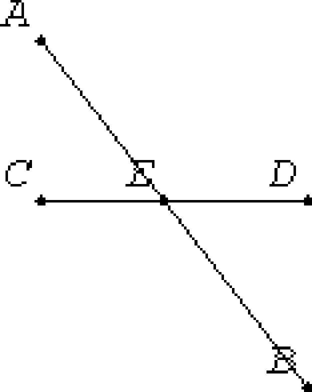
This is a proposition in plane geometry. If *A, B, C,* and *D* do not lie in a plane, then *CBD* cannot be a straight line. An ambient plane is necessary to talk about the sides of the line *AB*

The qualifying sentence, "Similarly we can prove that neither is any other straight line except *BD,*" is meant to take care of the cases when *E* does not lie inside the angle *ABD.*

Use of Proposition 14

This proposition is used in propositions I.45, I.47, and a few in Books VI and XI.

Next proposition: I.15 Select from Book I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Euclid's Elements

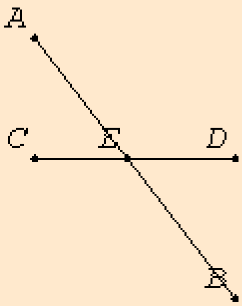
Book I

Proposition 15

If two straight lines cut one another, then they make the vertical angles equal to one another.

Let the straight lines *AB* and *CD* cut one another at the point *E.*

I say that the angle *CEA* equals the angle *DEB,* and the angle *BEC* equals the angle *AED.*



Since the straight line *AE* stands on the straight line *CD* making the angles *CEA* and *AED,* therefore the sum of the angles *CEA* and *AED* equals two right angles. I.13

Again, since the straight line *DE* stands on the straight line *AB* making the angles *AED* and *DEB,* therefore the sum of the angles *AED* and *DEB* equals two right angles. I.13

But the sum of the angles *CEA* and *AED* was also proved equal to two right angles, therefore the sum of the angles *CEA* and *AED* equals the sum of the angles *AED* and *DEB.* Subtract the angle *AED* from each. Then the remaining angle *CEA* equals the remaining angle *DEB.* Post.4 C.N.1 C.N.3

Similarly it can be proved that the angles *BEC* and *AED* are also equal.

Therefore *if two straight lines cut one another, then they make the vertical angles equal to one another.*

Q.E.D.

Corollary

From this it is manifest that, *if two straight lines cut one another, then they make the angles at the point of section equal to four right angles.*

Guide

Although the term "vertical angles" is not defined in the list of definitions at the beginning of Book I, its meaning is clear form its use in this proposition.

A *corollary* that follows a proposition is a statement that immediately follows from the proposition or the proof in the proposition. It is possible that this and the other corollaries in the *Elements* are interpolations inserted after Euclid wrote the *Elements.* During the writing, he could have either bundled the corollary into the proposition or made it a separate proposition.

Notes

Proclus includes another corollary: *If any number of straight lines intersect one another at one point, then the sum of all the angles so formed equals four right angles.*

Use of Proposition 15

This proposition is used in the next one, a few others in this book, II.10, IV.15

Next proposition: I.16

Previous: I.14

Book I introduction

Select from Book I

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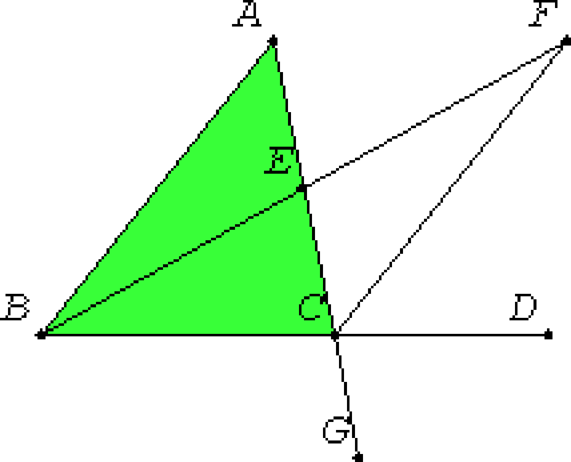
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Euclid's Elements

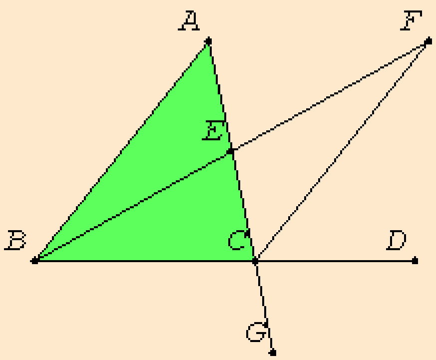
Book I

Proposition 16

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to *D.*

I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA* and *BAC.*



Bisect *AC* at *E*. Join *BE*, and produce it in a straight line to *F*. I.10 Post.1 Post.2

Make *EF* equal to *BE*, join *FC*, and draw *AC* through to *G*. I.3 Post.1 Post.2

Since *AE* equals *EC,* and *BE* equals *EF*, therefore the two sides *AE* and *EB* equal the two sides *CE* and *EF* respectively, and the angle *AEB* equals the angle *FEC,* for they are vertical angles. Therefore the base *AB* equals the base *FC,* the triangle *ABE* equals the triangle *CFE,* and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Therefore the angle *BAE* equals the angle *ECF.* I.15 I.4

But the angle *ECD* is greater than the angle *ECF,* therefore the angle *ACD* is greater than the angle *BAE.* C.N.5

Similarly, if *BC* is bisected, then the angle *BCG,* that is, the angle *ACD,* can also be proved to be greater than the angle *ABC.* I.15

Therefore *in any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.*

Q.E.D.

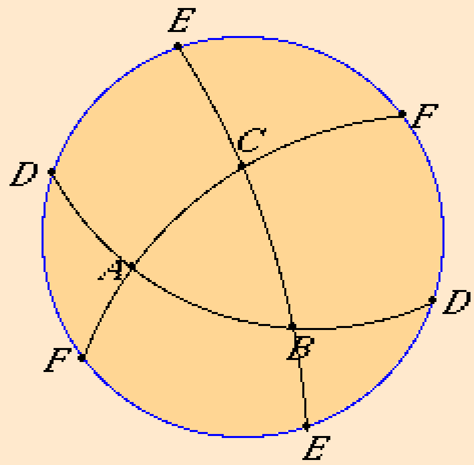
Guide

In the later proposition I.32, after he invokes the parallel postulate Post.5, Euclid shows the stronger result that the exterior angle of a triangle equals the sum of the interior, opposite angles.

Elliptic geometry

There are geometries besides Euclidean geometry. Two of the more important geometries are elliptic geometry and hyperbolic geometry, which were developed in the nineteenth century. The first 15 propositions in Book I hold in elliptic geometry, but not this one. (For more on hyperbolic geometry, see the note after Proposition I.29.)

Plane elliptic geometry is closely related to spherical geometry, but it differs in that antipodal points on the sphere are identified. Thus, a "point" in an elliptic plane is a pair of antipodal points on the sphere. A "straight line" in an elliptic plane is an arc of great circle on the sphere. When a "straight line" is extended, its ends eventually meet so that, topologically, it becomes a circle. This is very different from Euclidean geometry since here the ends of a line never meet when extended.



The illustration on the right shows the stereographic projection of one hemisphere. Since only one hemisphere is displayed, each "point" is represented by one point except those "points" such as *D*, *E*, and *F* on the blue bounding great circle which appear twice.

A "triangle" in elliptic geometry, such as *ABC*, is a spherical triangle (or, more precisely, a pair of antipodal spherical triangles). The internal angle sum of a spherical triangle is always greater than 180°, but less than 540°, whereas in Euclidean geometry, the internal angle sum of a triangle is 180° as shown in Proposition I.32.

Elliptic geometry satisfies some of the postulates of Euclidean geometry, but not all of them under all interpretations. Usually, Post.1, to draw a straight line from any point to any point, is interpreted to include the uniqueness of that line. But in elliptic geometry a completed "straight line" is topologically a circle so that any pair of points on it divide it into two arcs. Therefore, in elliptic geometry exactly two "straight lines" join any two given "points."

Also, Post.2, to produce a finite straight line continuously in a straight line, is sometimes interpreted to include the condition that its ends don't meet when extended. Under that interpretation, elliptic geometry fails Postulate 2.

Elliptic geometry fails Post.5, the parallel postulate, as well, since any two "straight lines" in an elliptic plane meet. That is, any two great circles on the sphere meet at a pair of antipodal points.

Finally, a completed "straight line" in the elliptic plane does not divide the plane into two parts as infinite straight lines do in the Euclidean plane. A completed "straight line" in the elliptic plane is a great circle on the sphere. Any two "points" not on that "straight line" include two points in the same hemisphere, and they can be joined by an arc that doesn't meet the great circle. Therefore two "points" lie on the same side of the completed "straight line."

The proof of this particular proposition fails for elliptic geometry, and the statement of the proposition is false for elliptic geometry. In particular, the statement "the angle *ECD* is greater than the angle *ECF*" is not true of all triangles in elliptic geometry. The line *CF* need not be contained in the angle *ACD.* All the previous propositions do hold in elliptic geometry and some of the later propositions, too, but some need different proofs.

Use of Proposition 16

This proposition is used in the proofs of the next two propositions, a few others in this book, and a couple in Book III.

Next proposition: I.17

Previous: I.15

Book I introduction

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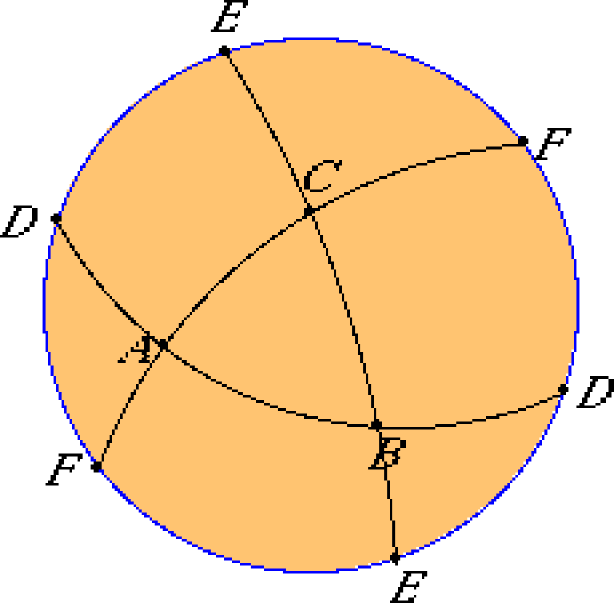
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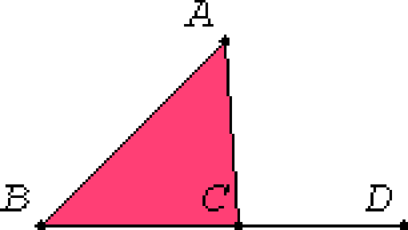
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Euclid's Elements

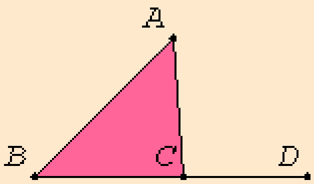
Book I

Proposition 17

In any triangle the sum of any two angles is less than two right angles.

Let *ABC* be a triangle.

I say that the sum of any two angles of the triangle *ABC* is less than two right angles.



Produce *BC* to *D*. Post.2

Since the angle *ACD* is an exterior angle of the triangle *ABC,* therefore it is greater than the interior and opposite angle *ABC.* Add the angle *ACB* to each. Then the sum of the angles *ACD* and *ACB* is greater than the sum of the angles *ABC* and *BCA.* I.16 C.N.

But the sum of the angles *ACD* and *ACB* is equal to two right angles. Therefore the sum of the angles *ABC* and *BCA* is less than two right angles. I.13

Similarly we can prove that the sum of the angles *BAC* and *ACB* is also less than two right angles, and so the sum of the angles *CAB* and *ABC* as well.

Therefore *in any triangle the sum of any two angles is less than two right angles.*

Q.E.D.

Guide

The statements

> ... the angle *ACD* ... is greater than the interior and opposite angle *ABC.* Add the angle *ACB* to each. Then the sum of the angles *ACD* and *ACB* is greater than the sum of the angles *ABC* and *BCA.*

use the property of magnitudes

> If $x > y$, then $x + z > y + z$.

This property is not listed among the Common Notions.

This proposition is strengthened in Proposition I.32 to say the sum of all three angles in a triangle equals two right angles.

Use of Proposition 17

This proposition is used in III.16 and a couple other propositions of Books III, and a few in Books VI and XI.

Next proposition: I.18 Select from Book I

Previous: I.16

Book I introduction

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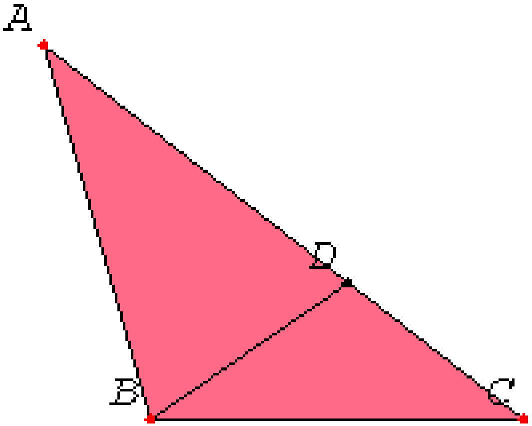
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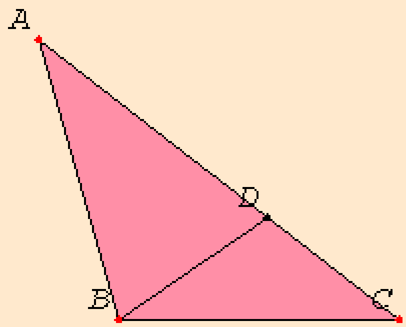
Euclid's Elements
Book I
Proposition 18

In any triangle the angle opposite the greater side is greater.

Let *ABC* be a triangle having the side *AC* greater than *AB.*

I say that the angle *ABC* is also greater than the angle *BCA.*



Since *AC* is greater than *AB,* make *AD* equal to *AB,* and join *BD.* I.3 Post.1

Since the angle *ADB* is an exterior angle of the triangle *BCD,* therefore it is greater than the interior and opposite angle *DCB.* I.16

But the angle *ADB* equals the angle *ABD,* since the side *AB* equals *AD,* therefore the angle *ABD* is also greater than the angle *ACB.* Therefore the angle *ABC* is much greater than the angle *ACB.* I.5

Therefore *in any triangle the angle opposite the greater side is greater.*

Q.E.D.

Guide

On word order

In this translation of Euclid's *Elements* the order of the words differs from the original Greek. In each of Euclid's Greek sentences, the data, that is the geometric objects given or already constructed, appear first, and the remaining geometric objects appear later. This is possible in Greek since it is an inflected language and the word order is very flexible. On the other hand, the word order in English is intrinsic to the syntax and semantics of the sentence and is not very flexible.

Take, for instance, the statements of this and the next proposition. Very literal translations of these are (I.18) "In any triangle, the greater side [as subject] the greater angle [as object] subtends," and (I.19) "In any triangle, the greater angle [as object] the greater side [as subject] subtends."

Heath keeps the word order in his translation but makes the second statement passive: (I.18) "In any triangle the greater side subtends the greater angle," and (I.19) "In any triangle the greater angle is subtended by the greater side." Without the understanding that the data come first, these two sentences are logically equivalent.

In this translation the original word order is abandoned in order to make for more readable sentences and to clarify the meaning. Thus, (I.18) "In any triangle the angle opposite the greater side is greater," and (I.19) "In any triangle the side opposite the greater angle is greater."

It may sound like these two propositions really do say the same thing, but they don't. They're actually disguised converses of each other. I.18 says "if side *AC* > side *AB*, then angle *ABC* > angle *BCA*" (but it hasn't yet been shown that there is no other way for angle *ABC* to be greater), while I.19 says "if angle *ABC* > angle *BCA*, then side *AC* > side *AB*."

Use of Proposition 18

This proposition is used in the proof of proposition I.19.

Next proposition: I.19

Previous: I.17

Book I introduction

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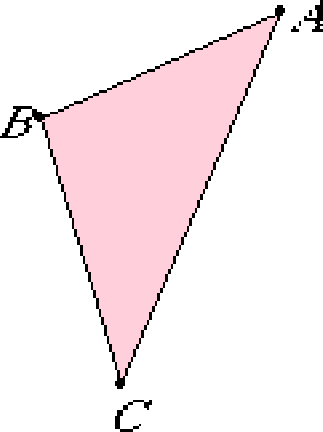
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Euclid's Elements

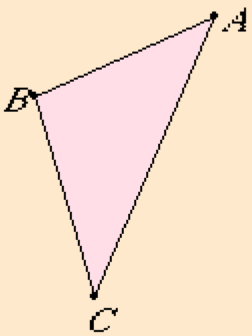
Book I

Proposition 19

In any triangle the side opposite the greater angle is greater.

Let *ABC* be a triangle having the angle *ABC* greater than the angle *BCA.*

I say that the side *AC* is greater than the side *AB.*



If not, either *AC* equals *AB* or it is less than it.

Now *AC* does not equal *AB,* for then the angle *ABC* would equal the angle *ACB,* but it does not. Therefore *AC* does not equal *AB.* I.5

Neither is *AC* less than *AB,* for then the angle *ABC* would be less than the angle *ACB,* but it is not. Therefore *AC* is not less than *AB.* I.18

And it was proved that it is not equal either. Therefore *AC* is greater than *AB.*

Therefore *in any triangle the side opposite the greater angle is greater.*

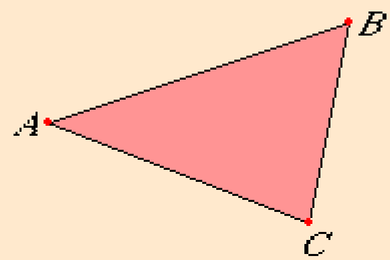
Q.E.D.

Guide

As mentioned before, this proposition is a disguised converse of the previous one. As Euclid often does, he uses a proof by contradiction involving the already proved converse to prove this proposition. It is not that there is a logical connection between this statement and its converse that makes this tactic work, but some kind of symmetry involved. In this case, if one side is less than another, then the other is greater than the one, and the previous proposition applies. So the relevant symmetry is between "less" and "greater."

The law of sines

Although some of the geometric underpinnings of trigonometry appear in the *Elements,* trigonometry itself does not. Trigonometry makes its appearance among later Greek mathematics where the the basic trigonometric function is the *chord,* which is related to the *sine.*



Without going into details, the law of sines contains more precise information about the relation between angles and sides of a triangle than this and the last proposition did. The law of sines states that

$$(\sin A)/BC = (\sin B)/AC = (\sin C)/AB.$$

Alternately, the first equation may be read a proportion

$$\sin A \text{ is to } \sin B \text{ as } BC \text{ is to } AC.$$

In other words, the sine of an angle in a triangle is proportional to the opposite side. (Proportions aren't defined in the *Elements* until Book V.)

Use of Proposition 19

This proposition is used in the proofs of propositions I.20, I.24, and some others in Book III.

Next proposition: I.20

Previous: I.18

Book I introduction

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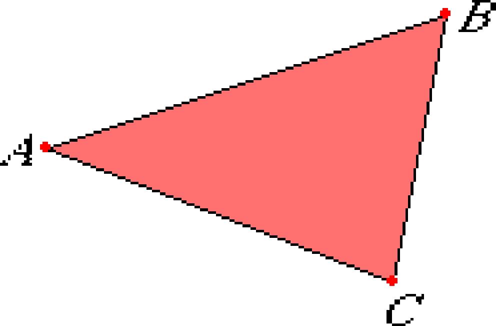
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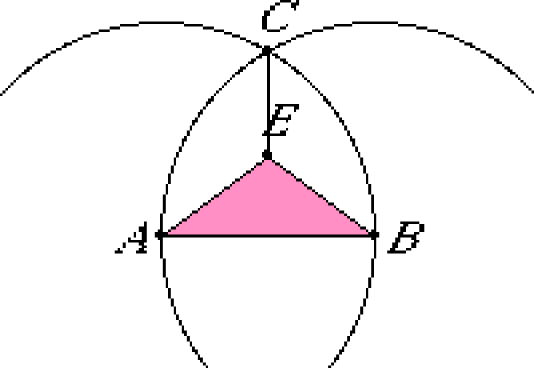
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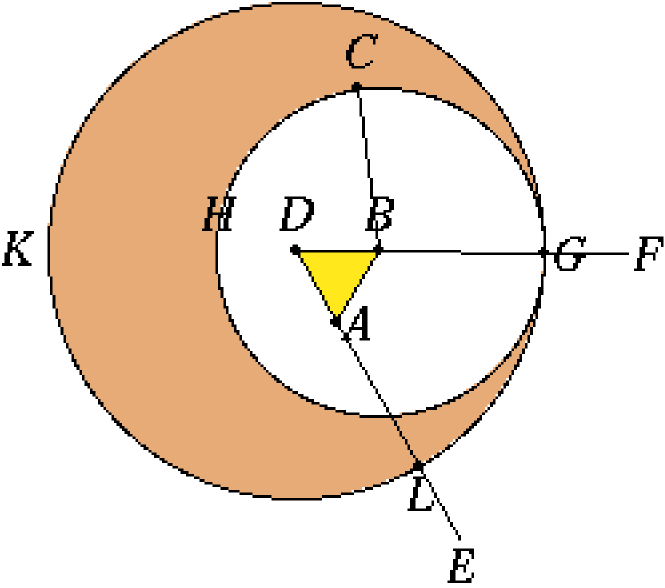
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Euclid's Elements

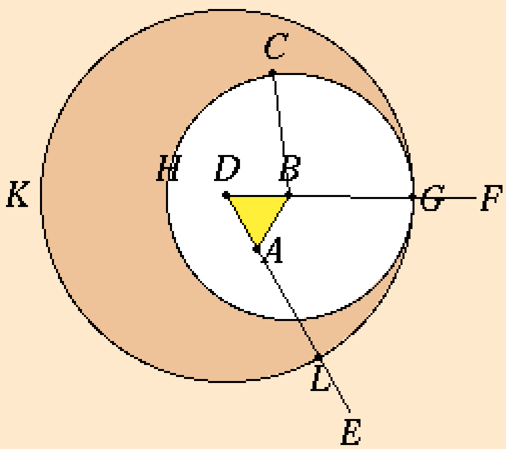
Book I

Proposition 2

To place a straight line equal to a given straight line with one end at a given point.

Let *A* be the given point, and *BC* the given straight line.

It is required to place a straight line equal to the given straight line *BC* with one end at the point *A*.



Join the straight line *AB* from the point *A* to the point *B*, and construct the equilateral triangle *DAB* on it. Post. 1 I.1.

Produce the straight lines *AE* and *BF* in a straight line with *DA* and *DB*. Describe the circle *CGH* with center *B* and radius *BC*, and again, describe the circle *GKL* with center *D* and radius *DG*. Post.2 Post.3

Since the point *B* is the center of the circle *CGH*, therefore *BC* equals *BG*. Again, since the point *D* is the center of the circle *GKL*, therefore *DL* equals *DG*. I.Def.15

And in these *DA* equals *DB*, therefore the remainder *AL* equals the remainder *BG*. C.N.3

But *BC* was also proved equal to *BG*, therefore each of the straight lines *AL* and *BC* equals *BG*. And things which equal the same thing also equal one another, therefore *AL* also equals *BC*. C.N.1

Therefore the straight line *AL* equal to the given straight line *BC* has been placed with one end at the given point *A*.

Q.E.F.

Guide

This is a very clever construction to solve what seems to be a simple problem. One would like simply to slide the line *BC* along so that one end coincides with the point *A*. But there is no motion in the geometry of Euclid. There is something like motion used in proposition I.4, but nothing is actually moved there. The only basic constructions that Euclid allows are those described in Postulates 1, 2, and 3. Euclid then builds new constructions (such as the one in this proposition) out of previously described constructions. So at this point, the only constructions available are those of the three postulates and the construction in proposition I.1, and Euclid uses all four here.

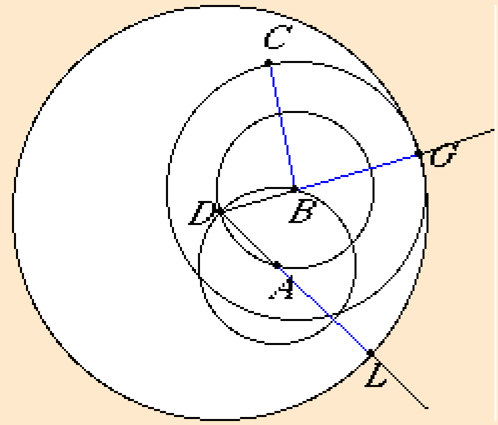
Another, different, expectation is that one might use a compass to transfer the distance *BC* over to the point *A*. It is clear from Euclid's use of postulate 3 that the point to be used for the center and a point that will be on the circumference must be constructed before applying the postulate; postulate 3 is not used to transfer distance. Sometimes postulate 3 is likened to a collapsing compass, that is, when the compass is lifted off the drawing surface, it collapses.

It could well be that in some earlier Greek geometric theory abstracted compasses that could transfer distances. If that speculation is correct, then this proposition would be a late addition to the theory. The construction of the proposition allows a weaker postulate (namely postulate 3) to be assumed.

Construction steps

When using a compass and a straightedge to perform this construction there are more circles drawn than shown in the diagram that accompanies the proposition. These are the two circles needed to construct the equilateral triangle *ABD.* One side, *AB,* of that triangle isn't necessary for the construction.

Altogether, four circles and two lines are required for this construction.



Use of Proposition 2

The construction in this proposition is only used in Proposition I.3.

Note that this constuction assumes that all the point *A* and the line *BC* lie in a plane. It may also be used in space, however, since Proposition XI.2 implies that *A* and *BC* do lie in a plane.

Next proposition: I.3

Previous: I.1

Book I introduction

Select from Book I

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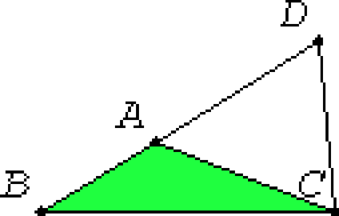

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Euclid's Elements

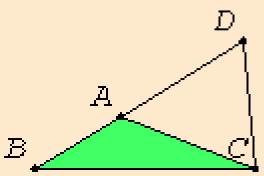
Book I

Proposition 20

In any triangle the sum of any two sides is greater than the remaining one.

Let *ABC* be a triangle.

I say that in the triangle *ABC* the sum of any two sides is greater than the remaining one, that is, the sum of *BA* and *AC* is greater than *BC*, the sum of *AB* and *BC* is greater than *AC*, and the sum of *BC* and *CA* is greater than *AB*.



Post.2

Draw *BA* through to the point *D*, and make *DA* equal to *CA*. Join *DC*. I.3 Post.1

Since *DA* equals *AC*, therefore the angle *ADC* also equals the angle *ACD*. Therefore the angle *BCD* is greater than the angle *ADC*. I.5 C.N.5

Since *DCB* is a triangle having the angle *BCD* greater than the angle *BDC*, and the side opposite the greater angle is greater, therefore *DB* is greater than *BC*. I.19

But *DA* equals *AC*, therefore the sum of *BA* and *AC* is greater than *BC*.

Similarly we can prove that the sum of *AB* and *BC* is also greater than *CA*, and the sum of *BC* and *CA* is greater than *AB*.

Therefore *in any triangle the sum of any two sides is greater than the remaining one.*

Q.E.D.

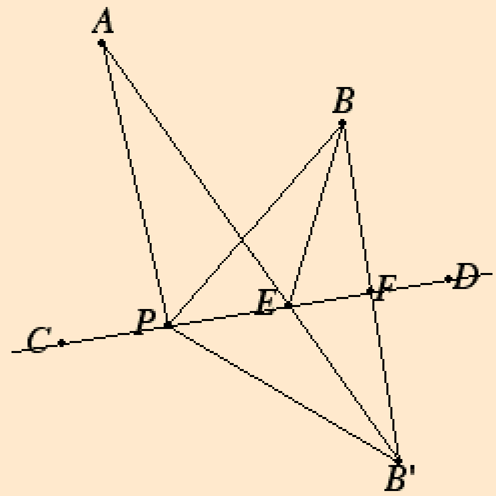
Guide

This proposition is known as "the triangle inequality." It is part of the statement that the shortest path between two points is a straight line, but there are many other conceivable paths besides broken lines.

A minimum distance

This proposition on the triangle inequality, along with I.15 on vertical angles, allows us to solve a problem on minimum distance, described and solved by Heron of Alexandria.

Suppose there are two points *A* and *B* on the same side of a line *CD*. The problem is to find the shortest path which goes first from the point *A* to some point *P* on the line *CD*, then from *P* to the point *B*. We will only consider paths that are made out of straight lines; call such a path a *bent line*. But that still leaves us the question of which point *P* to choose on the line *CD* to minimize the sum of the distances *AP* plus *PB*.



The solution is that the shortest path will be the path AEB where angle of incidence, namely, angle AEC, equals the angle of reflection, namely, angle BED.

First, we should show how to construct the bent line where the angle of incidence equals the angle of reflection. Draw a perpendicular BF from the point B to the line CD (I.12), and extend it to B' so that $FB' = BF$ (Post.2, I.3). Draw AB' and let E be the point where AB' intersects CD. (There will be a point of intersection since A and B' are on opposite sides of CD.) Draw BE.

Now, triangles BFE and $B'FE$ are congruent since they have two sides and the included angle equal (I.4), the included angles being right angles. Therefore, angles BFE and $BF'E$ are equal. The vertical angle AEC across from angle $B'ED$ also equals these angles (I.15). Thus, the angle of incidence AEC equals the angle of reflection BED.

We still have to show that the distance $AE + EB$ is less than any distance $AP + EP$ for any point P other than E that lies on the line CD. Let P be such a point and draw lines AP, BP, and $B'P$. Then by proposition I.20, above, $AP + EP$ is less than AB'. But $AB' = AE + EB'$, and $EB' = EB$, therefore $AP + EP$ is less than $AE + EB$.

Thus, the shortest bent line between two points on the same side of a line that meets that line is the one where the angle of incidence equals the angle of reflection.

Q.E.D.

Use of Proposition 20

This proposition is used in the next two propositions, several in Book III, and XI.20.

Next proposition: I.21

Previous: I.19

Book I introduction

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Select book

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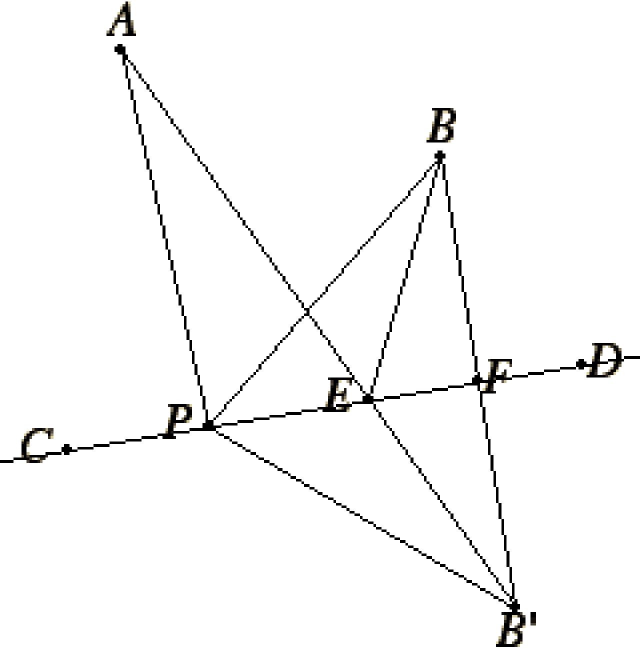
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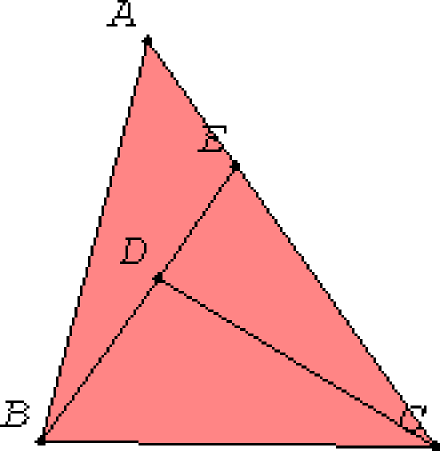
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Euclid's Elements

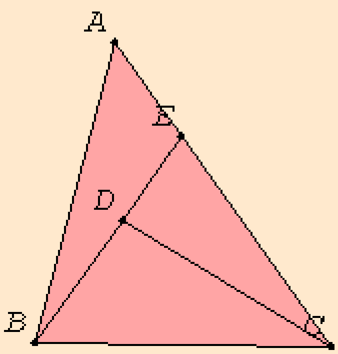
Book I

Proposition 21

If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

From the ends *B* and *C* of one of the sides *BC* of the triangle *ABC,* let the two straight lines *BD* and *DC* be constructed meeting within the triangle.

I say that the sum of *BD* and *DC* is less than the sum of the remaining two sides of the triangle *BA* and *AC,* but *BD* and *DC* contain an angle *BDC* greater than the angle *BAC.*



Draw *BD* through to *E.* Post.2

Since in any triangle the sum of two sides is greater than the remaining one, therefore, in the triangle *ABE,* the sum of the two sides *AB* and *AE* is greater than *BE.* I.20

Add *EC* to each. Then the sum of *BA* and *AC* is greater than the sum of *BE* and *EC.* C.N.

Again, since, in the triangle *CED,* the sum of the two sides *CE* and *ED* is greater than *CD,* add *DB* to each, therefore the sum of *CE* and *EB* is greater than the sum of *CD* and *DB.* I.20 C.N.

But the sum of *BA* and *AC* was proved greater than the sum of *BE* and *EC,* therefore the sum of *BA* and *AC* is much greater than the sum of *BD* and *DC.* C.N.

Again, since in any triangle the exterior angle is greater than the interior and opposite angle, therefore, in the triangle *CDE,* the exterior angle *BDC* is greater than the angle *CED.* I.16

For the same reason, moreover, in the triangle *ABE* the exterior angle *CEB* is greater than the angle *BAC.* But the angle *BDC* was proved greater than the angle *CEB,* therefore the angle *BDC* is much greater than the angle *BAC.* I.16 C.N.

Therefore *if from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.*

Q.E.D.

Guide

Pappus and others before him noticed that if the lines are not drawn from the ends of the side, then the sum of the the constructed straight lines can be greater than the sum of the remaining two sides of the triangle. In fact that sum can be made almost as large as twice the longest side of the triangle.

Use of Proposition 21

This proposition is used in proposition III.8.

Next proposition: I.22

Previous: I.20

Book I introduction

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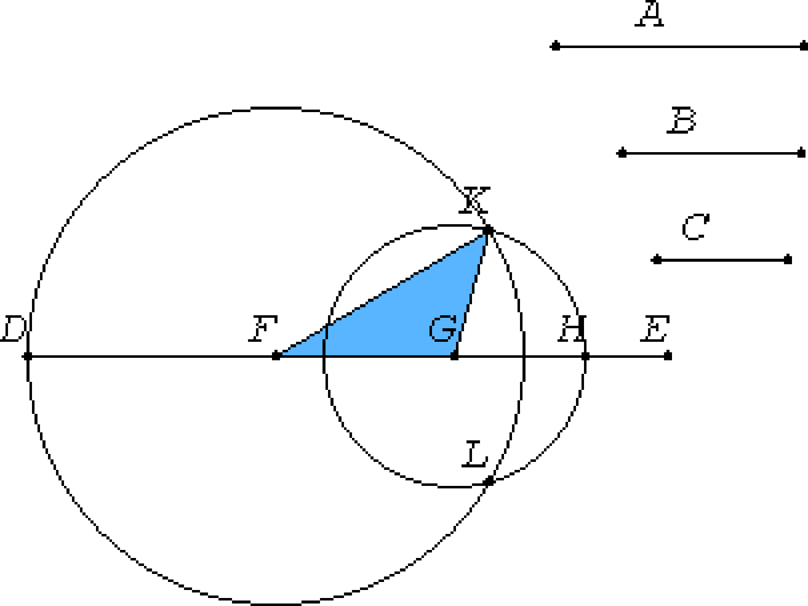

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Euclid's Elements

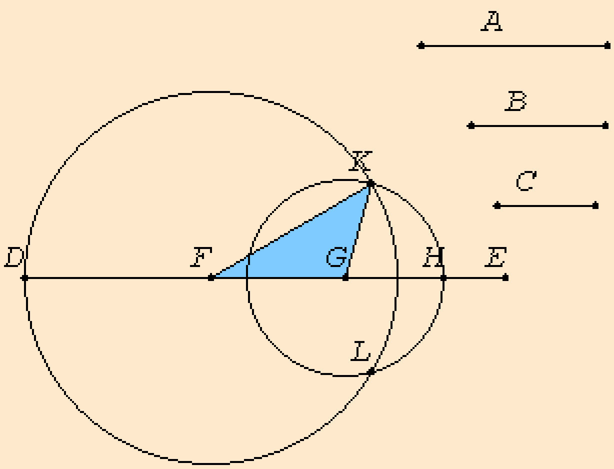
Book I

Proposition 22

To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one. I.20

Let the three given straight lines be *A*, *B*, and *C*, and let the sum of any two of these be greater than the remaining one, namely, *A* plus *B* greater than *C*, *A* plus *C* greater than *B*, and *B* plus *C* greater than *B*.

It is required to construct a triangle out of straight lines equal to *A*, *B*, and *C*.



Set out a straight line *DE*, terminated at *D* but of infinite length in the direction of *E*. Make *DF* equal to A, FG equal to *B*, and *GH* equal to *C*. Post.2 I.3

Describe the circle *DKL* with center *F* and radius *FD*. Again, describe the circle *KLH* with center *G* and radius *GH*. Join *KF* and *KG*. Post.3 Post.1

I say that the triangle *KFG* has been constructed out of three straight lines equal to *A*, *B*, and *C*.

Since the point *F* is the center of the circle *DKL*, therefore *FD* equals *FK*. But *FD* equals *A*, therefore *KF* also equals *A*. I.Def.16 C.N.1

Again, since the point *G* is the center of the circle *LKH*, therefore *GH* equals *GK*. But *GH* equals *C*, therefore *KG* also equals *C*. I.Def.16 C.N.1

And *FG* also equals *B*, therefore the three straight lines *KF*, *FG*, and GK equal the three straight lines *A*, *B*, and C.

Therefore out of the three straight lines *KF*, *FG*, and *GK*, which equal the three given straight lines *A*, *B*, and C, the triangle *KFG* has been constructed.

Q.E.F.

Guide

The qualifier in the statement of the proposition, "thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one," refers to the triangle inequality, Proposition I.20. This condition is, indeed, necessary. It is also sufficient, but Euclid failed to show that sufficiency.

This construction is actually a generalization of the very first proposition I.1 in which the three lines are all equal. There too, as was noted, Euclid failed to prove that the two circles intersected.

Use of Proposition 22

The construction in this proposition is used for the construction in proposition I.23. It is also used in XI.22

Next proposition: I.23

Previous: I.21

Book I introduction

Select from Book I

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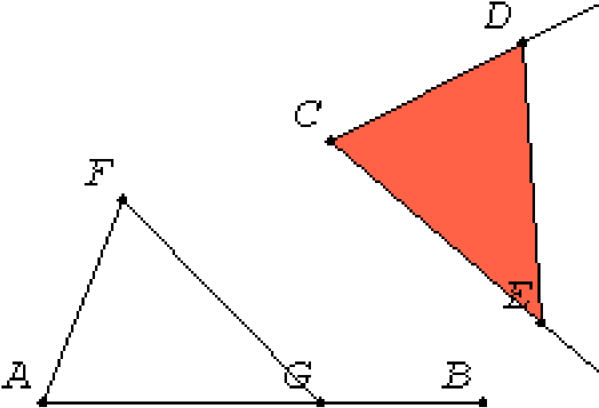
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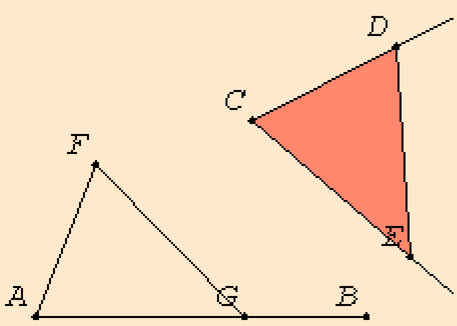
Euclid's Elements

Book I

Proposition 23

To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Let the angle *DCE* be the given rectilinear angle, *AB* the given straight line, and *A* the point on it.



It is required to construct a rectilinear angle equal to the given rectilinear angle *DCE* on the given straight line *AB* and at the point *A* on it.

Take the points *D* and *E* at random on the straight lines *CD* and *CE* respectively, and join *DE*. Out of three straight lines which equal the three straight lines *CD*, *DE*, and *CE* construct the triangle *AFG* in such a way that *CD* equals *AF*, *CE* equals *AG*, and *DE* equals *FG*. Post.1 I.22

Since the two sides *DC* and *CE* equal the two sides *FA* and *AG* respectively, and the base *DE* equals the base *FG*, therefore the angle *DCE* equals the angle *FAG*. I.8

Therefore on the given straight line *AB*, and at the point *A* on it, the rectilinear angle *FAG* has been constructed equal to the given rectilinear angle *DCE*.

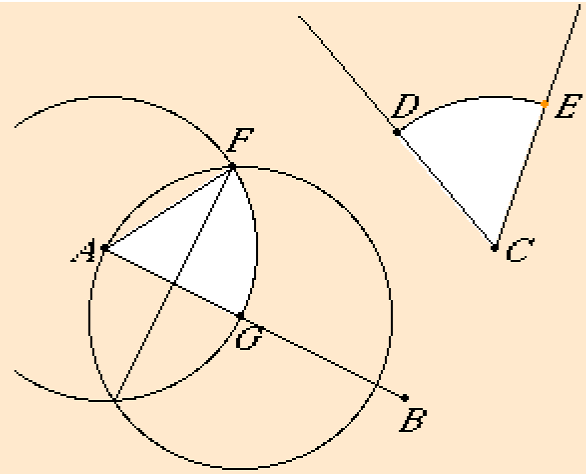
Q.E.F.

Guide

As Proclus and Heath point out, a very minor variant of the construction in I.22 is needed to make the triangle *AFG*. The problem is that in I.22 the triangle is placed not at the end of line, but somewhere beyond that, and in I.23, the triangle needs to be placed right at the end *A* of the line.

Construction steps

This construction that moves an angle requires a number of steps involving a straightedge and compass. Unless there is some special reason for selecting particular points *D* and *E* on the sides of the angle *C*, they might as well be taken equidistant from *C* as Apollonius suggested. Then only two distances need to be transferred instead of three.



In order to make *CE* equal to *CD*, one circle is required. Next, in order to transfer the distance *CD* to *A*, four circles (not shown) are required as per Propositions I.2 and I.1, and four more circles (also not shown) to transfer *ED* to *G*. Finally, two more circles are required, one with center *A* and radius *CD* (which has been transferred), the other with center *G* and radius *ED*. These last two circles meet at the point *F*, and the line *AF* is the other side of the required angle.

In all there are ten circles and one line that must be drawn. The lines in the intermediate stages may be suppressed as usual since they're only needed to verify the construction is correct.

Use of Proposition 23

The construction in this proposition is used in the next one and a couple others in Book I. It is also used frequently in the later books. It is also used frequently in BookS III and VI and occasionally in Books IV and XI.

Although it may appear that the triangles are to be in the same plane, that is not necessary. Indeed, the construction in this proposition is used to construct an angle in a different plane in proposition XI.31.

Next proposition: I.24

Previous: I.22

Book I introduction

Select from Book I

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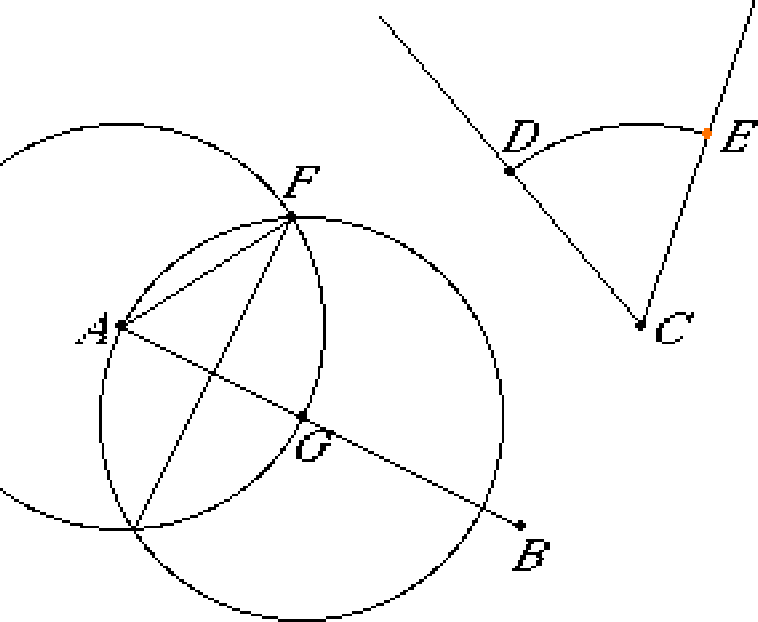

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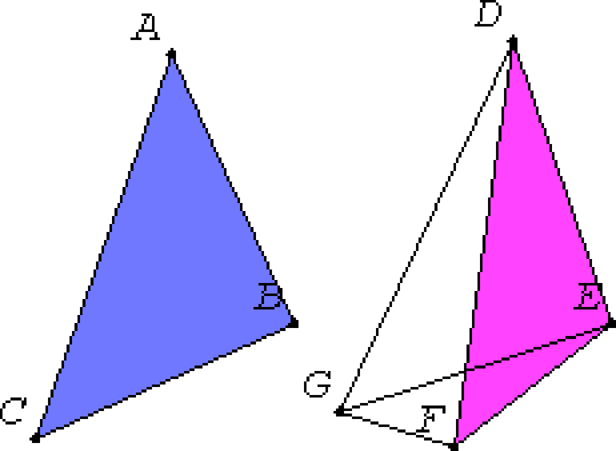
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Euclid's Elements

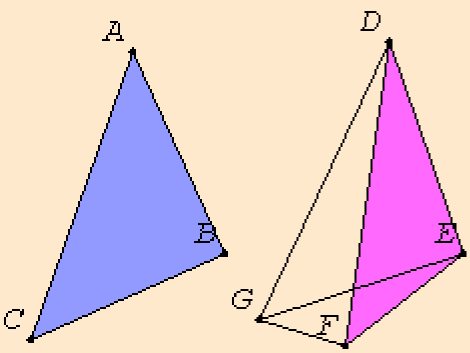
Book I

Proposition 24

If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Let *ABC* and *DEF* be two triangles having the two sides *AB* and *AC* equal to the two sides *DE* and *DF* respectively, so that *AB* equals *DE*, and *AC* equals *DF*, and let the angle at *A* be greater than the angle at *D*.

I say that the base *BC* is greater than the base *EF*.



Since the angle *BAC* is greater than the angle *EDF*, construct the angle *EDG* equal to the angle *BAC* at the point *D* on the straight line *DE*. Make *DG* equal to either of the two straight lines *AC* or *DF*. Join *EG* and *FG*. I.23 I.3 Post.1

Since *AB* equals *DE*, and *AC* equals *DG*, the two sides *BA* and *AC* equal the two sides *ED* and *DG*, respectively, and the angle *BAC* equals the angle *EDG*, therefore the base *BC* equals the base *EG*. I.4

Again, since *DF* equals *DG*, therefore the angle *DGF* equals the angle *DFG*. Therefore the angle *DFG* is greater than the angle *EGF*. I.5

Therefore the angle *EFG* is much greater than the angle *EGF*.

Since *EFG* is a triangle having the angle *EFG* greater than the angle *EGF*, and side opposite the greater angle is greater, therefore the side *EG* is also greater than *EF*. I.19

But *EG* equals *BC*, therefore *BC* is also greater than *EF*.

Therefore *if two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.*

Q.E.D.

Guide

Use of Proposition 24

This proposition is used in the next proposition as well as a few in Book III and XI.22.

Next proposition: I.25

Previous: I.23

Select from Book I

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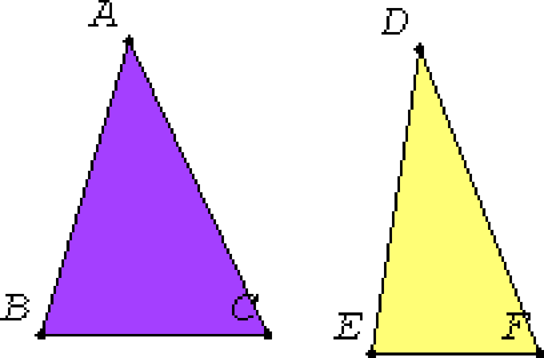
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Euclid's Elements

Book I

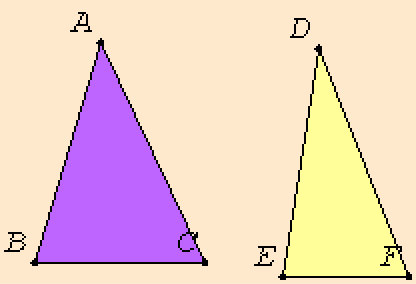
Proposition 25

If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

Let *ABC* and *DEF* be two triangles having two sides *AB* and *AC* equal to two sides *DE* and *DF* respectively, namely *AB* to *DE,* and *AC* to *DF,* and let the base *BC* be greater than the base *EF.*

I say that the angle *BAC* is also greater than the angle *EDF.*

If not, it either equals it or is less.



Now the angle *BAC* does not equal the angle *EDF,* for then the base *BC* would equal the base *EF,* but it is not. Therefore the angle *BAC* does not equal the angle *EDF.* I.4

Neither is the angle *BAC* less than the angle *EDF,* for then the base *BC* would be less than the base *EF,* but it is not. Therefore the angle *BAC* is not less than the angle *EDF.* I.24

But it was proved that it is not equal either. Therefore the angle *BAC* is greater than the angle *EDF.*

Therefore *if two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.*

Q.E.D.

Guide

The conclusions of this proposition and the previous are partial converses of each other. Together they say that if two triangles have two sides equal to two sides respectively, then the base greater than the base if and only if the one of the angles contained by the equal straight lines greater than the other.

Use of Proposition 25

This proposition is not used in the rest of Book I, but it is used in XI.20 and XI.23.

Next proposition: I.26

Previous: I.24

Book I introduction

Select from Book I

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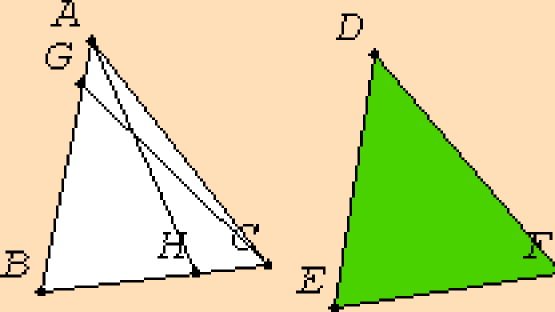
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Euclid's Elements

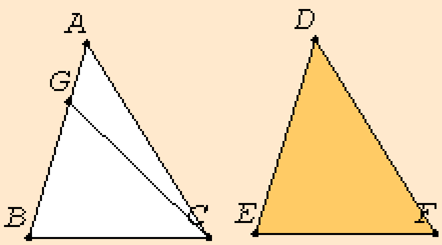
Book I

Proposition 26

If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Let *ABC* and *DEF* be two triangles having the two angles *ABC* and *BCA* equal to the two angles *DEF* and *EFD* respectively, namely the angle *ABC* to the angle *DEF*, and the angle *BCA* to the angle *EFD*, and let them also have one side equal to one side, first that adjoining the equal angles, namely *BC* equal to *EF*.

I say that the remaining sides equal the remaining sides respectively, namely *AB* equals *DE* and *AC* equals *DF*, and the remaining angle equals the remaining angle, namely the angle *BAC* equals the angle *EDF*.



If *AB* does not equal *DE*, then one of them is greater.

Let *AB* be greater. Make *BG* equal to *DE*, and join *GC*. I.3 Post.1

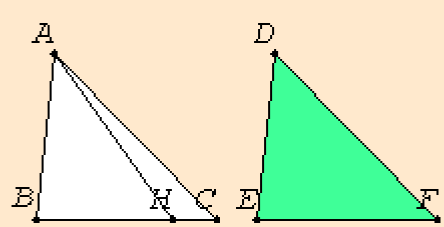
Since *BG* equals *DE*, and *BC* equals *EF*, the two sides *GB* and *BC* equal the two sides *DE* and *EF* respectively, and the angle *GBC* equals the angle *DEF*, therefore the base *GC* equals the base *DF*, the triangle *GBC* equals the triangle *DEF*, and the remaining angles equal the remaining angles, namely those opposite the equal sides. Therefore the angle *GCB* equals the angle *DFE*. But the angle *DFE* equals the angle *ACB* by hypothesis. Therefore the angle *BCG* equals the angle *BCA*, the less equals the greater, which is impossible. I.4 C.N.1

Therefore *AB* is not unequal to *DE*, and therefore equals it.

But *BC* also equals *EF*. Therefore the two sides *AB* and *BC* equal the two sides *DE* and *EF* respectively, and the angle *ABC* equals the angle *DEF*. Therefore the base *AC* equals the base *DF*, and the remaining angle *BAC* equals the remaining angle *EDF*. I.4

Next, let sides opposite equal angles be equal, as *AB* equals *DE*.

I say again that the remaining sides equal the remaining sides, namely *AC* equals *DF* and *BC* equals *EF*, and further the remaining angle *BAC* equals the remaining angle *EDF*.



If *BC* is unequal to *EF*, then one of them is greater.

Let *BC* be greater, if possible. Make *BH* equal to *EF*, and join *AH*. I.3 Post.1

Since *BH* equals *EF*, and *AB* equals *DE*, the two sides *AB* and *BH* equal the two sides *DE* and *EF* respectively, and they contain equal angles, therefore the base *AH* equals the base *DF*, the triangle *ABH* equals the triangle *DEF*, and the remaining angles equal the remaining angles, namely those opposite the I.4

equal sides. Therefore the angle *BHA* equals the angle *EFD.*

But the angle *EFD* equals the angle *BCA*, therefore, in the triangle *AHC*, the exterior angle *BHA* equals the interior and opposite angle *BCA*, which is impossible. C.N.1 I.16

Therefore *BC* is not unequal to *EF*, and therefore equals it.

But *AB* also equals *DE.* Therefore the two sides *AB* and *BC* equal the two sides *DE* and *EF* respectively, and they contain equal angles. Therefore the base *AC* equals the base *DF*, the triangle *ABC* equals the triangle *DEF*, and the remaining angle *BAC* equals the remaining angle *EDF*. I.4

Therefore *if two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.*

Q.E.D.

Guide

There are two statements in this theorem which are different only in their hypotheses. In one, the known side lies between the two angles, in the other, the known side lies opposite one of the angles. If this proposition had come after proposition I.32 which states the sum of the angles in a triangle equals two right angles, then these two hypotheses could have been merged into one, since then if two angles are known, then is the third. But proposition I.32 depends on the parallel postulate Post.5, which, it is apparent, Euclid did not want to use unless necessary. Thus, this proposition, I.26, appears where it is with two distinct hypotheses.

On congruence theorems

This is the last of Euclid's congruence theorems for triangles. Euclid's congruence theorems are I.4 (side-angle-side), I.8 (side-side-side), and this one, I.26 (side and two angles). Calling them congruence theorems is anachronistic, since Euclid did not explicitly use the concept of congruence. We would say that two triangles *ABC* and *DEF* are *congruent* if the angles *A*, *B*, and *C* equal the angles *D*, *E*, and *F* respectively, and the sides *AB*, *BC*, and *AC* equal the sides *DE*, *EF*, and *DF* respectively, and the triangle *ABC* equals the triangle *DEF* (by which is meant that they have the same area).

The remaining congruence theorem, side-side-angle, includes some ambiguous cases. Suppose triangles *ABC* and *DEF* are such that sides *AB* and *BC* are equal to sides *DE* and *EF* respectively, and angle *A* equals angle *D.* If it is also known that *AB* is less than or equal to *BC*, then it follows that the two triangles are congruent. If, however, *AB* is greater than *BC*, then the two triangles need not be congruent. Euclid does not include any form of a side-side-angle congruence theorem, but he does prove one special case, side-side-right angle, in the course of the proof of proposition III.14.

Although Euclid does not include a side-side-angle congruence theorem, he does have a side-side-angle similarity theorem, namely proposition VI.7. The analogous congruence theorem could be stated as follows: If two triangles have one angle equal to one angle, two sides adjoining the equal angles equal, namely, one side adjoining the equal angles, and one opposite the equal angles, and the remaining angles either both less or both not less than a right angle, then the remaining side equals the remaining side and the remaining angles equal the remaining angles.

Use of Proposition 26

This proposition is used in the proofs of proposition I.34 and several propositions in Books III, IV, XI, XII, and XIII.

As in propositions I.4 and I.8, it appears that the triangles are in the same plane, but, again, that is not necessary.

Indeed, this proposition is invoked in proposition XI.35 when two triangles do not lie in the same plane.

Next proposition: I.27

Previous: I.25

Book I introduction

Select from Book I

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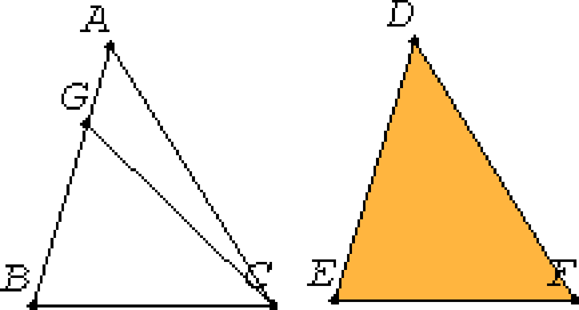
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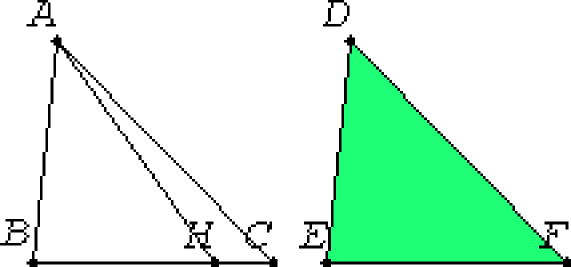
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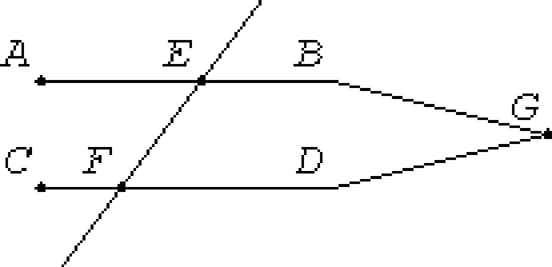
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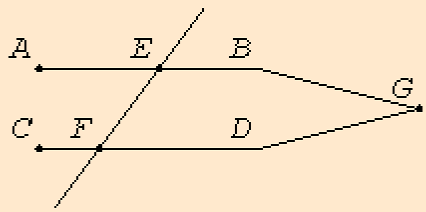
Euclid's Elements
Book I
Proposition 27

If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Let the straight line *EF* falling on the two straight lines *AB* and *CD* make the alternate angles *AEF* and *EFD* equal to one another.

I say that *AB* is parallel to *CD.*



If not, *AB* and *CD* when produced meet either in the direction of *B* and *D* or towards *A* and *C.*

Let them be produced and meet, in the direction of *B* and *D,* at *G.*

Then, in the triangle *GEF,* the exterior angle *AEF* equals the interior and opposite angle *EFG,* which is impossible. I.16

Therefore *AB* and *CD* when produced do not meet in the direction of *B* and *D.*

Similarly it can be proved that neither do they meet towards *A* and *C.*

But straight lines which do not meet in either direction are parallel. Therefore *AB* is parallel to *CD.* I.Def.23

Therefore *if a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.*

Q.E.D.

Guide

There is implicitly assumed an ambient plane. The term "alternate angles" doesn't have a meaning unless the lines all lie in a plane.

Note that Euclid does not consider two other possible ways that the two lines could meet, namely, in the directions *A* and *D* or toward *B* and *C.*

About logical inverses

Although this is the first proposition about parallel lines, it does not require the parallel postulate Post.5 as an assumption. This proposition I.27 and the parallel postulate can be made to look more similar if they are reworded (with the help of I.13).

Proposition 1.27.

> If a straight line falls on two straight lines, then if the alternate angles are equal, then the straight lines do not meet.

Post.5.

If a straight line falls on two straight lines, then if the alternate angles are not equal, then the straight lines meet [on a certain side of the line].

If the remark about the side is dropped, then the conclusions are logical inverses of each other, and the logical inverse of a statement is logically equivalent to the converse.

This little table summarizes the logical relations between similarly looking statements.

Statement	If *P* then *Q*.	
Converse	If *Q* then *P*.	Not logically equivalent to the statement.
Contrapositive	If not *Q* then not *P*.	Logically equivalent to the statement.
Inverse	If not *P* then not *Q*.	Logically equivalent to the converse.

Although the contrapositive is logically equivalent to the statement, Euclid always proves the contrapositive separately using a proof by contradiction and the original statement. Similarly, the inverse is proved using the converse. Sometimes all four statements appear in separate propositions as in propositions X.5 through X.8. Other times the four appear as four statements in one proposition as in X.9. More often than not, however, the contrapositive and inverse make no appearance, and, of course, the converse only appears when it can be proved.

Use of Proposition 27

At this point, parallel lines have yet to be constructed. That occurs in proposition I.31 which uses this proposition to verify that lines constructed there are parallel. This proposition is also used in the next one and in I.33.

Next proposition: I.28

Previous: I.26

Book I introduction

Select from Book I

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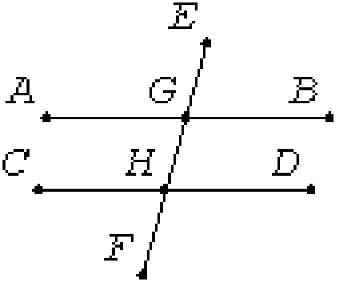
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Euclid's Elements

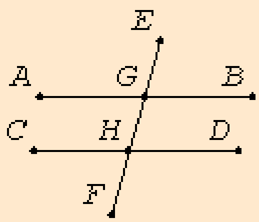
Book I

Proposition 28

If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Let the straight line *EF* falling on the two straight lines *AB* and *CD* make the exterior angle *EGB* equal to the interior and opposite angle *GHD*, or the sum of the interior angles on the same side, namely *BGH* and *GHD*, equal to two right angles.

I say that *AB* is parallel to *CD.*



Since the angle *EGB* equals the angle *GHD*, and the angle *EGB* equals the angle *AGH*, therefore the angle *AGH* equals the angle *GHD.* And they are alternate, therefore *AB* is parallel to *CD.* I.15 C.N.1 I.27

Next, since the sum of the angles *BGH* and *GHD* equals two right angles, and the sum of the angles *AGH* and *BGH* also equals two right angles, therefore the sum of the angles *AGH* and *BGH* equals the sum of the angles *BGH* and *GHD.* I.13 C.N.1 Post.4

Subtract the angle *BGH* from each. Therefore the remaining angle *AGH* equals the remaining angle *GHD.* And they are alternate, therefore *AB* is parallel to *CD.* C.N.3 I.27

Therefore *if a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.*

Q.E.D.

Guide

This proposition states two useful minor variants of the previous proposition. The three statements differ only in their hypotheses which are easily seen to be equivalent with the help of proposition I.13.

Use of Proposition 28

This proposition is used in IV.7, VI.4, and a couple times in Book XI.

Next proposition: I.29

Previous: I.27

Book I introduction

Select from Book I

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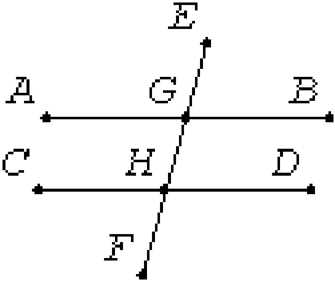
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Euclid's Elements

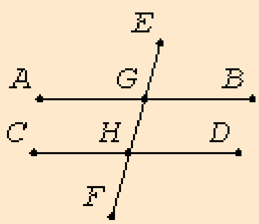
Book I

Proposition 29

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Let the straight line *EF* fall on the parallel straight lines *AB* and *CD.*

I say that it makes the alternate angles *AGH* and *GHD* equal, the exterior angle *EGB* equal to the interior and opposite angle *GHD*, and the sum of the interior angles on the same side, namely *BGH* and *GHD*, equal to two right angles.



If the angle *AGH* does not equal the angle *GHD*, then one of them is greater. Let the angle *AGH* be greater.

Add the angle *BGH* to each. Therefore the sum of the angles *AGH* and *BGH* is greater than the sum of the angles *BGH* and *GHD.*

But sum of the angles *AGH* and *BGH* equals two right angles. Therefore the sum of the angles *BGH* and *GHD* is less than two right angles. I.13

But straight lines produced indefinitely from angles less than two right angles meet. Therefore *AB* and *CD*, if produced indefinitely, will meet. But they do not meet, because they are by hypothesis parallel. Post.5

Therefore the angle *AGH* is not unequal to the angle *GHD*, and therefore equals it.

Again, the angle *AGH* equals the angle *EGB.* Therefore the angle *EGB* also equals the angle *GHD.* I.15 C.N.1

Add the angle *BGH* to each. Therefore the sum of the angles *EGB* and *BGH* equals the sum of the angles *BGH* and *GHD.* C.N.2

But the sum of the angles *EGB* and *BGH* equals two right angles. Therefore the sum of the angles *BGH* and *GHD* also equals two right angles. I.13 C.N.1

Therefore *a straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.*

Q.E.D.

Guide

The statement of this proposition includes three parts, one the converse of I.27, the other two the converse of I.28. Like those propositions, this one assumes an ambient plane containing all the three lines.

This is the first proposition which depends on the parallel postulate. As such it does not hold in hyperbolic geometry.

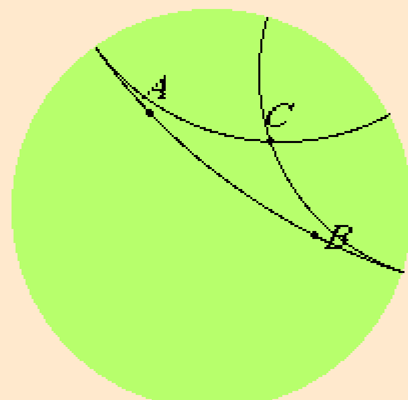
Hyperbolic geometry

Two important geometries alternative to Euclidean geometry are elliptic geometry and hyperbolic geometry. Elliptic geometry was discussed in the note after Proposition I.16, that being the first proposition which doesn't hold in elliptic geometry. This, I.29, is the first which doesn't hold in hyperbolic geometry.

These three geometries can be distinguished by the number of lines parallel to a given line passing through a given point. For elliptic geometry, there is no such parallel line; for Euclidean geometry (which may be called parabolic geometry), there is exactly one; and for hyperbolic geometry, there are infinitely many.

It is not possible to illustrate hyperbolic geometry with correct distances on a flat surface since a flat surface is Euclidean. Poincaré, however, described a useful model of hyperbolic geometry where the "points" in a hyperbolic plane are taken to be points inside a fixed circle (but not the points on the circumference). The "lines" in the hyperbolic plane are the parts of circles orthogonal, that is, at right angles to the fixed circle. And in this model, "angles" in the hyperbolic plane are angles between these arcs, or, more precisely, angles between the tangents to the arcs at the point of intersection. Since "angles" are just angles, this model is called a *conformal* model. Distances in the hyperbolic plane, however, are not measured by distances along the arcs. There is a more complicated relation between distances so that near the edge of the fixed circle a very short arc models a very long "line."

Once this model is accepted, it is easy to see why there are infinitely many "lines" parallel to a given "line" through a given "point." That is just that there are infinitely many circles orthogonal to the fixed circle which don't intersect the given circle orthogonal to the fixed circle but do pass through the given point.



In the diagram, *AB* is a "line" in the hyperbolic plane, that is, a circle orthogonal to the circumference of the shaded disk which represents the hyperbolic plane. A "point" *C* lies in that plane. Two "lines" are shown passing through *C*, one gets close to the line *AB* in the direction of *A*, the other gets close in the direction of *B*. But these two "lines" don't intersect *AB* since the arcs representing them only intersect on the circumference of the disk, and points on the circumference don't represent "points" in the hyperbolic plane.

These two parallel "lines" are called the *asymptotic* parallels of *AB* since they approach *AB* at one end or the other. There are infinitely many parallels between them. (In much of the literature on hyperbolic geometry, the word "parallels" is used for what are called "asymptotic parallels" here, while "nonintersecting lines" is used for what are called "parallels" here.)

Use of Proposition 29

This proposition is used in very frequently in Book I starting with the next proposition. It is also used frequently in Book II, VI, and XI, and once in Book XII.

Next proposition: I.30 Select from Book I

Previous: I.28

Book I introduction

Select book

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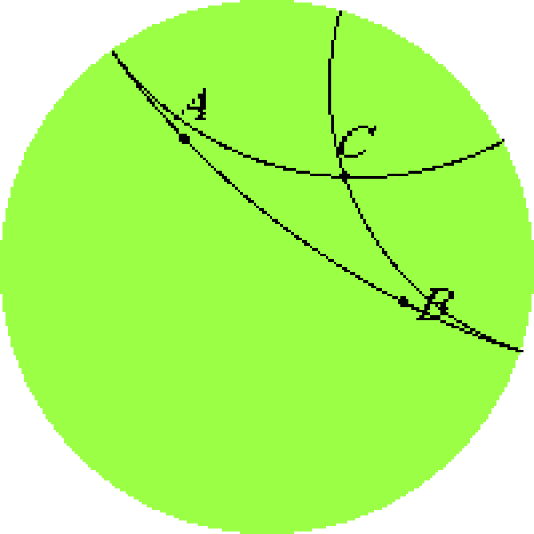
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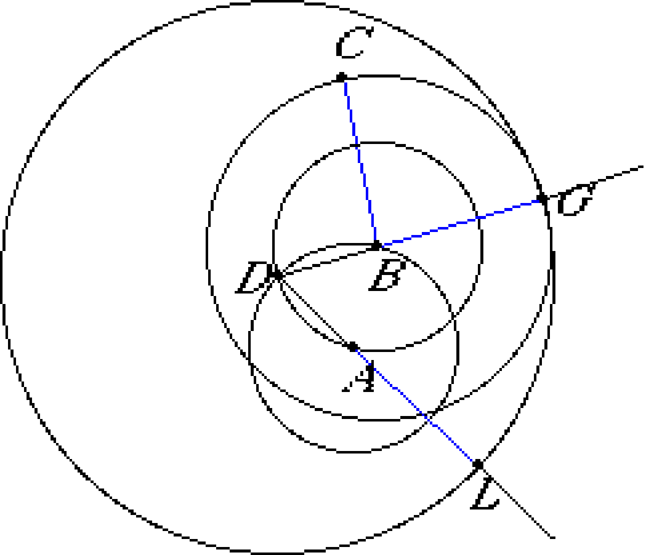

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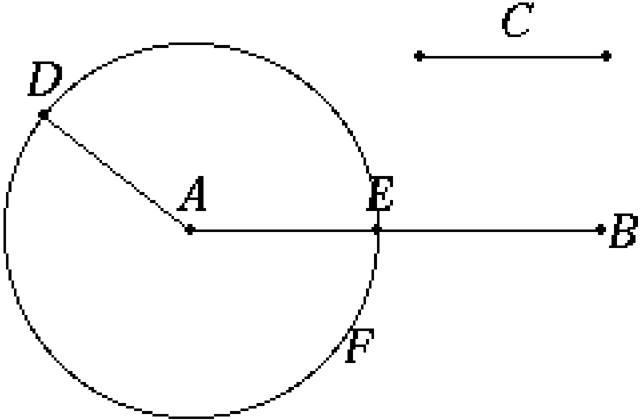
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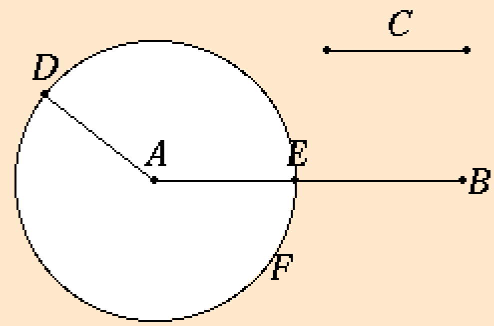
Euclid's Elements

Book I

Proposition 3

To cut off from the greater of two given unequal straight lines a straight line equal to the less.



Let *AB* and *C* be the two given unequal straight lines, and let *AB* be the greater of them.

It is required to cut off from *AB* the greater a straight line equal to *C* the less.

Place *AD* at the point *A* equal to the straight line *C*, and describe the circle *DEF* with center *A* and radius *AD*. I.2 Post. 3

Now, since the point *A* is the center of the circle *DEF*, therefore *AE* equals *AD*. I.Def.15

But *C* also equals *AD*, therefore each of the straight lines *AE* and *C* equals *AD*, so that *AE* also equals *C*. C.N.1

Therefore, given the two straight lines *AB* and *C*, *AE* has been cut off from *AB* the greater equal to *C* the less.

Q.E.F.

Guide

Now it is clear that the purpose of Proposition 2 is to effect the construction in this proposition.

According to Proclus (410-485 C.E.) in his *Commentary on Book I*, Hippocrates of Chios (fl. ca. 430 B.C.E.) was the first to write an *Elements*. Leon and Theudius also wrote versions before Euclid (fl. ca. 295 B.C.E.). These other *Elements* have all been lost since Euclid's replaced them. It is conceivable that in some of these earlier versions the construction in proposition I.2 was not known, so this proposition would instead have been a postulate (a stronger version of Post.3). Once the construction in I.2 was discovered, the current weaker Post.3 would do. Then again, I.2 might go back to the time of Hippocrates.

Construction steps

This construction takes one more step beyond that of I.2, and that is the final circle, the circle shown in the diagram accompanying this proposition. Altogether, therefore, five circles and two lines are required for this construction.

Frequently, though, one end of the line *C* is already placed at *A*, and then the construction of I.2 isn't required. In that case, only one circle needs to be drawn.

Use of Proposition 3

This proposition begins the geometric arithmetic of lines. Explicitly, it allows lines to be subtracted, but it can also be used to compare lines for equality and to add lines, that is, one line can be placed alongside another to determine if they are equal, or if not, which is greater. In other words, this construction justifies the law of trichotomy for lines.

The construction is use more often in the *Elements* than any other starting with proposition I.5. It is used in all the books on geometry, that is in Books I through IV, VI, and XI through XIII.

Naturally a construction of this sort is needed in the solid geometry of Books XI through XIII. Surprisingly, the construction given here also works in solid geometry, even the lines *AB* and *C* don't lie in the same plane. Since the point *A* and the line *C* lie in one plane, the construction of I.2 produces a line *AD* equal to *C* in that plane. Now *AD* and *AB* also lie in one plane, but not the same one, and the circle *AEF* can be drawn there.

Next proposition: I.4

Previous: I.2

Book I introduction

Select from Book I

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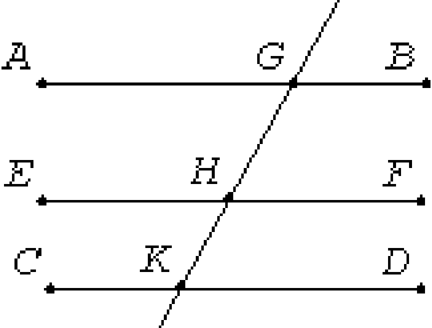
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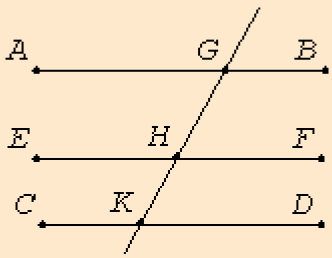
Euclid's Elements
Book I
Proposition 30

Straight lines parallel to the same straight line are also parallel to one another.

Let each of the straight lines *AB* and *CD* be parallel to *EF*.

I say that *AB* is also parallel to *CD.*



Let the straight line *GK* fall upon them. Since the straight line *GK* falls on the parallel straight lines *AB* and *EF*, therefore the angle *AGK* equals the angle *GHF*. I.29

Again, since the straight line *GK* falls on the parallel straight lines *EF* and *CD*, therefore the angle *GHF* equals the angle *GKD.* I.29

But the angle *AGK* was also proved equal to the angle *GHF*. Therefore the angle *AGK* also equals the angle *GKD,* and they are alternate. C.N.1

Therefore *AB* is parallel to *CD.*

Therefore *straight lines parallel to the same straight line are also parallel to one another*.

Q.E.D.

Guide

For this proposition it is supposed that the three lines lie in one plane. Proposition XI.9 applies to the case where the three lines do not lie in a plane.

Playfair's axiom

A number of the propositions in the *Elements* are equivalent to the parallel postulate Post.5 in the sense that if the rest of the postulates are assumed and any one of these propositions is assumed, then the parallel postulate can be proved as a proposition. This one I.30, the last I.29, either part of I.32, and almost any later one. Thus, Euclid had many statements to choose from to take as a postulate.

In many modern expositions of synthetic geometry, Playfair's axiom (John Playfair, 1748-1819) is chosen as that postulate instead of Euclid's parallel postulate Post.5. Playfair's axiom states that there is at most one line parallel to a given line passing through a given point. (That there is at least one follows from the next proposition I.31 which doesn't depend on the parallel postulate.)

Two advantages of Playfair's axiom over Euclid's parallel postulate are that it is a simpler statement, and it emphasizes the distinction between Euclidean and hyperbolic geometry.

Two disadvantages are that it does not have the historical importance of Euclid's parallel postulate, and the proof of the parallel postulate from Playfair's axiom is nonconstructive. That proof is a proof by contradiction that begins assuming that a point does not exist, deriving a contradiction, and concluding that the point must exist, but does not construct it.

It may well be that Euclid chose to make the construction an assumption of his parallel postulate rather rather than choosing some other equivalent statement for his postulate.

Use of Proposition 30

This proposition is used in I.45 and IV.7.

Next proposition: I.31

Previous: I.29

Book I introduction

Select from Book I

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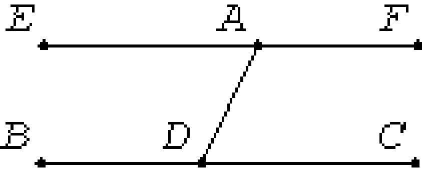
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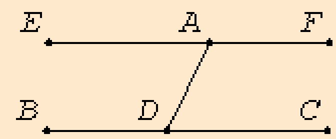
Euclid's Elements

Book I

Proposition 31

To draw a straight line through a given point parallel to a given straight line.



Let *A* be the given point, and *BC* the given straight line.

It is required to draw a straight line through the point *A* parallel to the straight line *BC*.

Take a point *D* at random on *BC*. Join *AD*. Construct the angle *DAE* equal to the angle *ADC* on the straight line *DA* and at the point *A* on it. Produce the straight line *AF* in a straight line with *EA*. Post.1 I.23 Post.2

Since the straight line *AD* falling on the two straight lines *BC* and *EF* makes the alternate angles *EAD* and *ADC* equal to one another, therefore *EAF* is parallel to *BC*. I.27

Therefore the straight line *EAF* has been drawn through the given point *A* parallel to the given straight line *BC*.

Q.E.F.

Guide

The parallel line *EF* constructed in this proposition is the only one passing through the point *A*. If there were another, then the interior angles on one side or the other of *AD* it makes with *BC* would be less than two right angles, and therefore by the parallel postulate Post.5, it would meet *BC*, a contradiction.

Incidentally, this construction also works in hyperbolic geometry, although different parallel lines through *A* are constructed for different points *D*.

Construction steps

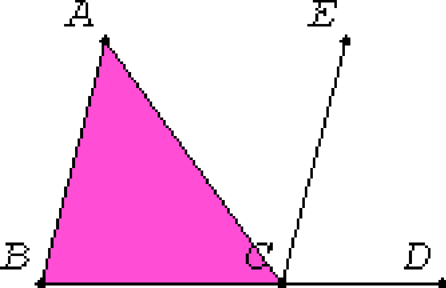
The construction needed is that of I.23 to construct an angle. That construction required ten circles and one line in general. In the specific case needed here, however, one of the distances does not have to be transferred, and that eliminates the need to construct four of the circles. Therefore this construction actually only requires six circles and a line.

Use of Proposition 31

This construction is frequently used in the remainder of Book I starting with the next proposition. It is also frequently used in Books II, IV, VI, XI, XII, and XIII.

Next proposition: I.32 Select from Book I

Previous: I.30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Euclid's Elements

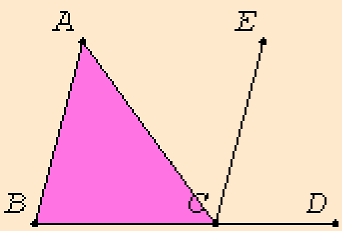
Book I

Proposition 32

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to *D.*

I say that the exterior angle *ACD* equals the sum of the two interior and opposite angles *CAB* and *ABC,* and the sum of the three interior angles of the triangle *ABC, BCA,* and *CAB* equals two right angles.



Draw *CE* through the point *C* parallel to the straight line *AB.* I.31

Since *AB* is parallel to *CE,* and *AC* falls upon them, therefore the alternate angles *BAC* and *ACE* equal one another. I.29

Again, since *AB* is parallel to *CE,* and the straight line *BD* falls upon them, therefore the exterior angle *ECD* equals the interior and opposite angle *ABC.* I.29

But the angle *ACE* was also proved equal to the angle *BAC.* Therefore the whole angle *ACD* equals the sum of the two interior and opposite angles *BAC* and *ABC.*

Add the angle *ACB* to each. Then the sum of the angles *ACD* and *ACB* equals the sum of the three angles *ABC, BCA,* and *CAB.* C.N.2

But the sum of the angles *ACD* and *ACB* equals two right angles. Therefore the sum of the angles *ABC, BCA,* and *CAB* also equals two right angles. I.13 C.N.1

Therefore *in any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.*

Q.E.D.

Guide

Corollaries of Proclus

There are two corollaries of this proposition given by Proclus.

Corollary 1. The sum of the interior angles of a convex rectilinear figure equals twice as many angles as the figure has sides, less four.

Corollary 2. The sum of the exterior angles of any convex rectilinear figure together equal four right angles.

Use of Proposition 32

Although this proposition isn't used in the rest of Book I, it is frequently used in the rest of the books on geometry, namely Books II, III, IV, VI, XI, XII, and XIII. The corollaries, however, are not used in the *Elements*.

Next proposition: I.33

Previous: I.31

Book I introduction

Select from Book I

Select book

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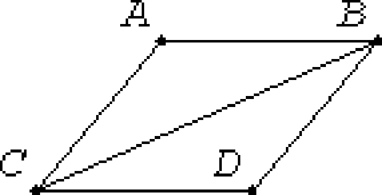
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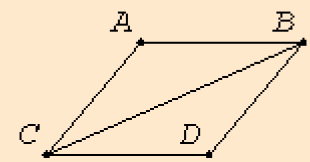
Euclid's Elements
Book I
Proposition 33

Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.



Let *AB* and *CD* be equal and parallel, and let the straight lines *AC* and *BD* join them at their ends in the same directions.

I say that *AC* and *BD* are also equal and parallel.

Join *BC*. Post.1

Since *AB* is parallel to *CD*, and *BC* falls upon them, therefore the alternate angles *ABC* and *BCD* equal one another. I.29

Since *AB* equals *CD*, and *BC* is common, the two sides *AB* and *BC* equal the two sides *DC* and *CB*, and the angle *ABC* equals the angle *BCD*, therefore the base *AC* equals the base *BD*, the triangle *ABC* equals the triangle *DCB*, and the remaining angles equals the remaining angles respectively, namely those opposite the equal sides. Therefore the angle *ACB* equals the angle *CBD*. I.4

Since the straight line *BC* falling on the two straight lines *AC* and *BD* makes the alternate angles equal to one another, therefore *AC* is parallel to *BD*. I.27

And it was also proved equal to it.

Therefore *straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.*

Q.E.D.

Guide

The qualifier "in the same directions" in the statement of this proposition is necessary since without it the lines *AD* and *BC* could join the endpoints of the parallel lines, and *AD* and *BC* are not parallel but intersect. But these words of Euclid words are informal, and it would take some work to determine geometrically which end of *AD* corresponds to which end of a parallel line *BC*.

In general, given four points *A*, *B*, *C*, and *D*, exactly one of the three pairs of lines, *AB* and *CD*, *AC* and *BD*, and *AD* and *BC*, intersects. (If extended to infinite lines, all three pairs of lines might intersect, but as line segments only one pair does.) This statement belongs to the fundamental part of plane geometry that includes betweenness and sides of lines that wasn't developed until the late nineteenth century.

Use of Proposition 33

This proposition is used in I.36, I.45, and a few prpositions in Books XI through XIII.

Next proposition: I.34

Previous: I.32

Book I introduction

Select from Book I

Select book

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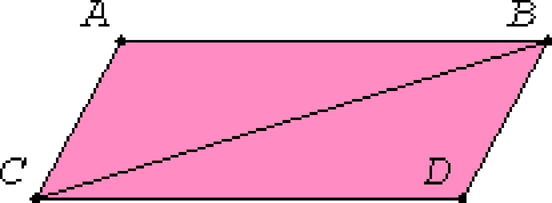
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Euclid's Elements

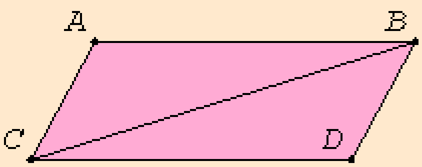
Book I

Proposition 34

In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

Let *ACDB* be a parallelogrammic area, and *BC* its diameter.

I say that the opposite sides and angles of the parallelogram *ACDB* equal one another, and the diameter *BC* bisects it.



Since *AB* is parallel to *CD,* and the straight line *BC* falls upon them, therefore the alternate angles *ABC* and *BCD* equal one another. I.29

Again, since *AC* is parallel to *BD,* and *BC* falls upon them, therefore the alternate angles *ACB* and *CBD* equal one another. I.29

Therefore *ABC* and *DCB* are two triangles having the two angles *ABC* and *BCA* equal to the two angles *DCB* and *CBD* respectively, and one side equal to one side, namely that adjoining the equal angles and common to both of them, *BC.* Therefore they also have the remaining sides equal to the remaining sides respectively, and the remaining angle to the remaining angle. Therefore the side *AB* equals *CD,* and *AC* equals *BD,* and further the angle *BAC* equals the angle *CDB.* I.26

Since the angle *ABC* equals the angle *BCD,* and the angle *CBD* equals the angle *ACB,* therefore the whole angle *ABD* equals the whole angle *ACD.* C.N.2

And the angle *BAC* was also proved equal to the angle *CDB.*

Therefore in parallelogrammic areas the opposite sides and angles equal one another.

I say, next, that the diameter also bisects the areas.

Since *AB* equals *CD,* and *BC* is common, the two sides *AB* and *BC* equal the two sides *DC* and *CB* respectively, and the angle *ABC* equals the angle *BCD.* Therefore the base *AC* also equals *DB,* and the triangle *ABC* equals the triangle *DCB.* I.4

Therefore the diameter *BC* bisects the parallelogram *ACDB.*

Therefore *in parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.*

Q.E.D.

Guide

In this proposition Euclid uses the term "parallelogrammic area" rather than the word "parallelogram" which first occurs in the next proposition. Proclus indicated that the word "parallelogram" was created by Euclid.

This proposition begins the study of areas of rectilinear figures. It is a modest beginning, but it allows the comparison of triangles and parallelograms so that problems and results concerning one can be converted to problems and results concerning the other.

Use of Proposition 34

This proposition is used in the next four propositions and some others in Book I, several in Book II, a few in Books IV, VI, X, XI, and XII.

Next proposition: I.35

Previous: I.33

Book I introduction

Select from Book I

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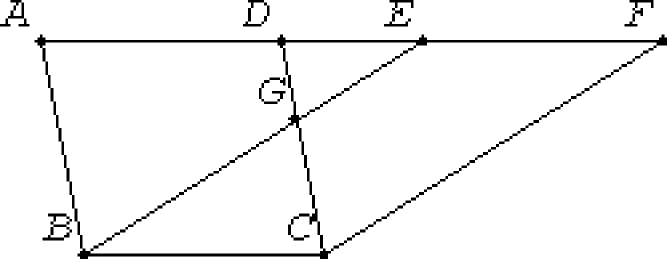
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Euclid's Elements
Book I
Proposition 35

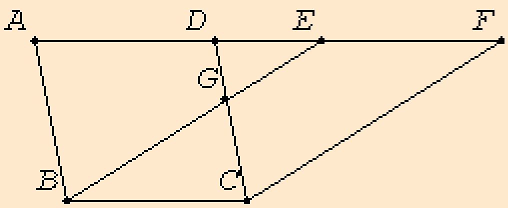
Parallelograms which are on the same base and in the same parallels equal one another.

Let *ABCD* and *EBCF* be parallelograms on the same base *BC* and in the same parallels *AF* and *BC.*

I say that *ABCD* equals the parallelogram *EBCF.*

Since *ABCD* is a parallelogram, therefore *AD* equals *BC.* I.34

For the same reason *EF* equals *BC,* so that *AD* also equals *EF.* And *DE* is common, therefore the whole *AE* equals the whole *DF.* C.N.1 C.N.2



But *AB* also equals *DC.* Therefore the two sides *EA* and *AB* equal the two sides *FD* and *DC* respectively, and the angle *FDC* equals the angle *EAB,* the exterior equals the interior. Therefore the base *EB* equals the base *FC,* and the triangle *EAB* equals the triangle *FDC.* I.34 I.29 I.4

Subtract *DGE* from each. Then the trapezium *ABGD* which remains equals the trapezium *EGCF* which remains. C.N.3

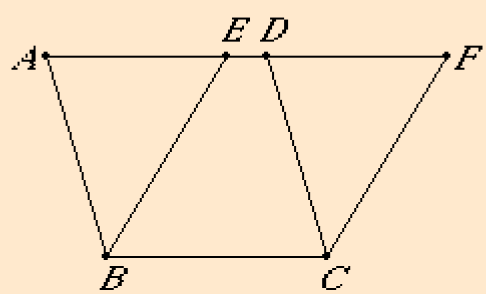
Add the triangle *GBC* to each. Then the whole parallelogram *ABCD* equals the whole parallelogram *EBCF.* C.N.2

Therefore *parallelograms which are on the same base and in the same parallels equal one another*.

Q.E.D.

Guide

Euclid's proof specifically treats the case when the point *D* lies between *A* and *E* in which case subtraction of a triangle is necessary. There are other cases to consider, for instance, when *E* lies between *A* and *D.* In that case the point *G* is irrelevant and the trapezium *BCED* may be added to the congruent triangles *ABE* and *DCF* to derive the conclusion. Euclid often supplies a proof for only one case, although occasionally he gives proofs for two or three cases.



Rectilinear figures as magnitudes

We see how Euclid treats figures as magnitudes by adding as subtracting them. The triangles *EAB* and *FDC* are shown directly to be equal. Then the triangle *DGE,* which is contained in each, is subtracted from each, and Euclid concludes that the remaining trapezia *ABGD* and *EGCF* are therefore equal. These trapezia are not congruent, but they do have the same area. Next, the triangle *GBC* is added to each trapezium to conclude the two parallelograms *ABCD* and *EBCF* are equal.

These are the same kinds of cut-and-paste operations that Euclid used on lines and angles earlier in Book I, but these are applied to rectilinear figures. In later books cut-and-paste operations will be applied to other kinds of magnitudes such as solid figures and parts of circumferences of circles.

Use of Proposition 35

This proposition is used in the next two propositions and in XI.31.

Next proposition: I.36

Previous: I.34

Book I introduction

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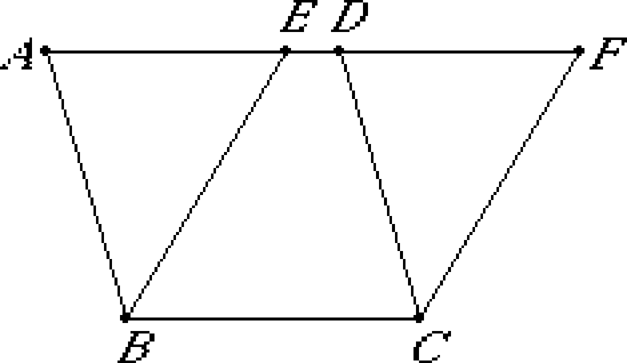
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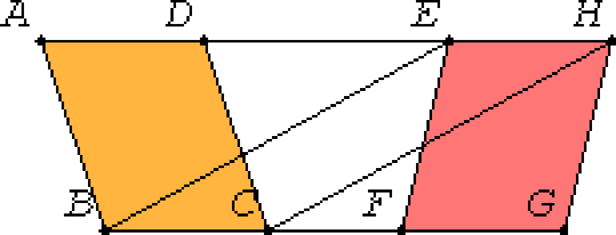
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Euclid's Elements

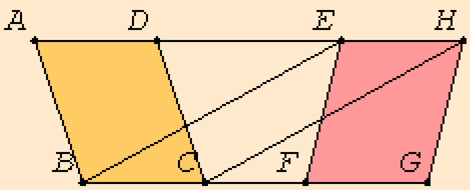
Book I

Proposition 36

Parallelograms which are on equal bases and in the same parallels equal one another.

Let *ABCD* and *EFGH* be parallelograms which are on the equal bases *BC* and *FG* and in the same parallels *AH* and *BG*.

I say that the parallelogram *ABCD* equals *EFGH*.



Join *BE* and *CH*. Post.1

Since *BC* equals *FG* and *FG* equals *EH*, therefore *BC* equals *EH*. I.34 C.N.1

But they are also parallel, and *EB* and *HC* join them. But straight lines joining equal and parallel straight lines in the same directions are equal and parallel, therefore *EBCH* is a parallelogram. I.33

And it equals *ABCD*, for it has the same base *BC* with it and is in the same parallels *BC* and *AH* with it. I.35

For the same reason also *EFGH* equals the same *EBCH*, so that the parallelogram *ABCD* also equals *EFGH*. C.N.1

Therefore *parallelograms which are on equal bases and in the same parallels equal one another*.

Q.E.D.

Guide

This proposition is a generalization of the previous proposition I.35, and its proof depends directly on it. Euclid could have bundled the two propositions into one. Then the special case of I.35 would have been proven first and then used to prove the general case of I.36. In an introductory book like Book I this separation makes it easier to follow the logic, but in later books special cases are often bundled into the general proposition.

Use of Proposition 36

This proposition is used in I.38, a few propositions in Books II and VI, and XI.29

Next proposition: I.37

Previous: I.35

Book I introduction

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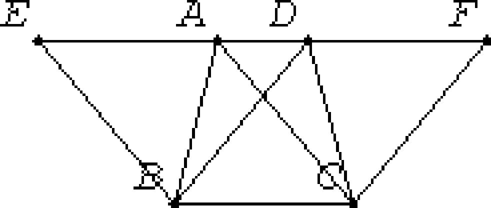
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Euclid's Elements

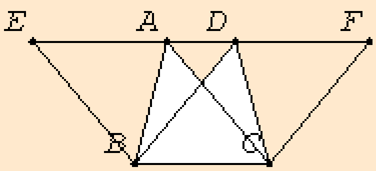
Book I

Proposition 37

Triangles which are on the same base and in the same parallels equal one another.

Let *ABC* and *DBC* be triangles on the same base *BC* and in the same parallels *AD* and *BC*.

I say that the triangle *ABC* equals the triangle *DBC*.



Produce *AD* in both directions to *E* and *F*. Draw *BE* through *B* parallel to *CA*, and draw *CF* through *C* parallel to *BD*. Post.2 I.31

Then each of the figures *EBCA* and *DBCF* is a parallelogram, and they are equal, for they are on the same base *BC* and in the same parallels *BC* and *EF*. I.35

Moreover the triangle *ABC* is half of the parallelogram *EBCA*, for the diameter *AB* bisects it. And the triangle *DBC* is half of the parallelogram *DBCF*, for the diameter *DC* bisects it. I.34

Therefore the triangle *ABC* equals the triangle *DBC*. C.N

Therefore *triangles which are on the same base and in the same parallels equal one another.*

Q.E.D.

Guide

In this proposition the triangles have the same base while in the next one the triangles have equal bases. Since the proofs are the same except that this depends on I.35 while the next depends on I.36, and the next is more general, there is no purpose to include this proposition.

The justification of the last conclusion is missing. From the statement that the doubles of two magnitudes are equal, we want to conclude that the magnitudes themselves are equal. Although Euclid included no such common notion, others inserted it later. See the commentary on Common Notions for a proof of this halving principle based on other properties of magnitudes.

Use of Proposition 37

This proposition is used in I.39, I.41, and VI.2.

Next proposition: I.38

Previous: I.36

Book I introduction

Select from Book I

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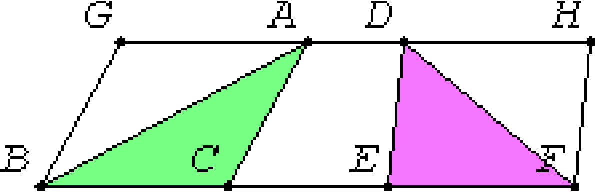
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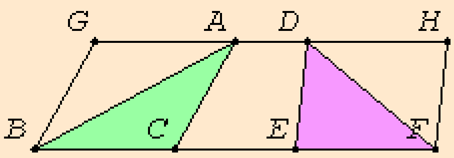
Euclid's Elements
Book I
Proposition 38

Triangles which are on equal bases and in the same parallels equal one another.

Let *ABC* and *DEF* be triangles on equal bases *BC* and *EF* and in the same parallels *BF* and *AD*.

I say that the triangle *ABC* equals the triangle *DEF*.



Produce *AD* in both directions to *G* and *H*. Draw *BG* through *B* parallel to *CA*, and draw *FH* through *F* parallel to *DE*. Post.2 I.31

Then each of the figures *GBCA* and *DEFH* is a parallelogram, and *GBCA* equals *DEFH*, for they are on equal bases *BC* and *EF* and in the same parallels *BF* and *GH*. I.36

Moreover the triangle *ABC* is half of the parallelogram *GBCA*, for the diameter *AB* bisects it. And the triangle *FED* is half of the parallelogram *DEFH*, for the diameter *DF* bisects it. I.34

Therefore the triangle *ABC* equals the triangle *DEF*. C.N.

Therefore *triangles which are on equal bases and in the same parallels equal one another*.

Q.E.D.

Guide

The idea of the argument is clear: since parallelograms on equal bases and in the same parallels are equal by I.36, and the triangles are half the parallelograms by I.34, therefore the triangles are also equal.

Use of Proposition 38

This proposition is used in I.40, I.42, and VI.1.

Next proposition: I.39

Previous: I.37

Book I introduction

Select from Book I

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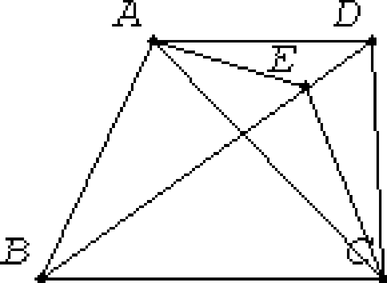
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Euclid's Elements

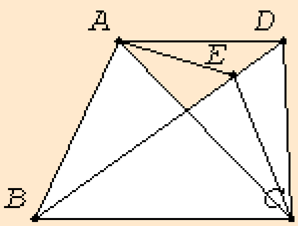
Book I

Proposition 39

Equal triangles which are on the same base and on the same side are also in the same parallels.

Let *ABC* and *DBC* be equal triangles which are on the same base *BC* and on the same side of it. Join *AD*. Post.1

I say that *AD* is parallel to *BC*.



If not, draw *AE* through the point *A* parallel to the straight line *BC*, and join *EC*. I.31 Post.1

Therefore the triangle *ABC* equals the triangle *EBC*, for it is on the same base *BC* with it and in the same parallels. I.37

But *ABC* equals *DBC*, therefore *DBC* also equals *EBC*, the greater equals the less, which is impossible. C.N.1

Therefore *AE* is not parallel to *BC*.

Similarly we can prove that neither is any other straight line except *AD*, therefore *AD* is parallel to *BC*.

Therefore *equal triangles which are on the same base and on the same side are also in the same parallels.*

Q.E.D.

Guide

This is a partial converse to proposition I.37, only partial since the two triangles *ABC* and *DBC* have to be on the same side of the line *BC*. If they weren't, then of course *AD* would not be parallel to *BC* but instead cross it at the midpoint

Use of Proposition 39

This proposition is used in VI.2.

Next proposition: I.40

Previous: I.38

Book I introduction

Select from Book I

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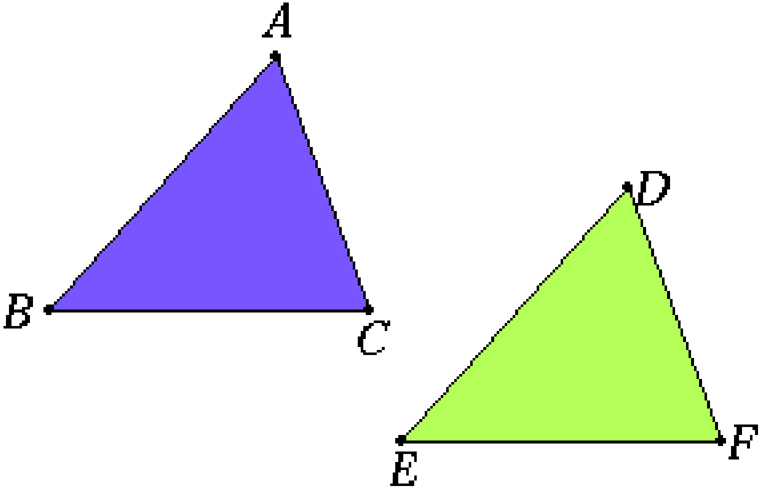
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Euclid's Elements

Book I

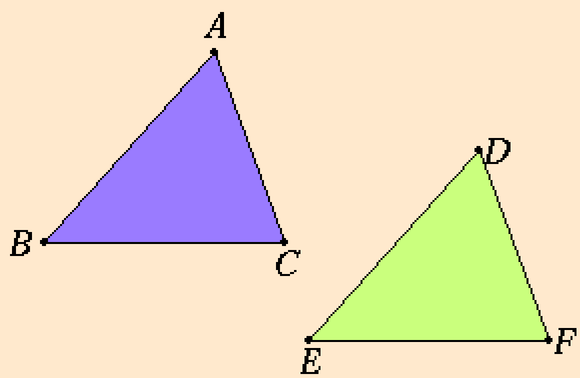
Proposition 4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Let *ABC* and *DEF* be two triangles having the two sides *AB* and *AC* equal to the two sides *DE* and *DF* respectively, namely *AB* equal to *DE* and *AC* equal to *DF*, and the angle *BAC* equal to the angle *EDF*.

I say that the base *BC* also equals the base *EF*, the triangle *ABC* equals the triangle *DEF*, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle *ABC* equals the angle *DEF*, and the angle *ACB* equals the angle *DFE*.

If the triangle *ABC* is superposed on the triangle *DEF*, and if the point *A* is placed on the point *D* and the straight line *AB* on *DE*, then the point *B* also coincides with *E*, because *AB* equals *DE*.



Again, *AB* coinciding with *DE*, the straight line *AC* also coincides with *DF*, because the angle *BAC* equals the angle *EDF*. Hence the point *C* also coincides with the point *F*, because *AC* again equals *DF*.

But *B* also coincides with *E*, hence the base *BC* coincides with the base *EF* and equals it. C.N.4

Thus the whole triangle *ABC* coincides with the whole triangle *DEF* and equals it. C.N.4

And the remaining angles also coincide with the remaining angles and equal them, the angle *ABC* equals the angle *DEF*, and the angle *ACB* equals the angle *DFE*.

Therefore *if two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

Q.E.D.

Guide

This is the first of the congruence propositions for triangles. Euclid did not explicitly use the concept of congruence, although it would have simplified his exposition a bit. The definition of congruence would include the hypotheses and conclusions of this proposition, that is, two triangles *ABC* and *DEF* are *congruent* if angles *A*, *B*, and *C* are equal to angles *D*, *E*, and *F* respectively, and sides *AB*, *BC*, and *AC* are equal to sides *DE*, *EF*, and *DF* respectively, and the triangle *ABC* equals the triangle *DEF* (by which is meant that they have the same area). In the books on solid geometry, Euclid uses the phrase "similar and equal" for congruence, but similarity is not defined until Book VI, so that phrase would be out of place in the first part of the *Elements*.

For more discussion of congruence theorems see the note after proposition I.26, the last of the congruence propositions.

Euclid frequently refers to one side of a triangle as its "base," leaving the other two named "sides." Any one of the sides might be chosen as the base, but once chosen, it remains the base for the rest of the discussion. This is simply a linguistic device to save words.

The method of superposition

The method of proof used in this proposition is sometimes called "superposition." It apparently is not a method that Euclid prefers since he so rarely uses it, only here in I.4 and in I.8 and III.24, but not in many other propositions in which he could have used it.

It is not entirely clear what is meant by "superposing a triangle on a triangle" means. It has been variously interpreted as actually moving one triangle to cover the other or as simply associating parts of one triangle with parts of the other. For the two triangles illustrated in the figure, you can actually slide one over the other in a continuous motion within the plane. Note, however, that if one triangle is the mirror image of the other, then any continuous motion would require moving one triangle outside of the plane. But the triangles don't have to be same plane to begin with, and they often are not in the same plane when this proposition is invoked in the books on solid geometry.

Whatever the intended meaning of superposition may be, there are no postulates to allow any conclusions based on superposition. One possibility is to add postulates based on a group of transformations of space, or if restricted to plane geometry, on a group of transformations of the plane. Charles Dodgson (a.k.a. Lewis Carroll) would have said that using group theory is not appropriate to an elementary exposition of Euclidean geometry. Heath has described a more elementary conservative basis in his commentary on this proposition.

Yet another alternative is to simply take this proposition as a postulate, or part of it as a postulate. For instance, Hilbert in his *Foundations of Geometry* takes as given that under the hypotheses of this proposition that the remaining angles equal the remaining angles. Then, Hilbert proves that the base equals the base.

Use of Proposition 4

Of the various congruence theorems, this one is the most used. This proposition is used frequently in Book I starting with the next two propositions, and it is often used in the rest of the books on geometry, namely, Books II, III, IV, VI, XI, XII, and XIII.

Although the two triangles in this proposition appear to be in the same plane, that is not necessary. In Proposition XI.4 and many others in Book XI this proposition is applied to pairs of triangles in different planes.

Next proposition: I.5

Previous: I.3

Book I introduction

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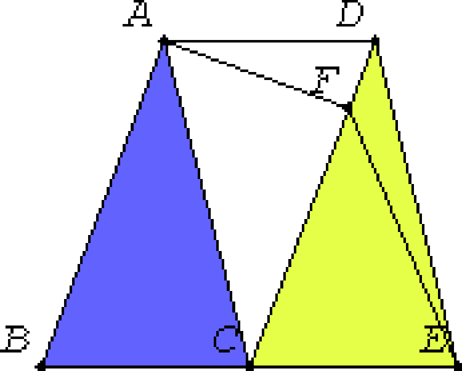
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Euclid's Elements

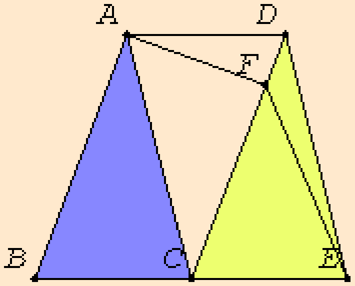
Book I

Proposition 40

Equal triangles which are on equal bases and on the same side are also in the same parallels.

Let *ABC* and *CDE* be equal triangles on equal bases *BC* and *CE* and on the same side.

I say that they are also in the same parallels.



Join *AD.* I say that *AD* is parallel to *BE.* Post.1

If not, draw *AF* through *A* parallel to *BE,* and join *FE.* I.31 Post.1

Therefore the triangle *ABC* equals the triangle *FCE,* for they are on equal bases *BC* and *CE* and in the same parallels *BE* and *AF.* I.38

But the triangle *ABC* equals the triangle *DCE,* therefore the triangle *DCE* also equals the triangle *FCE,* the greater equals the less, which is impossible. Therefore *AF* is not parallel to *BE.* C.N.1

Similarly we can prove that neither is any other straight line except *AD,* therefore *AD* is parallel to *BE.*

Therefore *equal triangles which are on equal bases and on the same side are also in the same parallels.*

Q.E.D.

Guide

The setting out of this proposition is not up to Euclid's standards. There is no justification for assuming that the point *C* is a common vertex of the two triangles. Fortunately, the proof works just as well if *C* is split into two points.

For some of the propositions and many of the lemmas and corollaries in the *Elements,* there is evidence that Euclid did not write them, but they were added later. The process of incorporating new material in textbooks was almost automatic when the books were copied by hand instead of printed. Scholars wrote comments (called "scholia") in the margins of the texts, and copyists (some of whom were later scholars) would include those comments as part of the text in their new copies.

Heiberg could show by means of an early papyrus fragment that this proposition was an early interpolation. For others, such as I.37 there is no direct evidence, only a doubt that a mathematician of Euclid's caliber would have included them.

Unlike the other propositions in Book I, this one is not used later in the *Elements.*

Next proposition: I.41 Select from Book I

Previous: I.39

Book I introduction

Select book

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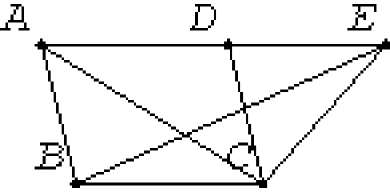
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Euclid's Elements

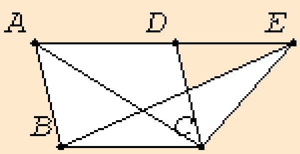
Book I

Proposition 41

If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

Let the parallelogram *ABCD* have the same base *BC* with the triangle *EBC*, and let it be in the same parallels *BC* and *AE*.

I say that the parallelogram *ABCD* is double the triangle *BEC*.



Join *AC*. Post.1

Then the triangle *ABC* equals the triangle *EBC*, for it is on the same base *BC* with it and in the same parallels *BC* and *AE*. I.37

But the parallelogram *ABCD* is double the triangle *ABC*, for the diameter *AC* bisects it, so that the parallelogram *ABCD* is also double the triangle *EBC*. I.34

Therefore *if a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.*

Q.E.D.

Guide

This partially generalizes I.34, that a parallelogram is twice the triangle by its diameter and two of its sides. A slightly more general statement would be that If a parallelogram has an equal base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

Use of Proposition 41

This proposition is used in the next one, I.47, VI.1, and X.38.

Next proposition: I.42

Previous: I.40

Book I introduction

Select from Book I

Select book

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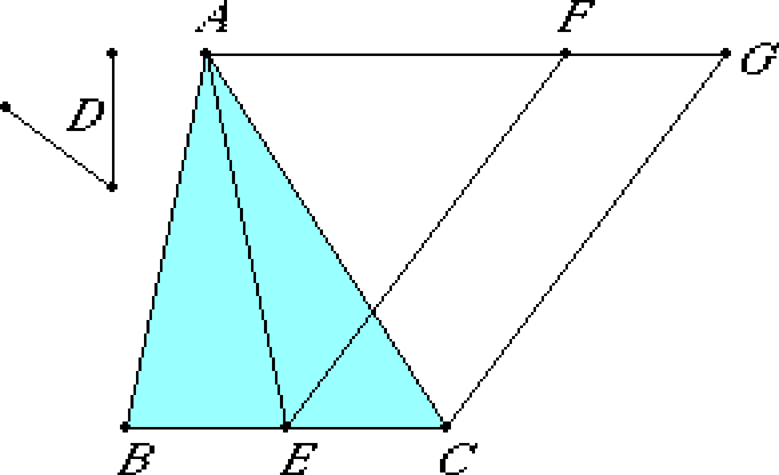
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Euclid's Elements

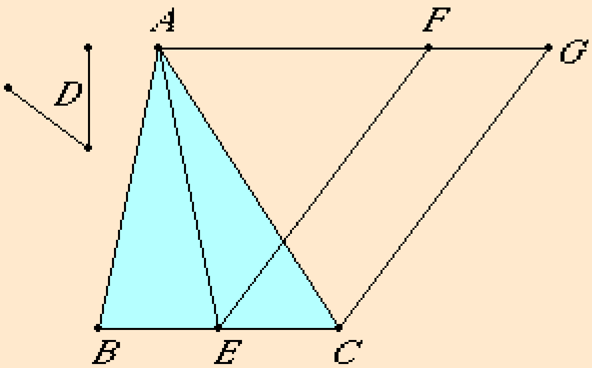
Book I

Proposition 42

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let *ABC* be the given triangle, and *D* the given rectilinear angle.

It is required to construct *D* a parallelogram equal to the triangle *ABC* in the rectilinear angle.



Bisect *BC* at *E*, and join *AE*. Construct the angle *CEF* on the straight line *EC* at the point *E* on it equal to the angle *D*. Draw *AG* through *A* parallel to *EC*, and draw *CG* through *C* parallel to *EF*. I.10 Post.1 I.23 I.31

Then *FECG* is a parallelogram.

Since *BE* equals *EC*, therefore the triangle *ABE* also equals the triangle *AEC*, for they are on equal bases *BE* and *EC* and in the same parallels *BC* and *AG*. Therefore the triangle *ABC* is double the triangle *AEC*. I.38

But the parallelogram *FECG* is also double the triangle *AEC*, for it has the same base with it and is in the same parallels with it, therefore the parallelogram *FECG* equals the triangle *ABC*. I.41 C.N.1

And it has the angle *CEF* equal to the given angle *D*.

Therefore the parallelogram *FECG* has been constructed equal to the given triangle *ABC*, in the angle *CEF* which equals *D*.

Q.E.F.

Guide

The idea of the construction is as follows. First make a triangle half the size of the given triangle. Next skew the half-size triangle to make one of its angles the desired angle without changing its area. Complete the resulting half-size triangle to a parallelogram. That's the desired parallelogram equal to the original triangle in the desired angle.

Application of areas

With this proposition Euclid moves to the next phase in his study of areas, the application of areas. Before this, he has exhibited various situations when triangles or parallelograms have equal areas, or when a triangle has half the area of a parallelogram. But now he's interested in constructing another figure with the same area as a given figure.

In this proposition, he constructs a parallelogram that has a given angle and has the same area as a given triangle. But his goals are coming up, application of areas in I.45 and quadrature in II.14. In proposition I.45, given a rectilinear figure an equal parallelogram is constructed on a given side within a given angle. This kind of construction is called "applying" an area to a side. The area is sort of laid along the line. It may be that before Euclid the area was always applied to a rectangle along the line, but Euclid generalized the construction to parallelograms. This extra generalization is not often used.

Later, in proposition II.14 a square is constructed equal to a given rectilinear figure, a process called "quadrature" (making into a square) of the figure. This square is a canonical measure of the area.

Use of Proposition 42

This construction is used as part of the constructions in the two propositions following the next one.

Next proposition: I.43

Previous: I.41

Book I introduction

Select from Book I

Select book

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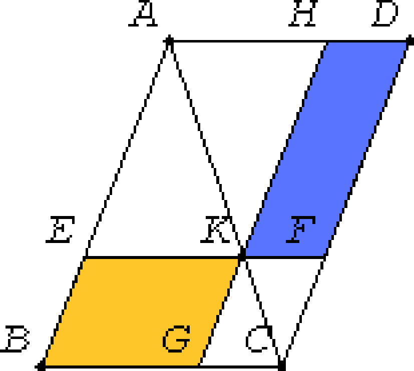
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Euclid's Elements

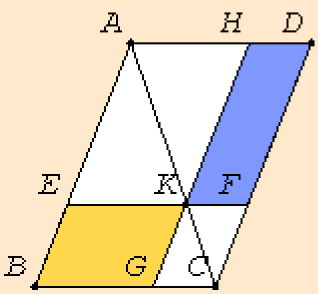
Book I

Proposition 43

In any parallelogram the complements of the parallelograms about the diameter equal one another.

Let *ABCD* be a parallelogram, and *AC* its diameter, and about *AC* let *EH* and *FG* be parallelograms, and *BK* and *KD* the so-called complements.

I say that the complement *BK* equals the complement *KD.*



Since *ABCD* is a parallelogram, and *AC* its diameter, therefore the triangle *ABC* equals the triangle *ACD.* I.34

Again, since *EH* is a parallelogram, and *AK* is its diameter, therefore the triangle *AEK* equals the triangle *AHK*. For the same reason the triangle *KFC* also equals *KGC.* I.34

Now, since the triangle *AEK* equals the triangle *AHK*, and *KFC* equals *KGC*, therefore the triangle *AEK* together with *KGC* equals the triangle *AHK* together with *KFC.* C.N.2

And the whole triangle *ABC* also equals the whole *ADC*, therefore the remaining complement *BK* equals the remaining complement *KD.* C.N.3

Therefore *in any parallelogram the complements of the parallelograms about the diameter equal one another*.

Q.E.D.

Guide

The meaning of the statement has to be found in its use. The term "the parallelograms about the diameter" refers to the two parallelograms having the same angles as the original parallelogram and with diameters *AK* and *KC* which are two parts of a diameter *AC* of the original parallelogram. The "complements" are the two parallelograms left over after removing those two parallelograms from the original parallelogram.

Use of Proposition 43

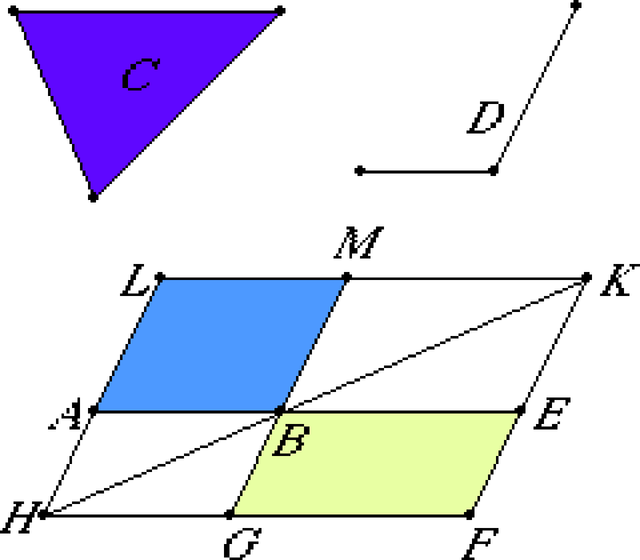
The immediate purpose of this proposition is to change the shape of a parallelogram (one of the complements) into an equal parallelogram with the same angles (the other complement). That's how it is used in the next proposition. It is also used in several propositions in Book II, and a couple in Book VI.

Next proposition: I.44

Select from Book I

Previous: I.42

Select book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Euclid's Elements

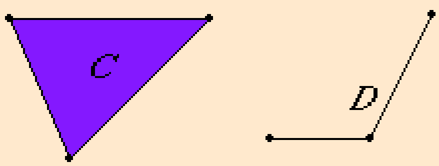
Book I

Proposition 44

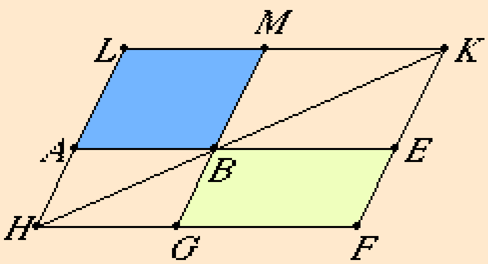
To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Let *AB* be the given straight line, *D* the given rectilinear angle, and *C* the given triangle.

It is required to apply a parallelogram equal to the given triangle *C* to the given straight line *AB* in an angle equal to *D*.



Construct the parallelogram *BEFG* equal to the triangle *C* in the angle *EBG* which equals *D*, and let it be placed so that *BE* is in a straight line with *AB*. I.42



Draw *FG* through to *H*, and draw *AH* through *A* parallel to either *BG* or *EF*. Join *HB*. Post.2 I.31 Post.1

Since the straight line *HF* falls upon the parallels *AH* and *EF*, therefore the sum of the angles *AHF* and *HFE* equals two right angles. Therefore the sum of the angles *BHG* and *GFE* is less than two right angles. And straight lines produced indefinitely from angles less than two right angles meet, therefore *HB* and *FE*, when produced, will meet. I.29 Post.5

Let them be produced and meet at *K*. Draw *KL* through the point *K* parallel to either *EA* or *FH*. Produce *HA* and *GB* to the points *L* and *M*. I.31

Then *HLKF* is a parallelogram, *HK* is its diameter, and *AG* and *ME* are parallelograms, and *LB* and *BF* are the so-called complements about *HK*. Therefore *LB* equals *BF*. I.43

But *BF* equals the triangle *C*, therefore *LB* also equals *C*. C.N.1

Since the angle *GBE* equals the angle *ABM*, while the angle *GBE* equals *D*, therefore the angle *ABM* also equals the angle *D*. I.15 C.N.1

Therefore the parallelogram *LB* equal to the given triangle *C* has been applied to the given straight line *AB*, in the angle *ABM* which equals *D*.

Q.E.F.

Guide

There are two steps in this construction. The first uses proposition I.42 to construct some parallelogram with the correct angle equal to the given triangle. The second uses I.43 to change its length to the proper length.

To "apply an area to a line in an angle" means just what this construction accomplishes, namely, to construct a parallelogram equal to that area with one side as the given line and one angle equal to the given angle.

In practice the angle is often a right angle. The given line may be thought of as a "unit" line. Then the length of the resulting rectangle represents the the area.

Use of Proposition 44

Besides being used in the next proposition, this construction is used in VI.25 to make a figure similar to one rectilinear figure but equal to another.

Next proposition: I.45

Previous: I.43

Book I introduction

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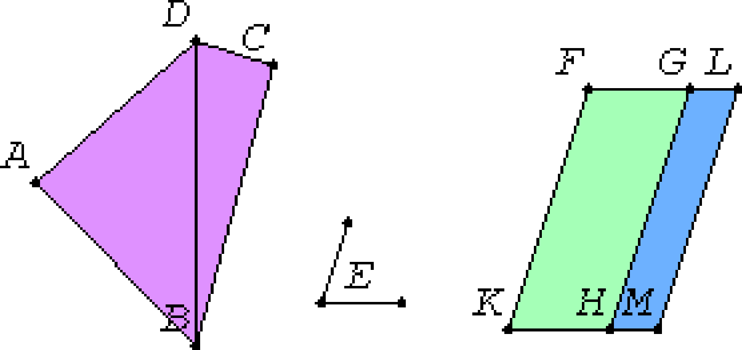

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Euclid's Elements

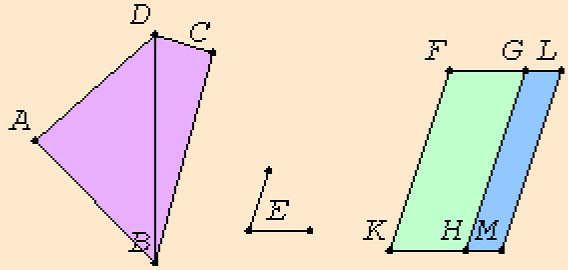
Book I

Proposition 45

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Let *ABCD* be the given rectilinear figure and *E* the given rectilinear angle.

It is required to construct a parallelogram equal to the rectilinear figure *ABCD* in the given angle *E*.



Join *DB*. Construct the parallelogram *FH* equal to the triangle *ABD* in the angle *HKF* which equals *E*. Apply the parallelogram *GM* equal to the triangle *DBC* to the straight line *GH* in the angle *GHM* which equals *E*. Post.1 I.42 I.44

Since the angle *E* equals each of the angles *HKF* and *GHM*, therefore the angle *HKF* also equals the angle *GHM*. C.N.1

Add the angle *KHG* to each. Therefore the sum of the angles *FKH* and *KHG* equals the sum of the angles *KHG* and *GHM*. C.N.2

But the sum of the angles *FKH* and *KHG* equals two right angles, therefore the sum of the angles *KHG* and *GHM* also equals two right angles. I.29 C.N.1

Thus, with a straight line *GH*, and at the point *H* on it, two straight lines *KH* and *HM* not lying on the same side make the adjacent angles together equal to two right angles, therefore *KH* is in a straight line with *HM*. I.14

Since the straight line *HG* falls upon the parallels *KM* and *FG*, therefore the alternate angles *MHG* and *HGF* equal one another. I.29

Add the angle *HGL* to each. Then the sum of the angles *MHG* and *HGL* equals the sum of the angles *HGF* and *HGL*. C.N.2

But the sum of the angles *MHG* and *HGL* equals two right angles, therefore the sum of the angles *HGF* and *HGL* also equals two right angles. Therefore *FG* is in a straight line with *GL*. I.29 C.N.1 I.14

Since *FK* is equal and parallel to *HG*, and *HG* equal and parallel to *ML* also, therefore *KF* is also equal and parallel to *ML*, and the straight lines *KM* and *FL* join them at their ends. Therefore *KM* and *FL* are also equal and parallel. Therefore *KFLM* is a parallelogram. I.34 I.30 C.N.1 I.33

Since the triangle *ABD* equals the parallelogram *FH*, and *DBC* equals *GM*, therefore the whole rectilinear figure *ABCD* equals the whole parallelogram *KFLM*. C.N.2

Therefore the parallelogram *KFLM* has been constructed equal to the given rectilinear figure *ABCD* in the angle *FKM* which equals the given angle *E*.

Q.E.F.

Guide

With this construction any rectilinear area can be applied to a line in an angle, that is, it can be transformed into a parallelogram with whatever angle you want and with one side whatever you want. That is a satisfactory solution to the question "what's the area of this figure?"

But the question "what's the area of a circle?" is not answered in the *Elements*. See the note on squaring the circle after proposition II.14 for more discussion of this question.

Use of Proposition 45

This construction is used in propositions II.14, VI.25, and XI.32. Like many of the other constructions in Book I, it is used to make constructions in different planes as is done in XI.32.

Next proposition: I.46

Previous: I.44

Book I introduction

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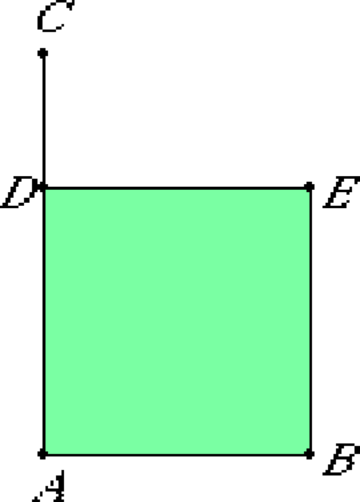
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Euclid's Elements

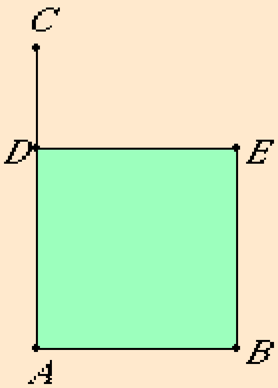
Book I

Proposition 46

To describe a square on a given straight line.

Let *AB* be the given straight line.

It is required to describe a square on the straight line *AB.*



Draw *AC* at right angles to the straight line *AB* from the point *A* on it. Make *AD* equal to *AB*. Draw *DE* through the point *D* parallel to *AB*, and draw *BE* through the point *B* parallel to *AD.* I.11 I.3 I.31

Then *ADEB* is a parallelogram. Therefore *AB* equals *DE*, and *AD* equals *BE.* I.34

But *AB* equals *AD*, therefore the four straight lines *BA*, *AD*, *DE*, and *EB* equal one another. Therefore the parallelogram *ADEB* is equilateral.

I say next that it is also right-angled.

Since the straight line *AD* falls upon the parallels *AB* and *DE*, therefore the sum of the angles *BAD* and *ADE* equals two right angles. I.29

But the angle *BAD* is right, therefore the angle *ADE* is also right.

And in parallelogrammic areas the opposite sides and angles equal one another, therefore each of the opposite angles *ABE* and *BED* is also right. Therefore *ADEB* is right-angled. I.34

And it was also proved equilateral.

Therefore it is a square, and it is described on the straight line *AB.* I.Def.22

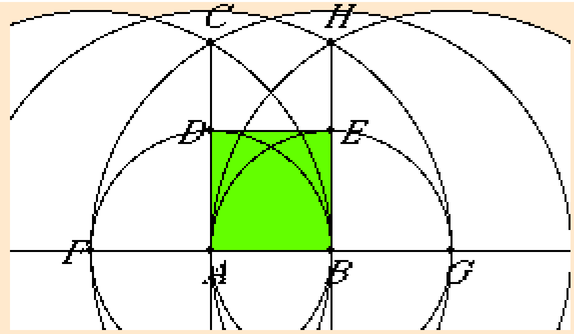
Q.E.F.

Guide

We now have the second regular polygon, the first being the equilateral triangle of proposition I.1. Regular polygons with 5, 6, and 15 sides are constructed in Book IV.

Consruction steps

There are quite a few steps needed to construct a square on *AB.* In order



to construct the perpendicular *AC*, first *AB* has to be extended in the direction of *A* and a point *F* on the far side the same distance from *A* as *B* is, then two more circles centered at *B* and *F* to get a perpendicular line, and then it needs to be cut off at length *C*, but fortunately, the needed circle has already been drawn.

Next, *EB* is to be drawn through *B* parallel to *AD*. In general that construction given in I.31 takes six circles, but in this case if *EB* is drawn perpendicular to *AB*, then it will be parallel to *AD*, too, and that construction only takes three circles with radii *BA*, *AG*, and *GA*.

This abbreviation of Euclid's construction requires six circles and four lines. There are alternate constructions that are a bit shorter. For instance, *E* may be found as the other intersection of the circles of radii *BA* and *DA*.

Use of Proposition 46

The construction of a square given in this proposition is used in the next proposition, numerous propositions in Book II, and others in Books VI, XII, and XIII.

Next proposition: I.47

Previous: I.45

Book I introduction

Select from Book I

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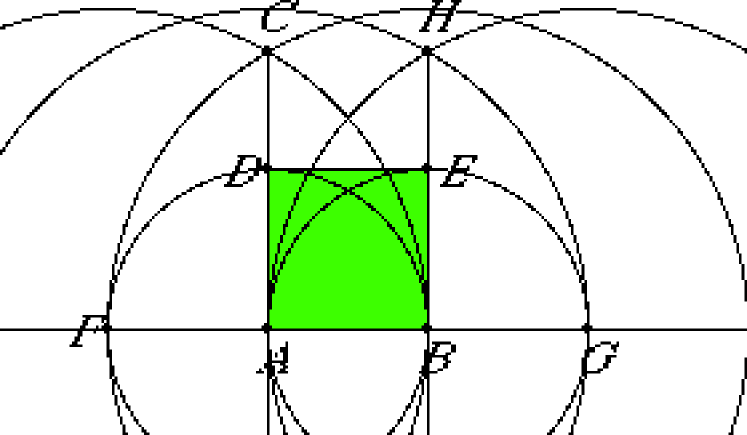
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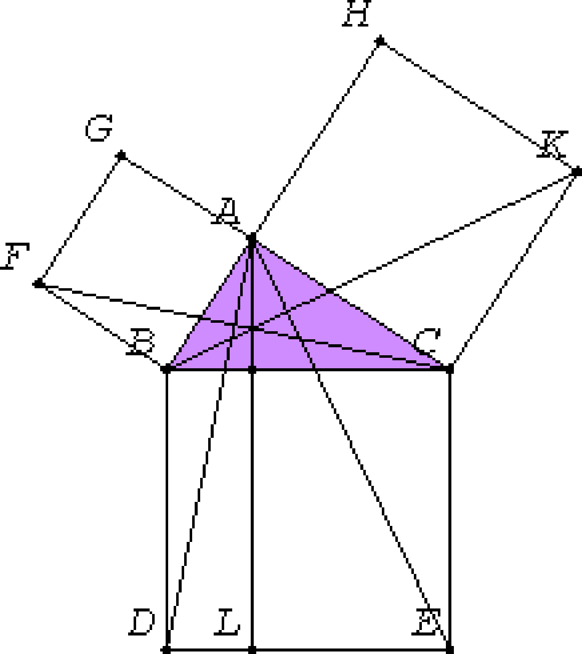
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Euclid's Elements

Book I

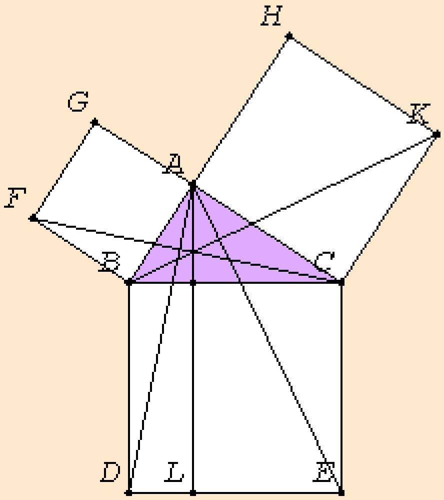
Proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right.

I say that the square on *BC* equals the sum of the squares on *BA* and *AC.*

Describe the square *BDEC* on *BC,* and the squares *GB* and *HC* on *BA* and *AC.* Draw *AL* through *A* parallel to either *BD* or *CE,* and join *AD* and *FC.* I.46 I.31 Post.1



Since each of the angles *BAC* and *BAG* is right, it follows that with a straight line *BA,* and at the point *A* on it, the two straight lines *AC* and *AG* not lying on the same side make the adjacent angles equal to two right angles, therefore *CA* is in a straight line with *AG.* I.Def.22 I.14

For the same reason *BA* is also in a straight line with *AH.*

Since the angle *DBC* equals the angle *FBA,* for each is right, add the angle *ABC* to each, therefore the whole angle *DBA* equals the whole angle *FBC.* I.Def.22 Post.4 C.N.2

Since *DB* equals *BC,* and *FB* equals *BA,* the two sides *AB* and *BD* equal the two sides *FB* and *BC* respectively, and the angle *ABD* equals the angle *FBC,* therefore the base *AD* equals the base *FC,* and the triangle *ABD* equals the triangle *FBC.* I.Def.22 I.4

Now the parallelogram *BL* is double the triangle *ABD,* for they have the same base *BD* and are in the same parallels *BD* and *AL.* And the square *GB* is double the triangle *FBC,* for they again have the same base *FB* and are in the same parallels *FB* and *GC.* I.41

Therefore the parallelogram *BL* also equals the square *GB.*

Similarly, if *AE* and *BK* are joined, the parallelogram *CL* can also be proved equal to the square *HC.* Therefore the whole square *BDEC* equals the sum of the two squares *GB* and *HC.* C.N.2

And the square *BDEC* is described on *BC,* and the squares *GB* and *HC* on *BA* and *AC.*

Therefore the square on *BC* equals the sum of the squares on *BA* and *AC.*

Therefore *in right-angled triangles the square on the side opposite the right angle equals the sum of the squares on* *the sides containing the right angle..*

Q.E.D.

Guide

This proposition is generalized in VI.31 to arbitrary similar figures placed on the sides of the triangle *ABC.* If the rectilinear figures on the sides of the triangle are similar, then that on the hypotenuse is the sum of the other two figures.

A bit of history

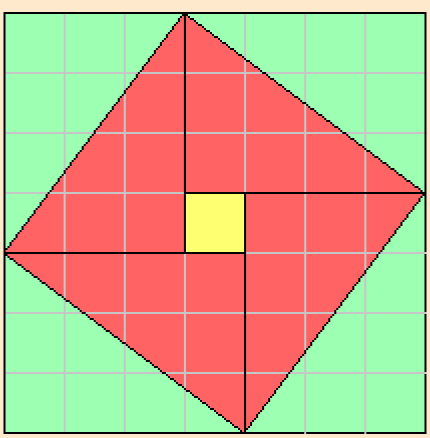
This proposition, I.47, is often called the "Pythagorean theorem," called so by Proclus and others centuries after Pythagoras and even centuries after Euclid. The statement of the proposition was very likely known to the Pythagoreans if not to Pythagoras himself. The Pythagoreans and perhaps Pythagoras even knew a proof of it. But the knowledge of this relation was far older than Pythagoras.

More than a millennium before Pythagoras, the Old Babylonians (ca. 1900-1600 B.C.E) used this relation to solve geometric problems involving right triangles. Moreover, the tablet known as Plimpton 322 shows that the Old Babylonians could construct all the so-called Pythagorean triples, those triples of numbers *a, b,* and *c* such that $a^2 + b^2 = c^2$ which describe triangles with integral sides. (The smallest of these is 3, 4, 5.)

The hypotenuse diagram in the *Zhou bi suan jing*

The rule for computing the hypotenuse of a right triangle was well known in ancient China. It is used in the *Zhou bi suan jing,* a work on astronomy and mathematics compiled during the Han period, and in the later important mathematical work *Jiu zhang suan shu* [*Nine Chapters*] to solve right triangles.

The *Zhou bi* includes a very interesting diagram known as the "hypotenuse diagram." This diagram may not have been in the original text but added by its primary commentator Zhao Shuang sometime in the third century C.E. A particular case of this proposition is illustrated by this diagram, namely, the 3-4-5 right triangle.



Place four 3 by 4 rectangles around a 1 by 1 square. A 7 by 7 square results. The four diagonals of the rectangles bound a tilted square as illustrated. The area of tilted square is 49 minus 4 times 6 (the 6 is the area of one right triangle with legs 3 and 4), which is 25. Therefore the tilted square is 5 by 5, and the diagonal of the original 3 by 4 rectangles is 5.

Zhao Shuang referred to the hypotenuse figure in a general way. He described all the ways the sides, the hypotenuse, and their squares can be found in terms of each other.

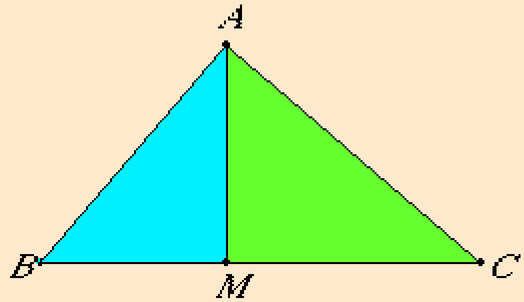
The *Zhou bi* has recently been translated into English with an excellent commentary. See *Astronomy and mathematics in ancient China: the* Zhou bi suan jing, by Christopher Cullin, Cambridge University Press, 1996.

Alternate methods of proof

According to Proclus, the specific proof of this proposition given in the *Elements* is Euclid's own. It is likely that older proofs depended on the theories of proportion and similarity, and as such this proposition would have to wait until after Books V and VI where those theories are developed. It appears that Euclid devised this proof so that the proposition could be placed in Book I.

Euclid presents a proof based on proportion and similarity in the lemma for proposition X.33. Compare it, summarized here, to the proof in I.47.

Let *ABC* be a right-angled triangle with a right angle at *A*. Draw *AM* perpendicular to *BC*.



According to VI.8, the triangles *ABM* and *AMC* are similar both to the whole *ABC* and to one another. (VI.8 concludes the triangles are similar after showing they have the same angles, see VI.4.)

Since triangle *ABC* is similar to triangle *ABD*, therefore, by the definition of similarity VI.Def.1, *CB* is to *BA* as *BA* is to *BM*.

Next VI.17 converts this proportion to a statement about areas, namely, the rectangle *CB* by *BM* (which is the parallelogram *BL* in the proof of I.47) equals the square on *AB*. For the same reason the rectangle *BC* by *CM* (which is the parallelogram *CL* in the proof of I.47) also equals the square on *AC*. Therefore the sum of the two rectangles *CB* by *BM* and *BC* by *CM*, which is the square on *BC*, equals the sum of the squares on *AC* and *BC*. Q.E.D.

(Actually, the final sentence is not part of the lemma, probably because Euclid moved that statement to the first Book as I.47.)

So, although Euclid's proof in I.47 may be more complicated than some others, we can see how it well it corresponds to a simpler proof that depends on the theories of proportion and similarity.

Generalizations of I.47

Propositions II.12 and II.13 consider triangles other than right triangles. In II.12 the right angle is replaced by an obtuse angle, while in II.13 the right angle is replaced by an acute angle. The resulting statements are actually geometric forms of the law of cosines.

Proposition VI.31 generalizes the figures that can be placed on the sides of the right triangle to any three similar figures instead of the three squares here in I.47.

Use of this proposition

This proposition is used in the next one, which its converse, in propositions II.9 through II.14 in Book II, and several propositions in the rest of the books on geometry.

Next proposition: I.48

Previous: I.46

Book I introduction

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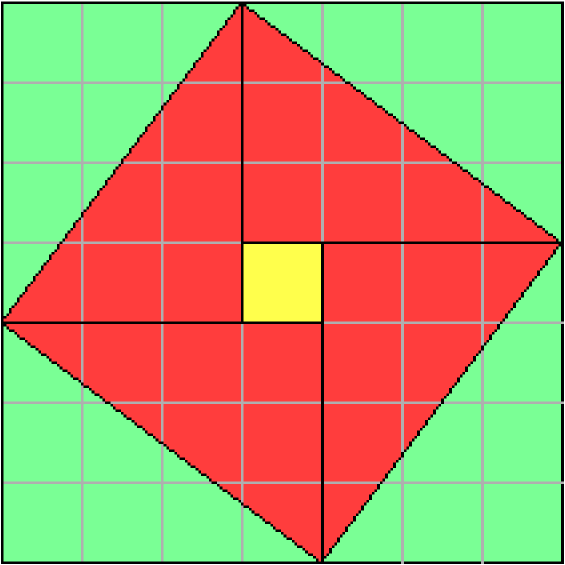

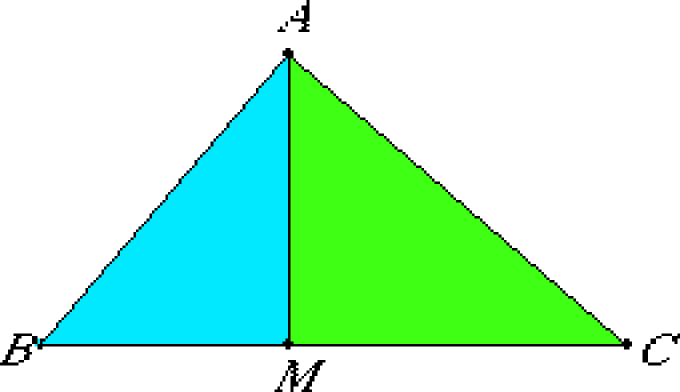
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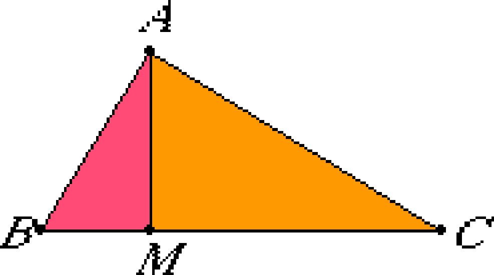
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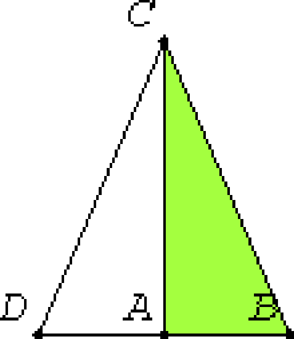
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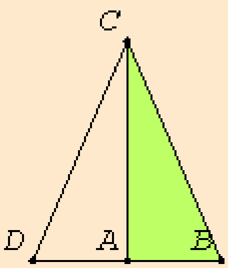
Euclid's Elements
Book I
Proposition 48

If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

In the triangle *ABC* let the square on one side *BC* equal the sum of the squares on the sides *BA* and *AC*

I say that the angle *BAC* is right.



Draw *AD* from the point *A* at right angles to the straight line *AC.* Make *AD* equal to *BA,* and join *DC.* I.11 I.3 Post.1

Since *DA* equals *AB,* therefore the square on *DA* also equals the square on *AB.*

Add the square on *AC* to each. Then the sum of the squares on *DA* and *AC* equals the sum of the squares on *BA* and *AC.* C.N.2

But the square on *DC* equals the sum of the squares on *DA* and *AC,* for the angle *DAC* is right, and the square on *BC* equals the sum of the squares on *BA* and *AC,* for this is the hypothesis, therefore the square on *DC* equals the square on *BC,* so that the side *DC* also equals *BC.* I.47 C.N.1

Since *DA* equals *AB,* and *AC* is common, the two sides *DA* and *AC* equal the two sides *BA* and *AC,* and the base *DC* equals the base *BC,* therefore the angle *DAC* equals the angle *BAC.* But the angle *DAC* is right, therefore the angle *BAC* is also right. I.8

Therefore *if in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.*

Q.E.D.

Guide

This proposition is the converse of the previous. It is used in proposition XI.35.

Next book: Book II introduction

Previous proposition: I.47

Book I introduction

Select from Book I

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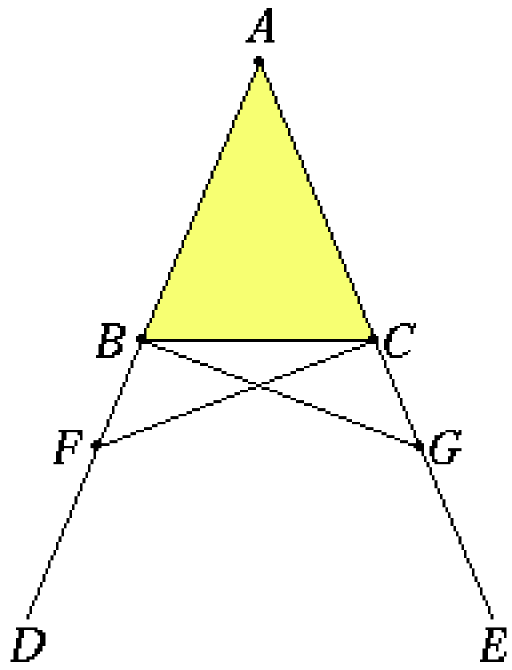
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Euclid's Elements

Book I

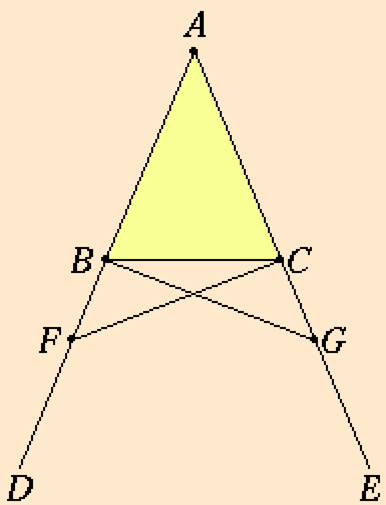
Proposition 5

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Let *ABC* be an isosceles triangle having the side *AB* equal to the side *AC*, and let the straight lines *BD* and *CE* be produced further in a straight line with *AB* and *AC*. I.Def.20 Post.2

I say that the angle *ABC* equals the angle *ACB*, and the angle *CBD* equals the angle *BCE*.

Take an arbitrary point *F* on *BD*. Cut off *AG* from *AE* the greater equal to *AF* the less, and join the straight lines *FC* and *GB*. I.3. Post.1



Since *AF* equals AG, and *AB* equals *AC*, therefore the two sides *FA* and *AC* equal the two sides *GA* and *AB*, respectively, and they contain a common angle, the angle *FAG*.

Therefore the base *FC* equals the base *GB*, the triangle *AFC* equals the triangle *AGB*, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle *ACF* equals the angle *ABG*, and the angle *AFC* equals the angle *AGB*. I.4

Since the whole *AF* equals the whole *AG*, and in these *AB* equals *AC*, therefore the remainder *BF* equals the remainder *CG*. C.N.3

But *FC* was also proved equal to *GB*, therefore the two sides *BF* and *FC* equal the two sides *CG* and *GB* respectively, and the angle *BFC* equals the angle *CGB*, while the base *BC* is common to them. Therefore the triangle *BFC* also equals the triangle *CGB*, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Therefore the angle *FBC* equals the angle *GCB*, and the angle *BCF* equals the angle *CBG*. I.4

Accordingly, since the whole angle *ABG* was proved equal to the angle *ACF*, and in these the angle *CBG* equals the angle *BCF*, the remaining angle *ABC* equals the remaining angle *ACB*, and they are at the base of the triangle *ABC*. But the angle *FBC* was also proved equal to the angle *GCB*, and they are under the base. C.N.3

Therefore *in isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another*.

Q.E.D.

Guide

There are two conclusions for this proposition, first that the internal base angles *ABC* and *ACB* are equal, second that the external base angles *FBC* and *GCB* are equal. From the diagram it looks like it would be easy to prove the second conclusion from the first by simply subtracting the equal angles *ABC* and *ACB* the straight angles *ABF* and *ACG*, respectively. But Euclid doesn't accept straight angles, and even if he did, he hasn't proved that all straight angles are equal. Proposition I.13 would be enough, since it implies the sum of angles *ABC* and *FBC* equals two right angles, and the sum of angles *ACB* and *GCB* also equals two right angles, and so the two sums are equal effectively saying all straight angles are equal.

Unfortunately, such an argument would be circular. I.13 depends on I.11, I.11 on I.8, I.8 on I.7, and I.7 on I.5. Thus, I.13 cannot be used in the proof of I.5. It may appear that I.7 only depends on the first conclusion of I.5, but a case of I.7 that Euclid does not discuss relies on the second conclusion of I.5.

This proposition has been called the Pons Asinorum, or Asses' Bridge. Whether this name is due to its difficulty (which it isn't) or the resemblance of its figure to a bridge is not clear. Very few of the propositions in the *Elements* are known by names.

Pappus' proof

Pappus (fl. ca. 320 C.E.) gave a much shorter proof of the first conclusion, but it is also conceptually more difficult. The two triangles *BAC* and *CAB* have two sides equal to two sides, namely side *BA* of the first triangle equals side *CA* of the second triangle, and side *AC* of the first triangle equal to side *AB* of the second, and the contained angles are equal, namely angle *BAC* of the first triangle equals angle *CAB* of the second, therefore, by I.4, the corresponding parts of the two triangles are equal, in particular, the angle *B* in the first triangle equals the angle *C* of the second.

The difficulty lies in treating one triangle as two, or in making a correspondence between a triangle and itself, but not the correspondence of identity. There is nothing wrong with this proof formally, but it might be more difficult for a student just learning geometry.

Use of Proposition 5

This proposition is used in Book I for the proofs of several propositions starting with I.7 It is also used frequently in Books II, III, IV, VI, and XIII.

Next proposition: I.6

Previous: I.4

Book I introduction

Select from Book I

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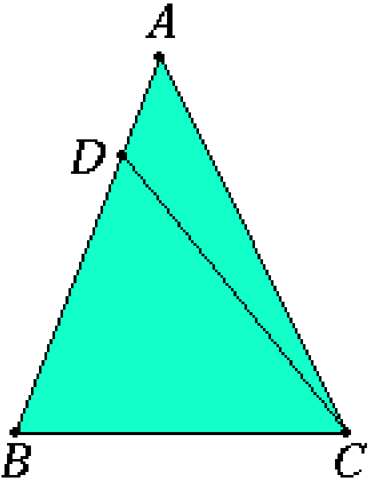
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Euclid's Elements

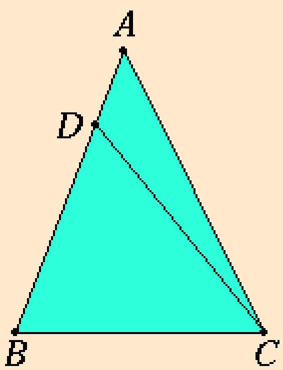
Book I

Proposition 6

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB.*

I say that the side *AB* also equals the side *AC.*



If *AB* does not equal *AC,* then one of them is greater. C.N

Let *AB* be greater. Cut off *DB* from *AB* the greater equal to *AC* the less, and join *DC.* I.3 Post.1

Since *DB* equals *AC,* and *BC* is common, therefore the two sides *DB* and *BC* equal the two sides *AC* and *CB* respectively, and the angle *DBC* equals the angle *ACB.* I.4 Therefore the base *DC* equals the base *AB,* and the triangle *DBC* equals the triangle *ACB,* the less equals the greater, which is absurd. C.N.5 Therefore *AB* is not unequal to *AC,* it therefore equals it.

Therefore *if in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

Q.E.D.

Guide

Converses of propositions

This is the converse of (part of) the previous proposition I.5. Proposition I.6 says that if angle *B* equals angle *C,* then side *AB* equals side *AC.* Proposition I.5 says that if side *AB* equals side *AC,* then angle *B* equals angle *C.* In general, the converse of a proposition of the form "If *P,* then *Q*" is the proposition "If *Q,* then *P.*" When both a proposition and its converse are valid, Euclid tends to prove the converse soon after the proposition, a practice that has continued to this day.

A proposition and its converse are *not* logically equivalent. There are examples where "If *P,* then *Q*" is valid, but "If *Q,* then *P*" is not valid. An example from the *Elements* is proposition III.5 which states "If two circles cut one another, then they do not have the same center." The converse would be "If two circles do not have the same center, then they cut one another" which is certainly not valid since if one circle lies entirely outside the other, then they don't have the same center.

Proofs by contradiction

This is the first "proof by contradiction," also called "reductio ad absurdum," in the *Elements.* In this proof, in order to prove *AB* equals *AC,* Euclid assumes they are unequal and derives a contradiction, namely, that the triangle *ACB* equals a part of itself, triangle *DBC,* which contradicts C.N.5, the whole is greater than the part. The contradiction is that triangle *ACB* both equals and does not equal triangle *DBC*.

In general, to prove a statement of the form "P" with a proof by contradiction, begin with an assumption "not P" and derive some contradiction "Q and not Q," and finally conclude "P."

Euclid often uses proofs by contradiction, but he does not use them to conclude the existence of geometric objects. That is, he does not use them in constructions. But he does use them to show what has been constructed is correct.

In modern mathematics nonconstructive proofs by contradiction do occur. Famous examples are Brouwer's fixed point theorems published in 1912. One of these states that any continuous transformation f of a circle (circular disk) to itself has a fixed point x, that is, a point such that $f(x) = x$. In his proof, he assumed that such a point did not exist and derived a contradiction. Although his proof is logically correct, he was not satisfied since the proof does not help in constructing a fixed point. Brouwer was an adherent of a philosophy of mathematics called "intuitionism" that holds, among other things, that mathematical objects have not been shown to exist until constructions have been given for them.

The law of trichotomy in practice

The proof uses the law of trichotomy for lines. "If *AB* does not equal *AC*, then one of them is greater." There are three cases: $AB < AC$, $AB = AC$, or $AB > AC$. If the middle possibility is excluded, then only the two others remain, so one of the lines is greater. The law of trichotomy is not explicitly stated as a Common Notion, but it is the sort of property of magnitudes listed as Common Notions.

Proposition I.3 can be read as a construction to determine whether one line is less than, equal to, or greater than another. Using I.3, one line is laid along another, and it will fall short, fall equal, or extend beyond the other. For this proposition I.6, the construction simplifies since the two lines *AB* and *AC* already have one end in common.

The other part of the law of trichotomy is also used in the proof, the part that says only one of the three cases can occur. "... the triangle *DBC* equals the triangle *ACB*, the less equals the greater, which is absurd." C.N.5, the whole is greater than the part, allows the conclusion that triangle *DBC* (the part) is less than triangle *ACB* (the whole). But the contradiction arises because only one of the two cases $DBC = ACB$ and $DBC < ACB$ can occur.

Use of Proposition 6

This proposition is not used in the proofs of any of the later propositions in Book I, but it is used in Books II, III, IV, VI, and XIII.

Next proposition: I.7

Previous: I.5

Book I introduction

Select from Book I

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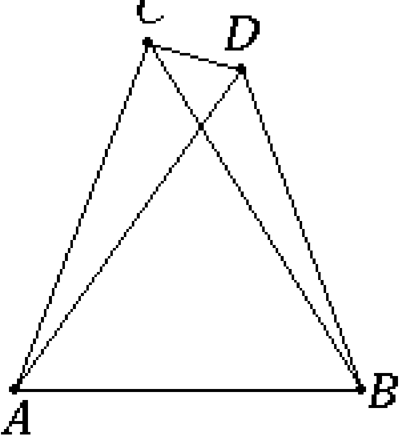
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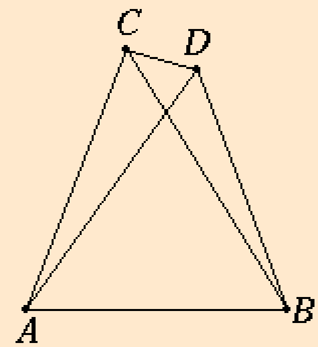
Euclid's Elements

Book I

Proposition 7

Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.



If possible, given two straight lines *AC* and *CB* constructed on the straight line *AB* and meeting at the point *C,* let two other straight lines *AD* and *DB* be constructed on the same straight line *AB,* on the same side of it, meeting in another point *D* and equal to the former two respectively, namely each equal to that from the same end, so that *AC* equals *AD* which has the same end *A,* and *CB* equals *DB* which has the same end *B.*

Join *CD.* Post.1

Since *AC* equals *AD,* therefore the angle *ACD* equals the angle *ADC.* Therefore the angle *ADC* is greater than the angle *DCB.* Therefore the angle *CDB* is much greater than the angle *DCB.* I.5 C.N.5 C.N.

Again, since *CB* equals *DB,* therefore the angle *CDB* also equals the angle *DCB.* But it was also proved much greater than it, which is impossible. I.5 C.N.

Therefore *given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.*

Q.E.D.

Guide

In order to conclude "the angle *ADC* is greater than the angle *DCB*" it is necessary for angle *ADC* to be greater than angle *DCB,* but that won't happen unless the point *D* lies outside the triangle *ABC.* Euclid hasn't considered the case when *D* lies inside triangle *ABC* as well as other special cases. This is not unusual as Euclid frequently treats only one case. Commentators over the centuries have inserted other cases in this and other propositions. It is usually easy to modify Euclid's proof for the remaining cases. In this proposition for the case when *D* lies inside triangle *ABC,* the second conclusion of I.5 may be used to justify the proof.

Hidden justifications

The sentences

> ... the angle *ACD* equals the angle *ADC*. Therefore the angle *ADC* is greater than the angle *DCB*. Therefore the angle *CDB* is much greater than the angle *DCB*.

use several properties of magnitudes. C.N.5 justifies the unstated angle *ACD* > *DCB* since *DCB* is part of *ACD*. The statement that *ADC* is greater than the angle *DCB* is justified by the property of magnitudes

If $x < y$ and $y = z$, then $x < z$.

This property is not among the listed Common Notions.

Next, transitivity of "less than"

If $x < y$ and $y < z$, then $x < z$.

justifies the last statement "*CDB* is much greater than the angle *DCB*." Transitivity is another property not listed as a Common Notion.

As in the proof of the last proposition and many to come, the law of trichotomy is also used. Here it's used to reach the final contradiction.

Use of Proposition 7

This proposition is used in the proof of the next proposition.

Next proposition: I.8

Previous: I.6

Book I introduction

Select from Book I

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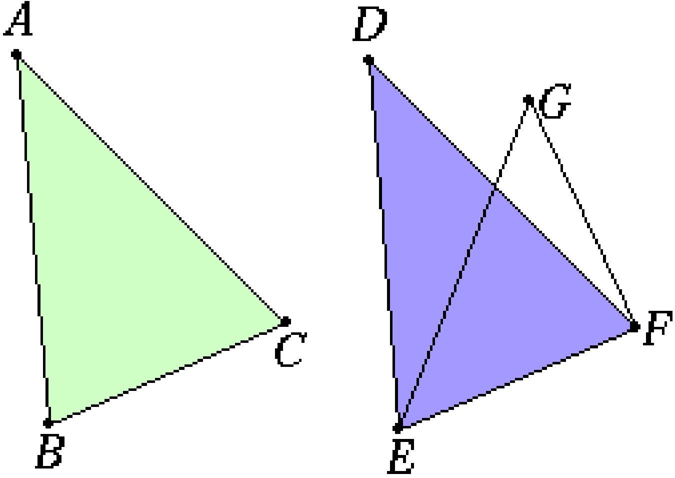
D.E.Joyce
Clark University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Euclid's Elements

Book I

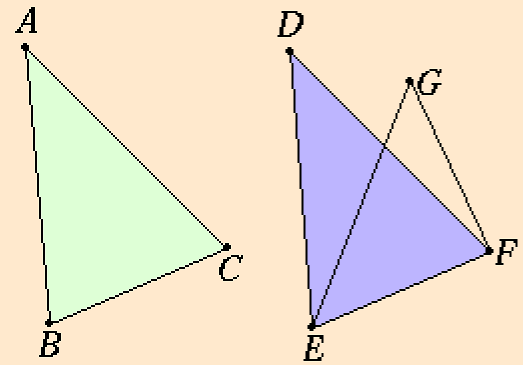
Proposition 8

If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Let *ABC* and *DEF* be two triangles having the two sides *AB* and *AC* equal to the two sides *DE* and *DF* respectively, namely *AB* equal to *DE* and *AC* equal to *DF*, and let them have the base *BC* equal to the base *EF*.

I say that the angle *BAC* also equals the angle *EDF*.

If the triangle *ABC* is applied to the triangle *DEF*, and if the point *B* is placed on the point *E* and the straight line *BC* on *EF*, then the point *C* also coincides with *F*, because *BC* equals *EF*.



Then, *BC* coinciding with *EF*, therefore *BA* and *AC* also coincide with *ED* and *DF*, for, if the base *BC* coincides with the base *EF*, and the sides *BA* and *AC* do not coincide with *ED* and *DF* but fall beside them as *EG* and *GF*, then given two straight lines constructed on a straight line and meeting in a point, there will have been constructed on the same straight line and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same end with it.

But they cannot be so constructed. I.7

Therefore it is not possible that, if the base *BC* is applied to the base *EF*, the sides *BA* and *AC* do not coincide with *ED* and *DF*. Therefore they coincide, so that the angle *BAC* coincides with the angle *EDF*, and equals it. C.N.4

Therefore *if two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.*

Q.E.D.

Guide

This, the "side-side-side" congruence theorem, is the second of Euclid's three congruence theorems for triangles. See the note on congruence theorems after proposition I.26.

As in the proof of I.4, this proof employs the hazy method of superposition.

Use of Proposition 8

This proposition is used for the a few of the propositions in Book I starting with the next one. It is also used several times in the Books III, IV, XI, and XIII.

As in I.4 the two triangles need not lie in one plane. Propositions such as XI.4 in Book XI apply this theorem to the case when the two triangles are not coplanar.

Next proposition: I.9

Previous: I.7

Book I introduction

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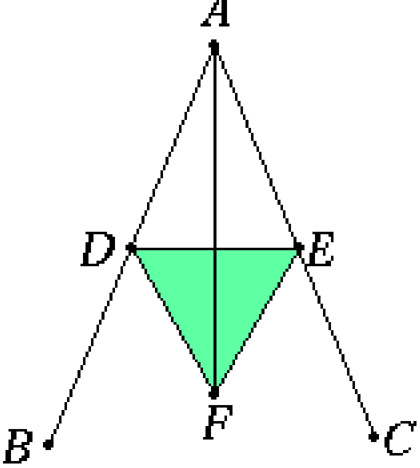
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Euclid's Elements

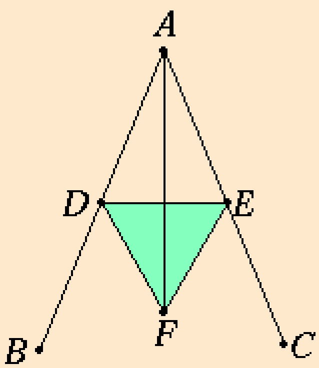
Book I

Proposition 9

To bisect a given rectilinear angle.

Let the angle *BAC* be the given rectilinear angle.

It is required to bisect it.



Take an arbitrary point *D* on *AB*. Cut off *AE* from *AC* equal to *AD*, and join *DE*. Construct the equilateral triangle *DEF* on *DE*, and join *AF*. I.3 Post.1 I.1

I say that the angle *BAC* is bisected by the straight line *AF*.

Since *AD* equals *AE*, and *AF* is common, therefore the two sides *AD* and *AF* equal the two sides *EA* and *AF* respectively.

And the base *DF* equals the base *EF*, therefore the angle *DAF* equals the angle *EAF*. I.Def.20 I.8

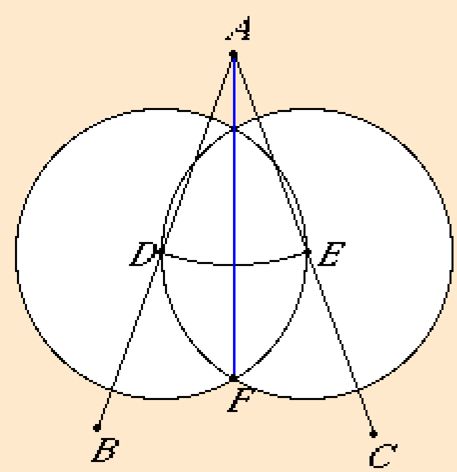
Therefore the given rectilinear angle *BAC* is bisected by the straight line *AF*.

Q.E.F.

Guide

Construction steps

When using a compass and a straightedge to perform this construction, three circles and the final bisecting line need to be drawn. One circle with center *A* and radius *AD* is needed to determine the point *E*. The other two circles with centers at *D* and *E* and common radius *DE* intersect to give the point *F*. The sides of the equilateral triangle aren't needed for the construction.



There is an alternate construction where the circles centered at *D* and *E* have a different radius, namely, *AD*, which equals *AE*. A different proof is required to show that this alternate construction works.

On angle trisection

Angle bisection is an easy construction to make using Euclidean tools of straightedge and compass. Also, line bisection is quite easy (see the next proposition I.10), and division of a line into any number of equal parts is not especially difficult (see proposition VI.9).

Dividing an angle into an odd number of equal parts is not so easy, in fact, it is impossible to trisect a 60°-angle using Euclidean tools (the Postulates 1 through 3). Euclid's predecessors employed a variety higher curves for this purpose. Archimedes, after Euclid, created two constructions: his spiral could divide an angle into any number of parts, and his neusis construction could trisect angles (see the note on Post.2). By Pappus' time it was believed that angle trisection was not possible using Euclidean tools, but that wasn't proven until 1837 when Wantzel published his proof.

Nevertheless, amateur geometers continue to search in vain for such a construction and frequently bother mathematicians with their purported solutions. Their solutions are of two forms. Sometimes they simply construct approximate trisections. Other times they use neusis or some other other tool that goes beyond Euclid's tools.

Students of geometry are cautioned not to waste their time on this problem and, if they do, not to bother others with their purported solutions. Much better would be to study Galois theory, the mathematics that proves the impossibility of angle trisection.

Use of Proposition 9

The construction of this proposition is used in the next one and a few propositions in Books IV, VI, and XIII.

Next proposition: I.10

Previous: I.8

Book I introduction

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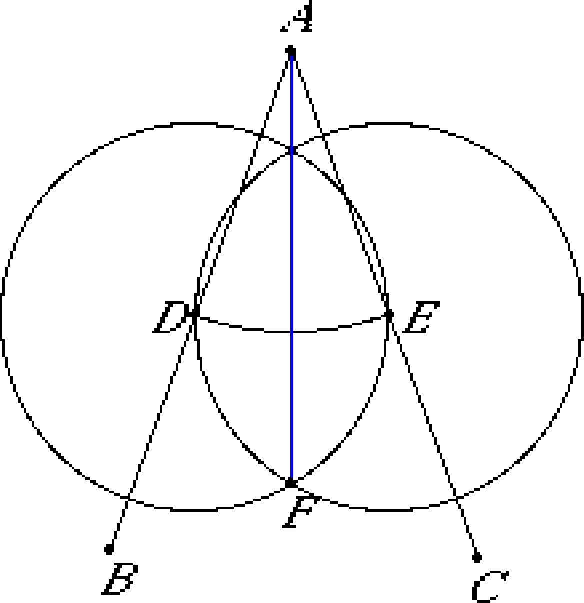
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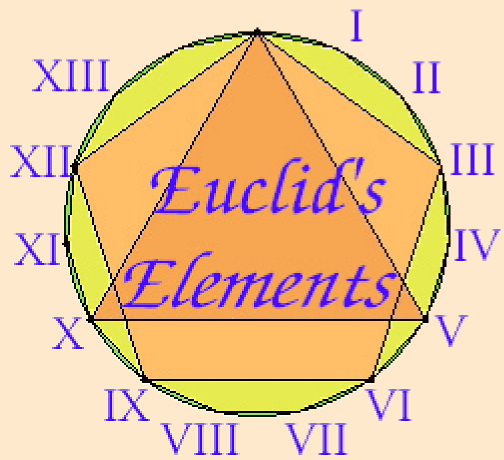
Book II



Book II

Table of contents

Definitions

Definition 1.
Any rectangular parallelogram is said to be *contained* by the two straight lines containing the right angle.

Definition 2
And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a *gnomon.*

Propositions

Proposition 1.
If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

Proposition 2.
If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.

Proposition 3.
If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment.

Proposition 4.

If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

Proposition 5.

If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.

Proposition 6.

If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.

Proposition 7.

If a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment.

Proposition 8.

If a straight line is cut at random, then four times the rectangle contained by the whole and one of the segments plus the square on the remaining segment equals the square described on the whole and the aforesaid segment as on one straight line.

Proposition 9.

If a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section.

Proposition 10.

If a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line.

Proposition 11.

To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

Proposition 12.

In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Proposition 13.

In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acutc angle.

Proposition 14.

To construct a square equal to a given rectilinear figure.

Guide to Book II

The subject matter of Book II is usually called "geometric algebra." The first ten propositions of Book II can be easily interpreted in modern algebraic notation. Of course, in doing so the geometric flavor of the propositions is lost. Nonetheless, restating them algebraically can aid in understanding them. The equations are all quadratic equations since the geometry is plane geometry.

II.1. If $y = y_1 + y_2 + ... + y_n$, then $xy = x\,y_1 + x\,y_2 + ... + x\,y_n$. This can be stated in a single identity as

$$x\,(y_1 + y_2 + ... + y_n) = x\,y_1 + x\,y_2 + ... + x\,y_n.$$

II.2. If $x = y + z$, then $x^2 = xy + xz$. This can be stated in various ways in an identity of two variables. For instance,

$$(y + z)^2 = (y + z)\,y + (y + z)\,z,$$

or

$$x^2 = xy + x\,(x - y).$$

II.3. If $x = y + z$, then $xy = yz + y^2$. Equivalent identities are

$$(y + z)y = yz + y^2,$$

and

$$xy = y(x - y) + y^2.$$

II.4. If $x = y + z$, then $x^2 = y^2 + z^2 + 2yz$. As an identity,

$$(y + z)^2 = y^2 + z^2 + 2yz.$$

II.5 and II.6. $(y + z)\,(y - z) + z^2 = y^2$.

II.7. if $x = y + z$, then $x^2 + z^2 = 2xz + y^2$. As an identity,

$$x^2 + z^2 = 2xz + (x - z)^2.$$

II.8. If $x = y + z$, then $4xy + z^2 = (x + y)^2$. As an identity,

$$4xy + (x - y)^2 = (x + y)^2.$$

II.9 and II.10. $(y + z)^2 + (y - z)^2 = 2\,(y^2 + z^2)$.

The remaining four propositions are of a slightly different nature. Proposition II.11 cuts a line into two parts which solves the equation $a\,(a - x) = x^2$ geometrically. Propositions II.12 and II.13 are recognizable as geometric forms of the law of cosines which is a generalization of I.47. The last propostion II.14 constructs a square equal to a given rectilinear figure thereby completeing the theory of areas begun in Book I.

Logical structure of Book II

The proofs of the propositions in Book II heavily rely on the propositions in Book I involving right angles and parallel lines, but few others. For instance, the important congruence theorems for triangles, namely I.4, I.8, and I.26, are not invoked even once. This is understandable considering Book II is mostly algebra interpreted in the theory of geometry.

The first ten propositions in Book II were written to be logically independent, but they could have easily been written in logical chains which, perhaps, would have shortened the exposition a little. The remaining four propositions each depend on one of the first ten.

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Previous book: Book I

Elements Introduction

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Select book

Select topic

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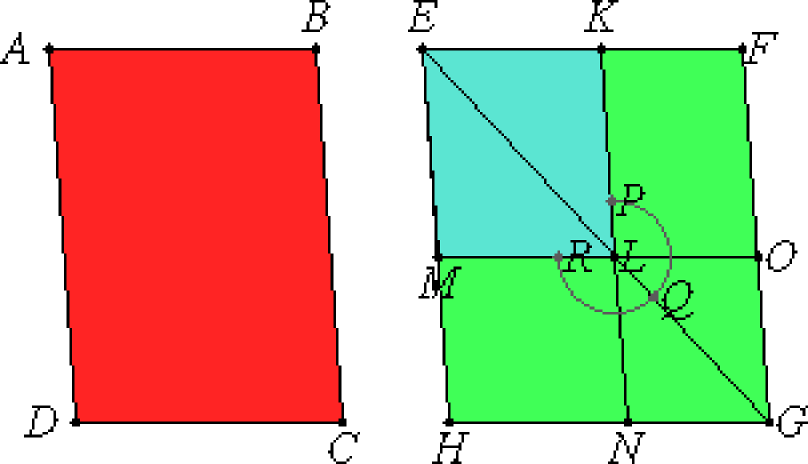
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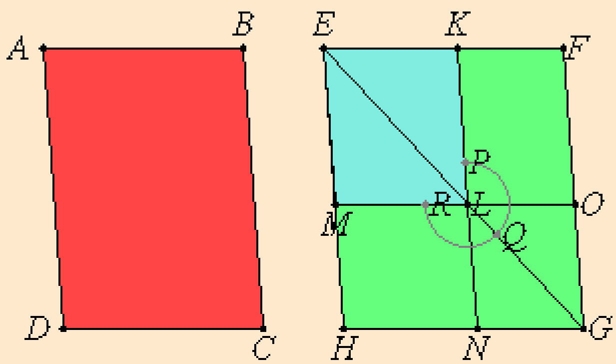
Euclid's Elements
Book II
Definitions

Def. 1. *Any rectangular parallelogram is said to be* contained *by the two straight lines containing the right angle.*

Def. 2. *And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a* gnomon.

Guide



According to the first definition, the rectangle *ABCD* illustrated on the left is contained by the lines *AB* and *BC*, and this rectangle can be called the rectangle *AB* by *BC*. Of course, it could also be called the rectangle *BC* by *CD*, or two other names.

On the right, in the parallelogram *EFGH*, there is a diameter *EG* with a parallelogram *LNGO* about it and the two complements *KLOF* and *MHNL*, and these three parallelograms together make up the gnomon. In other words a gnomon is an L-shaped figure made by removing a parallelogram from a larger similar parallelogram. (The "g" in "gnomon" is silent.)

Euclid illustrated gnomons by arcs of circles around the inner vertex. In this example, the gnomon is called the gnomon *PQR*.

First proposition: II.1

Book II introduction

Select from Book II

Select book

Select topic

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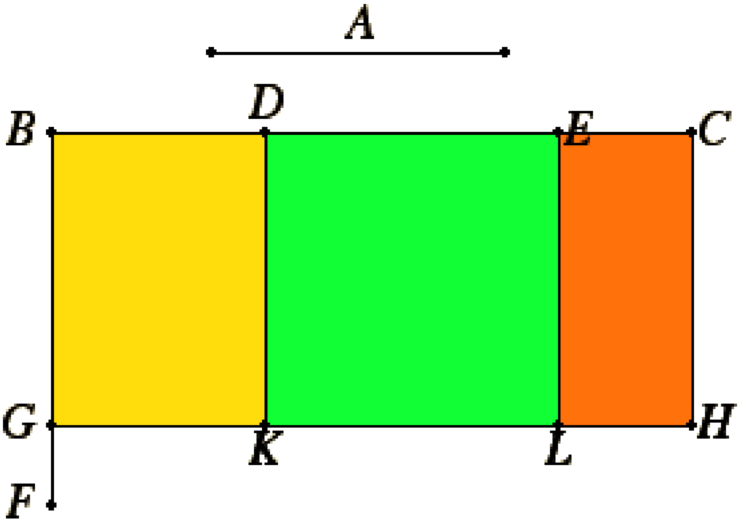
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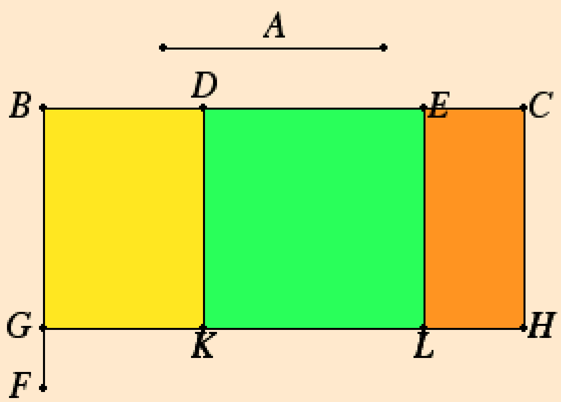
Euclid's Elements
Book II
Proposition 1

If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

Let *A* and *BC* be two straight lines, and let *BC* be cut at random at the points *D* and *E*.

I say that the rectangle *A* by *BC* equals the sum of the rectangle *A* by *BD*, the rectangle *A* by *DE*, and the rectangle *A* by *EC*.



Draw *BF* from *B* at right angles to *BC*. Make *BG* equal to *A*. Draw *GH* through *G* parallel to *BC*. Through *D*, *E*, and *C* draw *DK*, *EL*, and *CH* parallel to *BG*. I.11 I.3 I.31

Then *BH* equals the sum of *BK*, *DL*, and *EH*.

Now *BH* is the rectangle *A* by *BC*, for it is contained by *GB* and *BC*, and *BG* equals *A*; *BK* is the rectangle *A* by *BD*, for it is contained by *GB* and *BD*, and *BG* equals *A*; and *DL* is the rectangle *A* by *DE*, for *DK*, that is *BG*, equals *A*. Similarly also *EH* is the rectangle *A* by *EC*. II.Def.1 I.34

Therefore the rectangle *A* by *BC* equals the sum of the rectangle *A* by *BD*, the rectangle *A* by *DE*, and the rectangle *A* by *EC*.

Therefore *if there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.*

Q.E.D.

Guide

The phrase "the rectangle contained by the two straight lines" means any rectangle constructed with two sides equal to the two given sides. In some sense this is the product of the two lines. When the sides have names, such as *A* and *BC*, we will refer to that rectangle by "the rectangle *A* by *BC*" since that is a little clearer than Euclid's terse "the rectangle *A*, *BC*."

In this proposition Euclid proves that if

$$BC = BD + DE + EC,$$

then

$$(A \text{ by } BC) = (A \text{ by } BD) + (A \text{ by } DE) + (A \text{ by } EC).$$

In modern algebraic notation this could be stated as follows: If $y = y_1 + y_2 + ... + y_n$, then $xy = x\,y_1 + x\,y_2 + ... + x\,y_n$. This can also be stated in a single equation as

$$x\,(y_1 + y_2 + ... + y_n) = x\,y_1 + x\,y_2 + ... + x\,y_n.$$

Here x and the various y_i's are all lines, and n is an arbitrary number. In modern terminology this identity is called the *distributive law* for multiplication over addition.

Use of this proposition

This proposition is not specifically invoked in the rest of the *Elements.* The next two propositions, however, are special cases of it, and they are each explicitly used once.

Next proposition: II.2

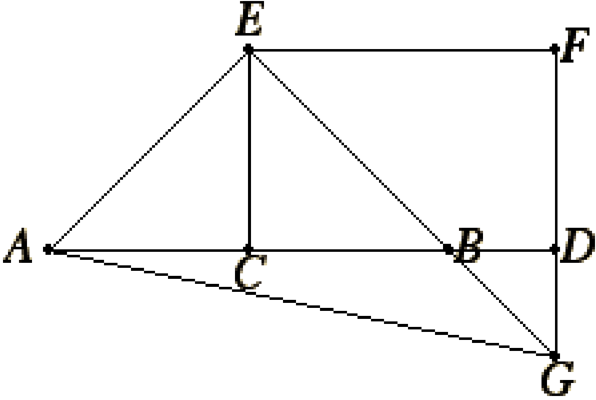
Previous: II.Def.1-2

Book II introduction

Select from Book II

Select book

Select topic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Euclid's Elements

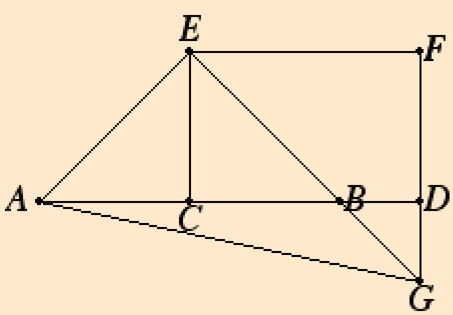
Book II

Proposition 10

If a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line.

Let a straight line *AB* be bisected at *C*, and let a straight line *BD* be added to it in a straight line.

I say that the sum of the squares on *AD* and *DB* is double the sum of the squares on *AC* and *CD*.



Draw *CE* from the point *C* at right angles to *AB* and equal to either *AC* or *CB*. Join *EA* and *EB*. Draw *EF* through *E* parallel to *AD*, and draw *FD* through *D* parallel to *CE*. I.11 I.3 I.31

Then, since a straight line *EF* falls on the parallel straight lines *EC* and *FD*, the sum of the angles *CEF* and *EFD* equals two right angles. Therefore the sum of the angles *FEB* and *EFD* is less than two right angles. I.29

But straight lines produced from angles less than two right angles meet. Therefore *EB* and *FD*, if produced in the direction *B* and *D*, will meet. Post.5

Let them be produced and meet at *G*, and join *AG*.

Then, since *AC* equals *CE*, the angle *EAC* also equals the angle *AEC*. The angle at *C* is right, therefore each of the angles *EAC* and *AEC* is half of a right angle. I.5 I.32

For the same reason each of the angles *CEB* and *EBC* is also half of a right angle, therefore the angle *AEB* is right.

And, since the angle *EBC* is half of a right angle, the angle *DBG* is also half of a right angle. But the angle *BDG* is also right, for it equals the angle *DCE*, since they are alternate. Therefore the remaining angle *DGB* is half of a right angle. Therefore the angle *DGB* equals the angle *DBG*, so that the side *BD* also equals the side *GD*. I.15 I.29 I.32 I.6

Again, since the angle *EGF* is half of a right angle, and the angle at *F* is right, for it equals the opposite angle, the angle at *C*, the remaining angle *FEG* is half of a right angle. Therefore the angle *EGF* equals the angle *FEG*, so that the side *GF* also equals the side *EF*. I.34 I.32 I.6

Now, since the square on *EC* equals the square on *CA*, the sum of the squares on *EC* and *CA* is double the square on *CA*. But the square on EA equals the sum of the squares on *EC* and *CA*, therefore the square on *EA* is double the square on *AC*. I.47

Again, since *FG* equals *EF*, the square on *FG* also equals the square on *FE*. Therefore the sum of the squares on *GF* and *FE* is double the square on *EF*. But the square on *EG* equals the sum of the squares on *GF* and *FE*, therefore the square on *EG* is double the square on *EF*. I.47

And *EF* equals *CD*, therefore the square on *EG* is double the square on *CD*. But the square on *EA* was also proved to be double the square on *AC,* therefore the sum of the squares on *AE* and *EG* is double the sum of the squares on *AC* and *CD.* I.34

And the square on *AG* equals the sum of the squares on *AE* and *EG,* therefore the square on *AG* is double the sum of the squares on *AC* and *CD.* But the sum of the squares on *AD* and *DG* equals the square on *AG,* I.47 therefore the sum of the squares on *AD* and *DG* is double the sum of the squares on *AC* and *CD.*

And *DG* equals *DB,* therefore the sum of the squares on *AD* and *DB* is double the sum of the squares on *AC* and *CD.*

Therefore *if a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line.*

Q.E.D.

Guide

This proposition can be interpreted as the same algebraic identity as the previous proposition if *y* is identified with *CD* and *z* is identified with *CB.* Then the proposition states that

$$(y + z)^2 + (y - z)^2 = 2\,(y^2 + z^2).$$

This proposition is not used in the rest of the *Elements.*

Next proposition: II.11

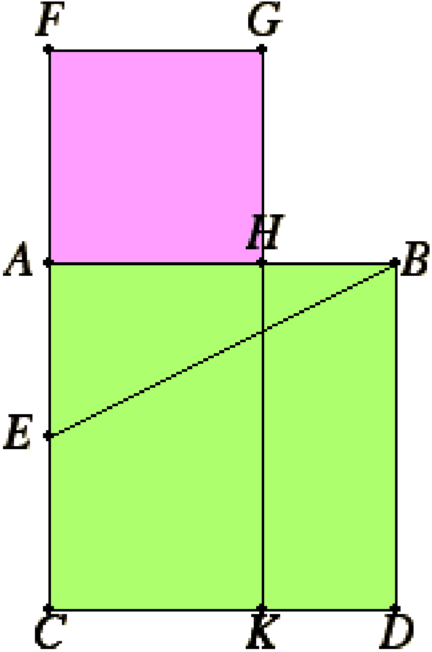
Previous: II.9

Book II introduction

Select from Book II

Select book

Select topic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Euclid's Elements

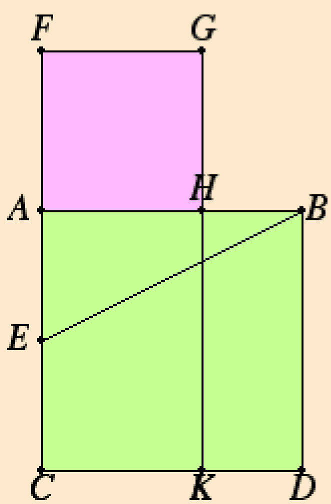
Book II

Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

Let *AB* be the given straight line.

It is required to cut *AB* so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.



I.46

Describe the square *ABDC* on *AB*. Bisect *AC* at the point *E*, and join *BE*. Draw *CA* through to *F*, and make *EF* equal to *BE*. Describe the square *FH* on *AF*, and draw *GH* through to *K*. I.10 I.3 I.46

I say that *AB* has been cut at *H* so that the rectangle *AB* by *BH* equals the square on *AH*.

Since the straight line *AC* has been bisected at *E*, and *FA* is added to it, the rectangle *CF* by *FA* together with the square on *AE* equals the square on *EF*. II.6

But *EF* equals *EB*, therefore the rectangle *CF* by *FA* together with the square on *AE* equals the square on *EB*.

But the sum of the squares on *BA* and *AE* equals the square on *EB*, for the angle at *A* is right, therefore the rectangle *CF* by *FA* together with the square on *AE* equals the sum of the squares on *BA* and *AE*. I.47

Subtract the square on *AE* from each. Therefore the remaining rectangle *CF* by *FA* equals the square on *AB*.

Now the rectangle *CF* by *FA* is *FK*, for *AF* equals *FG*, and the square on *AB* is *AD*, therefore *FK* equals *AD*.

Subtract *AK* from each. Therefore *FH*, which remains, equals *HD*.

And *HD* is the rectangle *AB* by *BH*, for *AB* equals *BD*, and *FH* is the square on *AH*, therefore the rectangle *AB* by *BH* equals the square on *HA*.

Therefore the given straight line *AB* has been cut at *H* so that the rectangle *AB* by *BH* equals the square on *HA*.

Q.E.F.

Guide

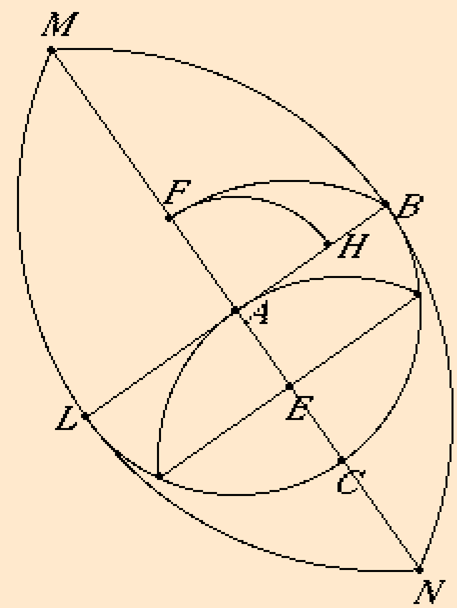
This construction cuts a line into two parts to solve the equation $a\,(a - x) = x^2$ geometrically.

This construction is used in the proof of IV.10, which is later used to construct a regular pentagon. It accomplishes the same thing as the construction of proposition VI.30, which cuts a line into extreme and mean ratio, defined in VI.Def.3, and that construction is used later in XIII.17.

The difference between this proposition and VI.30 is a matter of terminology. Propositions dealing with ratios of lines are postponed until Book VI, but any ratio concerning lines can be converted into a statement about areas of rectangles. Proposition VI.16 states that the line *A* is to the line *B* as the line *C* is to the line *D* is equivalent to the statement that the rectangle *A* by *D* equals the rectangle *B* by *C*.

The construction of this proposition cuts a line into two parts *A* and *B* so that the rectangle *A* + *B* by *A* equals the square *B* by *B*. The construction in VI.30 cuts a line so that $A + B : B = B : A$, which by VI.16, or by its special case VI.17, is the same thing.

Construction steps



For the purposes of cutting the line *AB*, the entire diagram does not have to be constructed. The points *D*, *G*, and *K* are unnecessary. In the diagram to the right, only those lines and circles necessary for the construction are shown, and only those parts of them that are relevant.

Altogether, there are six circles to be drawn, two lines connected, and one line extended. In order, they are as follows. Extend *BA* and draw a circle centered at *A* with radius *AB* to determine the point *L*. Draw circles centered at *B* and *L* with radius *BL* to determine points *M* and *N*. Draw the straight line *MN*. Then *C* is where it meets the circle centered at *A*. Draw a circle centered at *C* with radius *CA*, and connect the two points where it meets the circle centered at *A*. Then *E* is where that line meets *AC*. Draw the circle centered at *E* with radius *EB* to determine *F*. Draw the circle centered at *A* with radius *F* to determine *H*, the desired point to cut *AB*.

The golden ratio, the 36°-72°-72° triangle, and regular pentagons

This is the first of several propositions in the *Elements* that treats these concepts. At this point, ratios have not been introduced, so Euclid describes it in basic terms, that a given straight line is cut so that "the rectangle contained by the whole and one of the segments equals the square on the remaining segment."

Next proposition: II.12

Previous: II.10

Book II introduction

Select from Book II

Select book

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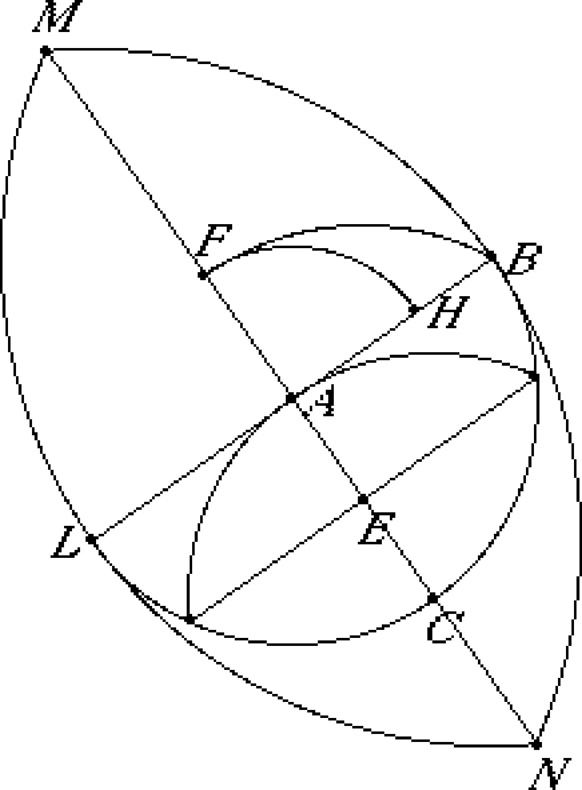
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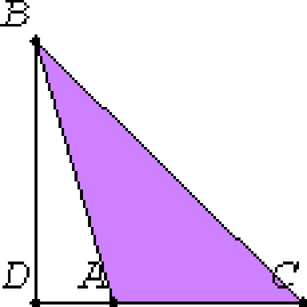
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Euclid's Elements

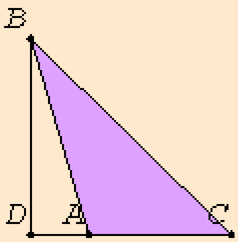
Book II

Proposition 12

In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Let *ABC* be an obtuse-angled triangle having the angle *BAC* obtuse, and draw *BD* from the point *B* perpendicular to *CA* produced. I.12

I say that the square on *BC* is greater than the sum of the squares on *BA* and *AC* by twice the rectangle *CA* by *AD*.



Since the straight line *CD* has been cut at random at the point *A*, the square on *DC* equals the sum of the squares on *CA* and *AD* and twice the rectangle *CA* by *AD*. II.4

Add the square on *DB* be added to each. Therefore the sum of the squares on *CD* and *DB* equals the sum of the squares on *CA*, *AD*, and *DB* plus twice the rectangle *CA* by *AD*.

But the square on *CB* equals the sum of the squares on *CD* and *DB*, for the angle at *D* is right, and the square on *AB* equals the sum of the squares on *AD* and *DB*, therefore the square on *CB* equals the sum of the squares on *CA* and *AB* plus twice the rectangle *CA* by *AD*, so that the square on *CB* is greater than the sum of the squares on *CA* and *AB* by twice the rectangle *CA* by *AD*. I.47

Therefore *in obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.*

Q.E.D.

Guide

This proposition for obtuse angles, together with the next one for acute triangles, complement the Pythagorean theorem, Proposition I.47, for right triangles. Prop.I.47 says that if triangle *ABC* has a right angle at *A*, then

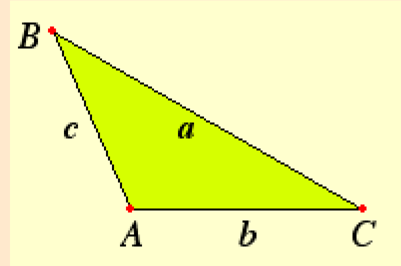
$$a^2 = b^2 + c^2$$

where a, b, and c are the sides opposite the angles A, B, and C, respectively.

In this proposition, II.12, the angle A is obtuse rather than right, and the conclusion is that

$$a^2 = b^2 + c^2 - 2ch$$

where h is the height of the triangle when c is taken as the base of the triangle. The



next proposition, II.13 has the same conclusion, but the hypothesis is that the angle at *A* is acute rather than obtuse.

This conclusion is very close to the law of cosines for oblique triangles.

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

since the height *h* equals *b* cos *A*. Trigonometry was developed some time after the *Elements* was written, and the negative numbers needed here (for the cosine of an obtuse angle) were not accepted until long after most of trigonometry was developed. Nonetheless, this proposition and the next may be considered geometric versions of the law of cosines.

Neither this nor the next is used in the rest of the *Elements*.

Next proposition: II.13

Previous: II.11

Book II introduction

Select from Book II

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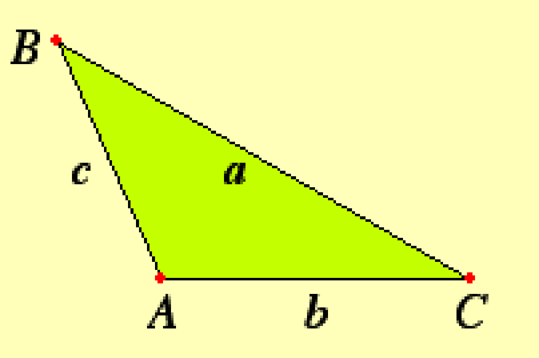
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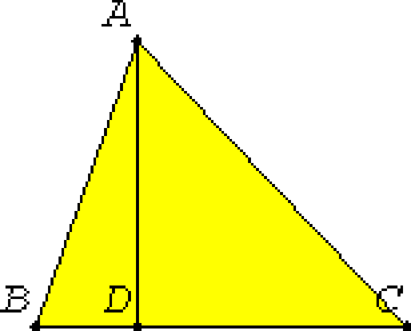
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Euclid's Elements

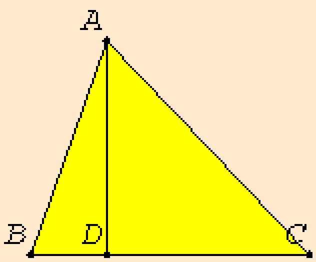
Book II

Proposition 13

In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Let *ABC* be a triangle having the angle at *B* acute, and draw *AD* from the point *A* perpendicular to *BC*. I.12

I say that the square on *AC* is less than the sum of the squares on *CB* and *BA* by twice the rectangle *CB* by *BD*.



Since the straight line *CB* has been cut at random at *D*, the sum of the squares on *CB* and *BD* equals twice the rectangle *CB* by *BD* plus the square on *DC*. II.7

Add the square on *DA* to each. Therefore the sum of the squares on *CB*, *BD*, and *DA* equals twice the rectangle *CB* by *BD* plus the sum of the squares on *AD* and *DC*.

But the square on *AB* equals the sum of the squares on *BD* and *DA*, for the angle at *D* is right, and the square on *AC* equals the sum of the squares on *AD* and *DC*, therefore the sum of the squares on *CB* and *BA* equals the square on *AC* plus twice the rectangle *CB* by *BD*, so that the square on *AC* alone is less than the sum of the squares on *CB* and *BA* by twice the rectangle *CB* by *BD*. I.47

Therefore *in acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.*

Q.E.D.

Guide

See the guide for the previous proposition II.12.

Next proposition: II.14

Previous: II.12

Book II introduction

Select from Book II

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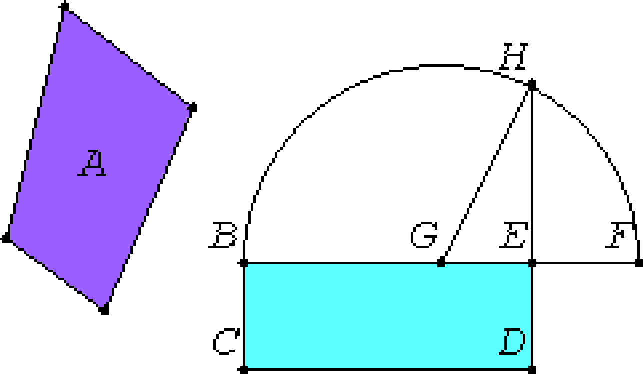
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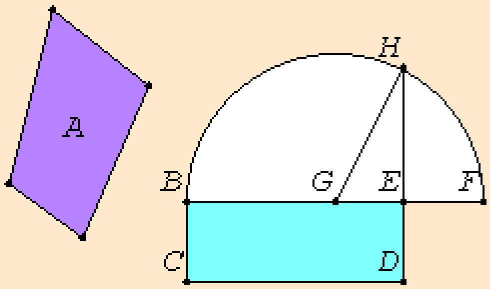
Euclid's Elements
Book II
Proposition 14

To construct a square equal to a given rectilinear figure.

Let *A* be the given rectilinear figure.

It is required to construct a square equal to the rectilinear figure *A.*

Construct the rectangular parallelogram *BD* equal to the rectilinear figure *A.* I.45



Then, if *BE* equals *ED,* then that which was proposed is done, for a square *BD* has been constructed equal to the rectilinear figure *A.*

But, if not, one of the straight lines *BE* or *ED* is greater.

Let *BE* be greater, and produce it to *F*. Make *EF* equal to *ED,* and bisect *BF* at *G.* I.3 I.10

Describe the semicircle *BHF* with center *G* and radius one of the straight lines *GB* or *GF*. Produce *DE* to *H,* and join *GH.* I.Def.18

Then, since the straight line *BF* has been cut into equal segments at *G* and into unequal segments at *E,* the rectangle *BE* by *EF* together with the square on *EG* equals the square on *GF.* II.5

But *GF* equals *GH,* therefore the rectangle *BE* by *EF* together with the square on *GE* equals the square on *GH.*

But the sum of the squares on *HE* and *EG* equals the square on *GH,* therefore the rectangle *BE* by *EF* together with the square on *GE* equals the sum of the squares on *HE* and *EG.* I.47

Subtract the square on *GE* from each. Therefore the remaining rectangle *BE* by *EF* equals the square on *EH.*

But the rectangle *BE* by *EF* is *BD,* for *EF* equals *ED,* therefore the parallelogram *BD* equals the square on *HE.*

And *BD* equals the rectilinear figure *A.*

Therefore the rectilinear figure *A* also equals the square which can be described on *EH.*

Therefore a square, namely that which can be described on *EH,* has been constructed equal to the given rectilinear figure *A.*

Q.E.F.

Guide

The construction of a square equal to a given is short as described in the proof. The verification that this construction works is also short with the help of Proposition II.5 and Proposition I.47, the Pythagorean theorem. First, Prop. II.5 allows us to convert the rectangle, *BE* by *ED,* into the difference of two squares, $GF^2 - GE^2$. Note that *GF* equals *GH,* the hypotenuse of a right triangle *GHE.* Using I.47 we can replace the difference of two squares, $GH^2 - GE^2$, by the single square, EH^2. Thus, the original rectangle equals the squre EH^2.

Quadrature of rectilinear figures

This proposition finishes the for quadrature of rectilinear figures. The narrow meaning of the word "quadrature" is to find a square with the same area of a given figure, also called "squaring" the figure. In a broader sense, "quadrature" means finding the area of a given figure.

Proposition I.45 on application of areas of rectilinear figures allows us to replace the figure under question with a rectangle of the same area. Now, the semicircle construction in this proposition finds what is called the "mean proportional" between the sides of the rectangle. If the sides of the rectangle are denoted *a* and *b,* then the mean proportional *x* between them satisfies the proportion $a:x = x:b$, and that's equivalent to an equality of areas $ab = x^2$, that is to say, the square on this mean proportional has the same area as the rectangle. Thus, any rectilinear figure can be squared.

This result is an end in itself. It is not used in the rest of the *Elements.*

There is another proof of this proposition that is based on similar triangles. Referring to the figure in the proposition, draw lines *BH* and *BF*, and you'll see three similar right triangles: *BFH, BHE,* and *HGE.* From their similarity it follows that $BE:EH = EH:EF$. That says *EH* is the required mean proportional.

Proportions aren't developed until Book V, and similar triangles aren't mentioned until Book VI. So in order to complete the theory of quadrature of rectilinear figures early in the *Elements,* Euclid chose a different proof that doesn't depend on similar triangles. Note that this same result appears in the garb of proportions in Proposition VI.13. Also in Book VI, Proposition VI.17 shows that the square on the mean proportional equals the rectangle on the two straight lines.

Squaring the circle

What about circles and other shapes? The general theory of circles is treated in Book III, but there are no propositions about the areas of circles until book XII. Proposition XII.2 says the areas of circles are proportional to the squares on their diameters. That allows the area of two circles to be compared, but it doesn't answer the question "what's the area of this circle?" in the same way that this proposition does for rectilinear figures. That would require finding a square equal to a given circle.

This problem of "quadrature of the circle" was one of three famous problems that goes back at least to the time of Anaxagoras, about 150 years before Euclid. It is equivalent to constructing a line segment of length pi (relative to a unit length). This problem was solved by ancient Greek geometers but not by means of the Euclidean tools of straightedge and compass; higher curves were required. In fact, by the time of Pappus it was believed that the circle could not be squared using only straightedge, compass, and, furthermore, couldn't be squared even with the help of the conic sections (parabola, hyperbola, and ellipse). But the ancient Greeks had no mathematical proof that it could not be squared.

That the circle could not be squared with Euclidean tools was not shown until 1882 when Lindemann proved that pi is a transcendental number.

Next book: Book III

Previous: II.13

Book II introduction

Select from Book II

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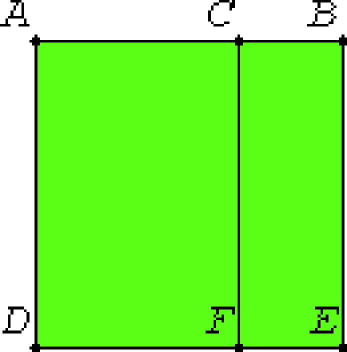
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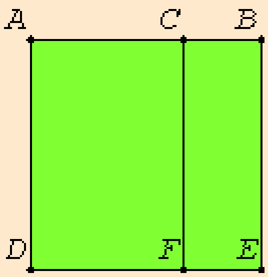
Euclid's Elements

Book II

Proposition 2

If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.



Let the straight line *AB* be cut at random at the point *C*.

I say that the sum of the rectangle *BA* by *AC* and the rectangle *AB* by *BC* equals the square on *AB*

Describe the square *ADEB* on *AB*, and draw *CF* through *C* parallel to either *AD* or *BE*. I.46
Then *AE* equals *AF* plus *CE*. I.31

Now *AE* is the square on *AB*; *AF* is the rectangle *BA* by *AC*, for it is contained by *DA* and *AC*, and *AD* equals *AB*; and *CE* is the rectangle *AB* by *BC*, for *BE* equals *AB*. II.Def.1

Therefore the sum of the rectangle *BA* by *AC* and the rectangle *AB* by *BC* equals the square on *AB*.

Therefore *if a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.*

Q.E.D.

Guide

This proposition is actually a special case of II.1. In II.1 Euclid shows that the product of one line by a sum of any number of lines is the sum of the products of that line by each of the lines. In this proposition, there are just two of those lines and their sum equals the one line. Rather than using II.1 to prove II.2, Euclid proves II.2 directly. This suggests that II.1 may have been inserted into the *Elements* after II.2 was included, either by Euclid or someone else.

In modern algebraic notation this proposition says that if $y = y_1 + y_2$, then $xy = x\,y_1 + x\,y_2$. This can also be stated in a single equation as

$$x\,(y_1 + y_2) = x\,y_1 + x\,y_2.$$

Use of this proposition

This proposition is used in the proof of proposition XIII.10 which shows that a certain relationship holds for the sides of a regular pentagon, regular hexagon, and regular decagon that are all inscribed in the same circle, namely, the square on the side of the pentagon equals the sum of the squares on the side of a hexagon and on the side of a decagon.

Next proposition: II.3

Previous: II.1

Select from Book II

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Book II introduction

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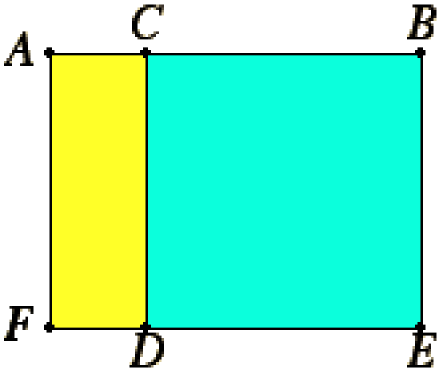
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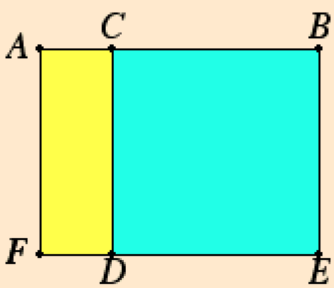
Euclid's Elements
Book II
Proposition 3

If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment.

Let the straight line *AB* be cut at random at *C.*



I say that the rectangle *AB* by *BC* equals the sum of the rectangle *AC* by *CB* and the square on *BC.*

Describe the square *CDEB* on *CB.* Draw *ED* through to *F,* and draw *AF* through *A* parallel to either *CD* or *BE.* I.46 I.31

Then *AE* equals *AD* plus *CE.*

Now *AE* is the rectangle *AB* by *BC,* for it is contained by *AB* and *BE,* and *BE* equals *BC*; *AD* is the rectangle *AC* by *CB,* for *DC* equals *CB*; and *DB* is the square on *CB.*

Therefore the rectangle *AB* by *BC* equals the sum of the rectangle *AC* by *CB* and the square on *BC.*

Therefore *if a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment.*

Q.E.D.

Guide

This proposition is another special case of II.1. In modern algebraic notation it says that if $x = y + z$, then $xy = y^2 + yz$. Identities that are logically equivalent to this implication can be found by eliminating one of the three variables x, y, or z. Here are two of them.

$$(y + z)\, y = y^2 + yz,$$

and

$$xy = y^2 + y\,(x - y).$$

Use of this proposition

This proposition refers to lines and rectangles, but the analogous statement for numbers is used in a proposition in one of the Euclid's books on number theory, namely, of proposition IX.15.

Next proposition: II.4

Select from Book II

Previous: II.2

Book II introduction

Select book

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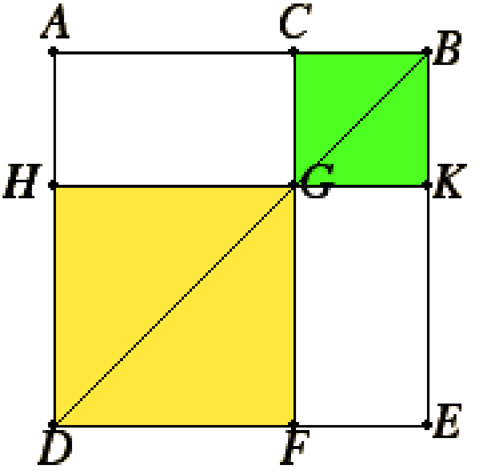
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Euclid's Elements

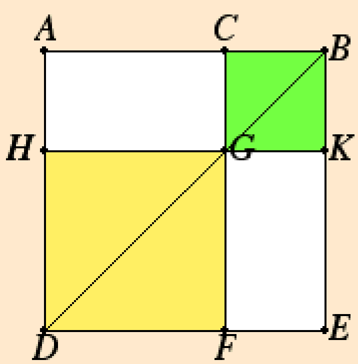
Book II

Proposition 4

If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

Let the straight line *AB* be cut at random at *C.*

I say that the square on *AB* equals the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB.*



Describe the square *ADEB* on *AB.* Join *BD.* Draw *CF* through *C* parallel to either *AD* or *EB,* and draw *HK* through *G* parallel to either *AB* or *DE.* I.46 I.31

Then, since *CF* is parallel to *AD,* and *BD* falls on them, the exterior angle *CGB* equals the interior and opposite angle *ADB.* I.29

But the angle *ADB* equals the angle *ABD,* since the side *BA* also equals *AD.* Therefore the angle *CGB* also equals the angle *GBC,* so that the side *BC* also equals the side *CG.* I.5 I.6

But *CB* equals *GK,* and *CG* to *KB.* Therefore *GK* also equals *KB.* Therefore *CGKB* is equilateral. I.34

I say next that it is also right-angled.

Since *CG* is parallel to *BK,* the sum of the angles *KBC* and *GCB* equals two right angles. I.29

But the angle *KBC* is right. Therefore the angle *BCG* is also right, so that the opposite angles *CGK* and *GKB* are also right. I.34

Therefore *CGKB* is right-angled, and it was also proved equilateral, therefore it is a square, and it is described on *CB.*

For the same reason *HF* is also a square, and it is described on *HG,* that is *AC.* Therefore the squares *HF* and *KC* are the squares on *AC* and *CB.* I.34

Now, since *AG* equals *GE,* and *AG* is the rectangle *AC* by *CB,* for *GC* equals *CB,* therefore *GE* also equals the rectangle *AC* by *CB.* Therefore the sum of *AG* and *GE* equals twice the rectangle *AC* by *CB.* I.43

But the squares *HF* and *CK* are also the squares on *AC* and *CB,* therefore the sum of the four figures *HF, CK, AG,* and *GE* equals the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB.*

But *HF, CK, AG,* and *GE* are the whole *ADEB,* which is the square on *AB.*

Therefore the square on *AB* equals the the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB.*

Therefore *if a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.*

Q.E.D.

Guide

The statement of the proposition can be interpreted in modern notation as saying that if $x = y + z$, then $x^2 = y^2 + z^2 + 2yz$. More simply, as an identity, it says

$$(y + z)^2 = y^2 + z^2 + 2yz.$$

Use of this proposition

This is one of the more frequently used propositions of Book II. It is used in II.12 later in this book, frequently in BookX, and in XIII.2. Also, the analogous statement for numbers is used in IX15,

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Next proposition: II.5

Previous: II.3

Book II introduction

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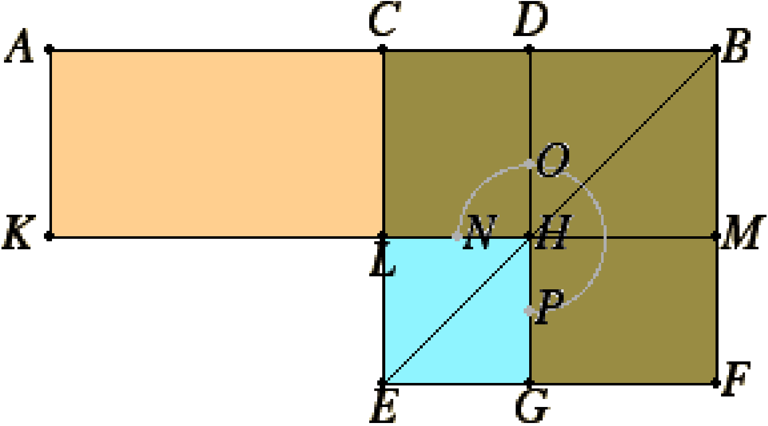
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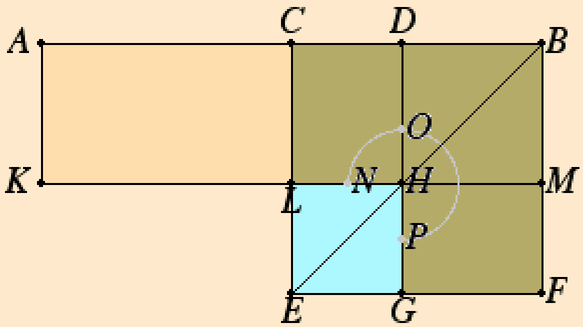
Euclid's Elements
Book II
Proposition 5

If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.

Let a straight line *AB* be cut into equal segments at *C* and into unequal segments at *D*.

I say that the rectangle *AD* by *DB* together with the square on *CD* equals the square on *CB*.



Describe the square *CEFB* on *CB*, and join *BE*. Draw *DG* through *D* parallel to either *CE* or *BF*, again draw *KM* through *H* parallel to either *AB* or *EF*, and again draw *AK* through *A* parallel to either *CL* or *BM*. I.46 I.31

Then, since the complement *CH* equals the complement *HF*, add *DM* to each. Therefore the whole *CM* equals the whole *DF*. I.43

But *CM* equals *AL*, since *AC* is also equal to *CB*. Therefore *AL* also equals *DF*. Add *CH* to each. Therefore the whole *AH* equals the gnomon *NOP*. I.36 II.Def.2

But *AH* is the rectangle *AD* by *DB*, for *DH* equals *DB*, therefore the gnomon *NOP* also equals the rectangle *AD* by *DB*.

Add *LG*, which equals the square on *CD*, to each. Therefore the sum of the gnomon *NOP* and *LG* equals the sum of the rectangle *AD* by *DB* and the square on *CD*.

But the gnomon *NOP* together with *LG* is the whole square *CEFB*, which is described on *CB*.

Therefore the rectangle *AD* by *DB* together with the square on *CD* equals the square on *CB*.

Therefore *if a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.*

Q.E.D.

Guide

In the figure there is a part of a circle denoted with the points *NOP*. This is supposed to indicate the gnomon which is three parts of the square *BCEF*, the only part left out for the gnomon being the square *EGHL*. Later versions of the *Elements* did not have this curve in the figure. Instead, they named the same gnomon as *LBG*. Either way is sufficient to specify the gnomon.

Explanation of the proof

We can represent a rectangle algebraically as *xy* where the sides are *x* and *y*. In the diagram above, take *x* as the line *AD* and *y* as the line *DH*, so that *xy* is the rectangle *AH*. According to this proposition, this product, or rectangle, is the difference of two squares, the large one being the square of $(x + y)/2$, which is the square on the line *BC* in the diagram, and the small one being the square of $(x – y)/2$, which is the square on the line *LH* (which equals the square on the line *CD*).

Using symbolic algebra, we can easily verify the identity

$$xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{y-x}{2}\right)^2$$

But Euclid was resticted to geometric arguments. The argument isn't difficult. The original rectangle *AH* is the sum of the rectangles *AL* and *CH*. By proposition I.43, the rectangles *CH* and *HF* are equal. And, of course, the rectangles *AL* and *CM* are equal. Therefore, $AH = AL + CH = CM + HF = CB^2 – LH^2$, as required.

Solution to a quadratic problem

This proposition is set up to help in the solution of a quadratic problem of the following form.

> Find two numbers *x* and *y* so that their sum is a known value *b* and their product is a known value c^2.

In terms of the single variable *x*, this is equivalent to solving the quadratic equation, $x(b – x) = c^2$. This equation can be written in a standard form as

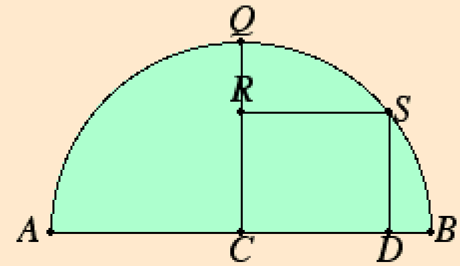
$$x^2 + c^2 = bx.$$

If *b* is represented as the line *AB* in the diagram, and with $x = AD$ and $y = BD$, the first condition $x + y = b$ is satisfied. This proposition says that the product *xy* equals the square on *BC* (which is $b/2$) minus the square on *CD*. Thus, the remaining condition reduces to finding *CD* so that $(b/2)^2 – CD^2 = c^2$. By I.47, if a right triangle is constructed with one side equal to $b/2$ and another equal to *c*, then the hypotenuse will equal the required value for *CD*. Algebraically, the solutions *AD* for *x* and *BD* for *y* have the values

$$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c^2} \qquad y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c^2}$$

This analysis yields a construction to solve the quadratic problem stated above.

To cut a given straight line so that the rectangle contained by the unequal segments equals a given square. Thus the given square must not be greater than the square described on the half of the given straight line.



Let *AB* be the given straight line. Bisect it at *C*. Draw the semicircle *AQB* with center *C* and radius *BC*. Construct a perpendicular *CR* to *AB* at *C* equal to the side of the given square. Draw *RS* parallel to *AB* intersecting the semicircle at *S*, and draw *SD* perpendicular to *AB*.

Then, as described above, *AB* has been cut at *G* so that *AG* times *GB* equals the given square.

This proposition is not found in the *Elements*, but a generalization is. After II.14 the given square could be replaced by any given rectilinear figure, since II.14 constructs a square equal to a given rectilinear figure. But the full generalization is not given until proposition VI.28. Not only has the given square become a general rectilinear figure, but all the rectangles and squares have been replaced by parallelograms. That requires generalizing II.4 to parallelograms. That's done in VI.25 which constructs a parallelgram similar to a given parallelogram and equal to a given rectilinear figure. It also requires a few technical propositions to carry out the proof.

Use of this proposition

This proposition is used in II.14, III.35, and occasionally in Book X.

Next proposition: II.6

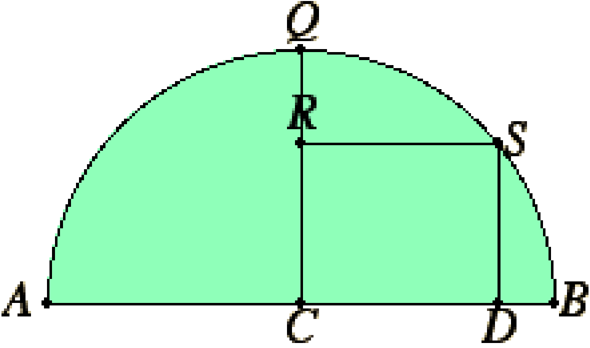
Previous: II.4

Book II introduction

Select from Book II

Select book

Select topic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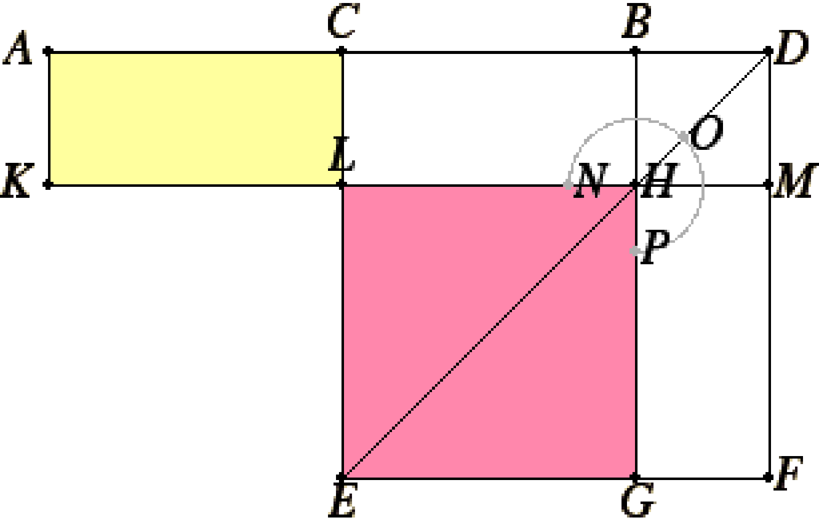
$$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c^2} \qquad y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c^2}$$



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Euclid's Elements

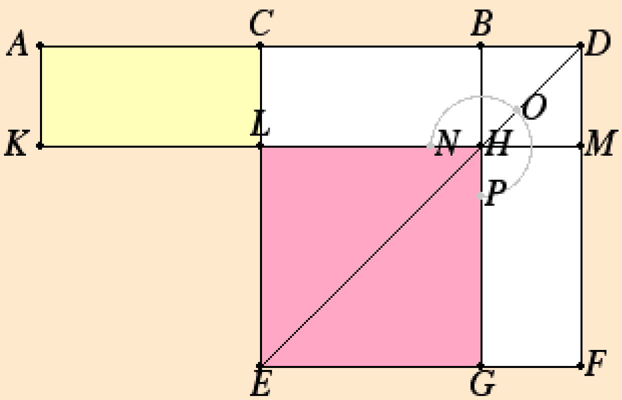
Book II

Proposition 6

If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.

Let a straight line *AB* be bisected at the point *C,* and let a straight line *BD* be added to it in a straight line.

I say that the rectangle *AD* by *DB* together with the square on *CB* equals the square on *CD.*



Describe the square *CEFD* on *CD,* and join *DE.* Draw *BG* through the point *B* parallel to either *EC* or *DF,* draw *KM* through the point *H* parallel to either *AB* or *EF,* and further draw *AK* through *A* parallel to either *CL* or *DM.* I.46 I.31

Then, since *AC* equals *CB, AL* also equals *CH.* But *CH* equals *HF.* Therefore *AL* also equals *HF.* I.36 I.43

Add *CM* to each. Therefore the whole *AM* equals the gnomon *NOP.* II.Def.2

But *AM* is the rectangle *AD* by *DB,* for *DM* equals *DB.* Therefore the gnomon *NOP* also equals the rectangle *AD* by *DB.*

Add *LG,* which equals the square on *BC,* to each. Therefore the rectangle *AD* by *DB* together with the square on *CB* equals the gnomon *NOP* plus *LG.*

But the gnomon *NOP* and *LG* are the whole square *CEFD,* which is described on *CD.*

Therefore the rectangle *AD* by *DB* together with the square on *CB* equals the square on *CD.*

Therefore *if a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.*

Q.E.D.

Guide

Explanation of the proof

This proposition is remarkably similar to the last one, II.5, except the point *D* does not lie on the line *AB* but on that line extended.

Let *b* denote the line *AB*, *x* denote *AD*, and *y* denote *BD* as in II.5. Then $x - y = b$ (as opposed to $x + y = b$ as in II.5). According to this proposition the rectangle *AD* by *DB*, which is the product *xy*, is the difference of two squares, the large one being the square on the line *CD*, that is the square of $x - b/2$, and the small one being the square on the line *CB*, that is, the square of $b/2$. Algebraically,

$$x(x - b) = (x - b/2)^2 - (b/2)^2.$$

This equation is easily verified with modern algebra, but it's also easily verified in geometry, as done here in the proof.

The geometric proof is primarily an exercise in cutting and pasting. The rectangle *AB* by *DB* is the rectangle *AM*, which is the sum of the rectangles *AL* and *CM*. But the rectangles *AL*, *CH*, and *HF* are all equal. Therefore, the rectangle *AB* by *DB* equals the gnomon formed by the rectangles *CM* and *HF*. That gnomon is the square *CF* minus the square *LG*, but the latter equals the square on *BC*. Thus, the rectangle *AB* by *DB* equals the square on *DB* minus the ssquare on *CB*.

Solution to a quadratic problem

As was II.5, this proposition is set up to help in the solution of a quadratic problem:

> Find two numbers *x* and *y* so that their difference $x - y$ is a known value *b* and their product is a known value c^2.

In terms of *x* alone, this is equivalent to solving the quadratic equation $x(x - b) = c^2$. Since this proposition says that $x(x - b) = (x - b/2)^2 - (b/2)^2$, the problem reduces to solving the equation

$$c^2 = (x - b/2)^2 - (b/2)^2,$$

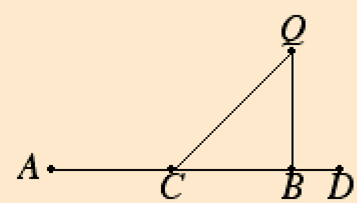
that is, finding *CD* so that $CD^2 = (b/2)^2 + c^2$. By I.47, if a right triangle is constructed with one side equal to $b/2$ and another equal to *c*, then the hypotenuse will equal the required value for *CD*. Algebraically, the solutions *AD* for *x* and *BD* for *y* have the values

$$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c^2} \qquad y = -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c^2}$$

This analysis yields a construction to solve the quadratic problem stated above.

To apply a rectangle equal to a given square to a given straight line but exceeding it by a square.

Let *AB* be the given straight line. Bisect it at *C*. Construct a perpendicular *BQ* to *AB* at *B* equal to the side of the given square. Draw *CQ*. Extend *AB* to *D* so that *BD* equals *CQ*.



Then, as described above, *AB* has been extended to *D* so that *AD* times *BD* equals the given square.

This construction is not found in the *Elements*, but a generalization of it to parallelograms is proposition VI.29.

Use of this proposition

This proposition is used in II.11, III.36, and a lemma for X.29.

Next proposition: II.7

Previous: II.5

Book II introduction

Select from Book II

Select book

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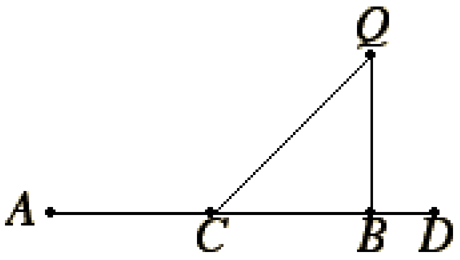

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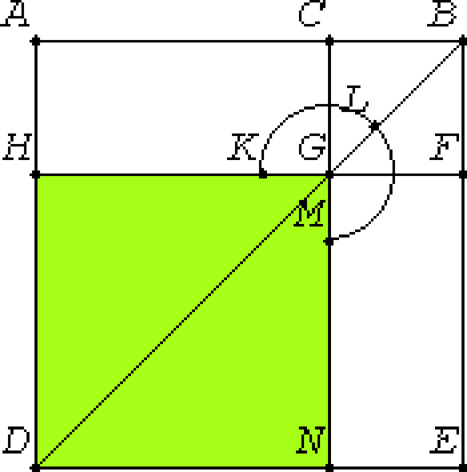
$$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c^2} \qquad y = -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c^2}$$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Euclid's Elements

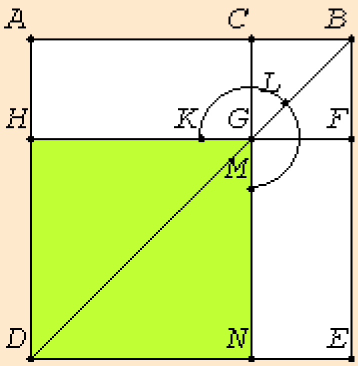
Book II

Proposition 7

If a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment.

Let a straight line *AB* be cut at random at the point *C*.

I say that the sum of the squares on *AB* and *BC* equals twice the rectangle *AB* by *BC* plus the square on *CA*.



Describe the square *ADEB* on *AB*, and let the figure be drawn. I.46 I.31

Then, since *AG* equals *GE*, add *CF* to each, therefore the whole *AF* equals the whole *CE*. I.43

Therefore the sum of *AF* and *CE* is double *AF*.

But the sum of *AF* and *CE* equals the gnomon *KLM* plus the square *CF*, therefore the gnomon *KLM* plus the square *CF* is double *AF*.

But twice the rectangle *AB* by *BC* is also double *AF*, for *BF* equals *BC*, therefore the gnomon *KLM* plus the square *CF* equal twice the rectangle *AB* by *BC*.

Add *DG*, which is the square on *AC*, to each. Therefore the gnomon *KLM* plus the sum of the squares *BG* and *GD* equals twice the rectangle *AB* by *BC* plus the square on *AC*.

But the gnomon *KLM* plus the sum of the squares *BG* and *GD* equals the whole *ADEB* plus *CF*, which are squares described on *AB* and *BC*.

Therefore the sum of the squares on *AB* and *BC* equals twice the rectangle *AB* by *BC* plus the square on *CA*.

Therefore *if a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment.*

Q.E.D.

Guide

We can interpret this algebraically with x for *AB*, y for *AC*, and z for *CB*. Then the proposition says that if $x = y + z$, then $x^2 + z^2 = 2xz + y^2$. This can be rewritten as various identities depending on which variable is eliminated, the simplest being

$$x^2 + z^2 = 2xz + (x - z)^2.$$

This proposition is used later in Book II to prove proposition II.13, and it is used repeatedly in Book X.

Next proposition: II.8

Previous: II.6

Book II introduction

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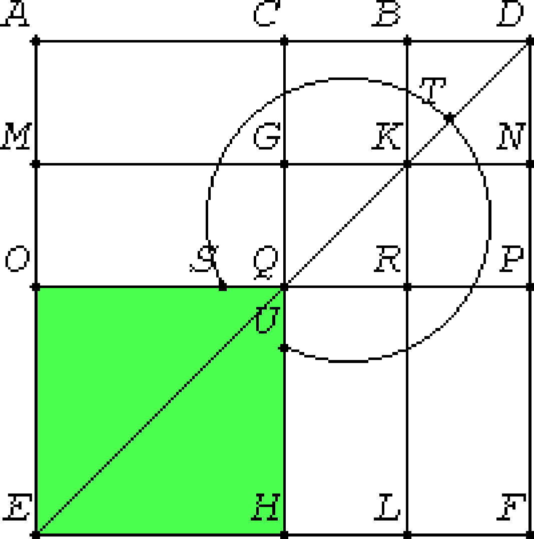
Select topic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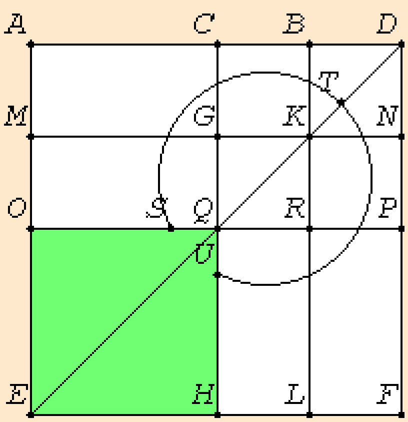
Euclid's Elements
Book II
Proposition 8

If a straight line is cut at random, then four times the rectangle contained by the whole and one of the segments plus the square on the remaining segment equals the square described on the whole and the aforesaid segment as on one straight line.

Let a straight line *AB* be cut at random at the point *C*.

I say that four times the rectangle *AB* by *BC* plus the square on *AC* equals the square described on *AB* and *BC* as on one straight line.



Produce the straight line *BD* in a straight line with *AB*, and make *BD* equal to *CB*. Describe the square *AEFD* on *AD*, and let the figure be drawn. I.3 I.46 I.31

Then, since *CB* equals *BD*, while *CB* equals *GK*, and *BD* equals *KN*, therefore *GK* also equals *KN*. I.34

For the same reason *QR* also equals *RP*.

And, since *BC* equals *BD*, and *GK* equals *KN*, therefore *CK* also equals *KD*, and *GR* equals *RN*. I.36

But *CK* equals *RN*, for they are complements of the parallelogram *CP*. Therefore *KD* also equals *GR*. Therefore the four areas *DK*, *CK*, *GR*, *RN* equal one another. Therefore the four are quadruple of *CK*. I.43

Again, since *CB* equals *BD*, while *BD* equals *BK*, that is *CG*, and *CB* equals *GK*, that is *GQ*, therefore *CG* also equals *GQ*. I.34

And, since *CG* equals *GQ*, and *QR* equals *RP*, *AG* also equals *MQ*, and *QL* equals *RF*. I.36

But *MQ* equals *QL*, for they are complements of the parallelogram *ML*, therefore *AG* also equals *RF*. Therefore the four areas *AG*, *MQ*, *QL*, *RF* equal one another. Therefore the four are quadruple of *AG*. But the four areas *CK*, *KD*, *GR*, *RN* were proved to be quadruple of *CK*, therefore the eight areas, which contain the gnomon *STU*, are quadruple of *AK*. I.43

Now, since *AK* is the rectangle *AB* by *BD*, for *BK* equals *BD*, therefore four times the rectangle *AB* by *BD* is quadruple of *AK*.

But the gnomon *STU* was also proved to be quadruple of *AK*, therefore four times the rectangle *AB* by *BD* equals the gnomon *STU*.

Add *OH*, which equals the square on *AC*, to each. Therefore four times the rectangle *AB* by *BD* plus the square on *AC* equals the gnomon *STU* plus *OH*.

But the gnomon *STU* and *OH* are the whole square *AEFD*, which is described on *AD*. Therefore four times the rectangle *AB* by *BD* plus the square on *AC* equals the square on *AD*.

But *BD* equals *BC*.

Therefore four times the rectangle *AB* by *BC* together with the square on *AC* equals the square on *AD*, that is to the square described on *AB* and *BC* as on one straight line.

Therefore *if a straight line is cut at random, then four times the rectangle contained by the whole and one of the segments plus the square on the remaining segment equals the square described on the whole and the aforesaid segment as on one straight line.*

Q.E.D.

Guide

Algebraically, if $x = y + z$, then $4xy + z^2 = (x + y)^2$. As an identity,

$$4xy + (x - y)^2 = (x + y)^2.$$

This proposition is not used in the rest of the *Elements*.

Next proposition: II.9

Previous: II.7

Book II introduction

Select from Book II

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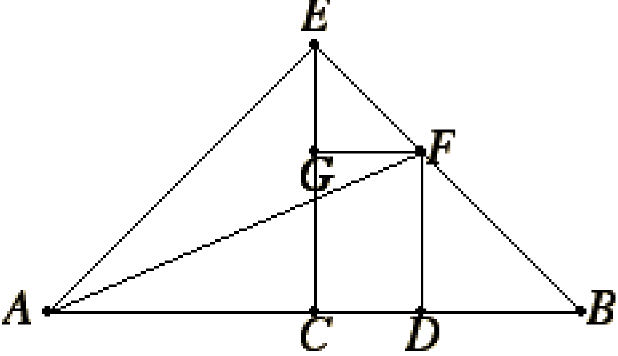
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Euclid's Elements

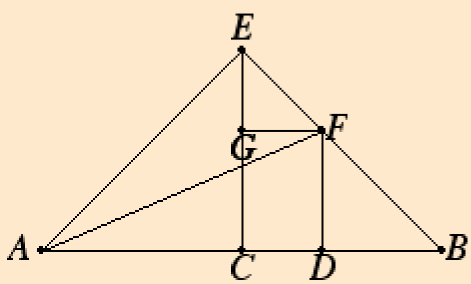
Book II

Proposition 9

If a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section.

Let a straight line *AB* be cut into equal segments at *C,* and into unequal segments at *D.*

I say that the sum of the squares on *AD* and *DB* is double the sum of the squares on *AC* and *CD.*



Draw *CE* from *C* at right angles to *AB,* and make it equal to either *AC* or *CB.* Join *EA* and *EB.* Draw *DF* through *D* parallel to *EC* and *FG* through *F* parallel to *AB.* Join *AF.* I.11 I.3 I.31

Then, since *AC* equals *CE,* the angle *EAC* also equals the angle *AEC.* I.5

And, since the angle at *C* is right, the sum of the remaining angles *EAC* and *AEC* equals one right angle. I.32

And they are equal, therefore each of the angles *CEA* and *CAE* is half of a right angle.

For the same reason each of the angles *CEB* and *EBC* is also half of a right angle, therefore the whole angle *AEB* is right.

And, since the angle *GEF* is half of a right angle, and the angle *EGF* is right, for it equals the interior and opposite angle *ECB,* the remaining angle *EFG* is half of a right angle. Therefore the angle *GEF* equals the angle *EFG,* so that the side *EG* also equals *GF.* I.29 I.32 I.6

Again, since the angle at *B* is half of a right angle, and the angle *FDB* is right, for it is again equal to the interior and opposite angle *ECB,* the remaining angle *BFD* is half of a right angle. Therefore the angle at *B* equals the angle *DFB,* so that the side *FD* also equals the side *DB.* I.29 I.32 I.6

Now, since *AC* equals *CE,* the square on *AC* also equals the square on *CE,* therefore the sum of the squares on *AC* and *CE* is double the square on *AC.*

But the square on *EA* equals the sum of the squares on *AC* and *CE,* for the angle *ACE* is right, therefore the square on *EA* is double the square on *AC.* I.47

Again, since *EG* equals *GF,* the square on *EG* also equals the square on *GF.* Therefore the sum of the squares on *EG* and *GF* is double the square on *GF.*

But the square on *EF* equals the sum of the squares on *EG* and *GF,* therefore the square on *EF* is double the square on *GF.* I.47

I.34

But *GF* equals *CD,* therefore the square on *EF* is double the square on *CD.*

But the square on *EA* is also double of the square on *AC*, therefore the sum of the squares on *AE* and *EF* is double the sum of the squares on *AC* and *CD.*

And the square on *AF* equals sum of the squares on *AE* and *EF*, for the angle *AEF* is right. Therefore the square on *AF* is double the sum of the squares on *AC* and *CD.* I.47

But the sum of the squares on *AD* and *DF* equals the square on *AF*, for the angle at *D* is right, therefore the sum of the squares on *AD* and *DF* is double the sum the squares on *AC* and *CD.* I.47

And *DF* equals *DB.*

Therefore the sum of the squares on *AD* and *DB* is double the sum of the squares on *AC* and *CD.*

Therefore *if a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section.*

Q.E.D.

Guide

Unlike the diagrams in the preceding propositions of Book II, for this one Euclid does not draw all the rectangles and squares.

As an identity,

$$(y + z)^2 + (y - z)^2 = 2\,(y^2 + z^2)$$

where $y = AC = CB$, and $z = CD$.

This proposition is used once in Book X to prove a lemma for X.60.

Next proposition: II.10

Previous: II.8

Book II introduction

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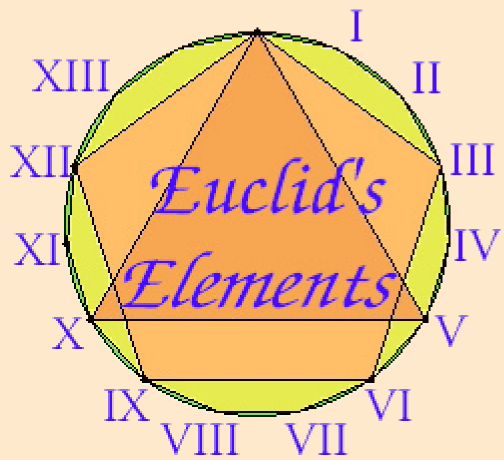
Book III



Book III

Table of contents

Definitions

Definition 1.
Equal circles are those whose diameters are equal, or whose radii are equal.

Definition 2.
A straight line is said to *touch* a circle which, meeting the circle and being produced, does not cut the circle.

Definition 3.
Circles are said to *touch* one another which meet one another but do not cut one another.

Definition 4.
Straight lines in a circle are said to be *equally distant* from the center when the perpendiculars drawn to them from the center are equal.

Definition 5.
And that straight line is said to be at a *greater distance* on which the greater perpendicular falls.

Definition 6.
A *segment* of a circle is the figure contained by a straight line and a circumference of a circle.

Definition 7.
An *angle of a segment* is that contained by a straight line and a circumference of a circle.

Definition 8.
An *angle in a segment* is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the ends of the straight line which is the base of the segment, is contained by

the straight lines so joined.

Definition 9.

And, when the straight lines containing the angle cut off a circumference, the angle is said to *stand upon* that circumference.

Definition 10.

A *sector* of a circle is the figure which, when an angle is constructed at the center of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

Definition 11.

Similar segments of circles are those which admit equal angles, or in which the angles equal one another.

Propositions

Proposition 1.

To find the center of a given circle.

Corollary. If in a circle a straight line cuts a straight line into two equal parts and at right angles, then the center of the circle lies on the cutting straight line.

Proposition 2.

If two points are taken at random on the circumference of a circle, then the straight line joining the points falls within the circle.

Proposition 3.

If a straight line passing through the center of a circle bisects a straight line not passing through the center, then it also cuts it at right angles; and if it cuts it at right angles, then it also bisects it.

Proposition 4.

If in a circle two straight lines which do not pass through the center cut one another, then they do not bisect one another.

Proposition 5.

If two circles cut one another, then they do not have the same center.

Proposition 6.

If two circles touch one another, then they do not have the same center.

Proposition 7.

If on the diameter of a circle a point is taken which is not the center of the circle, and from the point straight lines fall upon the circle, then that is greatest on which passes through the center, the remainder of the same diameter is least, and of the rest the nearer to the straight line through the center is always greater than the more remote; and only two equal straight lines fall from the point on the circle, one on each side of the least straight line.

Proposition 8.

If a point is taken outside a circle and from the point straight lines are drawn through to the circle, one of which is through the center and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the center is greatest, while of the rest the nearer to that through the center is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote; and only two equal straight lines fall on the circle from the point, one on each side of the least.

Proposition 9.

If a point is taken within a circle, and more than two equal straight lines fall from the point on the circle, then the point taken is the center of the circle.

Proposition 10.
A circle does not cut a circle at more than two points.

Proposition 11.
If two circles touch one another internally, and their centers are taken, then the straight line joining their centers, being produced, falls on the point of contact of the circles.

Proposition 12.
If two circles touch one another externally, then the straight line joining their centers passes through the point of contact.

Proposition 13.
A circle does not touch another circle at more than one point whether it touches it internally or externally..

Proposition 14.
Equal straight lines in a circle are equally distant from the center, and those which are equally distant from the center equal one another.

Proposition 15.
Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the center is always greater than the more remote.

Proposition 16.
The straight line drawn at right angles to the diameter of a circle from its end will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed, further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Corollary. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its end touches the circle.

Proposition 17.
From a given point to draw a straight line touching a given circle.

Proposition 18.
If a straight line touches a circle, and a straight line is joined from the center to the point of contact, the straight line so joined will be perpendicular to the tangent.

Proposition 19.
If a straight line touches a circle, and from the point of contact a straight line is drawn at right angles to the tangent, the center of the circle will be on the straight line so drawn.

Proposition 20.
In a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base.

Proposition 21.
In a circle the angles in the same segment equal one another.

Proposition 22.
The sum of the opposite angles of quadrilaterals in circles equals two right angles.

Proposition 23.

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

Proposition 24.
Similar segments of circles on equal straight lines equal one another.

Proposition 25.
Given a segment of a circle, to describe the complete circle of which it is a segment.

Proposition 26.
In equal circles equal angles stand on equal circumferences whether they stand at the centers or at the circumferences.

Proposition 27.
In equal circles angles standing on equal circumferences equal one another whether they stand at the centers or at the circumferences.

Proposition 28.
In equal circles equal straight lines cut off equal circumferences, the greater circumference equals the greater and the less equals the less.

Proposition 29.
In equal circles straight lines that cut off equal circumferences are equal.

Proposition 30.
To bisect a given circumference.

Proposition 31.
In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.

Proposition 32.
If a straight line touches a circle, and from the point of contact there is drawn across, in the circle, a straight line cutting the circle, then the angles which it makes with the tangent equal the angles in the alternate segments of the circle.

Proposition 33.
On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilinear angle.

Proposition 34.
From a given circle to cut off a segment admitting an angle equal to a given rectilinear angle.

Proposition 35.
If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other.

Proposition 36.
If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.

Proposition 37.
If a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts

the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.

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Previous: Book II

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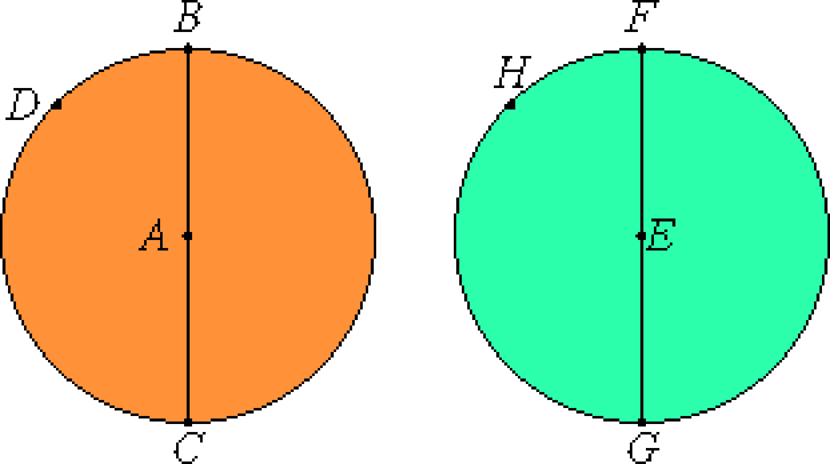
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Euclid's Elements

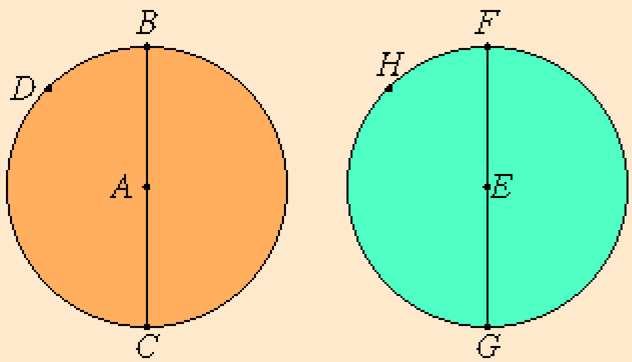
Book III

Definition 1

Equal circles are those whose diameters are equal, or whose radii are equal.

Guide

This should not be a definition but a postulate or a theorem. The subject of the area of circles is developed in Proposition XII.2 and this definition is not used there. Instead the concept of equality, or rather, inequality is the same as it is in the rest of the *Elements*. For instance, if one figure is the contained in another, then the first is less than the other. Thus, there is a prior defintion for the equality of two figures.



Two circles are illustrated, namely circle *BCD* and circle *FGH*. The center of circle *BCD* is *A*, while the center of circle *FGH* is *E*. They are equal by Euclid's definition since their diameters *BC* and *FG* are equal, or since their radii *AB* and *EF* are equal.

Next definition: III.Def.2-3

Book III introduction

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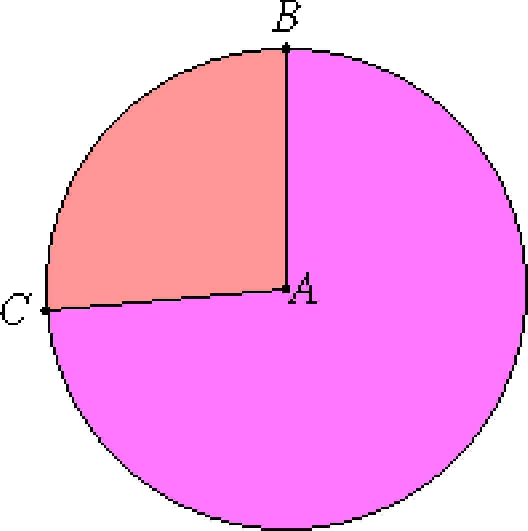
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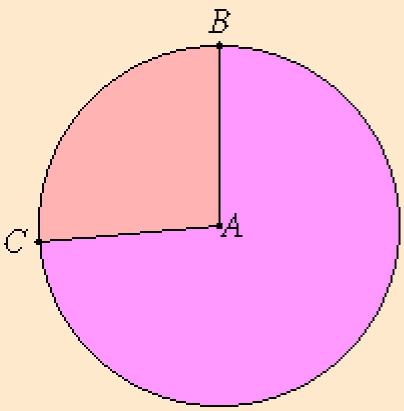
Euclid's Elements

Book III

Definition 10

A *sector* of a circle is the figure which, when an angle is constructed at the center of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

Guide



Here a sector *BAC* is illustrated. The angle *BAC* encloses the sector. Note that the remainder of the circle would not be considered a sector by Euclid since the angle at the center would be greater than 180°.

Next definition: III.Def.11

Previous: III.Def.6-9

Book III introduction

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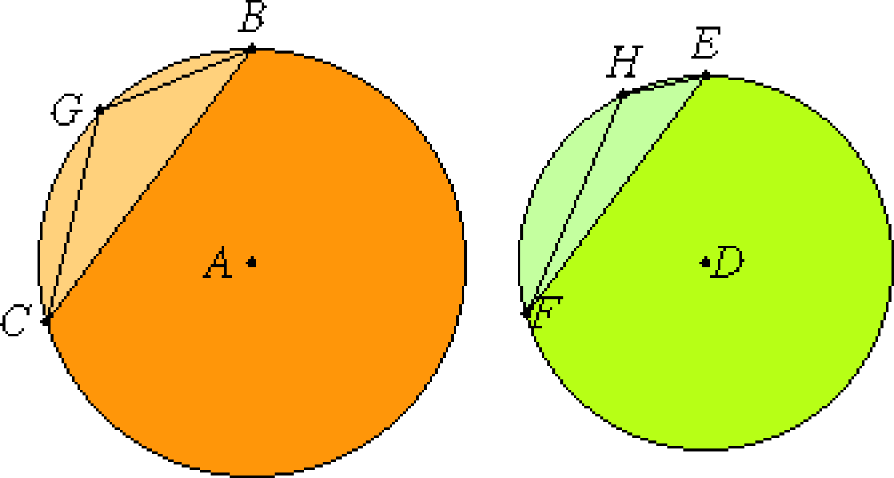
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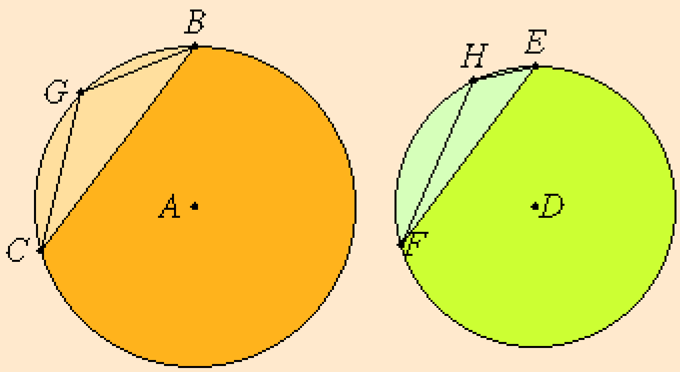
Euclid's Elements

Book III

Definition 11

Similar segments of circles are those which admit equal angles, or in which the angles equal one another.

Guide



Since the segments *BGC* and *EHF* have in them equal angles *BGC* and *EHF*, this definition declares them to be similar segments.

This is hardly a proper definition considering that proposition III.21 has yet to be proved in which it is shown that all the angles in one segment are equal.

Next: Proposition III.1

Previous: Definition III.10

Book III introduction

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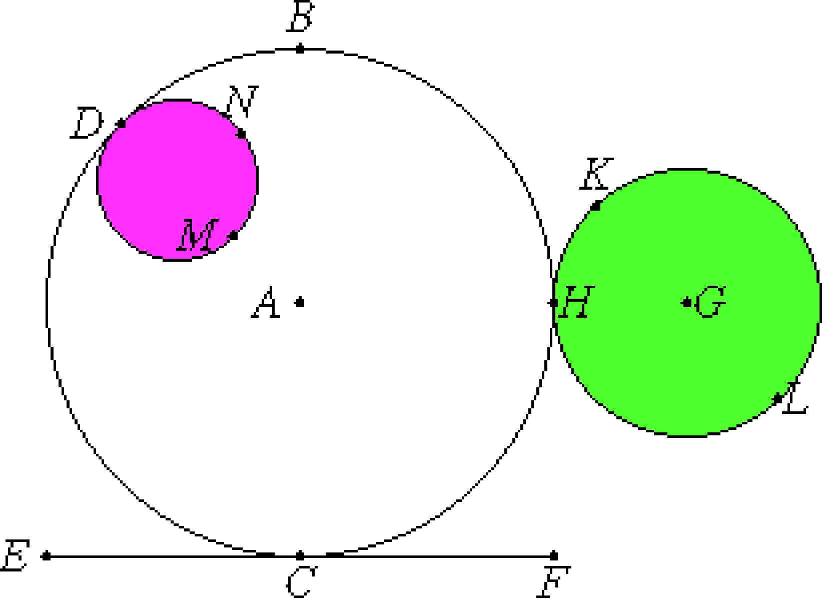
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Euclid's Elements

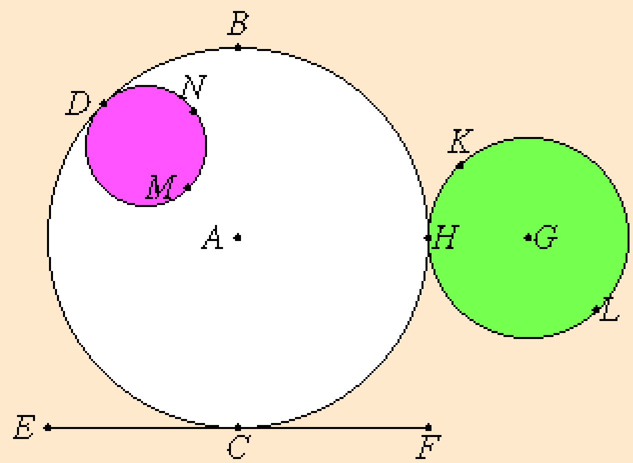
Book III

Definitions 2 and 3

Def. 2. A straight line is said to *touch* a circle which, meeting the circle and being produced, does not cut the circle.

Def. 3. Circles are said to *touch* one another which meet one another but do not cut one another.

Guide



Consider circle *BCD* in the figure. The line *EF* touches this circle at the point *C*. Another expression for the same thing is that *EF* is tangent to the circle at *C*.

Two circles can touch each other either internally or externally. Circle *HKL* touches circle *BCD* externally, while circle *DMN* touches circle *BCD* internally.

Next definitions: III.Def.4-5

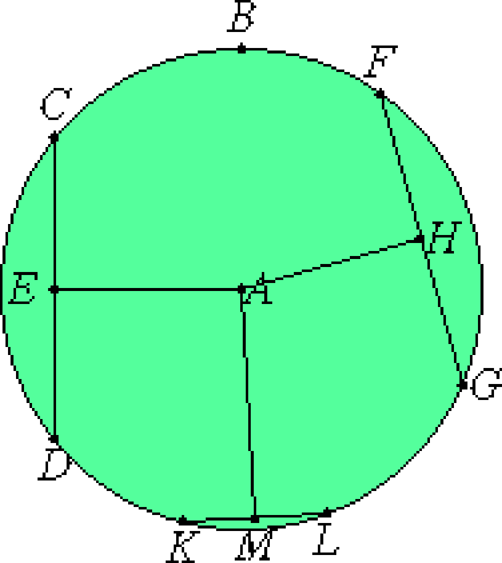
Previous: III.Def.1

Book III introduction

Select from Book III

Select book

Select topic


B
F
C
H
E
A
G
D
K
M
L

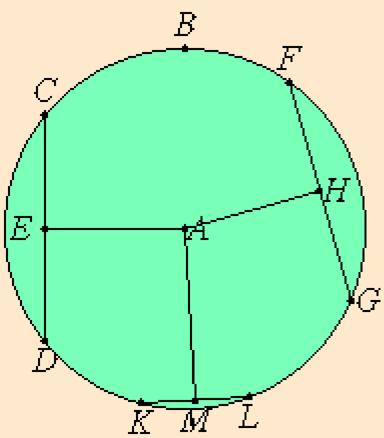
Euclid's Elements
Book III
Definitions 4 and 5

Def. 4. Straight lines in a circle are said to be *equally distant* from the center when the perpendiculars drawn to them from the center are equal.

Def. 5. And that straight line is said to be at a *greater distance* on which the greater perpendicular falls.

Guide

These definitions could have been broadened to distances from a line to a point, but Euclid's needs are for this situation.



The perpendiculars *AE* and *AH* drawn from the center *A* to the lines *CD* and *FG* respectively are equal, so the lines *CD* and *FG* are equally distant from the center. As the perpendicular *AM* is greater, the line *KL* is at a greater distance from the center.

Next definitions: III.Def.6-9

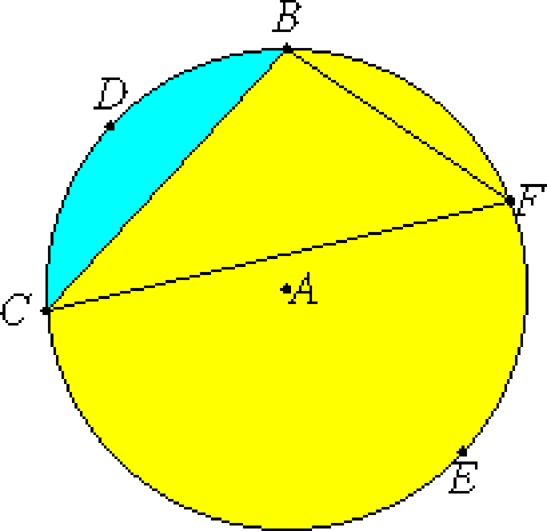
Previous: III.Def.2-3

Book III introduction

Select from Book III

Select book

Select topic



B
D
F
A
C
E

Euclid's Elements

Book III

Definitions 6 through 9

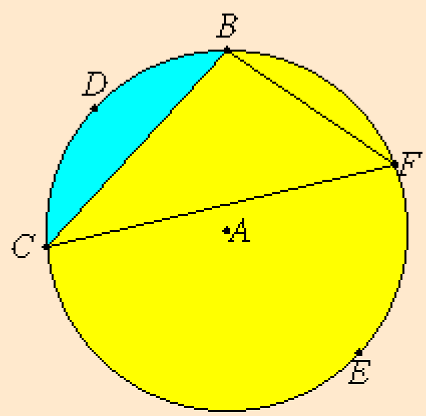
Def. 6. A *segment* of a circle is the figure contained by a straight line and a circumference of a circle.

Def. 7. An *angle of a segment* is that contained by a straight line and a circumference of a circle.

Def. 8. An *angle in a segment* is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the ends of the straight line which is the base of the segment, is contained by the straight lines so joined.

Def. 9. And, when the straight lines containing the angle cut off a circumference, the angle is said to *stand upon* that circumference.

Guide



A line in a circle, such as the line *BC*, divides the circle into two segments, the small blue segment *BDC*, and the large yellow segment *BEC*.

An angle *of* the segment *BDC* is not a rectilinear angle, since only one of its sides, *BC*, is a straight line. The other side is curved, namely, an arc of a circle. These angles of segments only appear in proposition III.16, and are not important in Euclid's development of geometry.

An example of an angle *in* a segment is the angle *BFC* in the yellow segment *BEC*. This angle *BFC* stands upon the circumference (arc) *BDC*. Angles *in* segments are rectilinear, and they are important. In proposition III.21, Euclid proves that all the angles in a given segment are equal.

Next definition: III.Def.10

Previous: III.Def.4-5

Book III introduction

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[D.E.Joyce](#)
[Clark University](#)

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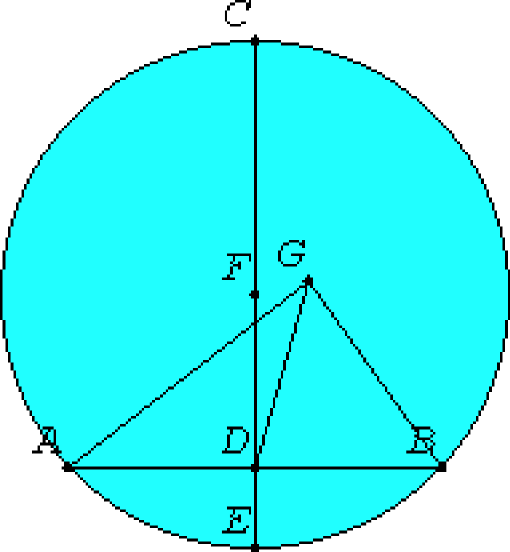
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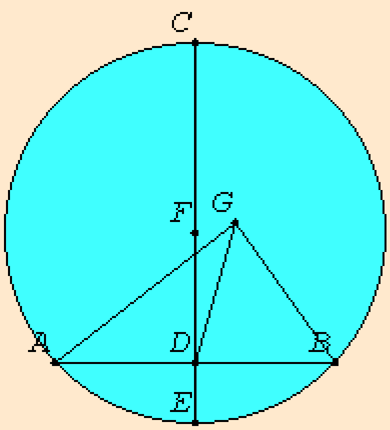
Euclid's Elements
Book III
Proposition 1

To find the center of a given circle.

Let *ABC* be the given circle.

It is required to find the center of the circle *ABC.*



Draw a straight line *AB* through it at random, and bisect it at the point *D.* Draw *DC* from *D* at right angles to *AB,* and draw it through to *E.* Bisect *CE* at *F.* I.10 I.11 I.10

I say that *F* is the center of the circle *ABC.*

For suppose it is not, but, if possible, let *G* be the center. Join *GA, GD,* and *GB.*

Then, since *AD* equals *DB,* and *DG* is common, the two sides *AD* and *DG* equal the two sides *BD* and *DG* respectively. And the base *GA* equals the base *GB,* for they are radii, therefore the angle *ADG* equals the angle *GDB.* I.Def.15 I.8

But, when a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, therefore the angle *GDB* is right. I.Def.10

But the angle *FDB* is also right, therefore the angle *FDB* equals the angle *GDB,* the greater equals the less, which is impossible. Therefore *G* is not the center of the circle *ABC.*

Similarly we can prove that neither is any other point except *F.*

Therefore the point *F* is the center of the circle *ABC.*

Q.E.F.

Corollary

From this it is manifest that *if in a circle a straight line cuts a straight line into two equal parts and at right angles, then the center of the circle lies on the cutting straight line.*

Guide

Since the definition of a circle, I.Def.15, includes the existence of a center, Euclid is justified in taking a point *G* as the center.

In this proof *G* is shown to lie on the perpendicular bisector of the line *AB.* He leaves to the reader to show that *G* actually is the point *F* on the perpendicular bisector, but that's clear since only the midpoint *F* is equidistant from the two points *C* and *E* on the circle. From that observation it also follows that the center of a circle is unique, although the uniqueness can easily be proved in other ways.

As Todhunter remarked, Euclid implicitly assumes that the perpendicular bisector of *AB* actually intersects the circle in points *C* and *E*.

Use of this proposition and its corollary

About half the proofs in Book III and several of those in Book IV begin with taking the center of a circle, but in plane geometry, it isn't necessary to invoke this proposition III.1 since the only way that circles can occur is if they are constructed around a center to begin with. Even in solid geometry, the center of a circle is usually known so that III.1 isn't necessary. Indeed, that is the case whenever the center is needed in Euclid's books on solid geometry (see XI.23, XIII.9 through XIII.13, and XIII.16). Sections of spheres cut by planes are also circles as are certain sections of cylinders and cones, but in these cases too, the centers can easily be found without recourse to III.1. Thus, III.1 redundant, although it is an interesting construction.

The corollary is used in propositions III.9 and III.10.

Next proposition: III.2

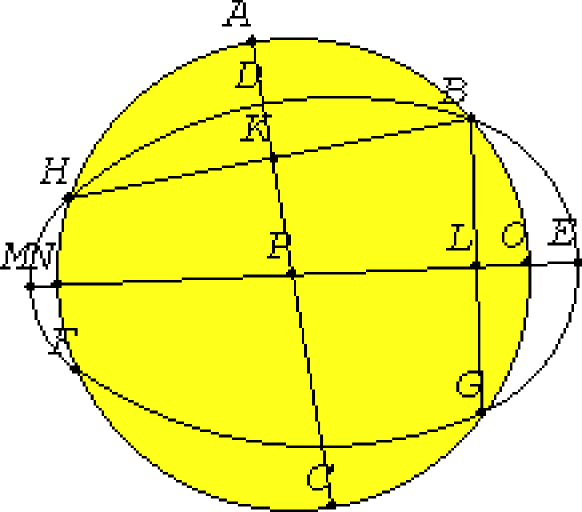
Previous: III.Def.11

Book III introduction

Select from Book III

Select book

Select topic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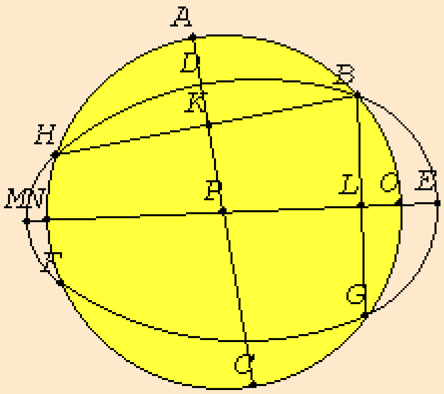
Euclid's Elements

Book III

Proposition 10

A circle does not cut a circle at more than two points.

For, if possible, let the circle *ABC* cut the circle *DEF* at more points than two, namely *B, G, F*, and *H*.



Join *BH* and *BG*, and bisect them at the points *K* and *L*. Draw *KC* and *LM* from *K* and *L* at right angles to *BH* and *BG*, and carry them through to the points *A* and *E*. I.10 I.11

Then, since in the circle *ABC* a straight line *AC* cuts a straight line *BH* into two equal parts and at right angles, the center of the circle *ABC* lies on *AC*. Again, since in the same circle *ABC* a straight line *NO* cuts a straight line *BG* into two equal parts and at right angles, the center of the circle *ABC* lies on *NO*. III.1,Cor

But it was also proved to lie on *AC*, and the straight lines *AC* and *NO* meet at no point except at *P*, therefore the point *P* is the center of the circle *ABC*.

Similarly we can prove that *P* is also the center of the circle *DEF*, therefore the two circles *ABC* and *DEF* which cut one another have the same center *P*, which is impossible. III.5

Therefore *a circle does not cut a circle at more than two points.*

Q.E.D.

Guide

The figure is another impossible figure. Both curves are supposed to be circumferences of circles, but, of course, they cannot both be drawn as circles since the situation is proved not to occur. Although Euclid names four points where the circles meet, only three, *B, G,* and *H,* are used in the proof.

The proof actually shows that the two circles cannot *meet* in more than two points, where "meet" could be either cut or touch.

Heath remarks that the lines bisecting *BG* and *BH* have not been shown to meet. In fact, they have, since the center of the circle *ABC* has been shown to be on both.

This proposition is used in III.24.

Next proposition: III.11

Previous: III.9

Book III introduction

Select from Book III

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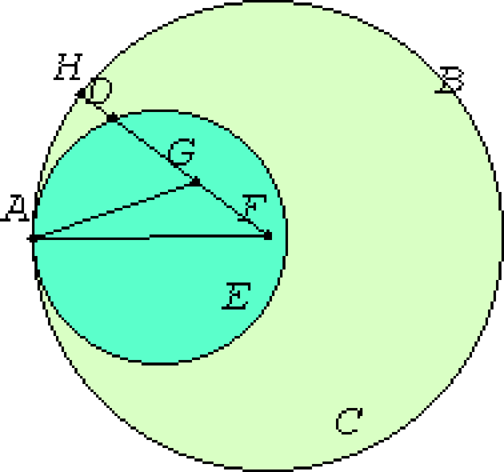
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Euclid's Elements

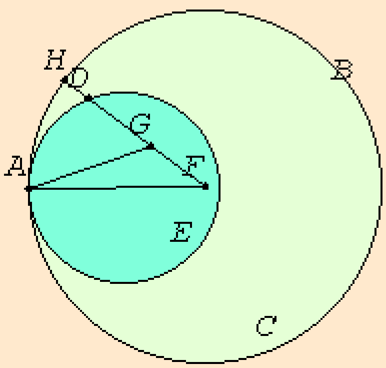
Book III

Proposition 11

If two circles touch one another internally, and their centers are taken, then the straight line joining their centers, being produced, falls on the point of contact of the circles.

Let the two circles *ABC* and *ADE* touch one another internally at the point *A*, and let the centers *F* and *G* of the circles *ABC* and *ADE* be taken. III.1

I say that the straight line joined from *G* to *F* and produced falls on *A*.



For suppose it does not, but, if possible, let it fall as *FGH*. Join *AF* and *AG*.

Then, since the sum of *AG* and *GF* is greater than *FA*, that is, than *FH*, subtract *FG* from each, therefore the remainder *AG* is greater than the remainder *GH*. I.20

But *AG* equals *GD*, therefore *GD* is also greater than *GH*, the less greater than the greater, which is impossible.

Therefore the straight line joined from *F* to *G* does not fall outside. Therefore it falls on *A*, the point of contact.

Therefore *if two circles touch one another internally, and their centers are taken, then the straight line joining their centers, being produced, falls on the point of contact of the circles.*

Q.E.D.

Guide

In order to carry through the proof, in particular so that *FA* = *FH*, the circle *ABC* needs to be the larger circle.

Various conclusions in the proof are based on the figure rather than rigorous deductive reasoning. Camerer and others have suggested ways of filling the gaps.

This proposition is used in III.13.

Next proposition: III.12

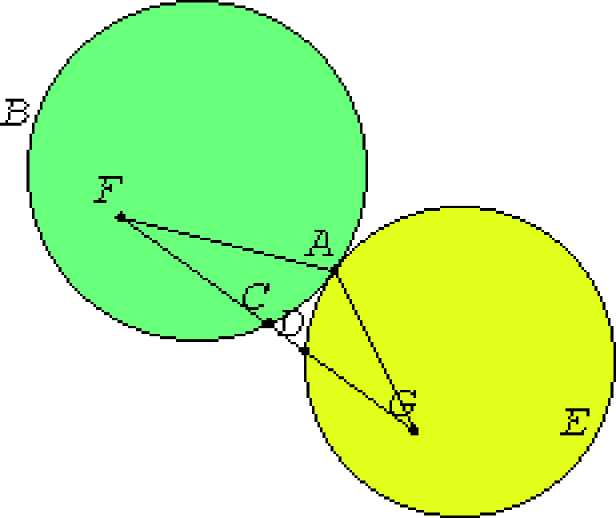
Previous: III.10

Book III introduction

Select from Book III

Select book

Select topic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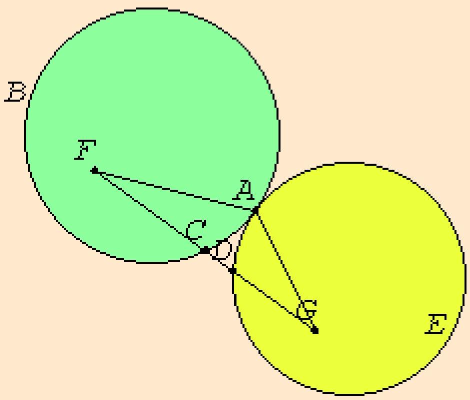
Euclid's Elements
Book III
Proposition 12

If two circles touch one another externally, then the straight line joining their centers passes through the point of contact.

Let the two circles *ABC* and *ADE* touch one another externally at the point *A*. Take the center *F* of *ABC*, and the center *G* of *ADE*. III.1

I say that the straight line joined from *F* to *G* passes through the point of contact at *A*.



For suppose it does not, but, if possible, let it pass as *FCDG*. Join *AF* and *AG*.

Then, since the point *F* is the center of the circle *ABC*, *FA* equals *FC*.

Again, since the point *G* is the center of the circle *ADE*, *GA* equals GD.

But *FA* was also proved equal to *FC*, therefore *FA* and *AG* equal *FC* and *GD*, so that the whole *FG* is greater than *FA* and *AG*, but it is also less, which is impossible. I.20

Therefore the straight line joined from *F* to *G* does not fail to pass through the point of contact at *A*, therefore it passes through it.

Therefore *if two circles touch one another externally, then the straight line joining their centers passes through the point of contact.*

Q.E.D.

Guide

This proposition was certainly added to the *Elements* after Euclid, perhaps by Heron or a later commentator.

This proposition is not used in the rest of the *Elements*.

Next proposition: III.13

Previous: III.11

Book III introduction

Select from Book III

Select book

Select topic

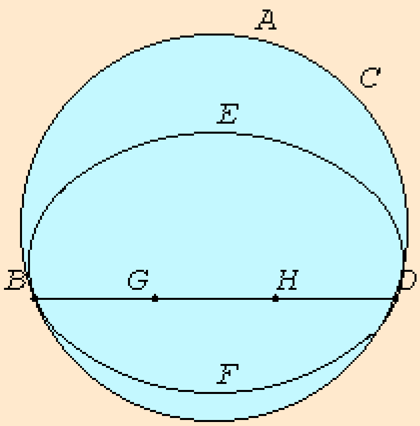
D.E.Joyce
Clark University

Euclid's Elements
Book III
Proposition 13

A circle does not touch another circle at more than one point whether it touches it internally or externally.

For, if possible, let the circle *ABDC* touch the circle *EBFD,* first internally, at more points than one, namely *D* and *B.*

Take the center *G* of the circle *ABDC* and the center *H* of *EBFD.* III.1



Therefore the straight line joined from *G* to *H* falls on *B* and *D.* III.11

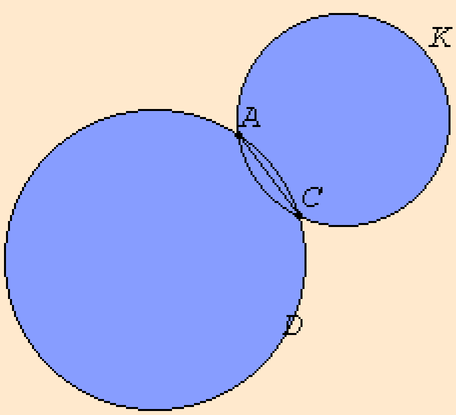
Let it so fall, as *BGHD.*

Then, since the point *G* is the center of the circle *ABCD* and *BG* equals *GD,* therefore *BG* is greater than *HD.* Therefore *BH* is much greater than HD.

Again, since the point *H* is the center of the circle *EBFD, BH* equals *HD,* but it was also proved much greater than it, which is impossible.

Therefore a circle does not touch a circle internally at more points than one.

I say further that neither does it so touch it externally.



For, if possible, let the circle *ACK* touch the circle *ABDC* at more points than one, namely *A* and *C.* Join *AC.*

Then, since on the circumference of each of the circles *ABDC* and *ACK* two points *A* and *C* have been taken at random, the straight line joining the points falls within each circle, but it fell within the circle *ABCD* and outside *ACK,* which is absurd. III.2 III.Def.3

Therefore a circle does not touch a circle externally at more points than one.

And it was proved that neither does it so touch it internally.

Therefore *a circle does not touch another circle at more than one point whether it touches it internally or externally.*

Q.E.D.

Guide

In the second impossible figure there are three curves connecting *A* to *C*. The two circles are not supposed to cut each other, but just to touch each other at the two points *A* and *C*, and the straight line *AC* should lie between the two circles and not within either one.

There are logical flaws in this proof similar to those in the last two proofs.

This proposition is not used in the rest of the *Elements*.

Next proposition: III.14

Previous: III.12

Book III introduction

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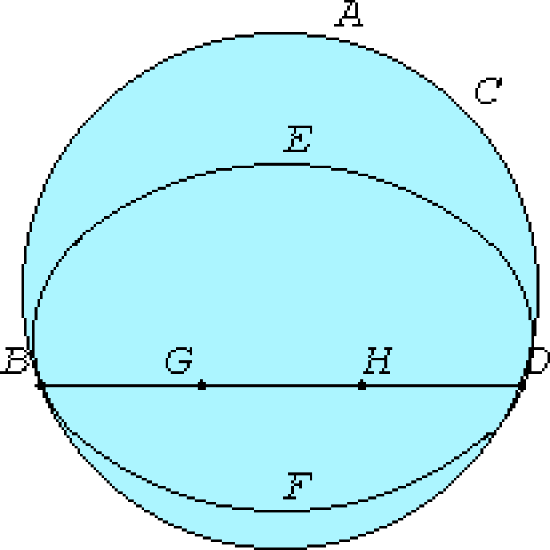
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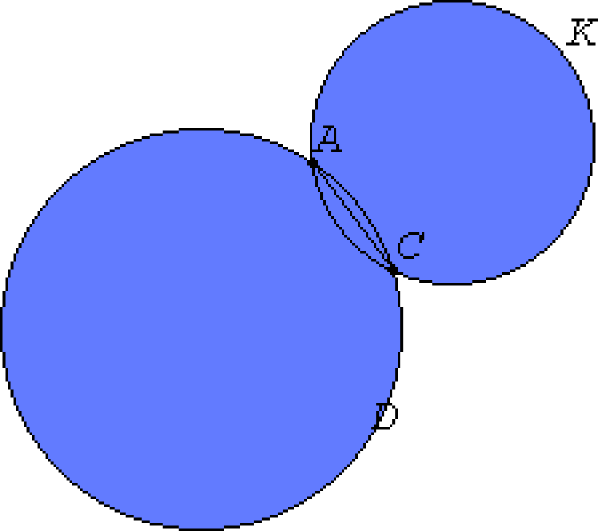

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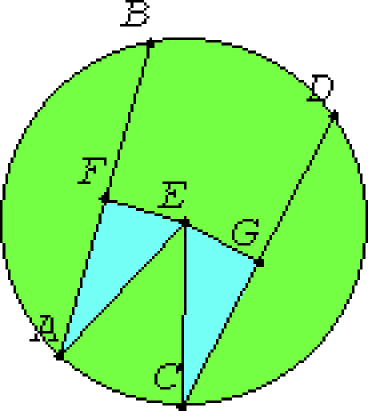
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Euclid's Elements

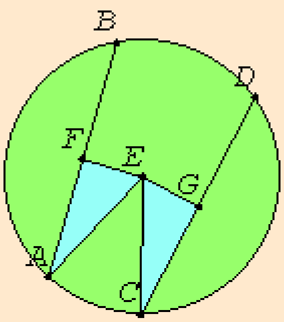
Book III

Proposition 14

Equal straight lines in a circle are equally distant from the center, and those which are equally distant from the center equal one another.

Let *AB* and *CD* be equal straight lines in a circle *ABDC.*

I say that *AB* and *CD* are equally distant from the center.



Take the center *E* of the circle *ABDC.* Draw *EF* and *EG* from *E* perpendicular to *AB* and *CD,* and join *AE* and *EC.* III.1 I.12

Then, since a straight line *EF* passing through the center cuts a straight line *AB* not passing through the center at right angles, it also bisects it. Therefore *AF* equals *FB.* Therefore *AB* is double *AF.* III.3

For the same reason *CD* is also double *CG.* But *AB* equals *CD,* therefore *AF* also equals *CG.*

Also, since *AE* equals *EC,* the square on *AE* also equals the square on *EC.* But the sum of the squares on *AF* and *EF* equals the square on *AE,* for the angle at *F* is right, and the sum of the squares on *EG* and *GC* equals the square on *EC,* for the angle at *G* is right. Therefore the sum of the squares on *AF* and *FE* equals the sum of the squares on *CG* and *GE,* of which the square on *AF* equals the square on *CG,* for *AF* equals *CG.* Therefore the remaining square on *FE* equals the square on *EG.* Therefore *EF* equals *EG.* I.47

But straight lines in a circle are said to be equally distant from the center when the perpendiculars drawn to them from the center are equal. Therefore *AB* and *CD* are equally distant from the center. III.Def.4

Next, let the straight lines *AB* and *CD* be equally distant from the center, that is, let *EF* equal *EG.*

I say that *AB* also equals *CD.*

For, with the same construction, we can prove, as before, that *AB* is double *AF,* and *CD* double *CG.* And, since *AE* equals *CE,* the square on *AE* equals the square on *CE.* But the sum of the squares on *EF* and *FA* equals the square on *AE,* and the sum of the squares on *EG* and *GC* equals the square on *CE.* I.47

Therefore the sum of the squares on *EF* and *FA* equals the sum of the squares on *EG* and *GC,* of which the square on *EF* equals the square on *EG,* for *EF* equals *EG.* Therefore the remaining square on *AF* equals the square on *CG.* Therefore *AF* equals *CG.* And *AB* is double *AF,* and *CD* double *CG,* therefore *AB* equals *CD.*

Therefore *equal straight lines in a circle are equally distant from the center, and those which are equally distant from the center equal one another.*

Q.E.D.

Guide

Note how Euclid has proved twice in the course of this proof the side-side-right angle congruence theorem. See the note after I.26 about congruence theorems for triangles.

This proposition is used in the next one.

Next proposition: III.15

Previous: III.13

Book III introduction

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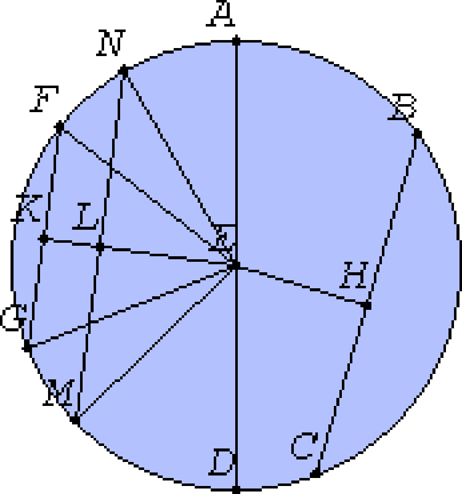
Clark University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Euclid's Elements

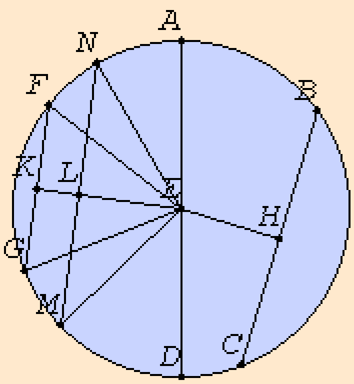
Book III

Proposition 15

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the center is always greater than the more remote.

Let *ABCD* be a circle, *AD* its diameter, and *E* its center. Let *BC* be nearer to the center *AD*, and *FG* more remote.

I say that *AD* is greatest and *BC* greater than *FG*.



Draw *EH* and *EK* from the center *E* perpendicular to *BC* and *FG*. I.12

Then, since *BC* is nearer to the center and *FG* more remote, *EK* is greater than *EH*. III.Def.5

Make *EL* equal to *EH*. Draw *LM* through *L* at right angles to *EK*, and carry it through to *N*. Join *ME*, *EN*, *FE*, and *EG*. I.3 I.11

Then, since *EH* equals *EL*, *BC* also equals *MN*. III.14

Again, since *AE* equals *EM*, and *ED* equals *EN*, *AD* equals the sum of *ME* and *EN*.

But the sum of *ME* and *EN* is greater than *MN*, and *MN* equals *BC*, therefore *AD* is greater than *BC*. I.20

And, since the two sides *ME* and *EN* equal the two sides *FE* and *EG*, and the angle *MEN* greater than the angle *FEG*, therefore the base *MN* is greater than the base *FG*. I.24

But *MN* was proved equal to *BC*.

Therefore the diameter *AD* is greatest and *BC* greater than *FG*.

Therefore *of straight lines in a circle the diameter is greatest, and of the rest the nearer to the center is always greater than the more remote.*

Q.E.D.

Guide

This proposition is not used in the rest of the *Elements.*

Next proposition: III.16

Previous: III.14

Select from Book III

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Book III introduction

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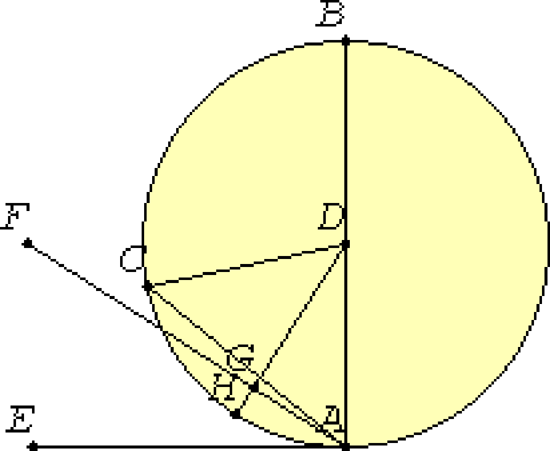
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Euclid's Elements

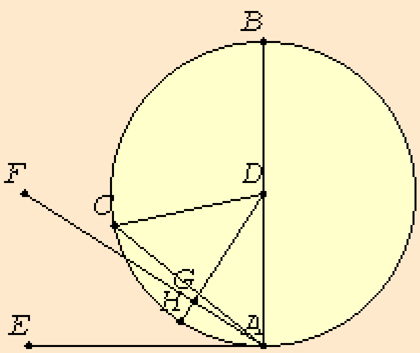
Book III

Proposition 16

The straight line drawn at right angles to the diameter of a circle from its end will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed, further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Let *ABC* be a circle about *D* as center and *AB* as diameter.

I say that the straight line drawn from *A* at right angles to *AB* from its end will fall outside the circle.



For suppose it does not, but, if possible, let it fall within as *CA*, and join *DC*.

Since *DA* equals *DC*, the angle *DAC* also equals the angle *ACD*. I.5

But the angle *DAC* is right, therefore the angle *ACD* is also right. Thus, in the triangle *ACD*, the two angles *DAC* and *ACD* equal two right angles, which is impossible. I.17

Therefore the straight line drawn from the point *A* at right angles to *BA* will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference, therefore it will fall outside.

Let it fall as *AE*.

I say next that into the space between the straight line *AE* and the circumference *CHA* another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as *FA*. Draw *DG* from the point *D* perpendicular to *FA*. I.12

Then, since the angle *AGD* is right, and the angle *DAG* is less than a right angle, *AD* is greater than *DG*. I.17 I.19

But *DA* equals *DH*, therefore *DH* is greater than *DG*, the less greater than the greater, which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line *BA* and the circumference *CHA* is greater than any acute rectilinear angle, and the remaining angle contained by the circumference *CHA* and the straight line *AE* is less than any acute rectilinear angle.

For, if there is any rectilinear angle greater than the angle contained by the straight line *BA* and the circumference *CHA*, and any rectilinear angle less than the angle contained by the circumference *CHA* and the straight line *AE*, then into the space between the circumference and the straight line *AE* a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line *BA* and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line *AE*.

But such a straight line cannot be interposed, therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line *BA* and the circumference *CHA*, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference *CHA* and the straight line *AE*. Above

Therefore *the straight line drawn at right angles to the diameter of a circle from its end will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed, further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.*

Q.E.D.

Corollary

From this it is manifest that *the straight line drawn at right angles to the diameter of a circle from its end touches the circle.*

Guide

This proposition is used in the proof of proposition IV.4 and two others in Book IV. The corollary is used in III.33, III.37, a few propositions in Book IV, and XII.16.

Next proposition: III.17

Previous: III.15

Book III introduction

Select from Book III

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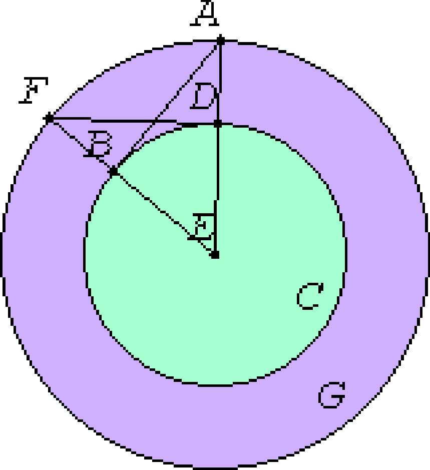
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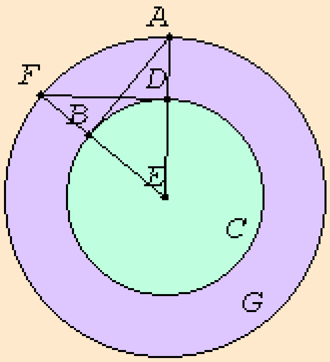
Euclid's Elements

Book III

Proposition 17

From a given point to draw a straight line touching a given circle.

Let *A* be the given point, and *BCD* the given circle.



It is required to draw from the point *A* a straight line touching the circle *BCD*.

Take the center *E* of the circle, and join *AE*. Describe the circle *AFG* with center *E* and radius *EA*. Draw *DF* from *D* at right angles to *EA*. Join *EF* and *AB*. III.1 I.11

I say that *AB* has been drawn from the point *A* touching the circle *BCD*.

For, since *E* is the center of the circles *BCD* and *AFG*, *EA* equals *EF*, and *ED* equals *EB*. Therefore the two sides *AE* and *EB* equal the two sides *FE* and *ED*, and they contain a common angle, the angle at *E*, therefore the base *DF* equals the base *AB*, and the triangle *DEF* equals the triangle *BEA*, and the remaining angles to the remaining angles, therefore the angle *EDF* equals the angle *EBA*. I.4

But the angle *EDF* is right, therefore the angle *EBA* is also right.

Now *EB* is a radius, and the straight line drawn at right angles to the diameter of a circle, from its end, touches the circle, therefore *AB* touches the circle *BCD*. III.16,Cor

Therefore from the given point *A* the straight line *AB* has been drawn touching the circle *BCD*.

Q.E.F.

Guide

The construction in this proposition is used in propositions III.34 and XII.2.

Next proposition: III.18

Previous: III.16

Book III introduction

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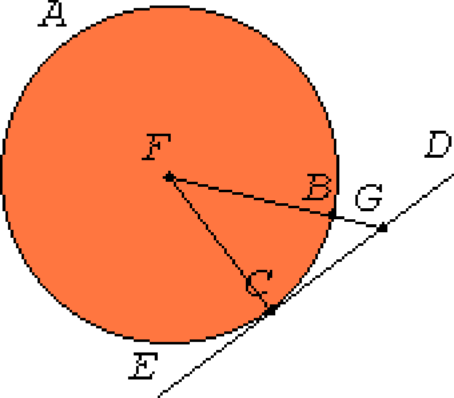
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Euclid's Elements

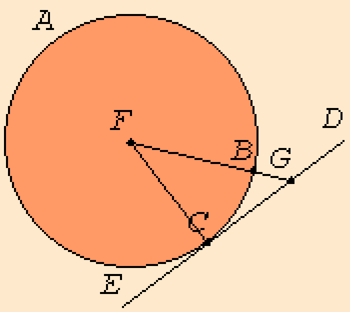
Book III

Proposition 18

If a straight line touches a circle, and a straight line is joined from the center to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line *DE* touch the circle *ABC* at the point *C*. Take the center *F* of the circle *ABC*, and join *FC* from *F* to *C*. III.1

I say that *FC* is perpendicular to *DE*.



For, if not, draw *FG* from *F* perpendicular to *DE*. I.12

Then, since the angle *FGC* is right, the angle *FCG* is acute, and the side opposite the greater angle is greater, therefore *FC* is greater than *FG*. I.17 I.19

But *FC* equals *FB*, therefore *FB* is also greater than *FG*, the less greater than the greater, which is impossible.

Therefore *FG* is not perpendicular to *DE*.

Similarly we can prove that neither is any other straight line except *FC*. Therefore *FC* is perpendicular to *DE*.

Therefore *if a straight line touches a circle, and a straight line is joined from the center to the point of contact, the straight line so joined will be perpendicular to the tangent.*

Q.E.D.

Guide

This proposition is used in a few propositions in Books III and IV beginning with III.36.

Next proposition: III.19

Previous: III.17

Book III introduction

Select from Book III

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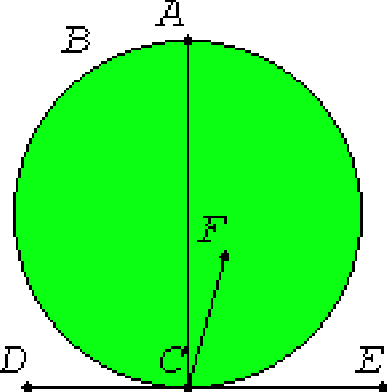
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Euclid's Elements

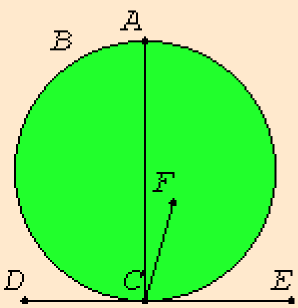
Book III

Proposition 19

If a straight line touches a circle, and from the point of contact a straight line is drawn at right angles to the tangent, the center of the circle will be on the straight line so drawn.

For let a straight line *DE* touch the circle *ABC* at the point *C*. Draw *CA* from *C* at right angles to *DE*. I.11

I say that the center of the circle is on *AC*.



For suppose it is not, but, if possible, let *F* be the center, and join *CF*.

Since a straight line *DE* touches the circle *ABC*, and *FC* has been joined from the center to the point of contact, *FC* is perpendicular to *DE*. Therefore the angle *FCE* is right. III.18

But the angle *ACE* is also right, therefore the angle *FCE* equals the angle *ACE*, the less equals the greater, which is impossible.

Therefore *F* is not the center of the circle *ABC*.

Similarly we can prove that neither is any other point except a point on *AC*.

Therefore *if a straight line touches a circle, and from the point of contact a straight line is drawn at right angles to the tangent, the center of the circle will be on the straight line so drawn.*

Q.E.D.

Guide

This proposition is used in proposition III.32.

Next proposition: III.20

Previous: III.18

Book III introduction

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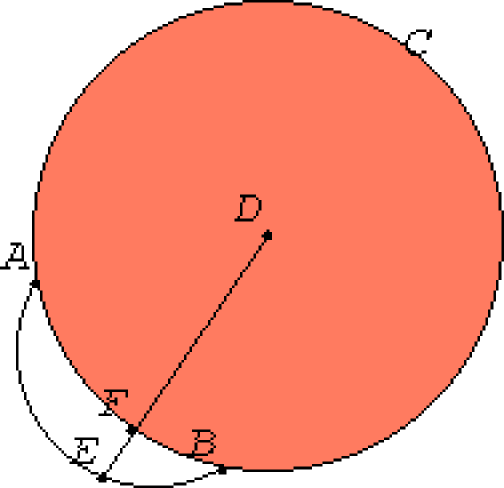
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Euclid's Elements

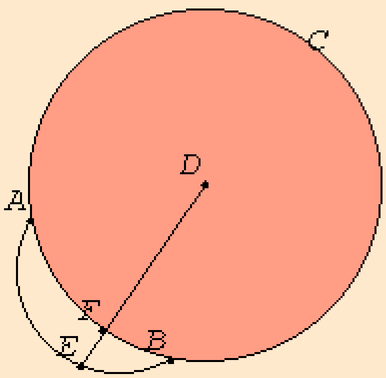
Book III

Proposition 2

If two points are taken at random on the circumference of a circle, then the straight line joining the points falls within the circle.

Let *ABC* be a circle, and let two points *A* and *B* be taken at random on its circumference.

I say that the straight line joined from *A* to *B* falls within the circle.



For suppose it does not, but, if possible, let it fall outside, as *AEB.* Take the center *D* of the circle *ABC.* Join *DA* and *DB,* and draw *DFE* through. III.1

I.Def.15

Then, since *DA* equals *DB,* the angle *DAE* also equals the angle *DBE.* I.5

And, since one side *AEB* of the triangle *DAE* is produced, the angle *DEB* is greater than the angle *DAE.* I.16

And the angle *DAE* equals the angle *DBE,* therefore the angle *DEB* is greater than the angle *DBE.* And the side opposite the greater angle is greater, therefore *DB* is greater than *DE.* But *DB* equals *DF,* therefore *DF* is greater than *DE,* the less greater than the greater, which is impossible. I.19 I.Def.15

Therefore the straight line joined from *A* to *B* does not fall outside the circle.

Similarly we can prove that neither does it fall on the circumference itself, therefore it falls within.

Therefore *if two points are taken at random on the circumference of a circle, then the straight line joining the points falls within the circle.*

Q.E.D.

Guide

The figure for this proposition is rather strange, but that is necessary since it refers to a hypothetical situation which is shown to be impossible. In this figure *AEB* is supposed to be a straight line that lies on outside the circle. There are other impossible figures in later propositions in this Book.

That Euclid even has this proposition is remarkable. Of course, it should be included, but there are equally obvious statements (but difficult to prove) left out in earlier books. For instance, that the two circles constructed in a plane on a line *AB* intersect is not proved although it is used in I.1. This indicates that more care has been given to the foundations for this book than for the previous books.

Euclid leaves to the reader to prove that *AB* cannot lie *on* the circumference, and that is not particularly difficult to prove.

This propostion is used in the next one.

Next proposition: III.3

Previous: III.1

Book III introduction

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Select book

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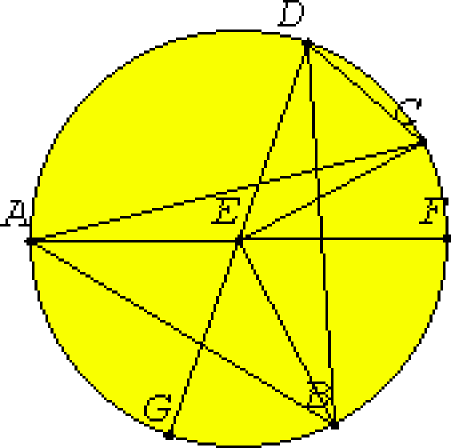
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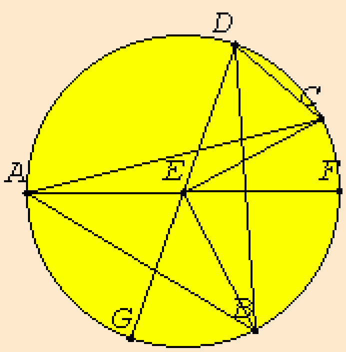
Euclid's Elements
Book III
Proposition 20

In a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base.

Let *ABC* be a circle, let the angle *BEC* be an angle at its center, and the angle *BAC* an angle at the circumference, and let them have the same circumference *BC* as base.

I say that the angle *BEC* is double the angle *BAC*.



Join *AE*, and draw it through to *F*.

Then, since *EA* equals *EB*, the angle *EAB* also equals the angle *EBA*. Therefore the sum of the angles the angles *EAB* and *EBA* is double the angle *EAB*. I.5

But the angle *BEF* equals the sum of the angles *EAB* and *EBA*, therefore the angle *BEF*, is also double the angle *EAB*. I.32

For the same reason the angle *FEC* is also double the angle *EAC*.

Therefore the whole angle *BEC* is double the whole angle *BAC*.

Again let another straight line be inflected, and let there be another angle *BDC*. Join *DE* and produced it to *G*.

Similarly then we can prove that the angle *GEC* is double the angle *EDC*, of which the angle *GEB* is double the angle *EDB*. Therefore the remaining angle *BEC* is double the angle *BDC*.

Therefore *in a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base.*

Q.E.D.

Guide

This proposition is used in the next one, III.27, and VI.33.

Next proposition: III.21

Previous: III.19

Book III introduction

Select from Book III

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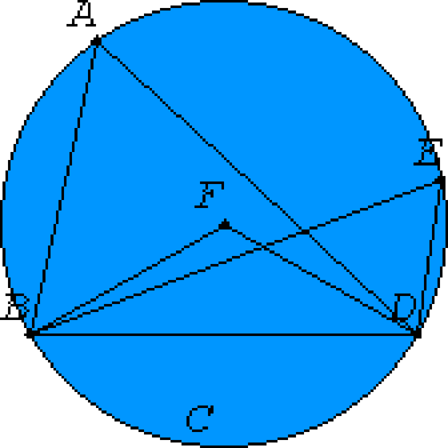
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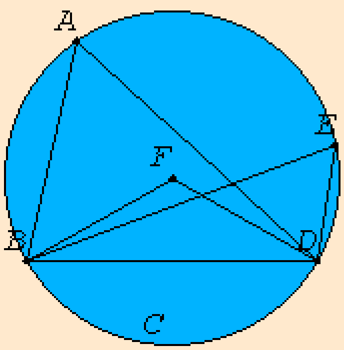
Euclid's Elements
Book III
Proposition 21

In a circle the angles in the same segment equal one another.

Let *ABCD* be a circle, and let the angles *BAD* and *BED* be angles in the same segment *BAED.*

I say that the angles *BAD* and *BED* equal one another.



Take the center *F* of the circle *ABCD,* and join *BF* and *FD.* III.1

Now, since the angle *BFD* is at the center, and the angle *BAD* at the circumference, and they have the same circumference *BCD* as base, therefore the angle *BFD* is double the angle *BAD.* III.20

For the same reason the angle *BFD* is also double the angle *BED.* Therefore the angle *BAD* equals the angle *BED.*

Therefore *in a circle the angles in the same segment equal one another.*

Q.E.D.

Guide

This proposition is used in the next one.

Next proposition: III.22

Previous: III.20

Book III introduction

Select from Book III

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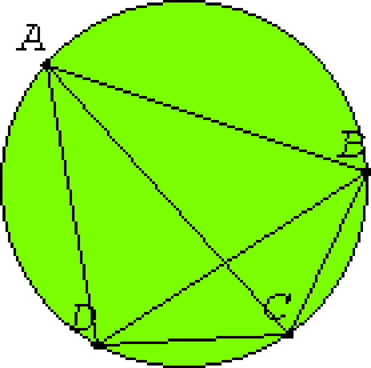
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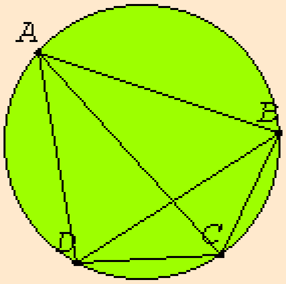
Euclid's Elements
Book III
Proposition 22

The sum of the opposite angles of quadrilaterals in circles equals two right angles.

Let *ABCD* be a circle, and let *ABCD* be a quadrilateral in it.

I say that the sum of the opposite angles equals two right angles.



Join *AC* and *BD.*

Then, since in any triangle the sum of the three angles equals two right angles, the sum of the three angles *CAB, ABC,* and *BCA* of the triangle *ABC* equals two right angles. I.32

But the angle *CAB* equals the angle *BDC*, for they are in the same segment *BADC*, and the angle *ACB* equals the angle *ADB,* for they are in the same segment *ADCB*, therefore the whole angle *ADC* equals the sum of the angles *BAC* and *ACB.* III.21

Add the angle *ABC* to each. Therefore the sum of the angles *ABC*, *BAC*, and *ACB* equals the sum of the angles *ABC* and *ADC*. But the sum of the angles *ABC*, *BAC*, and *ACB* equals two right angles, therefore the sum of the angles *ABC* and *ADC* also equal two right angles.

Similarly we can prove that the sum of the angles *BAD* and *DCB* also equals two right angles.

Therefore *the sum of the opposite angles of quadrilaterals in circles equals two right angles.*

Q.E.D.

Guide

This proposition is used in III.32.

Next proposition: III.23

Previous: III.21

Book III introduction

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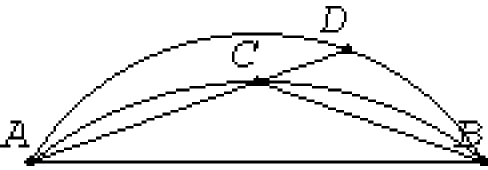
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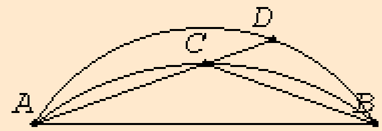
Euclid's Elements
Book III
Proposition 23

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possible, on the same straight line *AB* let two similar and unequal segments of circles *ACB* and *ADB* be constructed on the same side. Draw *ACD* through, and join *CB* and *DB.*



Then, since the segment *ACB* is similar to the segment *ADB,* and similar segments of circles are those which admit equal angles, the angle *ACB* equals the angle *ADB,* the exterior to the interior, which is impossible. III.Def.11 I.16

Therefore *on the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.*

Q.E.D.

Guide

This proposition is used in the next one.

Next proposition: III.24

Previous: III.22

Book III introduction

Select from Book III

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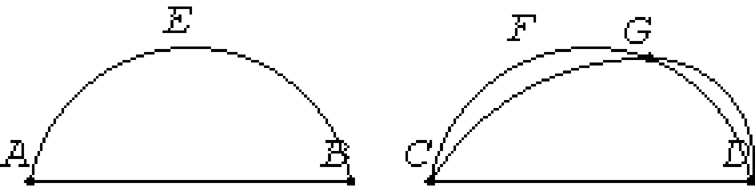
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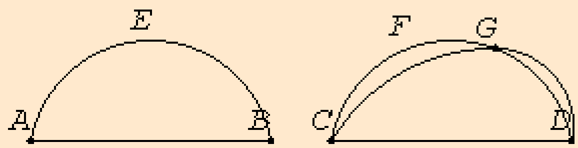
Euclid's Elements
Book III
Proposition 24

Similar segments of circles on equal straight lines equal one another.

Let *AEB* and *CFD* be similar segments of circles on equal straight lines *AB* and *CD*.

I say that the segment *AEB* equals the segment *CFD*.



For, if the segment *AEB* is superposed on *CFD*, and if the point *A* is placed on *C* and the straight line *AB* on *CD*, then the point *B* coincides with the point *D*, because *AB* equals *CD*, and, *AB* coinciding with *CD*, the segment *AEB* also coincides with *CFD*.

For, if the straight line *AB* coincides with *CD* but the segment *AEB* does not coincide with *CFD*, then it either falls within it, or outside it, or it falls awry, as *CGD*, and a circle cuts a circle at more points than two, which is impossible. III.23 III.10

Therefore, if the straight line *AB* is superposed on *CD*, then the segment *AEB* does not fail to coincide with *CFD* also, therefore it coincides with it and equals it. C.N.4

Therefore *similar segments of circles on equal straight lines equal one another.*

Q.E.D.

Guide

The proof here uses the method of superposition which was also used for I.4 and I.8.

Next proposition: III.25

Previous: III.23

Book III introduction

Select from Book III

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Euclid's Elements

Book III

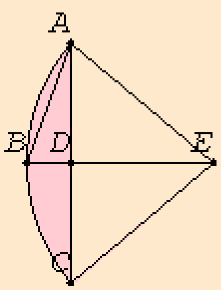
Proposition 25

Given a segment of a circle, to describe the complete circle of which it is a segment.

Let *ABC* be the given segment of a circle.

It is required to describe the complete circle belonging to the segment *ABC*, that is, of which it is a segment.

Bisect *AC* at *D*, draw *DB* from the point *D* at right angles to *AC*, and join *AB*. I.10 I.11



The angle *ABD* is then greater than, equal to, or less than the angle *BAD*.

First let it be greater. Construct the angle *BAE* on the straight line *BA*, and at the point *A* on it, equal to the angle *ABD*. Draw *DB* through to *E*, and join *EC*. I.23

Then, since the angle *ABE* equals the angle *BAE*, the straight line *EB* also equals *EA*. I.6

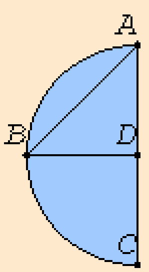
And, since *AD* equals *DC*, and *DE* is common, the two sides *AD* and *DE* equal the two sides *CD* and *DE* respectively, and the angle *ADE* equals the angle *CDE*, for each is right, therefore the base *AE* equals the base *CE*. I.4

But *AE* was proved equal to *BE*, therefore be also equals *CE*. Therefore the three straight lines *AE*, *EB*, and *EC* equal one another.

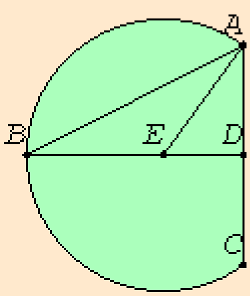
Therefore the circle drawn with center *E* and radius one of the straight lines *AE*, *EB*, or *EC* also passes through the remaining points and has been completed. III.9

Therefore, given a segment of a circle, the complete circle has been described.

And it is manifest that the segment *ABC* is less than a semicircle, because the center *E* happens to be outside it.



Similarly, even if the angle *ABD* equals the angle *BAD* and *AD* being equal to each of the two *BD* and *DC*, the three straight lines *DA*, *DB*, and *DC* will equal one another, *D* will be the center of the completed circle, and *ABC* will clearly be a semicircle.



But, if the angle *ABD* is less than the angle *BAD*, and if we construct, on the straight line *BA* and at the point *A* on it, an angle equal to the angle *ABD*, the center will fall on *DB* within the segment *ABC*, and the segment *ABC* will clearly be greater than a semicircle. I.23

Therefore, given a segment of a circle, the complete circle has been described.

Q.E.F.

Guide

The construction in this proposition is not used in the rest of the *Elements*.

Next proposition: III.26

Previous: III.24

Book III introduction

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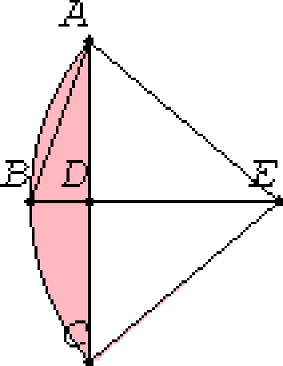
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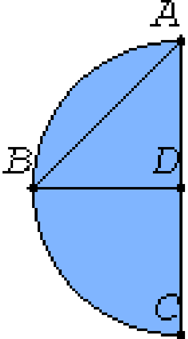
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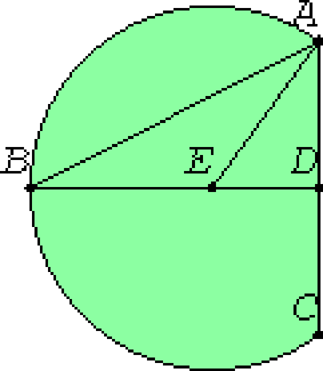
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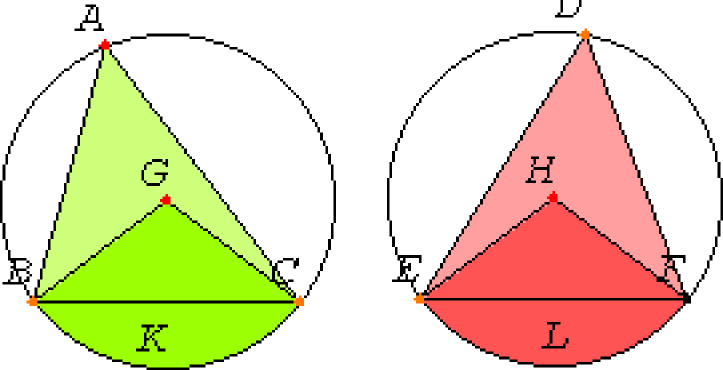
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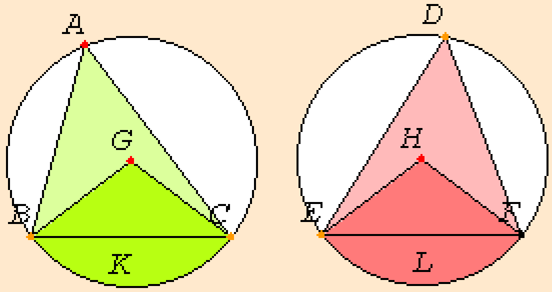
Euclid's Elements
Book III
Proposition 26

In equal circles equal angles stand on equal circumferences whether they stand at the centers or at the circumferences.

Let *ABC* and *DEF* be equal circles, and in them let there be equal angles, namely at the centers the angles *BGC* and *EHF*, and at the circumferences the angles *BAC* and *EDF*.

I say that the circumference *BKC* equals the circumference *ELF*.



Join *BC* and *EF*.

Now, since the circles *ABC* and *DEF* are equal, the radii are equal.

Thus the two straight lines *BG* and *GC* equal the two straight lines *EH* and *HF*, and the angle at *G* equals the angle at *H*, therefore the base *BC* equals the base *EF*. I.4

And, since the angle at *A* equals the angle at *D*, the segment *BAC* is similar to the segment *EDF*, and they are upon equal straight lines. III.Def.11

But similar segments of circles on equal straight lines equal one another, therefore the segment *BAC* equals *EDF*. But the whole circle *ABC* also equals the whole circle *DEF*, therefore the remaining circumference *BKC* equals the circumference *ELF*. III.24

Therefore *in equal circles equal angles stand on equal circumferences whether they stand at the centers or at the circumferences.*

Q.E.D.

Guide

This proposition is used in III.28, IV.11, IV,15, and XIII.10.

Next proposition: III.27

Previous: III.25

Book III introduction

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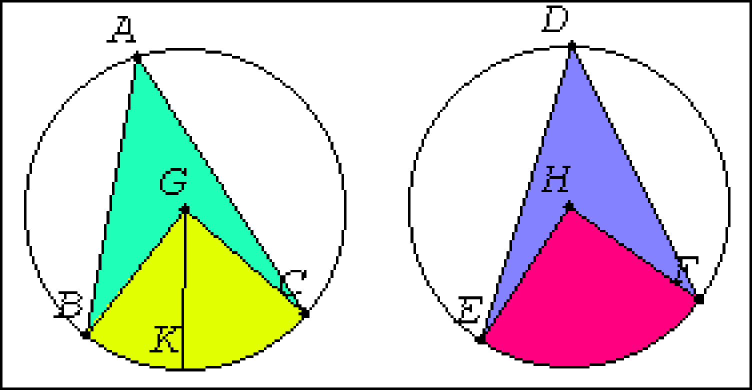

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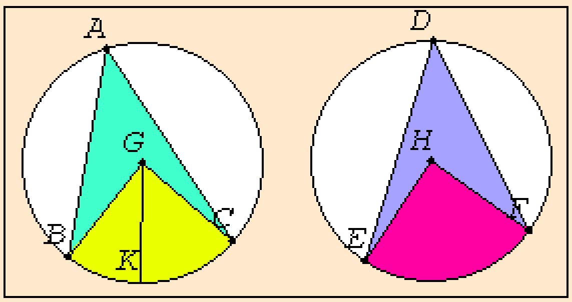
Book III

Proposition 27

In equal circles angles standing on equal circumferences equal one another whether they stand at the centers or at the circumferences.

For in equal circles *ABC* and *DEF,* on equal circumferences *BC* and *EF,* let the angles *BGC* and *EHF* stand at the centers *G* and *H,* and the angles *BAC* and *EDF* at the circumferences.

I say that the angle *BGC* equals the angle *EHF,* and the angle *BAC* equals the angle *EDF.*



For, if the angle *BGC* does not equal the angle *EHF,* one of them is greater. Let the angle *BGC* be greater. Construct the angle *BGK* equal to the angle *EHF* on the straight line *BG* and at the point *G* on it. I.23

Now equal angles stand on equal circumferences when they are at the centers, therefore the circumference *BK* equals the circumference *EF.* I.26

But *EF* equals *BC,* therefore *BK* also equals *BC,* the less equals the greater, which is impossible.

Therefore the angle *BGC* is not unequal to the angle *EHF,* therefore it equals it.

And the angle at *A* is half of the angle *BGC,* and the angle at *D* half of the angle *EHF,* therefore the angle at *A* also equals the angle at *D.* III.20

Therefore *in equal circles angles standing on equal circumferences equal one another whether they stand at the centers or at the circumferences.*

Q.E.D.

Guide

This proposition is used in a few propositions in Books III, IV, VI, and XII starting with III.29.

Next proposition: III.28

Previous: III.26

Book III introduction

Select from Book III

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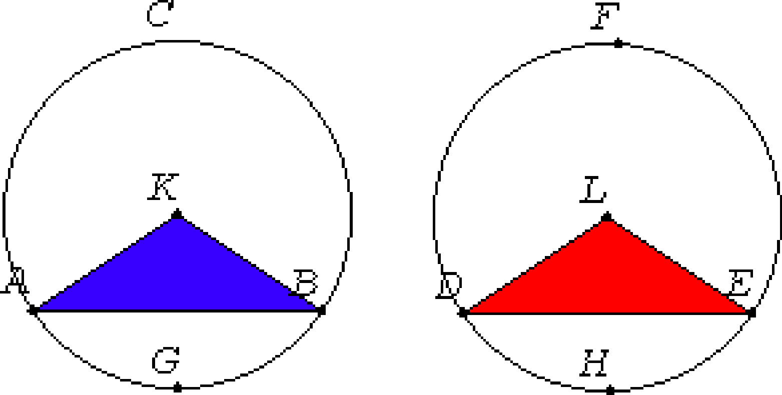
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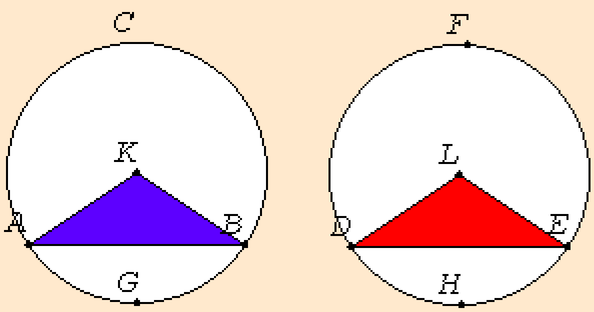
Euclid's Elements
Book III
Proposition 28

In equal circles equal straight lines cut off equal circumferences, the greater circumference equals the greater and the less equals the less.

Let *ABC* and *DEF* be equal circles, and in the circles let *AB* and *DE* be equal straight lines cutting off *ACB* and *DFE* as greater circumferences and *AGB* and *DHE* as lesser.



I say that the greater circumference *ACB* equals the greater circumference *DFE,* and the less circumference *AGB* equals *DHE.*

Take the centers *K* and *L* of the circles, and join *AK, KB, DL,* and *LE.* III.1

Now, since the circles are equal, the radii are also equal, therefore the two sides *AK* and *KB* equal the two sides *DL* and *LE,* and the base *AB* equals the base *DE,* therefore the angle *AKB* equals the angle *DLE.* I.8

But equal angles stand on equal circumferences when they are at the centers, therefore the circumference *AGB* equals *DHE.* III.26

And the whole circle *ABC* also equals the whole circle *DEF,* therefore the remaining circumference *ACB* also equals the remaining circumference *DFE.*

Therefore *in equal circles equal straight lines cut off equal circumferences, the greater circumference equals the greater and the less equals the less.*

Q.E.D.

Guide

This proposition is used in III.30 and XIII.18.

Next proposition: III.29

Previous: III.27

Book III introduction

Select from Book III

Select book

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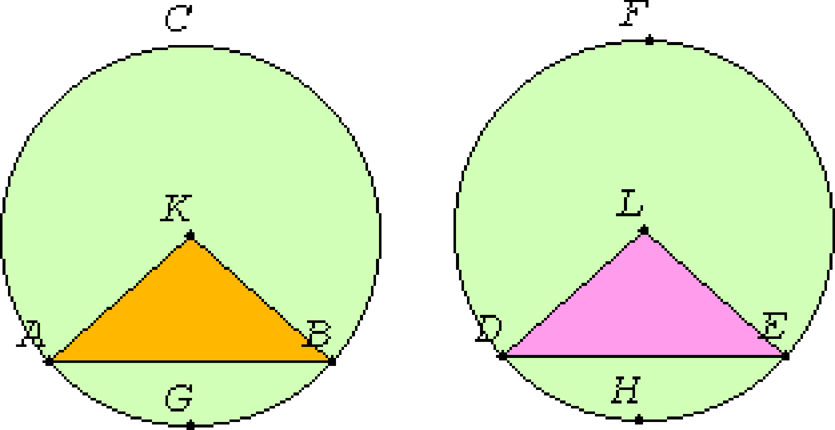
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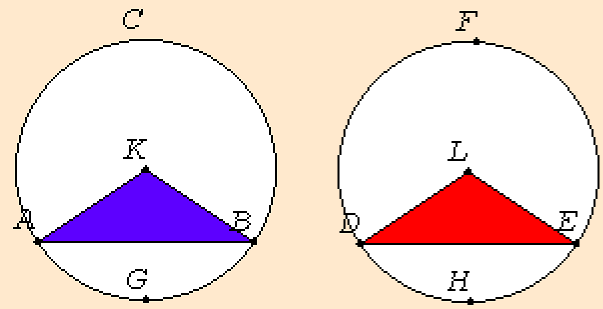
Euclid's Elements
Book III
Proposition 29

In equal circles straight lines that cut off equal circumferences are equal.



Let *ABC* and *DEF* be equal circles, and in them let equal circumferences *BGC* and *EHF* be cut off. Join the straight lines *BC* and *EF*.

I say that *BC* equals *EF*.

Take the centers *K* and *L* of the circles. Join *BK, KC, EL,* and *LF*. III.1

Now, since the circumference *BGC* equals the circumference *EHF,* the angle *BKC* also equals the angle *ELF*. III.27

And, since the circles *ABC* and *DEF* are equal, the radii are also equal, therefore the two sides *BK* and *KC* equal the two sides *EL* and *LF,* and they contain equal angles, therefore the base *BC* equals the base *EF*. I.4

Therefore *in equal circles straight lines that cut off equal circumferences are equal.*

Q.E.D.

Guide

This proposition is used in IV.11 and IV.15.

Next proposition: III.30

Previous: III.28

Book III introduction

Select from Book III

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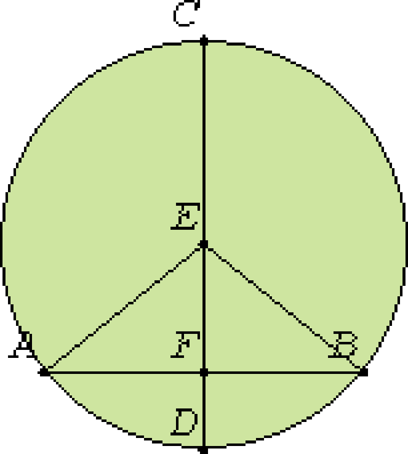
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Euclid's Elements

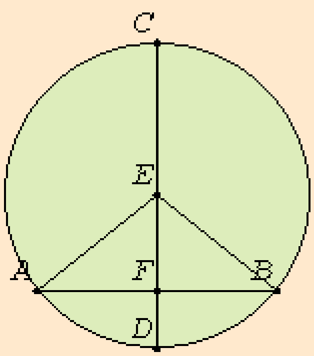
Book III

Proposition 3

If a straight line passing through the center of a circle bisects a straight line not passing through the center, then it also cuts it at right angles; and if it cuts it at right angles, then it also bisects it.

Let a straight line *CD* passing through the center of a circle *ABC* bisect a straight line *AB* not passing through the center at the point *F*.

I say that it also cuts it at right angles.



Take the center *E* of the circle *ABC*, and join *EA* and *EB*. III.1

Then, since *AF* equals *FB*, and *FE* is common, two sides equal two sides, and the base *EA* equals the base *EB*, therefore the angle *AFE* equals the angle *BFE*. I.Def.15 I.8

But, when a straight line standing on another straight line makes the adjacent angles equal to one another, each of the equal angles is right, therefore each of the angles *AFE* and *BFE* is right. I.Def.10

Therefore *CD*, which passes through the center and bisects *AB* which does not pass through the center, also cuts it at right angles.

Next, let *CD* cut *AB* at right angles.

I say that it also bisects it, that is, that *AF* equals *FB*.

For, with the same construction, since *EA* equals *EB*, the angle *EAF* also equals the angle *EBF*. I.5

But the right angle *AFE* equals the right angle *BFE*, therefore *EAF* and *EBF* are two triangles having two angles equal to two angles and one side equal to one side, namely *EF*, which is common to them, and opposite one of the equal angles. Therefore they also have the remaining sides equal to the remaining sides I.26

Therefore *AF* equals *FB*.

Therefore *if a straight line passing through the center of a circle bisects a straight line not passing through the center, then it also cuts it at right angles; and if it cuts it at right angles, then it also bisects it.*

Q.E.D.

Guide

Compare this statement to the corollary of proposition III.1.

This proposition is used in the next one, a few others in Book III, and XII.16.

Next proposition: III.4

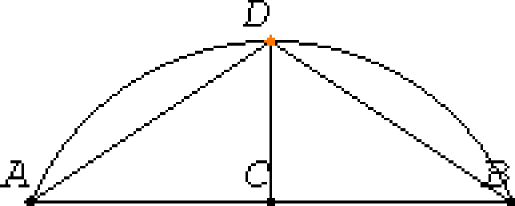
Previous: III.2

Book III introduction

Select from Book III

Select book

Select topic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Euclid's Elements

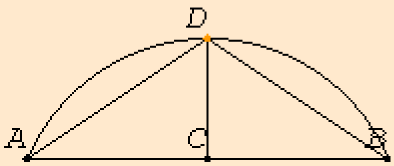
Book III

Proposition 30

To bisect a given circumference.

Let *ADB* be the given circumference.

It is required to bisect the circumference *ADB.*



Join *AB*, and bisect it at *C*. Draw *CD* from the point *C* at right angles to the straight line *AB.* Join *AD* and *DB.* I.10 I.11

Then, since *AC* equals *CB,* and *CD* is common, the two sides *AC* and *CD* equal the two sides *BC* and *CD*, and the angle *ACD* equals the angle *BCD,* for each is right, therefore the base *AD* equals the base *DB.* I.4

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less, and each of the circumferences *AD* and *DB* is less than a semicircle, therefore the circumference *AD* equals the circumference *DB.* III.28

Therefore the given circumference has been bisected at the point *D.*

Q.E.F.

Guide

The construction in this proposition is used in IV.16.

Next proposition: III.31

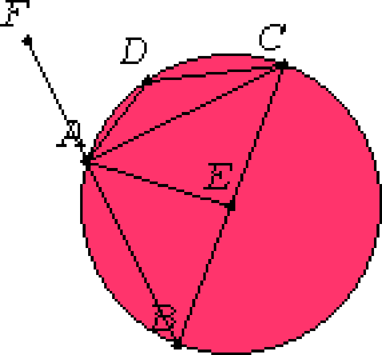
Previous: III.29

Book III introduction

Select from Book III

Select book

Select topic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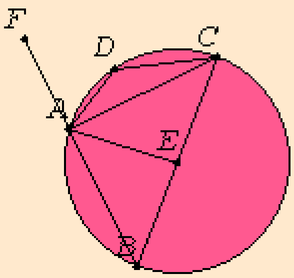
Euclid's Elements
Book III
Proposition 31

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.

Let *ABCD* be a circle, let *BC* be its diameter, and *E* its center. Join *BA, AC, AD,* and *DC.*

I say that the angle *BAC* in the semicircle *BAC* is right, the angle *ABC* in the segment *ABC* greater than the semicircle is less than a right angle, and the angle *ADC* in the segment *ADC* less than the semicircle is greater than a right angle.



Join *AE,* and carry *BA* through to *F.*

Then, since *BE* equals *EA,* the angle *ABE* also equals the angle *BAE.* Again, since *CE* equals *EA,* the angle *ACE* also equals the angle *CAE.* Therefore the whole angle *BAC* equals the sum of the two angles *ABC* and *ACB.* I.5

But the angle *FAC* exterior to the triangle *ABC* also equals the sum of the two angles *ABC* and *ACB.* Therefore the angle *BAC* also equals the angle *FAC.* Therefore each is right. Therefore the angle *BAC* in the semicircle *BAC* is right. I.32

Next, since in the triangle *ABC* the sum of the two angles *ABC* and *BAC* is less than two right angles, and the angle *BAC* is a right angle, the angle *ABC* is less than a right angle. And it is the angle in the segment *ABC* greater than the semicircle. I.17

Next, since *ABCD* is a quadrilateral in a circle, and the sum of the opposite angles of quadrilaterals in circles equals two right angles, while the angle *ABC* is less than a right angle, therefore the remaining angle *ADC* is greater than a right angle. And it is the angle in the segment *ADC* less than the semicircle. III.22

I say further that the angle of the greater segment, namely that contained by the circumference *ABC* and the straight line *AC,* is greater than a right angle, and the angle of the less segment, namely that contained by the circumference *ADC* and the straight line *AC,* is less than a right angle.

This is at once manifest. For, since the angle contained by the straight lines *BA* and *AC* is right, the angle contained by the circumference *ABC* and the straight line *AC* is greater than a right angle.

Again, since the angle contained by the straight lines *AC* and *AF* is right, the angle contained by the straight line *CA* and the circumference *ADC* is less than a right angle.

Therefore *in a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.*

Q.E.D.

Guide

This proposition is used in III.32 and in each of the rest of the geometry books, namely, Books IV, VI, XI, XII, XIII. It is also used in Book X.

Next proposition: III.32

Previous: III.30

Book III introduction

Select from Book III

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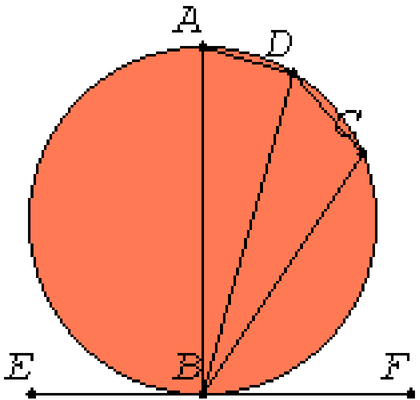
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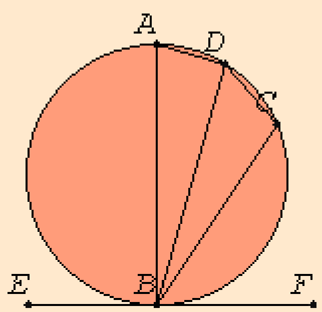
Euclid's Elements
Book III
Proposition 32

If a straight line touches a circle, and from the point of contact there is drawn across, in the circle, a straight line cutting the circle, then the angles which it makes with the tangent equal the angles in the alternate segments of the circle.

For let a straight line *EF* touch the circle *ABCD* at the point *B,* and from the point *B* let there be drawn across, in the circle *ABCD,* a straight line *BD* cutting it.

I say that the angles which *BD* makes with the tangent *EF* equal the angles in the alternate segments of the circle, that is, that the angle *FBD* equals the angle constructed in the segment *BAD,* and the angle *EBD* equals the angle constructed in the segment *DCB.*



Draw *BA* from *B* at right angles to *EF,* take a point *C* at random on the circumference *BD,* and join *AD, DC,* and *CB.* I.11

Then, since a straight line *EF* touches the circle *ABCD* at *B,* and *BA* has been drawn from the point of contact at right angles to the tangent, the center of the circle *ABCD* is on *BA.* III.19

Therefore *BA* is a diameter of the circle *ABCD.* Therefore the angle *ADB,* being an angle in a semicircle, is right. III.31

Therefore the sum of the remaining angles *BAD* and *ABD* equals one right angle. I.32

But the angle *ABF* is also right, therefore the angle *ABF* equals the sum of the angles *BAD* and *ABD.*

Subtract the angle *ABD* from each. Therefore the remaining angle *DBF* equals the angle *BAD* in the alternate segment of the circle.

Next, since *ABCD* is a quadrilateral in a circle, the sum of its opposite angles equals two right angles. III.22

But the sum of the angles *DBF* and *DBE* also equals two right angles, therefore the sum of the angles *DBF* and *DBE* equals the sum of the angles *BAD* and *BCD,* of which the angle *BAD* was proved equal to the angle *DBF,* therefore the remaining angle *DBE* equals the angle *DCB* in the alternate segment *DCB* of the circle.

Therefore *if a straight line touches a circle, and from the point of contact there is drawn across, in the circle, a straight line cutting the circle, then the angles which it makes with the tangent equal the angles in the alternate segments of the circle.*

Q.E.D.

Guide

This proposition is used in the next two propositions and a couple of the propositions in Book IV.

Next proposition: III.33

Previous: III.31

Book III introduction

Select from Book III

Select book

Select topic

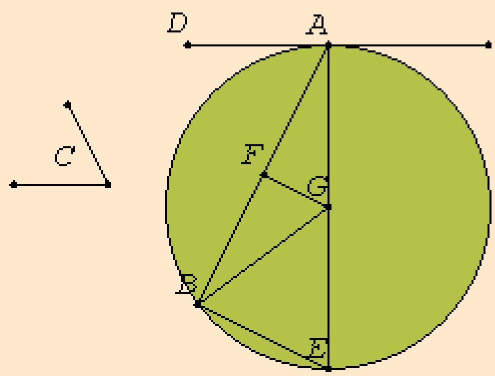
Euclid's Elements
Book III
Proposition 33

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilinear angle.

Let *AB* be the given straight line, and the angle at *C* the given rectilinear angle.

It is required to describe on the given straight line *AB* a segment of a circle admitting an angle equal to the angle at *C*.

The angle at *C* is then acute, or right, or obtuse.



First let it be acute as in the first figure. Construct the angle *BAD* equal to the angle at *C* on the straight line *AB* and at the point *A*. Therefore the angle *BAD* is also acute. I.23

Draw *AE* at right angles to *DA*. Bisect *AB* at *F*. Draw *FG* from the point *F* at right angles to *AB*, and join *GB*. I.10 I.12

Then, since *AF* equals *FB*, and *FG* is common, the two sides *AF* and *FG* equal the two sides *BF* and *FG*, and the angle *AFG* equals the angle *BFG*, therefore the base *AG* equals the base *BG*. I.4

Therefore the circle described with center *G* and radius *GA* passes through *B* also.

Draw it as *ABE*, and join *EB*.

Now, since *AD* is drawn from A, the end of the diameter *AE*, at right angles to *AE*, therefore *AD* touches the circle *ABE*. III.16,Cor.

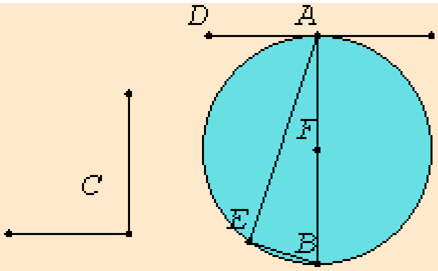
Since then a straight line *AD* touches the circle *ABE*, and from the point of contact at A a straight line *AB* has been drawn across in the circle *ABE*, the angle *DAB* equals the angle *AEB* in the alternate segment of the circle. III.32

But the angle *DAB* equals the angle at *C*, therefore the angle at *C* also equals the angle *AEB*.

Therefore on the given straight line *AB* the segment *AEB* of a circle has been described admitting the angle *AEB* equal to the given angle, the angle at *C*.

Next let the angle at *C* be right, and let it be again required to describe on *AB* a segment of a circle admitting an angle equal to the right angle at *C*.

Let the angle *BAD* be constructed equal to the right angle at *C*, as is the case in the second figure. Bisect *AB* at *F*. Describe the circle *AEB* with center *F* and radius either *FA* or *FB*. I.23 I.10



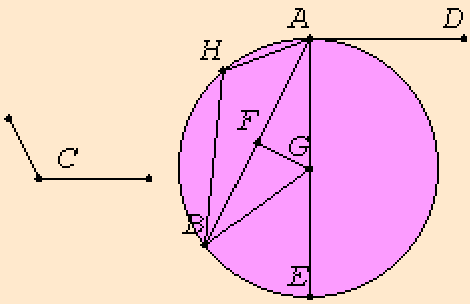
Therefore the straight line *AD* touches the circle *ABE,* because the angle at A is right. [III.16 Cor.]

And the angle *BAD* equals the angle in the segment *AEB,* for the latter too is itself a right angle, being an angle in a semicircle. [III.31]

But the angle *BAD* also equals the angle at *C,* therefore the angle *AEB* also equals the angle at *C.*

Therefore again the segment *AEB* of a circle has been described on *AB* admitting an angle equal to the angle at *C.*

Next, let the angle at *C* be obtuse.



Construct the angle *BAD* equal to *C* on the straight line *AB* and at the point A as is the case in the third figure. Draw *AE* at right angles to *AD.* Bisect *AB* again at *F.* Draw *FG* at right angles to *AB,* and join *GB.* [I.23 I.11 I.12]

Then, since *AF* again equals *FB,* and *FG* is common, the two sides *AF* and *FG* equal the two sides *BF* and *FG,* and the angle *AFG* equals the angle *BFG,* therefore the base *AG* equals the base *BG.* [I.4]

Therefore the circle described with center *G* and radius *GA* also passes through *B.* Let it so pass, as *AEB.*

Now, since *AD* is drawn at right angles to the diameter *AE* from its end, *AD* touches the circle *AEB.* [III.16 Cor.]

And *AB* has been drawn across from the point of contact at A, therefore the angle *BAD* equals the angle constructed in the alternate segment *AHB* of the circle. [III.32]

But the angle *BAD* equals the angle at *C.*

Therefore the angle in the segment *AHB* also equals the angle at *C.*

Therefore on the given straight line *AB* the segment *AHB* of a circle has been described admitting an angle equal to the angle at *C.*

Q.E.F.

Guide

This proposition is not used in the rest of the *Elements.*

Next proposition: III.34

Select from Book III

Previous: III.32

Book III introduction

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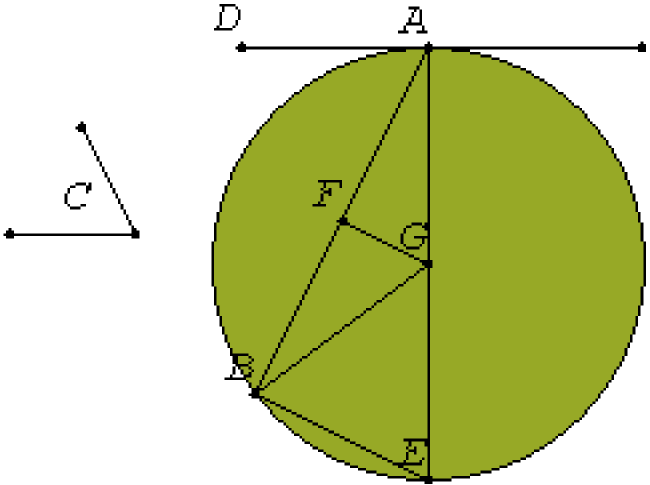
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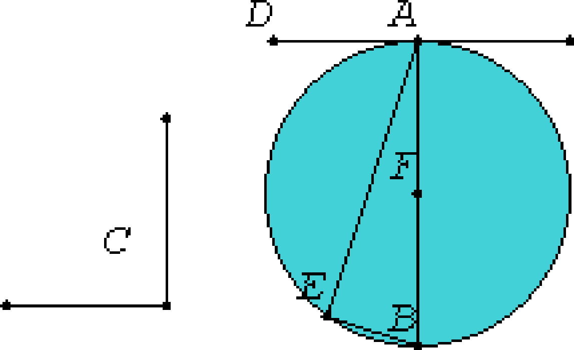

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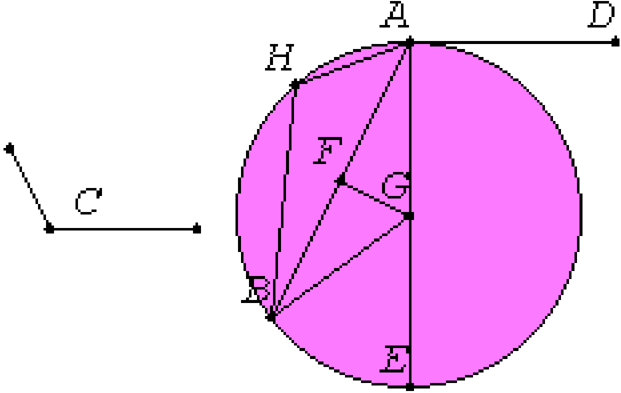
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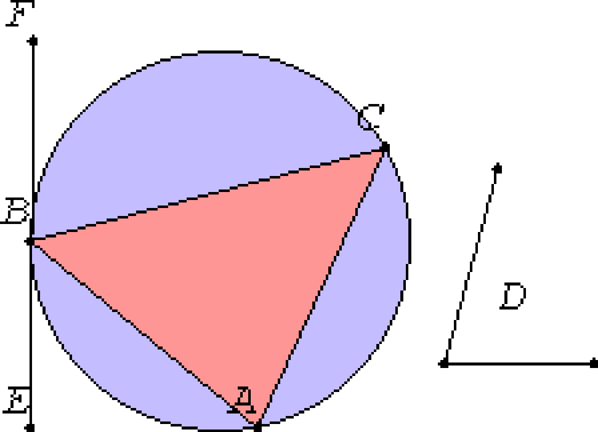
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Euclid's Elements

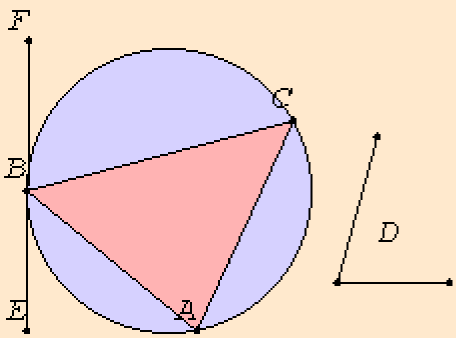
Book III

Proposition 34

From a given circle to cut off a segment admitting an angle equal to a given rectilinear angle.

Let *ABC* be the given circle, and the angle at *D* the given rectilinear angle.

It is required to cut off from the circle *ABC* a segment admitting an angle equal to the given rectilinear angle, the angle at *D*.



Draw *EF* touching *ABC* at the point *B*. Construct the angle *FBC* equal to the angle at *D* on the straight line *FB* and at the point *B* on it. III.17 I.23

Then, since a straight line *EF* touches the circle *ABC*, and *BC* has been drawn across from the point of contact at *B*, the angle *FBC* equals the angle constructed in the alternate segment *BAC*. III.32

But the angle *FBC* equals the angle at *D*, therefore the angle in the segment *BAC* equals the angle at *D*.

Therefore from the given circle *ABC* the segment *BAC* has been cut off admitting an angle equal to the given rectilinear angle, the angle at *D*.

Q.E.F.

Guide

This proposition is not used in the rest of the *Elements*.

Next proposition: III.35

Previous: III.33

Book III introduction

Select from Book III

Select book

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Euclid's Elements

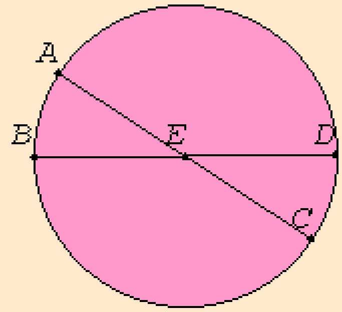
Book III

Proposition 35

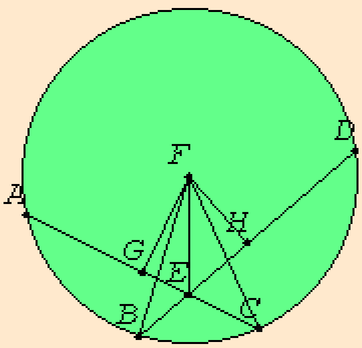
If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other.

For in the circle *ABCD* let the two straight lines *AC* and *BD* cut one another at the point *E.*

I say that the rectangle *AE* by *EC* equals the rectangle *DE* by *EB.*



If now *AC* and *BD* are through the center, so that *E* is the center of the circle *ABCD,* it is manifest that, *AE, EC, DE,* and *EB* being equal, the rectangle *AE* by *EC* also equals the rectangle *DE* by *EB.*



Next let *AC* and *DB* not be through the center. Take the center *F* let the center of *ABCD.* Draw *FG* and *FH* from *F* perpendicular to the straight lines *AC* and *DB.* Join *FB, FC,* and *FE.* III.1 I.12

Then, since a straight line *GF* through the center cuts a straight line *AC* not through the center at right angles, it also bisects it, therefore *AG* equals *GC.* III.3

Since, then, the straight line *AC* has been cut into equal parts at *G* and into unequal parts at *E,* the rectangle *AE* by *EC* together with the square on *EG* equals the square on *GC.* II.5

Add the square on *GF.* Therefore the rectangle *AE* by *EC* plus the sum of the squares on *GE* and *GF* equals the sum of the squares on *CG* and *GF.*

But the square on *FE* equals the sum of the squares on *EG* and *GF,* and the square on *FC* equals the sum of the squares on *CG* and *GF.* Therefore the rectangle *AE* by *EC* plus the square on *FE* equals the square on *FC.* I.47

And *FC* equals *FB,* therefore the rectangle *AE* by *EC* plus the square on *EF* equals the square on *FB.*

For the same reason, also, the rectangle *DE* by *EB* plus the square on *FE* equals the square on *FB.*

But the rectangle *AE* by *EC* plus the square on *FE* was also proved equal to the square on *FB,* therefore the rectangle *AE* by *EC* plus the square on *FE* equals the rectangle *DE* by *EB* plus the square on *FE.*

Subtract the square on *FE* from each. Therefore the remaining rectangle *AE* by *EC* equals the rectangle *DE* by *EB.*

Therefore *if in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other.*

Q.E.D.

Guide

By means of proposition VI.16, the statement, "the rectangle *AE* by *EC* equals the rectangle *DE* by *EB*," may be converted into one about ratios, namely, "*AE* : *EB* = *DE* : *EC*."

This proposition is not used in the rest of the *Elements.*

Next proposition: III.36

Previous: III.34

Book III introduction

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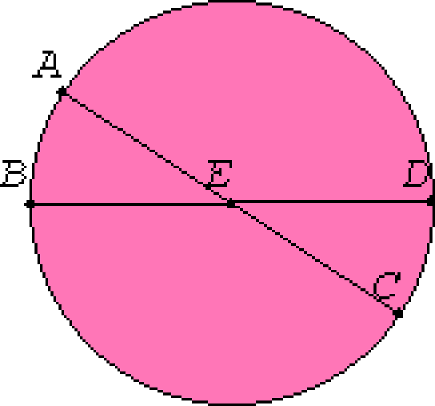

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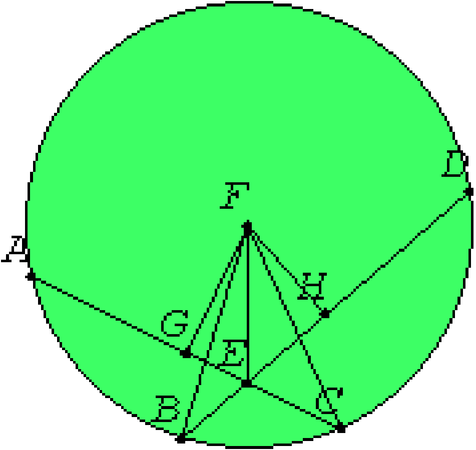

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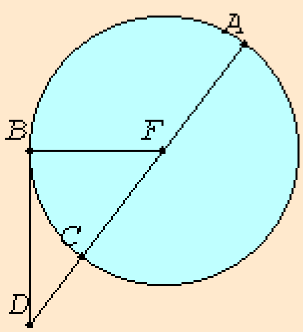
Euclid's Elements
Book III
Proposition 36

If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.

Let a point *D* be taken outside the circle *ABC*, and from *D* let the two straight lines *DCA* and *DB* fall on the circle *ABC*. Let *DCA* cut the circle *ABC*, and let *BD* touch it.

I say that the rectangle *AD* by *DC* equals the square on *DB*.

Then *DCA* is either through the center or not through the center.



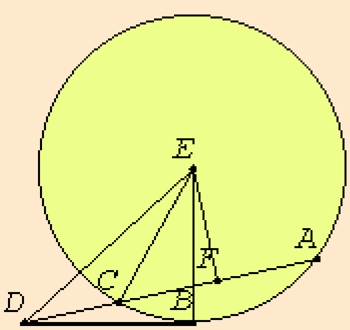
First let it be through the center, and let *F* be the center of the circle *ABC*. Join *FB*. Therefore the angle *FBD* is right. III.18

And, since *AC* has been bisected at *F*, and *CD* is added to it, the rectangle *AD* by *DC* plus the square on *FC* equals the square on *FD*. II.6

But *FC* equals *FB*, therefore the rectangle *AD* by *DC* plus the square on *FB* equals the square on *FD*.

And the sum of the squares on *FB* and *BD* equals the square on *FD*, therefore the rectangle *AD* by *DC* plus the square on *FB* equals the sum of the squares on *FB* and *BD*. I.47

Subtract the square on *FB* from each. Therefore the remaining rectangle *AD* by *DC* equals the square on the tangent *DB*.



Again, let *DCA* not be through the center of the circle *ABC*. Take the center *E*, and draw *EF* from *E* perpendicular to *AC*. Join *EB*, *EC*, and *ED*. III.1

Then the angle *EBD* is right. III.18

And, since a straight line *EF* through the center cuts a straight line *AC* not through the center at right angles, it also bisects it, therefore *AF* equals *FC*. III.3

Now, since the straight line *AC* has been bisected at the point *F*, and *CD* is added to it, the rectangle *AD* by *DC* plus the square on *FC* equals the square on *FD*. II.6

Add the square on *FE* to each. Therefore the rectangle *AD* by *DC* plus the sum of the squares on *CF* and *FE* equals the sum of the squares on *FD* and *FE*.

But the square on *EC* equals the sum of the squares on *CF* and *FE*, for the angle *EFC* is right, and the square on *ED* equals the sum of the squares on *DF* and *FE*, therefore the rectangle *AD* by *DC* plus the square on *EC* equals the square on *ED*. I.47

And *EC* equals *EB*, therefore the rectangle *AD* by *DC* plus the square on *EB* equals the square on *ED*.

But the sum of the squares on *EB* and *BD* equals the square on *ED*, for the angle *EBD* is right, therefore the rectangle *AD* by *DC* plus the square on *EB* equals the sum of the squares on *EB* and *BD*. I.47

Subtract the square on *EB* from each. Therefore the remaining rectangle *AD* by *DC* equals the square on *DB*.

Therefore *if a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.*

Q.E.D.

Guide

This proposition is used in the next one.

Next proposition: III.37

Previous: III.35

Book III introduction

Select from Book III

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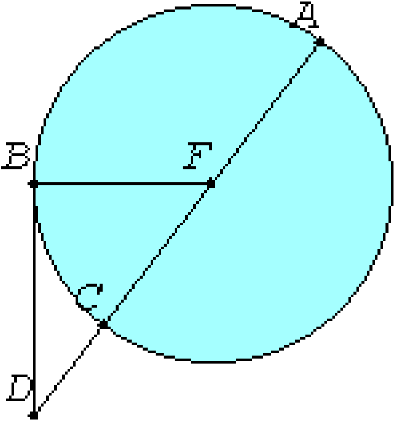
Select topic



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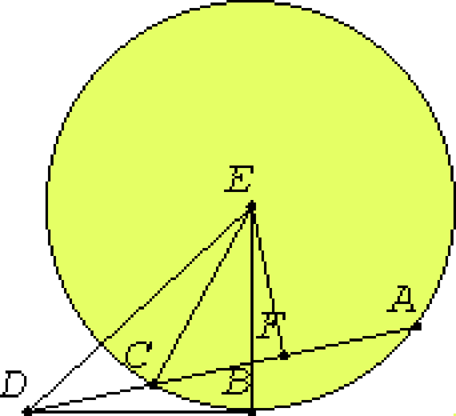
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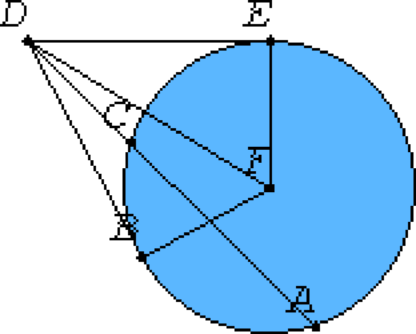

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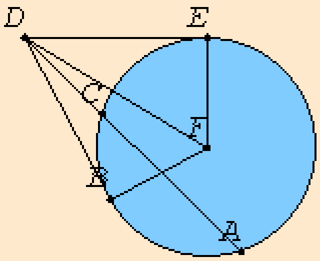
Euclid's Elements
Book III
Proposition 37

If a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.

Let a point *D* be taken outside the circle *ABC,* from *D* let the two straight lines *DCA* and *DB* fall on the circle *ACB,* let *DCA* cut the circle and *DB* fall on it, and let the rectangle *AD* by *DC* equal the square on *DB.*

I say that *DB* touches the circle *ABC.*



Draw *DE* touching *ABC.* Take the center *F* of the circle *ABC,* and join *FE, FB,* and *FD.* III.17 III.1

Thus the angle *FED* is right. III.18

Now, since *DE* touches the circle *ABC,* and *DCA* cuts it, the rectangle *AD* by *DC* equals the square on *DE.* III.36

But the rectangle *AD* by *DC* was also equal to the square on *DB,* therefore the square on *DE* equals the square on *DB.* Therefore *DE* equals *DB.*

And *FE* equals *FB,* therefore the two sides *DE* and *EF* equal the two sides *DB* and *BF,* and *FD* is the common base of the triangles, therefore the angle *DEF* equals the angle *DBF.* I.8

But the angle *DEF* is right, therefore the angle *DBF* is also right.

And *FB* produced is a diameter, and the straight line drawn at right angles to the diameter of a circle, from its end, touches the circle, therefore *DB* touches the circle. III.16 Cor

Similarly this can be proved to be the case even if the center is on *AC.*

Therefore *if a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.*

Q.E.D.

Guide

This proposition is used in IV.10.

Next: Book IV Introduction

Previous proposition: Proposition III.36

Book III introduction

Select from Book III

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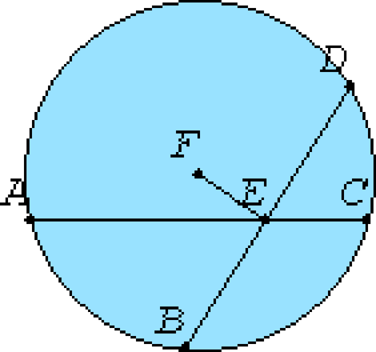
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Euclid's Elements

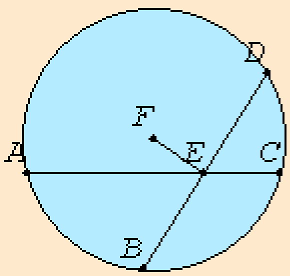
Book III

Proposition 4

If in a circle two straight lines which do not pass through the center cut one another, then they do not bisect one another.

Let *ABCD* be a circle, and in it let the two straight lines *AC* and *BD,* which do not pass through the center, cut one another at *E.*

I say that they do not bisect one another.



For, if so, let them bisect one another, so that *AE* equals *EC,* and *BE* equals *ED.* Take the center *F* of the circle *ABCD.* Join *FE.* III.1

Then, since a straight line *FE* passing through the center bisects a straight line *AC* not passing through the center, it also cuts it at right angles, therefore the angle *FEA* is right. III.3

Again, since a straight line *FE* bisects a straight line *BD,* it also cuts it at right angles. Therefore the angle *FEB* is right. III.3

But the angle *FEA* was also proved right, therefore the angle *FEA* equals the angle *FEB,* the less equals the greater, which is impossible.

Therefore *AC* and *BD* do not bisect one another.

Therefore *if in a circle two straight lines which do not pass through the center cut one another, then they do not bisect one another.*

Q.E.D.

Guide

This proposition is not used in the rest of the *Elements.* The contrapositive of this statement is more positive: if two straight lines in a circle bisect each other, then they meet at the center.

This proposition is not used in the rest of the *Elements.*

Next proposition: III.5

Previous: III.3

Book III introduction

Select from Book III

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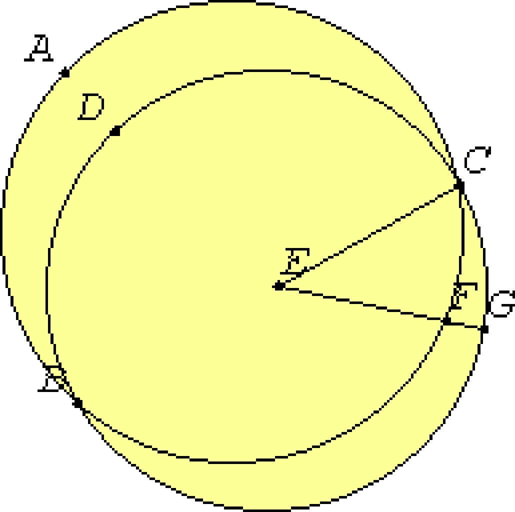
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Euclid's Elements

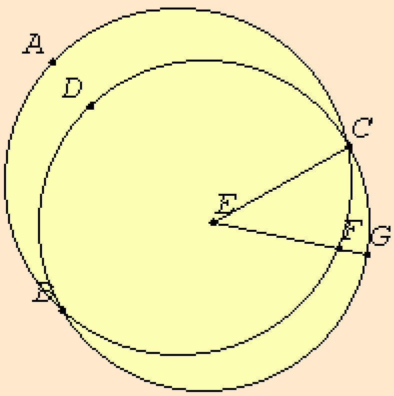
Book III

Proposition 5

If two circles cut one another, then they do not have the same center.

Let the circles *ABC* and *CDG* cut one another at the points *B* and *C.*

I say that they do not have the same center.



For, if possible, let it be *E.* Join *EC,* and draw *EFG* through at random.

Then, since the point *E* is the center of the circle *ABC, EC* equals *EF.* Again, since the point *E* is the center of the circle *CDG, EC* equals *EG.* I.Def.15

But *EC* was proved equal to *EF* also, therefore *EF* also equals *EG,* the less equals the greater which is impossible.

Therefore the point *E* is not the center of the circles *ABC* and *CDG.*

Therefore *if two circles cut one another, then they do not have the same center.*

Q.E.D.

Guide

Note that no use was made in the proof of the point *B.* That means the proof actually shows that *if two circles meet, then they do not have the same center,* and that covers not only this proposition but the next, too, where the two touch each other.

This proposition is used in III.10 which states that circles cannot intersect at more than two points.

Next proposition: III.6

Previous: III.4

Book III introduction

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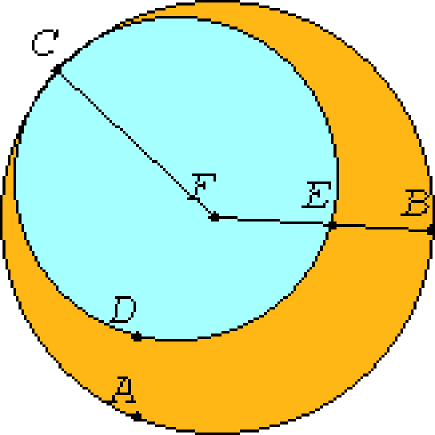
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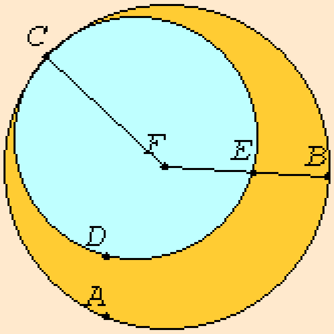
Euclid's Elements

Book III

Proposition 6

If two circles touch one another, then they do not have the same center.



Let the two circles *ABC* and *CDE* touch one another at the point *C*.

I say that they do not have the same center.

For, if possible, let it be *F*. Join *FC*, and draw *FEB* through at random.

Then, since the point *F* is the center of the circle *ABC*, *FC* equals *FB*. Again, since the point *F* is the center of the circle *CDE*, *FC* equals *FE*. I.Def.15

But *FC* was proved equal to *FB*, therefore *FE* also equals *FB*, the less equals the greater, which is impossible.

Therefore *F* is not the center of the circles *ABC* and *CDE*.

Therefore *if two circles touch one another, then they do not have the same center*.

Q.E.D.

Guide

As mentioned before, this proposition is almost the same as the previous. Both could be included in one statement: circles that meet don't have the same center, or the contrapositive: concentric circles don't meet.

This propostion is not used in the rest of the *Elements*.

Next proposition: III.7

Previous: III.5

Book III introduction

Select from Book III

Select book

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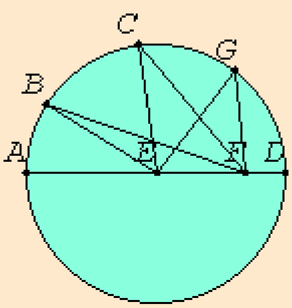
Euclid's Elements
Book III
Proposition 7

If on the diameter of a circle a point is taken which is not the center of the circle, and from the point straight lines fall upon the circle, then that is greatest on which passes through the center, the remainder of the same diameter is least, and of the rest the nearer to the straight line through the center is always greater than the more remote; and only two equal straight lines fall from the point on the circle, one on each side of the least straight line.

Let *ABCD* be a circle, and let *AD* be a diameter of it. Let *F* be a point *F* on *AD* which is not the center of the circle. Let *E* be the center of the circle. Let straight lines *FB, FC,* and *FG* fall upon the circle *ABCD* from *F.*

I say that *FA* is greatest, *FD* is least, and of the rest *FB* is greater than *FC,* and *FC* greater than *FG.*



Join *BE, CE,* and *GE.*

Then, since in any triangle the sum of any two sides is greater than the remaining one, the sum of *EB* and *EF* is greater than *BF.* I.20

But *AE* equals *BE,* therefore *AF* is greater than *BF.*

Again, since *BE* equals *CE,* and *FE* is common, the two sides *BE* and *EF* equal the two sides *CE* and *EF.* But the angle *BEF* is also greater than the angle CEF, therefore the base *BF* is greater than the base *CF.* I.24

For the same reason *CF* is also greater than *GF.*

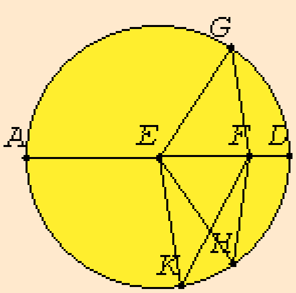
Again, since the sum of *GF* and *FE* is greater than *EG,* and *EG* equals *ED,* the sum of *GF* and *FE* is greater than *ED.* I.20

Subtract *EF* from each. Therefore the remainder *GF* is greater than the remainder *FD.*

Therefore *FA* is greatest, *FD* is least, *FB* is greater than *FC,* and *FC* greater than *FG.*

I say also that from the point *F* only two equal straight lines fall on the circle *ABCD,* one on each side of the least *FD.*

Construct the angle *FEH* equal to the angle *GEF* on the straight line *EF* and at the point *E* on it. Join *FH.* I.23



Then, since *GE* equals *EH,* and *EF* is common, the two sides *GE* and *EF* equal the two sides *HE* and *EF,* and the angle *GEF* equals the angle *HEF,* therefore the base *FG* equals the base *FH.* I.4

I say again that another straight line equal to *FG* does not fall on the circle from the point *F.*

For, if possible, let *FK* so fall.

Then, since *FK* equals *FG,* and *FH* equals *FG, FK* also equals *FH,* the nearer to the straight line through the center being thus equal to the more remote, which is impossible. Above

Therefore another straight line equal to *GF* does not fall from the point *F* upon the circle. Therefore only one straight line so falls.

Therefore *if on the diameter of a circle a point is taken which is not the center of the circle, and from the point straight lines fall upon the circle, then that is greatest on which passes through the center, the remainder of the same diameter is least, and of the rest the nearer to the straight line through the center is always greater than the more remote; and only two equal straight lines fall from the point on the circle, one on each side of the least straight line.*

Q.E.D.

Guide

The statement of this proposition is daunting. It concerns the distances from a point *F* inside a circle to the points on the circumference. The point *F* is assumed not to be the center. If a diameter *AD* is passed through *F*, then one of the points *A* is the point on the circumference furthest from *F* and the other *D* is the closest. As a point travels the circumference from *A* to *D* it gets closer to *F*. The final part of the statement is that if *G* is one point on the circumference, then there is exactly one other point *H* on the circumference the same distance from *F* (assuming, of course, that *G* is neither *A* nor *D*).

Note

There is some ambiguity in the statement of this proposition. It is not clear what the phrase "the nearer to the straight line through the center" means. It may well refer to the angle, so that *FB* is considered nearer to *FA* than *FC* since the angle *BFA* is less than the angle *CFA.* If so, there there is a missing detail in the proof, as De Morgan pointed out. It is declared that the angle *BEF* is greater than the angle *CEF,* but that hasn't been proved. DeMorgan and others have described various ways to fill this logical gap.

This propostion is not used in the rest of the *Elements.*

Next proposition: III.8

Previous: III.6

Book III introduction

Select from Book III

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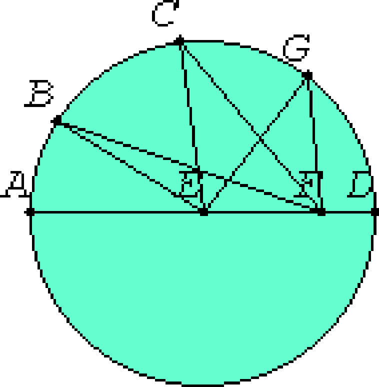

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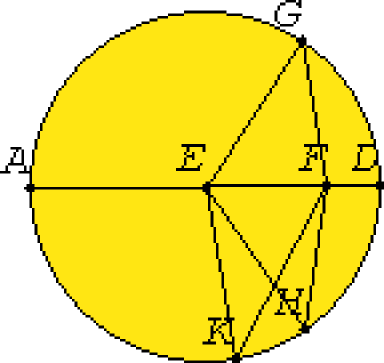
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Euclid's Elements

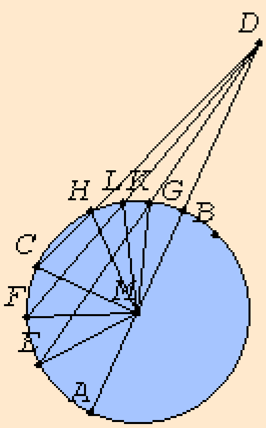
Book III

Proposition 8

If a point is taken outside a circle and from the point straight lines are drawn through to the circle, one of which is through the center and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the center is greatest, while of the rest the nearer to that through the center is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote; and only two equal straight lines fall on the circle from the point, one on each side of the least.

Let *ABC* be a circle, and let a point *D* be taken outside *ABC.* Let straight lines *DA, DE, DF,* and *DC* be drawn through from *D,* and let *DA* be drawn through the center.

I say that, of the straight lines falling on the concave circumference *AEFC*, the straight line *DA* through the center is greatest, while *DE* is greater than *DF,* and *DF* greater than *DC.* But, of the straight lines falling on the convex circumference *HLKG,* the straight line *DG* between the point and the diameter *AG* is least, and the nearer to the least *DG* is always less than the more remote, namely *DK* is less than *DL*, and *DL* is less than *DH.*



Take the center *M* of the circle *ABC.* Join *ME, MF, MC, MK, ML,* and *MH.* III.1

Then, since *AM* equals *EM,* add *MD* to each, therefore *AD* equals the sum of *EM* and *MD.*

But the sum of *EM* and *MD* is greater than *ED*, therefore *AD* is also greater than *ED.* I.20

Again, since *ME* equals *MF,* and *MD* is common, therefore *EM* and *MD* equal *FM* and *MD*, and the angle *EMD* is greater than the angle *FMD*, therefore the base *ED* is greater than the base *FD.* I.24

Similarly we can prove that *FD* is greater than *CD.* Therefore *DA* is greatest, while *DE* is greater than *DF,* and *DF* is greater than *DC.*

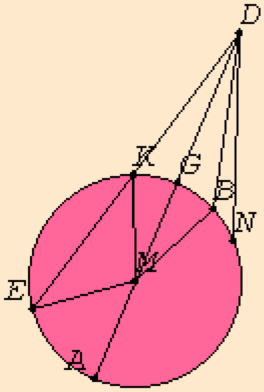
Next, since the sum of *MK* and *KD* is greater than *MD,* and *MG* equals *MK,* therefore the remainder *KD* is greater than the remainder *GD*, so that *GD* is less than *KD.* I.20

And, since on *MD,* one of the sides of the triangle *MLD,* two straight lines *MK* and *KD* are constructed meeting within the triangle, therefore the sum of *MK* and *KD* is less than the sum of *ML* and *LD.* And *MK* equals *ML,* therefore the remainder *DK* is less than the remainder *DL.* I.21

Similarly we can prove that *DL* is also less than *DH.* Therefore *DG* is least, while *DK* is less than *DL,* and *DL* is less than *DH.*

I say also that only two equal straight lines will fall from the point *D* on the circle, one on each side of the least *DG.*

Construct the angle *DMB* equal to the angle *KMD* on the straight line *MD* and at the



point *M* on it. Join *DB.* I.23

Then, since *MK* equals *MB,* and *MD* is common, the two sides *KM* and *MD* equal the two sides *BM* and *MD* respectively, and the angle *KMD* equals the angle *BMD,* therefore the base *DK* equals the base *DB.* I.4

I say that no other straight line equal to the straight line *DK* falls on the circle from the point *D.*

For, if possible, let a straight line so fall, and let it be *DN.* Then, since *DK* equals *DN,* and *DK* equals *DB, DB* also equals *DN,* that is, the nearer to the least *DG* equal to the more remote, which was proved impossible. Above

Therefore no more than two equal straight lines fall on the circle *ABC* from the point *D,* one on each side of *DG* the least.

Therefore *if a point is taken outside a circle and from the point straight lines are drawn through to the circle, one of which is through the center and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the center is greatest, while of the rest the nearer to that through the center is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote; and only two equal straight lines fall on the circle from the point, one on each side of the least.*

Q.E.D.

Guide

This proposition has a statement even more complicated than the previous proposition. This one deals with the distances from a point *D* outside a circle to the points on the circumference. If the diameter *AG* extended passes through *D*, then one of its endpoints *G* is the point on the circumference closest to *D* and the other *A* is furthest. As a point moves along the circumference from *A* to *D* it gets closer to *D.* Euclid considers two parts of the circumference, the convex part is the near part exposed to the point *D,* while the concave part is the part on the far side of the circle. The final part of the statement is that if *K* is one point on the circumference, then there is exactly one other point *B* on the circumference the same distance from *D* (assuming, of course, that *K* is neither *G* nor *A*).

Note

There is a logical gap in the proof of this proposition similar to that in the previous proposition. Again, various ways have been proposed to fill it.

This propostion is not used in the rest of the *Elements.*

Next proposition: III.9

Previous: III.7

Book III introduction

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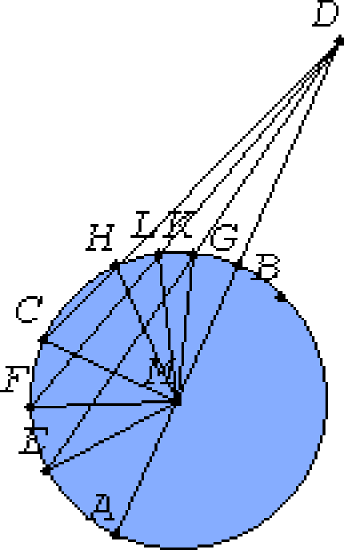
D.E.Joyce
Clark University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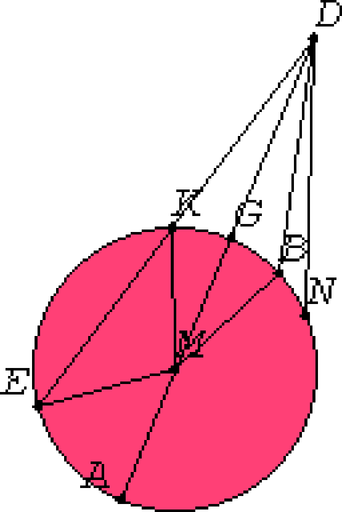
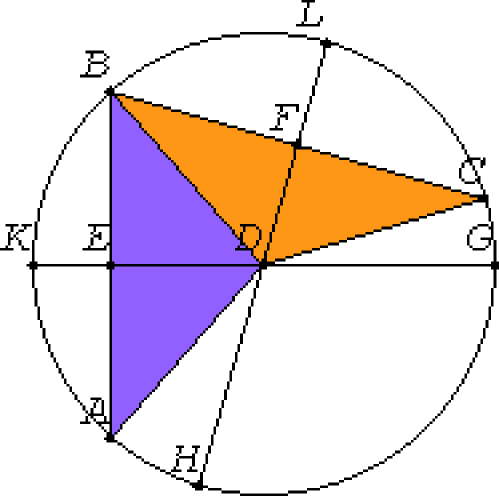

D
K
G
B
N
M
E
A



L
B
F
C
K
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D
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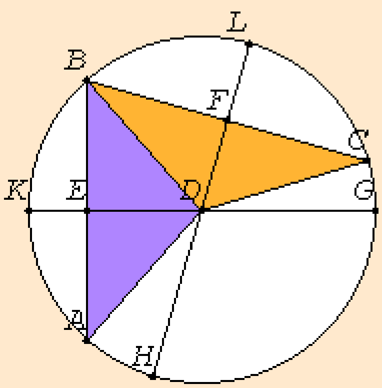
H

Euclid's Elements

Book III

Proposition 9

If a point is taken within a circle, and more than two equal straight lines fall from the point on the circle, then the point taken is the center of the circle.



Let *D* a point within a circle *ABC*, and from *D* let more than two equal straight lines, namely *DA* and *DB* and *DC*, fall on the circle *ABC*.

I say that the point *D* is the center of the circle *ABC*.

Join *AB* and *BC*, and bisect them at the points *E* and *F*. Join *ED* and *FD*, and draw them through to the points *G*, *K*, *H*, and *L*. I.10

Then, since *AE* equals *EB*, and *ED* is common, the two sides *AE* and *ED* equal the two sides *BE* and *ED*, and the base *DA* equals the base *DB*, therefore the angle *AED* equals the angle *BED*. I.8

Therefore the angles *AED* and *BED* are each right. Therefore *GK* cuts *AB* into two equal parts and at right angles.

And since, if in a circle a straight line cuts a straight line into two equal parts and at right angles, the center of the circle is on the cutting straight line, therefore the center of the circle is on *GK*. III.1,Cor

For the same reason the center of the circle *ABC* is also on *HL*.

And the straight lines *GK* and *HL* have no other point common but the point *D*, therefore the point *D* is the center of the circle *ABC*.

Therefore *if a point is taken within a circle, and more than two equal straight lines fall from the point on the circle, then the point taken is the center of the circle.*

Q.E.D.

Guide

The statement of this proposition is already covered by the last part of Proposition III.7 which says for a point in a circle that is not the center at most two points lie on the circumference at any distance from that point.

This proposition is used in III.25.

Next proposition: III.10

Select from Book III

Previous: III.8

Select book

Book III introduction

Select topic

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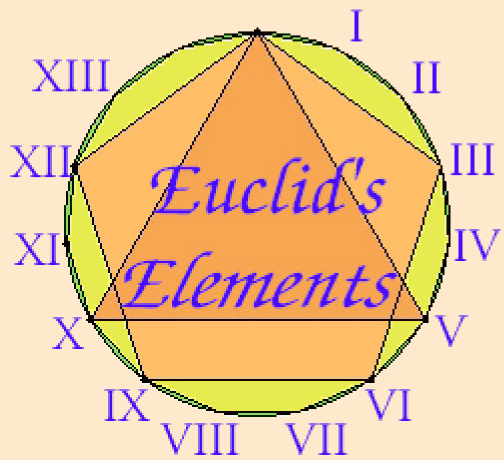
Book IV



Book IV

Table of contents

Definitions

Definition 1.
A rectilinear figure is said to be *inscribed in a rectilinear figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

Definition 2.
Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

Definition 3.
A rectilinear figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle.

Definition 4.
A rectilinear figure is said to be *circumscribed about a circle* when each side of the circumscribed figure touches the circumference of the circle.

Definition 5.
Similarly a circle is said to be *inscribed in a figure* when the circumference of the circle touches each side of the figure in which it is inscribed.

Definition 6.
A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each

angle of the figure about which it is circumscribed.

Definition 7.
A straight line is said to be *fitted into a circle* when its ends are on the circumference of the circle.

Propositions

Proposition 1.
To fit into a given circle a straight line equal to a given straight line which is not greater than the diameter of the circle.

Proposition 2.
To inscribe in a given circle a triangle equiangular with a given triangle.

Proposition 3.
To circumscribe about a given circle a triangle equiangular with a given triangle.

Proposition 4.
To inscribe a circle in a given triangle.

Proposition 5.
To circumscribe a circle about a given triangle.

Corollary. When the center of the circle falls within the triangle, the triangle is acute-angled; when the center falls on a side, the triangle is right-angled; and when the center of the circle falls outside the triangle, the triangle is obtuse-angled.

Proposition 6.
To inscribe a square in a given circle.

Proposition 7.
To circumscribe a square about a given circle.

Proposition 8.
To inscribe a circle in a given square.

Proposition 9.
To circumscribe a circle about a given square.

Proposition 10.
To construct an isosceles triangle having each of the angles at the base double the remaining one.

Proposition 11.
To inscribe an equilateral and equiangular pentagon in a given circle.

Proposition 12.
To circumscribe an equilateral and equiangular pentagon about a given circle.

Proposition 13.
To inscribe a circle in a given equilateral and equiangular pentagon.

Proposition 14.
To circumscribe a circle about a given equilateral and equiangular pentagon.

Proposition 15.

To inscribe an equilateral and equiangular hexagon in a given circle.

Corollary. The side of the hexagon equals the radius of the circle.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

Proposition 16.

To inscribe an equilateral and equiangular fifteen-angled figure in a given circle.

Corollary. And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled figure and circumscribe one about it.

Guide to Book IV

All but two of the propositions in this book are constructions to inscribe or circumscribe figures.

Figure	Inscribe figure in circle	Circumscribe figure about circle	Inscribe circle in figure	Circumscribe circle about figure
Triangle	IV.2	IV.3	IV.4	IV.5
Square	IV.6	IV.7	IV.8	IV.9
Regular pentagon	IV.11	IV.12	IV.13	IV.14
Regular hexagon	IV.15	IV.15,Cor	IV.15,Cor	IV.15,Cor
Regular 15-gon	IV.16	IV.16,Cor	IV.16,Cor	IV.16,Cor

There are only two other propositions. Proposition IV.1 is a basic construction to fit a line in a circle, and proposition IV.10 constructs a particular triangle needed in the construction of a regular pentagon.

Logical structure of Book IV

The proofs of the propositions in Book IV rely heavily on the propositions in Books I and III. Only one proposition from Book II is used and that is the construction in II.11 used in proposition IV.10 to construct a particular triangle needed in the construction of a regular pentagon.

Most of the propositions of Book IV are logically independent of each other. There is a short chain of deductions, however, involving the construction of regular pentagons.

Dependencies within Book IV	
1, 5	10
2, 10	11
11	12
1, 2, 11	16

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Previous: Book III

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Select book

Select topic

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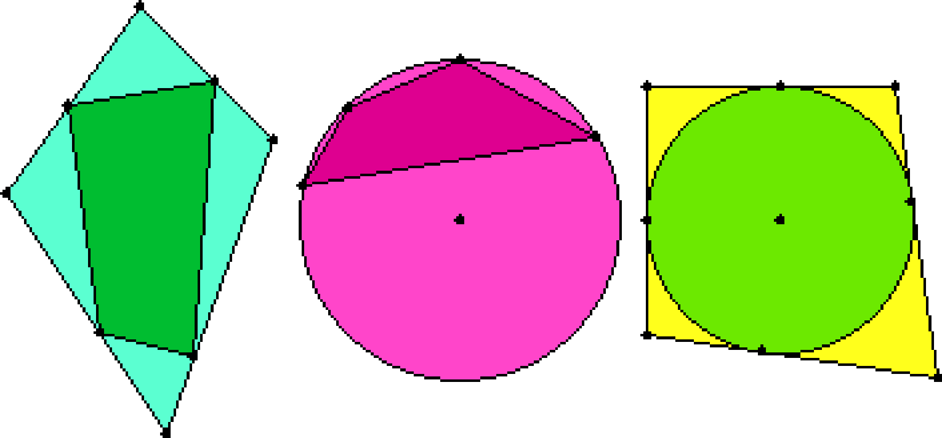


Euclid's Elements
Book IV
Definitions

Def. 1. A rectilinear figure is said to be *inscribed in a rectilinear figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

Def. 2. Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

Def. 3. A rectilinear figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle.

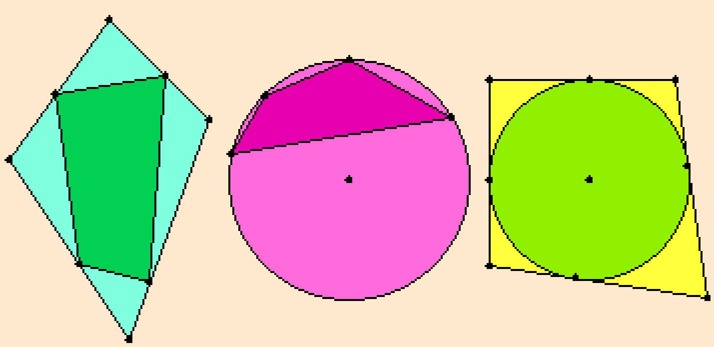
Def. 4. A rectilinear figure is said to be *circumscribed about a circle* when each side of the circumscribed figure touches the circumference of the circle.

Def. 5. Similarly a circle is said to be *inscribed in a figure* when the circumference of the circle touches each side of the figure in which it is inscribed.

Def. 6. A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each angle of the figure about which it is circumscribed.

Def. 7. A straight line is said to be *fitted into a circle* when its ends are on the circumference of the circle.

Guide



The first figure shows a smaller quadrilateral inscribed in a larger quadrilateral, and the larger one is circumscribed about the smaller one. The second figure shows a quadrilateral inscribed in a circle, and the circle is circumscribed about the quadrilateral. The third figure shows a circle inscribed in a quadrilateral, and the quadrilateral is circumscribed about the circle. Note also that in the second figure, each side of the quadrilateral is fitted into the circle.

First proposition: IV.1 Select from Book IV

Book IV introduction

Select book

Select topic

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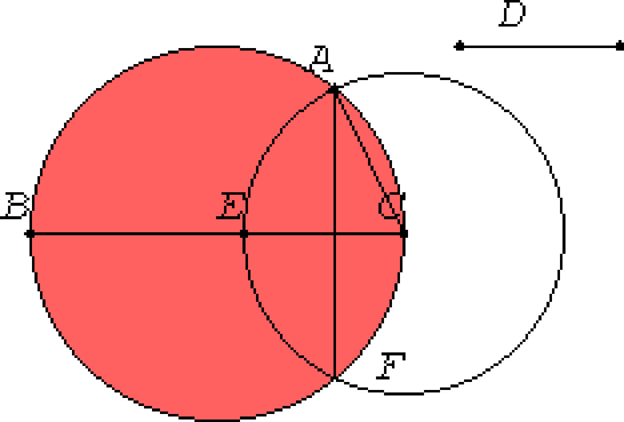
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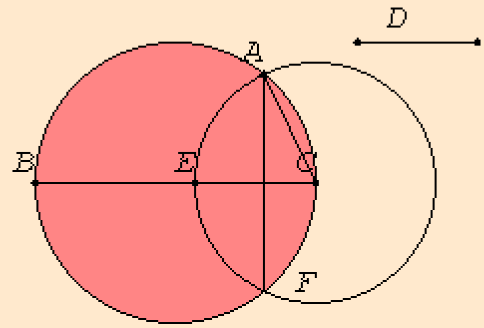
Euclid's Elements
Book IV
Proposition 1

To fit a straight line into a given circle equal to a given straight line which is not greater than the diameter of the circle.

Let *ABC* be the given circle, and *D* the given straight line not greater than the diameter of the circle.



It is required to fit a straight line into the circle *ABC* equal to the straight line *D*.

Draw a diameter *BC* of the circle *ABC*.

If *BC* equals *D*, then that which was proposed is done, for *BC* has been fitted into the circle *ABC* equal to the straight line *D*.

But, if *BC* is greater than *D*, make *CE* equal to *D*, describe the circle *EAF* with center *C* and radius *CE*, and join *CA*. I.3

Then, since the point *C* is the center of the circle *EAF*, *CA* equals *CE*.

But *CE* equals *D*, therefore *D* also equals *CA*.

Therefore *CA* has been fitted into the given circle *ABC* equal to the given straight line *D*. IV.Def.7

Q.E.F.

Guide

The hypothesis that the line to be fitted into the circle is no longer than the diameter of the circle is certainly necessary, but Euclid did not show it was sufficient. That is sufficient to conclude the two circles actually meet at a point *A* is never demonstrated. This logical gap has appeared before in the *Elements*, for instance in Propositions I.1 and I.22.

This proposition is used in the proofs of IV.10, IV.16, and occasionally in Books X, XI, and XII.

Next proposition: IV.2

Previous: Definitions

Book IV introduction

Select from Book IV

Select book

Select topic

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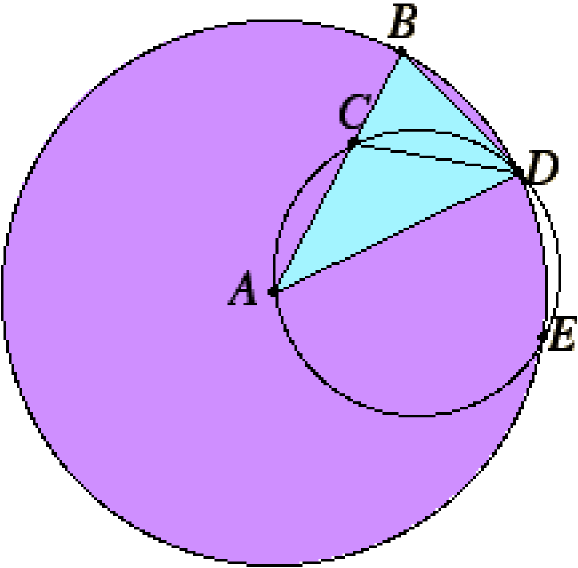
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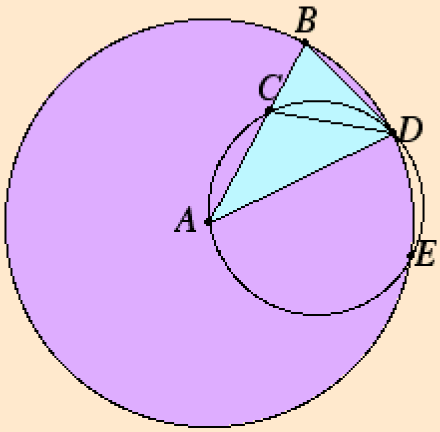
Euclid's Elements
Book IV
Proposition 10

To construct an isosceles triangle having each of the angles at the base double the remaining one.

Set out any straight line *AB,* and cut it at the point *C* so that the rectangle *AB* by *BC* equals the square on *CA.* Describe the circle *BDE* with center A and radius *AB.* Fit in the circle *BDE* the straight line *BD* equal to the straight line *AC* which is not greater than the diameter of the circle *BDE.* II.11 IV.1

Join *AD* and *DC,* and circumscribe the circle *ACD* about the triangle *ACD.* IV.5



Then, since the rectangle *AB* by *BC* equals the square on *AC*, and *AC* equals *BD,* therefore the rectangle *AB* by *BC* equals the square on *BD.*

And, since a point *B* was taken outside the circle *ACD,* and from *B* the two straight lines *BA* and *BD* fall on the circle *ACD*, and one of them cuts it while the other falls on it, and the rectangle *AB* by *BC* equals the square on *BD*, therefore *BD* touches the circle *ACD.* III.37

Since, then, *BD* touches it, and *DC* is drawn across from the point of contact at *D*, therefore the angle *BDC* equals the angle *DAC* in the alternate segment of the circle. III.32

Since, then, the angle *BDC* equals the angle *DAC,* add the angle *CDA* to each, therefore the whole angle *BDA* equals the sum of the two angles *CDA* and *DAC.*

But the exterior angle *BCD* equals the sum of the angles *CDA* and *DAC*, therefore the angle *BDA* also equals the angle *BCD.* I.32

But the angle *BDA* equals the angle *CBD,* since the side *AD* also equals *AB*, so that the angle *DBA* also equals the angle *BCD.* I.5

Therefore the three angles *BDA*, *DBA,* and *BCD* equal one another.

And, since the angle *DBC* equals the angle *BCD,* the side *BD* also equals the side *DC.* I.6

But *BD* equals *CA* by hypothesis, therefore *CA* also equals *CD*, so that the angle *CDA* also equals the angle *DAC.* Therefore the sum of the angles *CDA* and *DAC* is double the angle *DAC.* I.5

And the angle *BCD* equals the sum of the angles *CDA* and *DAC*, therefore the angle *BCD* is also double the angle *CAD.*

But the angle *BCD* equals each of the angles *BDA* and *DBA,* therefore each of the angles *BDA* and *DBA* is also double the angle *DAB.*

Therefore the isosceles triangle *ABD* has been constructed having each of the angles at the base *DB* double the remaining one.

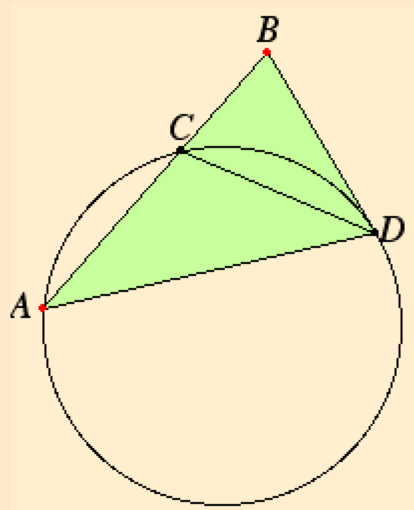
Q.E.F.

Guide

The goal of the proposition is to construct a 36°-72°-72° isosceles triangle *ABD.* It's actually constructed on a given side *AB.* The base will equal the larger part of *AB* when *AB* is cut at a point *C* so that *AB BC* = AC^2. The constuction for that cut was given in proposition II.11. The difficulty of the proof is showing that this construction results in the desired triangle. Cutting *AB* at that point is also called cutting the line "in extreme and mean ratio," see VI.Def.3 for the definition of "extreme and mean ratio," and see proposition VI.30 for details.

Euclid uses a surprizing amount of the theory of circles from Book III. After drawing circle *ACD,* he uses III.37 to conclude from *AB BC* = DB^2 that the line *DB* is tangent to the circle. Next, he uses III.32 to conclude that the angle *BDC* between the tangent *DB* and the chord *DC* equals the angle *CAD* which cuts off that chord.

At this point Euclid has shown that one of the two angles at *D,* namely angle *BDC,* equals the angle *A.* When he shows that the other, namely angle *CDA* also equals angle *A,* then since the triangle *ABD* is isosceles, he will have shown each of the base angles of triangle *ABD* is twice the vertex angle *A,* and the proof will be complete.



The rest is relatively easy. First, the small triangle *BCD* is isosceles, a fact that can be seen from the following equation about angles:

$$B = BDA = BDC + CDA = CAD + CDA = BCD.$$

Therefore, the sides *CD* and *BD* are equal, but from the original construction, *BD* = *CA.* Hence, the triangle *ADC* is also isosceles, so the two angles *CDA* and *A* are equal, as needed.

Comments

Euclid could have split the statement and the proof of this proposition into two. The first part would state that if an isosceles triangle has its base equal to a segment of its side so that square on the base equals the rectangle contained by the side and the remaining segment of the side, then each base angle of the triangle is twice the vertex angle. Most of the proof of this proposition IV.10 is actually a proof of this first part. The other part would be the constuction.

There is a converse of this proposition, one the Euclid did not state. Namely, if an isosceles triangle has each base angle equal to twice the vertex angle, then the base is equal to a segment of its side so that square on the base equals the rectangle contained by the side and the remaining segment of the side. In other words, 36°-72°-72° isosceles triangles are characterized by this property.

The triangle *ABD* constructed in this proposition is one of ten sectors of a regular decagon (10-gon). Thus, it is one short step from this proposition to the construction of a regular decagon inscribed in a circle. If alternate vertices of a regular decagon are connected, then a regular pentagon is formed which is inscribed in the circle. It is unclear why Euclid did not use such a construction rather than the one he chose in the next proposition

An alternate proof involving similar triangles

It was probably Euclid who made a concerted effort to include as many propositions that he could in the first four books that did not rely on proportions. The theory of similar triangles is not broached until Book VI which depends on the theory of proportion in Book V. The clever proof that Euclid gave to this proposition does not depend on similar triangles, and so it could be placed here in Book IV. There is, however, a simpler proof that does depend on similar triangles.

As Euclid does, begin by cutting a straight line *AB* at the point *C* so that the rectangle *AB* by *BC* equals the square on *CA* (II.11). Otherwise said, the straight line *AB* has been cut in extreme and mean ratio at *C* so that the proportion *AB*:*AC* = *AC*:*BC* holds. (See VI.Def.3, VI.17, and VI.30.) Next, construct an isosceles triangle with one side *AB*, a second side *AD* equal to side *AB*, and the base equal *BD* equal to *AC* (I.22). Then we have the proportion *AD*:*BD* = *BD*:*BC*. Therefore, two triangles *ADB* and *DBC* have one angle equal to one angle (angle *D* of the first triangle equals angle *B* of the second) and the sides about the equal angles proportional. Therefore, by VI.6, the triangles are equiangular. It easily follows that both triangles have their base angles each equal to twice their vertex angles.

Use of this proposition

This construction was designed to be used in the next proposition which inscribes a regular pentagon in a circle.

Next proposition: IV.11

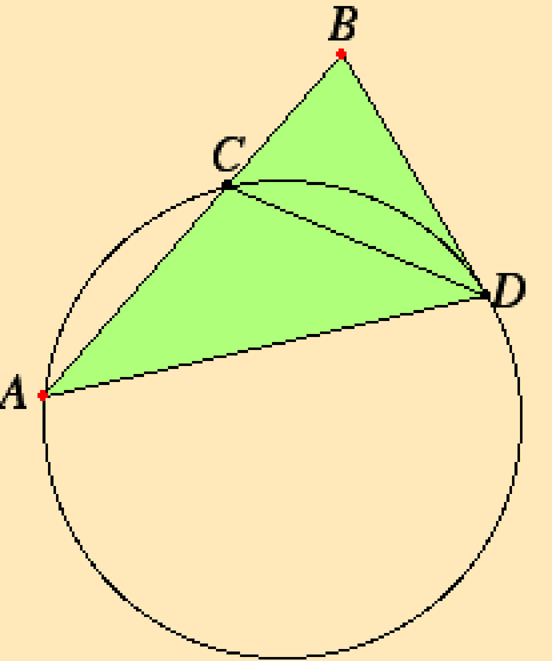
Previous: IV.9

Book IV introduction

Select from Book IV

Select book

Select topic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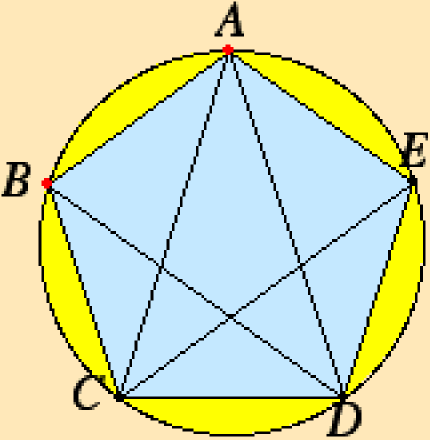
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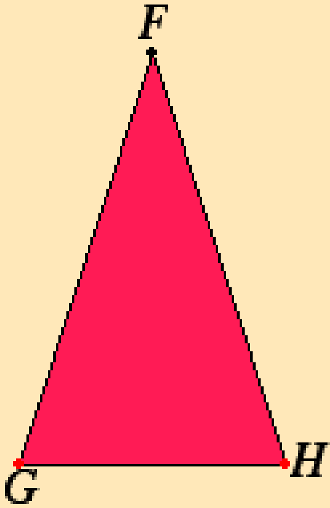
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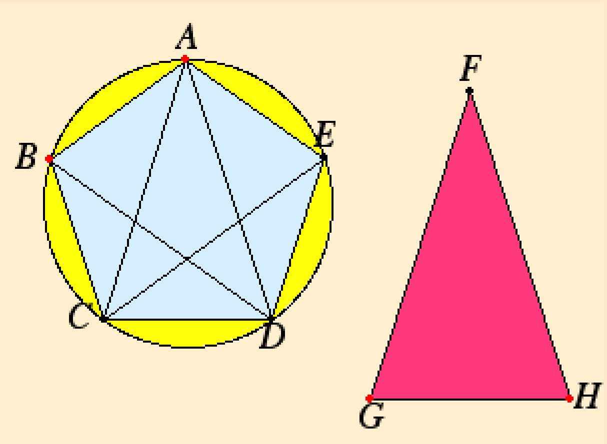
Euclid's Elements
Book IV
Proposition 11

To inscribe an equilateral and equiangular pentagon in a given circle.

Let *ABCDE* be the given circle.

It is required to inscribe an equilateral and equiangular pentagon in the circle *ABCDE.*

Set out the isosceles triangle *FGH* having each of the angles at *G* and *H* double the angle at *F*. Inscribe in the circle *ABCDE* the triangle *ACD* equiangular with the triangle *FGH*, so that the angles *CAD*, *ACD*, and *CDA* equal the angles at *F*, *G*, and *H* respectively. Therefore each of the angles *ACD* and *CDA* is also double the angle *CAD.* IV.10 IV.2



Now bisect the angles *ACD* and *CDA* respectively by the straight lines *CE* and *DB,* and join *AB, BC, DE,* and *EA.* I.9

Then, since each of the angles *ACD* and *CDA* is double the angle *CAD,* and they are bisected by the straight lines *CE* and *DB,* therefore the five angles *DAC*, *ACE*, *ECD*, *CDB,* and *BDA* equal one another.

But equal angles stand on equal circumferences, therefore the five circumferences *AB*, *BC*, *CD*, *DE*, and *EA* equal one another. III.26

But straight lines that cut off equal circumferences are equal, therefore the five straight lines *AB*, *BC*, *CD*, *DE*, and *EA* equal one another. Therefore the pentagon *ABCDE* is equilateral. III.29

I say next that it is also equiangular.

For, since the circumference *AB* equals the circumference *DE*, add *BCD* to each, therefore the whole circumference *ABCD* equals the whole circumference *EDCB.*

And the angle *AED* stands on the circumference *ABCD*, and the angle *BAE* on the circumference *EDCB*, therefore the angle *BAE* also equals the angle *AED.* III.27

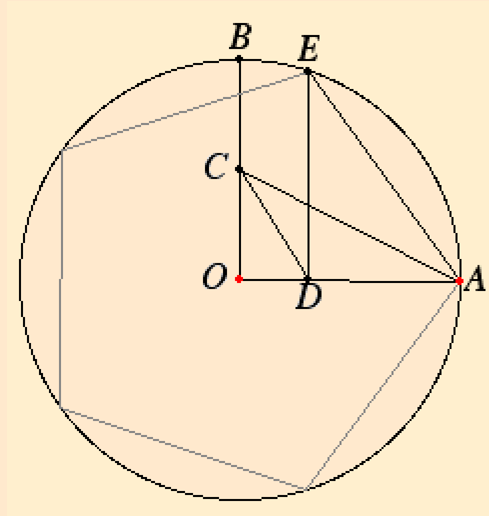
For the same reason each of the angles *ABC*, *BCD*, and *CDE* also equals each of the angles *BAE* and *AED*, therefore the pentagon *ABCDE* is equiangular.

But it was also proved equilateral, therefore an equilateral and equiangular pentagon has been inscribed in the given circle.

Q.E.F.

Guide

Richmond's construction



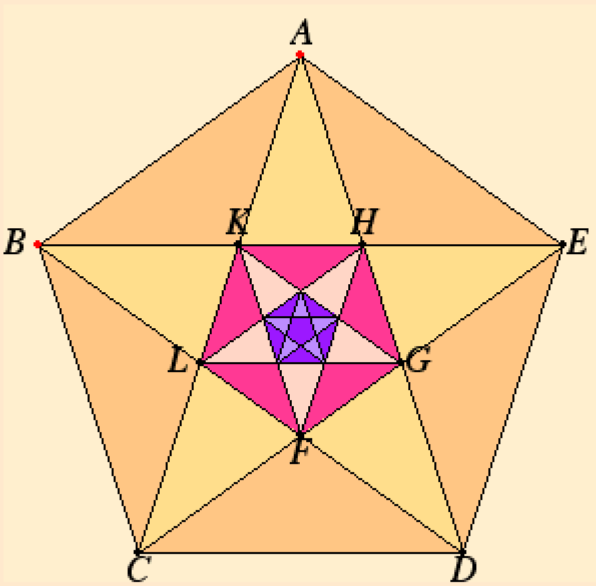
The construction of this proposition is rather tedious to carry out. First, a line has to be cut according to the construction in II.11. Next, that is used in IV.10 for the construction of a 36°-72°-72° isosceles triangle. Next, that triangle is fit into the given circle using the construction IV.2. Finally, a couple more lines are drawn to finish the pentagon.

Various alternatives have have been given by others, such as Ptolemy. One of the nicest was given in 1893 by H. W. Richmond. To inscribe a regular pentagon in a circle, first draw perpendicular radii *OA* and *OB* from the center *O* of a circle. Let *C* be the midpoint of *OB* and draw *AC*. Bisect angle *ACO* to meet *OA* at *D*. Draw a perpendicular *DE* to *OA* to the circle. Then *AE* is one side of the pentagon. The remaining sides can then be constructed.

The easiest way to verify that Richmond's construction works is by means of trigonometry.

The angles and sections in extreme and mean ratio in a pentagram

Consider the regular pentagon *ABCDE* constructed in this proposition.



Draw the diagonals of the pentagram to create a regular star pentagram *ACEBD* inside the pentagon. These diagonals meet forming a smaller regular pentagon in the center of the original pentagon. The diagonals of that pentagon can be drawn to make an inscribed pentagram which, in turn, bound a yet smaller regular pentagon. And so forth.

For purposes of analysis, let d_1 and s_1 denote the diagonal and side of the first regular pentagon *ABCDE.* Also let d_2 and s_2 denote the diagonal and side of the second regular pentagon *FGHKL.* And so forth.

It is evident that there are many lines parallel to the base *CD* of the triangle, namely *BE* and *LG*, as well as innumerable ones in the smaller pentagrams. That means that there will be many 36°-72°-72° triangles besides the large one *ACD.* The next smaller one is *ALG*, then *FKH*, then many smaller ones. Also, each of these various sized 36°-72°-72° triangles is congruent many others in the diagram. For instance, triangles *ALG* and *EAK* are congruent.

There are also a series of obtuse 36°-36°-108° isosceles triangles of varying sizes.

All these parallel lines and similar triangles yield numerous relationships among the various diagonals and sides of the pentagons. Some of these relationships are additive equations:

$$d_1 = s_1 + d_2$$
$$s_1 = d_2 + s_2$$

$d_2 = s_2 + d_3$
$s_2 = d_3 + s_3$
and so forth.

Other relationships are based on the property of 36°-72°-72° triangles used in their construction in IV.10, namely that the square of the base of such a triangle equals the product of a side and the difference between the side and the base. In terms of the diagonals and sides of the pentagons, this gives the equations:

$d_1 d_2 = s_1^2$
$s_1 s_2 = d_2^2$
$d_2 d_3 = s_2^2$
$s_2 s_3 = d_3^2$
and so forth.

After ratios are proportions are developed in Book V and Book VI, we can add the following continued proportion to the list of relationships:

$d_1:s_1 = s_1:d_2 = d_2:s_2 = s_2:d_3 = \ldots$

See the Guide to proposition X.2 which shows that diagonal and side of a regular pentagon are incommensurable. In more modern terms we would say that their ratio, which is called the "golden ratio," is an irrational number.

Use of this proposition

This construction is used in the next proposition to circumscribe a regular pentagon around a circle and later in IV.16 to construct a regular 15-gon. It is also used in XIII.16 for the construction of a regular icosahedron (a 20-sided polyhedron each of whose faces is an equilateral triangle). Surprizingly, it is not used in XIII.17 for construct a regular dodecahedron (a 12-sided polyhedron each of whose faces is a regular pentagon); the regular pentagons needed for it are constructed in space directly without the help of this proposition.

Next proposition: IV.12

Previous: IV.10

Book IV introduction

Select from Book IV

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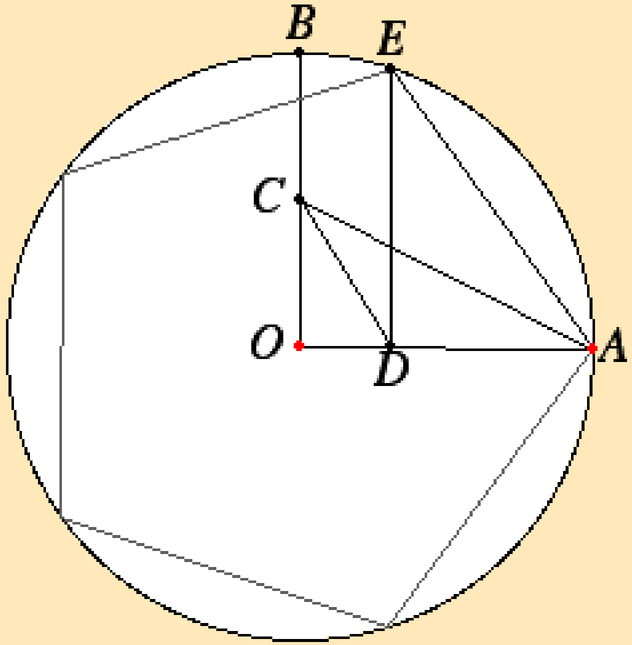
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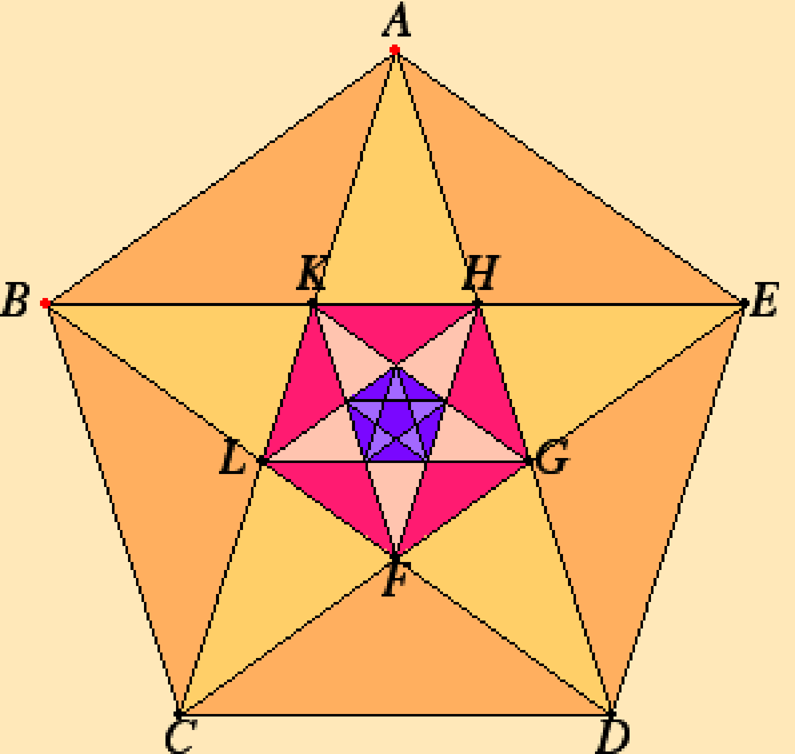
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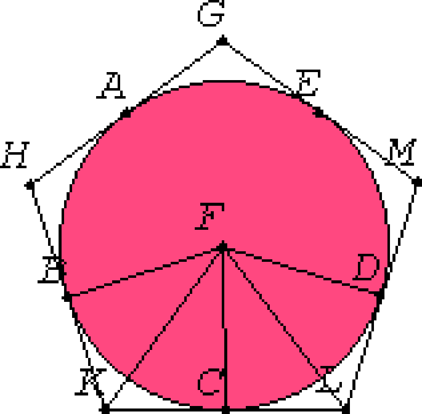
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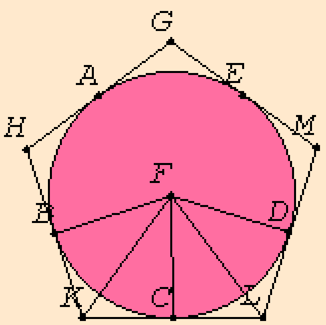
Euclid's Elements
Book IV
Proposition 12

To circumscribe an equilateral and equiangular pentagon about a given circle.

Let *ABCDE* be the given circle.

It is required to circumscribe an equilateral and equiangular pentagon about the circle *ABCDE*.

Let *A*, *B*, *C*, *D*, and *E* be conceived to be the angular points of the inscribed pentagon, so that the circumferences *AB*, *BC*, *CD*, *DE*, and *EA* are equal. Draw *GH*, *HK*, *KL*, *LM*, and *MG* through *A*, *B*, *C*, *D*, and *E* touching the circle. Take the center *F* of the circle *ABCDE*, and join *FB*, *FK*, *FC*, *FL*, and *FD*. IV.11 III.16,Cor III.1



Then, since the straight line *KL* touches the circle *ABCDE* at *C*, and *FC* has been joined from the center *F* to the point of contact at *C*, therefore *FC* is perpendicular to *KL*. Therefore each of the angles at *C* is right. III.18

For the same reason the angles at the points *B* and *D* are also right.

And, since the angle *FCK* is right, therefore the square on *FK* equals the sum of the squares on *FC* and *CK*. I.47

For the same reason the square on *FK* also equals the sum of the squares on *FB* and *BK*, so that the sum of the squares on *FC* and *CK* equals the sum of the squares on *FB* and *BK*, of which the square on *FC* equals the square on *FB*, therefore the remaining square on *CK* equals the square on *BK*. I.47

Therefore *BK* equals *CK*.

And, since *FB* equals *FC*, and *FK* is common, the two sides *BF* and *FK* equal the two sides *CF* and *FK*, and the base *BK* equals the base *CK*, therefore the angle *BFK* equals the angle *KFC*, and the angle *BKF* equals the angle *FKC*. Therefore the angle *BFC* is double the angle *KFC*, and the angle *BKC* double the angle *FKC*. I.8

For the same reason the angle *CFD* is also double the angle *CFL*, and the angle *DLC* double the angle *FLC*.

Now, since the circumference *BC* equals *CD*, the angle *BFC* also equals the angle *CFD*. III.27

And the angle *BFC* is double the angle *KFC*, and the angle *DFC* double the angle *LFC*, therefore the angle *KFC* also equals the angle *LFC*.

But the angle *FCK* also equals the angle *FCL*, therefore *FKC* and *FLC* are two triangles having two angles equal to two angles and one side equal to one side, namely *FC* which is common to them, therefore they will also have the remaining sides equal to the remaining sides, and the remaining angle to the remaining angle, therefore the straight line *KC* equals *CL*, and the angle *FKC* equals the angle *FLC*. I.26

And, since *KC* equals *CL*, therefore *KL* is double *KC*.

For the same reason it can be proved that *HK* is also double *BK*.

And *BK* equals *KC*, therefore *HK* also equals *KL*.

Similarly each of the straight lines *HG, GM,* and *ML* can also be proved equal to each of the straight lines *HK* and *KL,* therefore the pentagon *GHKLM* is equilateral.

I say next that it is also equiangular.

For, since the angle *FKC* equals the angle *FLC,* and the angle *HKL* was proved double the angle *FKC,* and the angle *KLM* double the angle *FLC,* therefore the angle *HKL* also equals the angle *KLM.*

Similarly each of the angles *KHG, HGM,* and *GML* can also be proved equal to each of the angles *HKL* and *KLM.* Therefore the five angles *GHK, HKL, KLM, LMG,* and *MGH* equal one another.

Therefore the pentagon *GHKLM* is equiangular.

And it was also proved equilateral, and it has been circumscribed about the circle *ABCDE.*

Q.E.F.

Guide

This construction depends on the last. First, inscribe a regular pentagon in the circle, then take tangents to the circle at the five vertices of the inscribed pentagon. The result will be a circumscribed pentagon. This method generally works to create a regular circumscribed *n*-gon given a regular inscribed *n*-gon.

Conversely, if you have a regular circumscribed *n*-gon, then you can connect the points of tangency in sequence to get a regular inscribed *n*-gon.

Next proposition: IV.13

Previous: IV.11

Book IV introduction

Select from Book IV

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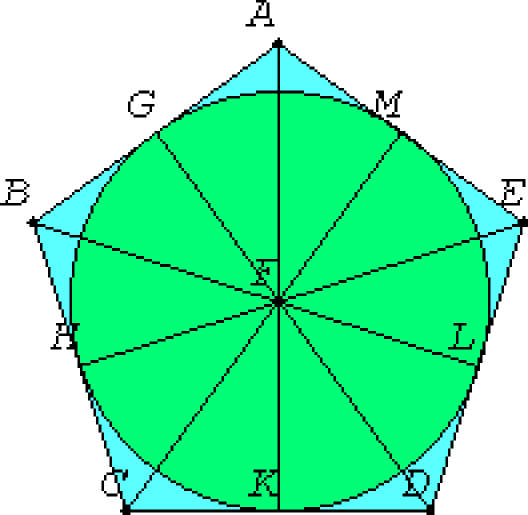

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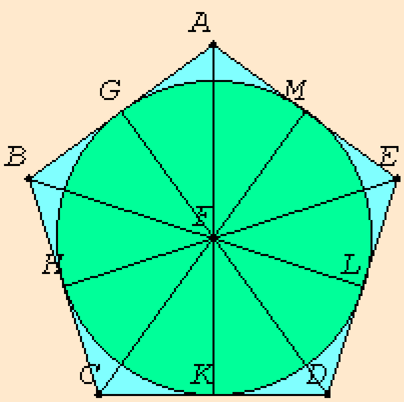
Euclid's Elements
Book IV
Proposition 13

To inscribe a circle in a given equilateral and equiangular pentagon.

Let *ABCDE* be the given equilateral and equiangular pentagon.

It is required to inscribe a circle in the pentagon *ABCDE.*

Bisect the angles *BCD* and *CDE* by the straight lines *CF* and *DF* respectively. Join the straight lines *FB, FA,* and *FE* from the point *F* at which the straight lines *CF* and *DF* meet one another. I.9



Then, since *BC* equals *CD,* and *CF* common, the two sides *BC* and *CF* equal the two sides *DC* and *CF,* and the angle *BCF* equals the angle *DCF,* therefore the base *BF* equals the base *DF,* and the triangle *BCF* equals the triangle *DCF,* and the remaining angles equal the remaining angles, namely those opposite the equal sides. I.4

Therefore the angle *CBF* equals the angle *CDF.*

And, since the angle *CDE* is double the angle *CDF,* and the angle *CDE* equals the angle *ABC,* while the angle *CDF* equals the angle *CBF,* therefore the angle *CBA* is also double the angle *CBF.* Therefore the angle *ABF* equals the angle *FBC.* Therefore the angle *ABC* is bisected by the straight line *BF.*

Similarly it can be proved that the angles *BAE* and *AED* are also bisected by the straight lines *FA* and *FE* respectively.

Now draw *FG, FH, FK, FL,* and *FM* from the point *F* perpendicular to the straight lines *AB, BC, CD, DE,* and *EA.* I.12

Then, since the angle *HCF* equals the angle *KCF,* and the right angle *FHC* also equals the angle *FKC, FHC* and *FKC* are two triangles having two angles equal to two angles and one side equal to one side, namely *FC* which is common to them and opposite one of the equal angles, therefore they also have the remaining sides equal to the remaining sides. Therefore the perpendicular *FH* equals the perpendicular *FK.* I.26

Similarly it can be proved that each of the straight lines *FL, FM,* and *FG* also equals each of the straight lines *FH* and *FK,* therefore the five straight lines *FG, FH, FK, FL,* and *FM* equal one another.

Therefore the circle described with center *F* and radius one of the straight lines *FG, FH, FK, FL,* or *FM* also passes through the remaining points, and it touches the straight lines *AB, BC, CD, DE,* and *EA,* because the angles at the points *G, H, K, L,* and *M* are right.

For, if it does not touch them. but cuts them, it will result that the straight line drawn at right angles to the diameter of the circle from its end falls within the circle, which was proved absurd. III.16

Therefore the circle described with center *F* and radius one of the straight lines *FG, FH, FK, FL,* or *FM* does not cut the straight lines *AB, BC, CD, DE,* and *EA.* Therefore it touches them.

Let it be described, as *GHKLM.*

Therefore a circle has been inscribed in the given equilateral and equiangular pentagon.

Q.E.F.

Guide

The method given here to inscribe a circle in a regular pentagon works in general to inscribe a circle in a regular *n*-gon. Simply bisect two of the angles of the *n*-gon to find the center of the circle. Then draw a perpendicular to one of the sides. The foot of the perpendicular gives a point on the circumference of the circle.

Next proposition: IV.14

Previous: IV.12

Book IV introduction

Select from Book IV

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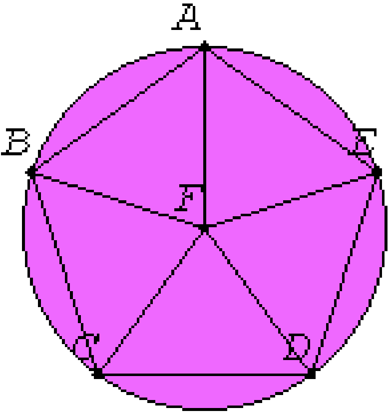
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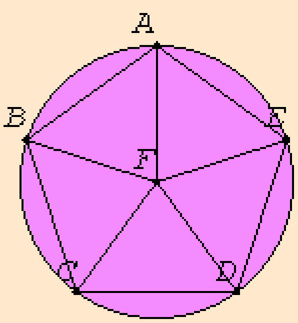
Euclid's Elements
Book IV
Proposition 14

To circumscribe a circle about a given equilateral and equiangular pentagon.

Let *ABCDE* be the given pentagon, which is equilateral and equiangular.

It is required to circumscribe a circle about the pentagon *ABCDE.*

Bisect the angles *BCD* and *CDE* by the straight lines *CF* and *DF* respectively. Join the straight lines *FB, FA,* and *FE* from the point *F* at which the straight lines meet to the points *B, A,* and *E.* I.9



Then in manner similar to the preceding it can be proved that the angles *CBA*, *BAE*, and *AED* are also bisected by the straight lines *FB*, *FA*, and *FE* respectively.

Now, since the angle *BCD* equals the angle *CDE*, and the angle *FCD* is half of the angle *BCD*, and the angle *CDF* half of the angle *CDE*, therefore the angle *FCD* also equals the angle *CDF*, so that the side *FC* also equals the side *FD.* I.6

Similarly it can be proved that each of the straight lines *FB*, *FA*, and *FE* also equals each of the straight lines *FC* and *FD.* Therefore the five straight lines *FA*, *FB*, *FC*, *FD*, and *FE* equal one another.

Therefore the circle described with center *F* and radius one of the straight lines *FA*, *FB*, *FC*, *FD*, or *FE* also passes through the remaining points, and is circumscribed.

Let it be circumscribed, and let it be *ABCDE.*

Therefore a circle has been circumscribed about the given equilateral and equiangular pentagon.

Q.E.F.

Guide

This construction is used in propositions XIII.8 and XIII.18.

Next proposition: IV.15

Previous: IV.13

Book IV introduction

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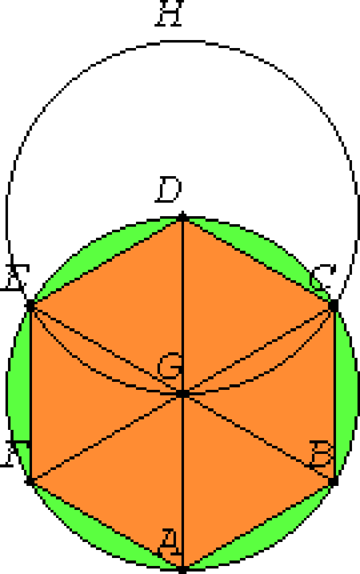

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Euclid's Elements
Book IV
Proposition 15

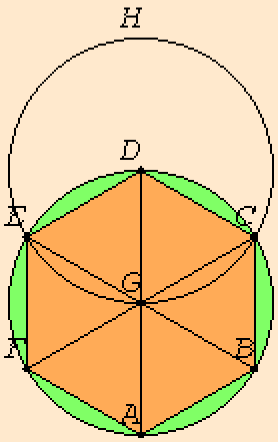
To inscribe an equilateral and equiangular hexagon in a given circle.

Let *ABCDEF* be the given circle.

It is required to inscribe an equilateral and equiangular hexagon in the circle *ABCDEF*.

Draw the diameter *AD* of the circle *ABCDEF*. Take the center *G* of the circle. Describe the circle *EGCH* with center *D* and radius *DG*. Join *EG* and *CG* and carry them through to the points *B* and *F*. Join *AB*, *BC*, *CD*, *DE*, *EF*, and *FA*. III.1

I say that the hexagon *ABCDEF* is equilateral and equiangular.



For, since the point *G* is the center of the circle *ABCDEF*, *GE* equals *GD*.

Again, since the point *D* is the center of the circle *GCH*, *DE* equals *DG*.

But *GE* was proved equal to *GD*, therefore *GE* also equals *ED*. Therefore the triangle *EGD* is equilateral, and therefore its three angles *EGD*, *GDE*, and *DEG* equal one another, inasmuch as, in isosceles triangles, the angles at the base equal one another. I.5

And the sum of the three angles of the triangle equals two right angles, therefore the angle *EGD* is one-third of two right angles. I.32

Similarly, the angle *DGC* can also be proved to be one third of two right angles.

And, since the straight line *CG* standing on *EB* makes the sum of the adjacent angles *EGC* and *CGB* equal to two right angles, therefore the remaining angle *CGB* is also one-third of two right angles. I.13

Therefore the angles *EGD*, *DGC*, and *CGB* equal one another, so that the angles vertical to them, the angles *BGA*, *AGF*, and *FGE*, are equal. I.15

Therefore the six angles *EGD*, *DGC*, *CGB*, *BGA*, *AGF*, and *FGE* equal one another.

But equal angles stand on equal circumferences, therefore the six circumferences *AB*, *BC*, *CD*, *DE*, *EF*, and *FA* equal one another. III.26

And straight lines that cut off equal circumferences are equal, therefore the six straight lines equal one another. Therefore the hexagon *ABCDEF* is equilateral. III.29

I say next that it is also equiangular.

For, since the circumference *FA* equals the circumference *ED*, add the circumference *ABCD* to each, therefore the whole *FABCD* equals the whole *EDCBA*. And the angle *FED* stands on the circumference *FABCD*, and the angle *AFE* on the circumference *EDCBA*, therefore the angle *AFE* equals the angle *DEF*. III.27

Similarly it can be proved that the remaining angles of the hexagon *ABCDEF* are also severally equal to each of the angles *AFE* and *FED*, therefore the hexagon *ABCDEF* is equiangular.

But it was also proved equilateral, and it has been inscribed in the circle *ABCDEF*.

Therefore an equilateral and equiangular hexagon has been inscribed in the given circle.

Q.E.F.

Corollary

From this it is manifest that *the side of the hexagon equals the radius of the circle.*

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

Guide

The corollary is used in several propositions in Book XIII starting with XIII.9.

Next proposition: IV.16

Previous: IV.14

Book IV introduction

Select from Book IV

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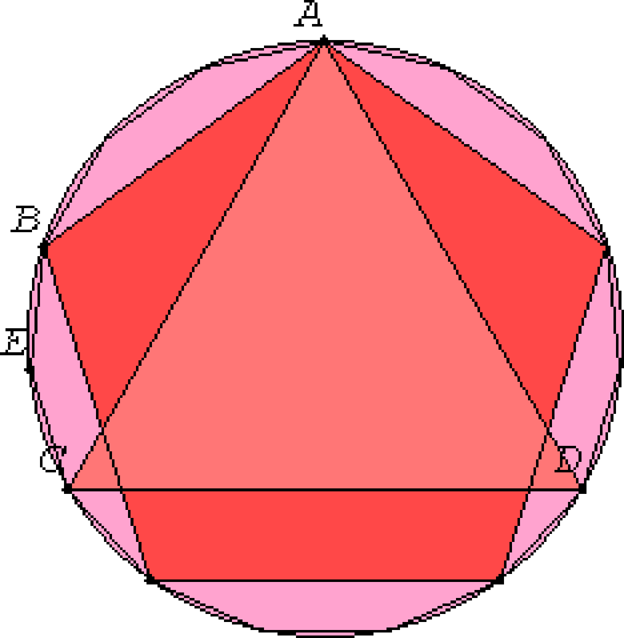
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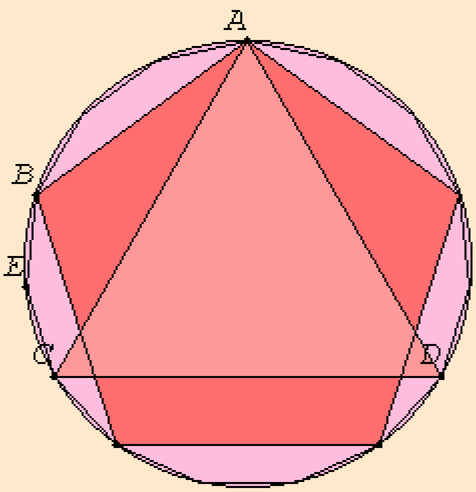
Euclid's Elements
Book IV
Proposition 16

To inscribe an equilateral and equiangular fifteen-angled figure in a given circle.

Let *ABCD* be the given circle.

It is required to inscribe in the circle *ABCD* a fifteen-angled figure which shall be both equilateral and equiangular.

Inscribe a side *AC* of an equilateral triangle and a side *AB* of an equilateral pentagon in in the circle *ABCD.* Therefore, of the equal segments of which there are fifteen in the circle *ABCD,* there will be five in the circumference *ABC* which is one-third of the circle, and there will be three in the circumference *AB* which is one-fifth of the circle. Therefore in the remainder *BC* there will be two of the equal segments. IV.2 IV.11



Inscribe a side *AC* of an equilateral triangle and a side *AB* of an equilateral pentagon in in the circle *ABCD.* Therefore, of the equal segments of which there are fifteen in the circle *ABCD*, there will be five in the circumference *ABC* which is one-third of the circle, and there will be three in the circumference *AB* which is one-fifth of the circle. Therefore in the remainder *BC* there will be two of the equal segments. IV.2 IV.11

Bisect *BC* at *E.* Therefore each of the circumferences *BE* and *EC* is a fifteenth of the circle *ABCD.* III.30

If therefore we join *BE* and *EC* and continually fit into the circle *ABCD* straight lines equal to them, a fifteen-angled figure which is both equilateral and equiangular will be inscribed in it. IV.1

Q.E.F.

Corollary

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled figure and circumscribe one about it.

Guide

The arc *AC* is 1/3 of the circle, since *A* and *B* are two of the three equally spaced vertices of a regular triangle. Likewise, the arc *AC* is 1/5 of the circle, since *A* and *C* are two adjacent points of a regular pentagon. Therefore, the difference of these two arcs, *AC* – *AB*, which is the arc *BC* is 1/3 –1/5 of the circle, that is 2/15 of the circle. Since *E* bisects that arc *BC*, therefore *BE* and *EC* are each 1/15 of the circle. The rest of the regular 15-gon can then easily be constructed.

Constructable regular polygons

Now, by the end of Book IV, Euclid has described how to construct many regular polygons. The regular 3-gon, known as the equilateral triangle, was constructed in I.1, while the regular 4-gon, known as the square, was constructed in I.46. In book IV, regular 5-gons and regular 6-gons have been constructed. An application of III.30 (which was used in this proposition) can double the number of sides of a regular polygon, and therefore regular polygons with 8, 10, 12, 16, 20, 24, etc., sides can be constructed. This proposition shows how to use a regular *m*-gon and a regular *n*-gon to produce a regular *mn*-gon, provided that *m* and *n* are relatively prime numbers. That produced a 15-gon, and from that we can produce regular polygons with 30, 60, 120, etc., sides. Thus, a regular *n*-gon can be constructed if the only prime numbers that divide *n* are 2, 3, and 5, where 2 can be a repeated factor, but 3 and 5 are not repeated.

But are there any others? What about regular polygons with 7, 9, 11, 13, 17, 18, 19, etc., sides? Euclid said nothing about them, but the ancient Greek mathematicians expected that they couldn't be constructed with only the Euclidean tools of straightedge and compass. There were constructions involving conic sections (hyperbolas, parabolas, ellipses) to trisect an angle. With such a construction a 9-gon can be made. But methods involving conic sections go beyond Euclidean tools. With the help of non-algebraic curves, like Archimedes' spiral, an angle can be divided into any number of equal parts, and with the aid of those curves any *n*-gon can be constructed. But, again, they go beyond Euclidean tools.

The problem of constructing other regular polygons with Euclidean tools remained just that, a problem, for over 2000 years. Finally, Carl Friedrich Gauss (1777-1855) made progress. He described in his *Disquitiones Arithmeticae*, a major work on number theory, how to construct a regular 17-gon with Euclidean tools. Thus, 17 can be added to 3 and 5 as prime numbers that can divide *n*, but at most once. Furthermore, he showed that any prime number which is of the form $2^{2^k} + 1$ can be included. Such prime numbers are called *Fermat primes*. The known Fermat primes are 3 (which is $2^{2^0} + 1$), 5 (which is $2^{2^1} + 1$), 17 (which is $2^{2^2} + 1$), 257 (which is $2^{2^3} + 1$), and 65537 (which is $2^{2^4} + 1$). Thus, 257 and 65537 can be appended to the list 3, 5, 17. It is not known whether there are any more Fermat primes.

Gauss was convinced that the *only* constructable *n*-gons were those where *n* was only divisible by 2 and the Fermat primes, where the Fermat primes were not repeated. But he had no proof of that, but in 1837 Wantzel did.

Next book: Book V Introduction

Previous proposition: IV.15

Book IV introduction

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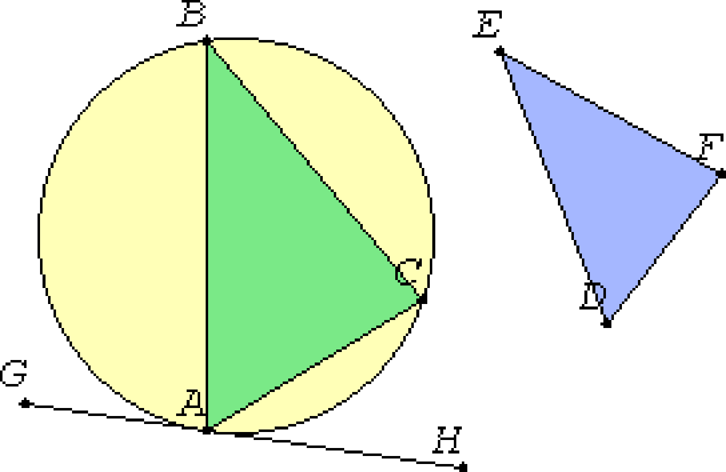
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Euclid's Elements

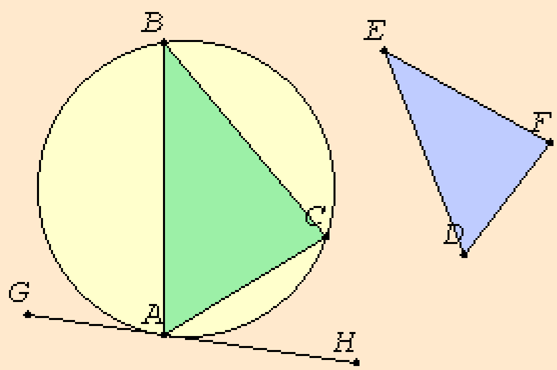
Book IV

Proposition 2

To inscribe a triangle equiangular with a given triangle in a given circle.

Let *ABC* be the given circle, and *DEF* the given triangle.

It is required to inscribe a triangle equiangular with the triangle *DEF* in the circle *ABC.*



Draw *GH* touching the circle *ABC* at *A.* Construct the angle *HAC* equal to the angle *DEF* on the straight line *AH* and at the point *A* on it, and construct the angle *GAB* equal to the angle *DFE* on the straight line *AG* and at the point *A* on it. Join *BC.* III.16,Cor I.23

Then, since a straight line *AH* touches the circle *ABC,* and from the point of contact at *A* the straight line *AC* is drawn across in the circle, therefore the angle *HAC* equals the angle *ABC* in the alternate segment of the circle. III.32

But the angle *HAC* equals the angle *DEF,* therefore the angle *ABC* also equals the angle *DEF.*

For the same reason the angle *ACB* also equals the angle *DFE,* therefore the remaining angle *BAC* also equals the remaining angle *EDF.* I.32

Therefore a triangle equiangular with the given triangle has been inscribed in the given circle. IV.Def.2

Q.E.F.

Guide

This construction is used in propositions IV.11, IV.16, and XIII.13.

Next proposition: IV.3

Previous: IV.1

Book IV introduction

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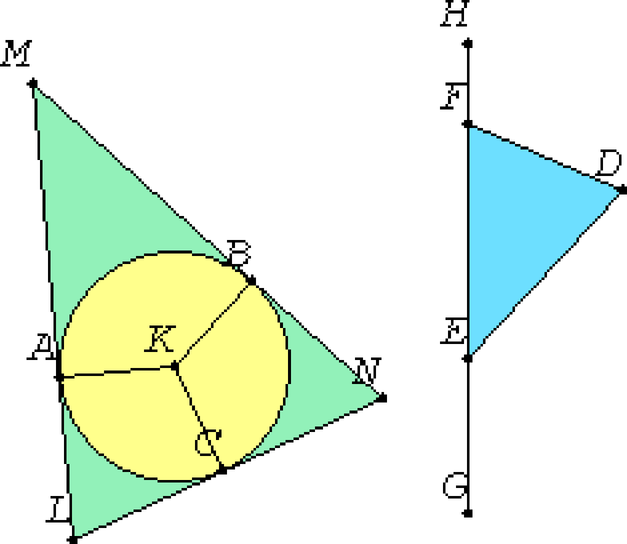
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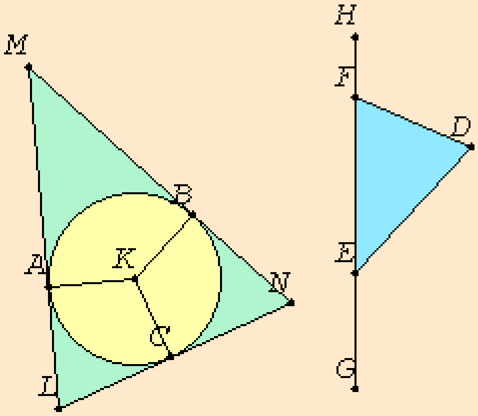
Euclid's Elements
Book IV
Proposition 3

To circumscribe a triangle equiangular with a given triangle about a given circle.

Let *ABC* be the given circle, and *DEF* the given triangle.

It is required to circumscribe a triangle equiangular with the triangle *DEF* about the circle *ABC*.



Produce *EF* in both directions to the points *G* and *H.* Take the center *K* of the circle *ABC*, and draw a radius *KB* at random. On the straight line *KB* and at the point *K* on it, construct the angle *BKA* equal to the angle *DEG*, and the angle *BKC* equal to the angle *DFH.* Through the points *A*, *B*, and *C* draw *LAM*, *MBN*, and *NCL* touching the circle *ABC.* III.1 I.23 III.16,Cor

Now, since *LM*, *MN*, and *NL* touch the circle *ABC* at the points *A*, *B*, and *C*, and *KA*, *KB*, and *KC* have been joined from the center *K* to the points *A*, *B*, and *C*, therefore the angles at the points *A*, *B*, and *C* are right. III.18

And, since the four angles of the quadrilateral *AMBK* equal four right angles, inasmuch as *AMBK* is in fact divisible into two triangles, and the angles *KAM* and *KBM* are right, therefore the sum of the remaining angles *AKB* and *AMB* equals two right angles.

But the sum of the angles *DEG* and *DEF* also equals two right angles, therefore the sum of the angles *AKB* and *AMB* equals the sum of the angles *DEG* and *DEF*, of which the angle *AKB* equals the angle *DEG*, therefore the remaining angle *AMB* equals the remaining angle *DEF*. I.13

Similarly it can be proved that the angle *LNB* also equals the angle *DFE*, therefore the remaining angle *MLN* equals the angle *EDF*. I.32

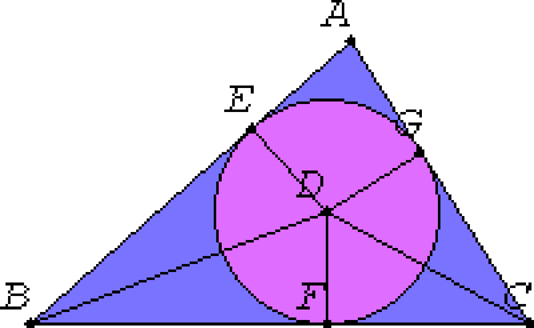
Therefore the triangle *LMN* is equiangular with the triangle *DEF*, and it has been circumscribed about the circle *ABC*. IV.Def.4

Therefore a triangle equiangular with the given triangle has been circumscribed about a given circle.

Q.E.F.

Guide

This proposition is not used elsewhere in the *Elements*, but is included as a mate to the previous proposition in which a triangle is inscribed inside rather than circumscribed outside a given circle.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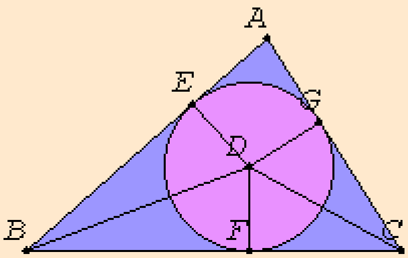
Euclid's Elements
Book IV
Proposition 4

To inscribe a circle in a given triangle.

Let *ABC* be the given triangle.

It is required to inscribe a circle in the triangle *ABC*.



Bisect the angles *ABC* and *ACB* by the straight lines *BD* and *CD*, and let these meet one another at the point *D*. Draw *DE*, *DF*, and *DG* from *D* perpendicular to the straight lines *AB*, *BC*, and *CA*. I.9 I.12

Now, since the angle *ABD* equals the angle *CBD*, and the right angle *BED* also equals the right angle *BFD*, *EBD* and *FBD* are two triangles having two angles equal to two angles and one side equal to one side, namely that opposite one of the equal angles, which is *BD* common to the triangles, therefore they will also have the remaining sides equal to the remaining sides, therefore *DE* equals *DF*. I.26

For the same reason *DG* also equals *DF*.

Therefore the three straight lines *DE*, *DF*, and *DG* equal one another. Therefore the circle described with center *D* and radius one of the straight lines *DE*, *DF*, or *DG* also passes through the remaining points and touches the straight lines *AB*, *BC*, and *CA*, because the angles at the points *E*, *F*, and *G* are right.

For, if it cuts them, the straight line drawn at right angles to the diameter of the circle from its end will be found to fall within the circle, which was proved absurd, therefore the circle described with center *D* and radius one of the straight lines *DE*, *DF*, or *DG* does not cut the straight lines *AB*, *BC*, and *CA* Therefore it touches them, and is the circle inscribed in the triangle *ABC*. III.16 IV.Def.5

Let it be inscribed as *FGE*.

Therefore the circle *EFG* has been inscribed in the given triangle *ABC*.

Q.E.F.

Guide

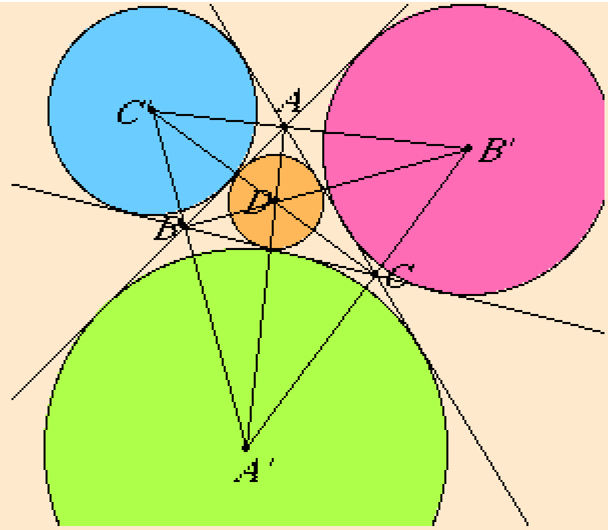
It is easy to supply the missing argument that the angle bisectors *BD* and *CD* do meet.

Incircles and excircles

This circle inscribed in a triangle has come to be known as the *incircle* of the triangle, its center the *incenter* of the triangle, and its radius the *inradius* of the triangle.

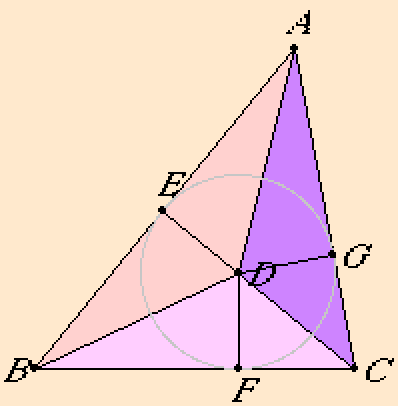
The incircle is a circle tangent to the three lines *AB*, *BC*, and *AC*. If these three lines are extended, then there are three other circles also tangent to them, but outside the triangle. They are called the *excircles.*

The points on the internal angle bisector *AD* are equidistant from the two sides of the triangle *AB* and *AC*. The line *KL* is perpendicular to *AD* at *A*, and the points on it are also equidistant from the extended sides *AB* and *AC*. The line *B'C'* is called the *external angle bisector* at *A*. Whereas the incenter *D* lies at the confluence of the three internal angle bisectors, the excenter *B'* lies at the confluence of two external angle bisectors *AB'* and *CB'* and one internal angle bisector *BB'*. Likewise for the other two excenters *A'* and *C'*.



Heron's formula

Heron of Alexandria (first century C.E.) was an important Greek mathematician who wrote, among other things, a commentary on the *Elements* which is lost now but was known to Proclus and an-Nairizi. In Heron's *Metrica,* which was rediscovered in 1896, there appears a proof of what is called Heron's formula. It states that the area of a triangle is the square root of $s(s-a)(s-b)(s-c)$ where $a = BC$, $b = AC$, and $c = AB$, the sides of the triangle, and s is the semiperimeter $(a + b + c)/2$. Archimedes may have known this formula, but but we don't have his proof. Heath gives Heron's complete proof, but here we'll just look at the first part that involves the incircle.



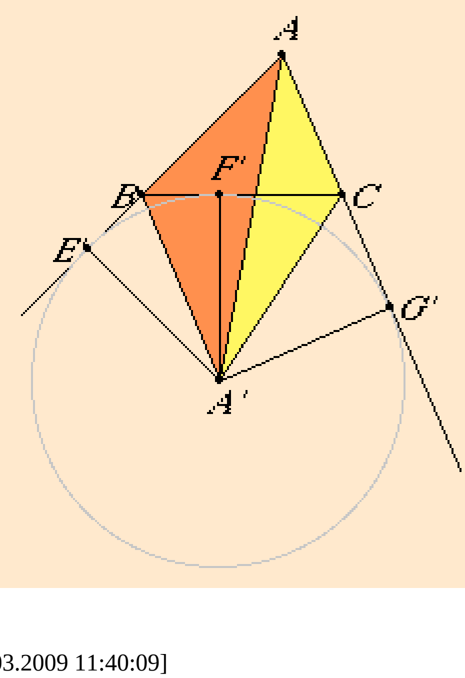
Let *D* be the incenter of the triangle *ABC*, and let *DE*, *DF*, and *DG* be perpendicular lines drawn to the sides as in Euclid's proof. These three lines are radii of the incircle, and therefore have length *r*, the inradius. The triangle *ABD* has base *AB* and height *r*, so its area is *r AB*/2. Likewise, the area of triangle *BCD* is *r BC*/2, and the area of triangle *CAD* is *r CA*/2. Adding these together we find the area of triangle *ABC* is *r* (*AB* + *BC* + *CA*)/2. Therefore we have

$$\text{Area}(ABC) = rs$$

an interesting result in itself.

We'll leave Heron's proof now and consider the corresponding statement for excircles.

Now let *A'* be the excenter on the bisector of the internal angle at *A*. Let *A'E'*, *A'F'*, and *A'G'* be the perpendiculars drawn from *A'* to the sides of the triangle. They are radii of the excircle of length r_A. Triangle *ABA'* has base *AB* and height *A'E'*, so its area is r_A *AB*/2. Likewise, the area of triangle *BCA'* is r_A *BC*/2, and the area of triangle *CAA'* is r_A *CA*/2. Triangle *ABC* is the sum of triangles *ABA'* and *ACA'* minus triangle *BCA'*, so its area is r_A (*AB* + *AC* – *CA*)/2 which equals $r_A(s - A)$.



From the other excircles we get two more equations. We then have

$$\text{Area}(ABC) = rs = r_A(s - A) = r_B(s - B) = r_C(s - C).$$

There are other relationships among these radii, for instance,

$$1/r = 1/r_A + 1/r_B + 1/r_C,$$

but let's stop here to go on to the circumcircle of a triangle.

Next proposition: IV.5

Previous: IV.3

Book IV introduction

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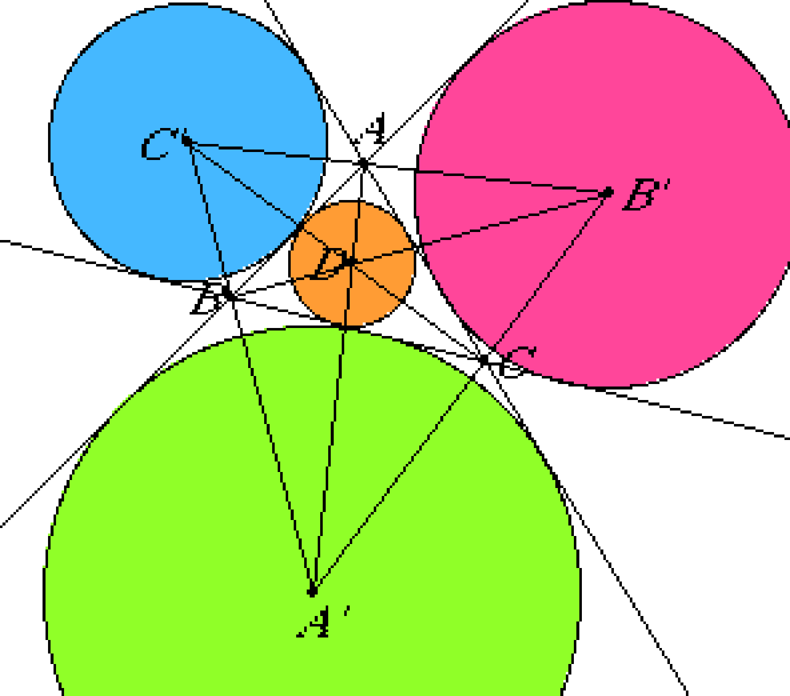

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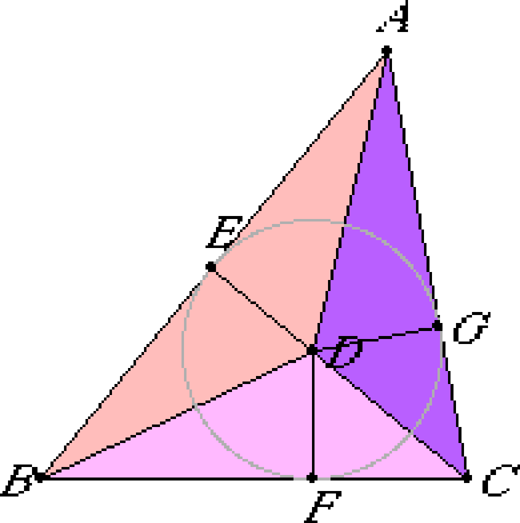
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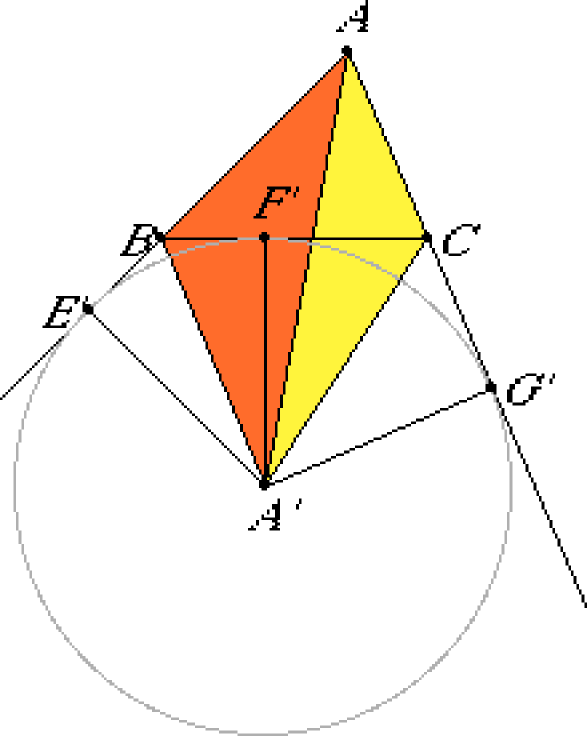

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Euclid's Elements

Book IV

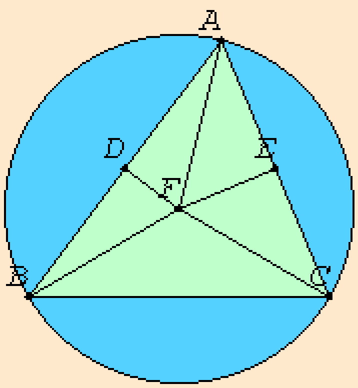
Proposition 5

To circumscribe a circle about a given triangle.

Let *ABC* be the given triangle.

It is required to circumscribe a circle about the given triangle *ABC.*

Bisect the straight lines *AB* and *AC* at the points *D* and *E.* Draw *DF* and *EF* from the points *D* and *E* at right angles to *AB* and *AC.* They will then meet within the triangle *ABC*, or on the straight line *BC*, or outside *BC.* I.10 I.11



First let them meet within at *F.* Join *FB*, *FC*, and *FA.*

Then, since *AD* equals *DB*, and *DF* is common and at right angles, therefore the base *AF* equals the base *FB.* I.4

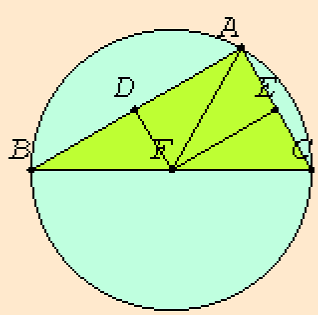
Similarly we can prove that *CF* also equals *AF*, so that *FB* also equals *FC*, therefore the three straight lines *FA* and *FB* and *FC* equal one another.

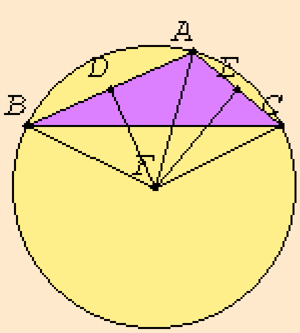
Therefore the circle described with center *F* and radius one of the straight lines *FA*, *FB*, or *FC* also passes through the remaining points, and the circle is circumscribed about the triangle *ABC.* IV.Def.6

Let it be circumscribed as *ABC.*

Next, let *DF* and *EF* meet on the straight line *BC* at *F*, as is the case in the second figure. Join *AF.*

Then, similarly, we can prove that the point *F* is the center of the circle circumscribed about the triangle *ABC.*





Next, let *DF* and *EF* meet outside the triangle *ABC* at *F*, as is the case in the third figure. Join *AF, BF*, and *CF.*

Then again, since *AD* equals *DB*, and *DF* is common and at right angles, therefore the base *AF* equals the base *BF.* I.4

Similarly we can prove that *CF* also equals *AF*, so that *BF* also equals *FC*. Therefore the circle described with center *F* and radius one of the straight lines *FA*, *FB*, or *FC* also passes through the remaining points, and is circumscribed about the triangle *ABC*. IV.Def.6

Therefore a circle has been circumscribed about the given triangle.

Q.E.F.

[Corollary]

And it is manifest that when the center of the circle falls within the triangle, the angle *BAC*, being in a segment greater than the semicircle, is less than a right angle, when the center falls on the straight line *BC*, the angle *BAC*, being in a semicircle, is right, and when the center of the circle falls outside the triangle, the angle *BAC*, being in a segment less than the semicircle, is greater than a right angle. III.31

Guide

As noted by Simson and others, Euclid does not justify the intersection of the perpendicular bisectors *DF* and *EF*. Such a justification is necessary but easy to supply.

The note following the proposition is not actually called a corollary in the Greek text. It is just a remark following the proposition.

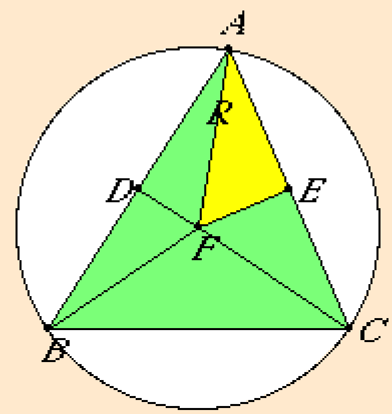
Circumcircles

This circle drawn about a triangle is called, naturally enough, the *circumcircle* of the triangle, its center the *circumcenter* of the triangle, and its radius the *circumradius*. Much has been discovered about the theory of incircles and circumcircles since Euclid.

The ratio that appears in the law of sines in trigonometry is the diameter of the circumcircle:

$$2R = BC / \sin A = CA / \sin B = AB / \sin C$$

where *R* is the circumradius.



This relation is easy to derive from the figure. Angle *AFC* is twice the angle at *B* [III.20], but it is also twice angle *AFE* since the triangles *AFE* and *CFE* are congruent. Therefore angle *AFE* equals the angle at *B*. Then the sine of *B* can be found in the right triangle *AFE* as the ratio of the side *AE* opposite angle *AFE* to the hypotenuse *AF*. Since *AE* is half of *AC*, it follows that $\sin B = AC/(2R)$ which yields one of the three equations for the law of sines.

There is also an equation relating the circumradius *R*, the inradius *r*, and the three exradii r_A, r_B, and r_C:

$$4R = r_A + r_B + r_C - r,$$

and a number of other interesting results about circumcircles, incircles, and other constructions based on an arbitrary triangle.

This construction is used in propositions IV.10 and XI.23.

Next proposition: IV.6

Previous: IV.4

Book IV introduction

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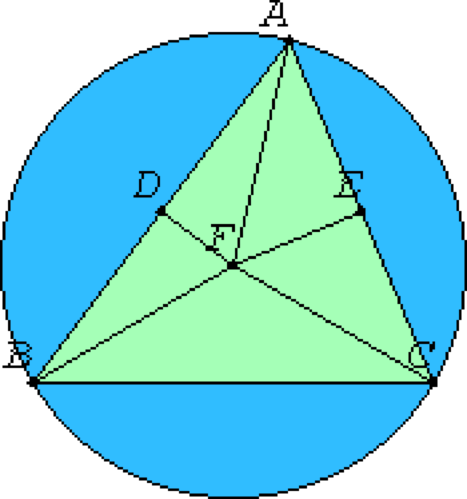
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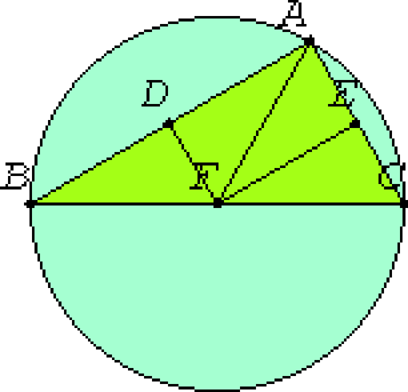
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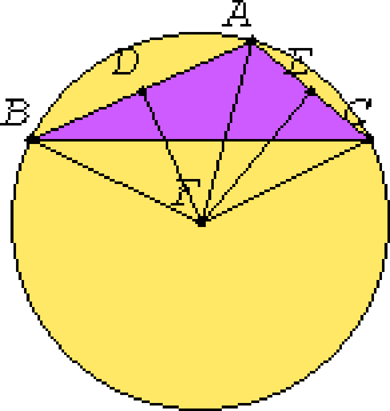
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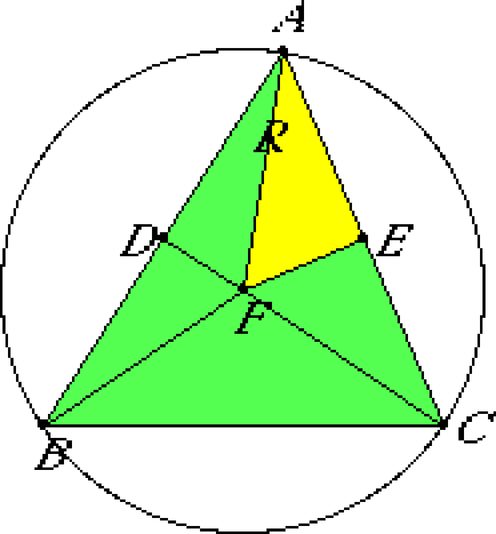

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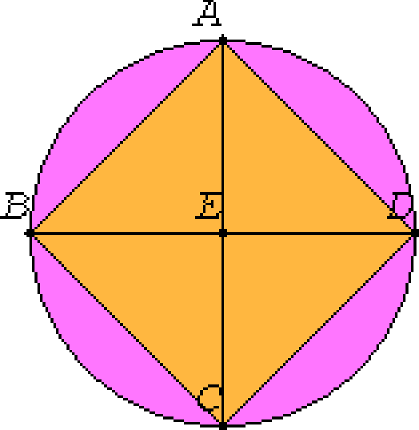
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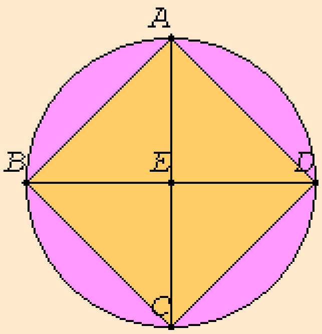
Euclid's Elements
Book IV
Proposition 6

To inscribe a square in a given circle.

Let *ABCD* be the given circle.

It is required to inscribe a square in the circle *ABCD.*

Draw two diameters *AC* and *BD* of the circle *ABCD* at right angles to one another, and join *AB, BC, CD,* and *DA.* III.1 I.11



Then, since *BE* equals *ED,* for *E* is the center, and *EA* is common and at right angles, therefore the base *AB* equals the base *AD.* I.4

For the same reason each of the straight lines *BC* and *CD* also equals each of the straight lines *AB* and *AD.* Therefore the quadrilateral *ABCD* is equilateral.

I say next that it is also right-angled.

For, since the straight line *BD* is a diameter of the circle *ABCD,* therefore *BAD* is a semicircle, therefore the angle *BAD* is right. III.31

For the same reason each of the angles *ABC, BCD,* and *CDA* is also right. Therefore the quadrilateral *ABCD* is right-angled.

But it was also proved equilateral, therefore it is a square, and it has been inscribed in the circle *ABCD.*

Therefore the square *ABCD* has been inscribed in the given circle.

Q.E.F.

Guide

This construction is used in a few propositions of Book XII, the first being XII.2.

Next proposition: IV.7

Previous: IV.5

Book IV introduction

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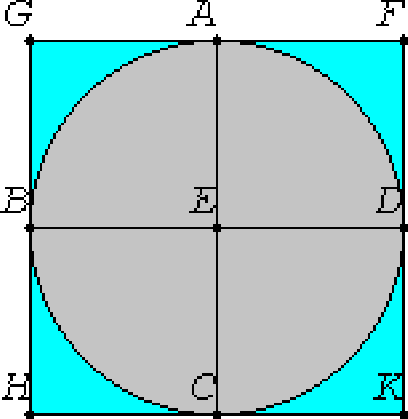
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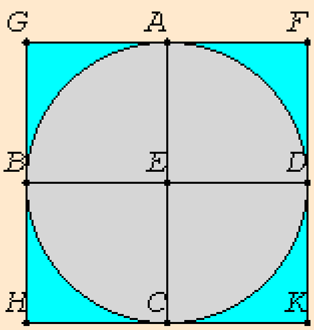
Euclid's Elements
Book IV
Proposition 7

To circumscribe a square about a given circle.

Let *ABCD* be the given circle.

It is required to circumscribe a square about the circle *ABCD.*

Draw two diameters *AC* and *BD* of the circle *ABCD* at right angles to one another. Draw *FG, GH, HK,* and *KF* through the points *A, B, C,* and *D* touching the circle *ABCD.* III.1 I.11 III.16,Cor



Then, since *FG* touches the circle *ABCD,* and *EA* has been joined from the center *E* to the point of contact at *A,* therefore the angles at A are right. III.18

For the same reason the angles at the points *B, C,* and *D* are also right.

Now, since the angle *AEB* is right, and the angle *EBG* is also right, therefore *GH* is parallel to *AC.* I.28

For the same reason *AC* is also parallel to *FK,* so that *GH* is also parallel to *FK.* I.30

Similarly we can prove that each of the straight lines *GF* and *HK* is parallel to *BED.*

Therefore *GK, GC, AK, FB,* and *BK* are parallelograms, therefore *GF* equals *HK,* and *GH* equals *FK.* I.34

And, since *AC* equals *BD,* and *AC* also equals each of the straight lines *GH* and *FK,* and *BD* equals each of the straight lines *GF* and *HK,* therefore the quadrilateral *FGHK* is equilateral. I.34

I say next that it is also right-angled.

For, since *GBEA* is a parallelogram, and the angle *AEB* is right, therefore the angle *AGB* is also right. I.34

Similarly we can prove that the angles at *H, K,* and *F* are also right.

Therefore *FGHK* is right-angled.

But it was also proved equilateral, therefore it is a square, and it has been circumscribed about the circle *ABCD.*

Therefore a square has been circumscribed about the given circle.

Q.E.F.

Guide

This proposition is used in XII.10.

Next proposition: IV.8

Previous: IV.6

Book IV introduction

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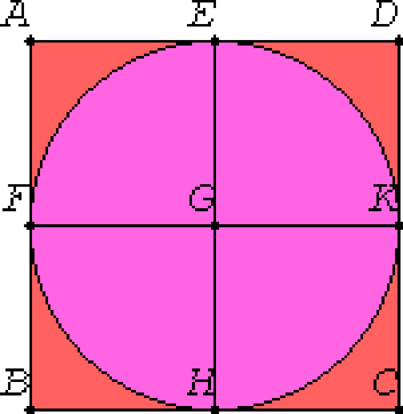

A
E
D
F
G
K
B
H
C

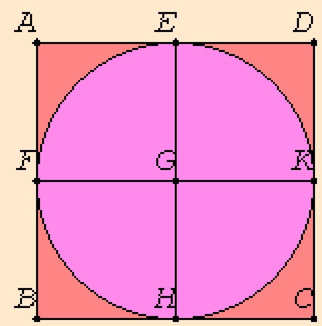
Euclid's Elements
Book IV
Proposition 8

To inscribe a circle in a given square.

Let *ABCD* be the given square.

It is required to inscribe a circle in the given square *ABCD.*

Bisect the straight lines *AD* and *AB* at the points *E* and *F* respectively. Draw *EH* through *E* parallel to either *AB* or *CD,* and draw *FK* through *F* parallel to either *AD* or *BC.* Therefore each of the figures *AK*, *KB*, *AH*, *HD*, *AG*, *GC*, *BG*, and *GD* is a parallelogram, and their opposite sides are evidently equal. I.10 I.31 I.34



Now, since *AD* equals *AB,* and *AE* is half of *AD,* and *AF* half of *AB,* therefore *AE* equals *AF,* so that the opposite sides are also equal, therefore *FG* equals *GE.*

Similarly we can prove that each of the straight lines *GH* and *GK* equals each of the straight lines *FG* and *GE.* Therefore the four straight lines *GE*, *GF*, *GH*, and *GK* equal one another.

Therefore the circle described with center *G* and radius one of the straight lines *GE*, *GF*, *GH*, or *GK* also passes through the remaining points.

And it touches the straight lines *AB*, *BC*, *CD*, and *DA*, because the angles at *E*, *F*, *H*, and *K* are right.

For, if the circle cuts *AB*, *BC*, *CD*, or *DA*, the straight line drawn at right angles to the diameter of the circle from its end will fall within the circle, which was proved absurd. Therefore the circle described with center *G* and radius one of the straight lines *GE*, *GF*, *GH*, or *GK* does not cut the straight lines *AB*, *BC*, *CD*, and *DA.* III.16

Therefore it touches them, and has been inscribed in the square *ABCD.*

Therefore a circle has been inscribed in the given square.

Q.E.F.

Guide

This is a straightforward proposition, one of four in the sequence IV.6 through IV.9 about circles and squares.

Next proposition: IV.9

Previous: IV.7

Book IV introduction

Select from Book IV

Select book

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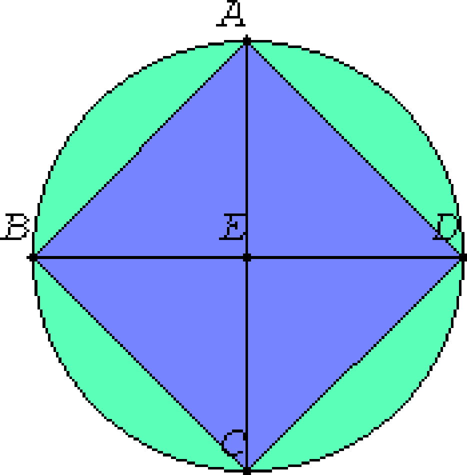
A
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Euclid's Elements

Book IV

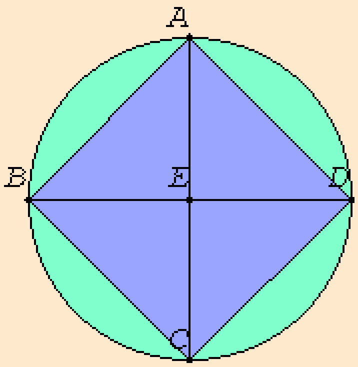
Proposition 9

To circumscribe a circle about a given square.

Let *ABCD* be the given square.

It is required to circumscribe a circle about the square *ABCD.*

Join *AC* and *BD,* and let them cut one another at *E.*



Then, since *DA* equals *AB,* and *AC* is common, therefore the two sides *DA* and *AC* equal the two sides *BA* and *AC,* and the base *DC* equals the base *BC,* therefore the angle *DAC* equals the angle *BAC.* I.8

Therefore the angle *DAB* is bisected by *AC.*

Similarly we can prove that each of the angles *ABC, BCD,* and *CDA* is bisected by the straight lines *AC* and *DB.*

Now, since the angle *DAB* equals the angle *ABC,* and the angle *EAB* is half of the angle *DAB,* and the angle *EBA* half of the angle *ABC,* therefore the angle *EAB* also equals the angle *EBA,* so that the side *EA* also equals *EB.* I.6

Similarly we can prove that each of the straight lines *EA* and *EB* equals each of the straight lines *EC* and *ED.*

Therefore the four straight lines *EA, EB, EC,* and *ED* equal one another.

Therefore the circle described with center *E* and radius one of the straight lines *EA, EB, EC,* or *ED* also passes through the remaining points, and it is circumscribed about the square *ABCD.*

Let it be circumscribed, as *ABCD.*

Therefore a circle has been circumscribed about the given square.

Q.E.F.

Guide

This is a straightforward proposition, one of four in the sequence IV.6 through IV.9 about circles and squares.

Next proposition: IV.10

Select from Book IV

Previous: IV.8

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Book IX

Euclid's Elements

Book IX

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Propositions

Proposition 1.
If two similar plane numbers multiplied by one another make some number, then the product is square.

Proposition 2.
If two numbers multiplied by one another make a square number, then they are similar plane numbers.

Proposition 3.
If a cubic number multiplied by itself makes some number, then the product is a cube.

Proposition 4.
If a cubic number multiplied by a cubic number makes some number, then the product is a cube.

Proposition 5.
If a cubic number multiplied by any number makes a cubic number, then the multiplied number is also cubic.

Proposition 6.
If a number multiplied by itself makes a cubic number, then it itself is also cubic.

Proposition 7.
If a composite number multiplied by any number makes some number, then the product is solid.

Proposition 8.
If as many numbers as we please beginning from a unit are in continued proportion, then the third from the unit is square as are also those which successively leave out one, the fourth is cubic as are also all those which leave out two, and the seventh is at once cubic and square are also those which leave out five.

Proposition 9.
If as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is square, then all the rest are also square; and if the number after the unit is cubic, then all the rest are also

cubic.

Proposition 10.

If as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is not square, then neither is any other square except the third from the unit and all those which leave out one; and, if the number after the unit is not cubic, then neither is any other cubic except the fourth from the unit and all those which leave out two.

Proposition 11.

If as many numbers as we please beginning from a unit are in continued proportion, then the less measures the greater according to some one of the numbers which appear among the proportional numbers.

Corollary. Whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.

Proposition 12.

If as many numbers as we please beginning from a unit are in continued proportion, then by whatever prime numbers the last is measured, the next to the unit is also measured by the same.

Proposition 13.

If as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is prime, then the greatest is not measured by any except those which have a place among the proportional numbers.

Proposition 14.

If a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.

Proposition 15.

If three numbers in continued proportion are the least of those which have the same ratio with them, then the sum of any two is relatively prime to the remaining number.

Proposition 16.

If two numbers are relatively prime, then the second is not to any other number as the first is to the second.

Proposition 17.

If there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then the last is not to any other number as the first is to the second.

Proposition 18.

Given two numbers, to investigate whether it is possible to find a third proportional to them.

Proposition 19.

Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Proposition 20.

Prime numbers are more than any assigned multitude of prime numbers.

Proposition 21.

If as many even numbers as we please are added together, then the sum is even.

Proposition 22.

If as many odd numbers as we please are added together, and their multitude is even, then the sum is even.

Proposition 23.
If as many odd numbers as we please are added together, and their multitude is odd, then the sum is also odd.

Proposition 24.
If an even number is subtracted from an even number, then the remainder is even.

Proposition 25.
If an odd number is subtracted from an even number, then the remainder is odd.

Proposition 26.
If an odd number is subtracted from an odd number, then the remainder is even.

Proposition 27.
If an even number is subtracted from an odd number, then the remainder is odd.

Proposition 28.
If an odd number is multiplied by an even number, then the product is even.

Proposition 29.
If an odd number is multiplied by an odd number, then the product is odd.

Proposition 30.
If an odd number measures an even number, then it also measures half of it.

Proposition 31.
If an odd number is relatively prime to any number, then it is also relatively prime to double it.

Proposition 32.
Each of the numbers which are continually doubled beginning from a dyad is even-times even only.

Proposition 33.
If a number has its half odd, then it is even-times odd only.

Proposition 34.
If an [even] number neither is one of those which is continually doubled from a dyad, nor has its half odd, then it is both even-times even and even-times odd.

Proposition 35.
If as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then the excess of the second is to the first as the excess of the last is to the sum of all those before it.

Proposition 36.
If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

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Elements Introduction –

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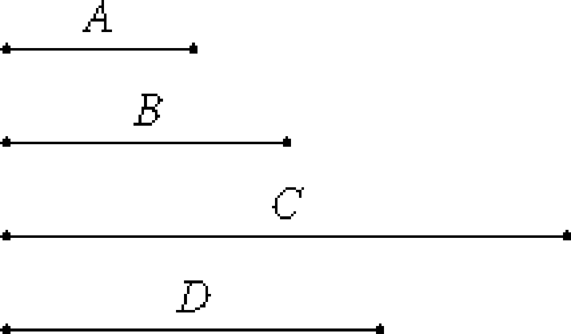
A
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C
D

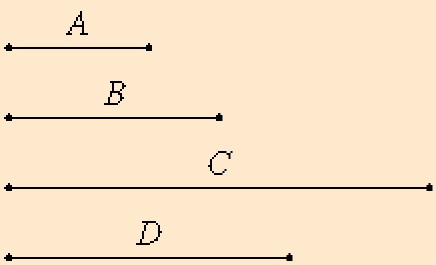
Euclid's Elements
Book IX
Proposition 1

If two similar plane numbers multiplied by one another make some number, then the product is square.

Let *A* and *B* be two similar plane numbers, and let *A* multiplied by *B* make *C*.

I say that *C* is square.

Multiply *A* by itself to make *D*. Then *D* is square.



Since then *A* multiplied by itself makes *D*, and multiplied by *B* makes *C*, therefore *A* is to *B* as *D* is to *C*. VII.17

And, since *A* and *B* are similar plane numbers, therefore one mean proportional number falls between *A* and *B*. VIII.18

Since as many number fall in continued proportion between those which have the same ratio, therefore one mean proportional number falls between *D* and *C* also. VIII.8

And *D* is square, therefore *C* is also square. VIII.22

Therefore, *if two similar plane numbers multiplied by one another make some number, then the product is square.*

Q.E.D.

Guide

Although this is the first proposition in Book IX, it and the succeeding propositions continue those of Book VIII without break.

To illustrate this proposition, consider the two similar plane numbers $a = 18$ and $b = 8$, as illustrated in the Guide to VII.Def.21. According to VIII.18, there is a mean proportional between them, namely, 12. And the square of the mean proportional is their product, $ab = 144$.

Outline of the proof

It is not clear why the proof of the proposition does not use the fact that the square of the mean proportional of two numbers equals their product, but instead uses slightly more complicated reasoning.

Let a and b be the given similar plane numbers. Then there is a mean proportional between them (VIII.18). And, since $a:b = a^2:ab$, therefore there is also a mean proportional between a^2 and ab (VIII.1). But since a^2 is a square, therefore ab is also a square (VIII.22). Thus, the product of the original similar plane numbers is a square.

Use of this proposition

The next proposition IX.2 is the converse of this one. This proposition is used in X.29.

Next proposition: IX.2

Book IX introduction

Select from Book IX

Select book

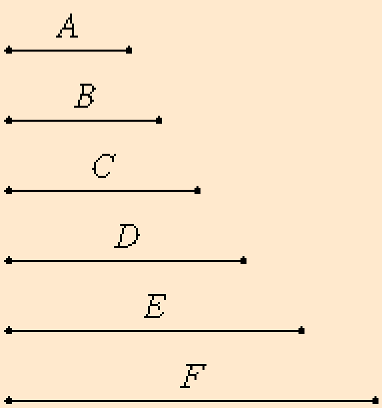
Select topic

Euclid's Elements
Book IX
Proposition 10

If as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is not square, then neither is any other square except the third from the unit and all those which leave out one; and, if the number after the unit is not cubic, then neither is any other cubic except the fourth from the unit and all those which leave out two.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, and *F*, beginning from a unit and in continued proportion, and let *A*, the number after the unit, not be square.

I say that neither are any other square except the third from the unit and those which leave out one.



If possible, let *C* be square. But *B* is also square, therefore *B* and *C* have to one another the ratio which a square number has to a square number. IX.8

And *B* is to *C* as *A* is to *B*, therefore *A* and *B* have to one another the ratio which a square number has to a square number, so that *A* and *B* are similar plane numbers. VIII.26 converse

And *B* is square, therefore *A* is also square, which is contrary to the hypothesis.

Therefore *C* is not square. Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one.

Next, let *A* not be a cube.

I say that neither are any other cubes except the fourth from the unit and those which leave out two.

If possible, let *D* be a cube.

Now *C* is also a cube, for it is fourth from the unit. And *C* is to *D* as *B* is to *C*, therefore *B* has to *C* the ratio which a cube has to a cube. And *C* is a cube, therefore *B* is also a cube. IX.8 VIII.25

And since the unit is to *A* as *A* is to *B*, and the unit measures *A* according to the units in it, therefore *A* also measures *B* according to the units in itself. Therefore *A* multiplied by itself makes the cubic number *B*.

But, if a number multiplied by itself makes cubic number, then it is itself a cube. Therefore *A* is also a cube, contrary to the hypothesis. Therefore *D* is not a cube. Similarly we can prove that neither is any other of the numbers a cube except the fourth from the unit and those which leave out two. IX.6

Therefore, *if as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is not square, then neither is any other square except the third from the unit and all those which leave out one; and, if the number after the unit is not cubic, then neither is any other cubic except the fourth from the unit and all those which leave out two.*

Q.E.D.

Guide

This is a converse of the previous theorem.

Next proposition: IX.11

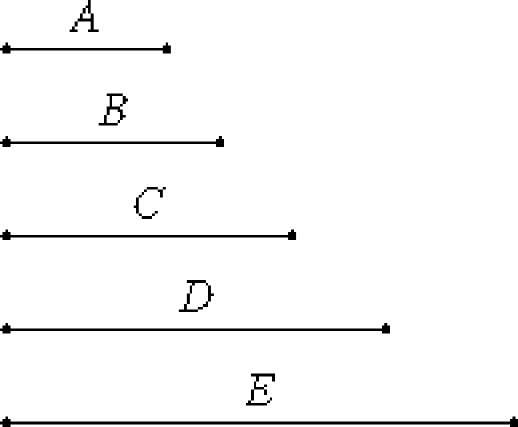
Previous: IX.9

Book IX introduction

Select from Book IX

Select book

Select topic


A
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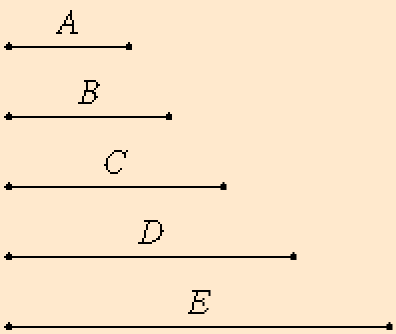
Euclid's Elements

Book IX

Proposition 11

If as many numbers as we please beginning from a unit are in continued proportion, then the less measures the greater according to some one of the numbers which appear among the proportional numbers.

Let there be as many numbers as we please, *B, C, D,* and *E,* beginning from the unit *A* and in continued proportion.



I say that *B,* the least of the numbers *B, C, D,* and *E,* measures *E* according to one of the numbers *C* or *D.*

Since the unit *A* is to *B* as *D* is to *E,* therefore the unit *A* measures the number *B* the same number of times as *D* measures *E.* Therefore, alternately, the unit *A* measures *D* the same number of times as *B* measures *E.* VII.15

But the unit *A* measures *D* according to the units in it, therefore *B* also measures *E* according to the units in *D,* so that *B* the less measures *E* the greater according to some number of those which have place among the proportional numbers.

Therefore, *if as many numbers as we please beginning from a unit are in continued proportion, then the less measures the greater according to some one of the numbers which appear among the proportional numbers.*

Q.E.D.

Corollary

And it is manifest that, *whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.*

Guide

This proposition, along with the comment make in the corollary, says that a^k divides a^n the number a^{n-k} times.

Use of this proposition

The corollary is used in the next proposition while the proposition itself is used in the one following that.

Next proposition: IX.12

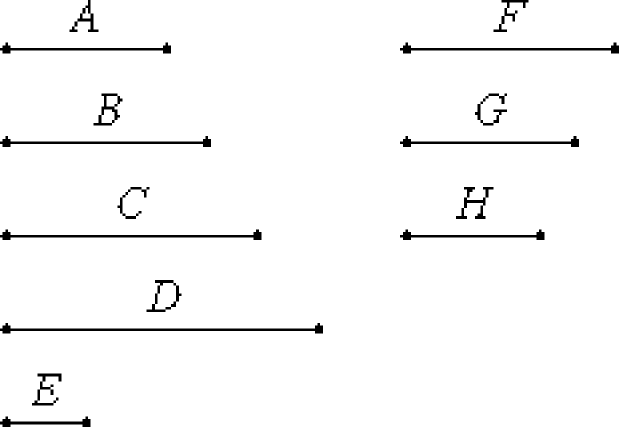
Select from Book IX

Previous: IX.10

Select book

[Book IX introduction](#)

Select topic


A
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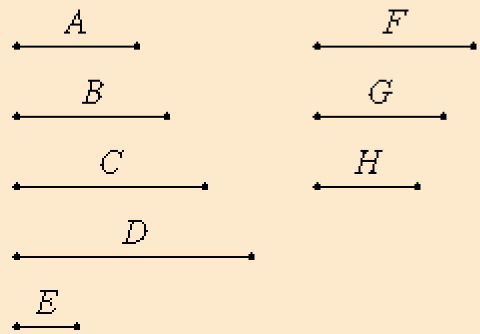
Euclid's Elements
Book IX
Proposition 12

If as many numbers as we please beginning from a unit are in continued proportion, then by whatever prime numbers the last is measured, the next to the unit is also measured by the same.

Let *A*, *B*, *C*, and *D* be as many numbers as we please beginning from a unit in continued proportion.

I say that, by whatever prime numbers *D* is measured, *A* is also measured by the same.



Let *D* be measured by any prime number *E*.

I say that *E* measures *A*.

Suppose it does not.

Now *E* is prime, and any prime number is relatively prime to any which it does not measure, therefore *E* and *A* are relatively prime. And, since *E* measures *D*, let it measure it according to *F*, therefore *E* multiplied by *F* makes *D*. VII.29

Again, since *A* measures *D* according to the units in *C*, therefore *A* multiplied by *C* makes *D*. But, further, *E* multiplied by *F* makes *D*, therefore the product of *A* and *C* equals the product of *E* and *F*. IX.11 and Cor.

Therefore *A* is to *E* as *F* is to *C*. But *A* and *E* are relatively prime, primes are also least, and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, therefore *E* measures *C*. Let it measure it according to *G*. VII.19 VII.21 VII.20

Therefore *E* multiplied by *G* makes *C*. But, further, by the previous theorem, *A* multiplied by *B* makes *C*. Therefore the product of *A* and *B* equals the product of *E* and *G*. IX.11 and Cor.

Therefore *A* is to *E* as *G* is to *B*. But *A* and *E* are relatively prime, primes are also least, and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent, therefore *E* measures *B*. Let it measure it according to *H*. Then *E* multiplied by *H* makes *B*. VII.19 VII.21 VII.20

But, further, *A* multiplied by itself makes *B*, therefore the product of *E* and *H* equals the square on *A*. Therefore *E* is to *A* as *A* is to *H*. IX.8 VII.19

But *A* and *E* are relatively prime, primes are also least, and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, therefore *E* measures *A* antecedent antecedent. But, again, it also does not measure it, which is impossible. VII.21 VII.20

Therefore *E* and *A* are not relatively prime. Therefore they are relatively composite. But numbers relatively composite are measured by some number. VII.Def.14

And, since *E* is by hypothesis prime, and a prime is not measured by any number other than itself, therefore *E* measures *A* and *E*, so that *E* measures *A*.

But it also measures *D*, therefore *E* measures *A* and *D*. Similarly we can prove that, by whatever prime numbers *D* is measured, *A* also is measured by the same.

Therefore, *if as many numbers as we please beginning from a unit are in continued proportion, then by whatever prime numbers the last is measured, the next to the unit is also measured by the same.*

Q.E.D.

Guide

This proposition says that if a prime number p divides a power a^k of a number a, then it divides the number a itself.

Outline of the proof

The proof is both elegant and inelegant. The elegant part is the reduction step from p dividing a^k to p dividing a^{k-1}. There are two inelegant parts. One is that the reduction step is applied three times starting with k equal to 4. The other is that three unnecessary statements are tacked on to the end of the proof after the goal is already reached.

Assume that a prime number p divides a power a^k of a number a. Suppose that p does not divide a. Then p is relatively prime to a (VII.29). From the proportion

$$(a^k/p):a^{k-1} = a:p,$$

we see that the ratio $(a^k/p):a^{k-1}$ reduces to $a:p$ in lowest terms (VII.21). Therefore, p divides a^{k-1} (VII.20).

Apply this reduction step repeatedly until the conclusion p divides a is reached.

Use of this proposition

This proposition is used in the next one.

Next proposition: IX.13

Previous: IX.11

Book IX introduction

Select from Book IX

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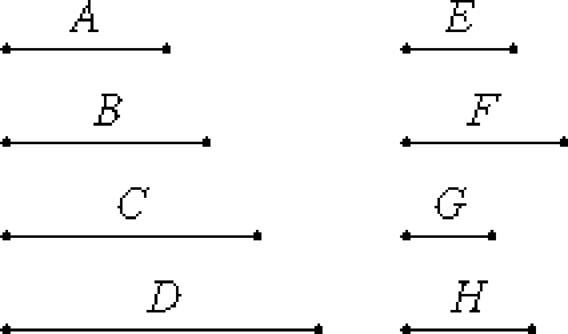
Select topic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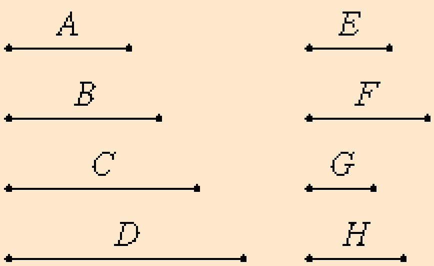
Euclid's Elements
Book IX
Proposition 13

If as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is prime, then the greatest is not measured by any except those which have a place among the proportional numbers.

Let there be as many numbers as we please, *A, B, C,* and *D,* beginning from a unit and in continued proportion, and let *A,* the number after the unit, be prime.

I say that *D,* the greatest of them, is not measured by any other number except *A, B,* or *C.*

If possible, let it be measured by *E,* and let *E* not be the same with any of the numbers *A, B,* or *C.*



It is then manifest that *E* is not prime, for if *E* is prime and measures *D,* then it also measures *A,* which is prime, though it is not the same with it, which is impossible. Therefore *E* is not prime, so it is composite. IX.12

But any composite number is measured by some prime number, therefore *E* is measured by some prime number. VII.31

I say next that it is no measured by any other prime except *A.* If *E* is measured by another, and *E* measures *D,* then that other measures *D,* so that it also measures *A,* which is prime, though it is not the same with it, which is impossible. Therefore [only the prime] *A* measures *E.* IX.12

And, since *E* measures *D,* let it measure it according to *F.*

I say that *F* is not the same with any of the numbers *A, B,* or *C.*

If *F* is the same with one of the numbers *A, B,* or *C,* and measures *D* according to *E,* then one of the numbers *A, B,* or *C* also measures *D* according to *E.* But one of the numbers *A, B,* or *C* measures *D* according to some one of the numbers *A, B,* or *C,* therefore *E* is also the same with one of the numbers *A, B* or *C,* which is contrary to the hypothesis. IX.11

Therefore *F* is not the same as any one of the numbers *A, B,* or *C.*

Similarly we can prove that *F* is measured by *A,* by proving again that *F* is not prime.

If it is, and measures *D,* then it also measures *A,* which is prime, though it is not the same with it, which is impossible. Therefore *F* is not prime, so it is composite. IX.12

But any composite number is measured by some prime number, therefore *F* is measured by some prime number. VII.31

I say next that it is not measured by any other prime except *A.*

If any other prime number measures *F,* and *F* measures *D,* then that other also measures *D,* so that it also measures *A,* which is prime, though it is not the same with it, which is impossible. Therefore [only the prime] IX.12

A measures *F*.

And, since *E* measures *D* according to *F*, therefore *E* multiplied by *F* makes *D*.

But, further, *A* multiplied by *C* makes *D*, therefore the product of *A* and *C* equals the product of *E* and *F*. IX.11

Therefore, proportionally *A* is to *E* as *F* is to *C*. VII.19

But *A* measures *E*, therefore *F* also measures *C*. Let it measure it according to *G*.

Similarly, then, we can prove that *G* is not the same with any of the numbers *A* or *B*, and that it is measured by *A*. And, since *F* measures *C* according to *G*, therefore *F* multiplied by *G* makes *C*.

But, further, *A* multiplied by *B* makes *C*, therefore the product of *A* and *B* equals the product of *F* and *G*. IX.11
Therefore, proportionally *A* is to *F* as *G* is to *B*. VII.19

But *A* measures *F*, therefore *G* also measures *B*. Let it measure it according to *H*.

Similarly then we can prove that *H* is not the same with *A*.

And, since *G* measures *B* according to *H*, therefore *G* multiplied by *H* makes *B*. But, further, *A* multiplied by itself makes *B*, therefore the product of *H* and *G* equals the square on *A*. IX.8

Therefore *H* is to *A* as *A* is to *G*. But *A* measures *G*, therefore *H* also measures *A*, which is prime, though it is not the same with it, which is absurd. VII.19

Therefore *D* the greatest is not measured by any other number except *A*, *B*, or *C*.

Therefore, *if as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is prime, then the greatest is not measured by any except those which have a place among the proportional numbers.*

Q.E.D.

Guide

This proposition says that the only numbers that can divide a power of a prime are smaller powers of that prime.

Outline of the proof

The proof involves a reduction step like that in the proof of the previous proposition.

Suppose a number *e* divides a power p^k of a prime number *p*, but *e* does not equal any lower power of *p*.

First note that *e* can't be prime itself, since then it would divide *p* (IX.12), which it doesn't.

Then *e* is composite. Then some prime number *q* divides *e* (VII.31). Then *q* also divides p^k, which it implies *q* divides *p*. Therefore, the only prime that can divide *e* is *p*.

The rest of the proof is repeated reduction of the power *k*. Since *e* is not 1, it is divisible by *p*. Let *g* be *e/p*. Then *g* divides p^{k-1}, but is not any lower power of *p*. Then the same argument can be applied. Continue in this manner until some number divides *p* but is not 1 or *p*, a contradiction. Thus, the only numbers that can divide a power of a prime are smaller powers of the prime.

Use of this proposition

This proposition is used in IX.32 and IX.36.

Next proposition: IX.14

Previous: IX.12

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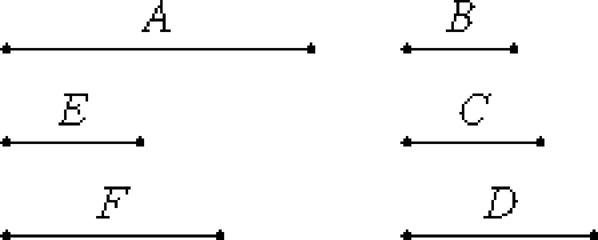
D.E.Joyce
Clark University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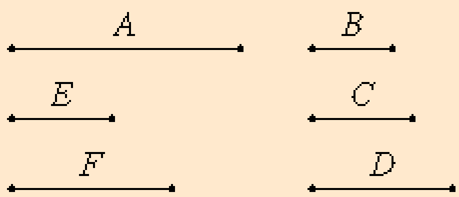
Euclid's Elements
Book IX
Proposition 14

If a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.

Let the number *A* be the least that is measured by the prime numbers *B*, *C*, and *D*.

I say that *A* is not measured by any other prime number except *B*, *C*, or *D*.

If possible, let it be measured by the prime number *E*, and let *E* not be the same as any one of the numbers *B*, *C*, or *D*.



Now, since *E* measures *A*, let it measure it according to *F*, therefore *E* multiplied by *F* makes *A*. And *A* is measured by the prime numbers *B*, *C*, and *D*.

But, if two numbers multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers, therefore each of *B*, *C*, and *D* measures one of the numbers *E* or *F*. VII.30

Now they do not measure *E*, for *E* is prime and not the same with any one of the numbers *B*, *C*, or *D*. Therefore they measure *F*, which is less than *A*, which is impossible, for *A* is by hypothesis the least number measured by *B*, *C*, and *D*.

Therefore no prime number measures *A* except *B*, *C*, and *D*.

Therefore, *if a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.*

Q.E.D.

Guide

This proposition states that the least common multiple of a set of prime numbers is not divisible by any other prime. The least common multiple is actually the product of those primes, but that isn't mentioned.

The proof is clear, and it depends on VII.30, that if a prime divides a product, then it divides one of the factors.

Next proposition: IX.15

Previous: IX.13

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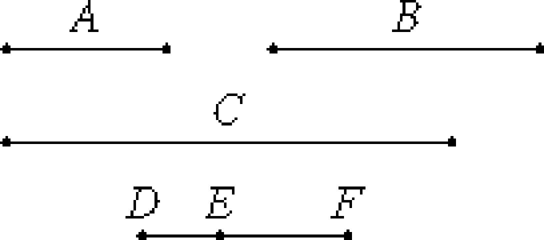
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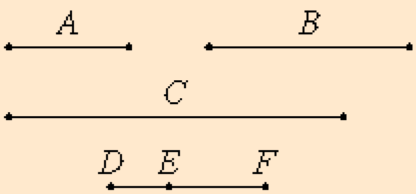
Euclid's Elements
Book IX
Proposition 15

If three numbers in continued proportion are the least of those which have the same ratio with them, then the sum of any two is relatively prime to the remaining number.

Let *A, B,* and *C,* three numbers in continued proportion, be the least of those which have the same ratio with them.

I say that the sum of any two of the numbers *A, B,* and *C* is relatively prime to the remaining number, that is, *A* plus *B* is relatively prime to *C, B* plus *C* is relatively prime to *A,* and *A* plus *C* is relatively prime to *B.*



Take two numbers *DE* and *EF* to be the least of those which have the same ratio with *A, B,* and *C.* VIII.2

It is then manifest that *DE* multiplied by itself makes *A,* and multiplied by *EF* makes *B,* and that *EF* multiplied by itself makes *C.* Cor. to VIII.2

Now, since *DE* and *EF* are least, therefore they are relatively prime. But, if two numbers are relatively prime, then their sum is also relatively prime to each, therefore *DF* is relatively prime to each of the numbers *DE* and *EF.* VII.22 VII.28

But, further, *DE* is also relatively prime to *EF,* therefore *DF* and *DE* are relatively prime to *EF.* But, if two numbers are relatively prime to any number, then their product is also relatively prime to the other, so that the product of *FD* and *DE* is relatively prime to *EF,* hence the product of *FD* and *DE* is also relatively prime to the square on *EF.* VII.24 VII.25

But the product of *FD* and *DE* is the square on *DE* together with the product of *DE* and *EF,* therefore the sum of the square on *DE* and the product of *DE* and *EF* is relatively prime to the square on *EF.* II.3

And the square on *DE* is *A,* the product of *DE* and *EF* is *B,* and the square on *EF* is *C,* therefore the sum of *A* and *B* is prime to *C.*

Similarly we can prove that the sum of *B* and *C* is relatively prime to *A.*

I say next that the sum of *A* and *C* is also relatively prime to *B.*

Since *DF* is relatively prime to each of the numbers *DE* and *EF,* therefore the square on *DF* is also relatively prime to the product of *DE* and *EF.* VII.24 VII.25

But the sum of the squares on *DE* and *EF* together with twice the product of *DE* and *EF* equals the square on *DF,* therefore the sum of the squares on *DE* and *EF* together with twice the product of *DE* and *EF* is relatively prime to the product of *DE* and *EF.* II.4

Taken separately, the sum of the squares on *DE* and *EF* together with the product of *DE* and *EF* is relatively prime to the product of *DE* and *EF.*

Therefore, taken separately again, the sum of the squares on *DE* and *EF* is relatively prime to the product of *DE* and *EF*.

And the square on *DE* is *A*, the product of *DE* and *EF* is *B*, and the square on *EF* is *C*.

Therefore the sum of *A* and *C* is relatively prime to *B*.

Therefore, *if three numbers in continued proportion are the least of those which have the same ratio with them, then the sum of any two is relatively prime to the remaining number*.

Q.E.D.

Guide

Outline of the proof

Let a, b, c be three numbers in continued proportion. Then according to VIII.2, they are of the form

$$a = d^2, \quad b = de, \quad c = e^2,$$

where d and e are relatively prime. Then the sum, $d + e$, is relatively prime to both d and e (VII.28).

Now, since both d and $d + e$ are relatively prime to e, so is their product $d^2 + de$ relatively prime to e (VII.24), and therefore to e^2 (VII.25). Thus, $a + b$ is relatively prime to c.

Likewise, $b + c$ is relatively prime to a.

Next, since $d + e$ is relatively prime to both d and e, so is its square $(d + e)^2$ relatively prime to the product de (VII.24 and VII.25). That is, $d^2 + e^2 + 2de$ is relatively prime to de. Subtract $2de$ to conclude that $d^2 + e^2$ is relatively prime to de. Thus, b is relatively prime to $a + c$.

Next proposition: IX.16

Previous: IX.14

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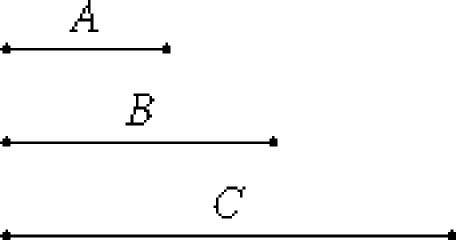
A
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Euclid's Elements

Book IX

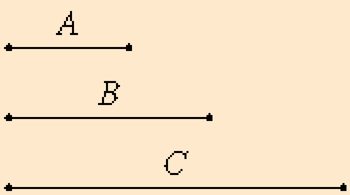
Proposition 16

If two numbers are relatively prime, then the second is not to any other number as the first is to the second.

Let the two numbers *A* and *B* be relatively prime.

I say that *B* is not to any other number as *A* is to *B*.

If possible as *A* is to *B*, let *B* be to *C*.



Now *A* and *B* are relatively prime, numbers which are relatively prime are also least, and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, therefore *A* measures *B* as antecedent antecedent. VII.21 VII.20

But it also measures itself, therefore *A* measures *A* and *B* which are relatively prime, which is absurd.

Therefore *B* is not to *C* as *A* is to *B*.

Therefore, *if two numbers are relatively prime, then the second is not to any other number as the first is to the second.*

Q.E.D.

Guide

Outline of the proof

Let a and b be relatively prime. Then he ratio $a:b$ is in lowest terms. Suppose that ratio is the same as the ratio $b:c$. Then the antecedent of the ratio $a:b$, namely a, divides the antecedent of the ratio $b:c$, namely b. But a cannot divide b since they're relatively prime.

Use of this proposition

This proposition is used in IX.18.

Next proposition: IX.17

Previous: IX.15

Book IX introduction

Select from Book IX

Select book

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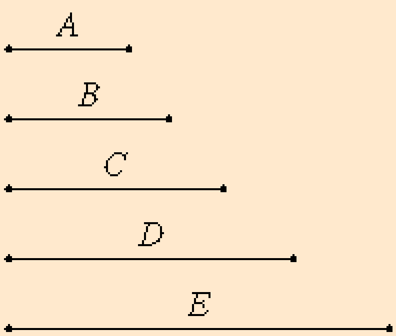
Euclid's Elements
Book IX
Proposition 17

If there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then the last is not to any other number as the first is to the second.

Let there be as many numbers as we please, *A, B, C,* and *D,* in continued proportion, and let the extremes of them, *A* and *D,* be relatively prime.

I say that *D* is not to any other number as *A* is to *B.*



If possible *A* is to *B,* so let *D* be to *E,* therefore, alternately *A* is to *D* as *B* is to *E.* VII.13

But *A* and *D* are prime, primes are also least, and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent. Therefore *A* measures *B.* And *A* is to *B* as *B* is to *C.* Therefore *B* also measures *C,* so that *A* also measures *C.* VII.21 VII.20

And since *B* is to *C* as *C* is to *D,* and *B* measures *C,* therefore *C* also measures *D.* But *A* measures *C,* so that *A* also measures *D.* But it also measures itself, therefore *A* measures *A* and *D* which are relatively prime, which is impossible.

Therefore *D* is not to any other number as *A* is to *B.*

Therefore, *if there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then the last is not to any other number as the first is to the second.*

Q.E.D.

Guide

This proposition generalizes the previous proposition from a ratio of two terms to a continued proportion of arbitrarily many. It says that a continued proportion in lowest terms cannot be extended.

Outline of the proof

Consider a continued proportion in lowest terms with the first term *a* relatively prime to the last term *d,* and having the ratio $a:b$. Suppose it can be extended to *e* so that $a:b = d:e$. Alternately, $a:d = b:e$. Since the first ratio $a:d$ is in lowest terms, therefore *a* divides *b.* Then each term of the continued proportion divides the next, hence *a* divides *d.* But that's impossible since *a* and *d* are relatively prime. Therefore, the continued proportion cannot be extended.

Next proposition: IX.18

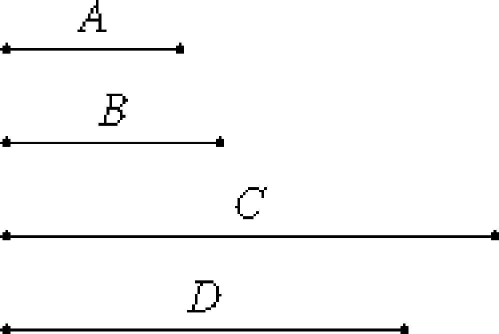
Select from Book IX

Previous: IX.16

Book IX introduction

Select book

Select topic


A
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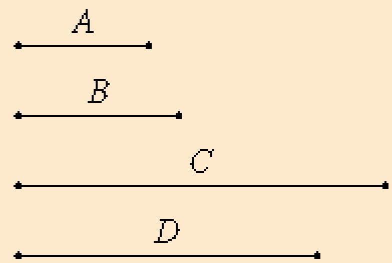
Euclid's Elements
Book IX
Proposition 18

Given two numbers, to investigate whether it is possible to find a third proportional to them.

Let *A* and *B* be the given two numbers. It is required to investigate whether it is possible to find a third proportional to them.

Now *A* and *B* are either relatively prime or not. And, if they are relatively prime, it was proved that it is impossible to find a third proportional to them. IX.16



Next, let *A* and *B* not be relatively prime, and let *B* multiplied by itself make *C*. Then *A* either measures *C* or does not measure it.

First, let it measure it according to *D*, therefore *A* multiplied by *D* makes *C*. But, further, *B* multiplied by itself makes *C*, therefore the product of *A* and *D* equals the square on *B*.

Therefore *A* is to *B* as *B* is to *D*, therefore a third proportional number *D* has been found to *A* and *B*. VII.19

Next, let *A* not measure *C*.

I say that it is impossible to find a third proportional number to *A* and *B*.

If possible, let *D* be such third proportional. Then the product of *A* and *D* equals the square on *B*. But the square on *B* is *C*, therefore the product of *A* and *D* equals *C*.

Hence *A* multiplied by *D* makes *C*, therefore *A* measures *C* according to *D*.

But, by hypothesis, it also does not measure it, which is absurd.

Therefore it is not possible to find a third proportional number to *A* and *B* when *A* does not measure *C*.

Q.E.D.

Guide

Note that a third proportional d to a and b has to satisfy $a:b = b:d$, so d would have to be b^2/a. So the third proportional exists when a divides b^2. This conclusion is just what Euclid discovers in this proposition.

Next proposition: IX.19

Previous: IX.17

Book IX introduction

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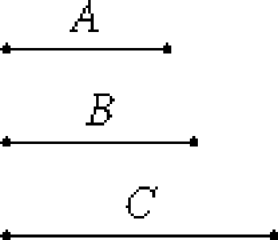

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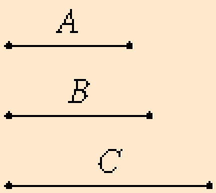
Euclid's Elements
Book IX
Proposition 19

[The Greek text of this Proposition is corrupt. However, analogously to Proposition 18 the condition that a fourth proportional to *A*, *B*, and *C* exists is that *A* measure the product of *B* and *C*.]

Given three numbers, to investigate when it is possible to find a fourth proportional to them.



Let *A*, *B*, and *C* be the given three numbers. It is required to investigate when it is possible to find a fourth proportional to them.

Q.E.D.

Guide

Note that a fourth proportional *d* to *a*, *b* and *c* has to satisfy $a:b = c:d$, so *d* would have to be bc/a. So the third proportional exists when *a* divides *bc*. No doubt that is what Euclid concludes in the missing part of this proposition.

Next proposition: IX.20

Previous: IX.18

Book IX introduction

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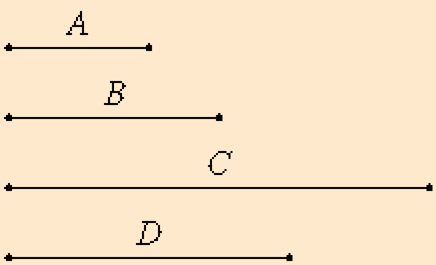
Euclid's Elements
Book IX
Proposition 2

If two numbers multiplied by one another make a square number, then they are similar plane numbers.

Let *A* and *B* be two numbers, and let *A* multiplied by *B* make the square number *C.*

I say that *A* and *B* are similar plane numbers.

Multiply *A* by itself to make *D.* Then *D* is square.



Now, since *A* multiplied by itself makes *D,* and multiplied by *B* makes *C,* therefore *A* is to *B* as *D* is to *C.* VII.17

And, since *D* is square, and *C* is so also, therefore *D* and *C* are similar plane numbers.

Therefore one mean proportional number falls between *D* and *C.* And *D* is to *C* as *A* is to *B,* therefore one mean proportional number falls between *A* and *B* also. VIII.18 VIII.8

But, if one mean proportional number falls between two numbers, then they are similar plane numbers, therefore *A* and *B* are similar plane numbers. VIII.20

Therefore, *if two numbers multiplied by one another make a square number, then they are similar plane numbers.*

Q.E.D.

Guide

This proposition is a converse of the previous one.

As an example to illustrate this proposition, take any square number, such as $20^2 = 400$. It can be factored as a product of two numbers in several ways. One such factorization is as *a* = 50 times *b* = 8. These two numbers have a mean proportional between them, namely, 20, so by VIII.20, they are similar plane numbers. (The actual shapes given by that proposition make 8 to be 2 by 4, and 50 to be 5 by 10.)

Outline of the proof

Let *a* and *b* be two numbers whose product *ab* is a square. Now, both a^2 and *ab* are square numbers which means that they're similar plane numbers. By VIII.8, there's a mean proportional between them. But $a^2:ab = a:b$, *so there is also a mean proportional between a and b (VIII.8). Therefore, a and b are similar plane figures (VIII.20).*

As in the last proof, this one can be shortened. When the product ab is a square, say e^2, then a mean proportional between a and b is e.

Next proposition: IX.3

Previous: IX.1

Book IX introduction

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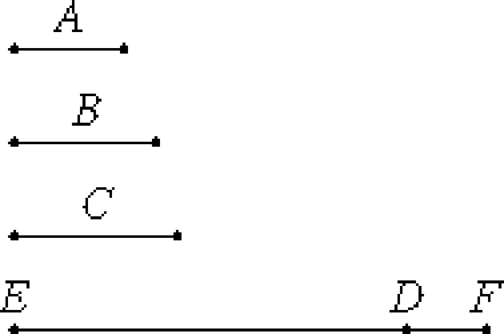
A
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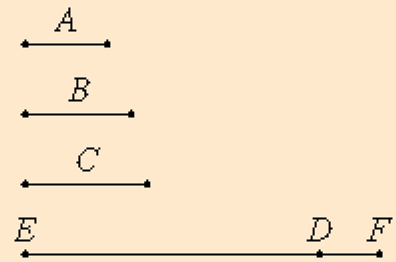
Euclid's Elements
Book IX
Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

Let *A*, *B*, and *C* be the assigned prime numbers.

I say that there are more prime numbers than *A*, *B*, and *C*.



Take the least number *DE* measured by *A*, *B*, and *C*. Add the unit *DF* to *DE*.

Then *EF* is either prime or not.

First, let it be prime. Then the prime numbers *A*, *B*, *C*, and *EF* have been found which are more than *A*, *B*, and *C*.

Next, let *EF* not be prime. Therefore it is measured by some prime number. Let it be measured by the prime number *G*. VII.31

I say that *G* is not the same with any of the numbers *A*, *B*, and *C*.

If possible, let it be so.

Now *A*, *B*, and *C* measure *DE*, therefore *G* also measures *DE*. But it also measures *EF*. Therefore *G*, being a number, measures the remainder, the unit *DF*, which is absurd.

Therefore *G* is not the same with any one of the numbers *A*, *B*, and *C*. And by hypothesis it is prime. Therefore the prime numbers *A*, *B*, *C*, and *G* have been found which are more than the assigned multitude of *A*, *B*, and *C*.

Therefore, *prime numbers are more than any assigned multitude of prime numbers.*

Q.E.D.

Guide

Outline of the proof

Suppose that there are only a finite number of prime numbers. Let *m* be the least common multiple of all of them. (This least common multiple was also considered in proposition IX.14. It wasn't noted in the proof of that proposition that the least common multiple is the product of the primes, and it isn't noted in this proof, either.)

Consider the number $m + 1$. It cannot be prime, since it's larger than all the primes.

So it is not prime. Then according to VII.31, some prime *g* divides it. But *g* cannot be any of the primes, since they all divide *m* and do not divide $m + 1$.

Thus, the assumption that there are a finite number of primes leads to a contradiction. Therefore, there are not a finite number of primes.

Next proposition: IX.21

Previous: IX.19

Book IX introduction

Select from Book IX

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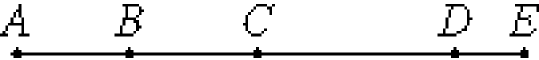
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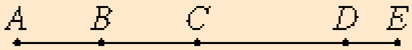
Euclid's Elements

Book IX

Proposition 21

If as many even numbers as we please are added together, then the sum is even.

Let as many even numbers as we please, *AB*, *BC*, *CD*, and *DE*, be added together.



I say that the sum *AE* is even.

Since each of the numbers *AB*, *BC*, *CD*, and *DE* is even, therefore each has a half part, so that the sum *AE* also has a half part. But an even number is that which is divisible into two equal parts, therefore *AE* is even. VII.Def.6

Therefore, *if as many even numbers as we please are added together, then the sum is even.*

Q.E.D.

Guide

With this proposition, Euclid commences the study of even and odd numbers. The study continues through proposition IX.34. The statements of these propositions probably constitute the oldest part of the *Elements* and date back to the Pythagoreans. Indeed, their proofs depend on no other propositions (except IX.31 which discusses prime numbers and may have been inserted among these propositions because it involves odd numbers), so that the statements together with the proofs may be the oldest part of the *Elements.*

Commutativity and associativity of addition

The proof of this proposition implicitly relies on a principle that the order that numbers are summed is irrelevant. For example, when showing that the sum of the two even numbers a and b is even, first a is divided into two equal parts, $a = c + c$, and b is divided into two equal parts, $b = d + d$, therefore

$$a + b = (c + c) + (d + d).$$

But to reach the conclusion that $a + b$ is divisible into two equal parts, we need

$$a + b = (c + d) + (c + d),$$

which adds the terms in a different order.

Of course the order that the terms are added has no effect on the sum. That is an implicit assumption made by Euclid and most everyone after him until the 19th century.

The modern way to deal with this question is to recognize two properties of addition of numbers. One of them is *commutativity*. Addition is commutative since for any two numbers a and b,

$$a + b = b + a.$$

Commutativity says we can two numbers in any order and get the same result.

The other property, *associativity,* is more subtle. When computing the sum $a + b + c$ of three numbers, there is still a choice of which numbers to add first. You can either add $a + b$ first to get d, then add $d + c$ to get the sum, or you can add $b + c$ first to get e, then add $a + e$ to get the sum. The same sum should result. As an equation, associativity says you can move the parentheses around:

$$(a + b) + c = a + (b + c).$$

These two properties, commutativity and associativity, are enough to guarantee that when you add any number of terms together, the order that they're added is irrelvant.

These properties should either be taken as postulates about numbers, or else proven from more basic assumptions. Besides these properties of addition, Euclid missed some other basic properties of the arithmetic operations.

Use of this proposition

This proposition is used in the next two and in IX.28.

Next proposition: IX.22

Previous: IX.20

Book IX introduction

Select from Book IX

Select book

Select topic

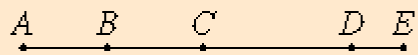
Euclid's Elements

Book IX

Proposition 22

If as many odd numbers as we please are added together, and their multitude is even, then the sum is even.

Let as many odd numbers as we please, *AB*, *BC*, *CD*, and *DE*, even in multitude, be added together.



I say that the sum *AE* is even.

Since each of the numbers *AB*, *BC*, *CD*, and *DE* is odd, if a unit is subtracted from each, then each of the remainders is even, so that the sum of them is even. But the multitude of the units is also even. Therefore the sum *AE* is also even. (VII.Def.7) IX.21

Therefore, *if as many odd numbers as we please are added together, and their multitude is even, then the sum is even.*

Q.E.D.

Guide

A critical step in the proof is the claim that if 1 is subracted from an odd number, then the remainder is even. This was mentioned in VII.Def.7, but never proved. See the Guide for that defintion for details. Unless that gap is filled, this proposition, along with many that depend upon it, are unjustified.

Use of this proposition

This proposition is used in the next one.

Next proposition: IX.23

Previous: IX.21

Book IX introduction

Select from Book IX

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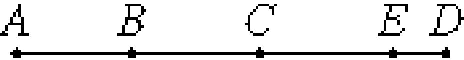
A
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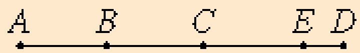
Euclid's Elements
Book IX
Proposition 23

If as many odd numbers as we please are added together, and their multitude is odd, then the sum is also odd.

Let as many odd numbers as we please, *AB*, *BC*, and *CD*, the multitude of which is odd, be added together.



I say that the sum *AD* is also odd.

Subtract the unit *DE* from *CD*, therefore the remainder *CE* is even. VII.Def.7

But *CA* is also even, therefore the sum *AE* is also even. IX.22 IX.21

And *DE* is a unit. Therefore *AD* is odd. VII.Def.7

Therefore, *if as many odd numbers as we please are added together, and their multitude is odd, then the sum is also odd.*

Q.E.D.

Guide

This proposition is used in propositions IX.29 and IX.30.

Next proposition: IX.24

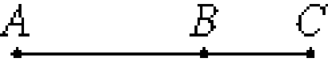
Previous: IX.22

Book IX introduction

Select from Book IX

Select book

Select topic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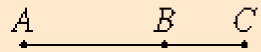
Euclid's Elements
Book IX
Proposition 24

If an even number is subtracted from an even number, then the remainder is even.



Let the even number *BC* be subtracted from the even number *AB.*

I say that the remainder *CA* is even.

Since *AB* is even, therefore it has a half part. For the same reason *BC* also has a half part, so that the remainder *CA* also has a half part, and *CA* is therefore even. VII.Def.6

Therefore, *if an even number is subtracted from an even number, then the remainder is even.*

Q.E.D.

Guide

This proposition is used in four of the next five propositions.

Next proposition: IX.25

Previous: IX.23

Book IX introduction

Select from Book IX

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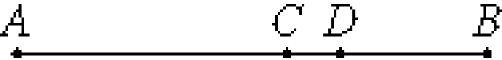
A
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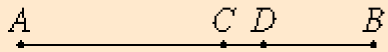
Euclid's Elements
Book IX
Proposition 25

If an odd number is subtracted from an even number, then the remainder is odd.

Let the odd number *BC* be subtracted from the even number.



I say that the remainder *CA* is odd.

Subtract the unit *CD* from *BC*, therefore *DB* is even. VII.Def.7

But *AB* is also even, therefore the remainder *AD* is also even. And *CD* is a unit, therefore *CA* is odd. IX.24 VII.Def.7

Therefore, *if an odd number is subtracted from an even number, then the remainder is odd.*

Q.E.D.

Guide

This is the second of four propositions that examine the result of subtracting even and odd numbers from each other.

Next proposition: IX.26

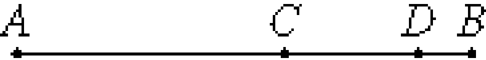
Previous: IX.24

Book IX introduction

Select from Book IX

Select book

Select topic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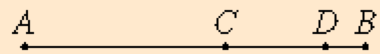
Euclid's Elements
Book IX
Proposition 26

If an odd number is subtracted from an odd number, then the remainder is even.

Let the odd number *BC* be subtracted from the odd number *AB*.



I say that the remainder *CA* is even.

Since *AB* is odd, subtract the unit *BD*, therefore the remainder *AD* is even. For the same reason *CD* is also even, so that the remainder *CA* is also even. VII.Def.7 IX.24

Therefore, *if an odd number is subtracted from an odd number, then the remainder is even.*

Q.E.D.

Guide

This proposition is used in IX.29.

Next proposition: IX.27

Previous: IX.25

Book IX introduction

Select from Book IX

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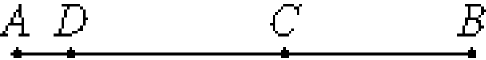
Select topic


A
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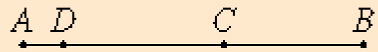
Euclid's Elements
Book IX
Proposition 27

If an even number is subtracted from an odd number, then the remainder is odd.



Let the even number *BC* be subtracted from the odd number *AB*.

I say that the remainder *CA* is odd.

Subtract the unit *AD*, therefore *DB* is even. VII.Def.7

But *BC* is also even, therefore the remainder *CD* is even. Therefore *CA* is odd. IX.24 VII.Def.7

Therefore, *if an even number is subtracted from an odd number, then the remainder is odd.*

Q.E.D.

Guide

This is the last of four propositions that examine the result of subtracting even and odd numbers from each other.

Next proposition: IX.28

Previous: IX.26

Book IX introduction

Select from Book IX

Select book

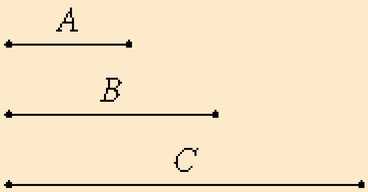
Select topic

Euclid's Elements
Book IX
Proposition 28

If an odd number is multiplied by an even number, then the product is even.

Let the odd number *A* multiplied by the even number *B* make *C*.

I say that *C* is even.



Since *A* multiplied by *B* makes *C*, therefore *C* is made up of as many numbers equal to *B* as there are units in *A*. And *B* is even, therefore *C* is made up of even numbers. VII.Def.15

But, if as many even numbers as we please be added together, the whole is even. Therefore *C* is even. IX.21

Therefore, *if an odd number is multiplied by an even number, then the product is even.*

Q.E.D.

Guide

This is one of two propositions that examine the result of multiplying even and odd numbers by each other. The third proposition, the product of two even numbers, is omitted.

Note that the proof for this theorem makes no use of the assumption that *A* is an odd number. The statement of this theorem might just as well be "if any number is multiplied by an even number, then the product is even."

Use of this proposition

The proof of proposition IX.31 concludes at one point that since a number divides a odd number, it must be even. Such a statement follows from this proposition.

Next proposition: IX.29

Previous: IX.27

Book IX introduction

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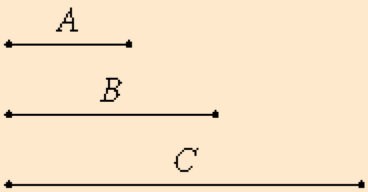
Select topic

Euclid's Elements
Book IX
Proposition 29

If an odd number is multiplied by an odd number, then the product is odd.

Let the odd number *A* multiplied by the odd number *B* make *C*.

I say that *C* is odd.



Since *A* multiplied by *B* makes *C*, therefore *C* is made up of as many numbers equal to *B* as there are units in *A*. And each of the numbers *A* and *B* is odd, therefore *C* is made up of odd numbers, the multitude of which is odd. Thus *C* is odd.

VII.Def.15
IX.23

Therefore, *if an odd number is multiplied by an odd number, then the product is odd.*

Q.E.D.

Guide

With the completion of this proposition, the study of addition, subtraction, and multiplication of even and odd numbers is also completed. There remain a few more propositions about even and odd numbers.

Next proposition: IX.30

Previous: IX.28

Book IX introduction

Select from Book IX

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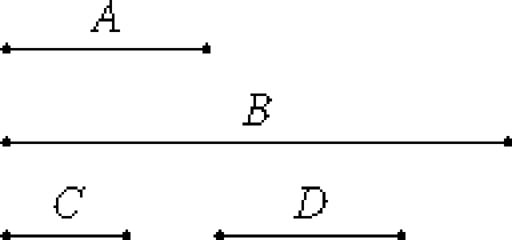
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Euclid's Elements

Book IX

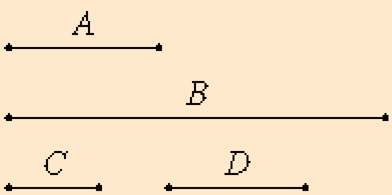
Proposition 3

If a cubic number multiplied by itself makes some number, then the product is a cube.

Let the cubic number *A* multiplied by itself make *B.*

I say that *B* is cubic.

Take *C*, the side of *A*. Multiply *C* by itself make *D.* It is then manifest that *C* multiplied by *D* makes *A.*



Now, since *C* multiplied by itself makes *D*, therefore *C* measures *D* according to the units in itself. But further the unit also measures *C* according to the units in it, therefore the unit is to *C* as *C* is to *D.* VII.Def.20

Again, since *C* multiplied by *D* makes *A*, therefore *D* measures *A* according to the units in *C.* But the unit also measures *C* according to the units in it, therefore the unit is to *C* as *D* is to *A.* But the unit is to *C* as *C* is to *D*, therefore the unit is to *C* as *C* is to *D*, and as *D* is to *A.*

Therefore between the unit and the number *A* two mean proportional numbers *C* and *D* have fallen in continued proportion.

Again, since *A* multiplied by itself makes *B,* therefore *A* measures *B* according to the units in itself. But the unit also measures *A* according to the units in it, therefore the unit is to *A* as *A* is to *B.* VII.Def.20

But between the unit and *A* two mean proportional numbers have fallen, therefore two mean proportional numbers also fall between *A* and *B.* VIII.8

But, if two mean proportional numbers fall between two numbers, and the first is a cube, then the second is also a cube. And *A* is a cube, therefore *B* is also a cube. VIII.23

Therefore, *if a cubic number multiplied by itself makes some number, then the product is a cube.*

Q.E.D.

Guide

Modern algebra certainly makes short work of this proposition: $(c^3)^2 = (c^2)^3$.

Use of this proposition

This proposition is used in the next two propositions and IX.9.

Next proposition: IX.4

Select from Book IX

Previous: IX.2

Book IX introduction

Select book

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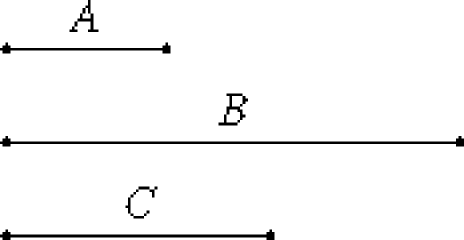
A
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Euclid's Elements
Book IX
Proposition 30

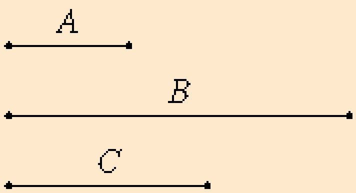
If an odd number measures an even number, then it also measures half of it.

Let the odd number *A* measure the even number *B.*

I say that it also measures the half of it.

Since *A* measures *B,* let it measure it according to *C.*

I say that *C* is not odd.



If possible, let it be so. Then, since *A* measures *B* according to *C,* therefore *A* multiplied by *C* makes *B.* Therefore *B* is made up of odd numbers the multitude of which is odd. Therefore *B* is odd, which is absurd, for by hypothesis it is even. Therefore *C* is not odd, therefore *C* is even. IX.23

Thus *A* measures *B* an even number of times. For this reason then it also measures the half of it.

Therefore, *if an odd number measures an even number, then it also measures half of it.*

Q.E.D.

Guide

This proposition is used in the next one.

Next proposition: IX.31

Previous: IX.29

Book IX introduction

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Select book

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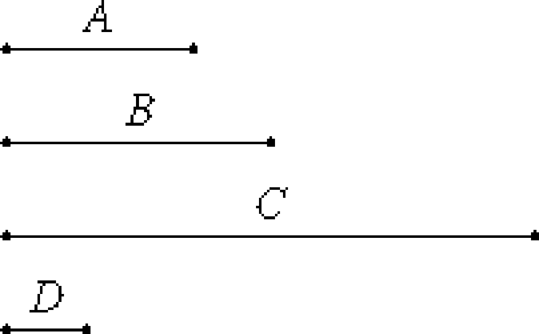
A
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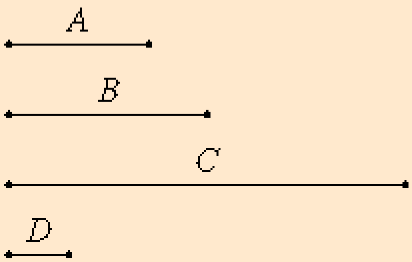
Euclid's Elements
Book IX
Proposition 31

If an odd number is relatively prime to any number, then it is also relatively prime to double it.

Let the odd number *A* be relatively prime to any number *B*, and let *C* be double of *B*.

I say that *A* is relatively prime to *C*.



If they are not relatively prime, then some number will measure them.

Let a number *D* measure them.

Now *A* is odd, therefore *D* is also odd. And since *D* which is odd measures *C*, and *C* is even, therefore *D* measures the half of *C* also. (IX.28) IX.30

But *B* is half of *C*, therefore *D* measures *B*. But it also measures *A*, therefore *D* measures *A* and *B* which are relatively prime, which is impossible.

Therefore *A* cannot but be relatively prime to *C*. Therefore *A* and *C* are relatively prime.

Therefore, *if an odd number is relatively prime to any number, then it is also relatively prime to double it.*

Q.E.D.

Guide

A generalization of this proposition would be "If two numbers (2 and *B* in this proposition) are relatively prime to to any number (*A*), then their product (2*B*) is also relatively prime to it (*A*)." That is proposition VII.24.

Next proposition: IX.32

Previous: IX.30

Book IX introduction

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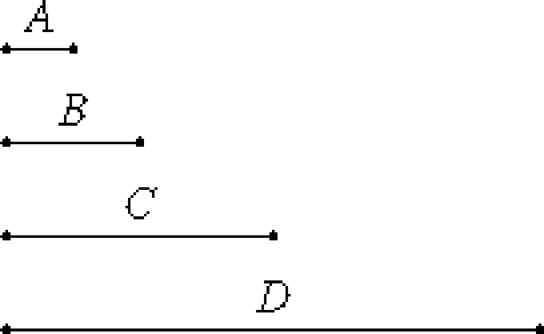
A
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Euclid's Elements
Book IX
Proposition 32

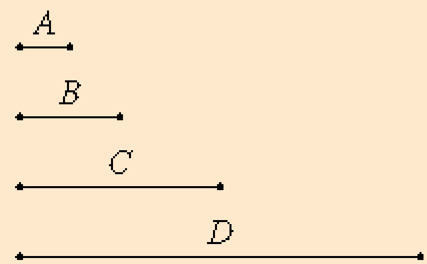
Each of the numbers which are continually doubled beginning from a dyad is even-times even only.

Let as many numbers as we please, *B, C,* and *D,* be continually doubled beginning from the dyad *A.*

I say that *B, C,* and *D* are even-times even only.

Now that each of the numbers *B, C,* and *D* is even-times even is manifest, for each is doubled from a dyad.

I say that it is also even-times even only.



Set out a unit. Since then as many numbers as we please beginning from a unit are in continued proportion, and the number *A* after the unit is prime, therefore *D,* the greatest of the numbers *A, B, C,* and *D,* is not measured by any other number except *A, B,* and *C.* And each of the numbers *A, B,* and *C* is even, therefore *D* is even-times even only. IX.13 VII.Def.8

Similarly we can prove that each of the numbers *B* and *C* is even-times even only.

Therefore, *each of the numbers which are continually doubled beginning from a dyad is even-times even only.*

Q.E.D.

Guide

Numbers which are even-times even only are just the powers of 2: 4, 8, 16, 32, etc.

An alternate proof of this proposition uses IX.30 rather than IX.13. If an odd number could divide *D,* then by IX.30, it would divide half of it, namely *C.* Then it would divide *B,* then *A,* the diad, which is absurd. Note that the reduction step mentioned in this alternate proof is much simpler than the reduction step used in the proof of IX.30.

Next proposition: IX.33

Previous: IX.31

Book IX introduction

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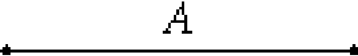
D.E.Joyce
Clark University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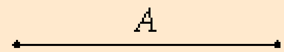
Euclid's Elements

Book IX

Proposition 33

If a number has its half odd, then it is even-times odd only.

Let the number *A* have its half odd.



I say that *A* is even-times odd only.

Now that it is even-times odd is manifest, for the half of it, being odd, measures it an even number of times. VII.Def.9

I say next that it is also even-times odd only.

If *A* is even-times even also, then it is measured by an even number according to an even number, so that the half of it is also measured by an even number though it is odd, which is absurd. VII.Def.8

Therefore *A* is even-times odd only.

Therefore, *if a number has its half odd, then it is even-times odd only.*

Q.E.D.

Guide

To say that a number is even-times odd only means that it is even-times odd, but it is not even-times even. As this proposition states, such numbers are the numbers which are twice odd numbers.

Next proposition: IX.34

Previous: IX.32

Book IX introduction

Select from Book IX

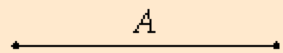
Select book

Select topic

Euclid's Elements
Book IX
Proposition 34

If an [even] number neither is one of those which is continually doubled from a dyad, nor has its half odd, then it is both even-times even and even-times odd.

Let the [even] number *A* neither be one of those doubled from a dyad, nor have its half odd.



I say that *A* is both even-times even and even-times odd.

Now that *A* is even-times even is manifest, for it has not its half odd. VII.Def.8

I say next that it is also even-times odd.

If we bisect *A*, then bisect its half, and do this continually, we shall come upon some odd number which measures *A* according to an even number. If not, we shall come upon a dyad, and *A* will be among those which are doubled from a dyad, which is contrary to the hypothesis.

Thus *A* is even-times odd.

But it was also proved even-times even. Therefore *A* is both even-times even and even-times odd.

Therefore, *if an [even] number neither is one of those which is continually doubled from a dyad, nor has its half odd, then it is both even-times even and even-times odd.*

Q.E.D.

Guide

This completes the series of propostions on even and odd numbers that started with IX.21.

Next proposition: IX.35

Previous: IX.33

Book IX introduction

Select from Book IX

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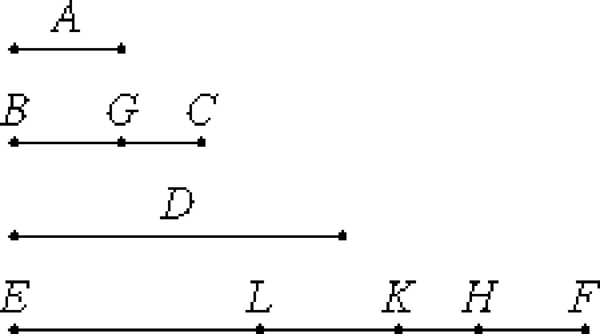
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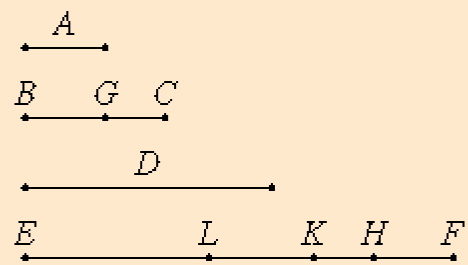
Euclid's Elements
Book IX
Proposition 35

If as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then the excess of the second is to the first as the excess of the last is to the sum of all those before it.

Let there be as many numbers as we please in continued proportion, *A, BC, D,* and *EF,* beginning from *A* as least, and let there be subtracted from *BC* and *EF* the numbers *BG* and *FH,* each equal to *A.*

I say that *GC* is to *A* as *EH* is to the sum of *A, BC,* and *D.*



Make *FK* equal to *BC,* and *FL* equal to *D.*

Then, since *FK* equals *BC,* and of these the part *FH* equals the part *BG,* therefore the remainder *HK* equals the remainder *GC.*

And since *EF* is to *D* as *D* is to *BC,* and as *BC* is to *A,* while *D* equals *FL, BC* equals *FK,* and *A* equals *FH,* therefore *EF* is to *FL* as *LF* is to *FK,* and as *FK* is to *FH.* Taken separately *EL* is to *LF* as *LK* is to *FK,* and as *KH* is to *FH.* VII.11 VII.13

Since one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents, therefore *KH* is to *FH* as the sum of *EL, LK,* and *KH* is to the sum of *LF, FK,* and *HF.* VII.12

But *KH* equals *CG, FH* equals *A,* and the sum of *LF, FK,* and *HF* equals the sum of *D, BC,* and *A,* therefore *CG* is to *A* as *EH* is to the sum of *D, BC,* and *A.*

Therefore the excess of the second is to the first as the excess of the last is to the sum of those before it.

Therefore, *if as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then the excess of the second is to the first as the excess of the last is to the sum of all those before it.*

Q.E.D.

Guide

This proposition says if a sequence of numbers $a_1, a_2, a_3, ..., a_n, a_{n+1}$ is in continued proportion

$$a_1:a_2 = a_2:a_3 = ... = a_n:a_{n+1}$$

then

$$(a_2 - a_1) : a_1 = (a_{n+1} - a_1) : (a_1 + a_2 + ... + a_n).$$

This conclusion gives a way of computing the sum of the terms in the continued proportion as

$$a_1 + a_2 + \ldots + a_n = a_1 \frac{a_{n+1} - a_1}{a_2 - a_1}.$$

If we denote the first term by *a* and the ratio of the terms by *r*, then this gives the familiar formula

$$a + ar + ar^2 + \ldots + ar^{n-1} = a \frac{r^n - 1}{r - 1}.$$

Summary of the proof

The proof is much more comprehisible when it's translated in to algebraic notation. The correspondence is as follows

$A = a_1$	$BG = FH = a_1$
$BC = a_2$	$GC = a_2 - a_1$
...	$EH = a_{n+1} - a_1\ a_2{:}a_1$.
$D = a_n$	
$EF = a_{n+1}$	

For each proportion, say the first,

$$a_{n+1}{:}a_n = a_n{:}a_{n-1},$$

take it separately according to VII.11 to get

$$(a_{n+1} - a_n){:}(a_n - a_{n-1}) = a_n{:}a_{n-1},$$

then alternately

$$(a_{n+1} - a_n){:}a_n = (a_n - a_{n-1}){:}a_{n-1}.$$

Stringing the conclusions together, we have

$$(a_{n+1} - a_n){:}a_n = (a_n - a_{n-1}){:}a_{n-1} = \ldots = (a_2 - a_1){:}a_1.$$

Next, using proposition VII.12, the sum of the antecedents is to the sum of the consequences equals this same ratio. Therefore

$$(a_{n+1}–a_n + a_n–a_{n-1} + \ldots + a_2–a_1) : (a_n + a_{n-1} + \ldots + a_2 + a_1) = (a_2 - a_1) : a_1.$$

But $a_{n+1} - a_n + a_n - a_{n-1} + \ldots + a_2 - a_1$ equals $a_{n+1} - a_1$. That gives us the conclusion of the proposition

$$(a_{n+1} - a_1) : (a_n + a_{n-1} + \ldots + a_2 + a_1) = (a_2 - a_1) : a_1.$$

Use of this propostion

This proposition is used in the next one.

Next proposition: IX.36

Previous: IX.34

Book IX introduction

Select from Book IX

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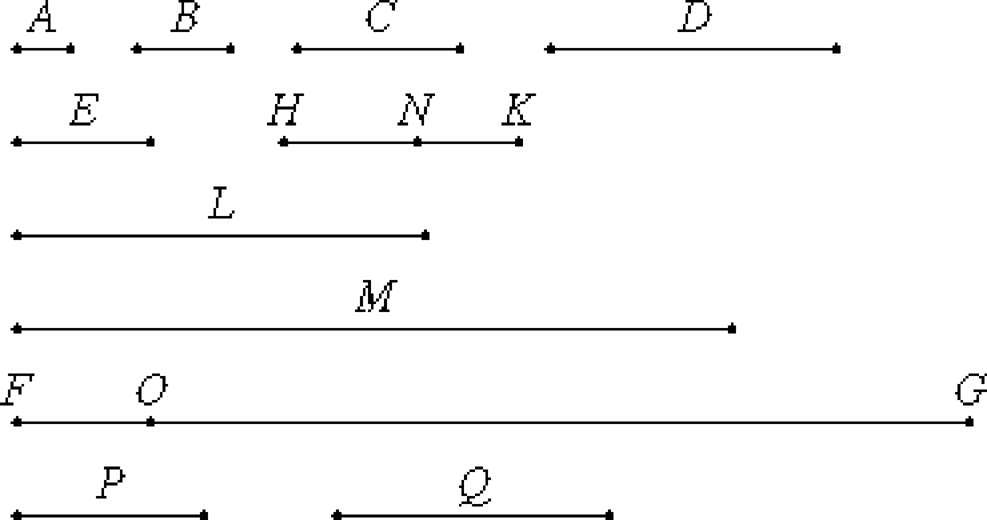
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Euclid's Elements

Book IX

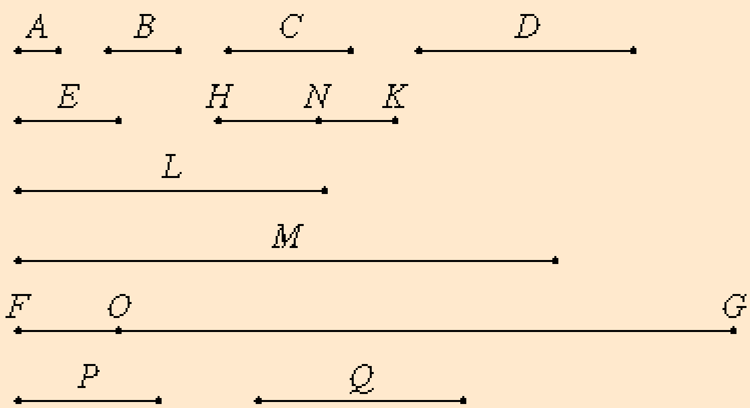
Proposition 36

If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

Let as many numbers as we please, *A, B, C,* and *D,* beginning from a unit be set out in double proportion, until the sum of all becomes prime, let *E* equal the sum, and let *E* multiplied by *D* make *FG.*

I say that *FG* is perfect.

For, however many *A, B, C,* and *D* are in multitude, take so many *E, HK, L,* and *M* in double proportion beginning from *E.*



Therefore, *ex aequali A* is to *D* as *E* is to *M.* Therefore the product of *E* and *D* equals the product of *A* VII.14
and *M.* And the product of *E* and *D* is *FG,* therefore the product of *A* and *M* is also *FG.* VII.19

Therefore *A* multiplied by *M* makes *FG.* Therefore *M* measures *FG* according to the units in *A.* And *A* is a dyad, therefore *FG* is double of *M.*

But *M, L, HK,* and *E* are continuously double of each other, therefore *E, HK, L, M,* and *FG* are continuously proportional in double proportion.

Subtract from the second *HK* and the last *FG* the numbers *HN* and *FO,* each equal to the first *E.*
Therefore the excess of the second is to the first as the excess of the last is to the sum of those before it. IX.35
Therefore *NK* is to *E* as *OG* is to the sum of *M, L, KH,* and *E.*

And *NK* equals *E,* therefore *OG* also equals *M, L, HK, E.* But *FO* also equals *E,* and *E* equals the sum of *A, B, C, D* and the unit. Therefore the whole *FG* equals the sum of *E, HK, L, M, A, B, C, D,* and the unit, and it is measured by them.

I say also that *FG* is not measured by any other number except *A, B, C, D, E, HK, L, M,* and the unit.

If possible, let some number *P* measure *FG,* and let *P* not be the same with any of the numbers *A, B, C, D, E, HK, L,* or *M.*

And, as many times as *P* measures *FG,* so many units let there be in *Q,* therefore *Q* multiplied by *P* makes *FG.*

But, further, *E* multiplied by *D* makes *FG,* therefore *E* is to *Q* as *P* is to *D.* VII.19

And, since *A, B, C,* and *D* are continuously proportional beginning from a unit, therefore *D* is not measured by any other number except *A, B,* or *C.* IX.13

And, by hypothesis, *P* is not the same with any of the numbers *A, B,* or *C,* therefore *P* does not measure *D.* But *P* is to *D* as *E* is to *Q,* therefore neither does *E* measure *Q.* VII.Def.20

And *E* is prime, and any prime number is prime to any number which it does not measure. Therefore *E* and *Q* are relatively prime. VII.29

But primes are also least, and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, and *E* is to *Q* as *P* is to *D,* therefore *E* measures *P* the same number of times that *Q* measures *D.* VII.21 VII.20

But *D* is not measured by any other number except *A, B,* or *C,* therefore *Q* is the same with one of the numbers *A, B,* or *C.* Let it be the same with *B.*

And, however many *B, C,* and *D* are in multitude, take so many *E, HK,* and *L* beginning from *E.*

Now *E, HK,* and *L* are in the same ratio with *B, C,* and *D,* therefore, *ex aequali B* is to *D* as *E* is to *L.* VII.14

Therefore the product of *B* and *L* equals the product of *D* and *E.* But the product of *D* and *E* equals the product of *Q* and *P,* therefore the product of *Q* and *P* also equals the product of *B* and *L.* VII.19

Therefore *Q* is to *B* as *L* is to *P.* And *Q* is the same with *B,* therefore *L* is also the same with *P,* which is impossible, for by hypothesis *P* is not the same with any of the numbers set out. VII.19

Therefore no number measures *FG* except *A, B, C, D, E, HK, L, M,* and the unit.

And *FG* was proved equal to the sum of *A, B, C, D, E, HK, L, M,* and the unit, and a perfect number is that which equals its own parts, therefore *FG* is perfect. VII.Def.22

Therefore, *if as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.*

Q.E.D.

Guide

Summary of the proof

Euclid begins by assuming that the sum of a number of powers of 2 (the sum beginning with 1) is a prime number. Let p be the number of powers of 2, and let s be their sum which is prime.

$$s = 1 + 2 + 2^2 + ... + 2^{p-1}$$

Note that the last power of 2 is 2^{p-1} since the sum starts with 1, which is 2^0.

In Euclid's proof, *A* represents 2, *B* represents 2^2, *C* represents 2^3, and *D* is supposed to be the last power of 2, so it represents 2^{p-1}. Also, *E* represents their sum s, and *FG* is the product of *E* and *D,* so it represents $s2^{p-1}$. Let's denote that last by n.

$$n = s2^{p-1}$$

The goal is to show that *n* is a perfect number.

In the first part of this proof, Euclid finds some proper divisors of *n* that sum to *n*. These come in two sequences:

$$1, 2, 2^2, ..., 2^{p-1}$$

and

$$s, 2s, 2^2s, ..., 2^{n-2}s$$

In his proof, the latter are represented by *E*, *HK*, *L*, and finally *M*.

It is clear that each of these is a proper divisor of *n*, and later in the proof Euclid shows that they are the only proper divisors of *n*.

Using the previous proposition, IX.35, Euclid finds the sum of the continued proportion,

$$s + 2s + 2^2s + ... + 2^{n-2}s,$$

to be $2^{n-1}s - s$. But *s* was the sum $1 + 2 + 2^2 + ... + 2^{p-1}$, hence,

$$n = 2^{n-1}s = 1 + 2 + 2^2 + ... + 2^{p-1}$$
$$+ s + 2s + 2^2s + ... + 2^{n-2}s$$

Thus, *n* is a sum of these proper divisors.

All that is left to do is to show that they are the only proper divisors of *n*, for then *n* will be the sum of all of its proper divisors, whence a perfect number.

The remainder of the proof is detailed and difficult to follow. It hinges on IX.13 which implies that the only factors of 2^{p-1} are powers of 2, so all the factors of 2^{p-1} have been found. Here's a not-too-faithful version of Euclid's argument. Suppose *n* factors as *ab* where *a* is not a proper divisor of *n* in the list above. In Euclid's proof, *P* represents *a* and *Q* represents *b*.

Since *a* divides $s\ 2^{p-1}$, but is not a power of 2, and *s* is prime, therefore *s* divides *a*. Then *b* has to be a power of 2. But then *a* has to be a power of 2 times *s*. But all the powers of 2 times *s* are on the list of known proper divisors. Therefore, the list includes all the proper divisors.

Mersenne primes and perfect numbers

Note that the sum, $s = 1 + 2 + 2^2 + ... + 2^{p-1}$, equals $2^p - 1$, by IX.35. As this fact is not needed in the proof, Euclid omits to mention it. Thus, we can restate the proposition as follows:

> If $2^p - 1$ is a prime number, then $(2^p - 1)\ 2^{p-1}$ is a perfect number.

Prime numbers of the form $2^p - 1$ have come to be called *Mersenne primes* named in honor of Marin Mersenne (1588-1648), one of many people who have studied these numbers. The four smallest perfect numbers, 6, 28, 496, and 8128, were known to the ancient Greek mathematicians. The Mersenne primes $2^p - 1$ corresponding to these four perfect numbers are 3, 7, 31, and 127, respectively, where the exponents *p* are 2, 3, 5, and 7, respectively.

The observation that these four exponents are all prime suggests the following two questions:

1. In order for $2^p - 1$ to be prime, is it sufficient for p to be prime?
2. In order for $2^p - 1$ to be prime, is it necessary for p to be prime?

Naturally, the next number to check for primality is $2^{11} - 1$, 2047, which, by a simple search for prime factors is found not to be prime. The number 2047 factors as 23 times 89. Therefore, primality of p is *not* sufficient.

In 1640 Pierre de Fermat (1601-1665) wrote to Mersenne with his investigation of these primes. Fermat found three conditions on p that were necessary for $2^p - 1$ to be prime. One of these conditions answers the second question above — p does have to be prime. Here's a quick argument for that. If p did factor, say as ab, then $2^p - 1$, which is $2^{ab} - 1$, would also factor, namely as

$$2^{ab} - 1 = (2^a - 1)\,(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a).$$

Many mathematicians have studied Mersenne primes since then. A fairly practical testing algorithm was constructed by Lucas in 1876. He showed that the the number $2^p - 1$ is prime if and only if it divides the number $S(p-1)$, where $S(p-1)$ is defined recursively: $S(1) = 4$, and $S(n+1) = S(n)^2 - 2$.

The search for more Mersenne primes, and therefore more perfect numbers, continues. It is not known if there are infinitely many or finitely many even perfect numbers. Mersenne primes are scarce, but more continue to be found. There are at least 39 of them, the largest known (as of 2002) is $2^{213466917} - 1$.

There is a also a question about odd perfect numbers: Are there any? It has been shown that there are no small odd perfect numbers; it is known that odd numbers with fewer then 300 digits are not perfect. It may well be that there are no odd perfect numbers, but to date there is no proof.

Next book: Book X introduction

Previous proposition: IX.35

Book IX introduction

Select from Book IX

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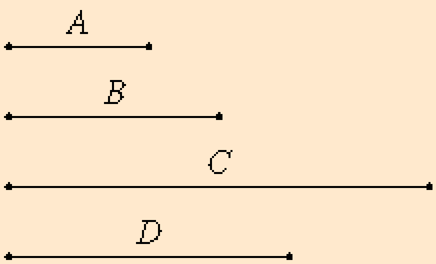
Euclid's Elements
Book IX
Proposition 4

If a cubic number multiplied by a cubic number makes some number, then the product is a cube.

Let the cubic number *A* multiplied by the cubic number *B* make *C*.

I say that *C* is cubic.

Multiply *A* by itself to make *D*. Then *D* is a cube. IX.3



Since *A* multiplied by itself makes *D*, and multiplied by *B* makes *C*, therefore *A* is to *B* as *D* is to *C*. And, since *A* and *B* are cubic numbers, therefore *A* and *B* are similar solid numbers. Therefore two mean proportional numbers fall between *A* and *B*, so that two mean proportional numbers fall between *D* and *C* also. VII.17 VIII.19 VIII.8

And *D* is a cube, therefore *C* is also a cube. VIII.23

Therefore, *if a cubic number multiplied by a cubic number makes some number, then the product is cubic.*

Q.E.D.

Guide

Of course, $m^3n^3 = (mn)^3$.

Next proposition: IX.5

Previous: IX.3

Book IX introduction

Select from Book IX

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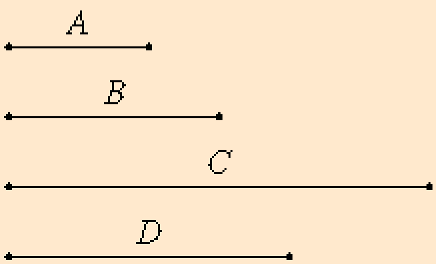
Euclid's Elements
Book IX
Proposition 5

If a cubic number multiplied by any number makes a cubic number, then the multiplied number is also cubic.

Let the cubic number *A* multiplied by any number *B* make the cubic number *C.*

I say that *B* is cubic.



Multiply *A* by itself to make *D.* Then *D* is a cube. IX.3

Now, since *A* multiplied by itself makes *D,* and multiplied by *B* makes *C,* therefore *A* is to *B* as *D* is to *C.* VII.17

And since *D* and *C* are cubes, therefore they are similar solid numbers. Therefore two mean proportional numbers fall between *D* and *C.* And *D* is to *C* as *A* is to *B,* therefore two mean proportional numbers fall between *A* and *B,* too. VIII.19 VIII.8

And *A* is a cube, therefore *B* is also a cube. VIII.23

Therefore, *if a cubic number multiplied by any number makes a cubic number, then the multiplied number is also cubic.*

Q.E.D.

Guide

This proposition is a converse of the previous one. When $ab = c$, and a is a cube, the previous propsition said that if b is a cube, then c is also, while this proposition says that if c is a cube, then b is also.

Next proposition: IX.6

Previous: IX.4

Book IX introduction

Select from Book IX

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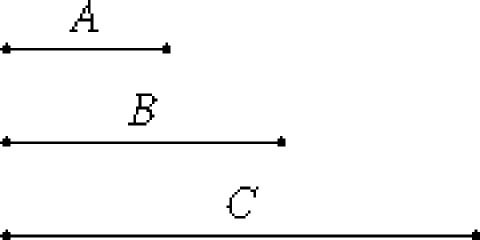
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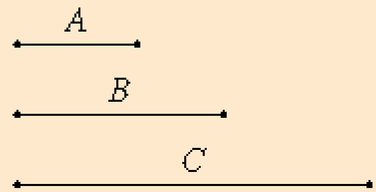
Euclid's Elements
Book IX
Proposition 6

If a number multiplied by itself makes a cubic number, then it itself is also cubic.

Let the number *A* multiplied by itself make the cubic number *B.*

I say that *A* is also cubic.



Multiply *A* by *B* to make *C.* Since, then, *A* multiplied by itself makes *B*, and multiplied by *B* makes *C*, therefore *C* is a cube. And, since *A* multiplied by itself makes *B*, therefore *A* measures *B* according to the units in itself.

But the unit also measures *A* according to the units in it. Therefore the unit is to *A* as *A* is to *B.* And, since *A* multiplied by *B* makes *C*, therefore *B* measures *C* according to the units in *A.* VII.Def.20

But the unit also measures *A* according to the units in it.

Therefore the unit is to *A* as *B* is to *C.* But the unit is to *A* as *A* is to *B*, therefore *A* is to *B* as *B* is to *C.* VII.Def.20

And, since *B* and *C* are cubes, therefore they are similar solid numbers.

Therefore there are two mean proportional numbers between *B* and *C.* And *B* is to *C* as *A* is to *B.* Therefore there are two mean proportional numbers between *A* and *B* also. VIII.19 VIII.8

And *B* is a cube, therefore *A* is also a cube. cf. VIII.23

Therefore, *if a number multiplied by itself makes a cubic number, then it itself is also cubic.*

Q.E.D.

Guide

Outline of the proof

Assume that a^2 is a cube. Since a^3 is also a cube, therefore there are two mean proportionals between them (VIII.19). But we have the proportion $a:a^2 = a^2:a^3$, so there are also two mean proportionals between a and a^2 (VIII.8). And since a^2 is a cube, therefore a is also a cube (VIII.23).

Use of this proposition

This proposition is used in IX.10.

Next proposition: IX.7 Select from Book IX

Previous: IX.5

Book IX introduction

Select book

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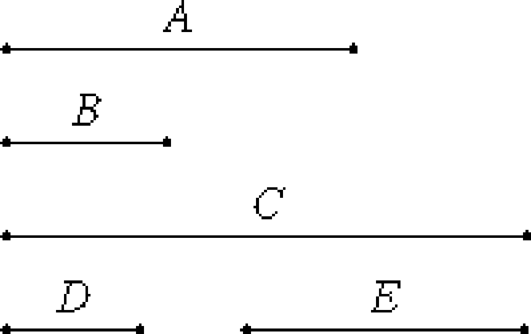
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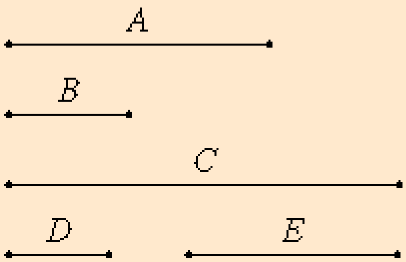
Euclid's Elements
Book IX
Proposition 7

If a composite number multiplied by any number makes some number, then the product is solid.

Let the composite number *A* multiplied by any number *B* make *C*.

I say that *C* is solid.



Since *A* is composite, it is measured by some number *D*. Let there be as many units in *E* as times that *D* measures *A* VII.Def.13

Since *D* measures *A* according to the units in *E*, therefore *E* multiplied by *D* makes *A*. And, since *A* multiplied by *B* makes *C*, and *A* is the product of *D* and *E*, therefore the product of *D* and *E* multiplied by *B* makes *C*. VII.Def.15

Therefore *C* is solid, and *D*, *E*, and *B* are its sides.

Therefore, *if a composite number multiplied by any number makes some number, then the product is solid.*

Q.E.D.

Guide

Numbers with at least two factors are plain numbers; those with at least three are solid numbers.

Perhaps Euclid takes extra steps that we would miss because he sees "*d* measures *a* a number *e* times" as saying something different from the product of *d* and *e* equals *a*."

Next proposition: IX.8

Previous: IX.6

Book IX introduction

Select from Book IX

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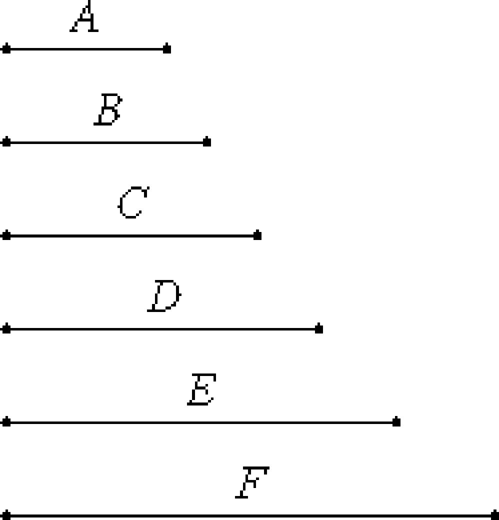
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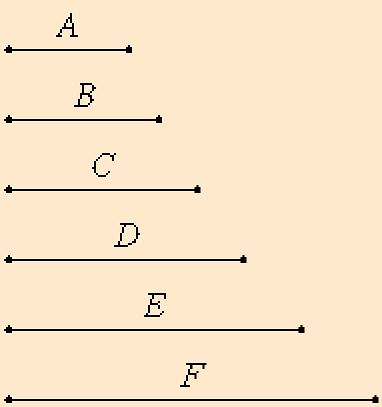
Euclid's Elements
Book IX
Proposition 8

If as many numbers as we please beginning from a unit are in continued proportion, then the third from the unit is square as are also those which successively leave out one, the fourth is cubic as are also all those which leave out two, and the seventh is at once cubic and square are also those which leave out five.

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, *E*, and *F*, beginning from a unit and in continued proportion.



I say that *B*, the third from the unit, is square as are all those which leave out one; *C*, the fourth, is cubic as are all those which leave out two; and *F*, the seventh, is at once cubic and square as are all those which leave out five.

Since the unit is to *A* as *A* is to *B*, therefore the unit measures the number *A* the same number of times that *A* measures *B*. But the unit measures the number *A* according to the units in it, therefore *A* also measures *B* according to the units in *A*. VII.Def.20

Therefore *A* multiplied by itself makes *B*, therefore *B* is square. And, since *B*, *C*, and *D* are in continued proportion, and *B* is square, therefore *D* is also square. For the same reason *F* is also square. VIII.22

Similarly we can prove that all those which leave out one are square.

I say next that *C*, the fourth from the unit, is cubic are also all those which leave out two.

Since the unit is to *A* as *B* is to *C*, therefore the unit measures the number *A* the same number of times that *B* measures *C*. But the unit measures the number *A* according to the units in *A*, therefore *B* also measures *C* according to the units in *A*. Therefore *A* multiplied by *B* makes *C*. Since then *A* multiplied by itself makes *B*, and multiplied by *B* makes *C*, therefore *C* is cubic.

And, since *C*, *D*, *E*, and *F* are in continued proportion, and *C* is cubic, therefore *F* is also cubic. But it was also proved square, therefore the seventh from the unit is both cubic and square. Similarly we can prove that all the numbers which leave out five are also both cubic and square. VIII.23

Therefore, *if as many numbers as we please beginning from a unit are in continued proportion, then the third from the unit is square as are also those which successively leave out one, the fourth is cubic as are also all those which leave out two, and the seventh is at once cubic and square are also those which leave out five.*

Q.E.D.

Guide

In the continued proportion

1, a, a^2, a^3, a^4, a^5, a^6, a^7, etc.,

every second, a^2, a^4, a^6, a^8, etc., is a square;

every third, a^3, a^6, a^9, a^{12}, etc., is a cube; and

every sixth, a^6, a^{12}, a^{18}, a^{24}, etc., is both a square and a cube.

Use of this proposition

This proposition is used in four of the next five propositions.

Next proposition: IX.9

Previous: IX.7

Book IX introduction

Select from Book IX

Select book

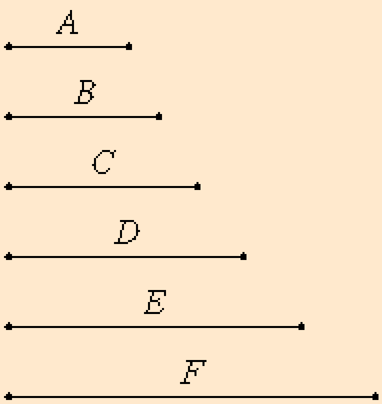
Select topic

Euclid's Elements

Book IX

Proposition 9

If as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is square, then all the rest are also square; and if the number after the unit is cubic, then all the rest are also cubic.



Let there be as many numbers as we please, *A, B, C, D, E,* and *F,* beginning from a unit and in continued proportion, and let *A,* the number after the unit, be square.

I say that all the rest are also square.

Now it was proved that *B,* the third from the unit, is square as are all those which leave out one. IX.8

I say that all the rest are also square.

Since *A, B,* and *C* are in continued proportion, and *A* is square, therefore *C* is also square. Again, since *B, C,* and *D* are in continued proportion, and *B* is square, therefore *D* is also square. Similarly we can prove that all the rest are also square. VIII.22

Next, let *A* be a cube.

I say that all the rest are also cubes.

Now it was proved that *C,* the fourth from the unit, is a cube as are all those which leave out two. IX.8

I say that all the rest are also cubic.

Since the unit is to *A* as *A* is to *B,* therefore the unit measures *A* the same number of times as *A* measures *B.* But the unit measures *A* according to the units in it, therefore *A* also measures *B* according to the units in itself, therefore *A* multiplied by itself makes *B.*

And *A* is cubic. But, if cubic number multiplied by itself makes some number, then the product is also a cube, therefore *B* is also a cube. IX.3

And, since the four numbers *A, B, C,* and *D* are in continued proportion, and *A* is a cube, therefore *D* also is a cube. VIII.23

For the same reason *E* is also a cube, and similarly all the rest are cubes.

Therefore, *if as many numbers as we please beginning from a unit are in continued proportion, and the number after the unit is square, then all the rest are also square; and if the number after the unit is cubic, then all the rest are also cubic.*

Q.E.D.

Guide

This proposition says that if a number is a square then all its powers are squares, too. Likewise for cubes.

The following theorem is a converse of this one.

Next proposition: IX.10

Previous: IX.8

Book IX introduction

Select from Book IX

Select book

Select topic

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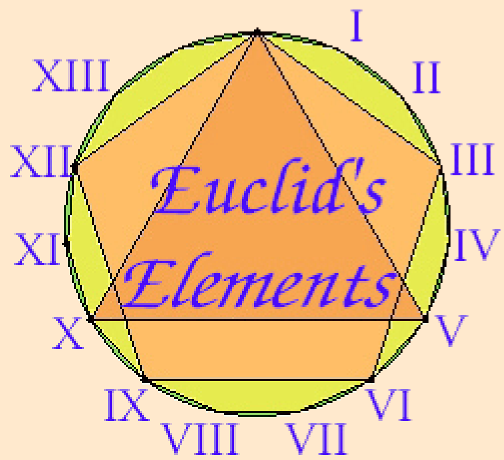
Book V



Book V

Table of contents

Definitions

Definition 1
A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.

Definition 2
The greater is a *multiple* of the less when it is measured by the less.

Definition 3
A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.

Definition 4
Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

Definition 5
Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Definition 6
Let magnitudes which have the same ratio be called *proportional.*

Definition 7
When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the

multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth.

Definition 8

A proportion in three terms is the least possible.

Definition 9

When three magnitudes are proportional, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

Definition 10

When four magnitudes are continuously proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion.

Definition 11

Antecedents are said to *correspond* to antecedents, and consequents to consequents.

Definition 12

Alternate ratio means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

Definition 13

Inverse ratio means taking the consequent as antecedent in relation to the antecedent as consequent.

Definition 14

A ratio *taken jointly* means taking the antecedent together with the consequent as one in relation to the consequent by itself.

Definition 15

A ratio *taken separately* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

Definition 16

Conversion of a ratio means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

Definition 17

A ratio *ex aequali* arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, the first is to the last among the first magnitudes as the first is to the last among the second magnitudes. Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

Definition 18

A *perturbed proportion* arises when, there being three magnitudes and another set equal to them in multitude, antecedent is to consequent among the first magnitudes as antecedent is to consequent among the second magnitudes, while, the consequent is to a third among the first magnitudes as a third is to the antecedent among the second magnitudes.

Propositions

Proposition 1

If any number of magnitudes are each the same multiple of the same number of other magnitudes, then the sum is that multiple of the sum.

Proposition 2

If a first magnitude is the same multiple of a second that a third is of a fourth, and a fifth also is the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth also is the same multiple of the second that the sum of the third and sixth is of the fourth.

Proposition 3

If a first magnitude is the same multiple of a second that a third is of a fourth, and if equimultiples are taken of the first and third, then the magnitudes taken also are equimultiples respectively, the one of the second and the other of the fourth.

Proposition 4

If a first magnitude has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

Proposition 5

If a magnitude is the same multiple of a magnitude that a subtracted part is of a subtracted part, then the remainder also is the same multiple of the remainder that the whole is of the whole.

Proposition 6

If two magnitudes are equimultiples of two magnitudes, and any magnitudes subtracted from them are equimultiples of the same, then the remainders either equal the same or are equimultiples of them.

Proposition 7

Equal magnitudes have to the same the same ratio; and the same has to equal magnitudes the same ratio.

Corollary If any magnitudes are proportional, then they are also proportional inversely.

Proposition 8

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Proposition 9

Magnitudes which have the same ratio to the same equal one another; and magnitudes to which the same has the same ratio are equal.

Proposition 10

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

Proposition 11

Ratios which are the same with the same ratio are also the same with one another.

Proposition 12

If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.

Proposition 13

If a first magnitude has to a second the same ratio as a third to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first also has to the second a greater ratio than the fifth to the sixth.

Proposition 14

If a first magnitude has to a second the same ratio as a third has to a fourth, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less.

Proposition 15

Parts have the same ratio as their equimultiples.

Proposition 16
If four magnitudes are proportional, then they are also proportional alternately.

Proposition 17
If magnitudes are proportional taken jointly, then they are also proportional taken separately.

Proposition 18
If magnitudes are proportional taken separately, then they are also proportional taken jointly.

Proposition 19
If a whole is to a whole as a part subtracted is to a part subtracted, then the remainder is also to the remainder as the whole is to the whole.

Corollary. If magnitudes are proportional taken jointly, then they are also proportional in conversion.

Proposition 20
If there are three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if *ex aequali* the first is greater than the third, then the fourth is also greater than the sixth; if equal, equal, and; if less, less.

Proposition 21
If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them is perturbed, then, if *ex aequali* the first magnitude is greater than the third, then the fourth is also greater than the sixth; if equal, equal; and if less, less.

Proposition 22
If there are any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio *ex aequali*.

Proposition 23
If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio *ex aequali*.

Proposition 24
If a first magnitude has to a second the same ratio as a third has to a fourth, and also a fifth has to the second the same ratio as a sixth to the fourth, then the sum of the first and fifth has to the second the same ratio as the sum of the third and sixth has to the fourth.

Proposition 25
If four magnitudes are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two.

Guide for Book V

Background on ratio and proportion

Book V covers the abstract theory of ratio and proportion. A ratio is an indication of the relative size of two magnitudes. The propositions in the following book, Book VI, are all geometric and depend on ratios, so the theory of ratios needs to be developed first. To get a better understanding of what ratios are in geometry, consider the first proposition VI.1. It states that triangles of the same height are proportional to their bases, that is to say, one triangle is to another as one base is to the other. (A proportion is simply an equality of two ratios.) A simple example is when one base is twice the other, therefore the triangle on that base is also twice the triangle on the other base. This ratio of 2:1 is fairly easy to comprehend. Indeed, any ratio equal to a ratio of two numbers is easy to comprehend. Given a proportion that says a ratio of lines equals a ratio of numbers, for instance, $A:B = 8:5$, we have two interpretations. One is that there is a shorter line $CA = 8C$ *while* $B = 5C$. *This interpretation is the definition of proportion that appears in Book VII. A second interpretation is that* $5A = 8B$. *Either interpretation will do if one of the ratios is a ratio of numbers, and if A:B equals a ratio of numbers that A and B are commensurable, that is, both are measured by a common measure.*

Many straight lines, however, are not commensurable. If A is the side of a square and B its diagonal, then A and B are not commensurable; the ratio A:B is not the ratio of numbers. This fact seems to have been discovered by the Pythagoreans, perhaps Hippasus of Metapontum, some time before 400 B.C.E., a hundred years before Euclid's Elements.

The difficulty is one of foundations: what is an adequate definition of proportion that includes the incommensurable case? The solution is that in V.Def.5. That definition, and the whole theory of ratio and proportion in Book V, are attributed to Eudoxus of Cnidus (died. ca. 355 B.C.E.)

Summary of the propositions

The first group of propositions, 1, 2, 3, 5, and 6 only mention multitudes of magnitudes, not ratios. They each either state, or depend strongly on, a distributivity or an associativity. In the following identities, m and n refer to numbers (that is, multitudes) while letters near the end of the alphabet refer to magnitudes.

V.1. *Multiplication by numbers distributes over addition of magnitudes.*

$$m(x_1 + x_2 + ... + x_n) = m x_1 + m x_2 + ... + m x_n.$$

V.2. *Multiplication by magnitudes distributes over addition of numbers.*

$$(m + n)x = mx + nx.$$

V.3. *An associativity of multiplication.*

$$m(nx) = (mn)x.$$

V.5. *Multiplication by numbers distributes over subtraction of magnitudes.*

$$m(x - y) = mx - my.$$

V.6. *Uses multiplication by magnitudes distributes over subtraction of numbers.*

$$(m - n)x = mx - nx.$$

The rest of the propositions develop the theory of ratios and proportions starting with basic properties and progressively becoming more advanced.

V.4. *If* $w:x = y:z$, *then for any numbers m and n,* $mw:mx = ny:nz$.

V.7. *Substitution of equals in ratios. If* $x = y$, *then* $x:z = y:z$ *and* $z:x = z:y$.

V.7.Cor. *Inverse proportions. If* $w:x = y:z$, *then* $x:w = z:y$.

V.8. *If* $x < y$, *then* $x:z < y:z$ *but* $z:x > z:y$.

V.9. *(A converse to V.7.) If* $x:z = y:z$, *then* $x = y$. *Also, if* $z:x = z:y$, *then* $x = y$.

V.10. (A converse to V.8.) If $x:z < y:z$, then $x < y$. But if $z:x < z:y$, then $x > y$

V.11. Transitivity of equal ratios. If $u:v = w:x$ and $w:x = y:z$, then $u:v = y:z$.

V.12. If $x_1:y_1 = x_2:y_2 = ... = x_n:y_n$, then each of these ratios also equals the ratio $(x_1 + x_2 + ... + x_n) : (y_1 + y_2 + ... + y_n)$.

V.13. Substitution of equal ratios in inequalities of ratios. If $u:v = w:x$ and $w:x > y:z$, then $u:v > y:z$.

V.14. If $w:x = y:z$ and $w > y$, then $x > z$.

V.15. $x:y = nx:ny$.

V.16. Alternate proportions. If $w:x = y:z$, then $w:y = x:z$.

V.17. Proportional taken jointly implies proportional taken separately. If $(w + x):x = (y + z):z$, then $w:x = y:z$.

V.18. Proportional taken separately implies proportional taken jointly. (A converse to V.17.) If $w:x = y:z$, then $(w + x):x = (y + z):z$.

V.19. If $(w + x):(y + z) = w:y$, then $(w + x):(y + z) = x:z$, too.

V.19.Cor. Proportions in conversion. If $(u + v):(x + y) = v:y$, then $(u + v):(x + y) = u:x$.

V.20 is just a preliminary proposition to V.22, and V.21 is just a preliminary proposition to V.23.

V.22. Ratios ex aequali. If $x_1:x_2 = y_1:y_2$, $x_2:x_3 = y_2:y_3$, ... , and $x_{n-1}:x_n = y_{n-1}:y_n$, then $x_1:x_n = y_1:y_n$.

V.23. Perturbed ratios ex aequali. If $u:v = y:z$ and $v:w = x:y$, then $u:w = x:z$.

V.24. If $u:v = w:x$ and $y:v = z:x$, then $(u + y):v = (w + z):x$.

V.25. If $w:x = y:z$ and w is the greatest of the four magnitudes while z is the least, then $w + z > x + y$.

Logical structure of Book V

Book V is on the foundations of ratios and proportions and in no way depends on any of the previous Books. Book VI contains the propositions on plane geometry that depend on ratios, and the proofs there frequently depend on the results in Book V. Also Book X on irrational lines and the books on solid geometry, XI through XIII, discuss ratios and depend on Book V. The books on number theory, VII through IX, do not directly depend on Book V since there is a different definition for ratios of numbers.

Although Euclid is fairly careful to prove the results on ratios that he uses later, there are some that he didn't notice he used, for instance, the law of trichotomy for ratios. These are described

Dependencies within Book V	
2	3, 6
3	4
1	5, 8*, 12
8*	9*
7, 8*	10
8*, 10, 13	14*
7, 12	15

in the Guides to definitions V.Def.4 through V.Def.7.

* Some of the propositions in Book V require treating definition V.Def.4 as an axiom of comparison. One side of the law of trichotomy for ratios depends on it as well as propositions 8, 9, 14, 16, 21, 23, and 25. Some of Euclid's proofs of the remaining propositions rely on these propositions, but alternate proofs that don't depend on an axiom of comparison can be given for them.

Propositions 1, 2, 7, 11, and 13 are proved without invoking other propositions. There are moderately long chains of deductions, but not so long as those in Book I.

The first six propositions excepting 4 have to do with arithmetic of magnitudes and build on the Common Notions. The next group of propositions, 4 and 7 through 15, use the earlier propositions and defintions 4 through 7 to develop the more basic properties of ratios. And the last 10 propositions depend on most of the preceeding ones to develop advanced properties.

11, 14*, 15	16*
1, 2	17
11, 14*, 17	18
11, 16*, 17	19
7.Cor, 8, 10, 13	20, 21*
4, 20	22
11, 15, 16*, 21*	23*
7.Cor, 18, 22	24
7, 11, (14), 19	25*

Next book: Book VI

Previous: Book IV

Select from Book V

Select book

Select topic

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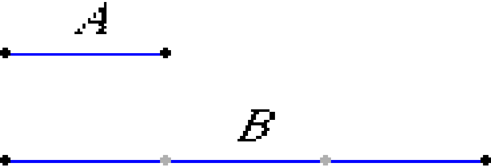

A
B

Euclid's Elements
Book V
Definitions 1 and 2

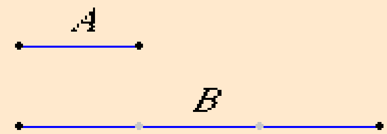
Def. 1. A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.

Def. 2. The greater is a *multiple* of the less when it is measured by the less.

Guide

The two magnitudes mentioned in each definition are of the same kind. Following Euclid, they are illustrated here as lines, but they could both be planar figures, or solids, or angles, or any other kind of magnitude so long as they are of the same kind.

The illustration shows two magnitudes, *A* and *B*, and *A* is one third of *B* since *A* measures *B* three times. Thus, *A* is a *part* of *B*, and *B* is a multiple of *A*.



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Book V introduction

Select from Book V

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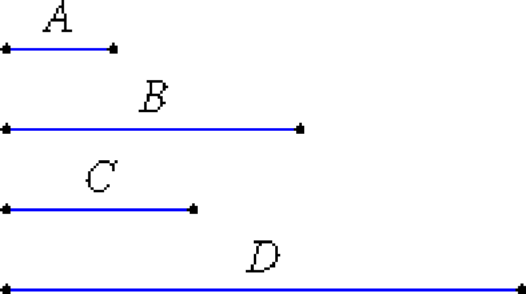

A
B
C
D

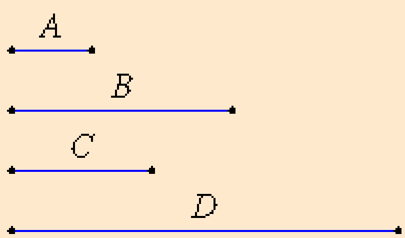
Euclid's Elements
Book V
Definitions 11 through 13

Def. 11. Antecedents are said to *correspond* to antecedents, and consequents to consequents.

Def. 12. *Alternate ratio* means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

Def. 13. *Inverse ratio* means taking the consequent as antecedent in relation to the antecedent as consequent.

Guide



The figure illustrates the proportion $A:B = C:D$. Thus, A and C are corresponding terms since they're the antecedents. Also, B and D are corresponding terms since they're the consequents.

Note that for alternate ratios to exist, all four magnitudes must be of the same kind. The alternate ratios in this proportion are $A:C$ and $B:D$. Euclid proves these are the same ratio in proposition V.16. Thereafter, given one proportion $A:B = C:D$, he concludes *alternately* the alternate proportion $A:C = B:D$.

The ratio inverse to $A:B$ is $B:A$. It is evident from the definition V.Def.5 that $A:B = C:D$ and $B:A = D:C$ reduce to the same conditions on A, B, C, and D. Therefore, if two ratios are the same, then their two inverse ratios are also the same. For some reason, this statement is misplaced as the corollary after proposition V.7.

Several of the propositions are stated using the antecedent terms but they apply as well for the consequent terms by inversion. For example, proposition V.24 says that if $u:v = w:x$ and $y:v = z:x$, then $(u + y):v = (w + z):x$. But the statement using consequents is valid, too: if $v:u = x:w$ and $v:y = x:z$, then $v:(u + y) = x(w + z)$.

The symmetry of the antecedent and consequent terms of a ratio $a:b$ is not, however, one of perfect parallelism. They're opposite in regard to order. Proposition V.8 shows that the ratio is $a:b$ increasing in a since if a increases then the ratio increases, but the ratio is decreasing in b since the if b increases then the ratio decreases. But that's still a kind of symmetry.

Next proposition: V.Def.14-16

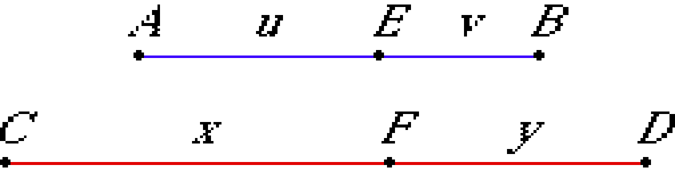
Previous: V.Def.8-10

Book V introduction

Select from Book V

Select book

Select topic


A
u
E
v
B
C
x
F
y
D

Euclid's Elements

Book V

Definitions 14 through 16

Def. 14. A ratio *taken jointly* means taking the antecedent together with the consequent as one in relation to the consequent by itself.

Def. 15. A ratio *taken separately* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

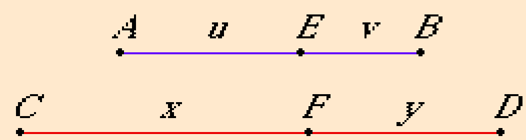
Def. 16. *Conversion of a ratio* means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

Guide

Taking jointly the ratio $u:v$ yields the ratio $(u + v):v$. Taking separately the ratio $(u + v):v$ returns the ratio $u:v$. Taking the ratio $(u + v):v$ in conversion yields the ratio $(u + v):u$.

These conversions are only important when the ratios are in proportions.

The following three proportions are shown to be equivalent in propositions V.17 and V.18.



1. $(u + v):v = (x + y):y$.
2. $(u + v):u = (x + y):x$.
3. $u:v = x:y$.

Proposition V.17 and V.18 show proportions 1 and 3 are equivalent. That means proportion 2 and the inverse of 3, $v:u = y:x$, are also equivalent. And of course, 3 and its inverse are equivalent, so all three proportions are equivalent.

Furthermore, when all the magnitudes are of the same kind, then the alternate proportions are also equivalent by V.16 making six equivalent statements.

4. $(u + v):(x + y) = v:y$.
5. $(u + v):(x + y) = u:x$
6. $u:x = v:y$

Proposition V.19 goes on to say that 4 implies 5, and its corollary says 1 implies 2.

Heath translates "taken jointly," "taken separately," and "in conversion" by the Latin words *componendo, separando, and convertendo*, respectively.

Next proposition: V.Def.17-18

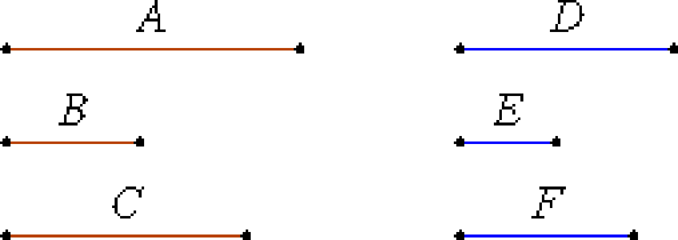
Select from Book V

Previous: V.Def.11-13

Select book

Book V introduction

Select topic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Euclid's Elements
Book V
Definitions 17 and 18

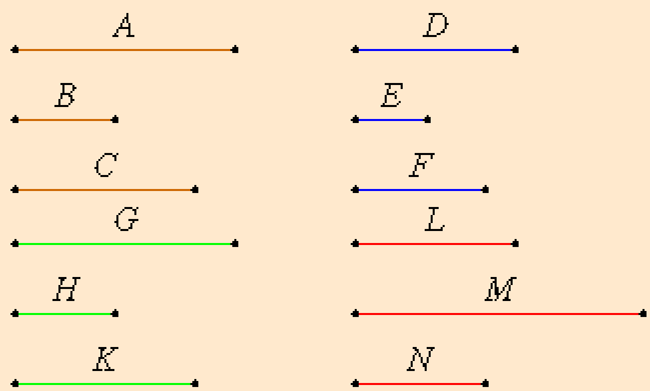
Def. 17. A ratio *ex aequali* arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, the first is to the last among the first magnitudes as the first is to the last among the second magnitudes. Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

Def. 18. A *perturbed proportion* arises when, there being three magnitudes and another set equal to them in multitude, antecedent is to consequent among the first magnitudes as antecedent is to consequent among the second magnitudes, while, the consequent is to a third among the first magnitudes as a third is to the antecedent among the second magnitudes.

Guide

If $A:B = D:E$, and $B:C = E:F$, then as shown in proposition V.22, *ex aequali*, $A:C = D:F$.

However, if $G:H = M:N$, and $H:K = L:M$, then a perturbed proportion holds as shown in proposition V.23, namely, $G:K = L:N$.



Next proposition: V.1

Previous: V.Def.14-16

Book V introduction

Select from Book V

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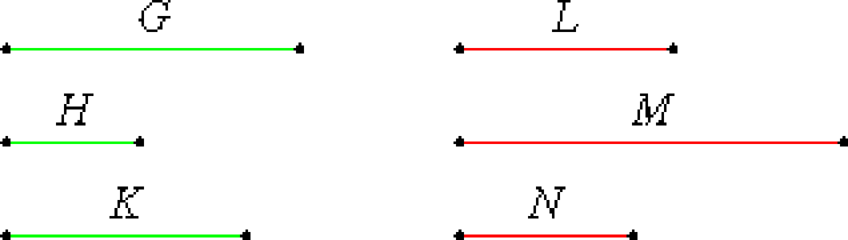
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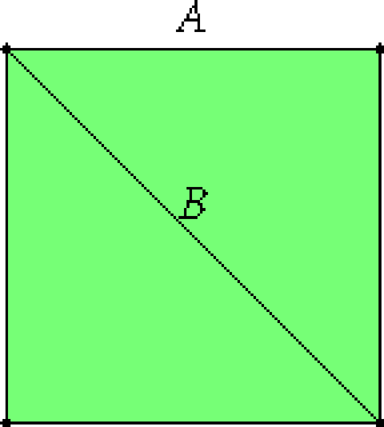
A
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Euclid's Elements
Book V
Definition 3

A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.

Guide

A convenient notation for a ratio of two magnitudes *A* and *B* of the same kind is *A*:*B*.

No mixed ratios

All of Euclid's ratios are pure ratios of two magnitudes of the *same* kind, in other words, there are no mixed ratios in the *Elements.* A familiar example of a mixed ratio is velocity, the ratio of a distance to a time, measured in units such as kilometers/hour.

That isn't to say that ratios of different kinds of magnitudes aren't equated. In fact, that's one of the more important aspects of ratios. For example, the fundamental proposition of Book VI, proposition VI.1, says that given two triangles of the same height, the ratio of the triangles *A*:*B* is the same as the ratio of their heights $A_h:B_h$. That says that the ratio of two plane figures equals the ratio of two lines.

Now, a common operation on proportions (equalities of ratios) is that of alternation (see V.Def.12 and V.16) which in its general form says that if *A*:*B* = *C*:*D*, then *A*:*C* = *B*:*D*. In the *Elements* alternation only applies when all four quantities are of the same kind. But if alternation is applied to the proportion of VI.1, then we get $A:A_h = B:B_h$, the equality of two mixed ratios, ratios of plane figures to lines. This step, and the acceptance of mixed ratios, which seems to us like a small thing, was not taken until centuries after Euclid.

The nature of ratios

A ratio is a pair of magnitudes of the same kind considered as a pair, but soon identified with other ratios. Definition 3 promises that ratios have sizes, that is, given two ratios *A*:*B* and *C*:*D*, either the first ratio is greater, equal, or less than the second ratio. That promise begins to be fulfilled in Definitions V.Def.5 and V.Def.7 where equality and order of ratios defined. Note that equality and order are defined for ratios, but they were assumed for numbers and magnitudes.

Since equality and order are defined, their expected properties are proved in propositions, or at least some of the properties. For example, proposition V.11 states that two ratios that are the same as a third are the same as each other, a statement analogous to C.N.1 for magnitudes.

Equivalence relations

(Equivalence relations were mentioned before in the guide for the Common Notions. It was mentioned there that equality of magnitudes of the same kinds is an equivalence relation.)

The process used for defining ratios of magnitudes was something new for Eudoxus and Euclid, but that process is now commonplace in mathematics to construct new kinds of things. The process starts with entities *x*, *y*, *z*, etc., that are well understood, such as pairs of magnitudes of the same kind. Then a relation *E* on these entities is found which is intended to be equality for them. For ratios, that is given in V.Def.5. Right now, let *xEy* denote that *x* is related to *y* by the relation *E.* Next, it is verified that the relation *E* is an *equivalence relation,* that is, a reflexive, symmetric, and transitive relation.

A relation *E* is *reflexive* if for any *x* it is the case that *xEx,* that is, anything is related to itself by *E.* A relation is *E* is *symmetric* if whenever *xEy,* then *yEx.* And it is *transitive* if whenever *xEy* and *yEz,* then *xEz.*

Once *E* is known to be an equivalence relation, new entities are conceived which are named by the old entities *x, y, z,* etc., but the new entities are taken to be equal, $x = y$, when their names are equivalent under the relation *E*, that is, *xEy.* Proportion as an equivalence relation is discussed in the Guide to definition V.Def.5.

Operations on ratios and proportions, compounded ratios

There are several operations on ratios and proportions defined soon. For instance, V.Def.9 defines duplicate ratios, under certain assumptions, which may be thought of as the squares of ratios. See also definitions V.Def.12 through V.18. But ratios are neither numbers nor magnitudes, and the usual operations of addition, subtraction, multiplication, and division that apply to numbers don't apply to ratios.

Numbers can be added and subtracted, and so can magnitudes of the same kind, but ratios cannot. Take for example a ratio *A*:*B* of plane figures and a ratio *C*:*D* of angles. What could be meant by their sum $(A{:}B) + (C{:}D)$? One obvious approach is to treat ratios as quotients. That suggests $A/B + C/D = (AD + BC)/BD$, but a product of a plane figure and an angle, such as *AD,* has no meaning, so the obvious approach has obvious difficulties.

Multiplication and division are not automatic for ratios. Ratios *A*:*B* and *B*:*C* are *compounded* to form *A*:*C,* which may be thought of as the product of the two ratios, and the duplicate ratio mentioned above is a special case of a compound ratio. But the compound of two ratios *A*:*B* and *C*:*D* depends on the middle terms *B* and *C* being the same. The proof of proposition V.18 assumes that fourth proportionals exist, a property unjustified by any postulate, but if fourth proportionals do exist, then the ratio *C*:*D* is equal to some ratio *B*:*E,* and then the compound of *A*:*B* and *C*:*D* is the compound of *A*:*B* and *B*:*E,* and that compound is *A*:*E.* Thus, multiplication is an operation when fourth proportionals exist. Division is also an operation when fourth proportionals exist since *D*:*C* may be thought of as the reciprocal of *C*:*D.*

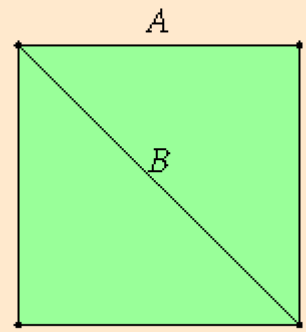
Ratios of various kinds

Several kinds of ratios appear in the *Elements.* There are ratios of numbers, ratios of lines constructable in plane geometry, ratios of rectilinear angles, ratios of plane figures constructable in plane geometry, and ratios of solids.

Numeric ratios, that is, ratios of numbers, are treated in the books on number theory, Books VII through VIII. In modern terminology these numeric ratios are called "positive rational numbers." Numeric ratios and proportions have a separate, simpler definition in VII.Def.20. That definition is compatible with the definitions here in Book V, but that compatibility is not demonstrated in the *Elements.*

The problem with numeric ratios is that there are not enough of them. That is ratios of magnitudes are not always equal to ratios of numbers.

The illustration to the right shows a square with side *A* and diameter *B.* The ratio *B* to *A* does exist according to the next definition V.Def.4 since some multiple of each is greater than the other. In modern terms this ratio would be identified with the square root of 2 and is known to be an irrational number, that is, it is not equal to a numeric ratio. It is, nonetheless, a ratio in Euclid's terminology. The ratio *B*:*A* is a ratio of lines, but it is not a ratio of numbers.



Since this and other ratios of lines are not ratios of numbers, a more general definition of ratio is required. That more general definition is the one given here and continuing through V.Def.6.

In modern terminology, the numeric ratios are positive rational numbers. The field of all rational numbers including 0 and the negative rational numbers is commonly denoted **Q**. The ratios of lines constructable in plane geometry form the field extended from **Q** by closure under square roots. A convenient notation for that field is $\mathbf{Q}^{\surd}$. It is a much larger field, but does not include all real numbers. For instance, the cube root of 2, needed for doubling a cube, the sine of 20°, needed for trisecting angles, and pi, needed for squaring the circle, all are missing from $\mathbf{Q}^{\surd}$.

The conic sections are part of solid geometry but they are not treated in the *Elements.* Cones are discussed in Book XII, but their sections (intersections with planes) which include ellipses, parabolas, and hyperbolas are not even defined in the *Elements*. Euclid's work on the *Conics* was superceded by Apollonius' and no longer exists. Intersections of conics lead to lines of new lengths that can be used to solve problems such as doubling a cube and trisecting an angle, but they don't help in squaring the circle. Thus, there are more ratios of lines constructable in solid geometry than ratios of lines constructable in plane geometry.

The ratios of rectilinear figures form the same field $\mathbf{Q}^{\surd}$ as the ratios of lines. This follows from the theory of application of areas developed in Book I, see proposition I.44. But there are other plane figures besides rectilinear ones: circles. The ratio of a circle to the square on its radius is pi. Thus, pi is a ratio of plane figures even though it is not a ratio of lines.

Ratios of more than two terms

Throughout Book V the only ratios that are considered are those with two terms in accordance with V.Def.3, but in Book VII ratios of three or more terms are used in proposition VI.33. In that proposition, a ratio A:B:C of three numbers is considered, and a certain proportion A:B:C = E:F:G is shown. These multiterm ratios and proportions are probably left over from an earlier time. In any case, the multiterm proportion may be interpreted as an abbreviation for two proportions A:B = E:F and B:C = F:G. Then it follows *ex aequali* that A:C = E:G.

Next defintion: V.Def.4

Previous: V.Def.1-2

Book V introduction

Select from Book V

Select book

Select topic

Euclid's Elements
Book V
Definition 4

Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

Guide

This definition limits the existence of ratios to comparable magnitudes of the same kind where comparable means each, when multiplied, can exceed the other. The ratio doesn't exist when one magnitude is so small or the other so large that no multiple of the one can exceed the other. This definition excludes the ratio of a finite straight line to an infinite straight line and the ratio of an infinitesimal straight line, should any exist, to a finite straight line.

The result on horn angles in proposition III.16 excludes ratios between horn angles and rectilinear angles. That proposition states that a horn angle is less than any rectilinear angle, hence no multiple of a horn angle is greater than a rectilinear angle. The situation of horn angles is much worse than that, however, since horn angles of different sizes aren't even comparable.

Definition 4 as an axiom of comparability

This definition is used repeatedly as a axiom for magnitudes rather than a definition. It is frequently invoked in this book, starting with proposition V.8 but also required for more fundamental properties, and elsewhere, such as the important proposition X.1. In the proofs of these propositions one magnitude is less than another, and it is asserted that some multiple of the smaller is greater than the larger. Euclid implicitly assumes that the magnitudes he discusses, except horn angles, are all comparable. Straight lines, rectilinear angles, plane figures, and solids are all comparable to any other of the same type.

This principle of comparability should be explicit in order to justify the principle of comparability for magnitudes of these kinds. One solution is to make it a postulate that straight lines are comparable. From that postulate comparability of each of the other kinds of magnitudes could be proved.

Several of the propositions, stated and unstated, depend on this principle. Without it, some are simply false for kinds of magnitude that have infinitesimals. If x and y are two magnitudes of the same kind, then x is *infinitesimal* with respect to y, or y is *infinite* with respect to x, if no multiple of x is greater than y. For example, horn angles are infinitesimal with respect to rectilinear angles. Although this definition excludes ratios between horn angles and rectilinear angles, it allows a ratio between a rectilinear angle B and the sum of a horn angle A and the rectilinear angle, and, according to the next three definitions, the two ratios $B:(A + B)$ and $B:B$ so not satisfy the law of trichotomy, that is, they aren't the same ratio but neither is greater than the other either. Examples involving infinitesimals can be useful to show which propositions require treating this definition as an axiom.

Next devinition: V.Def.5-6

Previous: Definition V.Def.3

Book V introduction

Select from Book V

Select book

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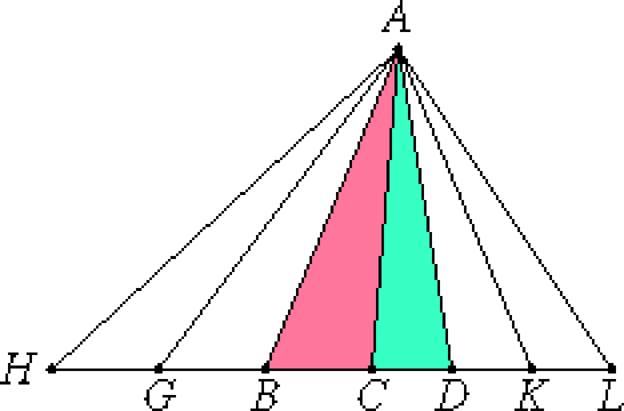
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Euclid's Elements
Book V
Definitions 5 and 6

Def. 5. Magnitudes are said to be *in the same ratio,* the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Def. 6. Let magnitudes which have the same ratio be called *proportional.*

Guide

Definition 5 defines two ratios $w:x$ and $y:z$ to be the same, written $w:x = y:z$, when for all numbers n and m it is the case that if nw is greater, equal, or less than mx, then ny is greater, equal, or less than mz, respectively, that is,

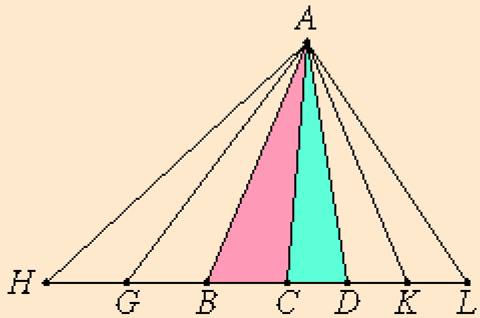
if $nw > mx$, then $ny > mz$,
if $nw = mx$, then $ny = mz$, and
if $nw < mx$, then $ny < mz$.

It is very convenient to use the shorter notation

if nw >=< mx, then ny >=< mz.

Note that whenever the symbol >=< is used there are three parallel statements being made.

The four magnitudes do not all have to be of the same kind, but the first pair w and x need to be of the one kind, and the second pair y and z of one kind, either the same kind as that of w and x or a different kind. Perhaps the best illustration of these definitions comes from proposition VI.1 in which Euclid first uses them to construct a proportion.



The goal in this proposition is to show that the lines are proportional to the triangles. More precisely, the line BC is to the line CD as the triangle ABC is to the triangle ACD, that is, the ratio $BC:CD$ of lines is the same as the ratio $ABC:ACD$ of triangles. Even though the ratios derive from different kinds of magnitudes, they are to be compared and shown equal.

According to Definition 5, in order to show the ratios are the same, Euclid takes any one multiple of BC and ABC (which he illustrates by taking three times each), and any one multiple of CD and ACD (which he also illustrates by taking three times each). Then he proceeds to show that the former equimultiples, namely HC and CL, alike exceed, are alike equal to, or alike fall short of, the latter equimultiples, namely, AHC and ACL.

Symbolically, in order to prove *BC*:*CD* = *ABC*:*ACD*, Euclid proves for any numbers *n* and *m* that the line *n BC* is greater, equal, or less than the line *m CD* when the triangle *n ABC* is greater, equal, or less than the triangle *m ACD.* We will abbreviate this condition symbolically as

if *n BC* >=< *m CD,* then *n ABC* >=< *m ACD.*

Note that in order to check this condition, it is only necessary to compare lines to lines and planar figures to planar figures. To see how Euclid does this, refer to VI.1.

Numerical ratios and commensurability

As it sometimes happens, a ratio of two magnitudes *A*:*B* is the same as a ratio of numbers *m*:*n.* Take for instance the case when *A* is a line that is twice a line *U* while *B* is a line that is three times the line *U.* Then, we could show that the ratio of magnitudes *A*:*B* is the same as the numrical ratio 2:3. Such ratios are studied in detail in Book X. That book begins by defining in X.Def.1. what it means for two quantities to be "commensurable." For instance, the two lines *A* and *B* are commensurable since there is a unit *U* that measures both. Later in Book X (propositions X.5 and X.6) it is explicitly shown that two magnitudes are commensurable if and only if their ratio is a numeric ratio.

Using modern concepts and notations, we can more easily see what the general definition of equality of two magnitudes means. If we treat ratios as real numbers, the a proportion such as the one described above, *BC*:*CD* = *ABC*:*ACD*, means that the ratio *BC*:*CD* compares to all numerical ratios (that is, rational numbers) *m*/*n* the same way that *ABC*:*ACD* does. Another way of saying this is that equality of two real numbers is determined by their relation to all rational numbers. This is often expressed by saying that the set of rational numbers is dense in the set of real numbers.

Of course, Euclid did not have what modern mathematicians call real numbers. Indeed, there is an ontological difference between real numbers and Euclid's ratios. Some real numbers are not ratios of the magnitudes of any kind mentioned in the *Elements.*

Proportions as equivalence relations

Equivalence relations were defined in the Guide for V.Def.3. Three things need to be checked to see if proportion is an equivalence relation: reflexivity, symmetry, and transitivity.

First, reflexivity. Is it the case for any pair of magnitudes of the same type *A* and *B* that *A* and *B* are in the same ratio as *A* and *B*? That means for any numbers *m* and *n,*

if *nA* >=< *mB*, then *nA* >=< *mB.*

That is trivial.

Second, symmetry. Is it the case that if *A* and *B* are in the same ratio as *C* and *D*, then *C* and *D* are in the same ratio as *A* and *B*? The first says

if *nA* >=< *mB*, then *nC* >=< *mD,*

while the second says

if *nC* >=< *mD,* then *nA* >=< *mB.*

This can be shown using the law of trichotomy for magnitudes. (Suppose *nC* > *mD.* If *nA* is not greater than *mB,* then it is less or equal, but then *nC* is less or equal to *mD,* contradicting *nC* > *mD.* etc.) Euclid missed symmetry, but he uses it very frequently.

Third, transitivity. Euclid states this explicitly in proposition V.11. The proof relies only on the definition.

Thus, proportion is an equivalence relation.

Are proportions equalities of ratios?

When *A* and *B* are in the same ratio as *C* and *D*, then the four magnitudes are said to be proportional, or in proportion, according to definition 6. Is that the same as saying the ratios *A*:*B* and *C*:*D* are equal?

A more fundamental question is "do ratios exist?" Are they some kind of mathematical object like numbers and magnitudes? The *Elements* do not require it. Instead, proportion is a relation held between one pair of magnitudes and another pair of magnitudes. Yet it is very easy to read Book V as though ratios are mathematical objects of some abstract variety. And it's easy to read "*A* and *B* have the same ratio as *C* and *D*" as saying that the ratio *A*:*B* is the same ratio as *C*:*D*.

Not every relation allows that reading, but equivalence relations do, and proportion is an equivalence relation.

The philosophical questions "do ratios exist?" and "is a proportion equality of ratios?" can be converted to the question "why do equivalence relations create entities?" or a little more conservatively, "why do equivalence relations allow us to think and act as if the entities exist?"

It is hard to imagine that Euclid did not think of ratios as things and proportions as equalities, especially since the next definition defines when one ratio is larger than another. Perhaps he did but continued to write noncommittally.

Proportions are written as equalities in the Guide.

Next definition: V.Def.7

Previous: V.Def.4

Book V introduction

Select from Book V

Select book

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Euclid's Elements
Book V
Definition 7

When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth.

Guide

Definition 5 explained when two ratios were equal, namely, $w:x = y:z$ when

for all numbers n and m, if nw >=< mx, then ny >=< mz.

Definition 7 now says $w:x > y:z$ when

there are numbers n and m such that $nw > mx$ but ny is not greater than mz.

Of course, $y:z$ is called the lesser ratio.

When defining greater and lesser, there are a number of properties that should be verified. These are various transitivities and the law of trichotomy, some of the same properities of greater and lesser that magnitudes have. (See the Guide for the Common Notions.)

If $u:v < w:x$, and $w:x = y:z$, then $u:v < y:z$.

If $u:v = w:x$, and $w:x < y:z$, then $u:v < y:z$.

If $u:v < w:x$, and $w:x < y:z$, then $u:v < y:z$.

Euclid only has the first property, which is is proposition V.13. Its proof depends only on the definitions. The second is so much like it that it isn't mentioned but it is used in the same way as the third. The third one is quite easy to prove.

The law of trichotomy for ratios

Stated for ratios, the law of trichotomy says that for any two ratios $w:x$ and $y:z$, exactly one of the following three cases holds: $w:x < y:z$, or $w:x = y:z$, or$w:x > y:z$.

Euclid missed the law of trichotomy for magnitudes in his list of Common Notions, and he missed it for ratios, too. The side of the law which says at most one of the three cases can occur is first used in proposition V.9, while the side which says at least one occurs is first used in V.10.

The side of the law which says at most one of the three cases can occur is fairly easy to prove.

From the definitions themselves it is clear that $w:x > y:z$ contradicts $w:x = y:z$. The first says there are n and m such that $nw > mx$ but ny is not greater than mz, while the second concludes from $nw > mx$ that $ny > mz$ yielding a contradiction based on the law of trichotomy for magnitudes. Similarly $w:x < y:z$ contradicts $w:x = y:z$.

Once transitivity has been shown, it can be shown that $w:x > y:z$ contradicts $w:x < y:z$, for then $w:x > w:x$, and that contradicts $w:x = w:x$. (There are also proofs that don't depend on transitivity.)

The other side of the law of trichotomy, the one that says at least one of the three cases holds, is a bit harder to prove, and it depends on treating V.Def.4 as an axiom of comparability. In fact, it is false without it. First, a proof using V.Def.4 as an axiom, then a counterexample to show that's necessary.

A proof of trichotomy. Let $w:x$ and $y:z$ be any two ratios. We need to show that one of the three cases holds. We'll assume the ratios aren't the same and show one of them is greater than the other. When they're not the same, then there are numbers m and n such that

$$nw >=< mx \text{ but not } ny >=< mz.$$

We have three cases to consider, and two of them are easy. In one case, $nw > mx$ but not $ny > mz$, so for that case $w:x > y:z$. In another case, $nw < mx$ but not $ny < mz$, so for that case $w:x < y:z$.

Consider now the last case: $nw = mx$ but ny does not equal mz. Then one of ny and mz is greater, say $ny > mz$. Now using V.Def.4 as an axiom, there is some some number k such that $k(ny - mz) > z$. Since $k(ny - mz) = kny - kmz$, therefore $kny > kmz + z$, that is, $kny > (km + 1)z$. But $knw = kmx$, and $kmx < (km + 1)x$. Therefore, $(kn)w < (km + 1)x$. But $(kn)y > (km + 1)z$. Therefore $w:x < y:z$. Q.E.D.

A counterexample to trichotomy. This counterexample has infinitesimals, so it doesn't satisfy the axiom of comparability, that is, V.Def.4 treated as an axiom. Let y be an infinitesimal with respect to x, that is, for any number n, $ny < x$. We'll show that the two ratios $x:x$ and $x:(x + y)$ do not satisfy the law of trichotomy. First, though, note that the second ratio does satisfy V.Def.4 as a definition since twice each of x and $x + y$ is greater than the other.

Now, the ratios $x:x$ and $x:(x + y)$ are not equal since $2x$ = $2x$ but $2x < 2(x + y)$. Next, $x:(x + y)$ is not greater than $x:x$, since $nx > m(x + y)$ implies $nx > mx$. Finally, $x:x$ is not greater than $x:(x + y)$, for if $nx > mx$, then $n > m$, so nx is not less than $mx > x$, and since y is an infinitesimal with respect to x, $x > my$, therefore $nx - mx > my$, that is, $nx > m(x + y)$.

In summary, since we have a proof of trichotomy that uses V.Def.4 as an axiom of comparability, and a counterexample of trichotomy that violates the axiom of comparability, we can conclude that any proof trichotomy requires the axiom of comparability.

Next definition: V.Def.8-10

Previous: V.Def.5-6

Book V introduction

Select from Book V

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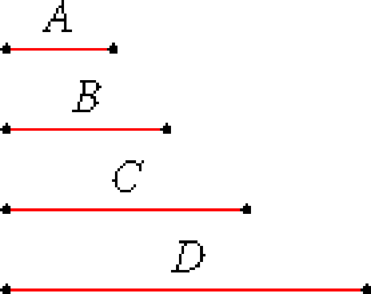
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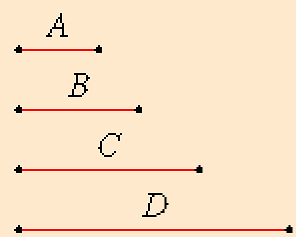
Euclid's Elements
Book V
Definitions 8 through 10

Def. 8. A proportion in three terms is the least possible.

Def. 9. When three magnitudes are proportional, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

Def. 10. When four magnitudes are continuously proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion.

Guide



In the illustration *A*, *B*, and *C* form three terms for the proportion *A*:*B* = *B*:*C*, therefore the ratio *A*:*C* is the duplicate ratio of *A*:*B*. For a numerical example, 9:4 is the duplicate ratio of 3:2.

The illustration also shows a continued proportion of four magnitudes, *A*, *B*, *C*, and *D*, since *A*:*B* = *B*:*C* = *C*:*D*. Also, *A*:*D* is the triplicate ratio of *A*:*B*. For a numerical example, 27:8 is the triplicate ratio of 3:2.

Next definition: V.Def.11-13

Previous: V.Def.7

Book V introduction

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defV17.gif	15-May-1997	22:07	1k
defV11.gif	15-May-1997	22:07	1k
bkV.gif	15-May-1997	22:07	1k

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Index of /~djoyce/java/elements/bookV

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Index of /~djoyce/java/elements/bookV

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defV7.html	16-Oct-2002	09:13	11k
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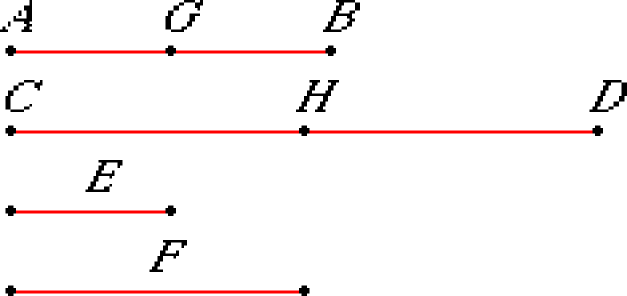

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Euclid's Elements
Book V
Proposition 1

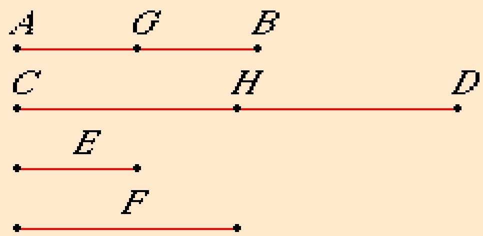
If any number of magnitudes are each the same multiple of the same number of other magnitudes, then the sum is that multiple of the sum.

Let any number of magnitudes *AB* and *CD* each be the same multiple of magnitudes *E* and *F* respectively. V.Def.2

I say that the sum of *AB* and *CD* is the same multiple of the sum of *E* and *F* that *AB* is of *E*.

Since *AB* is the same multiple of *E* that *CD* is of *F*, therefore there are as many magnitudes in *AB* equal to *E* as there are in *CD* equal to *F*.

Divide *AB* into magnitudes *AG* and *GB* equal to *E*, and divide *CD* into *CH* and *HD* equal to *F*. Then the number of the magnitudes *AG* and *GB* equals the number of the magnitudes *CH* and *HD*.



Now, since *AG* equals *E*, and *CH* equals *F*, therefore the sum of *AG* and *CH* equals the sum of *E* and *F*.

For the same reason *GB* equals *E*, and the sum of *GB* and *HD* equals the sum of *E* and *F*. Therefore, there are as many magnitudes in *AB* equal to *E* as there are in the sum of *AB* and *CD* equal to the sum of *E* and *F*. Therefore, the sum of *AB* and *CD* is the same multiple of the sum of *E* and *F* that *AB* is of *E*.

Therefore, *if any number of magnitudes are each the same multiple of the same number of other magnitudes, then the sum is that multiple of the sum.*

Q.E.D.

Guide

In modern terminology, this proposition states that multiplication by numbers distributes over addition of magnitudes, that is,

$$m\,(x_1 + x_2 + \ldots + x_n) = m\,x_1 + m\,x_2 + \ldots + m\,x_n.$$

Here, the m is a number, and all the x_i's are magnitudes of the same kind.

Euclid always displays his magnitudes as lines, but they could be magnitudes of other kinds, like plane regions, for instance. In this proposition, all the magnitudes are of the same kind.

Euclid's proof is only for the simplest nontrivial case. He takes the number n of magnitudes to be 2, and the multiple m also to be 2, so he proves that if $x_1 = m\,y_1$ and $x_2 = m\,y_2$, then $x_1 + x_2 = m\,(y_1 + y_2)$. Throughout Book V, Euclid proves the general numerical case by a particular case. The numbers he chooses are usually 2 and 3.

Use of this proposition

Proposition V.1 is used in the proofs of four other propositions, namely, V.5, V.8, V.12, and V.17.

Next proposition: V.2

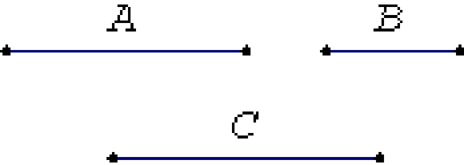
Previous: V.Def.17-18

Book V introduction

Select from Book V

Select book

Select topic


A
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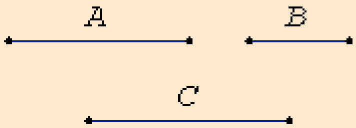
Euclid's Elements
Book V
Proposition 10

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

Let *A* have to *C* a greater ratio than *B* has to *C*.

I say that *A* is greater than *B*.

If not, then *A* either equals *B* or is less than it.



Now *A* does not equal *B*, for in that case each of the magnitudes *A* and *B* would have the same ratio to *C*, but they do not, therefore *A* does not equal *B*. V.7

Nor is *A* less than *B*, for in that case *A* would have to *C* a less ratio than *B* has to *C*, but it does not, therefore *A* is not less than *B*. V.8

But it was proved not to be equal either, therefore *A* is greater than *B*.

Next, let *C* have to *B* a greater ratio than *C* has to *A*.

I say that *B* is less than *A*.

If not, it is either equal or greater.

Now *B* does not equal *A*, for in that case *C* would have the same ratio to each of the magnitudes *A* and *B*, but it does not, therefore *A* does not equal *B*. V.7

Nor is *B* greater than *A*, for in that case *C* would have to *B* a less ratio than it has to *A*, but it does not, therefore *B* is not greater than *A*. V.8

But it was proved that it is not equal either, therefore *B* is less than *A*.

Therefore, *of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.*

Q.E.D.

Guide

This converse to proposition V.8 has two statements.

If $a:c > b:c$, then $a > b$.
If $c:b > c:a$, then $b < a$.

Part of the law of trichotomy for ratios is used in this proof, the part which says at most one of the three cases $a:c < b:c$, $a:c = b:c$, or $a:c > b:c$, can occur.

Euclid's proof relies on using V.Def.4 as an axiom of comparability since it uses proposition V.8 and the law of trichotomy for ratios. But the proposition can also be proved without the axiom.

> Suppose $a:c > b:c$. Then there are numbers m and n such that $na > mc$ but nb is not greater than mc. Therefore $na > nb$. Therefore $a > b$. Thus $a:c > b:c$ implies $a > b$.

The other implication of the proposition can be proved similarly.

This proposition is used a few times in book V starting with V.14.

Next proposition: V.11

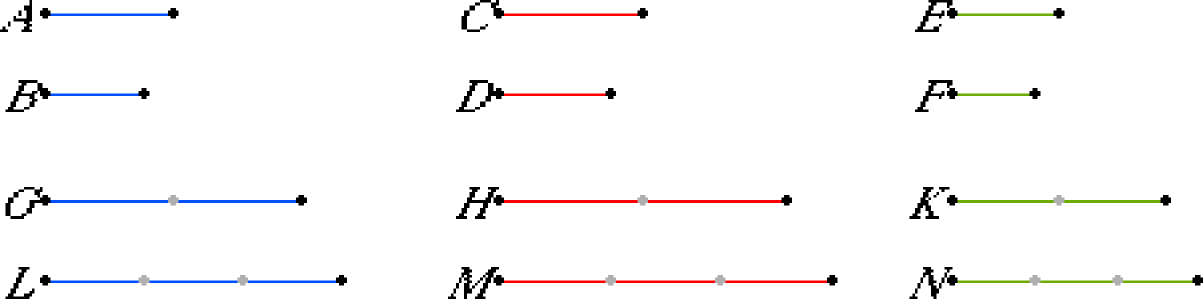
Previous: V.9

Book V introduction

Select from Book V

Select book

Select topic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Euclid's Elements
Book V

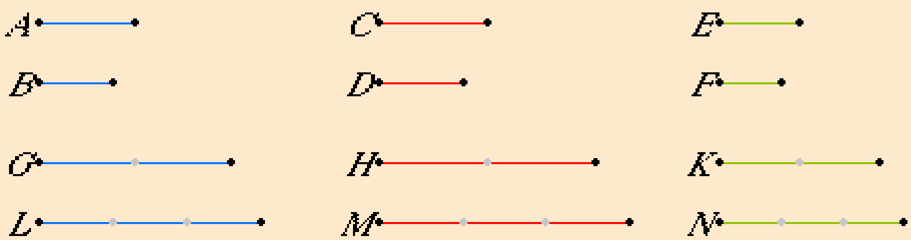
Proposition 11

Ratios which are the same with the same ratio are also the same with one another.

Let *A* be to *B* as *C* is to *D*, and let *C* be to *D* as *E* is to *F*.

I say that *A* is to *B* as *E* is to *F*.

Take equimultiples *G*, *H*, and *K* of *A*, *C*, and *E*, and take other, arbitrary, equimultiples *L*, *M*, and *N* of *B*, *D*, and *F*.



Then since *A* is to *B* as *C* is to *D*, and of *A* and *C* equimultiples *G* and *H* have been taken, and of *B* and *D* other, arbitrary, equimultiples *L* and *M*, therefore, if *G* is in excess of *L*, *H* is also in excess of *M*; if equal, equal; and if less, less. V.Def.5

Again, since *C* is to *D* as *E* is to *F*, and of *C* and *E* equimultiples *H* and *K* have been taken, and of *D* and *F* other, arbitrary, equimultiples *M* and *N*, therefore, if *H* is in excess of *M*, *K* is also in excess of *N*; if equal, equal; and if less, less.

But we saw that, if *H* was in excess of *M*, *G* was also in excess of *L*; if equal, equal; and if less, less, so that, in addition, if *G* is in excess of *L*, *K* is also in excess of *N*; if equal, equal; and if less, less. V.Def.5

And *G* and *K* are equimultiples of *A* and *E*, while *L* and *N* are other, arbitrary, equimultiples of *B* and *F*, therefore *A* is to *B* as *E* is to *F*. V.Def.5

Therefore, *ratios which are the same with the same ratio are also the same with one another*.

Q.E.D.

Guide

This proposition expresses the transitivity of the relation of being the same when applied to ratios. After this proposition (and the easily proved properties of reflexivity and symmetry, see the Guide to definition V.Def.6), the expression "two ratios are the same," or the equivalent expression "two ratios are equal," is justified. The proof follows directly from the definition. What is remarkable is that Eudoxus, or Euclid, recognized that this proposition needed to be proved.

The magnitudes may be of three different kinds with *A* and *B* of one kind, *C* and *D* of a second kind, and *E* and *F* of a third kind.

This proposition is used very frequently whenever ratios are used.

Next proposition: V.12

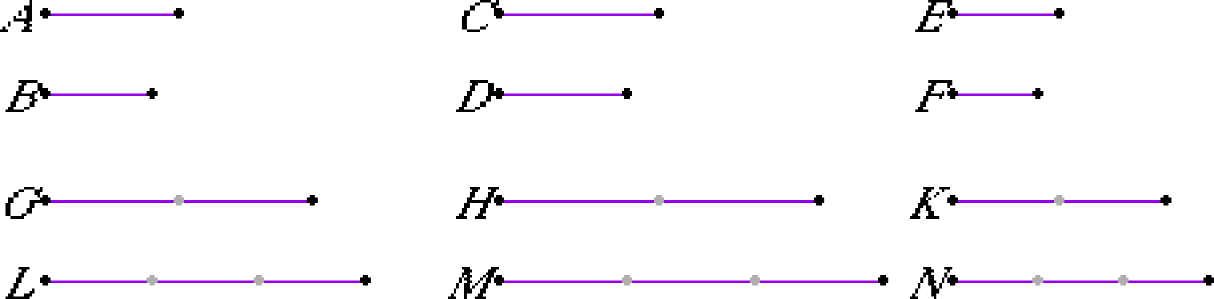
Previous: V.10

Book V introduction

Select from Book V

Select book

Select topic



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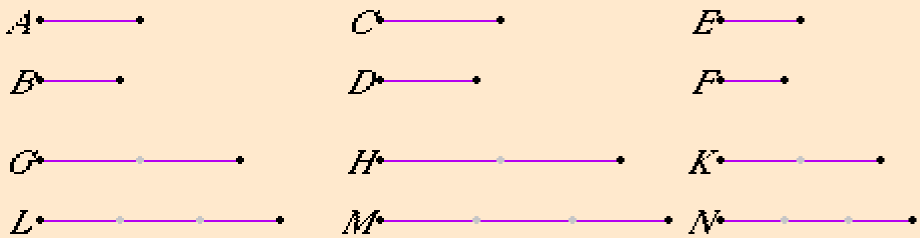
Euclid's Elements
Book V
Proposition 12

If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.

Let any number of magnitudes *A, B, C, D, E,* and *F* be proportional, so that *A* is to *B* as *C* is to *D,* and as *E* is to *F.*

I say that *A* is to *B* as the sum of *A, C,* and *E* is to the sum of *B, D,* and *F.*



Take equimultiples *G, H,* and *K* of *A, C,* and *E,* and take other, arbitrary, equimultiples *L, M,* and *N* of *B, D,* and *F.*

Then since *A* is to *B* as *C* is to *D,* and as *E* is to *F,* and of *A, C,* and *E* equimultiples *G, H,* and *K* have been taken, and of *B, D,* and *F* other, arbitrary, equimultiples *L, M,* and *N,* therefore, if *G* is in excess of *L,* then *H* is also in excess of *M,* and *K* of *N*; if equal, equal; and if less, less. So that, in addition, if *G* is in excess of *L,* then the sum of *G, H,* and *K* is in excess of the sum of *L, M,* and *N*; if equal, equal; and if less, less. V.Def.5

Now *G* and the sum of *G, H,* and *K* are equimultiples of *A* and the sum of *A, C,* and *E,* since, if any number of magnitudes are each the same multiple the same number of other magnitudes, then the sum is that multiple of the sum. V.1

For the same reason *L* and the sum of *L, M,* and *N* are also equimultiples of *B* and the sum of *B, D,* and *F,* therefore *A* is to *B* as the sum of *A, C,* and *E* is to the sum of *B, D,* and *F.* V.Def.5

Therefore, *if any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.*

Q.E.D.

Guide

The general form for this proposition is that if $x_1:y_1 = x_2:y_2 = \ldots = x_n:y_n$, then each of these ratios also equals the ratio $(x_1 + x_2 + \ldots + x_n) : (y_1 + y_2 + \ldots + y_n)$.

This proposition is used in V.15 and a few other propositions in books VI, X, and XII.

Next proposition: V.13

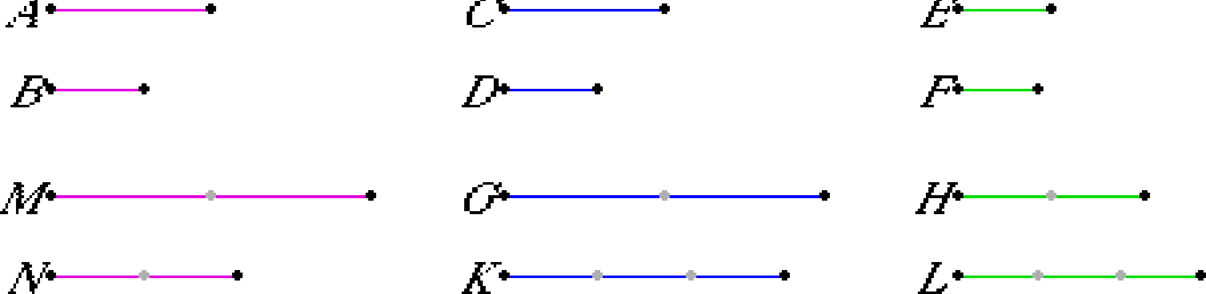
Previous: V.11

Select from Book V

Select book

[Book V introduction](#)

Select topic



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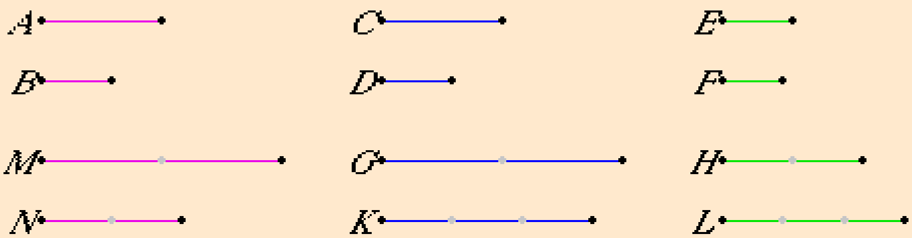
Euclid's Elements
Book V
Proposition 13

If a first magnitude has to a second the same ratio as a third to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first also has to the second a greater ratio than the fifth to the sixth.

Let a first magnitude *A* have to a second *B* the same ratio as a third *C* has to a fourth *D*, and let the third *C* have to the fourth *D* a greater ratio than a fifth *E* has to a sixth *F*.

I say that the first *A* also has to the second *B* a greater ratio than the fifth *E* to the sixth *F*.



Since there are some equimultiples of *C* and *E*, and of *D* and *F* other equimultiples, such that the multiple of *C* is in excess of the multiple of *D*, while the multiple of *E* is not in excess of the multiple of *F*, let them be taken. Let *G* and *H* be equimultiples of *C* and *E*, and *K* and *L* other, arbitrary, equimultiples of *D* and *F*, so that *G* is in excess of *K*, but *H* is not in excess of *L*. Whatever multiple *G* is of *C*, let *M* also be that multiple of *A*, and, whatever multiple *K* is of *D*, let *N* also be that multiple of *B*. V.Def.7

Now, since *A* is to *B* as *C* is to *D*, and of *A* and *C* equimultiples *M* and *G* have been taken, and of *B* and *D* other, arbitrary, equimultiples *N* and *K*, therefore, if *M* is in excess of *N*, *G* is also in excess of *K*; if equal, equal; and if less, less. V.Def.5

But *G* is in excess of *K*, therefore *M* is also in excess of *N*.

But *H* is not in excess of *L*, and *M* and *H* are equimultiples of *A* and *E*, and *N* and *L* other, arbitrary, equimultiples of *B* and *F*, therefore A has to *B* a greater ratio than *E* has to *F*. V.Def.7

Therefore, *if a first magnitude has to a second the same ratio as a third to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first also has to the second a greater ratio than the fifth to the sixth.*

Q.E.D.

Guide

This proposition states that if two ratios are equal, and one is greater than a third, then so is the other. That is, if $a:b = c:d$ and $c:d > e:f$, then $a:b > e:f$. The magnitudes may be of three different kinds with *a* and *b* of one kind, *c* and *d* of a second kind, and *e* and *f* of a third kind.

The analogous statement for lesser ratios isn't stated, but, of course, it holds as well. Euclid uses it as well as this proposition, in V.20.

So does transitivity: if $a:b > c:d$ and $c:d > e:f$, then $a:b > e:f$. The proof isn't difficult, but without symbolic algebra it becomes unwieldly. Euclid would have required 20 lines in his diagram.

This proposition is used in the next one as well as V.20 and V.21.

Next proposition: V.14

Previous: V.12

Book V introduction

Select from Book V

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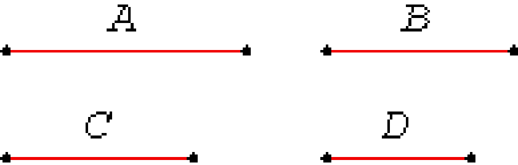

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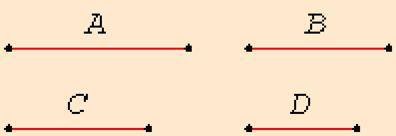
Euclid's Elements
Book V
Proposition 14

If a first magnitude has to a second the same ratio as a third has to a fourth, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less.

Let a first magnitude *A* have the same ratio to a second *B* as a third *C* has to a fourth *D*, and let *A* be greater than *C*.

I say that *B* is also greater than *D*.



Since *A* is greater than *C*, and *B* is another, arbitrary, magnitude, therefore *A* has to *B* a greater ratio than *C* has to *B*. V.8

But *A* is to *B* as *C* is to *D*, therefore *C* has to *D* a greater ratio than *C* has to *B*. V.13

But that to which the same has a greater ratio is less, therefore *D* is less than *B*, so that *B* is greater than *D*. V.10

Similarly we can prove that, if *A* equals *C*, then *B* equals *D*, and, if *A* is less than *C*, then *B* is less than *D*.

Therefore, *if a first magnitude has to a second the same ratio as a third has to a fourth, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less.*

Q.E.D.

Guide

The statement is that

> if $a:b = c:d$ and $a >=< c$, then $b >=< d$.

In this form all four magnitudes need to be of the same kind.

The alternate form of the proposition

Curiously, sometimes the alternate form

> if $a:b = c:d$ and $a >=< b$, then $c >=< d$

is used. This other form is more general since *a* and *b* may be of one kind while *c* and *d* can be of a different kind. (See definition V.Def.12 and proposition V.16 for alternate proportions.) For example, in proposition VI.25 there are the statements:

> ...the triangle *ABC* is to the triangle *KGH* as the parallelogram *BE* is to the parallelogram *EF*. Therefore, alternately, the triangle *ABC* is to the parallelogram *BE* as the triangle *KGH* is to the parallelogram *EF*. But the triangle *ABC* equals the parallelogram *BE*, therefore the triangle *KGH* also equals the parallelogram *EF*.

First, the proportion is converted to its alternate form by V.16. Then, it is claimed that since the first equals the second, therefore the third equals the fourth. Clearly, V.14 is not being invoked otherwise the alternate form of the proportion would not be mentioned.

Another example comes from X.112.

> ... the rectangle *BC* by *EF* equals the rectangle *BD* by *G*, therefore *CB* is to *BD* as *G* is to *EF*. But *CB* is greater than *BD*, therefore *G* is also greater than *EF*.

Here the first is greater than the second, so the third is greater than the fourth. Proposition VI.16 (if the rectangle contained by the extremes equals the rectangle contained by the means, then the four straight lines are proportional) was used to derive the proportion $CB:BD = G:EF$ from the equality of the rectangles, but it would have been just as easy to conclude $CB:G = BD:EF$, and then V.14 could be used.

Clearly, this proposition V.14 is not being invoked in either of these propositions, but the alternate form is used instead. That suggests that the proofs of VI.25 and X.112 were written when V.14 wasn't available.

The proof of the statement

> if $a:b = c:d$ and a >=< b, then c >=< d

is not difficult using the definition V.Def.5. Since a >=< b, therefore $2a$ >=< $2b$. From the proportion $a:b = c:d$ it follows that $2c$ >=< $2d$. Therefore c >=< d. Q.E.D. The proof is even easier when 1 is considered to be a number.

Although this alternate form does not rely on using V.Def.4 as an axiom of comparability, the original form does. The statement of the proposition is false when infinitesimals are allowed. For a particular example, take the proportion $x:(x + y) = x:(x + 2y)$ or its inverse.

Use of this proposition

This proposition is used in V.16 and a few other propositions in Books V, VI, X, XII, and XIII.

Next proposition: V.15

Previous: V.13

Book V introduction

Select from Book V

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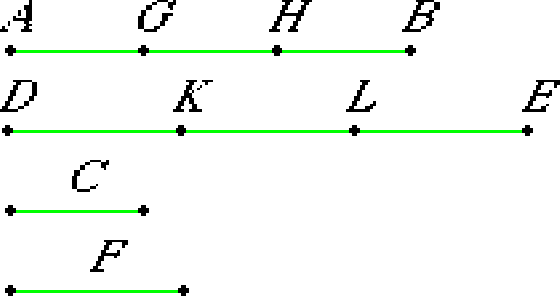
Select topic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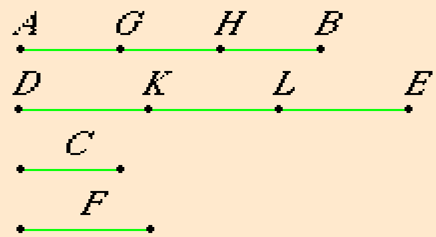
Euclid's Elements
Book V
Proposition 15

Parts have the same ratio as their equimultiples.

Let *AB* be the same multiple of *C* that *DE* is of *F*.

I say that *C* is to *F* as *AB* is to *DE*.

Since *AB* is the same multiple of *C* that *DE* is of *F*, as many magnitudes as there are in *AB* equal to *C*, there are also in *DE* equal to *F*.



Divide *AB* into the magnitudes *AG*, *GH*, and *HB* equal to *C*, and divide *DE* into the magnitudes *DK*, *KL*, and *LE* equal to *F*. Then the number of the magnitudes *AG*, *GH*, and *HB* equals the number of the magnitudes *DK*, *KL*, and *LE*.

And, since *AG*, *GH*, and *HB* equal one another, and *DK*, *KL*, and *LE* also equal one another, therefore *AG* is to *DK* as *GH* is to *KL*, and as *HB* is to *LE*. V.7

Therefore one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents. Therefore *AG* is to *DK* as *AB* is to *DE*. V.12

But *AG* equals *C* and *DK* equals *F*, therefore *C* is to *F* as *AB* is to *DE*.

Therefore, *parts have the same ratio as their equimultiples.*

Q.E.D.

Guide

This proposition states that if *n* is any number and *c* and *f* any magnitudes of the same kind, then $c:f = nc:nf$.

This proposition is used in the next one and a few others in Books V, VI, and XIII.

Next proposition: V.16

Previous: V.14

Book V introduction

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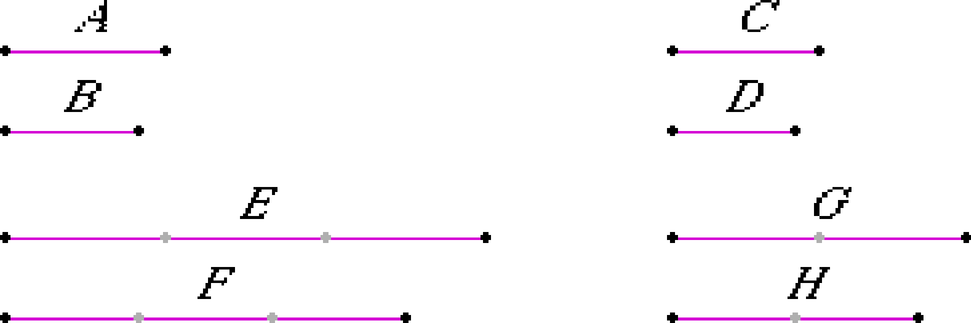
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Euclid's Elements

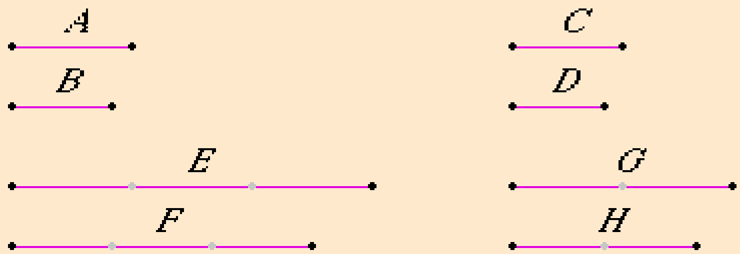
Book V

Proposition 16

If four magnitudes are proportional, then they are also proportional alternately.

Let *A*, *B*, *C*, and *D* be four proportional magnitudes, so that *A* is to *B* as *C* is to *D*.

I say that they are also so alternately, that is *A* is to *C* as *B* is to *D*. V.Def.12



Take equimultiples *E* and *F* of *A* and *B*, and take other, arbitrary, equimultiples *G* and *H* of *C* and *D*.

Then, since *E* is the same multiple of *A* that *F* is of *B*, and parts have the same ratio as their equimultiples, therefore *A* is to *B* as *E* is to *F*. V.15

But *A* is to *B* as *C* is to *D*, therefore *C* is to *D* also as *E* is to *F*. V.11

Again, since *G* and *H* are equimultiples of *C* and *D*, therefore *C* is to *D* as *G* is to *H*. V.15

But *C* is to *D* as *E* is to *F*, therefore as *E* is to *F* also as *G* is to *H*. V.11

But, if four magnitudes are proportional, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less. V.14

Therefore, if *E* is in excess of *G*, *F* is also in excess of *H*; if equal, equal; and if less, less.

Now *E* and *F* are equimultiples of *A* and *B*, and *G* and *H* other, arbitrary, equimultiples of *C* and *D*, therefore *A* is to *C* as *B* is to *D*. V.Def.5

Therefore, *if four magnitudes are proportional, then they are also proportional alternately*.

Q.E.D.

Guide

The four magnitudes *A*, *B*, *C*, and *D* need to be of the same kind. If *A* and *B* are of a different kind than *C* and *D*, then the alternate ratios *A*:*C* and *B*:*D* would be "mixed." The Greek geometers did not accept mixed ratios, but modern physicists and engineers routinely use them, as do we all since such a common measurement as velocity is made out of a ratio of a distance to a time.

This proposition requires using V.Def.4 as an axiom of comparability.

Use of this proposition

This proposition is used in V.19 and a couple others in Book V, and frequently in Books VI, X, XI, and XII.

Occasionally it is used when the magnitudes need not be all of the same kind, as it ought not.

Next proposition: V.17

Previous: V.15

Book V introduction

Select from Book V

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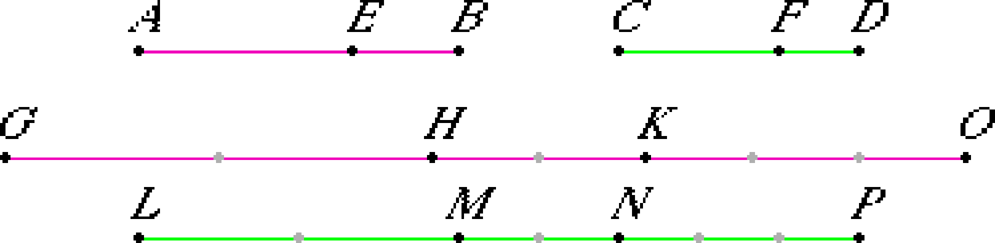
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Euclid's Elements

Book V

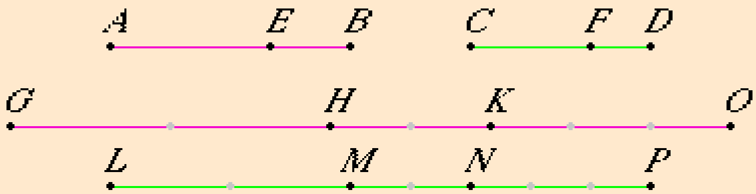
Proposition 17

If magnitudes are proportional taken jointly, then they are also proportional taken separately.

Let *AB*, *BE*, *CD*, and *DF* be magnitudes proportional taken jointly, so that *AB* is to *BE* as *CD* is to *DF*. V.Def.14

I say that they are also proportional taken separately, that is, *AE* is to *EB* as *CF* is to *DF*. V.Def.15

Take equimultiples *GH*, *HK*, *LM*, and *MN* of *AE*, *EB*, *CF*, and *FD*, and take other, arbitrary, equimultiples, *KO* and *NP* of *EB* and *FD*.



Then, since *GH* is the same multiple of *AE* that *HK* is of *EB*, therefore *GH* is the same multiple of *AE* that *GK* is of *AB*. V.1

But *GH* is the same multiple of *AE* that *LM* is of *CF*, therefore *GK* is the same multiple of *AB* that *LM* is of *CF*.

Again, since *LM* is the same multiple of *CF* that *MN* is of *FD*, therefore *LM* is the same multiple of *CF* that *LN* is of *CD*. V.1

But *LM* was the same multiple of *CF* that *GK* is of *AB*, therefore *GK* is the same multiple of *AB* that *LN* is of *CD*.

Therefore *GK* and *LN* are equimultiples of *AB* and *CD*.

Again, since *HK* is the same multiple of *EB* that *MN* is of *FD*, and *KO* is also the same multiple of *EB* that *NP* is of *FD*, therefore the sum *HO* is also the same multiple of *EB* that *MP* is of *FD*. V.2

And, since *AB* is to *BE* as *CD* is to *DF*, and of *AB* and *CD* equimultiples *GK* and *LN* have been taken, and of *EB* and *FD* equimultiples *HO* and *MP*, therefore, if *GK* is in excess of *HO*, and *LN* is also in excess of *MP*; if equal, equal; and if less, less.

Let *GK* be in excess of *HO*. Subtract *HK* from each. Therefore *GH* is also in excess of *KO*.

But we saw that, if *GK* was in excess of *HO*, then *LN* was also in excess of *MP*, therefore *LN* is also in excess of *MP*, and, if *MN* is subtracted from each, then *LM* is also in excess of *NP*, so that, if *GH* is in excess of *KO*, then *LM* is also in excess of *NP*.

Similarly we can prove that, if *GH* equals *KO*, then *LM* also equals *NP*; and if less, less.

And *GH* and *LM* are equimultiples of *AE* and *CF*, while *KO* and *NP* are other, arbitrary, equimultiples of *EB* and *FD*, therefore *AE* is to *EB* as *CF* is to *FD*. V.Def.5

Therefore, *if magnitudes are proportional taken jointly, then they are also proportional taken separately.*

Q.E.D.

Guide

The proposition says that if $(w + x):x = (y + z):z$, then $w:x = y:z$. Two of the magnitudes w and x can be of one kind while the other two y and z are of another kind.

The converse is given in the next proposition.

This proposition is used in the next two propositions and a couple in Book X.

Next proposition: V.18

Previous: V.16

Book V introduction

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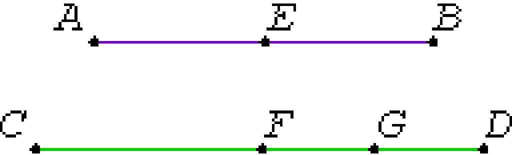
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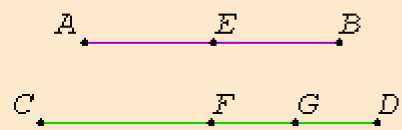
Euclid's Elements
Book V
Proposition 18

If magnitudes are proportional taken separately, then they are also proportional taken jointly.

Let *AE*, *EB*, *CF*, and *FD* be magnitudes proportional taken separately, so that *AE* is to *EB* as *CF* is to *FD*. V.Def.15

I say that they are also proportional taken jointly, that is, *AB* is to *BE* as *CD* is to *FD*. V.Def.14



For, if *CD* is not to *DF* as *AB* is to *BE*, then *AB* is to *BE* as *CD* is either to some magnitude less than *DF* or to a greater.

First, let it be in that ratio to a less magnitude *DG*.

Then, since *AB* is to *BE* as *CD* is to *DG*, they are magnitudes proportional taken jointly, so that they are also proportional taken separately. Therefore *AE* is to *EB* as *CG* is to *GD*. V.17

But also, by hypothesis, *AE* is to *EB* as *CF* is to *FD*. Therefore *CG* is to *GD* as *CF* is to *FD*. V.11

But the first *CG* is greater than the third *CF*, therefore the second *GD* is also greater than the fourth *FD*. V.14

But it is also less, which is impossible. Therefore *AB* is to *BE* as *CD* is not to a less magnitude than *FD*.

Similarly we can prove that neither is it in that ratio to a greater, it is therefore in that ratio to *FD* itself.

Therefore, *if magnitudes are proportional taken separately, then they are also proportional taken jointly.*

Q.E.D.

Guide

This proposition is the converse of the last one. It says that if $w:x = y:z$, then $(w + x):x = (y + z):z$. As in the last proposition, two of the magnitudes w and x can be of one kind while the other two y and z are of another.

On the existence of fourth proportionals

At the beginning of this proof we have, paraphrased:

> If $CD:DF$ does not equal $AB:BE$, then $AB:BE = CD:DG$ where DG is some magnitude greater or less than DF.

Given the other three magnitudes, a fourth proportional *DG* is assumed. It is not asserted that the fourth proportional can be constructed; it is only hypothetical. This is the beginning of a proof by contradiction.

This technique of assuming the existence of a fourth proportional to derive a contradiction is also used in Book XII to prove various proportionalities of areas and volumes, for example, in proposition XII.2 which shows circles are proportional to the squares on their diameters. Eudoxus, who developed the techniques of both Books V and XII, or Euclid, or both of them, accepted this technique as valid.

The problem is: do fourth proportionals exist? They certainly can't be constructed in all cases. The problems of doubling a cube, squaring a circle, and trisecting an angle cannot be solved by plane Euclidean methods, and they all involve inconstructable fourth proportionals. Take doubling a cube for example. If *C* is a cube with an edge *A*, then the inconstructable edge *B* of a cube with double the volume of *C* is the fourth proportional in $C:(C+C) = A:B$.

Is there a difference between existence and constructibility? Constructibility is a fairly clear concept since there are postulates for what can be constructed. There are no postulates for things that exist but aren't constructed, but the existence of a fourth proportional is a good candidate for a such a postulate.

There is a similar situation in modern mathematics with the axiom of choice for set theory. That axiom says that in certain situations there is at least one set satisfying certain criteria. It does not construct anything in the usual sense of "construct," and it doesn't even specify a particular set. Although it is useful in many situations, mathematicians prefer not to use it unless it's necessary.

For this proposition, the assumption of the existence of fourth proportionals is unnecessary.

An alternate proof

Proposition: If $w:x = y:z$, then $(w + x):x = (y + z):z$.

Proof: Suppose $w:x = y:z$. Let n and m be any numbers. Either $n < m$ or not.

Case 1: $n < m$.

> Suppose $n(w + x)$ >=< mx. Subtract nx to get nw >=< $(m - n)x$. But $w:x = y:z$, so ny >=< $(m - n)z$. Add nz to get $n(y + z)$ >=< mz.

Case 2: n is not less than m.

> Then both $n(w + x) > mx$, and $n(y + z) > mz$.

In any case $n(w + x)$ >=< mx implies $n(y + z)$ >=< mz. Therefore $(w + x):x = (y + z):z$. Q.E.D.

Use of this proposition

This proposition is used in proposition V.24 and a few in Books VI, X, XII, and XIII.

Next proposition: V.`9

Previous: V.17

Book V introduction

Select from Book V

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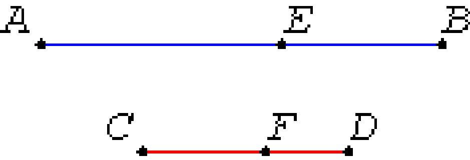
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Euclid's Elements

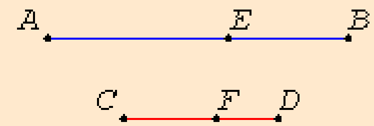
Book V

Proposition 19

If a whole is to a whole as a part subtracted is to a part subtracted, then the remainder is also to the remainder as the whole is to the whole.

Let the whole *AB* be to the whole *CD* as the part *AE* subtracted is to the part *CF* subtracted.

I say that the remainder *EB* is also to the remainder *FD* as the whole *AB* is to the whole *CD.*



Since *AB* is to *CD* as *AE* is to *CF,* therefore alternately, *BA* is to *AE* as *DC* is to *CF.* V.16

And, since the magnitudes are proportional taken jointly, they are also proportional taken separately, that is, *BE* is to *EA* as *DF* is to *CF,* and, alternately, *BE* is to *DF* as *EA* is to *FC.* V.17 V.16

But, by hypothesis, *AE* is to *CF* as is the whole *AB* to the whole *CD.*

Therefore the remainder *EB* is also to the remainder *FD* as the whole *AB* is to the whole *CD.* V.11

Therefore *if a whole is to a whole as a part subtracted is to a part subtracted, then the remainder is also to the remainder as the whole is to the whole.*

Q.E.D.

Corollary

From this it is manifest that, *if magnitudes are proportional taken jointly, then they are also proportional in conversion.* V.Def.16

Guide

This proposition says that if $(u + v):(x + y)$ equals $v:y$, then it also equals $u:x$.

The transformations of proportions taken jointly, taken separately, and in conversion are summarized in the Guide for V.Def.14.

The magnitudes in this proposition must all be of the same kind, but those in the corollary can be of two different kinds. Thus, the corollary is out of place. It should probably be after the last proposition since it follows from the previous two propositions by inversion. As Heiberg and Heath agree, the corollary was probably interpolated before Theon's time.

This proposition relies on using V.Def.4 as an axiom of comparability. (Infinitesimal counterexample: when y is infinitesimal with respect to x, then $2x:(2x + 2y)$ equals $x:(x + 2y)$ but does not equal $x:x$.) The corollary, however, does not rely on an axiom of comparability.

Use of the proposition and the corollary

This proposition is used in V.25 and a few propositions in Book X. The corollary is used once in each of Books VI and XIII and fairly often in Book X.

Next proposition: V.20

Previous: V.18

Book V introduction

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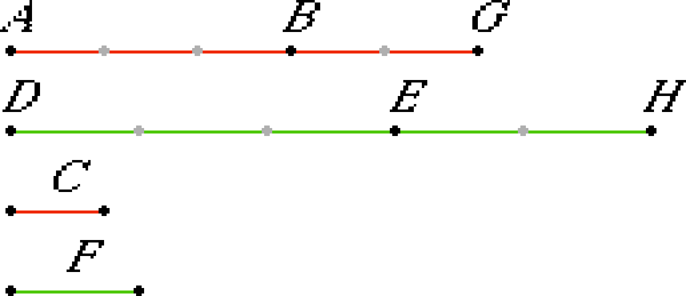
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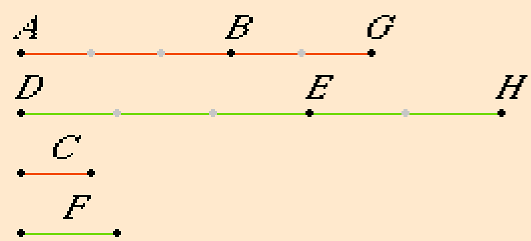
Euclid's Elements
Book V
Proposition 2

If a first magnitude is the same multiple of a second that a third is of a fourth, and a fifth also is the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth also is the same multiple of the second that the sum of the third and sixth is of the fourth.

Let a first magnitude *AB* be the same multiple of a second *C* that a third *DE* is of a fourth *F*, and let a fifth *BG* be the same multiple of the second *C* that a sixth *EH* is of the fourth *F*. V.Def.2

I say that the sum *AG* of the first and fifth is the same multiple of the second, *C*, that the sum *DH* of the third and sixth is of the fourth, *F*.

Since *AB* is the same multiple of *C* that *DE* is of *F*, therefore there are as many magnitudes in *AB* equal to *C* as there are in *DE* equal to *F*.



For the same reason there are as many magnitudes in *BG* equal to *C* as there are in *EH* equal to *F*. Therefore, there are as many magnitudes in the whole *AG* equal to *C* as there are in the whole *DH* equal to *F*.

Therefore, *AG* is the same multiple of *C* that *DH* is of *F*.

Therefore the sum *AG* of the first and fifth is the same multiple of the second, *C*, that the sum *DH* of the third and sixth is of the fourth, *F*.

Therefore, *if a first magnitude is the same multiple of a second that a third is of a fourth, and a fifth also is the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth also is the same multiple of the second that the sum of the third and sixth is of the fourth.*

Q.E.D.

Guide

This proposition simply states that sums of equimultiples are equimultiples, that is, if mc and mf are equimultiples of c and f, and nc and nf are also equimultiples of c and f, then the sums $mc + nc$ and $mf + nf$ are also equimultiples of c and f. The proof depends on a form of distributivity, namely, that multiplication by magnitudes distributes over addition of numbers.

$$(m + n)c = mc + nc.$$

Note that the magnitudes don't all have to be of the same kind. Different colors are used in the figures here to indicate different kinds of magnitudes.

Use of this proposition

Proposition V.2 is used in the proofs of three other propositions, namely, V.3, V.6, and V.17.

Next proposition: V.3

Previous: V.1

Book V introduction

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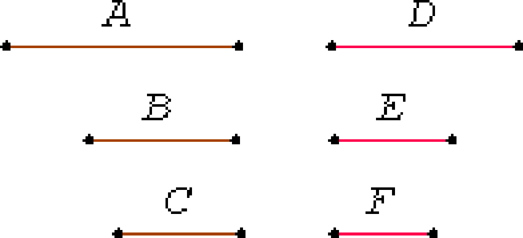
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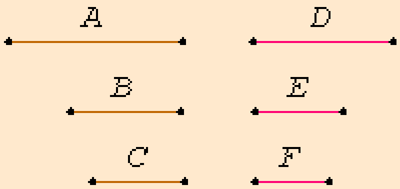
Euclid's Elements
Book V
Proposition 20

If there are three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali the first is greater than the third, then the fourth is also greater than the sixth; if equal, equal, and; if less, less.

Let there be three magnitudes *A, B,* and *C,* and others *D, E,* and *F* equal to them in multitude, which taken two and two are in the same ratio, so that *A* is to *B* as *D* is to *E,* and *B* is to *C* as *E* is to *F.*

Let *A* be greater than *C ex aequali.*

I say that *D* is also greater than *F*; if *A* equals *C*, equal; and, if less, less.



Since *A* is greater than *C*, and *B* is some other magnitude, and the greater has to the same a greater ratio than the less has, therefore *A* has to *B* a greater ratio than *C* has to *B.* V.8

But *A* is to *B* as *D* is to *E,* and, *C* is to *B,* inversely, as *F* is to *E,* therefore *D* has to *E* a greater ratio than *F* has to *E.* V.7.Cor V.13

But, of magnitudes which have a ratio to the same, that which has a greater ratio is greater, therefore *D* is greater than *F.* V.10

Similarly we can prove that, if *A* equals *C,* then *D* also equals *F,* and if less, less.

Therefore, *if there are three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali the first is greater than the third, then the fourth is also greater than the sixth; if equal, equal, and; if less, less.*

Q.E.D.

Guide

This proposition is in preparation for V.22, and its proof is clear.

Next proposition: V.21

Previous: V.19

Book V introduction

Select from Book V

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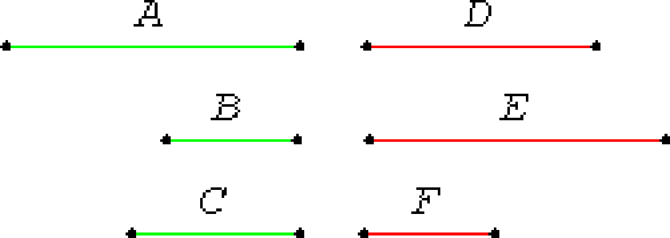
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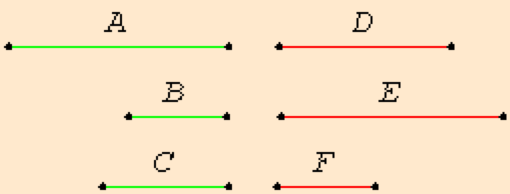
Euclid's Elements
Book V
Proposition 21

If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them is perturbed, then, if ex aequali the first magnitude is greater than the third, then the fourth is also greater than the sixth; if equal, equal; and if less, less.

Let there be three magnitudes *A*, *B*, and *C*, and others *D*, *E*, and *F* equal to them in multitude, which taken two and two are in the same ratio, and let the proportion of them be perturbed, so that *A* is to *B* as *E* is to *F*, and *B* is to *C* as *D* is to *E*. V.Def.18

Let *A* be greater than *C ex aequali.*

I say that *D* is also greater than *F*; if *A* equals *C*, equal; and if less, less.



Since *A* is greater than *C*, and *B* is some other magnitude, therefore *A* has to *B* a greater ratio than *C* has to *B*. V.8

But *A* is to *B* as *E* is to *F*, and, inversely, *C* is to *B* as *E* is to *D*. Therefore also *E* has to *F* a greater ratio than *E* has to *D*. V.7.Cor V.13

But that to which the same has a greater ratio is less, therefore *F* is less than *D*, therefore *D* is greater than *F*. V.10

Similarly we can prove that, if *A* equals *C*, then *D* also equals *F*; and if less, less.

Therefore, *if there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them is perturbed, then, if ex aequali the first magnitude is greater than the third, then the fourth is also greater than the sixth; if equal, equal; and if less, less.*

Q.E.D.

Guide

This proposition is in preparation for V.23. Both this and V.23 rely on treating V.Def.4 as an axiom of comparability.

Next proposition: V.22

Previous: V.20

Book V introduction

Select from Book V

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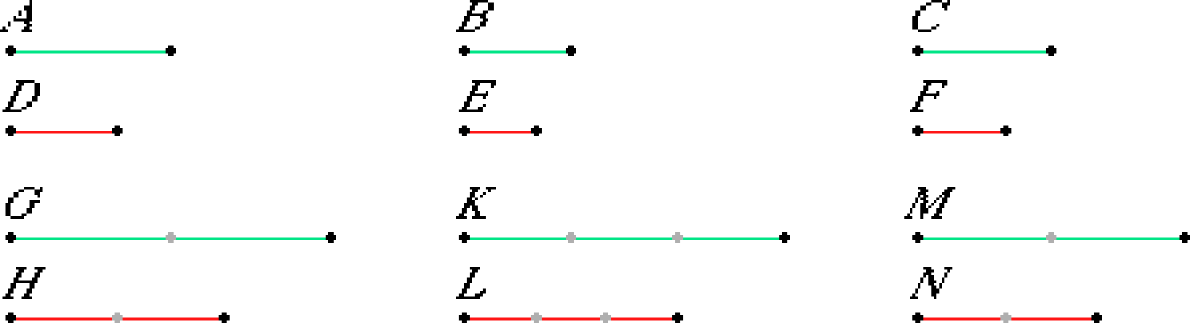
Clark University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Euclid's Elements

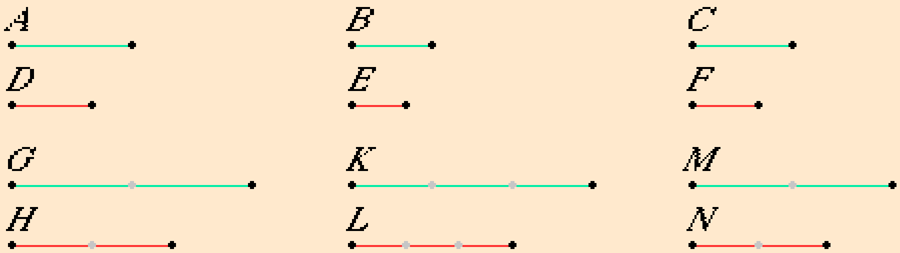
Book V

Proposition 22

If there are any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio ex aequali.

Let there be any number of magnitudes *A*, *B*, and *C*, and others *D*, *E*, and *F* equal to them in multitude, which taken two and two together are in the same ratio, so that *A* is to *B* as *D* is to *E*, and *B* is to *C* as *E* is to *F*.

I say that they are also in the same ratio *ex aequali*, that is, *A* is to *C* as *D* is to *F*. V.Def.17



Take equimultiples *G* and *H* of *A* and *D*, and take other, arbitrary, equimultiples *K* and *L* of *B* and *E*, and, further, take other, arbitrary, equimultiples *M* and *N* of *C* and *F*.

Then, since *A* is to *B* as *D* is to *E*, and of *A* and *D* equimultiples *G* and *H* have been taken, and of *B* and *E* other, arbitrary, equimultiples *K* and *L*, therefore *G* is to *K* as *H* is to *L*. V.4

For the same reason also *K* is to *M* as *L* is to *N*.

Since, then, there are three magnitudes *G*, *K*, and *M*, and others *H*, *L*, and *N* equal to them in multitude, which taken two and two together are in the same ratio, therefore, *ex aequali*, if *G* is in excess of *M*, *H* is also in excess of *N*; if equal, equal; and if less, less. V.20

And *G* and *H* are equimultiples of *A* and *D*, and *M* and *N* other, arbitrary, equimultiples of *C* and *F*.

Therefore *A* is to *C* as *D* is to *F*. V.Def.5

Therefore, *if there are any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio ex aequali.*

Q.E.D.

Guide

The general statement for this proposition is that for magnitudes x_1, x_2, ... , and x_n of one kind, and magnitudes y_1, y_2, ... , and y_n of the same or another kind, if $x_1:x_2 = y_1:y_2$, $x_2:x_3 = y_2:y_3$, ... , and $x_{n-1}:x_n = y_{n-1}:y_n$, then $x_1:x_n = y_1:y_n$.

The proof builds on proposition V.20. Assume $A:B = D:E$, and $B:C = E:F$. To show $A:C = D:F$.
Let *n*, *m*, and *k* be three numbers. By V.4, $nA:mB = nD:mE$, and $mA:kC = mD:kF$. By V.20,

nA >=< kC implies nD >=< kF.

Therefore, $A:C = D:F$. Q.E.D.

This proposition can also be proved directly from the definition Def.V.5 very easily.

The analogous proposition for ratios of numbers is given in proposition VII.14. The proof given there works for magnitudes as well, but they all have to be of the same kind.

This proposition is used in V.24 and several propositions in Books VI, X, and XII.

Next proposition: V.23

Previous: V.21

Book V introduction

Select from Book V

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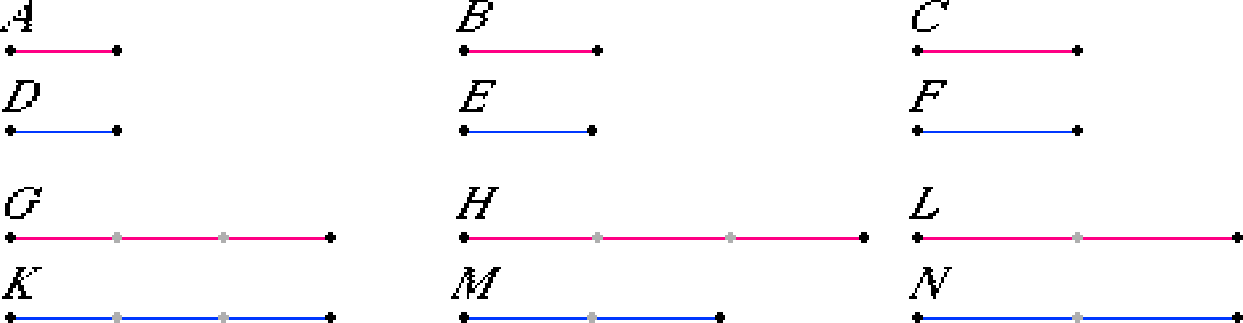
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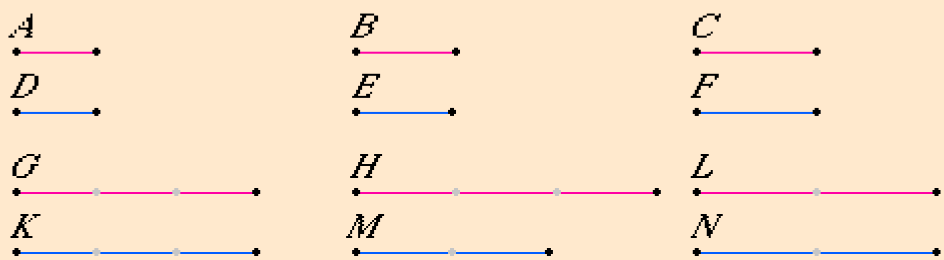
Euclid's Elements
Book V
Proposition 23

If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio ex aequali.

Let there be three magnitudes *A*, *B*, and *C*, and others *D*, *E*, and *F*, equal to them in multitude, which, taken two and two together, are in the same proportion, and let the proportion of them be perturbed, so that *A* is to *B* as *E* is to *F*, and *B* is to *C* as *D* is to *E*. V.Def.18

I say that *A* is to *C* as *D* is to *F*.

Take equimultiples *G*, *H*, and *K* of *A*, *B*, and *D*, and take other, arbitrary, equimultiples *L*, *M*, and *N* of *C*, *E*, and *F*.



Then, since *G* and *H* are equimultiples of *A* and *B*, and parts have the same ratio as their multiples, therefore *A* is to *B* as *G* is to *H*. V.15

For the same reason *E* is to *F* as *M* is to *N*. And *A* is to *B* as *E* is to *F*, therefore *G* is also to *H* as *M* is to *N*. V.11

Next, since *B* is to *C* as *D* is to *E*, alternately, also, *B* is to *D* as *C* is to *E*. (V.16)

And, since *H* and *K* are equimultiples of *B* and *D*, and parts have the same ratio as their equimultiples, therefore *B* is to *D* as *H* is to *K*. V.15

But *B* is to *D* as *C* is to *E*, therefore also, *H* is to *K* as *C* is to *E*. V.11

Again, since *L* and *M* are equimultiples of *C* and *E*, therefore *C* is to *E* as *L* is to *M*. V.15

But *C* is to *E* as *H* is to *K*, therefore also, *H* is to *K* as *L* is to *M*, and, alternately, *H* is to *L* as *K* is to *M*. V.11 (V.16)

But it was also proved that *G* is to *H* as *M* is to *N*.

Since, then, there are three magnitudes *G*, *H*, and *L*, and others equal to them in multitude *K*, *M*, and *N*, which taken two and two together are in the same ratio, and the proportion of them is perturbed, therefore, *ex aequali*, if *G* is in excess of *L*, *K* is also in excess of *N*; if equal, equal; and if less, less. V.21

And *G* and *K* are equimultiples of *A* and *D*, and *L* and *N* of *C* and *F*.

Therefore *A* is to *C* as *D* is to *F*. V.Def.5

Therefore, *if there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio ex aequali.*

Q.E.D.

Guide

This proposition says that when *a*, *b*, and *c* are of one kind, and *d*, *e*, and *f* are of the same or another kind, if $a:b = e:f$ and $b:c = d:e$, then $a:c = d:f$.

The proof given here uses proposition V.16 and alternate ratios, and that means it only applies when all six magnitudes are of the same kind. There is a shorter variation of the proof that uses V.4 instead of V.16 and applies in the general situation.

It is not used in the rest of the *Elements*.

Book V Introduction - Proposition V.22 - Proposition V.24.

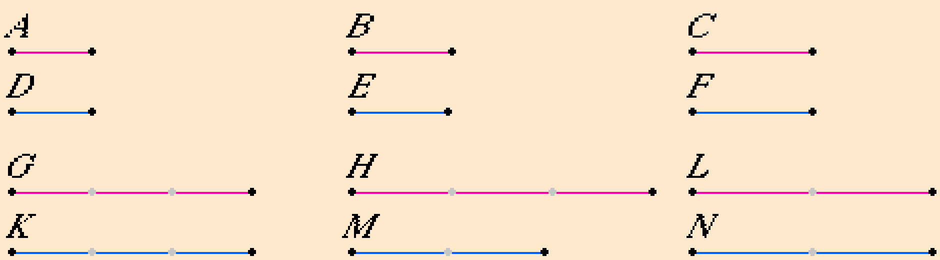
Euclid's Elements
Book V
Proposition 23

If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio ex aequali.

Let there be three magnitudes *A, B,* and *C,* and others *D, E,* and *F,* equal to them in multitude, which, taken two and two together, are in the same proportion, and let the proportion of them be perturbed, so that *A* is to *B* as *E* is to *F,* and *B* is to *C* as *D* is to *E.* V.Def.18

I say that *A* is to *C* as *D* is to *F.*

Take equimultiples *G, H,* and *K* of *A, B,* and *D,* and take other, arbitrary, equimultiples *L, M,* and *N* of *C, E,* and *F.*



Then, since *G* and *H* are equimultiples of *A* and *B,* and parts have the same ratio as their multiples, therefore *A* is to *B* as *G* is to *H.* V.15

For the same reason *E* is to *F* as *M* is to *N.* And *A* is to *B* as *E* is to *F,* therefore *G* is to *H* as *M* is to *N.* V.11

Next, since *B* is to *C* as *D* is to *E,* alternately, also, *B* is to *D* as *C* is to *E.* (V.16)

And, since *H* and *K* are equimultiples of *B* and *D,* and parts have the same ratio as their equimultiples, therefore *B* is to *D* as *H* is to *K.* V.15

But *B* is to *D* as *C* is to *E,* therefore also, *H* is to *K* as *C* is to *E.* V.11

Again, since *L* and *M* are equimultiples of *C* and *E,* therefore *C* is to *E* as *L* is to *M.* V.15

But *C* is to *E* as *H* is to *K,* therefore also, *H* is to *K* as *L* is to *M,* and, alternately, *H* is to *L* as *K* is to *M.* V.11 (V.16)

But it was also proved that *G* is to *H* as *M* is to *N.*

Since, then, there are three magnitudes *G, H,* and *L,* and others equal to them in multitude *K, M,* and *N,* which taken two and two together are in the same ratio, and the proportion of them is perturbed, therefore, *ex aequali,* if *G* is in excess of *L, K* is also in excess of *N*; if equal, equal; and if less, less. V.21

And *G* and *K* are equimultiples of *A* and *D,* and *L* and *N* of *C* and *F.*

Therefore *A* is to *C* as *D* is to *F.* V.Def.5

Therefore, *if there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio ex aequali.*

Q.E.D.

Guide

This proposition says that when a, b, and c are of one kind, and d, e, and f are of the same or another kind, if $a:b = e:f$ and $b:c = d:e$, then $a:c = d:f$.

The proof given here uses proposition V.16 and alternate ratios, and that means it only applies when all six magnitudes are of the same kind. It is also rather clumsy, since it uses V.15 and V.11 instead of V.4 as the previous proposition V.22 did. It doesn't seem likely that this proof would be written when the better proof of V.22 could serve as a guide, so it seems likely that V.4 was inserted later and an older proof of V.22 was cleaned up, but that of V.23 wasn't for some reason such as its relative unimportance. After all, it is not used in the rest of the *Elements*.

Here's a summary of the proof as given.

> Assume $A:B = E:F$, and $B:C = D:E$. To show $A:C = D:F$.
>
> Let n and m be two arbitrary numbers. By V.15, both $A:B = nA:nB$, and $E:F = mE:mF$. Therefore, by V.11, $nA:nB = mE:mF$. (The last two sentences would reduce to one with V.4.)
>
> Using alternation V.16 on the other proportion $B:C = D:E$ yields $B:D = C:E$ (but that requires that all the magnitudes are of the same kind).
>
> For similar reasons $nB:nD = B:D = C:E = mC:mE$. Therefore, alternately, $nB:mC = nD:mE$.
>
> Now use V.21 on the two proportions $nB:mC = nD:mE$ and $nA:nB = mE:mF$, to conclude
>
> $$nA >=< mC \text{ implies } nD >=< mF$$
>
> Therefore, $A:C = D:F$. Q.E.D.

Although the last proposition on proportions *ex aequali* did not depend on treating V.Def.4 as an axiom of comparability, this proposition on perturbed proportions *ex aequali* does. For a counterexample involving infinitesimals, take $a = c = e$, $b = e = (a + y)$, and $f = (a + 2y)$, where y is infinitesimal with respect to a.

Next proposition: V.24

Previous: V.22

Book V introduction

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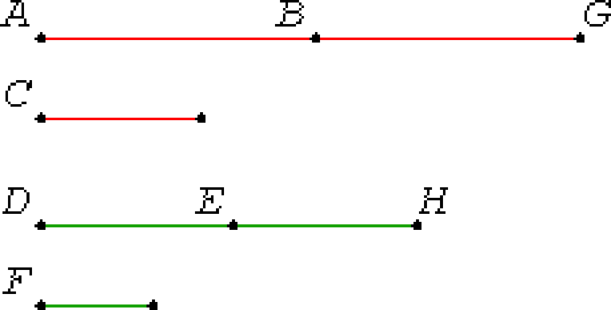
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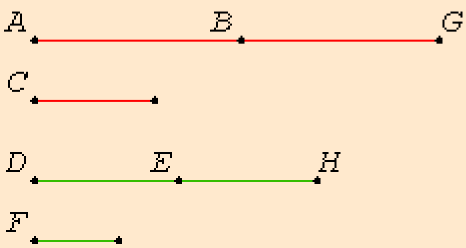
Euclid's Elements
Book V
Proposition 24

If a first magnitude has to a second the same ratio as a third has to a fourth, and also a fifth has to the second the same ratio as a sixth to the fourth, then the sum of the first and fifth has to the second the same ratio as the sum of the third and sixth has to the fourth.

Let a first magnitude *AB* have to a second *C* the same ratio as a third *DE* has to a fourth *F*, and let also a fifth *BG* have to the second *C* the same ratio as a sixth *EH* has to the fourth *F*.

I say that the sum of the first and fifth, *AG*, has to the second *C* the same ratio as the sum of the third and sixth, *DH*, has to the fourth *F*.



Since *BG* is to *C* as *EH* is to *F*, inversely, *C* is to *BG* as *F* is to *EH*. V.7.Cor

Then, since *AB* is to *C* as *DE* is to *F*, and *C* is to *BG* as *F* is to *EH*, therefore, *ex aequali*, *AB* is to *BG* as *DE* is to *EH*. V.22

And, since the magnitudes are proportional taken separately, they are also proportional taken jointly, therefore *AG* is to *GB* as *DH* is to *HE*. V.18

But also *BG* is to *C* as *EH* is to *F*, therefore, *ex aequali*, *AG* is to *C* as *DH* is to *F*. V.22

Therefore, *if a first magnitude has to a second the same ratio as a third has to a fourth, and also a fifth has to the second the same ratio as a sixth to the fourth, then the sum of the first and fifth has to the second the same ratio as the sum of the third and sixth has to the fourth.*

Q.E.D.

Guide

This proposition says that if $u:v = w:x$ and $y:v = z:x$, then $(u + y):v = (w + z):x$.

Although the proposition is stated using the antecedent terms of the proportions, by inversion it applies to the consequent terms as well.

This proposition is used in proposition VI.31.

Next proposition: V.25

Previous: V.23

Book V introduction

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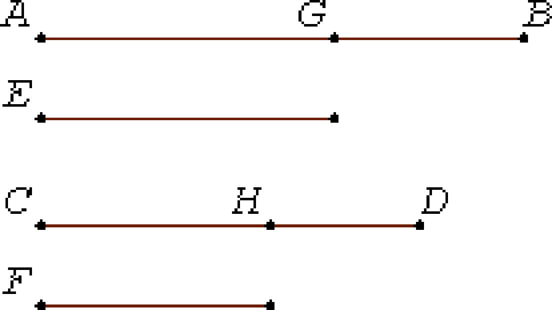
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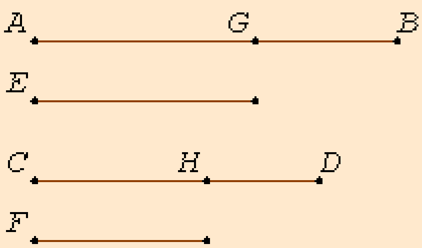
Euclid's Elements
Book V
Proposition 25

If four magnitudes are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two.

Let the four magnitudes *AB*, *CD*, *E*, and *F* be proportional so that *AB* is to *CD* as *E* is to *F*, and let *AB* be the greatest of them and *F* the least.

I say that the sum of *AB* and *F* is greater than the sum of *CD* and *E*.



Make *AG* equal to *E*, and *CH* equal to *F*.

Since *AB* is to *CD* as *E* is to *F*, and *E* equals *AG*, and *F* equals *CH*, therefore *AB* is to *CD* as *AG* is to *CH*. V.7 V.11

And since the whole *AB* is to the whole *CD* as the part *AG* subtracted is to the part *CH* subtracted, therefore the remainder *GB* is also to the remainder *HD* as the whole *AB* is to the whole *CD*. V.19

But *AB* is greater than *CD*, therefore *GB* is also greater than *HD*. (V.14)

And, since *AG* equals *E*, and *CH* equals *F*, therefore the sum of *AG* and *F* equals the sum of *CH* and *E*.

And if, *GB* and *HD* being unequal, and *GB* greater, the sum of *AG* and *F* is added to *GB*, and the sum of *CH* and *E* is added to *HD*, it follows that the sum of *AB* and *F* is greater than the sum of *CD* and *E*.

Therefore, *if four magnitudes are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two.*

Q.E.D.

Guide

This proposition says that if $w:x = y:z$ and w is the greatest of the four magnitudes while z is the least, then $w + z > x + y$. All four magnitudes must be of the same kind.

This proposition is not used in the rest of the *Elements* but is an end in itself.

Arithmetic and geometric means

A special case of it is when the middle terms are the same: $x:y = y:z$. In that case y is the mean proportional, equivalent to the geometric mean for real numbers and described as the square root of the product xz. The conclusion of the proposition, after dividing by 2, says $(x + z)/2 > y$. The arithmetic mean, or average, of two magnitudes is half their sum. Thus, this proposition gives as a corollary

> The arithmetic mean of two magnitudes is less than their geometric mean.

This proposition relies on treating V.Def.4 as an axiom of comparability. Infinitesimal counterexample: when y is infinitesimal with respect to x, consider the proportion $(x + 5y):(x + 2y) = (x + 4y):x$.

Next book: Book VI Introduction

Previous: V.24

Book V introduction

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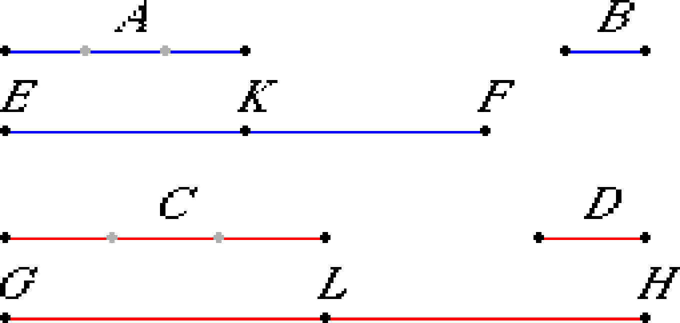
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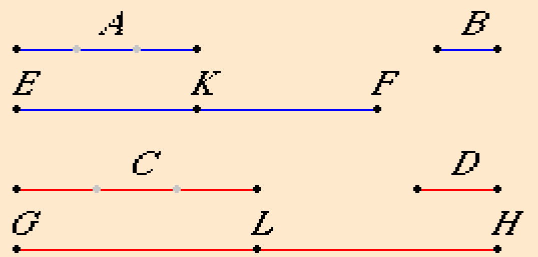
Euclid's Elements
Book V
Proposition 3

If a first magnitude is the same multiple of a second that a third is of a fourth, and if equimultiples are taken of the first and third, then the magnitudes taken also are equimultiples respectively, the one of the second and the other of the fourth.

Let a first magnitude *A* be the same multiple of a second *B* that a third *C* is of a fourth *D*, and let equimultiples *EF* and *GH* be taken of *A* and *C*. V.Def.2

I say that *EF* is the same multiple of *B* that *GH* is of *D*.

Since *EF* is the same multiple of *A* that *GH* is of *C*, therefore there are as many magnitudes as in *EF* equal to *A* as there are in *GH* equal to *C*.



Divide *EF* into the magnitudes *EK* and *KF* equal to *A*, and divide *GH* into the magnitudes *GL* and *LH* equal to *C*. Then the number of the magnitudes *EK* and *KF* equals the number of the magnitudes *GL* and *LH*.

And, since *A* is the same multiple of *B* that *C* is of *D*, while *EK* equals *A*, and *GL* equals *C*, therefore *EK* is the same multiple of *B* that *GL* is of *D*.

For the same reason *KF* is the same multiple of *B* that *LH* is of *D*.

Since a first magnitude *EK* is the same multiple of a second *B* that a third *GL* is of a fourth *D*, and a fifth *KF* is the same multiple of the second *B* that a sixth *LH* is of the fourth *D*, therefore the sum *EF* of the first and fifth is the same multiple of the second *B* that the sum *GH* of the third and sixth is of the fourth *D*. V.2

Therefore, *if a first magnitude is the same multiple of a second that a third is of a fourth, and if equimultiples are taken of the first and third, then the magnitudes taken also are equimultiples respectively, the one of the second and the other of the fourth.*

Q.E.D.

Guide

This proposition says that equimultiples of equimultiples are equimultiples, that is, if *w* and *x* are equimultiples of *y* and *z*, and *u* and *v* are equimultiples of *w* and *x*, then *u* and *v* are equimultiples of *y* and *z*. The proof depends on an associativity of multiplication: $m\,(ny) = (mn)\,y$. In Euclid's proof, *n* is 3 and *m* is 2.

As in the last proposition, the magnitudes need not all be of the same kind.

Although this proposition is not actually a statement about ratios, it can be interpreted as one. The hypotheses that *A* and *C* are equimultiples of *B* and *D* can be interpreted as a proportion $A:B = C:D$, and the conclusion that *mA* and *mC* are equimultiples of *B* and *D* can be interpreted as a proportion $mA:B = mC:D$. Under these interpretations this proposition becomes a special case of the next, and it is the special case that is used to prove the general case in the next proposition.

Next proposition: V.4

Previous: V.2

Book V introduction

Select from Book V

Select book

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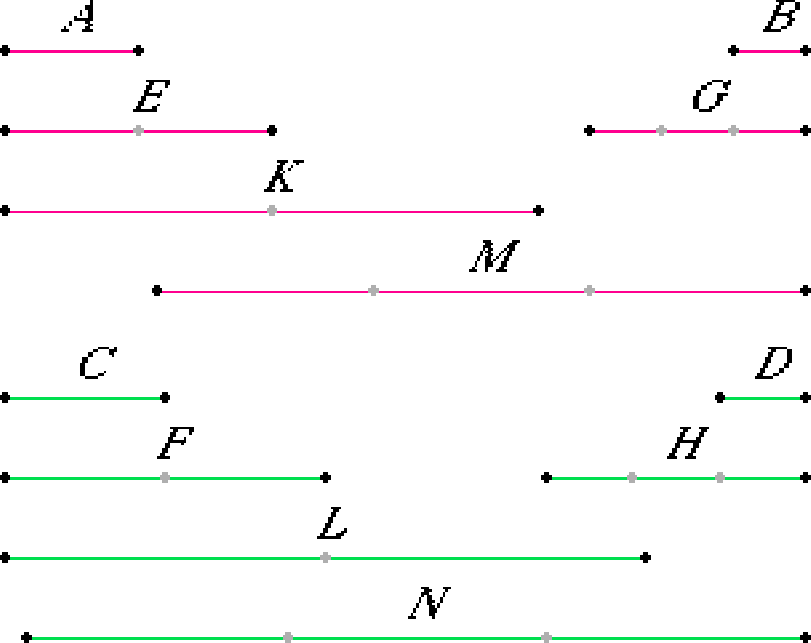
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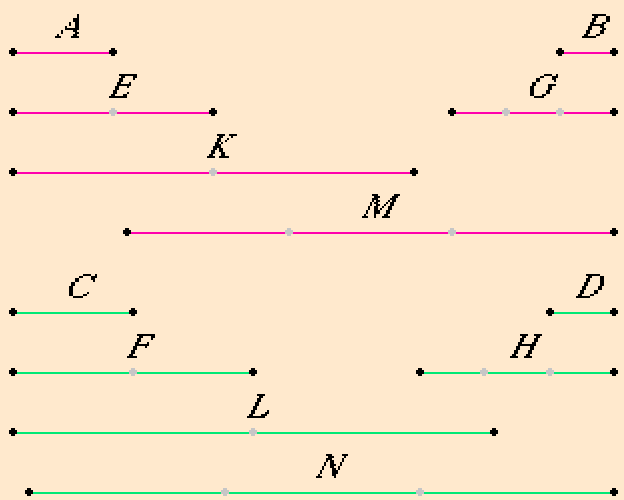
Euclid's Elements
Book V
Proposition 4

If a first magnitude has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

Let a first magnitude *A* have to a second *B* the same ratio as a third *C* to a fourth *D*, and let equimultiples *E* and *F* be taken of *A* and *C*, and *G* and *H* other, arbitrary, equimultiples of *B* and *D*.

I say that *E* is to *G* as *F* is to *H*.



Take equimultiples *K* and *L* of *E* and *F*, and other, arbitrary, equimultiples *M* and *N* of *G* and *H*.

Since *E* is the same multiple of *A* that *F* is of *C*, and equimultiples *K* and *L* of *E* and *F* have been taken, therefore *K* is the same multiple of *A* that *L* is of *C*. For the same reason *M* is the same multiple of *B* that *N* is of *D*. V.3

And, since *A* is to *B* as *C* is to *D*, and equimultiples *K* and *L* have been taken of *A* and *C*, and other, arbitrary, equimultiples *M* and *N* of *B* and *D*, therefore, if *K* is in excess of *M*, then *L* is in excess of *N*; if it is equal, equal; and if less, less. V.Def.5

And *K* and *L* are equimultiples of *E* and *F*, and *M* and *N* are other, arbitrary, equimultiples of *G* and *H*, therefore *E* is to *G* as *F* is to *H*. V.Def.5

Therefore, *if a first magnitude has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.*

Q.E.D.

Guide

Note how Euclid uses the definition to prove that the two ratios $pa:qb$ and $pc:qd$ are the same. (Here, *a* and *b* are magnitudes of one kind, and *c* and *d* are magnitudes of another kind, but *p* and *q* are numbers.) We are given $a:b = c:d$. That means for any numbers *m* and *n* that

if *ma* >=< *nb*, then *mc* >=< *nd*.

We have to prove that $pa:qb = pc:qd$ for any numbers *p* and *q*. That means, we have to prove that for any *m* and *n*, if *mpa* >=< *nqb*, then *mpc* >=< *nqd*.

But that's just a special case of the given relation

if *ma* >=< *nb*, then *mc* >=< *nd*.

Use of this proposition

Proposition V.4 is used in the proof of one other proposition, namely, V.22.

Next proposition: V.5

Previous: V.3

Book V introduction

Select from Book V

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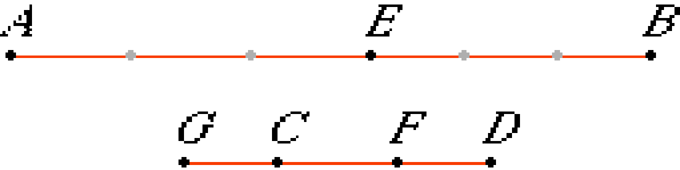
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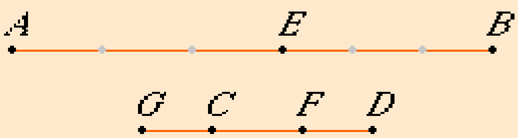
Euclid's Elements
Book V
Proposition 5

If a magnitude is the same multiple of a magnitude that a subtracted part is of a subtracted part, then the remainder also is the same multiple of the remainder that the whole is of the whole.

Let the magnitude *AB* be the same multiple of the magnitude *CD* that the subtracted part *AE* is of the subtracted part *CF*.

I say that the remainder *EB* is also the same multiple of the remainder *FD* that the whole *AB* is of the whole *CD*.

Make *CG* so that *EB* is the same multiple of *CG* that *AE* is of *CF*.



Then, since *AE* is the same multiple of *CF* that *EB* is of *GC*, therefore *AE* is the same multiple of *CF* that *AB* is of *GF*. V.1

But, by the assumption, *AE* is the same multiple of *CF* that *AB* is of *CD*.

Therefore *AB* is the same multiple of each of the magnitudes *GF* and *CD*. Therefore *GF* equals *CD*.

Subtract *CF* from each. Then the remainder *GC* equals the remainder *FD*.

And, since *AE* is the same multiple of *CF* that *EB* is of *GC*, and *GC* equals *DF*, therefore *AE* is the same multiple of *CF* that *EB* is of *FD*.

But, by hypothesis, *AE* is the same multiple of *CF* that *AB* is of *CD*, therefore *EB* is the same multiple of *FD* that *AB* is of *CD*.

That is, the remainder *EB* is the same multiple of the remainder *FD* that the whole *AB* is of the whole *CD*.

Therefore, *If a magnitude is the same multiple of a magnitude that a subtracted part is of a subtracted part, then the remainder also is the same multiple of the remainder that the whole is of the whole.*

Q.E.D.

Guide

This proposition is analogous to V.1 but involves differences rather than sums. It states that multiplication by numbers distributes over subtraction of magnitudes: $m(x - y) = mx - my$.

Note that all the magnitudes in this proposition are of the same kind.

The problem of parts

There's a construction at the beginning of the proof to make a part of a magnitude:

> Make *CG* so that *EB* is the same multiple of *CG* that *AE* is of *CF*.

so that, if for example, *AE* is a third of *CF*, then *CG* is to be made a third of *EB.* Such constructions cannot be made for all kinds of magnitudes, in particular, angles and arcs.

Alternative proofs that don't require constructions of parts are relatively easy to find. A more interesting problem of general constructions for magnitudes is discussed in the Guide for proposition V.18.

This proposition is not used in the rest of the *Elements.*

Next proposition: V.6

Previous: V.4

Book V introduction

Select from Book V

Select book

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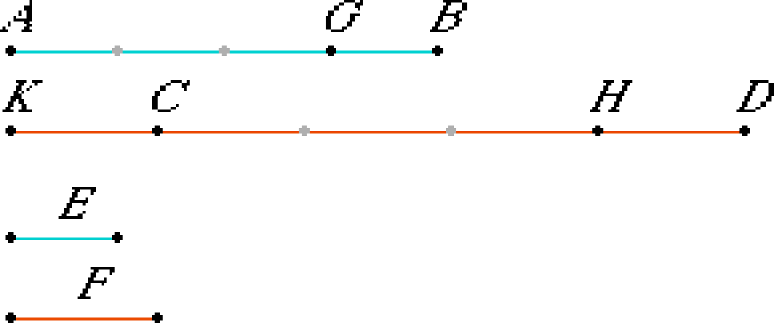
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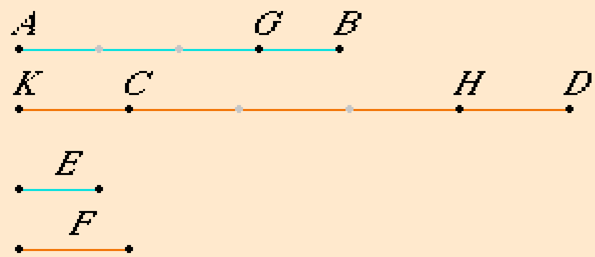
Euclid's Elements
Book V
Proposition 6

If two magnitudes are equimultiples of two magnitudes, and any magnitudes subtracted from them are equimultiples of the same, then the remainders either equal the same or are equimultiples of them.

Let two magnitudes *AB* and *CD* be equimultiples of two magnitudes *E* and *F*, and let *AG* and *CH* subtracted from them be equimultiples of the same two *E* and *F*.

I say that the remainders *GB* and *HD* either equal *E* and *F* or are equimultiples of them.



First, let *GB* equal *E*.

I say that *HD* also equals *F*.

Make *CK* equal to *F*.

Since *AG* is the same multiple of *E* that *CH* is of *F*, while *GB* equals *E*, and *KC* equals *F*, therefore *AB* is the same multiple of *E* that *KH* is of *F*. V.2

But, by hypothesis, *AB* is the same multiple of *E* that *CD* is of *F*, therefore *KH* is the same multiple of *F* that *CD* is of *F*.

Since then each of the magnitudes *KH* and *CD* is the same multiple of *F*, therefore *KH* equals *CD*.

Subtract *CH* from each. Then the remainder *KC* equals the remainder *HD*.

But *F* equals *KC*, therefore *HD* also equals *F*.

Hence, if *GB* equals *E*, *HD* also equals *F*.

Similarly we can prove that, even if *GB* is a multiple of *E*, *HD* is also the same multiple of *F*.

Therefore, *if two magnitudes are equimultiples of two magnitudes, and any magnitudes subtracted from them are equimultiples of the same, then the remainders either equal the same or are equimultiples of them.*

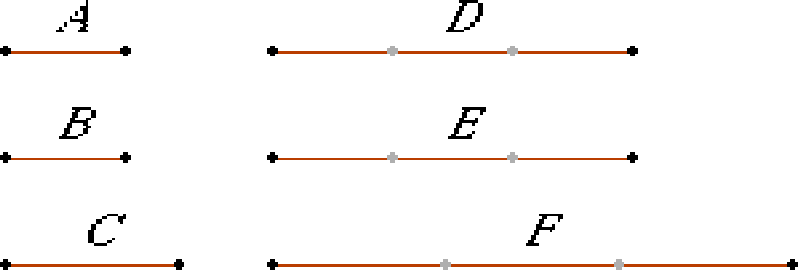
Q.E.D.

Guide

The proposition states that if *ma* and *mb* are equimultiples of *a* and *b*, and *na* and *nb* are also equimultiples, then the differences, *ma* - *na* and *mb* - *nb* are more equimultiples. It's analogous to proposition V.2 which was for addition.

Its proof depends on a distributivity, namely that multiplication by magnitudes distributes over subtraction of numbers: $(m - n)a = ma - na$. Euclid takes 4 as *m* and 3 as *n*. He has two cases since since he doesn't take 1 to be a number.

This proposition is not used in the rest of the *Elements.*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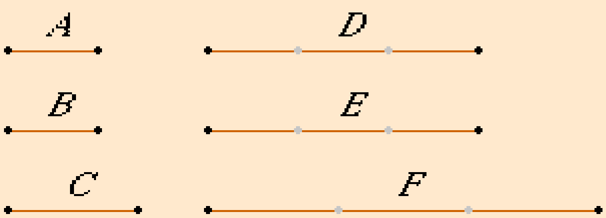
Euclid's Elements
Book V
Proposition 7

Equal magnitudes have to the same the same ratio; and the same has to equal magnitudes the same ratio.

Let *A* and *B* be equal magnitudes and *C* an arbitrary magnitude.

I say that each of the magnitudes *A* and *B* has the same ratio to *C*, and *C* has the same ratio to each of the magnitudes *A* and *B*.



Take equimultiples *D* and *E* of *A* and *B*, and take an arbitrary multiple *F* of *C*.

Then, since *D* is the same multiple of *A* that *E* is of *B*, and *A* equals *B*, therefore *D* equals *E*.

But *F* is another, arbitrary, magnitude. If therefore *D* is in excess of *F*, then *E* is also in excess of *F*; if equal, equal; and, if less, less.

And *D* and *E* are equimultiples of *A* and *B*, while *F* is another, arbitrary, multiple of *C*, therefore *A* is to *C* as *B* is to *C*. V.Def.5

I say next that *C* also has the same ratio to each of the magnitudes *A* and *B*.

With the same construction, we can prove similarly that *D* equals *E*, and *F* is some other magnitude. If therefore *F* is in excess of *D*, it is also in excess of *E*; if equal, equal; and, if less, less.

And *F* is a multiple of *C*, while *D* and *E* are other, arbitrary, equimultiples of *A* and *B*, therefore *C* is to *A* as *C* is to *B*. V.Def.5

Therefore, *equal magnitudes have to the same the same ratio; and the same has to equal magnitudes the same ratio.*

Q.E.D.

Corollary

From this it is manifest that, *if any magnitudes are proportional, then they are also proportional inversely.*

Guide

This proposition says that if $a = b$, then $a:c = b:c$, and $c:a = c:b$. The proposition is evident. Its converse is given in V.9.

The corollary is misplaced. There is nothing relevant in the proposition. There's no way it could yield the corollary since the proposition requires all the magnitudes to be of the same kind and the corollary doesn't. But the corollary is valid, and it follows easily from definition V.Def.5.

Use of this proposition

Such a basic property of ratios as this is used frequently when ratios are mentioned. It is used in a few times in Book V starting with V.10, frequently in Book VI, and ocassionally in later books.

Next proposition: V.8

Previous: V.6

Book V introduction

Select from Book V

Select book

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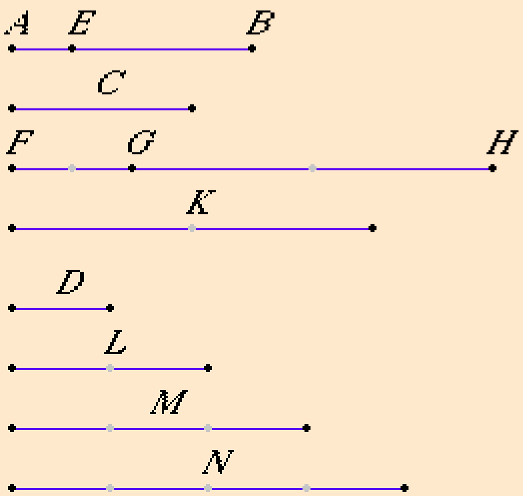
Euclid's Elements
Book V
Proposition 8

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Let *AB* and *C* be unequal magnitudes, and let *AB* be greater, and let *D* be another, arbitrary, magnitude.

I say that *AB* has to *D* a greater ratio than *C* has to *D*, and *D* has to *C* a greater ratio than it has to *AB*.



Since *AB* is greater than *C*, make *EB* equal to *C*. Then the less of the magnitudes *AE* and *EB*, if multiplied, will eventually be greater than *D*. (V.Def.4)

First, let *AE* be less than *EB*. Let *AE* be multiplied, and let *FG* be a multiple of it which is greater than *D*. Make *GH* the same multiple of *EB* and *K* the same multiple of *C* that *FG* is of *AE*.

Take *L* double of *D* and *M* triple of it, and successive multiples increasing by one, until what is taken is the first multiple of *D* that is greater than *K*. Let it be taken, and let it be *N* which is quadruple of *D* and the first multiple of it greater than *K*. (V.Def.4)

Since *K* is less than *N* first, therefore *K* is not less than *M*.

And, since *FG* is the same multiple of *AE* that *GH* is of *EB*, therefore *FG* is the same multiple of *AE* that *FH* is of *AB*. V.1

But *FG* is the same multiple of *AE* that *K* is of *C*, therefore *FH* is the same multiple of *AB* that *K* is of *C*. Therefore *FH* and *K* are equimultiples of *AB* and *C*.

Again, since *GH* is the same multiple of *EB* that *K* is of *C*, and *EB* equals *C*, therefore *GH* equals *K*.

But *K* is not less than *M*, therefore neither is *GH* less than *M*.

And *FG* is greater than *D*, therefore the whole *FH* is greater than the sum of *D* and *M*.

But the sum of *D* and *M* equals *N*, inasmuch as *M* is triple *D*, and the sum of *M* and *D* is quadruple *D*, while *N* is also quadruple *D*, therefore the sum of *M* and *D* equals *N*.

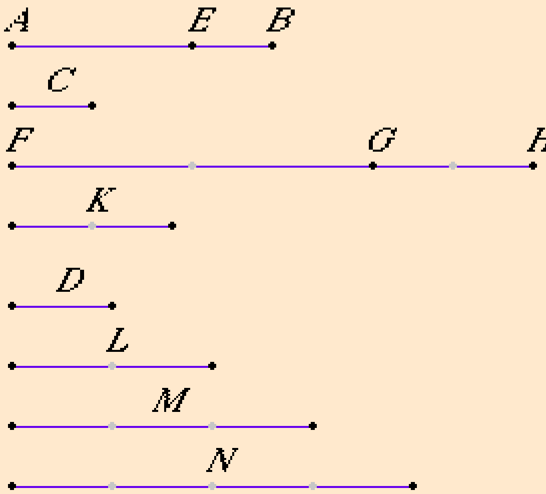
But *FH* is greater than the sum of *M* and *D*, therefore *FH* is in excess of *N*, while *K* is not in excess of *N*.

And *FH* and *K* are equimultiples of *AB* and *C*, while *N* is another, arbitrary, multiple of *D*, therefore *AB* has to *D* a greater ratio than *C* has to *D*. V.Def.7

I say next, that *D* has to *C* a greater ratio than *D* has to *AB*.

With the same construction, we can prove similarly that *N* is in excess of *K*, while *N* is not in excess of *FH*.

And *N* is a multiple of *D*, while *FH* and *K* are other, arbitrary, equimultiples of *AB* and *C*, therefore *D* has to *C* a greater ratio than *D* has to *AB*. V.Def.7



Next, let *AE* be greater than *EB*.

Then the less, *EB*, if multiplied, will eventually be greater than *D*. (V.Def.4)

Let it be multiplied, and let *GH* be a multiple of *EB* and greater than *D*. Make *FG* the same multiple of *AE*, and *K* the same multiple of *C* that *GH* is of *EB*.

Then we can prove similarly that *FH* and *K* are equimultiples of *AB* and *C*, and, similarly, take *N* the first multiple of *D* that is greater than *FG*, so that *FG* is again not less than *M*. (V.Def.4)

But *GH* is greater than *D*, therefore the whole *FH* is in excess of the sum of *D* and *M*, that is, of *N*.

Now *K* is not in excess of *N*, inasmuch as *FG* also, which is greater than *GH*, that is, than *K*, is not in excess of *N*.

And in the same manner, by following the above argument, we complete the demonstration.

Therefore, *of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.*

Q.E.D.

Guide

Although the statement of this proposition is easy to comprehend, its proof is difficult. It says that if $x > y$, then $x:z > y:z$ but $z:x < z:y$. Its converse is proposition V.10.

At four points in the proof V.Def.4 is used as an axiom of comparabililty rather than a definition. The first instance:

> Then the less of the magnitudes *AE* and *EB*, if multiplied, will eventually be greater than *D*.

In fact the axiom of comparability is required for this proposition since it is false when infinitesimals are allowed. When y is infinitesimal with respect to x, then the first statement of the proposition doesn't hold since $x > x - y$ but it is not the case that $x:x > (x - y):x$, and the second statement doesn't hold since $x + y > x$, but not $x:x > x:(x + y)$.

Explanation of the proof

The proof is slightly more comprehensible when modern algebraic notation is used since that clarifies its overall structure. Every magnitude in Euclid's proof is represented by a name and illustrated by a line. With an algebraic notation, we can refer to a magnitude by a formula. For instance, if we let a be *AB* and c be *C*, then we can use $a - c$

for *AE*, thus reducing the number of variables and easing comprehension. We can also have variables for numbers, instead of having to choose a specific number as Euclid does when he takes N to be $4D$.

But algebra obscures much, too. Euclid carefully proved distributivity of multiplication by numbers over addition of magnitudes in V.1, which is used in this proof. We manipulate algebraic expressions almost automatically. In order to be as correct as Euclid, we should verify the rules of algebra and be aware when we use them.

With these preliminary qualifications, let's look at a translation of the proof into symbolic algebra.

To prove: if $a > c$, then $a:d > c:d$, but $d:c > d:a$.	$a = AB$ $c = C = EB$ $d = D$
Let $a > c$. Either $a - c < c$, or $a - c > c$, [or $a - c = c$].	$a - c = AE$
Case 1: Suppose $a - c < c$. Let m be a number such that $m(a - c) > d$.	$m(a - c) = FG$ $mc = GH = K$
Let n be the smallest number such that $nd > mc$. [What happens when $n = 1$ to Euclid's proof?]	$nd = N$ $(n - 1)d = M$
Since mc is not less than $(n - 1)d$, and $m(a - c) > d$, therefore, by adding, $ma > nd$. But mc is not greater than nd. Therefore $a:d > c:d$.	$ma = FH$
Also $nd > mc$ but nd is not greater than ma. Therefore $d:c > d:a$.	
Case 2: Suppose $c < a - c$. Let Let m be a number such that $mc > d$.	(Same as above)
Let n be the smallest number such that $nd > m(a - c)$.	
Since $m(a - c)$ is not less than $(n - 1)d$, and $mc > d$, therefore, by adding, $ma > nd$. [Euclid says to do the rest in the same manner: Since $a - c > c$, therefore $m(a - c) > mc$. But $nd > m(a - c)$, therefore $nd > mc$. But $ma > nd$, therefore $a:d > c:d$.	
Also $nd > mc$ but nd is not greater than ma. Therefore $d:c > d:a$.]	
[Case 3 when $a - c = c$ is left to the reader.]	
Thus, the conclusion is reached in any case. Q.E.D.	

Use of this proposition

Proposition V.8 is used a few times in Book V starting with the next proposition.

Next proposition: V.9

Select from Book V

Previous: V.7

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Book V introduction

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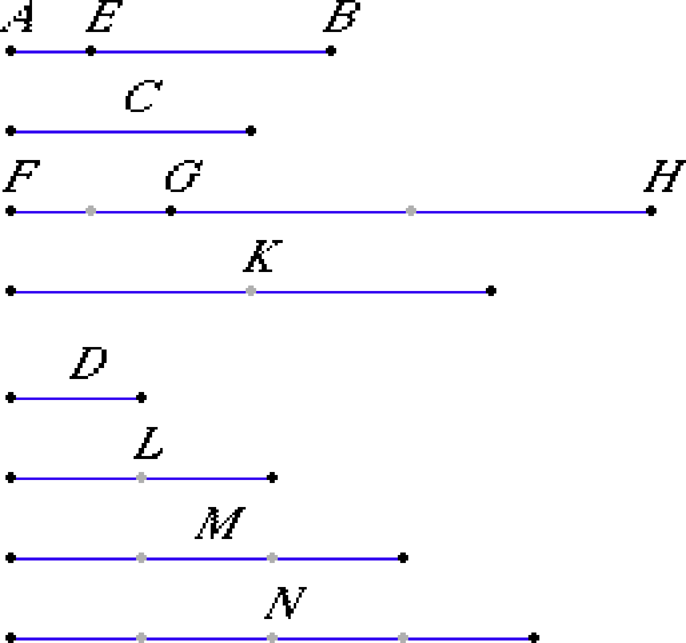

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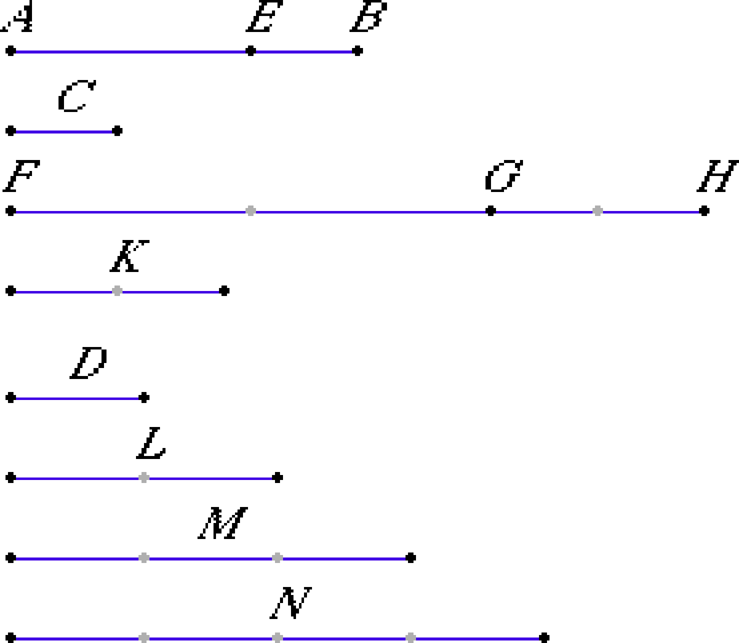
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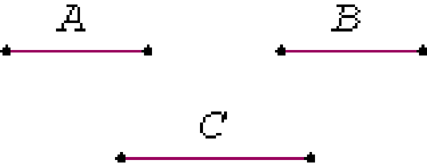
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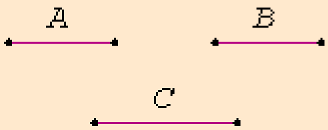
Euclid's Elements
Book V
Proposition 9

Magnitudes which have the same ratio to the same equal one another; and magnitudes to which the same has the same ratio are equal.

Let each of the magnitudes *A* and *B* have the same ratio to *C*.

I say that *A* equals *B*.



Otherwise, each of the magnitudes *A* and *B* would not have the same ratio to *C*, but they do, therefore *A* equals *B*. V.8

Next, let *C* have the same ratio to each of the magnitudes *A* and *B*.

I say that *A* equals *B*.

Otherwise, *C* would not have the same ratio to each of the magnitudes *A* and *B*, but it does, therefore *A* equals *B*. V.8

Therefore, *magnitudes which have the same ratio to the same equal one another; and magnitudes to which the same has the same ratio are equal.*

Q.E.D.

Guide

This converse to proposition V.7 has two statements:

If $a:c = b:c$, then $a = b$.

If $c:a = c:b$, then $a = b$.

Besides the previous proposition, the proof relies on the law of trichotomy for ratios, the part which says that $a:b < a:c$ and $a:b = a:c$ cannot both occur. Although Euclid didn't prove that, it follows easily from the definitions in V.Def.5 and V.Def.7.

This proposition relies on using V.Def.4 as an axiom of comparability through its use of the previous proposition. The axiom is required since the statement of the proposition is false when a is $c + y$, and b is $c + 2y$ where y is an infinitesimal with respect to c.

This proposition is used occasionally in Books VI, VII, X, XI, and XII to conclude equality of geometric magnitudes.

Next proposition: V.10

Select from Book V

Previous: V.8

Select book

Book V introduction

Select book

Select topic

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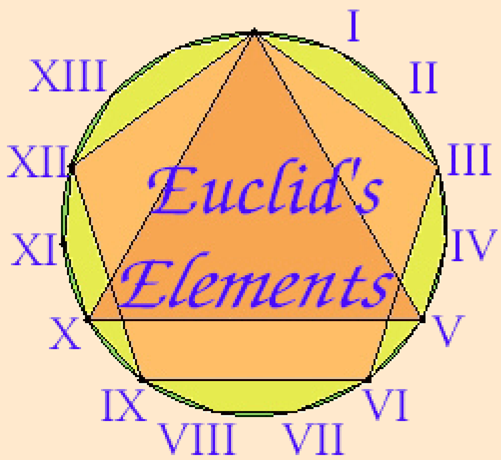
Book VI



Book VI

Table of contents

Definitions

Definition 1.
Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.

Definition 2.
Two figures are *reciprocally related* when the sides about corresponding angles are reciprocally proportional.

Definition 3.
A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less.

Definition 4.
The *height* of any figure is the perpendicular drawn from the vertex to the base.

Propositions

Proposition 1.
Triangles and parallelograms which are under the same height are to one another as their bases.

Proposition 2.
If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle.

Proposition 3.

If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle.

Proposition 4.
In equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.

Proposition 5.
If two triangles have their sides proportional, then the triangles are equiangular with the equal angles opposite the corresponding sides.

Proposition 6.
If two triangles have one angle equal to one angle and the sides about the equal angles proportional, then the triangles are equiangular and have those angles equal opposite the corresponding sides.

Proposition 7.
If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, then the triangles are equiangular and have those angles equal the sides about which are proportional.

Proposition 8.
If in a right-angled triangle a perpendicular is drawn from the right angle to the base, then the triangles adjoining the perpendicular are similar both to the whole and to one another.

Corollary. If in a right-angled triangle a perpendicular is drawn from the right angle to the base, then the straight line so drawn is a mean proportional between the segments of the base.

Proposition 9.
To cut off a prescribed part from a given straight line.

Proposition 10.
To cut a given uncut straight line similarly to a given cut straight line.

Proposition 11.
To find a third proportional to two given straight lines.

Proposition 12.
To find a fourth proportional to three given straight lines.

Proposition 13.
To find a mean proportional to two given straight lines.

Proposition 14.
In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Proposition 15.
In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

Proposition 16.
If four straight lines are proportional, then the rectangle contained by the extremes equals the rectangle contained

by the means; and, if the rectangle contained by the extremes equals the rectangle contained by the means, then the four straight lines are proportional.

Proposition 17.
If three straight lines are proportional, then the rectangle contained by the extremes equals the square on the mean; and, if the rectangle contained by the extremes equals the square on the mean, then the three straight lines are proportional.

Proposition 18.
To describe a rectilinear figure similar and similarly situated to a given rectilinear figure on a given straight line.

Proposition 19.
Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Corollary. If three straight lines are proportional, then the first is to the third as the figure described on the first is to that which is similar and similarly described on the second.

Proposition 20.
Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.

Corollary. Similar rectilinear figures are to one another in the duplicate ratio of the corresponding sides.

Proposition 21.
Figures which are similar to the same rectilinear figure are also similar to one another.

Proposition 22.
If four straight lines are proportional, then the rectilinear figures similar and similarly described upon them are also proportional; and, if the rectilinear figures similar and similarly described upon them are proportional, then the straight lines are themselves also proportional.

Proposition 23.
Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Proposition 24.
In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Proposition 25.
To construct a figure similar to one given rectilinear figure and equal to another.

Proposition 26.
If from a parallelogram there is taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, then it is about the same diameter with the whole.

Proposition 27.
Of all the parallelograms applied to the same straight line falling short by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the difference.

Proposition 28.
To apply a parallelogram equal to a given rectilinear figure to a given straight line but falling short by a parallelogram similar to a given one; thus the given rectilinear figure must not be greater than the parallelogram described on the half of the straight line and similar to the given parallelogram.

Proposition 29.
To apply a parallelogram equal to a given rectilinear figure to a given straight line but exceeding it by a parallelogram similar to a given one.

Proposition 30.
To cut a given finite straight line in extreme and mean ratio.

Proposition 31.
In right-angled triangles the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle.

Proposition 32.
If two triangles having two sides proportional to two sides are placed together at one angle so that their corresponding sides are also parallel, then the remaining sides of the triangles are in a straight line.

Proposition 33.
Angles in equal circles have the same ratio as the circumferences on which they stand whether they stand at the centers or at the circumferences.

Logical structure of Book VI

Proposition VI.1 is the basis for the entire of Book VI except the last proposition VI.33. Only these two propositions directly use the definition of proportion in Book V. Proposition VI.1 constructs a proportion between lines and figures while VI.33 constructs a proportion between angles and circumferences. The intervening propositions use other properties of proportions developed in Book V, but they do not construct new proportions using the definition of proportion.

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Previous: Book V

Book VI introduction

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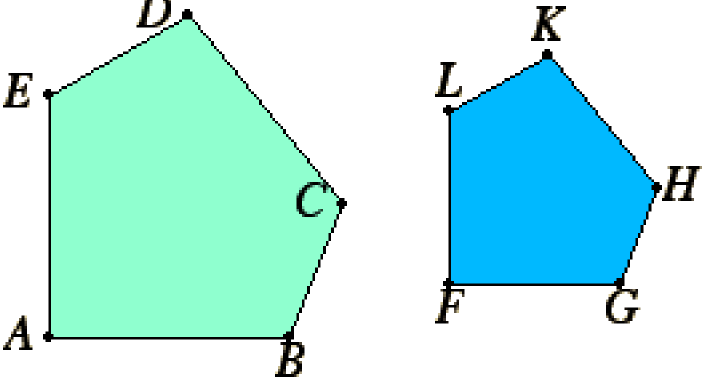
D
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C
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K
L
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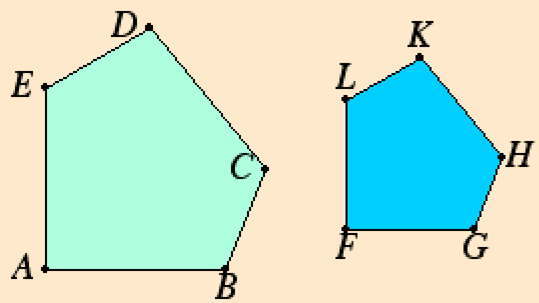
F
G

Euclid's Elements
Book VI
Definition 1

Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.

Guide

The words in this definition do not quite express its entire intent. It is apparent from its use that the notion of similarity assumes a specific correspondence of consecutive vertices and sides. Consider, for instance, pentagons.



In order for the pentagons *ABCDE* and *FGHKL* to be similar, it is required that

1. corresponding angles taken in order are equal, that is, $A = F$, $B = G$, $C = H$, $D = K$, and $E = L$, and
2. the sides about their equal angles are proportional in the same order:

$$EA:AB = LF:FG,$$
$$AB:BC = FG:GH,$$
$$BC:CD = GH:HK,$$
$$CD:EF = HK:KL, \text{ and}$$
$$EA:AB = KL:LF.$$

It wouldn't be allowed, for instance, if the angles of one figure equalled the angles of the other, but in some haphazard order. And it wouldn't be allowed for the orders of the terms in the proportions to be permuted, or inverted, for instance, the second proportion could not be $AB:BC = GH:FG$.

Use of this definition

Propositions VI.4 and VI.5 give two criteria for two triangles to be similar. Proposition VI.4 says that condition 1 implies similarity, while VI.5 says condition 2 implies similarity. Proposition VI.6 is a side-angle-side similarity theorem, and VI.7 is a side-side-angle similarity theorem. Many of the other propositions in this and later books involve similarity in one way or another.

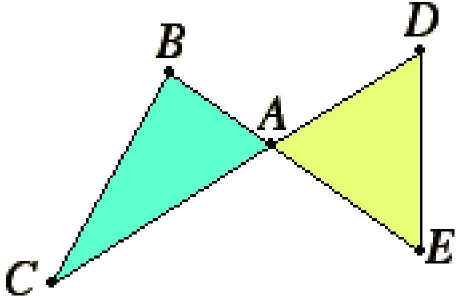
Next definition: VI.Def.2

Select from Book VI

Book VI introduction

Select book

Select topic



B
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E
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Euclid's Elements

Book VI

Definition 2

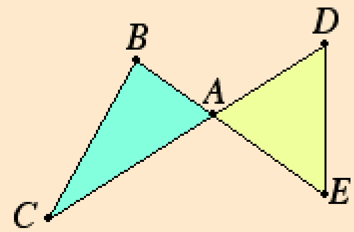
Two figures are reciprocally related when the sides about corresponding angles are reciprocally proportional.

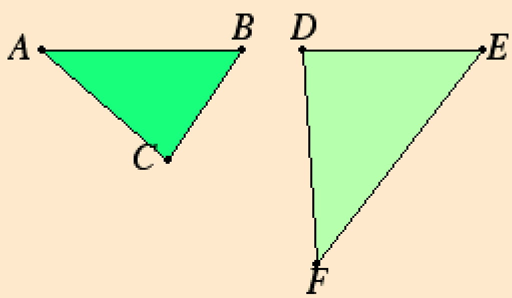
Guide

This isn't the actual definition that appears, but an approximation of its intent. A literal translation is incomplete, and this definition may have been added after Euclid.

The intention can be seen in proposition VI.15 as illustrated here.

The proposition states that if two triangles have one angle equal to one angle, then the triangles are equal if and only if the sides about the equal angles are reciprocally proportional. In the figure *BAC* and *DAE* are equal angles. So the two triangles are equal if and only if *CA*:*AD* = *EA*:*AB*. Euclid doesn't define the term "reciprocally proportional," but the meaning of the term is clear from its use.





Although Euclid doesn't address the question, it would be interesting to characterize which triangles are reciprocally related, as shown to the left. The conditions are that *AB*:*DE* = *DF*:*AC*, *BC*:EF = *DE*:*AB*, and *AC*:*DF* = *DF*:*BC*. Multiplicatively, *AB DE* = *BC EF* = *AC DF*.

Next definition: VI.Def.3

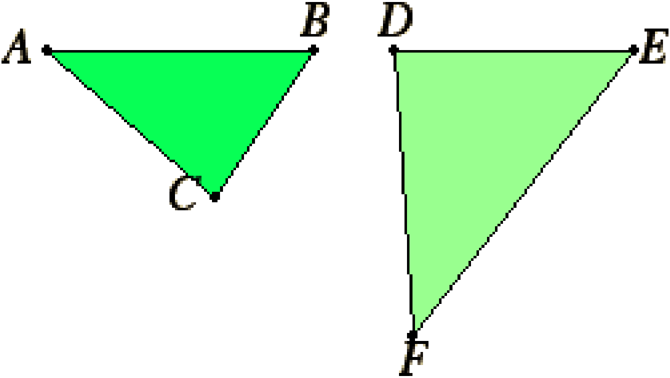
Previous: VI.Def.1

Book VI introduction

Select from Book VI

Select book

Select topic


A
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Euclid's Elements
Book VI
Definition 3

A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less.

Guide

The line *AB* is cut in extreme and mean ratio at *C* since *AB*:*AC* = *AC*:*CB*.

A construction to cut a line in this manner first appeared in Book II, proposition II.11. Of course that was before ratios were defined, and there an equivalent condition was stated in terms of rectangles, namely, that the square on *AC* equal the rectangle *AB* by *BC*. That construction was later used in Book IV in order to construct regular pentagons and 15-sided polygons (propositions IV.10 through 12 and 16).

Now that the theory of ratios and proportions has been developed, it is time to define this section as a ratio, rather than using rectangles. An alternate construction is given in proposition VI.30.

Next definition: VI.Def.4

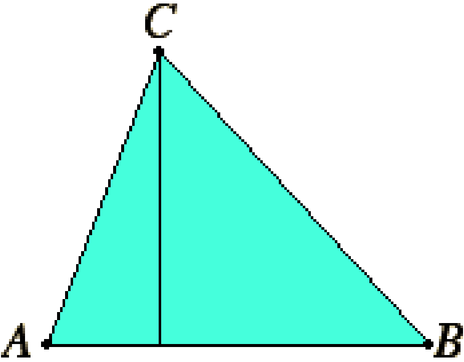
Previous: VI.Def.3

Book VI introduction

Select from Book VI

Select book

Select topic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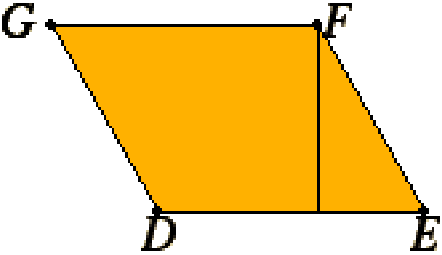
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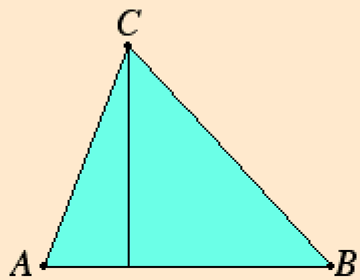
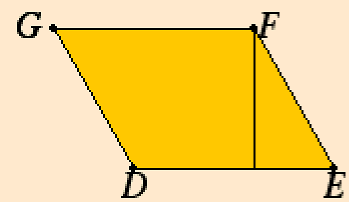
Euclid's Elements
Book VI
Definition 4

The *height* of any figure is the perpendicular drawn from the vertex to the base.

Guide



Evidently, what is meant by "vertex" is the highest point in the figure, or a highest point when many points are equally high. But to define "height" in terms of "highest point" would be a circular definition. Indeed, this definition only suggests what "height" might mean without defining it at all. Still, by the way the term is used in the *Elements,* we can determine its meaning. The only planar figures where heights are used in the *Elements* are triangles and parallelograms.



If the figure is a triangle, and one side has been declared the base, then the height is the expected line, the line drawn from the opposite vertex perpendicular to the base. If the figure is a parallelogram, and one side has been declared the base, then the height may be taken to be a perpendicular from either of the two vertices not on the base.

In the later books on solid geometry, other figures also can have bases and heights such as parallelepipeds, pyramids, prisms, cones, and cylinders.

Different sides of a figure may be selected as the base depending on the application. In proposition XI.39 there are two triangular prisms. A triangle is chosen taken to be the base of one, while the base of the other is a parallelogram. The height of the first is a perpendicular drawn between two triangular opposite faces, but the height of the other is a perpendicular drawn between the parallelogram taken as the base and the opposite parallelogram.

Next proposition: VI.1

Previous: VI.Def.3

Book VI introduction

Select from Book VI

Select book

Select topic

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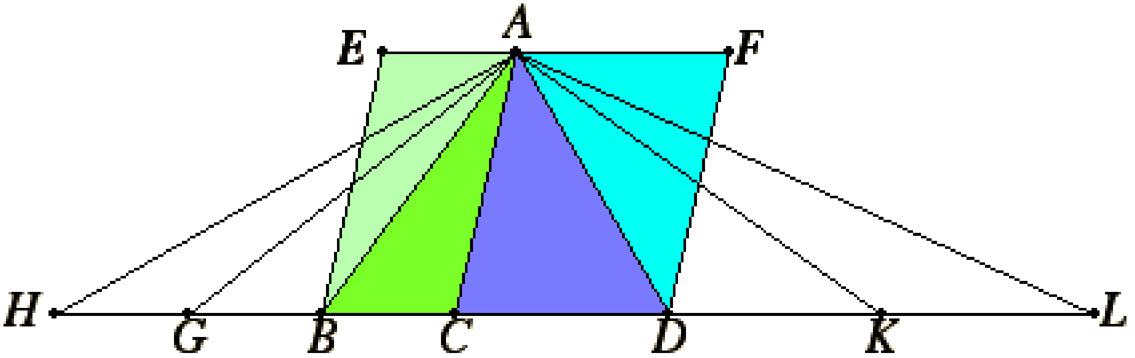
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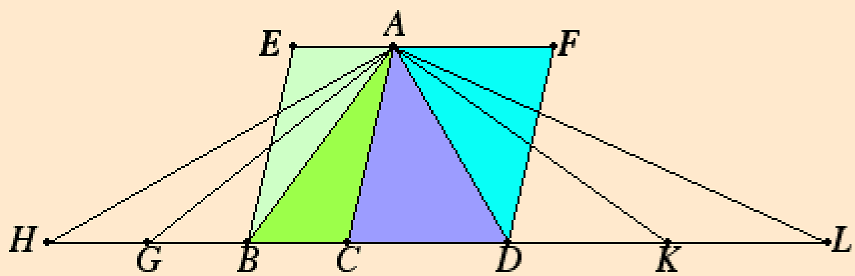
Euclid's Elements
Book VI
Proposition 1

Triangles and parallelograms which are under the same height are to one another as their bases.

Let *ACB* and *ACD* be triangles, and let *CE* and *CF* be parallelograms under the same height.

I say that the base *CB* is to the base *CD* as the triangle *ACB* is to the triangle *ACD*, and as the parallelogram *CE* is to the parallelogram *CF*.

Produce *BD* in both directions to the points *H* and *L*. Make any number of straight lines *BG* and *GH* equal to the base *CB*, and any number of straight lines *DK* and *KL* equal to the base *CD*. Join *AG*, *AH*, *AK*, and *AL*. I.3



Then, since *CB*, *BG*, and *GH* equal one another, the triangles *ACB*, *ABG*, and *AGH* also equal one another. I.38

Therefore, whatever multiple the base *CH* is of the base *CB*, the triangle *ACH* is also that multiple of the triangle *ACB*.

For the same reason, whatever multiple the base *CL* is of the base *CD*, the triangle *ACL* is also that multiple of the triangle *ACD*. And, if the base *CH* equals the base *CL*, then the triangle *ACH* also equals the triangle *ACL*; if the base *CH* is in excess of the base *CL*, the triangle *ACH* is also in excess of the triangle *ACL*; and, if less, less. I.38

Thus, there being four magnitudes, namely two bases *CB* and *CD*, and two triangles *ACB* and *ACD*, equimultiples have been taken of the base *CB* and the triangle *ACB*, namely the base *CH* and the triangle *ACH*, and other, arbitrary, equimultiples of the base *CD* and the triangle *ADC*, namely the base *CL* and the triangle *ACL*, and it has been proved that, if the base *CH* is in excess of the base *CL*, the triangle *ACH* is also in excess of the triangle *ACL*; if equal, equal; and, if less, less. Therefore the base *CB* is to the base *CD* as the triangle *ACB* is to the triangle *ACD*. V.Def.5

Next, since the parallelogram *CE* is double the triangle *ACB*, and the parallelogram *FC* is double the triangle *ACD*, and parts have the same ratio as their equimultiples, therefore the triangle *ACB* is to the triangle *ACD* as the parallelogram *CE* is to the parallelogram *FC*. I.41 V.15

Since, then, it was proved that the base *CB* is to *CD* as the triangle *ACB* is to the triangle *ACD*, and the triangle *ACB* is to the triangle *ACD* as the parallelogram *CE* is to the parallelogram *CF*, therefore also the base *CB* is to the base *CD* as the parallelogram *CE* is to the parallelogram *FC*. V.11

Therefore, *triangles and parallelograms which are under the same height are to one another as their bases.*

Q.E.D.

Guide

In a more proper setting out of the proposition, the triangles under the same height would not have a common side, and the parallelograms would not have a common base and side with the triangles. Since triangles on equal bases and in the same parallels are equal (I.36), and parallelograms on equal bases and in the same parallels are equal (I.35), and equals may be substituted in proportions (V.7), Euclid's simplified setting out is sufficient. Nonetheless, a proper setting out does not require a more complicated proof.

The goal of the proof is to show that three ratios, namely the ratio of the lines *CB* to *CD*, the ratio of the triangles *ACB* to *ACD*, and the ratio of the parallelograms *CE* to *CF*, are all the same ratio. That is

$$CB:CD = ACB:ACD = CE:CF.$$

The first stage of the proof shows that *CB*:*CD* = *ACB*:*ACD*. By the definition of proportion, V.Def.5, that means for any number *m* and any number *n* that

$$m\ BC >=< n\ CD \text{ when } m\ ABC >=< n\ ACD.$$

Note that Euclid takes both *m* and *n* to be 3 in his proof. Now *m BC* equals the line *CH, n CD* equals the line *CL, m ABC* equals the triangle *ACH*, and *n ACD* equals the triangle *ACL.* So what has to be shown is that

$$CH >=< CL \text{ when } ACH >=< ACL.$$

But that follows from proposition I.38. So the first stage of the proof is complete.

The second stage is easier. Since the parallelograms are twice the triangles, they also have the same ratio.

Other propositions that state fundamental proportions use the same outline for their proofs. Proposition VI.33: arcs of circles are proportional to angles on which they stand; XI.25: parallelepipeds are proportional to their bases; and XII.13: cylinders are proportional to their axes.

On the method of modern analysis

Heath remarked that "some American and German text-books adopt the less rigorous method of appealing to the theory of limits" for the foundation for the theory of proportion used here in geometry. Heath preferred Eudoxus' theory of proportion in Euclid's Book V as a foundation.

It is remarkable how much mathematics has changed over the last century. In the beginning of the 20th century Heath could still gloat over the superiority of synthetic geometry, although he may have been one of the last to do so. Now, in the 21st century, synthetic geometry has receded into near oblivion while analysis, based on various concepts of limits, is preeminent.

It took some time to find a foundation for mathematical analysis as solid, or more solid, than geometry. In the 17th century, the time of the creation of differential and integral calculus, geometry was seen as the most dependable justification for calculus. In the first half of the 19th century, the concept of limit was clarified and limits became the foundation of mathematical analysis. Heath's complaint would have been valid then since the theory of real numbers was still without any foundation except a geometric one, which, ultimately was based on Eudoxus' theory of proportion in Euclid's Book V. In the later 19th century Weierstrass, Cantor, and Dedekind succeeded in founding the theory of real numbers on that of natural numbers and a bit of set theory, so that by the beginning of the 20th century, there was a modern foundation for mathematical analysis. All the same, this new foundation could still be called Eudoxus' since the modern definition of real number is the same as his, but in a modern guise.

Use of this proposition

This is one of the most used propositions in the *Elements*. It is used frequently in Book VI starting with the next proposition, dozens of times in Book X, and and a few times in Books XI and XIII.

Next proposition: VI.2

Previous: VI.Def.4

Book VI introduction

Select from Book VI

Select book

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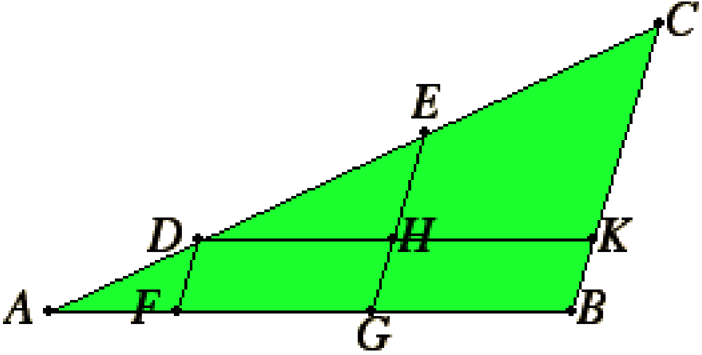

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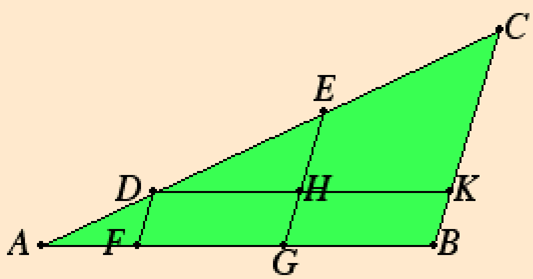
Euclid's Elements
Book VI
Proposition 10

To cut a given uncut straight line similarly to a given cut straight line.

Let *AB* be the given uncut straight line, and *AC* the straight line cut at the points *D* and *E,* and let them be so placed as to contain any angle. Join *CB,* and draw *DF* and *EG* through *D* and *E* parallel to *CB,* and draw *DHK* through *D* parallel to *AB.* I.31



Therefore each of the figures *FH* and *HB* is a parallelogram. Therefore *DH* equals *FG* and *HK* equals *GB.* I.34

Now, since the straight line *EH* is parallel to a side *CK* of the triangle *DCK,* therefore, proportionally, *DE* is to *EC* as *DH* is to *HK.* VI.2

But *DH* equals *FG,* and *HK* equals *GB,* therefore *DE* is to *EC* as *FG* is to *GB.* V.7

Again, since *DF* is parallel to a side *EG* of the triangle *AEG,* therefore, proportionally, *AD* is to *DE* as *AF* is to *FG.* VI.2

But it was also proved that *DE* is to *EC* as *FG* is to *GB,* therefore *DE* is to *EC* as *FG* is to *GB,* and *AD* is to *DE* as *AF* is to *FG.*

Therefore the given uncut straight line *AB* has been cut similarly to the given cut straight line *AC.*

Q.E.F.

Guide

In a sense, this proposition is a generalization of the last one VI.9. Prop. VI.9 cut a line into two parts whose ratio was a given numerical ratio. This proposition cuts a line into two parts whose ratio is a given ratio of two other lines. Both propositions rely on VI.2 as a basis to make any conclusion about the ratio of two lines.

Use of this construction

This proposition is not used later in the *Elements,* but it is a basic construction of geometry.

Next proposition: VI.11

Previous: VI.9

Book VI introduction

Select from Book VI

Select book

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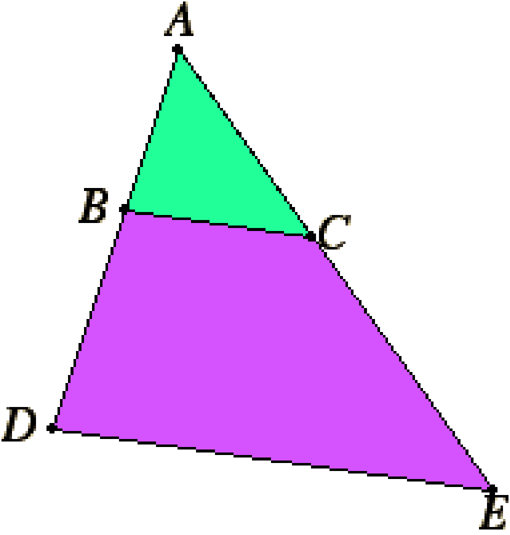
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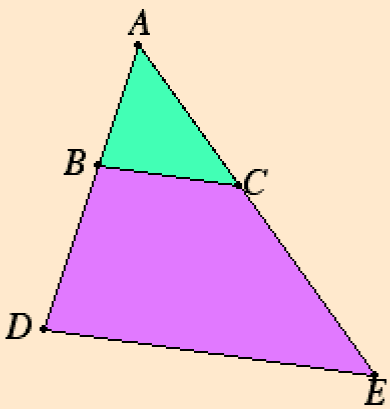
Euclid's Elements
Book VI
Proposition 11

To find a third proportional to two given straight lines.

Let *AB* and *AC* be the two given straight lines, and let them be placed so as to contain any angle.

It is required to find a third proportional to *AB* and *AC*.



Produce them to the points *D* and *E*, and make *BD* equal to *AC*. Join *BC*, and draw *DE* through *D* parallel to it. I.3 I.31

Then since *BC* is parallel to a side *DE* of the triangle *ADE*, therefore, proportionally, *AB* is to *BD* as *AC* is to *CE*. VI.2

But *BD* equals *AC*, therefore *AB* is to *AC* as *AC* is to *CE*. V.7

Therefore a third proportional *CE* has been found to two given straight lines *AB* and *AC*.

Q.E.F.

Guide

If *a* and *b* are two magnitudes, then their third proportional is a magnitude *c* such that $a:b = b:c$. The third proportional is needed whenever a duplicate ratio is needed when the ratio itself is known. The duplicate ratio for $a:b$ is $a:c$.

Use of this proposition

This construction is used in propositions VI.19, VI.22, and a few propositions in Book X.

Next proposition: VI.12

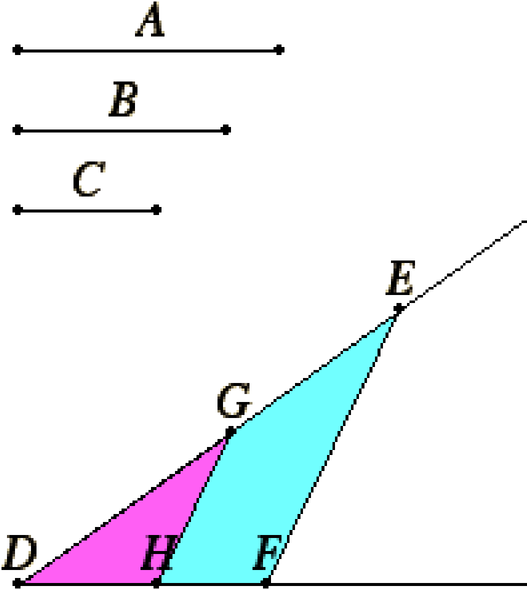
Previous: VI.10

Book VI introduction

Select from Book VI

Select book

Select topic



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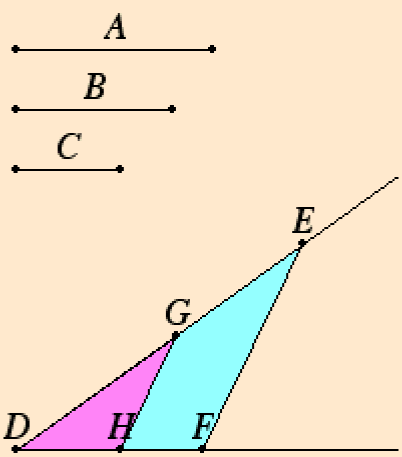
Euclid's Elements
Book VI
Proposition 12

To find a fourth proportional to three given straight lines.

Let *A* and *B* and *C* be the three given straight lines.

It is required to find a fourth proportional to *A*, *B*, and *C*.



Set out two straight lines *DE* and *DF* containing any angle *EDF*. Make *DG* equal to *A*, *GE* equal to *B*, and *DH* equal to *C*. Join *GH*, and draw *EF* through *E* parallel to it. I.3 I.31

Then since *GH* is parallel to a side *EF* of the triangle *DEF*, therefore *DG* is to *GE* as *DH* is to *HF*. VI.2

But *DG* equals *A* and *GE* to *B*, and *DH* to *C*, therefore *A* is to *B* as *C* is to *HF*. V.7

Therefore a fourth proportional *HF* has been found to the three given straight lines *A*, *B*, and *C*.

Q.E.F.

Guide

Of course, the previous proposition is a special case of this one.

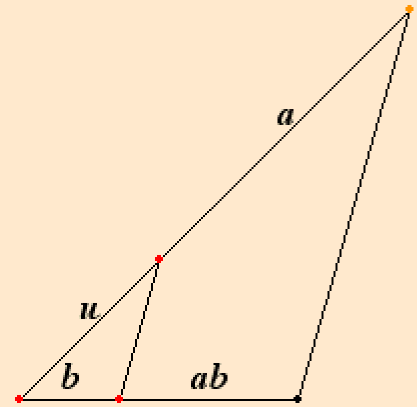
Descartes' geometric algebra

Descartes (1591-1661) is well known for his coordinate geometry which he and Fermat developed in the 16th century. This subject, also called analytic geometry, places an *x*-*y*-coordinate system on a plane so that a curve in the plane corresponds to an equation in two variables *x* and *y*. The usual way this correspondence is used is to convert a problem in geometry into an algebraic problem about equations.

Descartes was equally interested is using geometry to solve algebraic problems, but using a method quite distinct from that in the *Elements* which began in Book II.

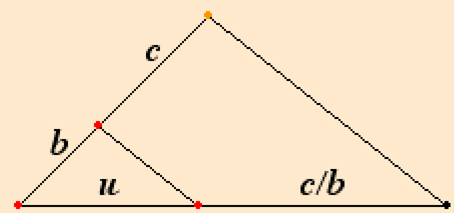
His idea was to take an equation in one variable and find a geometric figure which can be used to solve the equation. The idea wasn't particularly new as even Menaechmus (fl. about 350 B.C.E) had about 50 years before Euclid intersected two parabolas to find cube roots, which are solutions to particular cubic equations. Furthermore, about 1100, Omar Khayyam solved all cubic equations by means of parabolas and hyperbolas. But Descartes was systematic and was able to use the relatively recent invention of symbolic algebra to make more connections.

Descartes began by interpreting the algebraic operations of addition, subtraction, multiplication, division, and extraction of square roots as geometric constructions on lines. He represented each (positive) magnitude by a line. Addition and subtraction were the same as Euclid's. To add two lines, just extend one by the length of the other. To subtract one line from another, just take the remainder after cutting it off the other. Multiplication and division, however, were different from Euclid's. Euclid represented the product of two lines by a rectangle, the product of three lines by a box in space, and Euclid didn't represent the product of four lines.



But Descartes took the product of two lines to be another line. That required selecting a unit line, that is, a line of length 1. Then to find the product *ab* of two quantities *a* and *b,* he only needed to find the fourth proportional of 1, *a*, and *b*. This proposition VI.10 does that.

In the diagram to the right, *u* is the unit line, *a* and *b* are to be multiplied, and *ab* is their product, the fourth proportional.



This same proposition works to construct the quotient of two quantities. If *b* and *c* are two quantities, then the fourth proportional for *b, c,* and 1 is the quotient *c/b.*

Descartes achieved the fifth operation, extraction of square roots, by means of the semicircle and right angle construction described in the next proposition VI.13.

Use of this proposition

This proposition is used in the proofs of VI.22, VI.23, and half a dozen propositions in Book X.

Next proposition: VI.13

Previous: VI.11

Book VI introduction

Select from Book VI

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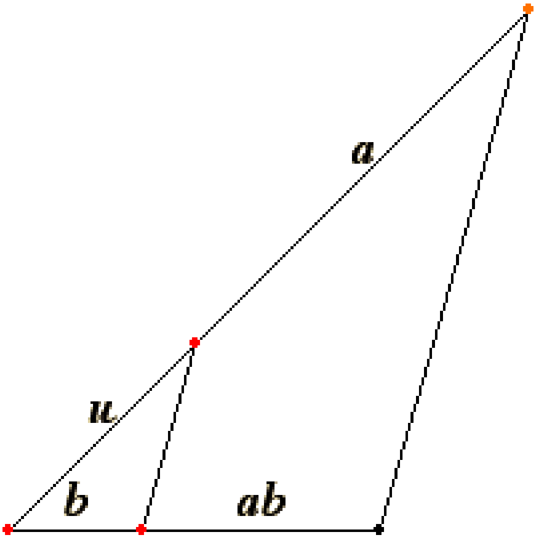

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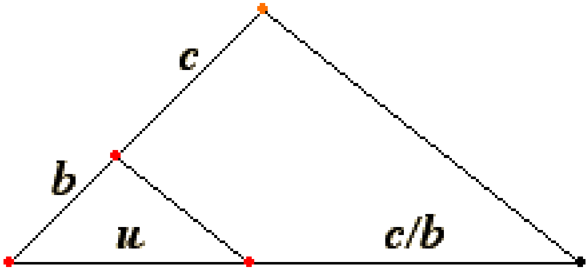
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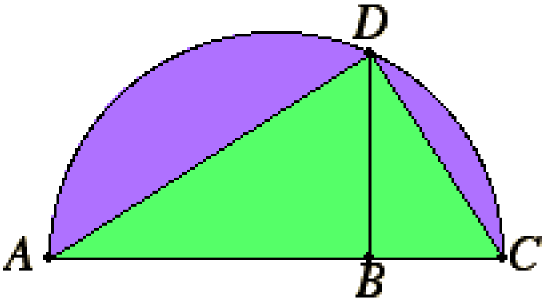

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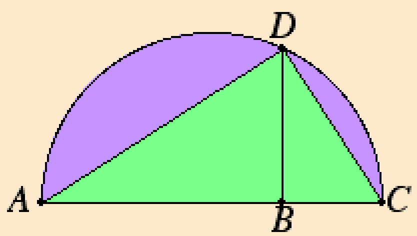
Euclid's Elements
Book VI
Proposition 13

To find a mean proportional to two given straight lines.

Let *AB* and *BC* be the two given straight lines.

It is required to find a mean proportional to *AB* and *BC.*



Place them in a straight line, and describe the semicircle *ADC* on *AC*. Draw *BD* from the point *B* at right angles to the straight line *AC*, and join *AD* and *DC.* I.11

Since the angle *ADC* is an angle in a semicircle, it is right. III.31

And, since, in the right-angled triangle *ADC, BD* has been drawn from the right angle perpendicular to the base, therefore *BD* is a mean proportional between the segments of the base, *AB* and *BC.* VI.8,Cor

Therefore a mean proportional *BD* has been found to the two given straight lines *AB* and *BC.*

Q.E.F.

Guide

This construction of the mean proportional was used before in II.4 to find a square equal to a given rectangle. By proposition VI.17 coming up, the two constructions are equivalent. That is the mean proportional between two lines is the side of a square equal to the rectangle contained by the two lines. Algebraically, $a : x = x : b$ if and only if $ab = x^2$. Thus, x is the square root of ab.

When b is taken to have unit length, this construction gives the construction for the square root of a.

Use of this proposition

This construction is used in the proofs of propositions VI.25, X.27, and X.28.

Next proposition: VI.14

Previous: VI.12

Book VI introduction

Select from Book VI

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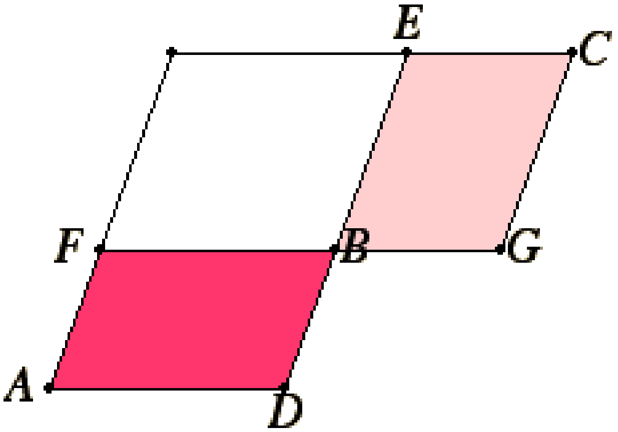

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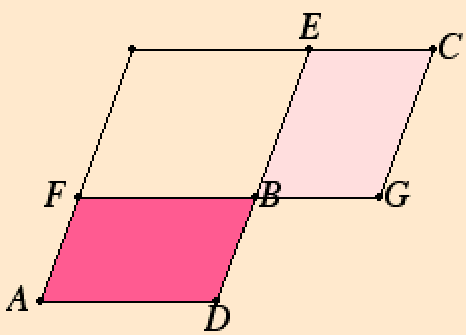
Euclid's Elements
Book VI
Proposition 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let *AB* and *BC* be equal and equiangular parallelograms having the angles at *B* equal, and let *DB* and *BE* be placed in a straight line. Therefore *FB* and *BG* are also in a straight line. I.14

I say that, in *AB* and *BC*, the sides about the equal angles are reciprocally proportional, that is to say, *DB* is to *BE* as *BG* is to *BF*.



Complete the parallelogram *FE*. I.31

Then since the parallelogram *AB* equals the parallelogram *BC*, and *FE* is another parallelogram, therefore *AB* is to *FE* as *BC* is to *FE*. V.7

But *AB* is to *FE* as *DB* is to *BE*, and *BC* is to *FE* as *BG* is to *BF*. Therefore *DB* is to *BE* as *BG* is to *BF*. VI.1 V.11

Therefore in the parallelograms *AB* and *BC* the sides about the equal angles are reciprocally proportional.

Next, let *DB* be to *BE* as *BG* is to *BF*.

I say that the parallelogram *AB* equals the parallelogram *BC*.

Since *DB* is to *BE* as *BG* is to *BF*, while *DB* is to *BE* as the parallelogram *AB* is to the parallelogram *FE*, and, *BG* is to *BF* as the parallelogram *BC* is to the parallelogram *FE*, therefore also *AB* is to *FE* as *BC* is to *FE*. VI.1 V.11

Therefore the parallelogram *AB* equals the parallelogram *BC*. V.9

Therefore, *in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.*

Q.E.D.

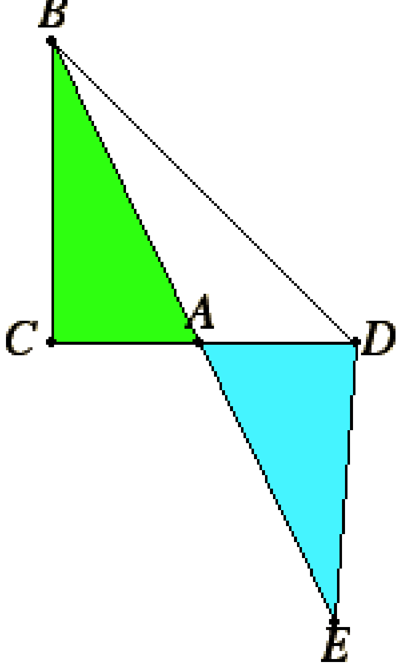
Guide

This proposition is used in the proofs of propositions VI.16, VI.30, and X.22.

Next proposition: VI.15

Select from Book VI

Previous: VI.13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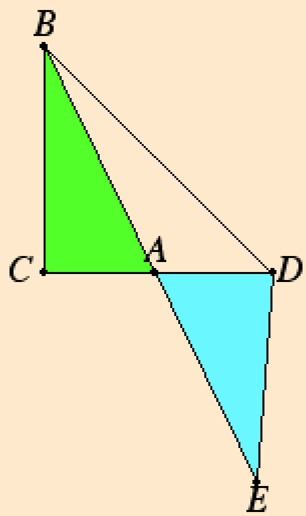
Euclid's Elements
Book VI
Proposition 15

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

Let *ABC* and *ADE* be equal triangles having one angle equal to one angle, namely the angle *BAC* equal to the angle *DAE.*

I say that in the triangles *ABC* and *ADE* the sides about the equal angles are reciprocally proportional, that is to say, that *CA* is to *AD* as *EA* is to *AB.*



Place them so that *CA* is in a straight line with *AD.* Therefore *EA* is also in a straight line with *AB.* I.14

Join *BD.*

Since then the triangle *ABC* equals the triangle *ADE*, and *ABD* is another triangle, therefore the triangle *ABC* is to the triangle *ABD* as the triangle *ADE* is to the triangle *ABD.* V.7

But *ABC* is to *ABD* as *AC* is to *AD*, and *ADE* is to *ABD* as *AE* is to *AB.* VI.1

Therefore also *AC* is to *AD* as *AE* is to *AB.* V.11

Therefore in the triangles *ABC* and *ADE* the sides about the equal angles are reciprocally proportional.

Next, let the sides of the triangles *ABC* and *ADE* be reciprocally proportional, that is to say, let *AE* be to *AB* as *CA* is to *AD.*

I say that the triangle *ABC* equals the triangle *ADE.*

If *BD* is again joined, since *AC* is to *AD* as *AE* is to *AB*, while *AC* is to *AD* as the triangle *ABC* is to the triangle *ABD*, and *AE* is to *AB* as the triangle *ADE* is to the triangle *ABD*, therefore the triangle *ABC* is to the triangle *ABD* as the triangle *ADE* is to the triangle *ABD.* VI.1 V.11

Therefore each of the triangles *ABC* and *ADE* has the same ratio to *ABD.*

Therefore the triangle *ABC* equals the triangle *ADE.* V.9

Therefore, *in equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides* *about the equal angles are reciprocally proportional, are equal.*

Q.E.D.

Guide

This proposition is used in the proof of proposition VI.19.

Next proposition: VI.16

Previous: VI.14

Book VI introduction

Select from Book VI

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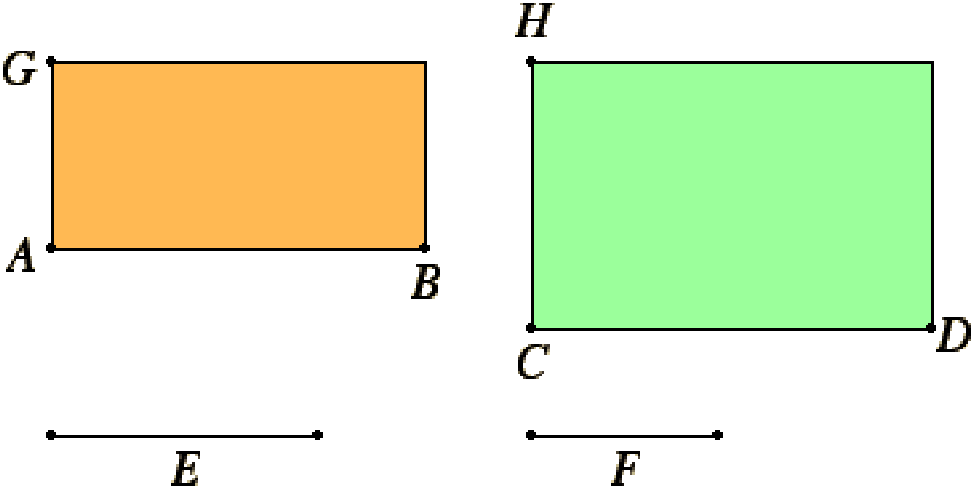
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Euclid's Elements
Book VI
Proposition 16

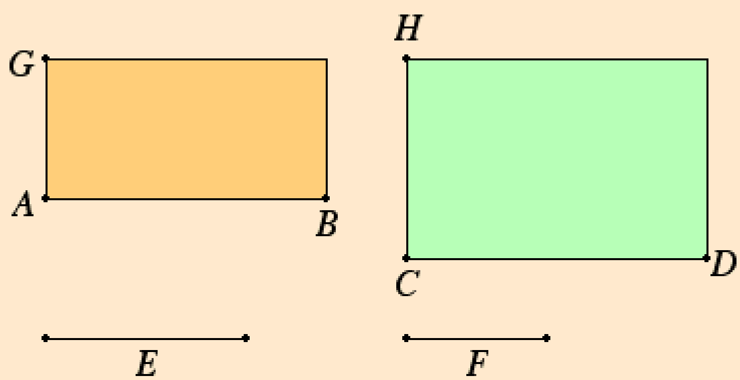
If four straight lines are proportional, then the rectangle contained by the extremes equals the rectangle contained by the means; and, if the rectangle contained by the extremes equals the rectangle contained by the means, then the four straight lines are proportional.

Let the four straight lines *AB, CD, E,* and *F* be proportional, so that *AB* is to *CD* as *E* is to *F*.

I say that the rectangle *AB* by *F* equals the rectangle *CD* by *E.*

Draw *AG* and *CH* from the points *A* and *C* at right angles to the straight lines *AB* and *CD,* and make *AG* equal to *F,* and *CH* equal to *E.* I.11 I.3

Complete the parallelograms *BG* and *DH.* I.31



Then since *AB* is to *CD* as *E* is to *F,* while *E* equals *CH,* and *F* equals *AG,* therefore *AB* is to *CD* as *CH* is to *AG.* V.7

Therefore in the parallelograms *BG* and *DH* the sides about the equal angles are reciprocally proportional.

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal, therefore the parallelogram *BG* equals the parallelogram *DH.* VI.14

And *BG* is the rectangle *AB* by *F,* for *AG* equals *F,* and *DH* is the rectangle *CD* by *E,* for *E* equals *CH,* therefore the rectangle *AB* by *F* equals the rectangle *CD* by *E.*

Next, let the rectangle *AB* by *F* be equal to the rectangle *CD* by *E.*

I say that the four straight lines are proportional, so that *AB* is to *CD* as *E* is to *F.*

With the same construction, since the rectangle *AB* by *F* equals the rectangle *CD* by *E,* and the rectangle *AB* by *F* is *BG,* for *AG* equals *F,* and the rectangle *CD* by *E* is *DH,* for *CH* equals *E,* therefore *BG* equals *DH.*

And they are equiangular. But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. VI.14

Therefore *AB* is to *CD* as *CH* is to *AG.* V.7

But *CH* equals *E*, and *AG* to *F*, therefore *AB* is to *CD* as *E* is to *F*.

Therefore, *if four straight lines are proportional, then the rectangle contained by the extremes equals the rectangle contained by the means; and, if the rectangle contained by the extremes equals the rectangle contained by the means, then the four straight lines are proportional.*

Q.E.D.

Guide

This proposition is a special case of VI.14. It hardly needs such a protracted proof. It is used occasionally in Book X, but the special case when the means are equal and the second figure is a square, as enunciated in the next proposition, is used throughout Book X and frequently in Book XIII.

Next proposition: VI.17

Previous: VI.16

Book VI introduction

Select from Book VI

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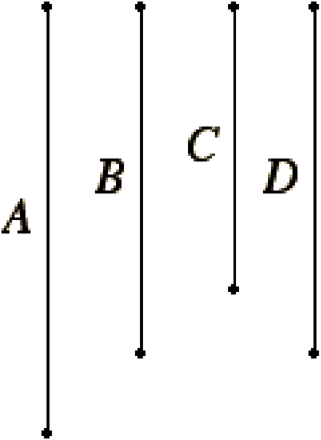
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Euclid's Elements

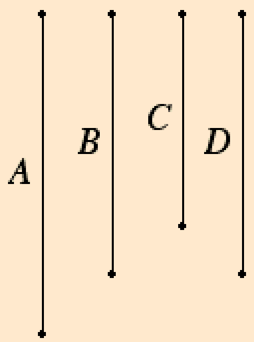
Book VI

Proposition 17

If three straight lines are proportional, then the rectangle contained by the extremes equals the square on the mean; and, if the rectangle contained by the extremes equals the square on the mean, then the three straight lines are proportional.

Let the three straight lines *A* and *B* and *C* be proportional, so that *A* is to *B* as *B* is to *C*.

I say that the rectangle *A* by *C* equals the square on *B*.



Make *D* equal to *B*. I.3

Then, since *A* is to *B* as *B* is to *C*, and *B* equals *D*, therefore *A* is to *B* as *D* is to *C*. V.7 V.11

But, if four straight lines are proportional, then the rectangle contained by the extremes equals the rectangle contained by the means. VI.16

Therefore the rectangle *A* by *C* equals the rectangle *B* by *D*. But the rectangle *B* by *D* is the square on *B*, for *B* equals *D*, therefore the rectangle *A* by *C* equals the square on *B*.

Next, let the rectangle *A* by *C* equal the square on *B*.

I say that *A* is to *B* as *B* is to *C*.

With the same construction, since the rectangle *A* by *C* equals the square on *B*, while the square on *B* is the rectangle *B* by *D*, for *B* equals *D*, therefore the rectangle *A* by *C* equals the rectangle *B* by *D*.

But, if the rectangle contained by the extremes equals that contained by the means, then the four straight lines are proportional. VI.16

Therefore *A* is to *B* as *D* is to *C*.

But *B* equals *D*, therefore *A* is to *B* as *B* is to *C*.

Therefore, *if three straight lines are proportional, then the rectangle contained by the extremes equals the square on the mean; and, if the rectangle contained by the extremes equals the square on the mean, then the three straight lines are proportional.*

Q.E.D.

Guide

This is obviously a special case of the previous proposition. It is used very frequently in Books X and XIII.

Next proposition: VI.18

Previous: VI.16

Book VI introduction

Select from Book VI

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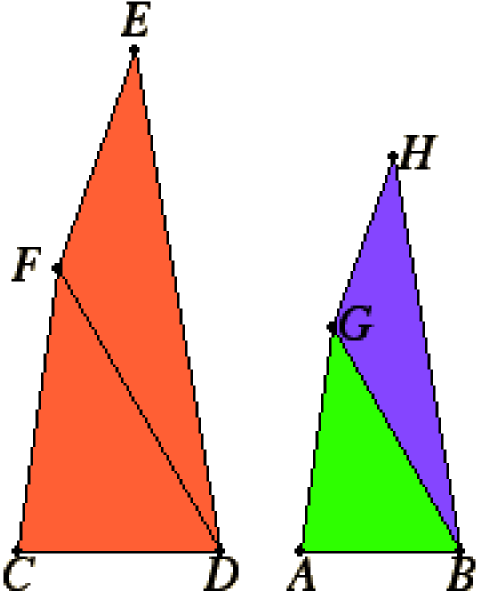
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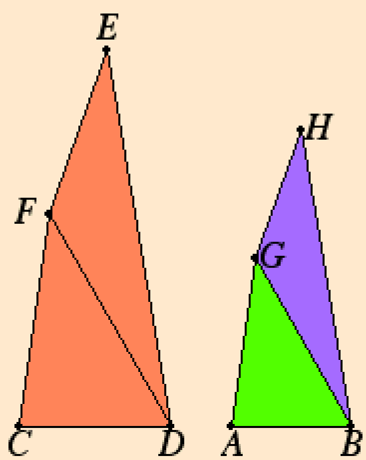
Euclid's Elements
Book VI
Proposition 18

To describe a rectilinear figure similar and similarly situated to a given rectilinear figure on a given straight line.

Let *AB* be the given straight line and *CE* the given rectilinear figure.

It is required to describe on the straight line *AB* a rectilinear figure similar and similarly situated to the rectilinear figure *CE.*



Join *DF*. Construct the angle *GAB* equal to the angle at *C*, and the angle *ABG* equal to the angle *CDF*, on the straight line *AB* at the points *A* and *B* on it. I.23

Therefore the remaining angle *CFD* equals the angle *AGB.* Therefore the triangle *FCD* is equiangular with the triangle *GAB.* I.32

Therefore, proportionally, *FD* is to *GB* as *FC* is to *GA*, and as *CD* is to *AB.* VI.4 V.16

Again, construct the angle *BGH* equal to the angle *DFE,* and the angle *GBH* equal to the angle *FDE*, on the straight line *BG* and at the points *B* and *G* on it. I.23

Therefore the remaining angle at *E* equals the remaining angle at *H.* Therefore the triangle *FDE* is equiangular with the triangle *GBH.* Therefore, proportionally, *FD* is to *GB* as *FE* is to *GH*, and as *ED* is to *HB.* I.32 VI.4 V.16

But it was also proved that *FD* is to *GB* as *FC* is to *GA*, and as *CD* is to *AB.* Therefore *FC* is to *AG* as *CD* is to *AB*, and as *FE* is to *GH*, and further as *ED* is to *HB.* V.11

And, since the angle *CFD* equals the angle *AGB,* and the angle *DFE* equals the angle *BGH,* therefore the whole angle *CFE* equals the whole angle *AGH.*

For the same reason the angle *CDE* also equals the angle *ABH.*

And the angle at *C* also equals the angle at *A*, and the angle at *E* equals the angle at *H.*

Therefore *AH* is equiangular with *CE*, and they have the sides about their equal angles proportional. Therefore the rectilinear figure *AH* is similar to the rectilinear figure *CE.* VI.Def.1

Therefore the rectilinear figure *AH* has been described similar and similarly situated to the given rectilinear figure *CE* on the given straight line *AB.*

Q.E.F.

Guide

It is evident from the diagram, if not from the text, that *AB* should correspond to *CD*.

Although the figure has only four sides, it is clear that the method applies to figures with more than four sides.

This proposition is used in the proofs of propositions VI.22, VI.25, and VI.28, and the corollary is used in XII.17.

Next proposition: VI.19

Previous: VI.17

Book VI introduction

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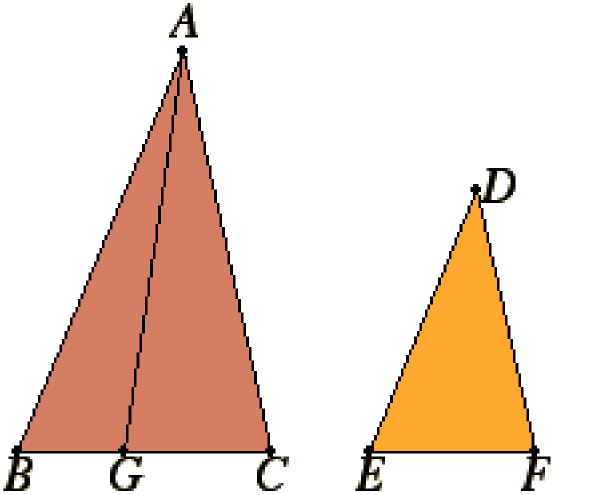
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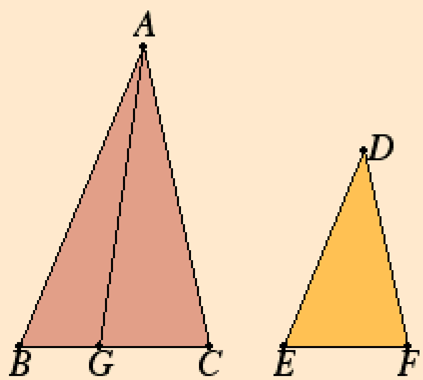
Euclid's Elements
Book VI
Proposition 19

Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Let *ABC* and *DEF* be similar triangles having the angle at *B* equal to the angle at *E*, and such that *AB* is to *BC* as *DE* is to *EF*, so that *BC* corresponds to *EF*. V.Def.11

I say that the triangle *ABC* has to the triangle *DEF* a ratio duplicate of that which *BC* has to *EF*.



Take a third proportional *BG* to *BC* and *EF* so that *BC* is to *EF* as *EF* is to *BG*, and join *AG*. VI.11

Since *AB* is to *BC* as *DE* is to *EF*, therefore, alternately, *AB* is to *DE* as *BC* is to *EF*. V.16

But *BC* is to *EF* as *EF* is to *BG*, therefore also *AB* is to *DE* as *EF* is to *BG*. V.11

Therefore in the triangles *ABG* and *DEF* the sides about the equal angles are reciprocally proportional.

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal. Therefore the triangle *ABG* equals the triangle *DEF*. VI.15

Now since *BC* is to *EF* as *EF* is to *BG*, and, if three straight lines are proportional, the first has to the third a ratio duplicate of that which it has to the second, therefore *BC* has to *BG* a ratio duplicate of that which *BC* has to *EF*. V.Def.9

But *BC* is to *BG* as the triangle *ABC* is to the triangle *ABG*, therefore the triangle *ABC* also has to the triangle *ABG* a ratio duplicate of that which *BC* has to *EF*. VI.1 V.11

But the triangle *ABG* equals the triangle *DEF*, therefore the triangle *ABC* also has to the triangle *DEF* a ratio duplicate of that which *BC* has to *EF*. V.7

Therefore, *similar triangles are to one another in the duplicate ratio of the corresponding sides.*

Q.E.D.

Corollary

From this it is manifest that *if three straight lines are proportional, then the first is to the third as the figure described on the first is to that which is similar and similarly described on the second.*

Guide

This proposition is used in the proof of the next one, and the corollary is used in the proofs of VI.22, VI.31, and X.6.

Next proposition: VI.20

Previous: VI.18

Book VI introduction

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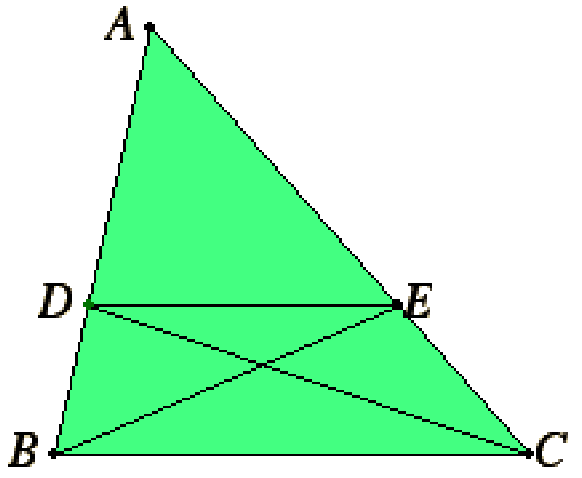
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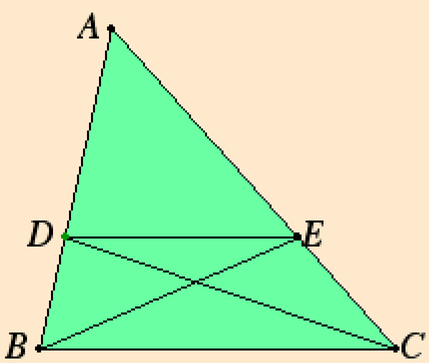
Euclid's Elements
Book VI
Proposition 2

If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle.

Let *DE* be drawn parallel to *BC,* one of the sides of the triangle ABC.

I say that *BD* is to *AD* as *CE* is to *AE.*



Join *BE* and *CD.*

Therefore the triangle *BDE* equals the triangle *CDE,* for they are on the same base *DE* and in the same parallels *DE* and *BC.* I.37

And *ADE* is another triangle.

But equals have the same ratio to the same, therefore the triangle *BDE* is to the triangle *ADE* as the triangle *CDE* is to the triangle *ADE.* V.7

But the triangle *BDE* is to *ADE* as *BD* is to *AD,* for, being under the same height, the perpendicular drawn from *E* to *AB,* they are to one another as their bases. VI.1

For the same reason, the triangle *CDE* is to *ADE* as *CE* is to *AE.*

Therefore *BD* is to *AD* also as *CE* is to *AE.* V.11

Next, let the sides *AB* and *AC* of the triangle *ABC* be cut proportionally, so that *BD* is to *AD* as *CE* is to *AE.* Join *DE.*

I say that *DE* is parallel to *BC.*

With the same construction, since *BD* is to *AD* as *CE* is to *AE,* but *BD* is to *AD* as the triangle *BDE* is to the triangle *ADE,* and *CE* is to *AE* as the triangle *CDE* is to the triangle *ADE,* therefore the triangle *BDE* is to the triangle *ADE* as the triangle *CDE* is to the triangle *ADE.* VI.1 V.11

Therefore each of the triangles *BDE* and *CDE* has the same ratio to *ADE.*

Therefore the triangle *BDE* equals the triangle *CDE,* and they are on the same base *DE.* V.9

But equal triangles which are on the same base are also in the same parallels. I.39

Therefore *DE* is parallel to *BC.*

Therefore, *if a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle.*

Q.E.D.

Guide

Euclid prefers to prove a pair of converses in two stages, but in some propositions, as this one, the proofs in the two stages are almost inverses of each other, so both could be proved at once.

In this proposition we have a given triangle *ABC* and a line *DE* joining a point *D* on the side *BC* to a point *E* on the side *AC*. The claim is that

$$BD:AD = CE:AE \text{ if and only if } DE \parallel BC.$$

By the previous proposition VI.1 we know in any case that

$$BD:AD = \text{triangle } BDE : \text{triangle } ADE, \text{ and}$$
$$CE:AE = \text{triangle } CDE : \text{triangle } ADE.$$

Hence,

$$BD:AD = CE:AE \text{ if and only if } BDE:ADE = CDE:ADE.$$

By propositions V.7 and V.9 the latter condition is equivalent to *BDE* = *CDE*, and that, in turn, by propositions I.37 and I.39 is equivalent to *DE* || *BC*.

Note

It should be noted that a proportion such as *BD*:*AD* = *AE*:*CE* is not intended. In that case the sides are cut proportionally, but the correspondence is not the intended one.

Use of this theorem

This proposition is frequently used in the rest of Book VI starting with the next proposition. It is also used in Books XI and XII.

Next proposition: VI.3

Previous: VI.1

Book VI introduction

Select from Book VI

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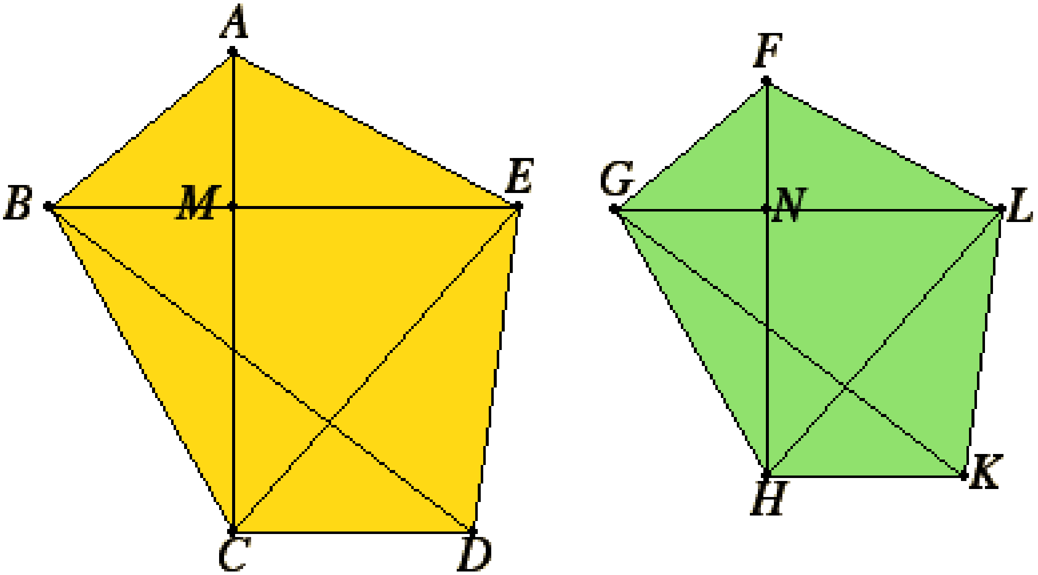
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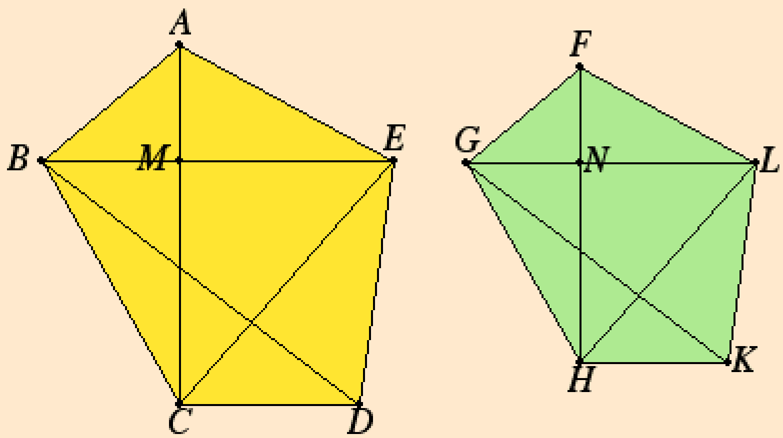
Euclid's Elements
Book VI
Proposition 20

Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.

Let *ABCDE* and *FGHKL* be similar polygons, and let *AB* correspond to *FG*.

I say that the polygons *ABCDE* and *FGHKL* are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon *ABCDE* has to the polygon *FGHKL* a ratio duplicate of that which *AB* has to *FG*.

Join *BE*, *CE*, *GL*, and *HL*.



Now, since the polygon *ABCDE* is similar to the polygon *FGHKL*, therefore the angle *BAE* equals the angle *GFL*, and *AB* is to *AE* as *GF* is to *FL*. VI.Def.1

Since then *ABE* and *FGL* are two triangles having one angle equal to one angle and the sides about the equal angles proportional, therefore the triangle *ABE* is equiangular with the triangle *FGL*, so that it is also similar, therefore the angle *ABE* equals the angle *FGL*. VI.6 VI.4 VI.Def.1

But the whole angle *ABC* also equals the whole angle *FGH* because of the similarity of the polygons, therefore the remaining angle *EBC* equals the angle *LGH*.

And, since the triangles *ABE* and *FGL* are similar, *BE* is to *AB* as *GL* is to *GF*. Also, since the polygons are similar, *AB* is to *BC* as *FG* is to *GH*. Therefore, *ex aequali*, *BE* is to *BC* as *GL* is to *GH*, that is, the sides about the equal angles *EBC* and *LGH* are proportional. Therefore the triangle *EBC* is equiangular with the triangle *LGH*, so that the triangle *EBC* is also similar to the triangle *LGH*. V.22 VI.6 VI.4 VI.Def.1

For the same reason the triangle *ECD* is also similar to the triangle *LHK*.

Therefore the similar polygons *ABCDE* and *FGHKL* have been divided into similar triangles, and into triangles equal in multitude.

I say that they are also in the same ratio as the wholes, that is, in such manner that the triangles are proportional, and *ABE*, *EBC*, and *ECD* are antecedents, while *FGL*, *LGH*, and *LHK* are their consequents, and that the polygon *ABCDE* has to the polygon *FGHKL* a ratio duplicate of that which the corresponding side has to the corresponding side, that is *AB* to *FG*.

Join *AC* and *FH*.

Then since the polygons are similar, the angle *ABC* equals the angle *FGH*, and *AB* is to *BC* as *FG* is to *GH*, the triangle *ABC* is equiangular with the triangle *FGH*, therefore the angle *BAC* equals the angle *GFH*, and the angle *BCA* to the angle *GHF*. VI.6

And, since the angle *BAM* equals the angle *GFN*, and the angle *ABM* also equals the angle *FGN*, therefore the remaining angle *AMB* also equals the remaining angle *FNG*. Therefore the triangle *ABM* is equiangular with the triangle *FGN*. I.32

Similarly we can prove that the triangle *BMC* is also equiangular with the triangle *GNH*.

Therefore, proportionally, *AM* is to *MB* as *FN* is to *NG*, and *BM* is to *MC* as *GN* is to *NH*. So that, in addition, *ex aequali*, *AM* is to *MC* as *FN* is to *NH*. V.22

But *AM* is to *MC* as the triangle *ABM* is to *MBC*, and as *AME* is to *EMC*, for they are to one another as their bases. VI.1

Therefore also one of the antecedents is to one of the consequents as are all the antecedents to all the consequents, therefore the triangle *AMB* is to *BMC* as *ABE* is to *CBE*. V.12

But *AMB* is to *BMC* as *AM* is to *MC*, therefore *AM* is to *MC* as the triangle *ABE* is to the triangle *EBC*. V.11

For the same reason also *FN* is to *NH* as the triangle *FGL* is to the triangle *GLH*.

And *AM* is to *MC* as *FN* is to *NH*, therefore the triangle *ABE* is to the triangle *BEC* as the triangle *FGL* is to the triangle *GLH*, and, alternately, the triangle *ABE* is to the triangle *FGL* as the triangle *BEC* is to the triangle *GLH*. V.11 V.16

Similarly we can prove, if *BD* and *GK* are joined, that the triangle *BEC* is to the triangle *LGH* as the triangle *ECD* is to the triangle *LHK*.

And since the triangle *ABE* is to the triangle *FGL* as *EBC* is to *LGH*, and further as *ECD* is to *LHK*, therefore also one of the antecedents is to one of the consequents as the sum of the antecedents to the sum of the consequents. Therefore the triangle *ABE* is to the triangle *FGL* as the polygon *ABCDE* is to the polygon *FGHKL*. V.12

But the triangle *ABE* has to the triangle *FGL* a ratio duplicate of that which the corresponding side *AB* has to the corresponding side *FG*, for similar triangles are in the duplicate ratio of the corresponding sides. VI.19

Therefore the polygon *ABCDE* also has to the polygon *FGHKL* a ratio duplicate of that which the corresponding side *AB* has to the corresponding side *FG*. V.11

Therefore, *similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.*

Q.E.D.

Corollary

Similarly also it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the corresponding sides. And it was also proved in the case of triangles, therefore also, generally, *similar rectilinear figures are to one another in the duplicate ratio of the corresponding sides.*

Guide

This proposition and its corollary are used occassionally in Books X, XII, and XIII, in particular, XII.1 and XII.8.

Next proposition: VI.21

Previous: VI.19

Book VI introduction

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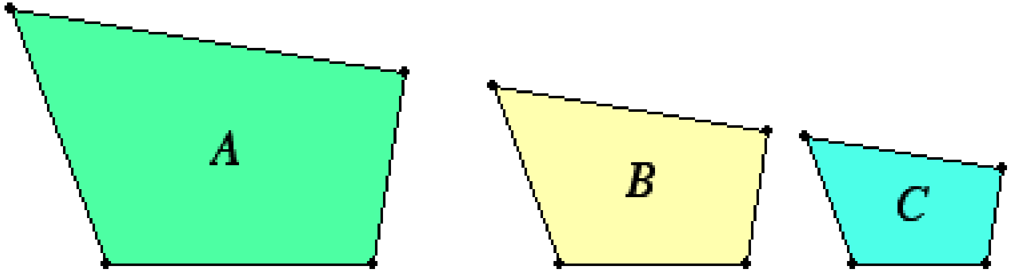
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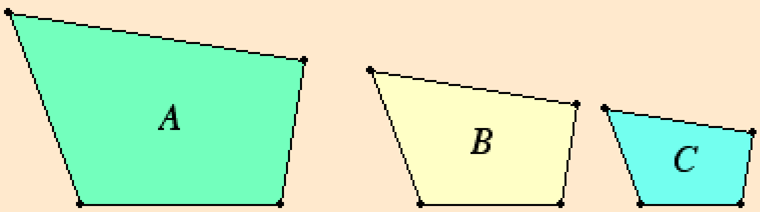
Euclid's Elements
Book VI
Proposition 21

Figures which are similar to the same rectilinear figure are also similar to one another.

Let each of the rectilinear figures *A* and *B* be similar to *C*.

I say that A is also similar to *B*.



Since *A* is similar to *C*, it is equiangular with it and has the sides about the equal angles proportional. VI.Def.1

Again, since *B* is similar to *C*, it is equiangular with it and has the sides about the equal angles proportional.

Therefore each of the figures *A* and *B* is equiangular with *C* and with *C* has the sides about the equal angles proportional, therefore *A* is similar to *B*. V.11

Therefore, *figures which are similar to the same rectilinear figure are also similar to one another*.

Q.E.D.

Guide

This proposition is used in the proofs of propositions VI.24, VI.28, and VI.29. It also would have been useful in the proof of VI.8.

Next proposition: VI.22

Previous: VI.20

Book VI introduction

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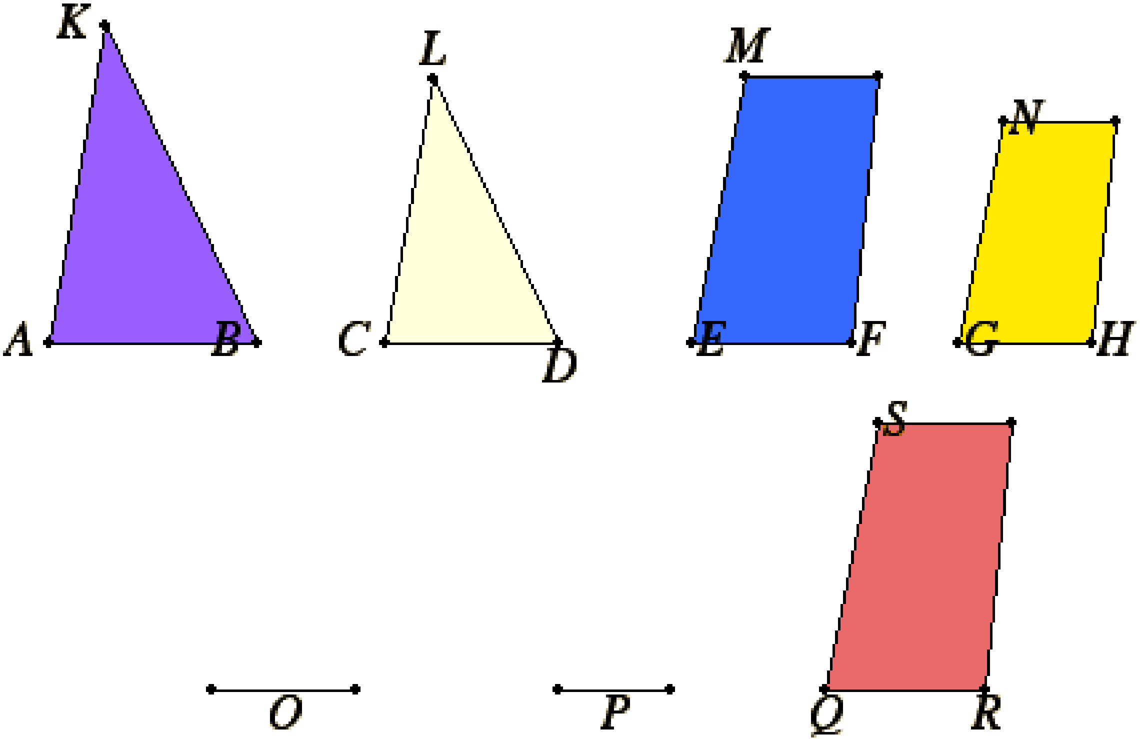
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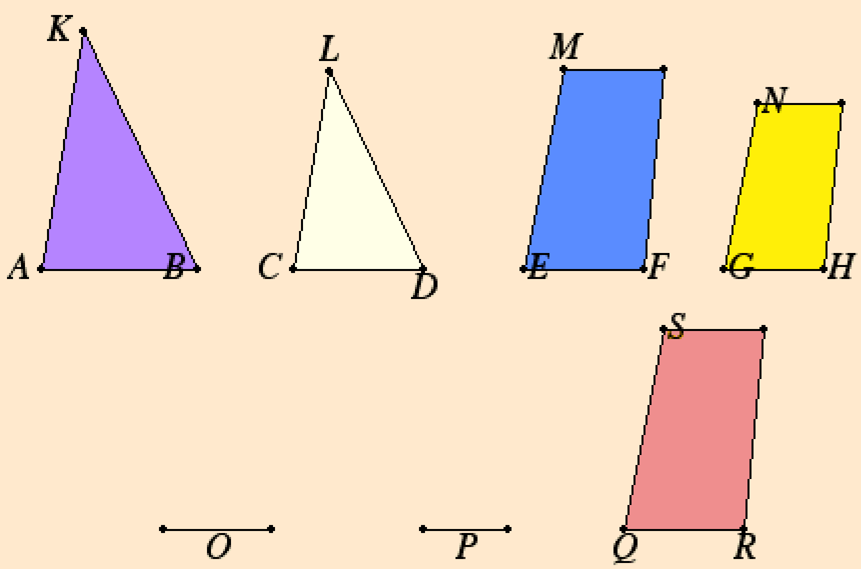
Euclid's Elements
Book VI
Proposition 22

If four straight lines are proportional, then the rectilinear figures similar and similarly described upon them are also proportional; and, if the rectilinear figures similar and similarly described upon them are proportional, then the straight lines are themselves also proportional.

Let the four straight lines *AB, CD, EF,* and *GH* be proportional, so that *AB* is to *CD* as *EF* is to *GH.* Let the similar and similarly situated rectilinear figures *KAB* and *LCD* be described on *AB* and *CD,* and the similar and similarly situated rectilinear figures *MF* and *NH* be described on *EF* and *GH.*

I say that *KAB* is to *LCD* as *MF* is to *NH.*

Take a third proportional *O* to *AB* and *CD,* and a third proportional *P* to *EF* and *GH.* VI.11



Then since *AB* is to *CD* as *EF* is to *GH,* therefore *CD* is to *O* as *GH* is to *P.* Therefore, *ex aequali, AB* is to *O* as *EF* is to *P.* V.11 V.22

But *AB* is to *O* as *KAB* is to *LCD,* and *EF* is to *P* as *MF* is to *NH,* therefore *KAB* is to *LCD* also as *MF* is to *NH.* VI.19,Cor V.11

Next, let *KAB* be to *LCD* as *MF* is to *NH.*

I say also that *AB* is to *CD* as *EF* is to *GH.*

For, if *EF* is not to *GH* as *AB* is to *CD,* let *EF* be to *QR* as *AB* is to *CD.* Describe the rectilinear figure *SR* similar and similarly situated to either of the two *MF* or *NH* on *QR.* VI.12 VI.18

Since then *AB* is to *CD* as *EF* is to *QR,* and there have been described on *AB* and *CD* the similar and similarly situated figures *KAB* and *LCD,* and on *EF* and *QR* the similar and similarly situated figures *MF* and *SR,* therefore *KAB* is to *LCD* as *MF* is to *SR.* Above

But also, by hypothesis, *KAB* is to *LCD* as *MF* is to *NH*, therefore also *MF* is to *SR* as *MF* is to *NH*. V.11

Therefore *MF* has the same ratio to each of the figures *NH* and *SR*, therefore *NH* equals *SR*. V.9

But it is also similar and similarly situated to it, therefore *GH* equals *QR*.

And, since *AB* is to *CD* as *EF* is to *QR*, while *QR* equals *GH*, therefore *AB* is to *CD* as *EF* is to *GH*.

Therefore, *if four straight lines are proportional, then the rectilinear figures similar and similarly described upon them are also proportional; and, if the rectilinear figures similar and similarly described upon them are proportional, then the straight lines are themselves also proportional.*

Q.E.D.

Guide

There is a missing step near the end of the proof, namely, the justification of the statement that *GH* equals *QR* is missing. Just before it we have *NH* and *SR* are similar equal rectilinear figures, and we want to conclude the corresponding sides *GH* and *QR* are equal. The needed demonstration is not difficult to supply.

Use of this proposition

This proposition is used in the proofs of several propositions in Book X and in XII.4 in Book XII.

Next proposition: VI.23

Previous: VI.21

Book VI introduction

Select from Book VI

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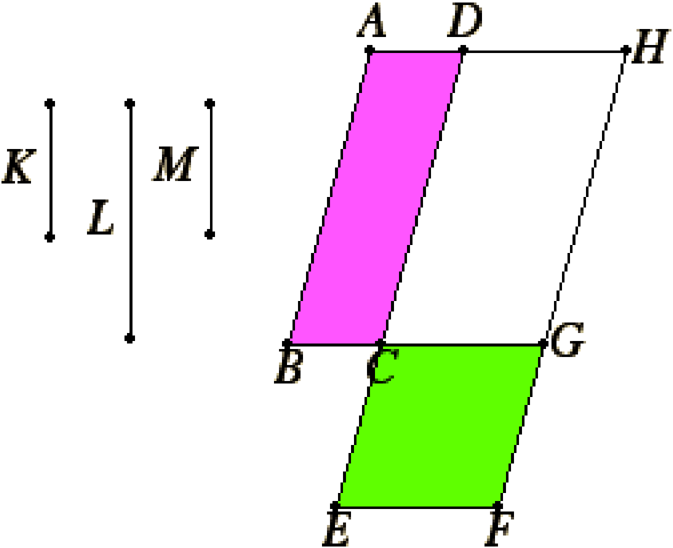
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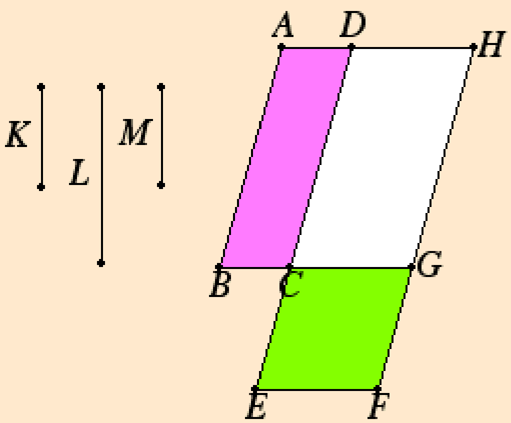
Euclid's Elements
Book VI
Proposition 23

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Let *AC* and *CF* be equiangular parallelograms having the angle *BCD* equal to the angle *ECG.*

I say that the parallelogram *AC* has to the parallelogram *CF* the ratio compounded of the ratios of the sides.

Let them be placed so that *BC* is in a straight line with *CG.* Then *DC* is also in a straight line with *CE.* I.14



Complete the parallelogram *DG.* Set out a straight line *K*, and make it so that *BC* is to *CG* as *K* is to *L*, and *DC* is to *CE* as *L* is to *M.* I.31 VI.12

Then the ratios of *K* to *L* and of *L* to *M* are the same as the ratios of the sides, namely of *BC* to *CG* and of *DC* to *CE.*

But the ratio of *K* to *M* is compounded of the ratio of *K* to *L* and of that of *L* to *M*, so that *K* has also to *M* the ratio compounded of the ratios of the sides.

Now since *BC* is to *CG* as the parallelogram *AC* is to the parallelogram *CH*, and *BC* is to *CG* as *K* is to *L*, therefore *K* is to *L* as *AC* is to *CH.* VI.1 V.11

Again, since *DC* is to *CE* as the parallelogram *CH* is to *CF*, and *DC* is to *CE* as *L* is to *M*, therefore *L* is to *M* as the parallelogram *CH* is to the parallelogram *CF.* VI.1 V.11

Since then it was proved that *K* is to *L* as the parallelogram *AC* is to the parallelogram *CH*, and *L* is to *M* as the parallelogram *CH* is to the parallelogram *CF*, therefore, *ex aequali K* is to *M* as *AC* is to the parallelogram *CF.* V.22

But *K* has to *M* the ratio compounded of the ratios of the sides, therefore *AC* also has to *CF* the ratio compounded of the ratios of the sides.

Therefore, *equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.*

Q.E.D.

Guide

This proposition is a generalization of the basic formula for the area of a rectangle, that is, the area of a rectangle is the product of its length and width. Such a formula depends on predetermined units of length and area so that the unit area is the area of a square whose sides have length equal to the unit length. Euclid and other Greek mathematicians did not use predetermined units of length or area, so they expressed this formula as a proportion. We would state that proportion as saying the ratio of the area of a given rectangle to the area of a given square is the product of the ratios of the lengths of the sides of the rectangle to the length of a side of the square. Of course, Euclid would say that without using the words 'area' and 'length' as follows: the ratio of the a given rectangle to a given square is the product of the ratios of the sides of the rectangle to a side of the square. Note that his terminology for a product of ratios involves "compounding the ratios." A natural generalization of the ratio of a rectangle to a square is the ratio of a rectangle to a rectangle. A broader generalization is the ratio of one parallelogram to another parallelogram having the same angles. That gives the generalization as stated in this proposition.

Areas of rectangles and parallelograms

These areas have been treated earlier in the *Elements.* Back in Book I and II the basic concept was "quadrature," that is, finding a square or other shaped figure of the same area as the given rectangle or parallelogram. That began with Proposition I.35 which said two triangles on the same base and with the same height are equal, and ended with I.14 which constructed a square equal to a given rectangle.

Early in this book was the proposition VI.1 generalizing I.35 which said that parallelograms with the same height are proportional to their bases. Finally, in this proposition we have the full statement about areas of rectangles and parallelograms.

Analogous statements in other books

Proposition VIII.5 states that plane numbers have to one another the ratio compounded of the ratios of their sides. That proposition is probably a much older version that may go back to the Pythagoreans when "all was number." The discovery of incommensurable lines showed there were serious limitations to that version of the proposition.

In Book XI there are analogous statements for volumes of parallelopipeds. For instance, Proposition XI.33 states that similar parallelepipeds are to one another in the triplicate ratio of their corresponding sides. That statement for parallelepipeds is analogous to this one for parallelograms.

Use of this theorem

Although this is a basic proposition on areas, it is actually not used in the rest of the *Elements.*

Next proposition: VI.24

Previous: VI.22

Book VI introduction

Select from Book VI

Select book

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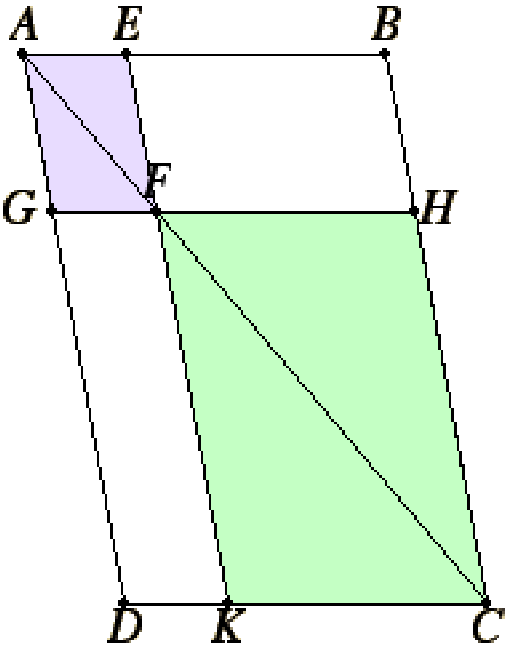
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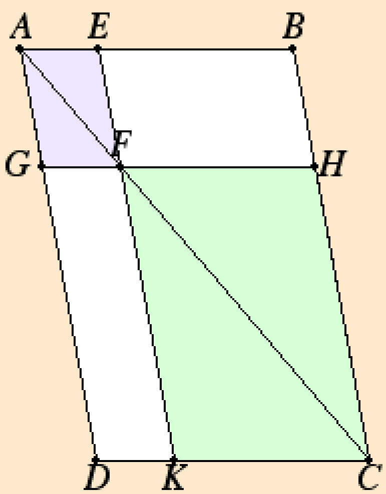
Euclid's Elements
Book VI
Proposition 24

In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Let *ABCD* be a parallelogram, and *AC* its diameter, and let *EG* and *HK* be parallelograms about *AC.*

I say that each of the parallelograms *EG* and *HK* is similar both to the whole *ABCD* and to the other.



For, since *EF* is parallel to a side *BC* of the triangle *ABC*, proportionally, *BE* is to *EA* as *CF* is to *FA.* VI.2

Again, since *FG* is parallel to a side *CD* of the triangle *ACD*, proportionally, *CF* is to *FA* as *DG* is to *GA.* VI.2

But it was proved that *CF* is to *FA* as *BE* is to *EA*, therefore *BE* is to *EA* as *DG* is to *GA.* Therefore, taken jointly, *BA* is to *AE* as *DA* is to *AG*, and, alternately, *BA* is to *AD* as *EA* is to *AG.* V.18 V.16

Therefore in the parallelograms *ABCD* and *EG*, the sides about the common angle *BAD* are proportional.

And, since *GF* is parallel to *DC*, the angle *AFG* equals the angle *ACD*, and the angle *DAC* is common to the two triangles *ADC* and *AGF*, therefore the triangle *ADC* is equiangular with the triangle *AGF.* I.29

For the same reason the triangle *ACB* is also equiangular with the triangle *AFE*, and the whole parallelogram *ABCD* is equiangular with the parallelogram *EG.*

Therefore, proportionally, *AD* is to *DC* as *AG* is to *GF*, *DC* is to *CA* as *GF* is to *FA*, *AC* is to *CB* as *AF* is to *FE*, and *CB* is to *BA* as *FE* is to *EA.*

And, since it was proved that *DC* is to *CA* as *GF* is to *FA*, and *AC* is to *CB* as *AF* is to *FE*, therefore, *ex aequali*, *DC* is to *CB* as *GF* is to *FE.* V.22

Therefore in the parallelograms *ABCD* and *EG* the sides about the equal angles are proportional. Therefore the parallelogram *ABCD* is similar to the parallelogram *EG.* VI.Def.1

For the same reason the parallelogram *ABCD* is also similar to the parallelogram *KH.* Therefore each of the parallelograms *EG* and *HK* is similar to *ABCD.*

But figures similar to the same rectilinear figure are also similar to one another, therefore the parallelogram *EG* is also similar to the parallelogram *HK.* VI.21

Therefore, *in any parallelogram the parallelograms about the diameter are similar both to the whole and to one* *another*.

Q.E.D.

Guide

With this proposition Euclid returns to applications of areas. Back in proposition I.45 rectilinear areas were applied to lines. In the upcoming propositions VI.28 and VI.29, rectilinear areas will be applied to lines but the areas will fall short of or extend beyond the end of the lines. Those propositions geometric solve two kinds of quadratic equations. This proposition is preporatory to them. It is used in the proofs of VI.26 (its converse) and VI.29.

Next proposition: VI.25

Previous: VI.23

Book VI introduction

Select from Book VI

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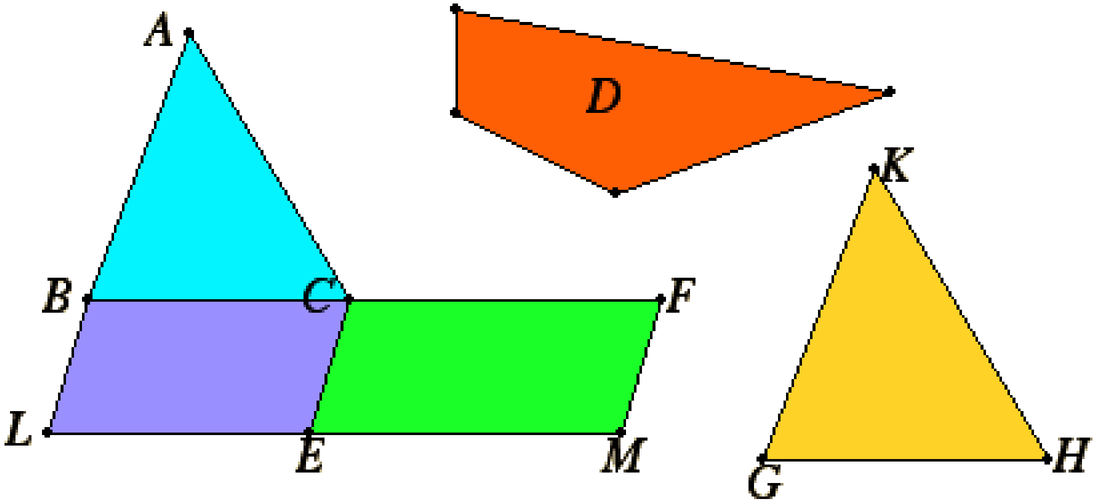
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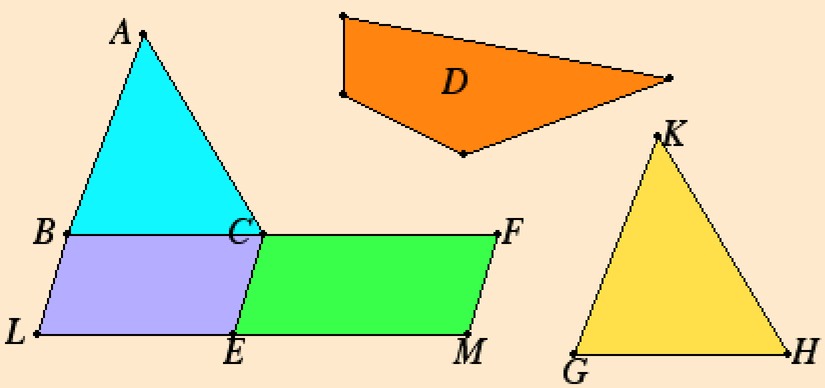
Euclid's Elements
Book VI
Proposition 25

To construct a figure similar to one given rectilinear figure and equal to another.

Let *ABC* be the given rectilinear figure to which the figure to be constructed must be similar, and *D* that to which it must be equal.

It is required to construct one figure similar to *ABC* and equal to *D.*

Let there be applied to *BC* the parallelogram *BE* equal to the triangle *ABC,* and to *CE* the parallelogram *CM* equal to *D* in the angle *FCE* which equals the angle *CBL.* I.44 I.45



Then *BC* is in a straight line with *CF,* and *LE* with *EM.*

Take a mean proportional *GH* to *BC* and *CF,* and describe *KGH* similar and similarly situated to *ABC* on *GH.* VI.13 VI.18

Then, since *BC* is to *GH* as *GH* is to *CF,* and, if three straight lines are proportional, then the first is to the third as the figure on the first is to the similar and similarly situated figure described on the second, therefore *BC* is to *CF* as the triangle *ABC* is to the triangle *KGH.* V.19,Cor

But *BC* is to *CF* as the parallelogram *BE* is to the parallelogram *EF.* VI.1

Therefore also the triangle *ABC* is to the triangle *KGH* as the parallelogram *BE* is to the parallelogram *EF.* Therefore, alternately, the triangle *ABC* is to the parallelogram *BE* as the triangle *KGH* is to the parallelogram *EF.* V.11 V.16

But the triangle *ABC* equals the parallelogram *BE,* therefore the triangle *KGH* also equals the parallelogram *EF.* And the parallelogram *EF* equals *D,* therefore *KGH* also equals *D.* (V.14)

And *KGH* is also similar to *ABC.* Therefore this figure *KGH* has been constructed similar to the given rectilinear figure *ABC* and equal to the other given figure *D.*

Q.E.F.

Guide

Note that it isn't proposition V.14 being invoked near the end of the proof, but an alternate form of it. See the Guide to V.14.

This proposition solves a similar problem, to find a figure with the size of one figure but the shape of another, a problem reputedly solved by Pythagoras. It is used in the proofs of propositions VI.28 and VI.29

Next proposition: VI.26

Previous: VI.24

Book VI introduction

Select from Book VI

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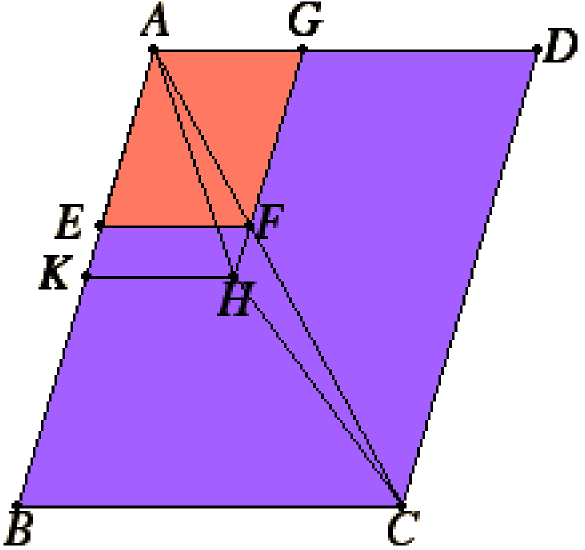
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Euclid's Elements

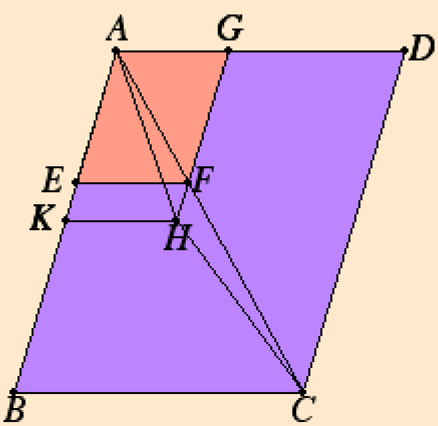
Book VI

Proposition 26

If from a parallelogram there is taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, then it is about the same diameter with the whole.

From the parallelogram *ABCD* let there be taken away the parallelogram *AF* similar and similarly situated to *ABCD*, and having the angle *DAB* common with it.

I say that *ABCD* is about the same diameter with *AF*.



For suppose it is not, but, if possible, let *AHC* be the diameter. Produce *GF* and carry it through to *H*. Draw *HK* through *H* parallel to either of the straight lines *AD* or *BC*. I.31

Since, then, *ABCD* is about the same diameter with *KG*, therefore *DA* is to *AB* as *GA* is to *AK*. VI.24

But also, since *ABCD* and *EG* are similar, therefore *DA* is to *AB* as *GA* is to *AE*. Therefore *GA* is to *AK* as *GA* is to *AE*. VI.Def.1 V.11

Therefore *GA* has the same ratio to each of the straight lines *AK* and *AE*.

Therefore *AE* equals *AK* the less equals the greater, which is impossible. V.9

Therefore *ABCD* cannot fail to be about the same diameter with *AF*. Therefore the parallelogram *ABCD* is about the same diameter with the parallelogram *AF*.

Therefore, *if from a parallelogram there is taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, then it is about the same diameter with the whole.*

Q.E.D.

Guide

This proposition is the converse of VI.24. It is used in the proofs of the next three and for a few propositions in Book X.

Next proposition: VI.27

Select from Book VI

Previous: VI.23

Book VI introduction

Select book

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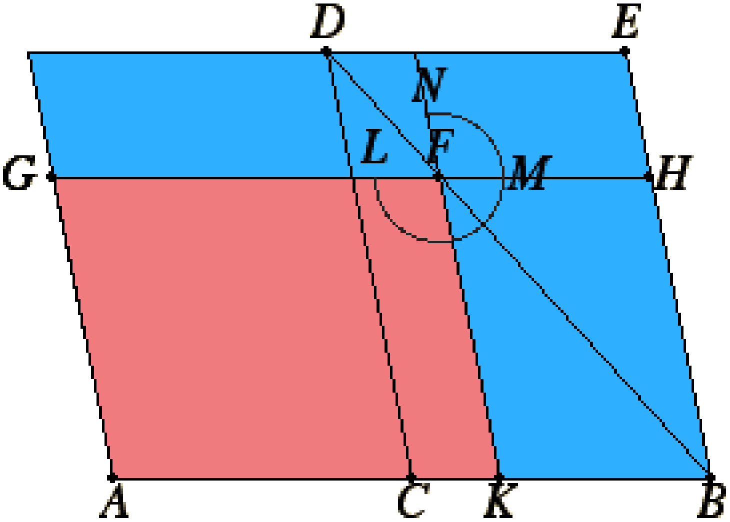
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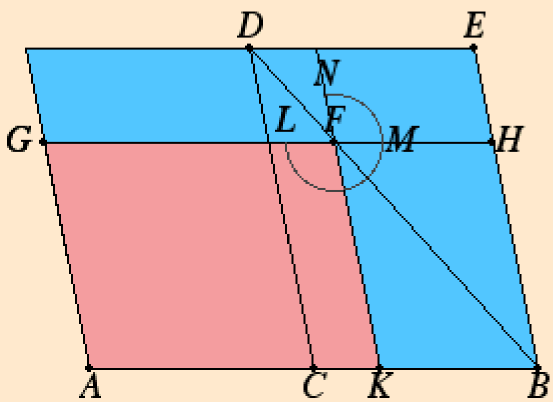
Euclid's Elements
Book VI
Proposition 27

Of all the parallelograms applied to the same straight line falling short by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the difference.

Let *AB* be a straight line and let it be bisected at *C.* Let there be applied to the straight line *AB* the parallelogram *AD* falling short by the parallelogrammic figure *DB* described on the half of *AB,* that is, *CB.*

I say that, of all the parallelograms applied to *AB* falling short by parallelogrammic figures similar and similarly situated to *DB, AD* is greatest.



Let there be applied to the straight line *AB* the parallelogram *AF* falling short by the parallelogrammic figure *FB* similar and similarly situated to *DB.*

I say that *AD* is greater than *AF.*

Since the parallelogram *DB* is similar to the parallelogram *FB,* therefore they are about the same diameter. VI.26

Draw their diameter *DB,* and describe the figure.

Then, since *CF* equals *FE,* and *FB* is common, therefore the whole *CH* equals the whole *KE.* I.43

But *CH* equals *CG,* since *AC* also equals *CB.* I.36

Therefore *CG* also equals *KE.*

Add *CF* to each. Therefore the whole *AF* equals the gnomon *LMN,* so that the parallelogram *DB,* that is, *AD,* is greater than the parallelogram *AF.*

Therefore, *of all the parallelograms applied to the same straight line falling short by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the difference.*

Q.E.D.

Guide

This proposition clarifies the limitations of the next one, VI.28. In VI.28 a construction is made to apply a parallelogram equal to a given rectilinear figure to a line falling short by a parallelogrammic figure. This proposition implies that that construction cannot be made if the given rectilinear figure is too large.

When that proposition is applied, the part which falls short is usually a square, not just any parallelogram, and this and the next proposition are much more easily understood in that case. In that case, the next proposition applies a rectangle equal to a given area to a line but falling short by a square. And this proposition implies that can only be done if the given area is at least the square on half the line, since that square is the greatest rectangle that can be so applied.

Next proposition: VI.28

Previous: VI.26

Book VI introduction

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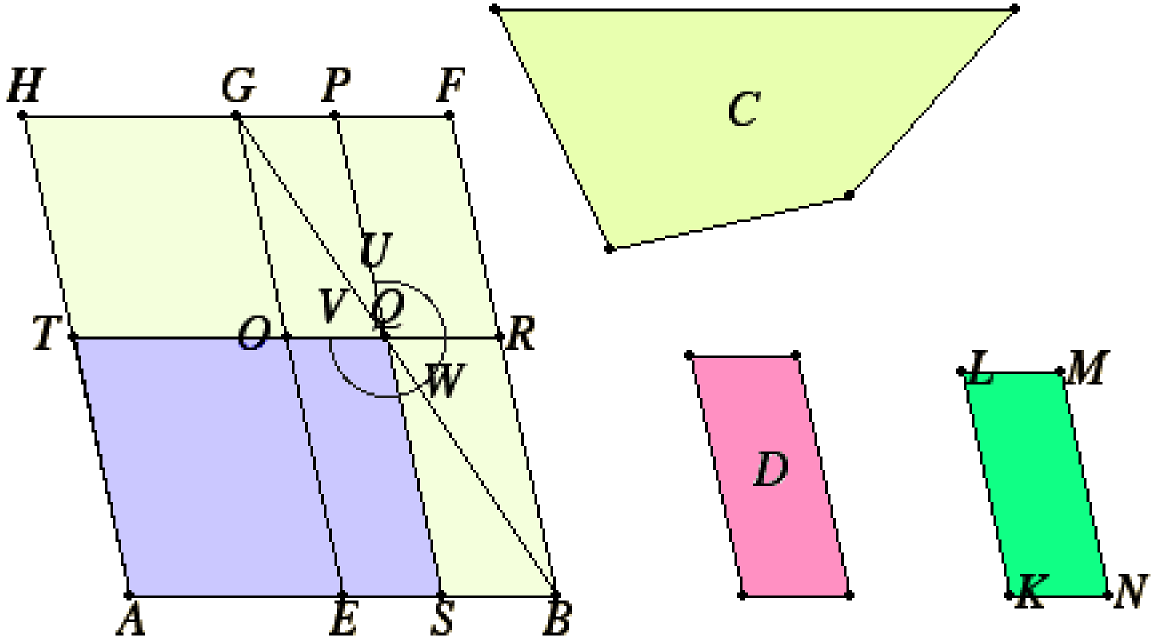
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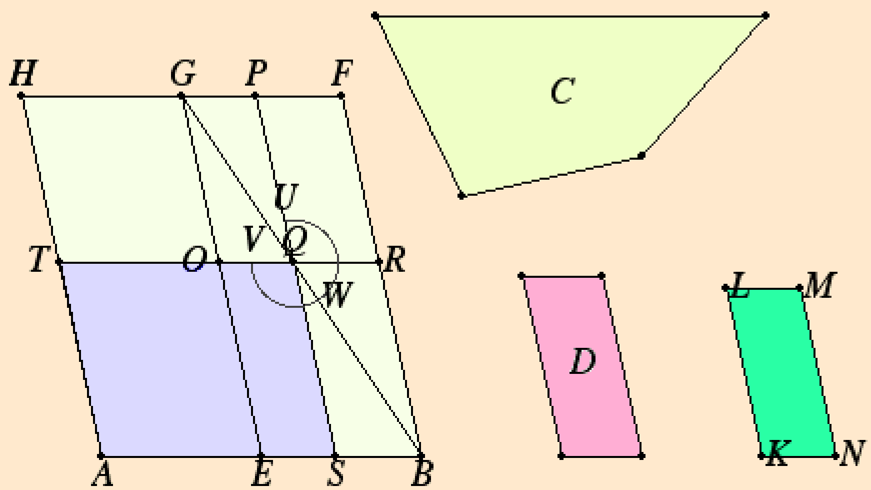
Euclid's Elements
Book VI
Proposition 28

To apply a parallelogram equal to a given rectilinear figure to a given straight line but falling short by a parallelogram similar to a given one; thus the given rectilinear figure must not be greater than the parallelogram described on the half of the straight line and similar to the given parallelogram. VI.27

Let *C* be the given rectilinear figure, *AB* the given straight line, and *D* the given parallelogram, and let *C* not be greater than the parallelogram described on the half of *AB* similar to the given parallelogram *D.*

It is required to apply a parallelogram equal to the given rectilinear figure *C* to the given straight line *AB* but falling short by a parallelogram similar to *D.*



Bisect *AB* at the point *E.* Describe *EBFG* similar and similarly situated to *D* on *EB,* and complete the parallelogram *AG.* I.9 VI.18

If then *AG* equals *C,* that which was proposed is done, for the parallelogram *AG* equal to the given rectilinear figure *C* has been applied to the given straight line *AB* but falling short by a parallelogram *GB* similar to *D.*

But, if not, let *HE* be greater than *C.*

Now *HE* equals *GB,* therefore *GB* is also greater than *C.*

Construct *KLMN* equal to *GB* minus *C* and similar and similarly situated to *D.* VI.25

But *D* is similar to *GB,* therefore *KM* is also similar to *GB.* VI.21

Let, then, *KL* correspond to *GE,* and *LM* to *GF.*

Now, since *GB* equals *C* and *KM,* therefore *GB* is greater than *KM,* therefore also *GE* is greater than *KL,* and *GF* than *LM.*

Make *GO* equal to *KL,* and *GP* equal to *LM,* and let the parallelogram *OGPQ* be completed, therefore it is equal and similar to *KM.*

Therefore *GQ* is also similar to *GB*, therefore *GQ* is about the same diameter with *GB*. VI.21 VI.26

Let *GQB* be their diameter, and describe the figure.

Then, since *BG* equals *C* and *KM*, and in them *GQ* equals *KM*, therefore the remainder, the gnomon *UWV*, equals the remainder *C*.

And, since *PR* equals *OS*, add *QB* to each, therefore the whole *PB* equals the whole *OB*.

But *OB* equals *TE*, since the side *AE* also equals the side *EB*, therefore *TE* also equals *PB*. I.36

Add *OS* to each. Therefore the whole *TS* equals the whole, the gnomon *VWU*.

But the gnomon *VWU* was proved equal to *C*, therefore *TS* also equals *C*.

Therefore there the parallelogram *ST* equal to the given rectilinear figure *C* has been applied to the given straight line *AB* but falling short by a parallelogram *QB* similar to *D*.

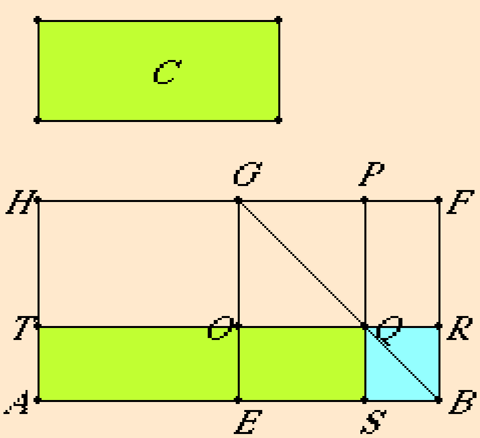
Q.E.F.

Guide

When this proposition is used, the given parallelgram *D* usually is a square. Then the problem is to cut the line *AB* at a point *S* so that the rectangle *AS* by *SB* equals the given rectilinear figure *C*. This special case can be proved with the help of the propositions in Book II. See the Guide to proposition II.5 for more details.

The outline of a simplified proof for rectangles

The proof of the current propostion is difficult to follow. It is simplified when we take the special case mentioned above, namely, when the given parallelogram *D* is a square. The simplified proof is easier to follow since the rest of the parallelograms mentioned all become rectangles.



The construction is as follows. Bisect *AB* at *E*, construct a square *GFBE*.

The next stage is to construct a square *GPQO* equal to the square *GFBE* minus the figure *C*. For an alternate construction of *GPQO*, see the lemma for X.14 which applies I.47 to do that.)

Complete the figure.

We can understand the meaning of this construction more easily if we interpret it algebraically. Let *a* stand for the known quantity *AB*, and *c* the known quantity *C*. Then let *x* and *y* stand for the unknown quantities *SB* and *SA*. Then this construction finds *x* and *y* so that their sum is *a* and their product is *c*.

In terms of the single variable *x*, the construction solves the quadratic equation $ax - x^2 = C$.

The next proposition solves a similar quadratic equation: $ax + x^2 = C$.

Use of this proposition

This constuction in this proposition is used in propositions X.33 and X.34.

Next proposition: VI.29

Previous: VI.27

Book VI introduction

Select from Book VI

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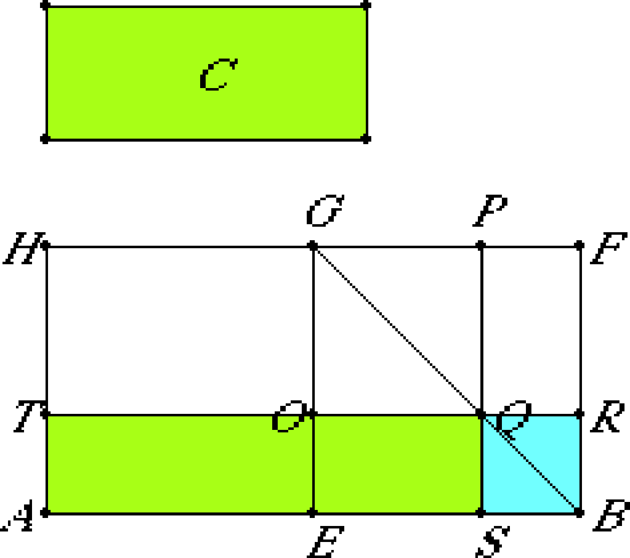

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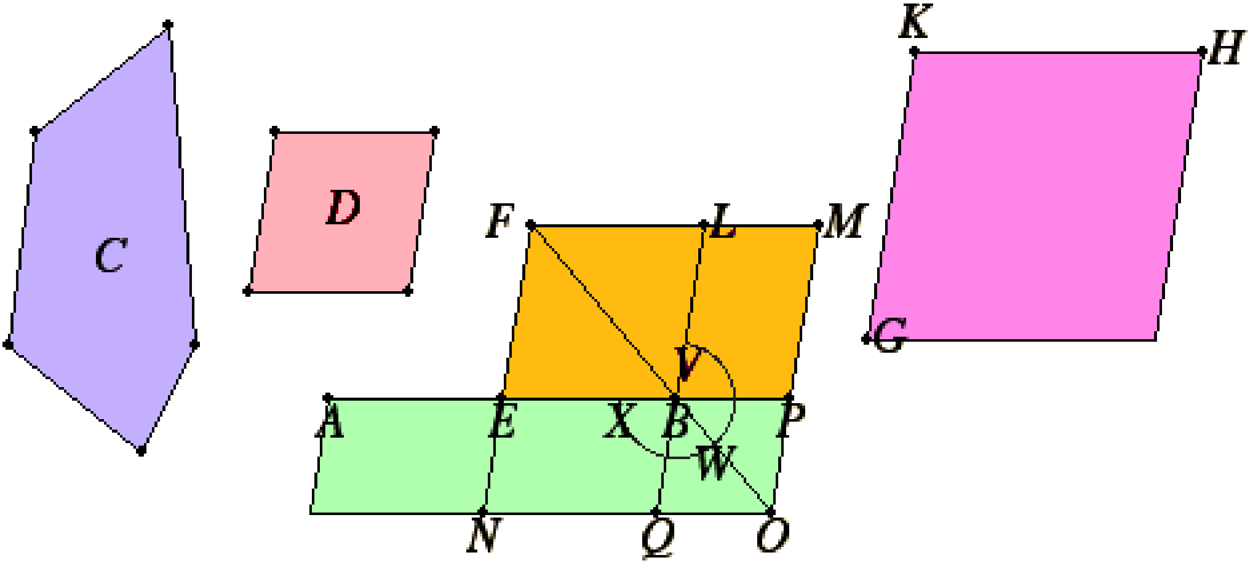
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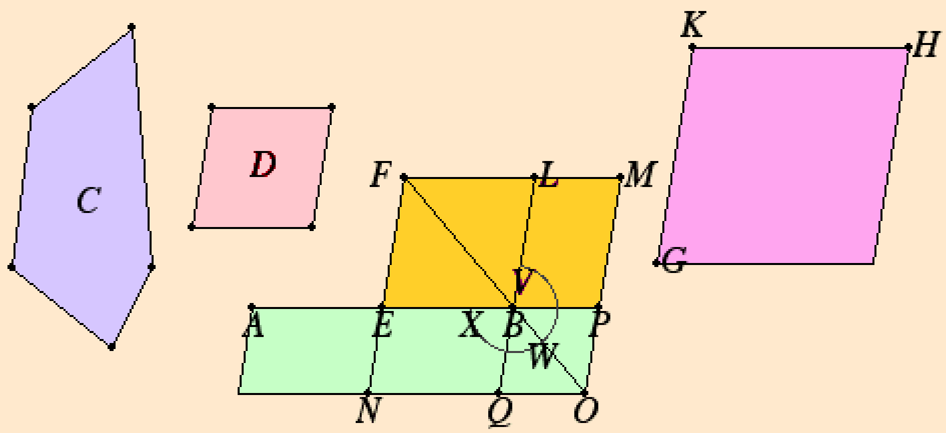
Euclid's Elements
Book VI
Proposition 29

To apply a parallelogram equal to a given rectilinear figure to a given straight line but exceeding it by a parallelogram similar to a given one.

Let *C* be the given rectilinear figure, *AB* be the given straight line, and *D* the parallelogram to which the excess is required to be similar.

It is required to apply a parallelogram equal to the the rectilinear figure *C* to the straight line *AB* but exceeding it by a parallelogram similar to *D.*



Bisect *AB* at *E.* Describe the parallelogram *BF* on *EB* similar and similarly situated to *D,* and construct *GH* equal to the sum of *BF* and *C* and similar and similarly situated to *D.* VI.25

Let *KH* correspond to *FL* and *KG* to *FE.*

Now, since *GH* is greater than *FB,* therefore *KH* is also greater than *FL,* and *KG* greater than*FE.*

Produce *FL* and *FE.* Make *FLM* equal to *KH,* and *FEN* equal to *KG.* Complete *MN.* Then *MN* is both equal and similar to *GH.*

But *GH* is similar to *EL,* therefore *MN* is also similar to *EL,* therefore *EL* is about the same diameter with *MN.* VI.21 VI.26

Draw their diameter *FO,* and describe the figure.

Since *GH* equals the sum of *EL* and *C,* while *GH* equals *MN,* therefore *MN* also equals the sum of *EL* and *C.*

Subtract *EL* from each. Therefore the remainder, the gnomon *XWV,* equals *C.*

Now, since *AE* equals *EB,* therefore *AN* equals *NB,* that is, *LP.* I.36 I.43

Add *EO* to each. Therefore the whole *AO* equals the gnomon *VWX.*

But the gnomon *VWX* equals *C,* therefore *AO* also equals *C.*

Therefore the parallelogram *AO* equal to the given rectilinear figure *C* has been applied to the given straight line *AB* but exceeding it by a parallelogram *QP* similar to *D*, since *PQ* is also similar to *EL*. VI.24

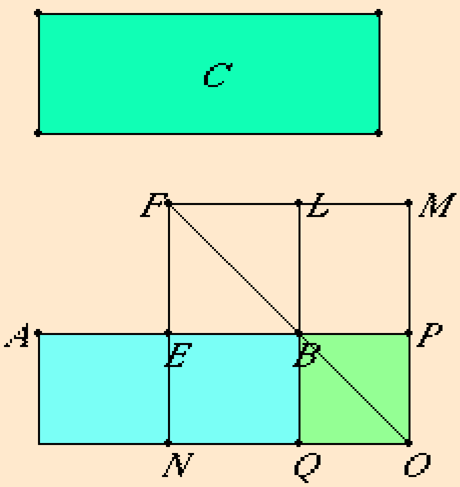
Q.E.F.

Guide

The construction in this proposition is a generalization of that described in the Guide for II.6. In that proposition, the figure *D* is a square.

The outline of a simplified proof for rectangles

Like the last proposition, this one is more easily understood when the given parallelogram *D* is a square.



In that case of this proposition a rectangle *AO* equal to a given rectilinear figure *C* is applied to a given straight line *AB* but exceeds it by a square (*BQOP* in the figure). So the rectangle being laid alongside the line extends past the end of the line *AB*, but the part that extends beyond the end is a square.

For the construction, bisect *AB* at *E*, construct the square *BEFL*, then make the square *FNOM* equal to the square *BEFL* plus the figure *C*, and complete the figure.

As in the last proposition we can understand the meaning of this construction more easily if we interpret it algebraically. Let *a* stand for the known quantity *AB* and *x* stand for the unknown quantity *BP*. Then this construction finds *x* so that $(a + x)\, x = C$. In other words it solves the quadratic equation $ax + x^2 = C$.

The construction of this proposition is used in the next proposition to cut a straight line in extreme and mean ratio.

Next proposition: VI.30

Previous: VI.28

Book VI introduction

Select from Book VI

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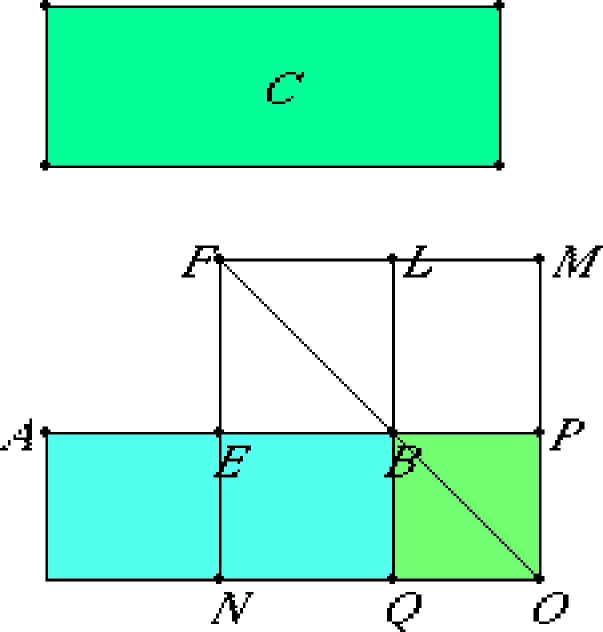
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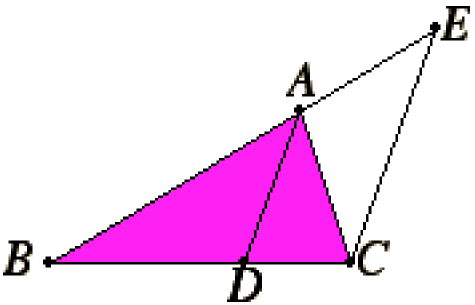
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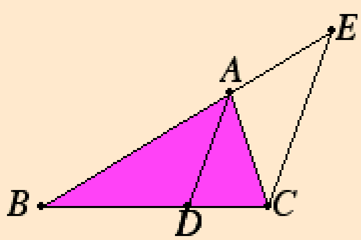
Euclid's Elements
Book VI
Proposition 3

If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle.

Let *ABC* be a triangle, and let the angle *BAC* be bisected by the straight line *AD.*

I say that *DB* is to *DC* as *AB* is to *AC.*



Draw *CE* through *C* parallel to *DA,* and carry *AB* through to meet it at *E.* I.31

Then, since the straight line *AC* falls upon the parallels *AD* and *EC,* the angle *ACE* equals the angle *CAD.* I.29

But the angle *CAD* equals the angle *BAD* by hypothesis, therefore the angle *BAD* also equals the angle *ACE.*

Again, since the straight line *BAE* falls upon the parallels *AD* and *EC,* the exterior angle *BAD* equals the interior angle *AEC.* I.29

But the angle *ACE* was also proved equal to the angle *BAD,* therefore the angle *ACE* also equals the angle *AEC,* so that the side *AE* also equals the side *AC.* I.6

And, since *AD* is parallel to *EC,* one of the sides of the triangle *BCE,* therefore, proportionally *DB* is to *DC* as *AB* is to *AE.* VI.2

But *AE* equals *AC,* therefore *DB* is to *DC* as *AB* is to *AC.* V.7

Next, let *DB* be to *DC* as *AB* is to *AC.* Join *AD.*

I say that the straight line *AD* bisects the angle *BAC.*

With the same construction, since *DB* is to *DC* as *AB* is to *AC,* and also *DB* is to *DC* as *AB* is to *AE,* for *AD* is parallel to *EC,* one of the sides of the triangle *BCE,* therefore also *AB* is to *AC* as *AB* is to *AE.* VI.2 V.11

Therefore *AC* equals *AE,* so that the angle *AEC* also equals the angle *ACE.* V.9 I.5

But the angle *AEC* equals the exterior angle *BAD,* and the angle *ACE* equals the alternate angle *CAD,* therefore the angle *BAD* also equals the angle *CAD.* I.29

Therefore the straight line *AD* bisects the angle *BAC.*

Therefore, *if an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle.*

Q.E.D.

Guide

This proposition characterizes an angle bisector of an angle in a triangle as the line that partitions the base into parts proportional to the adjacent sides. The second part of the statement of the proposition is the converse of the first part of the statement. The proof relies on basic properties of triangles and parallel lines developed in Book I along with the result of the previous proposition VI.2.

Use of this proposition

This proposition is not used in the remainder of the *Elements*.

Next proposition: VI.4

Previous: VI.2

Book VI introduction

Select from Book VI

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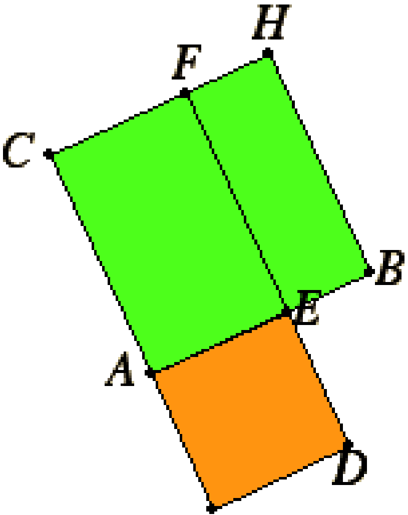

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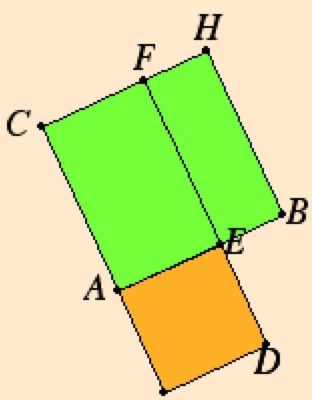
Euclid's Elements
Book VI
Proposition 30

To cut a given finite straight line in extreme and mean ratio.

Let *AB* be the given finite straight line.

It is required to cut *AB* in extreme and mean ratio.



Describe the square *BC* on *AB*. Apply the parallelogram *CD* to *AC* equal to the sum of *BC* and the figure *AD* similar to *BC*. I.46 VI.29

Now *BC* is a square, therefore *AD* is also a square.

And, since *BC* equals *CD*, subtract *CE* from each, therefore the remainder *BF* equals the remainder *AD*.

But it is also equiangular with it, therefore in *BF* and *AD* the sides about the equal angles are reciprocally proportional. Therefore *FE* is to *ED* as *AE* is to *EB*. VI.14

But *FE* equals *AB*, and *ED* equals *AE*.

Therefore *AB* is to *AE* as *AE* is to *EB*. V.7

And *AB* is greater than *AE*, therefore *AE* is also greater than *EB*.

Therefore the straight line *AB* has been cut in extreme and mean ratio at *E*, and the greater segment of it is *AE*. VI.Def.3

Q.E.F.

Guide

The construction given here cuts a line into two parts *A* and *B* so that (*A* + *B*):*B* = *B*:*A*. By proposition VI.17, that condition is equivalent to making the rectangle *A* + *B* by *A* equal the square *B* by *B*, and that is the construction of proposition II.11.

Use of this proposition

This construction is used in XIII.17 to construct a pentagonal face of a regular dodecahedron.

Next proposition: VI.31 Select from Book VI

Previous: VI.29

Book VI introduction

Select book

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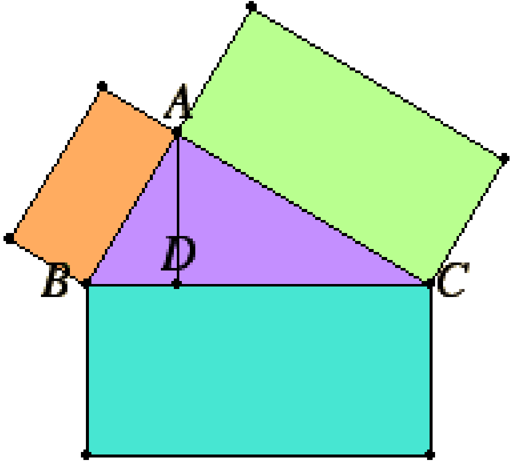
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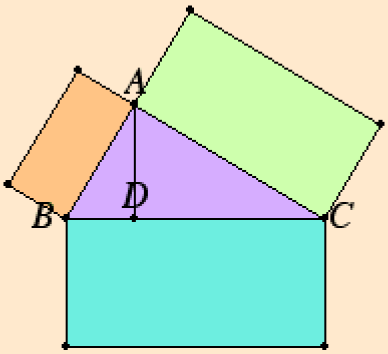
Euclid's Elements
Book VI
Proposition 31

In right-angled triangles the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right.

I say that the figure on *BC* equals the sum of the similar and similarly described figures on *BA* and *AC.*

Draw the perpendicular *AD.* I.12



Then, since in the right-angled triangle *ABC, AD* has been drawn from the right angle at *A* perpendicular to the base *BC,* therefore the triangles *DBA* and *DAC* adjoining the perpendicular are similar both to the whole *ABC* and to one another. VI.8

And, since *ABC* is similar to *DBA,* therefore *BC* is to *BA* as *BA* is to *BD.* VI.Def.1

And, since three straight lines are proportional, the first is to the third as the figure on the first is to the similar and similarly described figure on the second. VI.19,Cor

Therefore *BC* is to *BD* as the figure on *BC* is to the similar and similarly described figure on *BA.*

For the same reason also, *BC* is to *CD* as the figure on *BC* is to that on *CA,* so that, in addition, *BC* is to the sum of *BD* and *DC* as the figure on *BC* is to the sum of the similar and similarly described figures on *BA* and *AC.* V.24

But *BC* equals the sum of *BD* and *DC,* therefore the figure on *BC* equals the sum of the similar and similarly described figures on *BA* and *AC.*

Therefore, *in right-angled triangles the figure on the side opposite the right angle equals the sum of the similar and similarly described figures on the sides containing the right angle.*

Q.E.D.

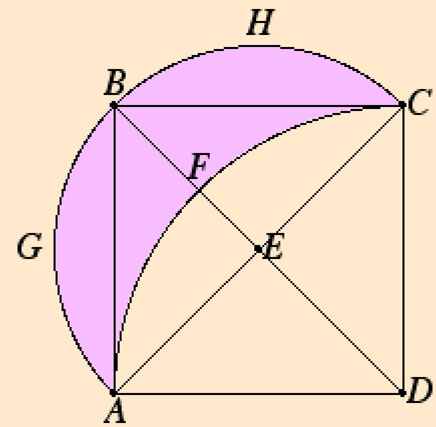
Guide

This proposition is a generalization of I.47 where the squares in I.47 are replaced by any similar rectilinear figures.

Hippocrates' quadrature of lunes

Proclus says that this proposition is Euclid's own, and the proof may be his, but the result, if not the proof, was known long before Euclid, at least in the time of Hippocrates. idea was known to Hippocrates a century before Euclid.

The broad problem Hippocrates was investigating was that of quadrature of a circle, also called squaring a circle, which is to find a square (or any other polygon) with the same area as a given circle. Hippocrates did not solve that problem, but he did solve a related one involving lunes. A *lune* (also called a *crescent*) is a region of nonoverlap of two intersecting circles. Hippocrates did not succeed in squaring an arbitrary lune, but he did succeed in a couple special cases. Here is a summary of the simplist case.



Draw a square *ABCD* with diameters *AC* and *BD* meeting at *E*. Circumscribe a semicircle *AGBHC* about the right isosceles triangle *ABC*. Draw the arc *AFC* from *A* to *C* of a circle with center *D* and radius *DA*.

Hippocrates finds the area of the lune formed between the semicircle *AGBHC* and the arc *AF* as follows.

Note that there are three segments of circles in the diagram; two of them are small, namely, *AGB* with base *AB*, and *BHC* with base *BC*, and one is large, namely, *AFC* with base *AC*. The first two are congruent, and the third is similar to them since all three are segments in quarters of circles.

Hippocrates then uses a version of this proposition VI.31—generalized so the figures don't have to be rectilinear but may have curved sides— to conclude that the sum of the two small segments, *AGB* + *BHC*, equals the large segment *AFC*, since the bases of the small segments are sides of a right triangle while the base of the large segment is the triangle's hypotenuse.

Therefore, the lune, which is the semicircle minus the large segment, equals the semicircle minus the sum of the small segments. But the semicircle minus the sum of the small segments is just the right triangle *ABC*. Thus, a rectilinear figure (the triangle) has been found equal to the lune, as required.

Note that at Hippocrates' time, Eudoxus' theory of proportion had not been developed, so the understanding of the theory of similar figures (Book V) was not as complete as it was after Eudoxus. Also, Eudoxus' principle of exhaustion (see Book XI and proposition X.1) for finding areas of curved figures was still to come, so the concept of area of curved figures was on shaky ground, too. Such a situation is common in mathematics—mathematics advances into new territory long before the foundations of mathematics are developed to logically justify those advances.

Next proposition: VI.32

Previous: VI.30

Book VI introduction

Select from Book VI

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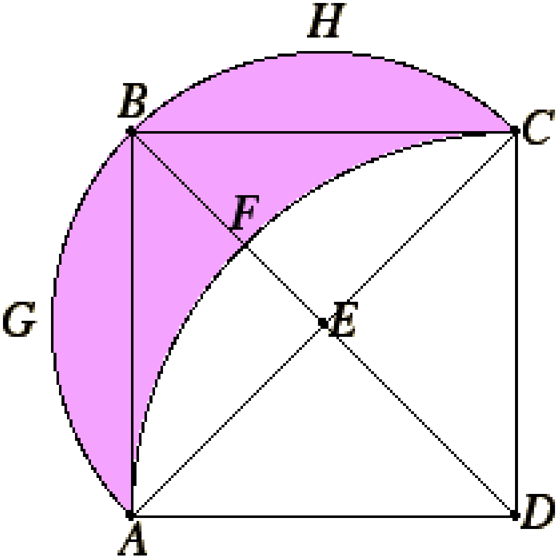
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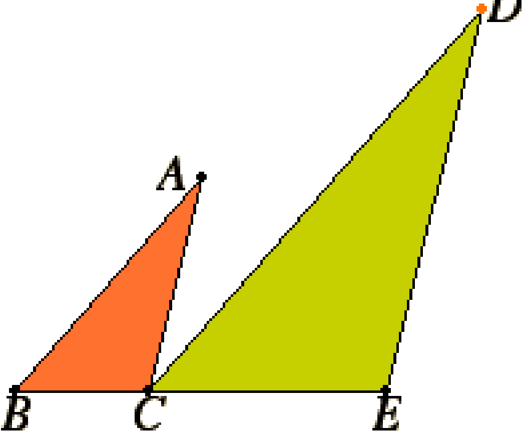
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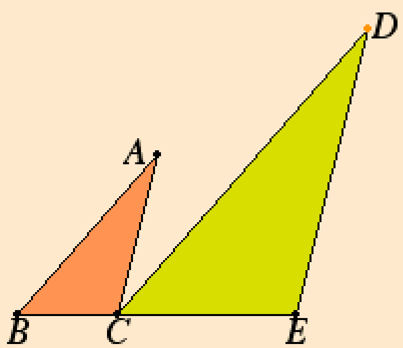
Euclid's Elements
Book VI
Proposition 32

If two triangles having two sides proportional to two sides are placed together at one angle so that their corresponding sides are also parallel, then the remaining sides of the triangles are in a straight line.

Let *ABC* and *DCE* be two triangles having the two sides *AB* and *AC* proportional to the two sides *DC* and *DE,* so that *AB* is to *AC* as *DC* is to *DE,* and *AB* parallel to *DC,* and *AC* parallel to *DE.*

I say that *BC* is in a straight line with *CE.*



Since *AB* is parallel to *DC,* and the straight line *AC* falls upon them, therefore the alternate angles *BAC* and *ACD* equal one another. I.29

For the same reason the angle *CDE* also equals the angle *ACD,* so that the angle *BAC* equals the angle *CDE.*

And, since *ABC* and *DCE* are two triangles having one angle, the angle at *A,* equal to one angle, the angle at *D,* and the sides about the equal angles proportional, so that *AB* is to *AC* as *DC* is to *DE,* therefore the triangle *ABC* is equiangular with the triangle *DCE.* Therefore the angle *ABC* equals the angle *DCE.* VI.6

But the angle *ACD* was also proved equal to the angle *BAC,* therefore the whole angle *ACE* equals the sum of the two angles *ABC* and *BAC.*

Add the angle *ACB* to each. Therefore the sum of the angles *ACE* and *ACB* equals the sum of the angles *BAC, ACB,* and *CBA.*

But the sum of the angles *BAC, ABC,* and *ACB* equals two right angles, therefore the sum of the angles *ACE* and *ACB* also equals two right angles. I.32

Therefore with a straight line *AC,* and at the point *C* on it, the two straight lines *BC* and *CE* not lying on the same side make the sum of the adjacent angles *ACE* and *ACB* equal to two right angles. Therefore *BC* is in a straight line with *CE.* I.14

Therefore, *if two triangles having two sides proportional to two sides are placed together at one angle so that their corresponding sides are also parallel, then the remaining sides of the triangles are in a straight line.*

Q.E.D.

Guide

The corresponding sides mentioned in the statement of the proposition are supposed to be directed in the same direction, even though that is not explicitly stated.

This proposition is used in the proof of proposition XIII.17 which inscribes a regular dodecahedron in a sphere.

Next proposition: VI.33

Previous: VI.31

Book VI introduction

Select from Book VI

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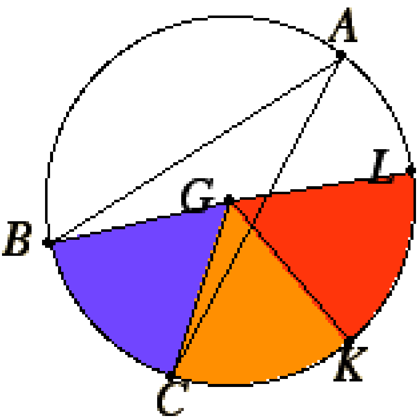
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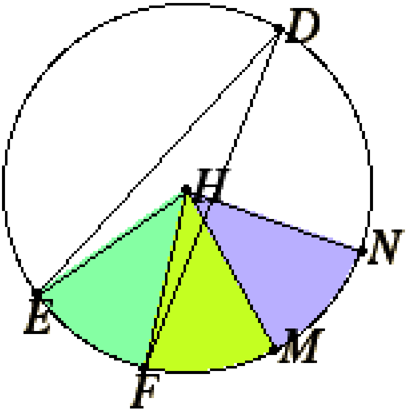
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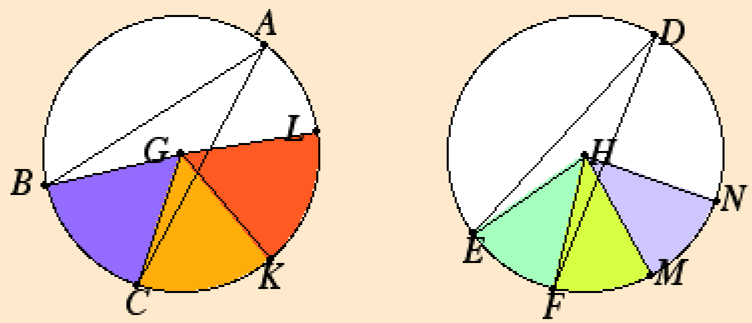
Euclid's Elements
Book VI
Proposition 33

Angles in equal circles have the same ratio as the circumferences on which they stand whether they stand at the centers or at the circumferences.

Let *ABC* and *DEF* be equal circles, and let the angles *BGC* and *EHF* be angles at their centers *G* and *H,* and the angles *BAC* and *EDF* angles at the circumferences.

I say that the circumference *BC* is to the circumference *EF* as the angle *BGC* is to the angle *EHF*, and as the angle *BAC* is to the angle *EDF.*



Make any number of consecutive circumferences *CK* and *KL* equal to the circumference *BC,* and any number of consecutive circumferences *FM, MN* equal to the circumference *EF,* and join *GK* and *GL* and *HM* and *HN.*

Then, since the circumferences *BC, CK,* and *KL* equal one another, the angles *BGC, CGK,* and *KGL* also equal one another. Therefore, whatever multiple the circumference *BL* is of *BC*, the angle *BGL* is also that multiple of the angle *BGC.* III.27

For the same reason, whatever multiple the circumference *NE* is of *EF,* the angle *NHE* is also that multiple of the angle *EHF.*

If the circumference *BL* equals the circumference *EN,* then the angle *BGL* also equals the angle *EHN*; if the circumference *BL* is greater than the circumference *EN,* then the angle *BGL* is also greater than the angle *EHN*; and, if less, less. III.27

There being then four magnitudes, two circumferences *BC* and *EF,* and two angles *BGC* and *EHF*, there have been taken, of the circumference *BC* and the angle *BGC* equimultiples, namely the circumference *BL* and the angle *BGL,* and of the circumference *EF* and the angle *EHF* equimultiples, namely the circumference *EN* and the angle *EHN.*

And it has been proved that, if the circumference *BL* is in excess of the circumference *EN*, the angle *BGL* is also in excess of the angle *EHN*; if equal, equal; and if less, less.

Therefore the circumference *BC* is to *EF* as the angle *BGC* is to the angle *EHF.* V.Def.5

But the angle *BGC* is to the angle *EHF* as the angle *BAC* is to the angle *EDF,* for they are doubles respectively. V.15 III.20

Therefore also the circumference *BC* is to the circumference *EF* as the angle *BGC* is to the angle *EHF*, and the angle *BAC* to the angle *EDF*.

Therefore, *angles in equal circles have the same ratio as the circumferences on which they stand whether they stand at the centers or at the circumferences.*

Q.E.D.

Guide

This proposition stands apart from the rest in this book since it depends on none of them, but it is like the first proposition VI.1 which established a proportion between lines and plane figures since it establishes a proportion between angles and portions of circumferences cut off by those angles.

This proposition is used in three consecutive propositions in Book XIII starting with XIII.8 to convert statements about arcs to statements about angles. Incidentally, all three have to do with regular pentagons inscribed in circles.

Next book: Book VII

Previous proposition: VI.32

Book VI introduction

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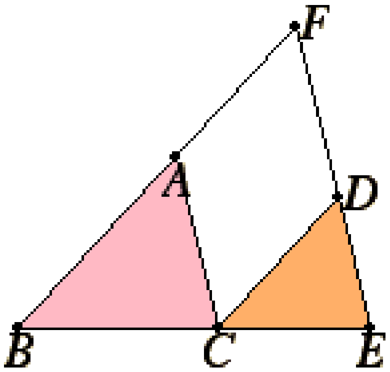
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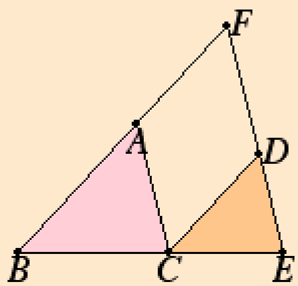
Euclid's Elements
Book VI
Proposition 4

In equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.

Let *ABC* and *DCE* be equiangular triangles having the angle *ABC* equal to the angle *DCE,* the angle *BAC* equal to the angle *CDE,* and the angle *ACB* equal to the angle *CED.*

I say that in the triangles *ABC* and *DEC* the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.



Let *BC* be placed in a straight line with *CE.*

Then, since the sum of the angles *ABC* and *ACB* is less than two right angles, and the angle *ACB* equals the angle *DEC,* therefore the sum of the angles *ABC* and *DEC* is less than two right angles. Therefore *BA* and *ED,* when produced, will meet. Let them be produced and meet at *F.* I.17 I.Post.5

Now, since the angle *DCE* equals the angle *ABC, DC* is parallel to *FB.* Again, since the angle *ACB* equals the angle *DEC, AC* is parallel to *FE.* I.28

Therefore *FACD* is a parallelogram, therefore *FA* equals *DC,* and *AC* equals *FD.* I.34

And, since *AC* is parallel to a side *FE* of the triangle *FBE,* therefore *BA* is to *AF* as *BC* is to *CE.* VI.2

But *FD* equals *AC,* therefore *BC* is to *CE* as *AC* is to *DE,* and alternately *BC* is to *CA* as *CE* is to *ED.* V.7 V.16

Since then it was proved that *AB* is to *BC* as *DC* is to *CE,* and *BC* is to *CA* as *CE* is to *ED,* therefore, *ex aequali, BA* is to *AC* as *CD* is to *DE.* V.22

Therefore, *in equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.*

Q.E.D.

Guide

In the enunciation of this proposition the term "equiangular triangles" refers to two triangles whose corresponding angles are equal, not to two triangles each of which is equiangular (equilateral).

Euclid has placed the triangles in particular positions in order to employ this particular proof. Such positioning is common in Book VI and is easily justified.

This proposition implies that equiangular triangles are similar, a fact proved in detail in the proof of proposition VI.8. It also implies that triangles similar to the same triangle are similar to each other, also proved in detail in VI.8. The latter statement is generalized in VI.21 to rectilinear figures in general.

This proposition is frequently used in the rest of Book VI starting with the next proposition, its converse. It is also used in Books X through XIII.

Next proposition: VI.5

Previous: VI.3

Book VI introduction

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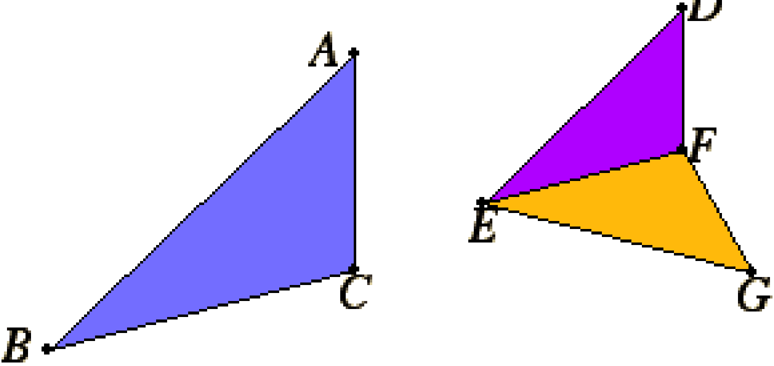
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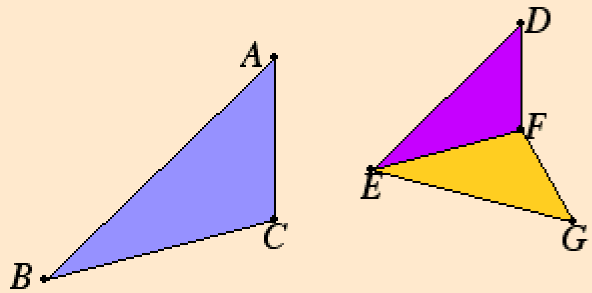
Euclid's Elements
Book VI
Proposition 5

If two triangles have their sides proportional, then the triangles are equiangular with the equal angles opposite the corresponding sides.

Let *ABC* and *DEF* be two triangles having their sides proportional, so that *AB* is to *BC* as *DE* is to *EF*, *BC* is to *CA* as *EF* is to *FD*, and further *BA* is to *AC* as *ED* is to *DF*.

I say that the triangle *ABC* is equiangular with the triangle *DEF* where the equal angles are opposite the corresponding sides, namely the angle *ABC* equals the angle *DEF*, the angle *BCA* equals the angle *EFD*, and the angle *BAC* equals the angle *EDF*.



Construct the angle *FEG* equal to the angle *CBA* and the angle *EFG* equal to the angle *BCA* on the straight line *EF* and at the points *E* and *F* on it. Therefore the remaining angle at *A* equals the remaining angle at *G*. I.23 I.32

Therefore the triangle *ABC* is equiangular with the triangle *GEF*.

Therefore in the triangles *ABC* and *GEF* the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles, therefore *AB* is to *BC* as *GE* is to *EF*. VI.4

But, by hypothesis, *AB* is to *BC* as *DE* to *EF*, therefore *DE* is to *EF* as *GE* is to *EF*. V.11

Therefore each of the straight lines *DE* and *GE* has the same ratio to *EF*, therefore *DE* equals *GE*. V.9

For the same reason *DF* also equals *GF*.

Then since *DE* equals *GE*, and *EF* is common, the two sides *DE* and *EF* equal the two sides *GE* and *EF*, and the base *DF* equals the base *GF*, therefore the angle *DEF* equals the angle *GEF*, and the triangle *DEF* equals the triangle *GEF*, and the remaining angles equal the remaining angles, namely those opposite the equal sides. I.8 I.4

Therefore the angle *DFE* also equals the angle *GFE*, and the angle *EDF* equals the angle *EGF*.

And, since the angle *DEF* equals the angle *GEF*, and the angle *GEF* equals the angle *ABC*, therefore the angle *ABC* also equals the angle *DEF*.

For the same reason the angle *ACB* also equals the angle *DFE*, and further, the angle at *A* equals the angle at *D*, therefore the triangle *ABC* is equiangular with the triangle *DEF*.

Therefore, *if two triangles have their sides proportional, then the triangles are equiangular with the equal angles opposite the corresponding sides.*

Q.E.D.

Guide

Of course, this proposition is the converse of the previous. We now have two characterizations of similar triangles, either as equiangular triangles or as triangles with proportional sides. The next two propositions give two more characterizations corresponding to characterizations of congruent triangles.

As in VI.2, a certain order is assumed for the proportionality. It is not intended, for instance, that *AB*:*BC* = *DE*:*EF* while *BC*:*CA* = *FD*:*EF*. See the remark about VI.Def.1.

This proposition is used in the proof of proposition XII.12.

Next proposition: VI.6

Previous: VI.4

Book VI introduction

Select from Book VI

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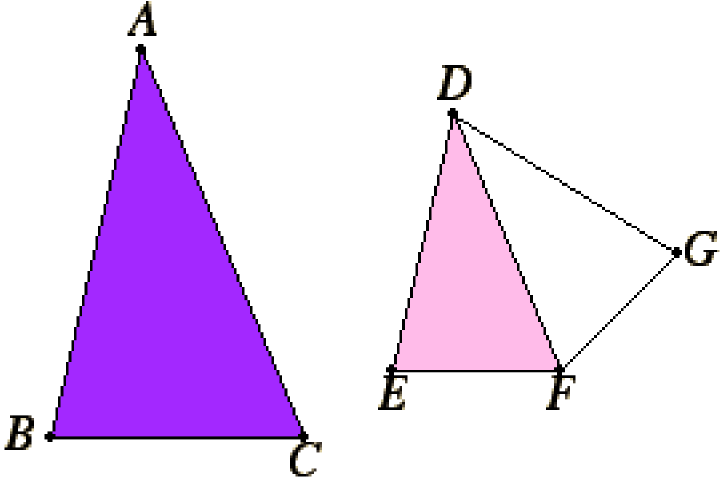
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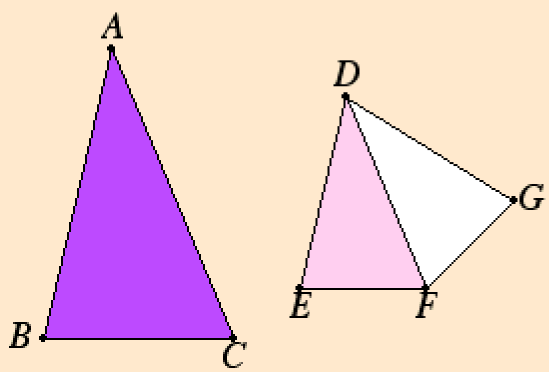
Euclid's Elements
Book VI
Proposition 6

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, then the triangles are equiangular and have those angles equal opposite the corresponding sides.

Let *ABC* and *DEF* be two triangles having one angle *BAC* equal to one angle *EDF* and the sides about the equal angles proportional, so that *BA* is to *AC* as *ED* is to *DF*.

I say that the triangle *ABC* is equiangular with the triangle *DEF*, and has the angle *ABC* equal to the angle *DEF*, and the angle *ACB* equal to the angle *DFE.*



On the straight line *DF* and at the points *D* and *F* on it, construct the angle *FDG* equal to either of the angles *BAC* or *EDF*, and the angle *DFG* equal to the angle *ACB.* I.23

Therefore the remaining angle at *B* equals the remaining angle at *G*. Therefore the triangle *ABC* is equiangular with the triangle DGF. I.32

Therefore, proportionally *BA* is to *AC* as *GD* is to *DF*. VI.4

But, by hypothesis, *BA* is to *AC* also as *ED* is to *DF*, therefore also *ED* is to *DF* as *GD* is to *DF*. V.11

Therefore *ED* equals *GD.* And *DF* is common, therefore the two sides *ED* and *DF* equal the two sides *GD* and *DF*, and the angle *EDF* equals the angle *GDF*, therefore the base *EF* equals the base *GF*, the triangle *DEF* equals the triangle *DGF*, and the remaining angles equal the remaining angles, namely those opposite the equal sides. V.9 I.4

Therefore the angle *DFG* equals the angle *DFE*, and the angle *DGF* equals the angle *DEF*.

But the angle *DFG* equals the angle *ACB*, therefore the angle *ACB* also equals the angle *DFE.*

And, by hypothesis, the angle *BAC* also equals the angle *EDF*, therefore the remaining angle at *B* also equals the remaining angle at *E*. Therefore the triangle *ABC* is equiangular with the triangle *DEF*. I.32

Therefore, *if two triangles have one angle equal to one angle and the sides about the equal angles proportional, then the triangles are equiangular and have those angles equal opposite the corresponding sides.*

Q.E.D.

Guide

This is a side-angle-side similarity theorem analogus to side-angle-side congruence theorem I.4.

Here's a summary of the proof. Construct a triangle *DGF* equiangular with triangle *ABC*. Then triangle *DGF* is similar to triangle *ABC* (VI.4), and that gives us the proportion

$$BA:AC = GD:DF.$$

But we have assumed the proportion

$$BA:AC = ED:DF,$$

and these two proportions together give us

$$GD:DF = ED:DF$$

(V.11), from which it follows that *GD* = *ED* (V.9). Therefore triangles *DEF* and *DGF* are congruent, and the rest follows easily.

Use of this proposition

This proposition is used in the proofs of propositions VI.20, VI.32, XII.1, and several times in XII.12.

Next proposition: VI.7

Previous: VI.5

Book VI introduction

Select from Book VI

Select book

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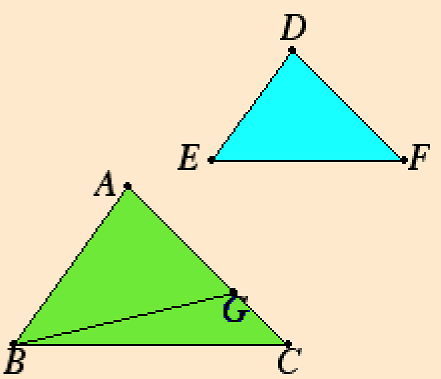
Euclid's Elements
Book VI
Proposition 7

If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, then the triangles are equiangular and have those angles equal the sides about which are proportional.

Let *ABC* and *DEF* be two triangles having one angle equal to one angle, the angle *BAC* equal to the angle *EDF*, the sides about other angles *ABC* and *DEF* proportional, so that *AB* is to *BC* as *DE* is to *EF*. And, first, each of the remaining angles at *C* and *F* less than a right angle.

I say that the triangle *ABC* is equiangular with the triangle *DEF*, the angle *ABC* equals the angle *DEF*, and the remaining angle, namely the angle at *C*, equals the remaining angle, the angle at *F*.



If the angle *ABC* does not equal the angle *DEF*, then one of them is greater.

Let the angle *ABC* be greater. Construct the angle *ABG* equal to the angle *DEF* on the straight line *AB* and at the point *B* on it. I.23

Then, since the angle *A* equals *D*, and the angle *ABG* equals the angle *DEF*, therefore the remaining angle *AGB* equals the remaining angle *DFE*. I.32

Therefore the triangle *ABG* is equiangular with the triangle *DEF*.

Therefore *AB* is to *BG* as *DE* is to *EF*. VI.4

But, by hypothesis, *DE* is to *EF* as *AB* is to *BC*, therefore *AB* has the same ratio to each of the straight lines *BC* and *BG*. Therefore *BC* equals *BG*, so that the angle at *C* also equals the angle *BGC*. V.11 V.9 I.5

But, by hypothesis, the angle at *C* is less than a right angle, therefore the angle *BGC* is also less than a right angle, so that the angle *AGB* adjacent to it is greater than a right angle. I.13

And it was proved equal to the angle at *F*, therefore the angle at *F* is also greater than a right angle. But it is by hypothesis less than a right angle, which is absurd.

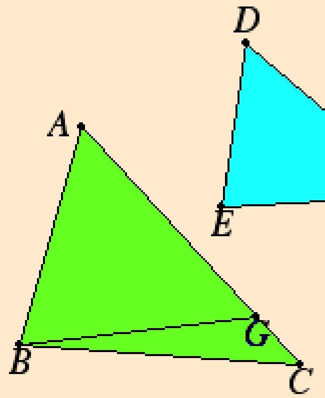
Therefore the angle *ABC* is not unequal to the angle *DEF*. Therefore it equals it.

But the angle at *A* also equals the angle at *D*, therefore the remaining angle at *C* equals the remaining angle at *F*. I.32

Therefore the triangle *ABC* is equiangular with the triangle *DEF*.

Next let each of the angles at *C* and *F* be supposed not less than a right angle.

I say again that, in this case too, the triangle *ABC* is equiangular with the triangle *DEF*.



With the same construction, we can prove similarly that *BC* equals *BG*, so that the angle at *C* also equals the angle *BGC*. I.5

But the angle at *C* is not less than a right angle, therefore neither is the angle *BGC* less than a right angle.

Thus in the triangle *BGC* the sum of two angles is not less than two right angles, which is impossible. I.17

Therefore, once more, the angle *ABC* is not unequal to the angle *DEF*. Therefore it equals it.

But the angle at *A* also equals the angle at *D*, therefore the remaining angle at *C* equals the remaining angle at *F*. I.32

Therefore the triangle *ABC* is equiangular with the triangle *DEF*.

Therefore, *if two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, then the triangles are equiangular and have those angles equal the sides about which are proportional.*

Q.E.D.

Guide

This is a side-side-angle similarity proposition for triangles. The *Elements* does not have the analogous side-side-angle congruence proposition for triangles. See the note on congruence theorems after I.26 for more about congruence theorems.

Next proposition: VI.8

Previous: VI.6

Book VI introduction

Select from Book VI

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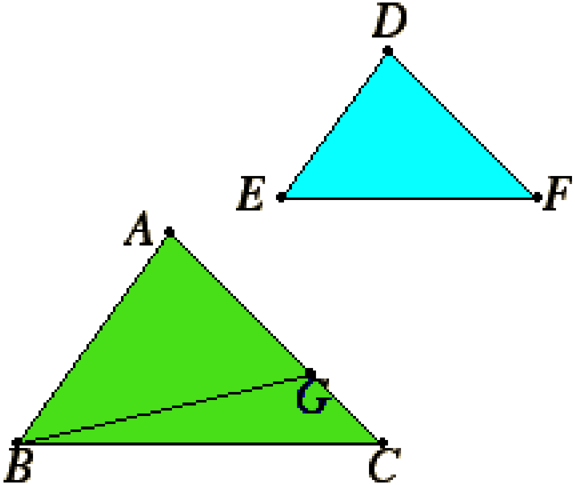
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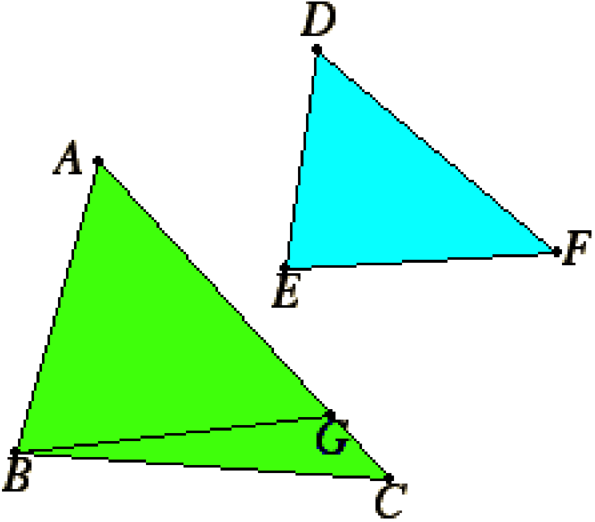
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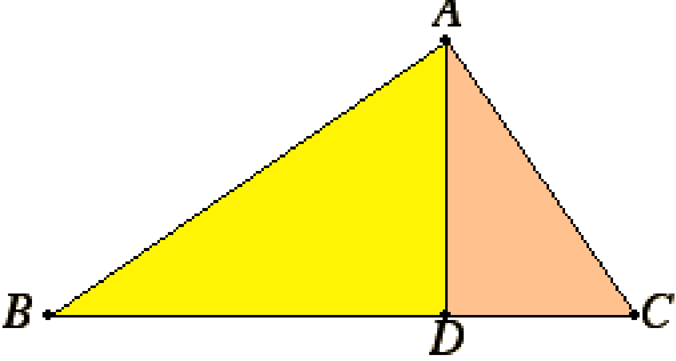
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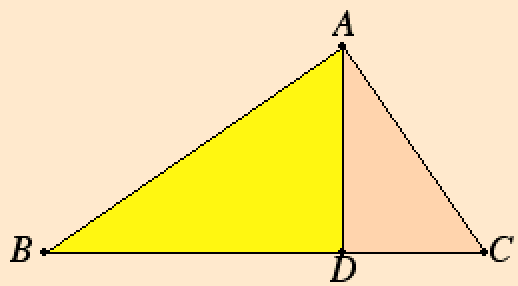
Euclid's Elements
Book VI
Proposition 8

If in a right-angled triangle a perpendicular is drawn from the right angle to the base, then the triangles adjoining the perpendicular are similar both to the whole and to one another.

Let *ABC* be a right-angled triangle having the angle *BAC* right, and let *AD* be drawn from *A* perpendicular to *BC*.

I say that each of the triangles *DBA* and *DAC* is similar to the whole *ABC*, and, further, they are similar to one another.



Since the angle *BAC* equals the angle *BDA*, for each is right, and the angle at *B* is common to the two triangles *ABC* and *DBA*, therefore the remaining angle *ACB* equals the remaining angle *DAB*. Therefore the triangle *ABC* is equiangular with the triangle *DBA*. I.32

Therefore *BC*, which is opposite the right angle in the triangle *ABC*, is to *BA*, which is opposite the right angle in the triangle *DBA*, as *AB*, which is opposite the angle at *C* in the triangle *ABC*, is to *DB*, which is opposite the equal angle *BAD* in the triangle *DBA*, and also as *AC* is to *DA*, which is opposite the angle at *B* common to the two triangles. VI.4

Therefore the triangle *ABC* is both equiangular to the triangle *DBA* and has the sides about the equal angles proportional.

Therefore the triangle *ABC* is similar to the triangle *DBA*. VI.Def.1

In the same manner we can prove that the triangle *DAC* is also similar to the triangle *ADC*. Therefore each of the triangles *DBA* and *DAC* is similar to the whole *ABC*.

I say next that the triangles *DBA* and *DAC* are also similar to one another.

Since the right angle *BDA* equals the right angle *ADC*, and moreover the angle *DAB* was also proved equal to the angle at *C*, therefore the remaining angle at *B* also equals the remaining angle *DAC*. Therefore the triangle *DBA* is equiangular with the triangle *ADC*. I.32

Therefore *BD*, which is opposite the angle *DAB* in the triangle *DBA*, is to *AD*, which is opposite the angle at *C* in the triangle *DAC* equal to the angle *DAB*, as *AD*, itself which is opposite the angle at *B* in the triangle *DBA*, is to *CD*, which is opposite the angle *DAC* in the triangle *DAC* equal to the angle at *B*, and also as *BA* is to *AC*, these sides opposite the right angles. Therefore the triangle *DBA* is similar to the triangle *DAC*. VI.4 VI.Def.1

Therefore, *if in a right-angled triangle a perpendicular is drawn from the right angle to the base, then the triangles adjoining the perpendicular are similar both to the whole and to one another.*

Q.E.D.

Corollary

From this it is clear that, *if in a right-angled triangle a perpendicular is drawn from the right angle to the base, then the straight line so drawn is a mean proportional between the segments of the base.*

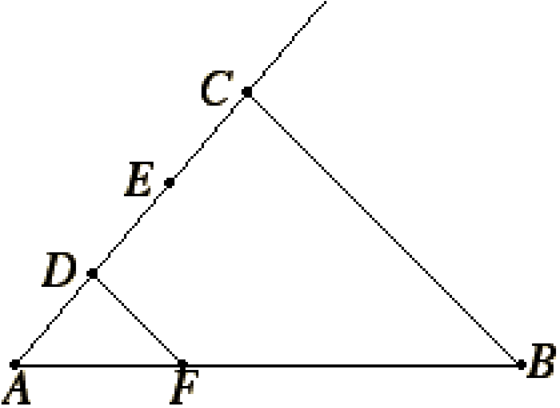
Guide

Essentially, triangles *ABC* and *DBA* are equiangular since they are right triangles with a common angle. Therefore, they are similar. Likewise, triangles *ABC* and *DAC* are similar.

Note that Euclid verbosely draws from proposition VI.4 the conclusions that equiangular triangles are similar and that triangles similar to the same triangle are similar to each other. The general proposition that figures similar to the same figure are also similar to one another is proposition VI.21. There is no reason why that proposition could not have been placed before this one.

This proposition may be used to give an alternate proof of proposition I.47. Indeed, Euclid presents such a proof in the lemma for X.33. That proof is probably older than Euclid's as given in I.47, but Euclid's proof has the advantage of not being dependent on Eudoxus' theory of proportion in Book V.

This proposition and its corollary are used in propositions VI.13, VI.31, X.33, and often in Book XIII.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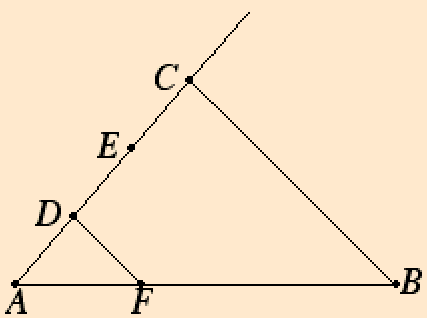
Euclid's Elements
Book VI
Proposition 9

To cut off a prescribed part from a given straight line.

Let *AB* be the given straight line.

It is required to cut off from *AB* a prescribed part.

Let the third part be that prescribed.



Draw a straight line *AC* through from A containing with *AB* any angle. Take a point *D* random on *AC*, and make *DE* and *EC* equal to *AD*. I.3

Join *CB*, and draw *DF* through *D* parallel to it. I.31

Then, since *DF* is parallel to a side *CB* of the triangle *ABC*, therefore, proportionally, *AD* is to *DC* as *AF* is to *FB*. VI.2

But *DC* is double *AD*, therefore *FB* is also double *AF*, therefore *AB* is triple of *AF*.

Therefore from the given straight line *AB* the prescribed third part *AF* has been cut off.

Q.E.F.

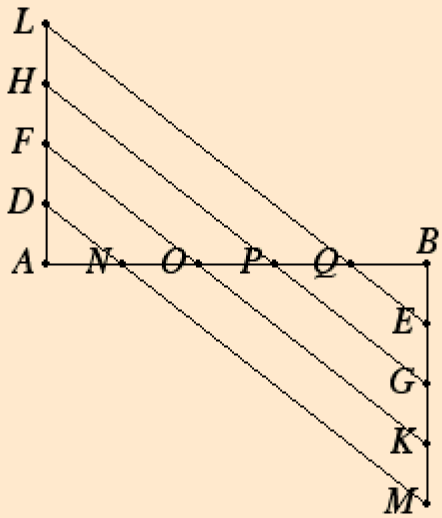
Guide

The word "part" in this proposition means submultiple. The problem here is to divide a line *AB* into some given number *n* of equal parts, or actually, to to find just one of these parts. Euclid takes the case $n = 3$ in his proof.

Simson complained that proving the general case by using a specific case, the one-third part, "is not at all like Euclid's manner." But it is very much Euclid's manner throughout books V and VI to prove a general numerical statement with a specific numerical value.

Al-Nayrizi's construction

Abu'l-Abbas al-Fadl ibn al-Nayrizi (fl. c. 897, d. c. 922) wrote a commentary on the first ten books of the *Elements.* He gives another construction to divide a line *AB* into *n* equal parts. First, construct equal perpendiculars at *A* and *B* in opposite directions, mark off $n - 1$ equal parts on each of them, and connect the



points as illustrated. The diagram shows *AB* divided into five equal parts.

Al-Nayrizi's construction takes considerably less work than Euclid's. The proof that this construction is valid is about the same length as that for Euclid's construction.

Use of this construction

This construction is used in a few propositions in Book XIII to find a third or a fifth of a line.

Next proposition: VI.10

Previous: VI.8

Book VI introduction

Select from Book VI

Select book

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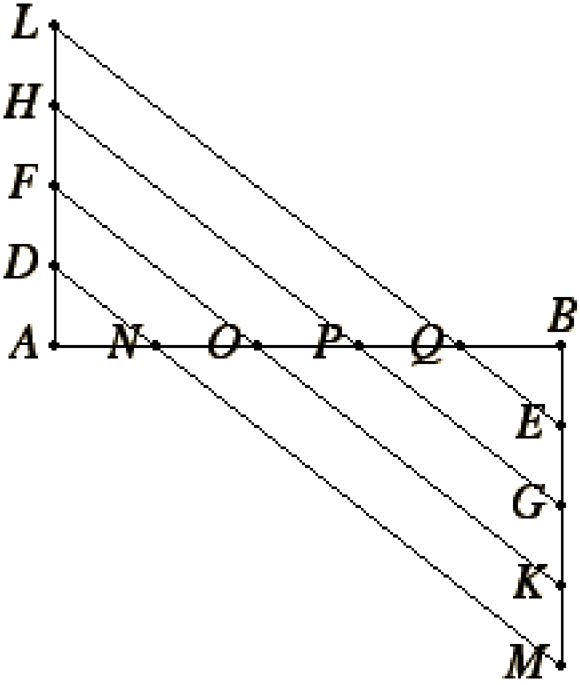
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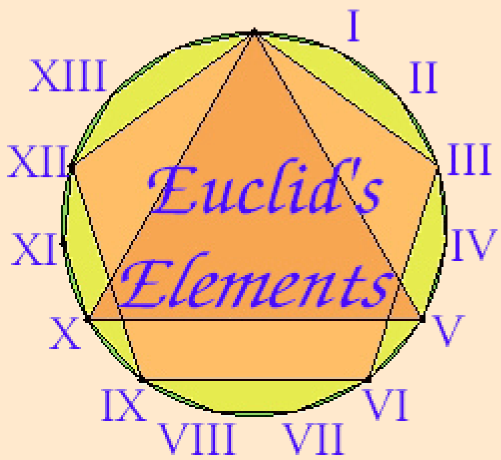
Book VII



Book VII

Table of contents

Definitions

Definition 1

A *unit* is that by virtue of which each of the things that exist is called one.

Definition 2

A *number* is a multitude composed of units.

Definition 3

A number is *a part* of a number, the less of the greater, when it measures the greater;

Definition 4

But *parts* when it does not measure it.

Definition 5

The greater number is a *multiple* of the less when it is measured by the less.

Definition 6

An *even number* is that which is divisible into two equal parts.

Definition 7

An *odd number* is that which is not divisible into two equal parts, or that which differs by a unit from an even number.

Definition 8

An *even-times even number* is that which is measured by an even number according to an even number.

Definition 9

An *even-times odd number* is that which is measured by an even number according to an odd number.

Definition 10

An *odd-times odd number* is that which is measured by an odd number according to an odd number.

Definition 11

A *prime number* is that which is measured by a unit alone.

Definition 12

Numbers *relatively prime* are those which are measured by a unit alone as a common measure.

Definition 13

A *composite number* is that which is measured by some number.

Definition 14

Numbers *relatively composite* are those which are measured by some number as a common measure.

Definition 15

A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other.

Definition 16

And, when two numbers having multiplied one another make some number, the number so produced be called *plane*, and its *sides* are the numbers which have multiplied one another.

Definition 17

And, when three numbers having multiplied one another make some number, the number so produced be called *solid*, and its *sides* are the numbers which have multiplied one another.

Definition 18

A *square number* is equal multiplied by equal, or a number which is contained by two equal numbers.

Definition 19

And a *cube* is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

Definition 20

Numbers are *proportional* when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

Definition 21

Similar plane and *solid* numbers are those which have their sides proportional.

Definition 22

A *perfect number* is that which is equal to the sum its own parts.

Propositions

Proposition 1

When two unequal numbers are set out, and the less is continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, then the original numbers are relatively prime.

Proposition 2
To find the greatest common measure of two given numbers not relatively prime.

Corollary. If a number measures two numbers, then it also measures their greatest common measure.

Proposition 3
To find the greatest common measure of three given numbers not relatively prime.

Proposition 4
Any number is either a part or parts of any number, the less of the greater.

Proposition 5
If a number is part of a number, and another is the same part of another, then the sum is also the same part of the sum that the one is of the one.

Proposition 6
If a number is parts of a number, and another is the same parts of another, then the sum is also the same parts of the sum that the one is of the one.

Proposition 7
If a number is that part of a number which a subtracted number is of a subtracted number, then the remainder is also the same part of the remainder that the whole is of the whole.

Proposition 8
If a number is the same parts of a number that a subtracted number is of a subtracted number, then the remainder is also the same parts of the remainder that the whole is of the whole.

Proposition 9
If a number is a part of a number, and another is the same part of another, then alternately, whatever part of parts the first is of the third, the same part, or the same parts, the second is of the fourth.

Proposition 10
If a number is a parts of a number, and another is the same parts of another, then alternately, whatever part of parts the first is of the third, the same part, or the same parts, the second is of the fourth.

Proposition 11
If a whole is to a whole as a subtracted number is to a subtracted number, then the remainder is to the remainder as the whole is to the whole.

Proposition 12
If any number of numbers are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.

Proposition 13
If four numbers are proportional, then they are also proportional alternately.

Proposition 14
If there are any number of numbers, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio *ex aequali*.

Proposition 15
If a unit number measures any number, and another number measures any other number the same number of times, then alternately, the unit measures the third number the same number of times that the second measures the fourth.

Proposition 16
If two numbers multiplied by one another make certain numbers, then the numbers so produced equal one another.

Proposition 17
If a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.

Proposition 18
If two number multiplied by any number make certain numbers, then the numbers so produced have the same ratio as the multipliers.

Proposition 19
If four numbers are proportional, then the number produced from the first and fourth equals the number produced from the second and third; and, if the number produced from the first and fourth equals that produced from the second and third, then the four numbers are proportional.

Proposition 20
The least numbers of those which have the same ratio with them measure those which have the same ratio with them the same number of times; the greater the greater; and the less the less.

Proposition 21
Numbers relatively prime are the least of those which have the same ratio with them.

Proposition 22
The least numbers of those which have the same ratio with them are relatively prime.

Proposition 23
If two numbers are relatively prime, then any number which measures one of them is relatively prime to the remaining number.

Proposition 24
If two numbers are relatively prime to any number, then their product is also relatively prime to the same.

Proposition 25
If two numbers are relatively prime, then the product of one of them with itself is relatively prime to the remaining one.

Proposition 26
If two numbers are relatively prime to two numbers, both to each, then their products are also relatively prime.

Proposition 27
If two numbers are relatively prime, and each multiplied by itself makes a certain number, then the products are relatively prime; and, if the original numbers multiplied by the products make certain numbers, then the latter are also relatively prime.

Proposition 28
If two numbers are relatively prime, then their sum is also prime to each of them; and, if the sum of two numbers is relatively prime to either of them, then the original numbers are also relatively prime.

Proposition 29
Any prime number is relatively prime to any number which it does not measure.

Proposition 30
If two numbers, multiplied by one another make some number, and any prime number measures the product,

then it also measures one of the original numbers.

Proposition 31
Any composite number is measured by some prime number.

Proposition 32
Any number is either prime or is measured by some prime number.

Proposition 33
Given as many numbers as we please, to find the least of those which have the same ratio with them.

Proposition 34
To find the least number which two given numbers measure.

Proposition 35
If two numbers measure any number, then the least number measured by them also measures the same.

Proposition 36
To find the least number which three given numbers measure.

Proposition 37
If a number is measured by any number, then the number which is measured has a part called by the same name as the measuring number.

Proposition 38
If a number has any part whatever, then it is measured by a number called by the same name as the part.

Proposition 39
To find the number which is the least that has given parts.

Guide

Book VII is the first of the three books on number theory. It begins with the 22 definitions used in these books. The important definitions being those for unit and number, part and multiple, even and odd, prime and relatively prime, proportion, and perfect number. The topics in Book VII are antenaresis and the greatest common divisor, proportions of numbers, relatively prime numbers and prime numbers, and the least common multiple.

The basic construction for Book VII is *antenaresis,* also called the *Euclidean algorithm,* a kind of reciprocal subtraction. Beginning with two numbers, the smaller, whichever it is, is repeatedly subtracted from the larger until a single number is left. This algorithm, studied in propositions VII.1 through VII.3, results in the greatest common divisor of two or more numbers.

Propositions V.5 through V.10 develop properties of fractions, that is, they study how many parts one number is of another in preparation for ratios and proportions. The next group of propositions VII.11 through VII.19 develop the theory of proportions for numbers.

Propositions VII.20 through VII.29 discuss representing ratios in lowest terms as relatively prime numbers and properties of relatively prime numbers. Properties of prime numbers are presented in propositions VII.30 through VII.32. Book VII finishes with least common multiples in propositions VII.33 through VII.39.

Postulates for numbers

Postulates are as necessary for numbers as they are for geometry. Missing postulates occurs as early as proposition VII.2. In its proof, Euclid constructs a decreasing sequence of whole positive numbers, and, apparently, uses a principle that conclude that the sequence must stop, that is, there cannot be an infinite decreasing sequence of numbers. If that is the principle he uses, then it ought to be stated as a postulate for numbers.

Numbers are so familiar that it hardly occurs to us that the theory of numbers needs axioms, too. In fact, that field was one of the last to receive a careful scrutiny, and axioms for numbers weren't developed until the late 19th century. By that time foundations for the rest of mathematics were laid upon either geometry or number theory or both, and only geometry had axioms. About the same time that foundations for number theory were developed, a new subject, set theory, was created by Cantor, and mathematics was refounded in terms of set theory.

The foundations of number theory will be discussed in the Guides to the various definitions and propositions.

Next book: Book VIII

Previous: Book VI

Book VII introduction

Select from Book VII

Select book

Select topic

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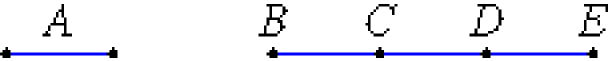
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Euclid's Elements
Book VII
Definitions 1 and 2

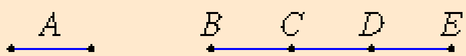
Def. 1. A *unit* is that by virtue of which each of the things that exist is called one.

Def. 2. A *number* is a multitude composed of units.

Guide

These 23 definitions at the beginning of Book VII are the definitions for all three books VII through IX on number theory. Some won't be used until Books VIII or IX.

These first two definitions are not very constructive towards a theory of numbers. The numbers in definition 2 are meant to be whole positive numbers greater than 1, and definition 1 is meant to define the unit as 1. The word "monad," derived directly from the Greek, is sometimes used instead of "unit."



Throughout these three books on number theory Euclid exhibits numbers as lines. In the diagram above, if *A* is the unit, then *BE* is the number 3. But, just because he draws them as lines does not mean they are lines, and he never calls them lines.

It is not clear what the nature of these numbers is supposed to be. But their nature is irrelevant. Euclid could illustrate the unit as a line or as any other magnitude, and numbers would then be illustrated as multiples of that unit.

There is a major distinction between lines and numbers. Lines are infinitely divisible, but numbers are not, in particular, the unit is not divisible into smaller numbers.

Euclid has no postulates to elaborate the concept of number (other than the Common Notions which are meant to apply to numbers as well as magnitudes of various kinds). Indeed, mathematicians did not develop foundations for number theory until the late nineteenth century. Peano's axioms for numbers are the best known. The most important of Peano's axioms is the principle of mathematical induction which states that

1. if a property of numbers holds for 1,
2. and whenever property holds for n then it also holds for $n + 1$,
3. then the property holds for all numbers.

Euclid does not use the principle of mathematical induction, but he does implicitly use a similar property of numbers, namely, that any decreasing sequence of numbers is finite. That property is known variously as the "well-ordering principle" for numbers and the "descending chain condition." We will discuss it later in more detail.

Next definitions: VII.3-5

Select from Book VII

Book VII introduction

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Euclid's Elements
Book VII
Definitions 11 through 14

Def. 11. A *prime number* is that which is measured by a unit alone.

Def. 12. Numbers *relatively prime* are those which are measured by a unit alone as a common measure.

Def. 13. A *composite number* is that which is measured by some number.

Def. 14. Numbers *relatively composite* are those which are measured by some number as a common measure.

Guide

Prime numbers form a very important class of numbers, and much of number theory is devoted to their analysis. The only proper divisor of a prime number is 1. The first few prime numbers are, of course, 2, 3, 5, 7, 11. Those numbers that aren't prime are composite, for instance, 4, 6, 8, 9, 10. The number 1 holds a special position. For Euclid, it was the unit rather than a number. For modern mathematicians 1 is also a unit, but in a different sense of the word, since it has a reciprocal, namely, itself.

Numbers are relatively prime if their only common divisor is 1. For example, 6 and 35 are relatively prime (although neither is a prime number in itself). This situation is also phrased as "6 is prime to 35." For another example, the three numbers 6, 10, and 15 are relatively prime since no number (except 1) divides all three. If the numbers aren't relatively prime, then they're called "relatively composite," a term rarely used now.

Next definitions: VII.Def.15-19

Previous: VII.Def.6-10

Book VII introduction

Select from Book VII

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Euclid's Elements
Book VII
Definitions 15 through 19

Def. 15. A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other.

Def. 16. And, when two numbers having multiplied one another make some number, the number so produced be called *plane,* and its *sides* are the numbers which have multiplied one another.

Def. 17. And, when three numbers having multiplied one another make some number, the number so produced be called *solid,* and its *sides* are the numbers which have multiplied one another.

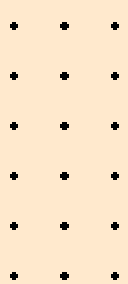
Def. 18. A *square number* is equal multiplied by equal, or a number which is contained by two equal numbers.

Def. 19. And a *cube* is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

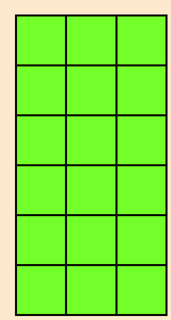
Guide

Notice that Euclid doesn't define addition and subtraction. Those operations are assumed to be understood. But multiplication and proportion are defined, and proportion is defined next in VII.Def.20. Definition 15 defines multiplication in terms of addition as a kind of composition. For instance, if 3 is multiplied by 6, then since 6 is 1+1+1+1+1+1, therefore, 3 multiplied by 6 is 3+3+3+3+3+3. The first proposition on multiplication is VII.16 which says multiplication is commutative. For our example, that would say 3 multiplied by 6 equals 6 multiplied by 3, which is 6+6+6.

Figurate numbers

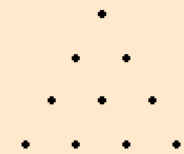


Although Euclid never displays numbers except as lines, the Pythagoreans before him evidently did, that is, they displayed numbers as figures. The figures were in various shapes, such as triangles, squares, and so forth. Definitions 16 through 19 deal with figurate numbers, but without the figures. Euclid defines a plane number as a number which is the product of two numbers. Remember that for Euclid, 1 is the unit, not a number, so a prime number is not a plane number, even though it is a product of 1 and itself.



Plane numbers are the composite numbers. Each composite number can be a plane number in at least one way, but most in more than one way. For instance, 16 can be viewed as a plane number either with sides 2 and 8 or with sides 4 and 4, that is, as a square number.

Plane numbers can be displayed as rectangular configurations of dots. Alternatively, these "rectangular numbers" can be displayed as a configuration of squares. But most of the other figurate numbers, such as triangular numbers, could only easily be displayed by dots.



Perhaps for the Pythagoreans, the most important figures were the triangular numbers: 3, 6, especially 10, 15, 21, etc. Each could be formed from the previous by adding a new row one unit longer. So, for instance, 10 = 1 + 2 + 3 + 4. For some reason, Euclid doesn't mention triangular numbers. Indeed, he doesn't address sums of arithmetic progressions at all, a subject of interest in many ancient cultures. Euclid does give the sum of a geometric progression, that is, a continued proportion, in proposition IX.35.

Definition 18 defines solid numbers. For example, if 18 is presented as 3 times 3 times 2, then it is given as a solid number with three sides 3, 3, and 2. Solid numbers can be represented as a configuration of dots or cubes in three dimensions.

Squares and cubes are are described as certain symmetric plane and solid numbers. Of course, some numbers, such as 64, can be simultaneously squares and cubes.

Next definitions: VII.Def.20

Previous: VII.Def.11-14

Book VII introduction

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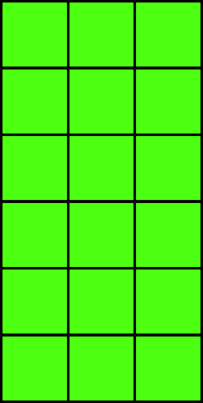

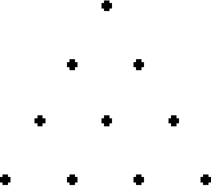

Euclid's Elements
Book VII
Definition 20

Def. 20. Numbers are *proportional* when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

Guide

Definition V.20 for proportionality of numbers is not the same as the definition of proportionality for magnitudes in Book V given in V.Def.5. This definition for numbers was probably the earlier one, but as not all magnitudes are commensurable, it cannot adequately define proportionality for magnitudes.

This definition VII.20 is given by cases. The various cases correspond to defintions VII.Def.3 through VII.Def.6 for part, parts, and multiple.

When four numbers, *j, k, m,* and *n,* are proportional, we'll write that symbolically as $j:k = m:n$,

In the first case, *j* is the same multiple of *k* as *m* is of *n.* An example of this is the proportion 12:6 = 22:11, where 12 is twice 6 and 22 is twice 11.

The second case is inverse to the first, *j* is the same part of *k* as *m* is of *n.* For an example take the proportion 6:12 = 22:11, where 6 is one half of 12, and 11 is one half of 22.

For an example of the third case, consider 12:16 = 21:28. Since the first is the same parts of the second, namely 3 parts of 4, as the the third is of the fourth, the proportion holds. Actually, there should be a fourth case (inverse to the third case) when the second is the same parts of the first as the fourth is of the third, as 16:12 = 28:21. Of course, these cases could be merged into one by considering 1 to be a number and not distinguishing when the first is greater or less than the second.

Ratios of numbers

Although the word "ratio" doesn't appear in this definition, it appears frequently beginning in proposition VII.14. In book VII ratio is restricted to the use of saying when one ratio is the same as another, that is, there is a proportion as defined in this definition. In Book VIII, duplicate ratios, triplicate ratios, and other compounded ratios appear. Definitions for these concepts are not explicitly given, but once the concept of proportion has been defined, they have the same defintion given in Book V for duplicate and triplicate ratio in V.Def.9-10. Compound ratios aren't defined in Book V, but they can be understood by their use. See the Guide to V.Def.3. Various other definitions that go along with ratios and proportions were given in Book V, for instance, alternate ratios, inverse ratios, taken jointly, taken separately, and *ex aequali.* These definitions are also not repeated here in Book VII.

Very soon in these books on number theory Euclid begins to rely on properties of proportion proved in Book V using the other definition of proportion. That these are valid for proportions of numbers could be verified individually or by showing that the two definitions of proportion are equivalent for numbers.

Use of this defintion

Proportions of numbers first appear in proposition VII.11, but all the propositions from VII.4 through VII.10 are in preparation for the study of numeric proportions.

Next definition: VII.Def.21

Previous: VII.Def.15-19

Book VII introduction

Select from Book VII

Select book

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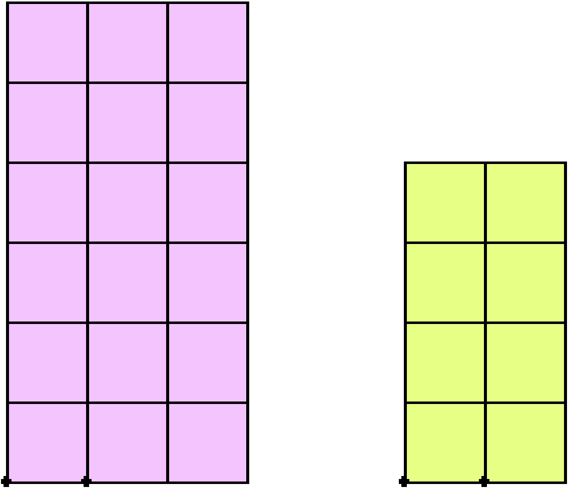


Euclid's Elements
Book VII
Definition 21

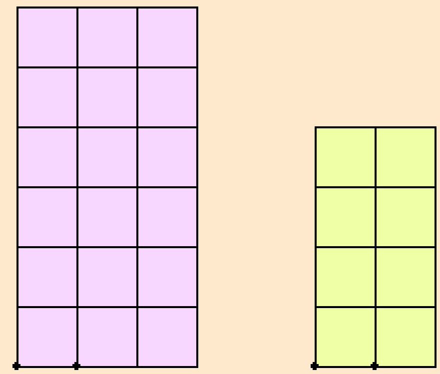
Def. 21. *Similar plane* and *solid* numbers are those which have their sides proportional.

Guide

The numbers 18 and 8 are similar plane numbers.

When 18 is interpreted as a plane number with sides 6 and 3, and 8 has sides 4 and 2, then the sides are proportional.

Proposition VIII.18 shows that the ratio of two similar plane numbers is the duplicate ratio of the corresponding sides. In this example, the ratio 18:8 is duplicate of the ratio 6:4.



To illustrate similar solid numbers, consider the two numbers 240 and 810 when represented as 4 times 6 times 10 and 6 times 9 times 15, respectively.

Proposition VIII.19 shows that the ratio of two similar solid numbers is the triplicate ratio of the corresponding sides. In this example, the ratio 240:810 is triplicate of the ratio 4:6.

Next definition: VII.Def.22

Previous: VII.Def.20

Book VII introduction

Select from Book VII

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Euclid's Elements
Book VII
Definition 22

A *perfect number* is that which is equal to the sum of its own parts.

Guide

For example, the number 28 is perfect because its parts (that is, proper divisors) 1, 2, 4, 7, and 14 sum to 28. The four smallest perfect numbers were known to the ancient Greek mathematicians. They are 6, 28, 496, and 8128. In proposition IX.36 Euclid gives a construction of even perfect numbers.

The divisors of these even perfect numbers can be listed in two columns, illustrated here for the divisors of 496.

1	31
2	62
4	124
8	248
16	(496)

The first column lists powers of 2 from 2^0 up through 2^4. The sum of these powers of 2 is 31, which is one less than 2^5. That number 31 appears at the top of the second column, and its repeated doubles up through 496 appear on the second column. In such a tableau, the sum of all the numbers, except the last, will equal the last.

The question of odd perfect numbers was not solved by Euclid. Probably the oldest open conjecture in mathematics is that there are no ood perfect numbers. There is no proof yet, but it is known that if there is an odd perfect number, then it has to be immensely huge.

Next proposition: VII.1

Previous: VII.Def.21

Book VII introduction

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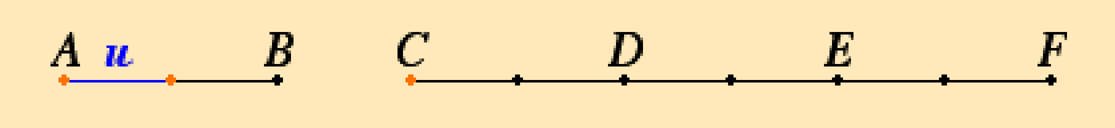
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Euclid's Elements

Book VII

Definitions 3 through 5

Def. 3. A number is *a part* of a number, the less of the greater, when it measures the greater;

Def. 4. But *parts* when it does not measure it.

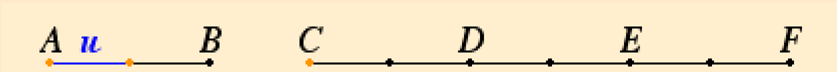
Def. 5. The greater number is a *multiple* of the less when it is measured by the less.

Guide

These definitions are in preparation for the definition of proportion of numbers given in VII.Def.20. In the current definitions, the possible relations between a pair of numbers, *m* and *n,* are classified. Later in Book VII, the term "ratio" will be used for this relation.

In all three of these definitions, the concept of "measures" is assumed to be understood. There is more to these definitions than meets the eye, though, at least part of the intent is evident.

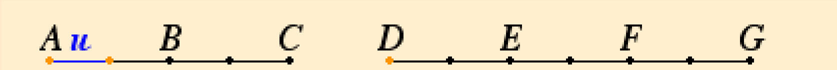
To illustrate VII.Def.3, take 2, which is a part of 6, namely, the one-third part of 6.



If *u* is the unit, then 2 is represented as *AB* while 6 is represented by *CF.* As *AB* measures *CF* three times by *CD, DE,* and *DF,* therefore 2 is a part of 6, namely, the one-third part since it measures 6 three times.

We can also use the same figure as an illustration of VII.Def.5 to see that 6 is a multiple of 2, in particular, the third multiple of 2.

Definition VII.Def.4 is less clear, but its intent can be read from the use to which it's put in VII.Def.20 for proportions of numbers. For an example, consider the numbers 4 and 6. The number 4 does not measure the number 6, but it is parts of 6.



Here, 4 is represented as *AC* while 6 is represented as *DG.* Clearly, *AC* does not measure *DG.* The way this definition is used in VII.Def.4, just the knowledge that "4 is parts of 6" is not enough, what is also needed is how many parts of 6 is 4. This will be needed to define a proportion such as 4:6 = 6:9. That proportion is supposed to hold since 4 is the same parts of 6 as 6 is of 9, namely 2 third parts. Thus, one number being parts of another also carries along with it how many of what parts.

There is one more difficulty with this definition. It seems obvious that when one number *m* is less than another *n,* then in all cases *m* would be parts of *n,* namely *m* consists of *m* one-n^{th} parts of *n.* Yet, the proposition VII.4 has a proof to show that *m* is either a part or parts of *n.*

Divisors

Where Euclid would say that *m* is a part of *n*, modern mathematicians would say that *m* is a proper divisor of *n*. A *divisor* of *n* is any whole number *m* (including 1) that divides *n* in the sense that there is another number *k* such that *mk* = *n*. A *proper divisor* of *n* is any divisor except *n* itself.

For example, the proper divisors of the number 12 are 1, 2, 3, 4, and 6.

Next definitions: VII.Def.6-10

Previous: VII.Def.1-2

Book VII introduction

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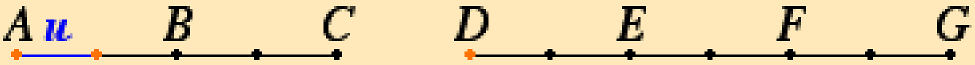
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Euclid's *Elements* Book VII
Definitions 6 through 10

Def. 6. An *even number* is that which is divisible into two equal parts.

Def. 7. An *odd number* is that which is not divisible into two equal parts, or that which differs by a unit from an even number.

Def. 8. An *even-times even number* is that which is measured by an even number according to an even number.

Def. 9. An *even-times odd number* is that which is measured by an even number according to an odd number.

Def. 10. An *odd-times odd number* is that which is measured by an odd number according to an odd number.

Guide

Definition 6 for "even number" is clear: the number n is even if it is of the form $m + m$.

Definition 7 for "odd number" has two statements. The first can be taken as a definition of odd number, a number which is not divisible into two equal parts, that is to say not an even number.

The other statement is not a definition for odd number, since one has already been given, but an unproved statement. It is easy to recognize that something has to be proved, since if we make the analogous definitions for another number, say 10, then analogous statement is false. Suppose we say a "decade number" is one divisible by 10, and and "undecade number' is one not divisible by 10. Then it is not the case that an undecade number differs by a unit from a decade number; the number 13, for instance, is not within 1 of a decade number.

The unproved statement that a number differing from an even number by 1 is an odd number ought to be proved. That statment is used in proposition IX.22 and several propositions that follow it. It could be proved using, for instance, a principle that any decreasing sequence of numbers is finite.

Definitions 8-10 are also clear. A product of two even numbers is an even-times even number; a product of an even and an odd number is an even-times odd number; and a product of of two odd numbers is an odd-times odd number. Note that a number like 12 is both even-times even and even-times odd being at the same time 2 times 6 and 4 times 3.

The numbers which are even-times even but not even-times odd are just the powers of 2: 4, 8, 16, 32, etc. These are the numbers which are even-times even only, and they occur in proposition IX.32.

Next definitions: VII.Def.11-14

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Previous: VII.Def.3-5

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propVII7.gif	16-May-1997	10:12	1k
propVII9.gif	16-May-1997	10:12	1k

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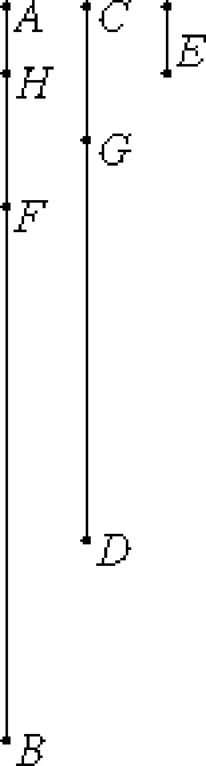
A
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F
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C
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D
E

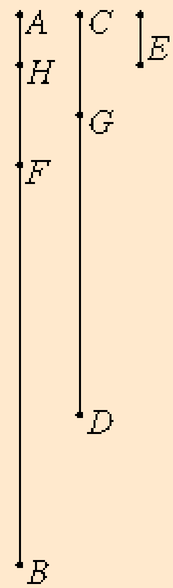
Euclid's Elements
Book VII
Proposition 1

When two unequal numbers are set out, and the less is continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, then the original numbers are relatively prime.

The less of two unequal numbers *AB* and *CD* being continually subtracted from the greater, let the number which is left never measure the one before it until a unit is left.

I say that *AB* and *CD* are relatively prime, that is, that a unit alone measures *AB* and *CD*. VII.Def.12

If *AB* and *CD* are not relatively prime, then some number *E* measures them. Let *CD,* measuring *BF,* leave *FA* less than itself, let *AF,* measuring *DG,* leave *GC* less than itself, and let *GC,* measuring *FH,* leave a unit *HA.*



Since, then, *E* measures *CD,* and *CD* measures *BF,* therefore *E* also measures *BF.*

But it also measures the whole *BA,* therefore it measures the remainder *AF.* But *AF* measures *DG,* therefore *E* also measures *DG.* But it also measures the whole *DC,* therefore it also measures the remainder *CG.*

But *CG* measures *FH,* therefore *E* also measures *FH.* But it also measures the whole *FA,* therefore it measures the remainder, the unit *AH,* though it is a number, which is impossible.

Therefore no number measures the numbers *AB* and *CD.* Therefore *AB* and *CD* are relatively prime. VII.Def.12

Therefore, *when two unequal numbers are set out, and the less is continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, then the original numbers are relatively prime.*

Q.E.D.

Guide

Modern terminology uses the word "divides" rather than "measures," and the notation $a \mid b$ is used to abbreviate the phrase "*a* divides *b.*"

This proposition assumes that 1 is the result of an antenaresis process. *Antenaresis,* also called the *Euclidean algorithm,* is a kind of reciprocal subtraction. Beginning with two numbers, the smaller, whichever it is, is repeatedly subtracted from the larger.

If the initial two numbers are a_1 (*AB* in the proof) and a_2 (*CD*), with a_1 greater than a_2, then the first stage is to repeatedly subtract a_2 from a_1 until a remainder a_3 (*AF*) less than a_2 is found. That can be stated algebraically as

$$a_1 = m_1 a_2 + a_3$$

where m_1 is the number of times that a_2 was subtracted from a_1.

The next stage repeatedly subtracts a_3 from a_2 leaving a remainder a_4 (*CG*):

$$a_2 = m_2 a_3 + a_4.$$

For the hypotheses of this proposition, the algorithm stops when a remainder of 1 occurs:

$$a_{n-1} = m_{n-1} a_n + 1.$$

(In Euclid's proof, a_n is a_5 which is *AH*.) The conclusion is that a_1 and a_2 are relatively prime.

The proof is not difficult. It depends on the observation that if b divides (that is, measures) both c and d, then b divides their difference $c - d$. So, if some number b divides both a_1 and a_2, then it divides the remainder a_3, too. And since it divides both a_2 and a_3, it divides the remainder a_4. And so forth, with the final conclusion that b divides the last remainder 1.

Since there is no number b (and by "number" is meant a number greater than 1) which divides 1, there is no number that divides both a_1 and a_2. Therefore a_1 and a_2 are relatively prime.

Compare this proposition to X.2, a somewhat analogous statement about magnitudes.

This proposition is used in the proof of the next one.

Next proposition: VII.2

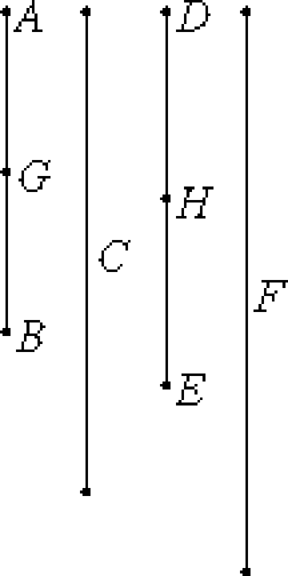
Previous: VII.Def.22

Book VII introduction

Select from Book VII

Select book

Select topic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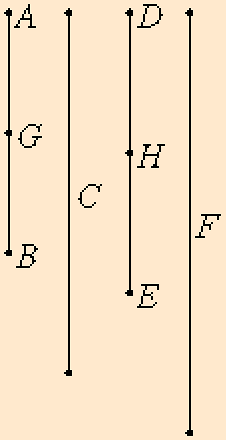
Euclid's Elements
Book VII
Proposition 10

If a number is parts of a number, and another is the same parts of another, then alternately, whatever part of parts the first is of the third, the same part, or the same parts, the second is of the fourth.

Let the number *AB* be parts of the number *C*, and another number *DE* be the same parts of another number *F*.

I say that, alternately, *C* is the same parts or part of *F* that *AB* is of *DE*.



Since *DE* is the same parts of *F* as *AB* is of *C*, therefore *F* is the same parts of *DE* as *C* is of *AB*.

Divide *AB* into the parts of *C*, namely *AG* and *GB*, and divide *DE* into the parts of *F*, namely *DH* and *HE*. Then the multitude of *AG* and *GB* equals the multitude of *DH* and *HE*.

Now since *DH* is the same part of *F* as *AG* is of *C*, therefore, alternately, *C* is the same part or the same parts of *F* as *AG* is of *DH*. VII.9

VII.9

For the same reason, *C* is the same part or the same parts of *F* as *GB* is of *HE*, so that, in addition, *C* is the same part or the same parts of *F* as *AB* is of *DE*. VII.5

VII.6

Therefore, *if a number is parts of a number, and another is the same parts of another, then alternately, whatever part of parts the first is of the third, the same part, or the same parts, the second is of the fourth.*

Q.E.D.

Guide

In this proposition, Euclid shows that if $a = (m/n)b$, and $d = (m/n)e$, and if $a = (p/q)d$, then $b = (p/q)e$. The sample value taken for m/n in the proof is 2/3.

Use of this proposition

This proposition is used in VII.13.

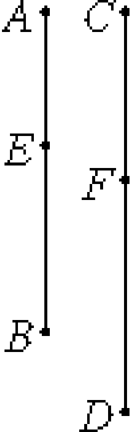
Next proposition: VII.11 Select from Book VII

Previous: VII.9

Book VII introduction

Select book

Select topic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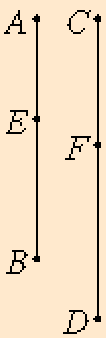
Euclid's Elements
Book VII
Proposition 11

If a whole is to a whole as a subtracted number is to a subtracted number, then the remainder is to the remainder as the whole is to the whole.

Let the whole *AB* be to the whole *CD* as *AE* subtracted is to *CF* subtracted.

I say that the remainder *EB* is to the remainder *FD* as the whole *AB* is to the whole *CD.*



Since *AB* is to *CD* as *AE* is to *CF,* therefore *AE* is the same part or parts of *CF* as *AB* is of *CD.* VII.Def.20

Therefore the remainder *EB* is the same part or parts of *FD* that *AB* is of *CD.* VII.7 VII.8

Therefore *EB* is to *FD* as *AB* is to *CD.* VII.Def.20

Therefore, *if a whole is to a whole as a subtracted number is to a subtracted number, then the remainder is to the remainder as the whole is to the whole.*

Q.E.D.

Guide

This proposition is the numerical analogue of proposition V.19 for general magnitudes. Algebraically, if $a:c = e:f$, then $a - e:c - f = a:c$.

Note that Euclid only deals with two cases, when *AB* is a part or parts of *CD,* and leaves out the other two, when *CD* is a part or parts of *AB.*

This proposition is used in IX.35.

Next proposition: VII.12

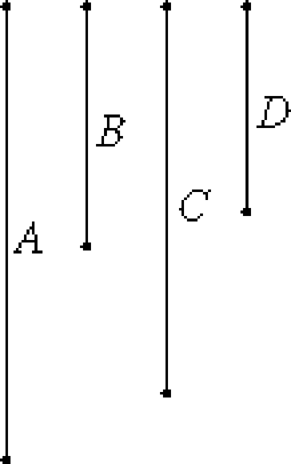
Previous: VII.10

Book VII introduction

Select from Book VII

Select book

Select topic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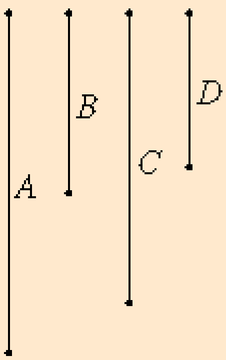
Euclid's Elements
Book VII
Proposition 12

If any number of numbers are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.

Let *A*, *B*, *C*, and *D* be as many numbers as we please in proportion, so that *A* is to *B* as *C* is to *D*.

I say that *A* is to *B* as the sum of *A* and *C* is to the sum of *B* and *D*.



Since *A* is to *B* as *C* is to *D*, therefore *A* is the same part or parts of *B* as *C* is of *D*. Therefore the sum of *A* and *C* is the same part or parts of the sum of *B* and *D* that *A* is of *B*. VII.Def.20 VII.5 VII.6

Therefore *A* is to *B* as the sum of *A* and *C* is to the sum of *B* and *D*. VII.Def.20

Therefore, *if any number of numbers are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.*

Q.E.D.

Guide

This proposition is the numerical analogue of V.12. Algebraically, If

$$x_1:y_1 = x_2:y_2 = ... = x_n:y_n,$$

then each of these ratios also equals the ratio

$$(x_1 + x_2 + ... + x_n) : (y_1 + y_2 + ... + y_n).$$

Euclid takes *n* to be 2 in his proof.

This proposition is used in VII.15 , VII.20, and IX.35.

Next proposition: VII.13

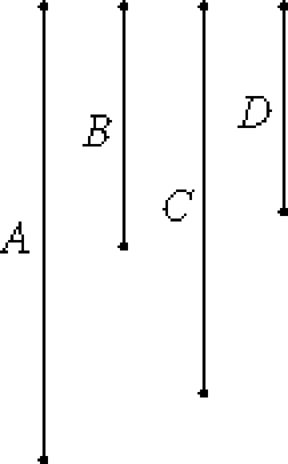
Previous: VII.11

Book VII introduction

Select from Book VII

Select book

Select topic



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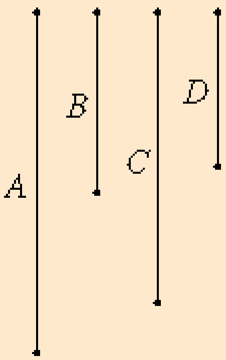
Euclid's Elements
Book VII
Proposition 13

If four numbers are proportional, then they are also proportional alternately.

Let the four numbers *A, B, C,* and *D* be proportional, so that *A* is to *B* as *C* is to *D.*

I say that they are also proportional alternately, so that *A* is to *C* as *B* is to *D.*



Since *A* is to *B* as *C* is to *D,* therefore, *A* is the same part or parts of *B* as *C* is of *D.* VII.Def.20

Therefore, alternately, *A* is the same part or parts of *C* as *B* is of *D.* VII.10

Therefore *A* is to *C* as *B* is to *D.* VII.Def.20

Therefore, *if four numbers are proportional, then they are also proportional alternately.*

Q.E.D.

Guide

This is the numerical analogue of proposition V.16 for magnitudes. It says that if $a : b = c : d$, then $a : c = b : d$.

This proposition is used frequently in Books VII through IX starting with the next proposition.

Next proposition: VII.14

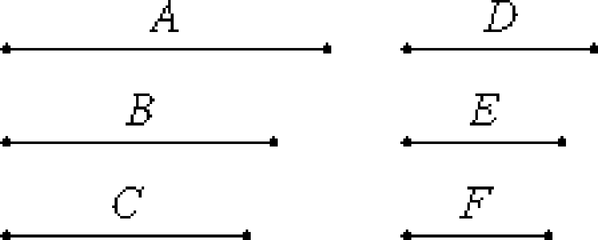
Previous: VII.12

Book VII introduction

Select from Book VII

Select book

Select topic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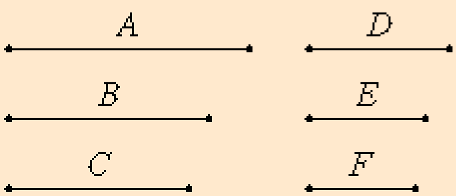
Euclid's Elements
Book VII
Proposition 14

If there are any number of numbers, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio ex aequali.

Let there be as many numbers as we please *A*, *B*, and *C*, and others equal to them in multitude *D*, *E*, and *F*, which taken two and two are in the same ratio, so that *A* is to *B* as *D* is to *E*, and *B* is to *C* as *E* is to *F*.

I say that, *ex aequali* *A* is to *C* as *D* is to *F*.



Since *A* is to *B* as *D* is to *E*, therefore, alternately *A* is to *D* as *B* is to *E*. VII.13

Again, since *B* is to *C* as *E* is to *F*, therefore, alternately *B* is to *E* as *C* is to *F*. But *B* is to *E* as *A* is to *D*, therefore *A* is to *D* as *C* is to *F*. VII.13

Therefore, alternately *A* is to *C* as *D* is to *F*. (V.11)

Therefore, *if there are any number of numbers, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio ex aequali.*

Q.E.D.

Guide

This is the numerical analogue of V.22 for magnitudes. It says that if $x_1:x_2 = y_1:y_2$, $x_2:x_3 = y_2:y_3$, ... , and $x_{n-1}:x_n = y_{n-1}:y_n$, then $x_1:x_n = y_1:y_n$. Euclid takes *n* to be 3 in his proof.

The proof is straightforward, and a simpler proof than the one given in V.22 for magnitudes. Note that at one point, the missing analogue of proposition V.11 is used: from the two proportions *B* : *E* = *C* : *F* and *B* : *E* = *A* : *D*, the conclusion *A* : *D* = *C* : *F* is drawn. Similar missing analogues of propositions from Book V are used in other proofs in book VII. See, for instance, VII.19 where V.7 and V.9 are used.

This proposition is used occasionally in Books VIII and IX starting with VIII.1.

Next proposition: VII.15

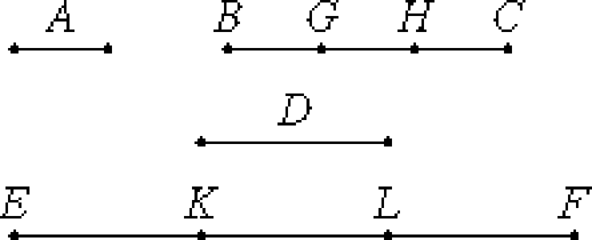
Previous: VII.13

Book VII introduction

Select from Book VII

Select book

Select topic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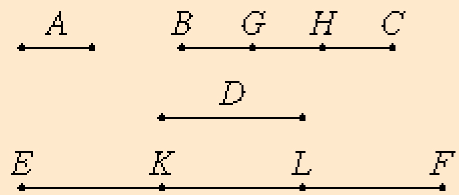
Euclid's Elements
Book VII
Proposition 15

If a unit measures any number, and another number measures any other number the same number of times, then alternately, the unit measures the third number the same number of times that the second measures the fourth.

Let the unit *A* measure any number *BC,* and let another number *D* measure any other number *EF* the same number of times.

I say that, alternately also, the unit measures the number *D* the same number of times that *BC* measures *EF.*



Since the unit *A* measures the number *BC* the same number of times that *D* measures *EF,* therefore there are as many numbers equal to *D* in *EF* as there are units in *BC.*

Divide *BC* into the units in it, *BG, GH,* and *HC,* and divide *EF* into the numbers *EK, KL,* and *LF* equal to *D.* Then the multitude of *BG, GH,* and *HC* equals the multitude of *EK, KL,* and *LF.*

And, since the units *BG, GH,* and *HC* equal one another, and the numbers *EK, KL,* and *LF* also equal one another, while the multitude of the units *BG, GH,* and *HC* equals the multitude of the numbers *EK, KL,* and *LF,* therefore the unit *BG* is to the number *EK* as the unit *GH* is to the number *KL,* and as the unit *HC* is to the number *LF.*

Since one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents, therefore the unit *BG* is to the number *EK* as *BC* is to *EF.* VII.12

But the unit *BG* equals the unit *A,* and the number *EK* equals the number *D.* Therefore the unit *A* is to the number *D* as *BC* is to *EF.* Therefore the unit *A* measures the number *D* the same number of times that *BC* measures *EF.*

Therefore, *if a unit number measures any number, and another number measures any other number the same number of times, then alternately, the unit measures the third number the same number of times that the second measures the fourth.*

Q.E.D.

Guide

This proposition expresses the commutativity of multiplication. If a number *e* is *b* times *d,* that is, 1 measures *b* the same number of times that *b* measures *d,* then *e* also is *d* times *b.* In other words, *bd* = *db.* The next proposition states this commutativity more explicitly.

This proposition can be viewed as a special case of proposition VII.9.

This proposition is used in the next proposition and a few others in Books VII and IX.

Next proposition: VII.16

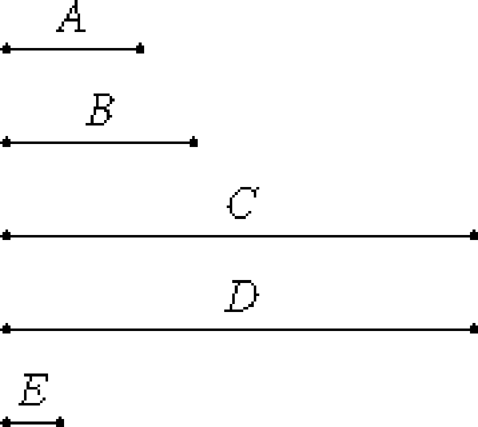
Previous: VII.14

Book VII introduction

Select from Book VII

Select book

Select topic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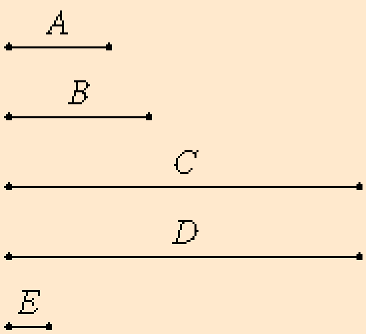
Euclid's Elements
Book VII
Proposition 16

If two numbers multiplied by one another make certain numbers, then the numbers so produced equal one another.

Let *A* and *B* be two numbers, and let *A* multiplied by *B* make *C*, and *B* multiplied by *A* make *D*.

I say that *C* equals *D*.



Since *A* multiplied by *B* makes *C*, therefore *B* measures *C* according to the units in *A*.

But the unit *E* also measures the number *A* according to the units in it, therefore the unit *E* measures *A* the same number of times that *B* measures *C*.

Therefore, alternately, the unit *E* measures the number *B* the same number of times that *A* measures *C*. VII.15

Again, since *B* multiplied by *A* makes *D*, therefore *A* measures *D* according to the units in *B*. But the unit *E* also measures *B* according to the units in it, therefore the unit *E* measures the number *B* the same number of times that *A* measures *D*.

But the unit *E* measures the number *B* the same number of times that *A* measures *C*, therefore *A* measures each of the numbers *C* and *D* the same number of times.

Therefore *C* equals *D*.

Therefore, *if two numbers multiplied by one another make certain numbers, then the numbers so produced equal one another.*

Q.E.D.

Guide

This proposition describes the commutativity mentioned in the last proposition more explicitly, *ab* = *ba*. It is used in VII.18 and a few others in Book VII.

Next proposition: VII.17

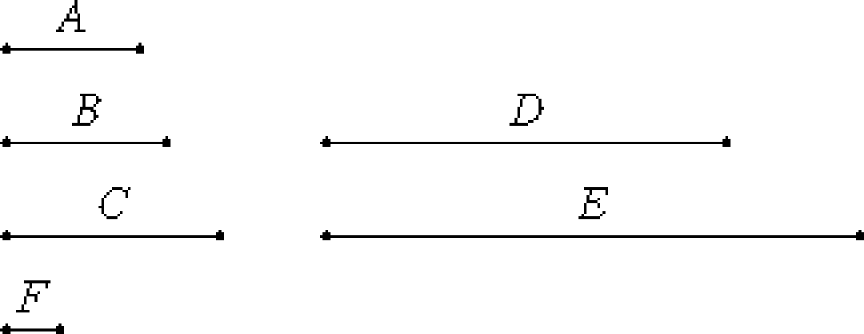
Previous: VII.15

Book VII introduction

Select from Book VII

Select book

Select topic



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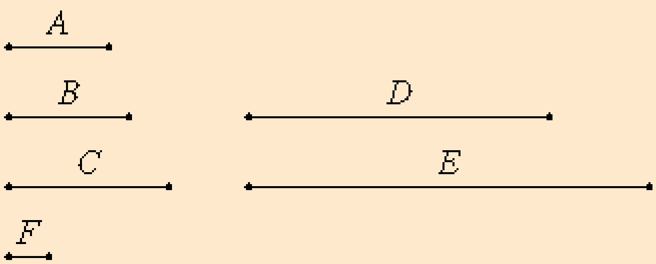
Euclid's Elements
Book VII
Proposition 17

If a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.

Let the number *A* multiplied by the two numbers *B* and *C* make *D* and *E*.

I say that *B* is to *C* as *D* is to *E*.



Since *A* multiplied by *B* makes *D*, therefore *B* measures *D* according to the units in *A*.

But the unit *F* also measures the number *A* according to the units in it, therefore the unit *F* measures the number *A* the same number of times that *B* measures *D*. Therefore the unit *F* is to the number *A* as *B* is to *D*. VII.Def.20

VII.Def.20

For the same reason the unit *F* is to the number *A* as *C* is to *E*, therefore *B* is to *D* as *C* is to *E*.

(V.11)

Therefore, alternately *B* is to *C* as *D* is to *E*. VII.13

Therefore, *if a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.*

Q.E.D.

Guide

Algebraically, $b : c = ab : ac$.

This proposition is used very frequently in Books VII through IX starting with the next proposition.

Next proposition: VII.18

Previous: VII.16

Book VII introduction

Select from Book VII

Select book

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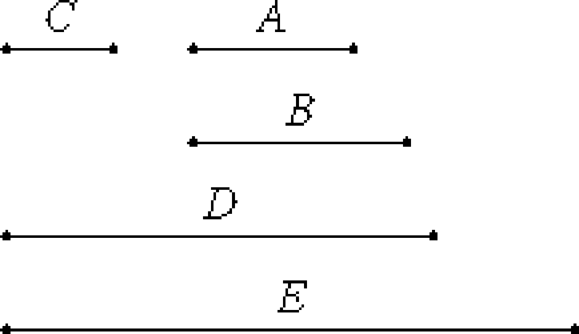
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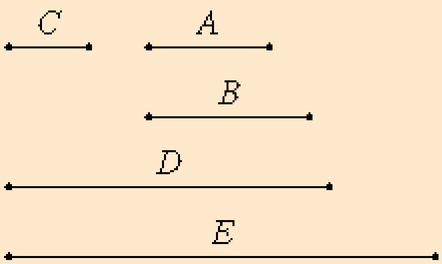
Euclid's Elements
Book VII
Proposition 18

If two numbers multiplied by any number make certain numbers, then the numbers so produced have the same ratio as the multipliers.

Let two numbers *A* and *B* multiplied by any number *C* make *D* and *E*.

I say that *A* is to *B* as *D* is to *E*.



Since *A* multiplied by *C* makes *D*, therefore *C* multiplied by *A* makes *D*. For the same reason also *C* multiplied by *B* makes *E*. VII.16

Therefore the number *C* multiplied by the two numbers *A* and *B* makes *D* and *E*. Therefore *A* is to *B* as *D*is to *E*. VII.17

Therefore, *if two numbers multiplied by any number make certain numbers, then the numbers so produced have the same ratio as the multipliers.*

Q.E.D.

Guide

Whereas the last proposition stated

$$b : c = ab : ac,$$

this one says

$$b : c = ba : ca.$$

The only difference is the order of multiplication, but VII.16 says multiplication is commutative, so that order is irrelevant.

This proposition is used in the next one and occasionally in Book VIII.

Next proposition: VII.19

Previous: VII.17

Book VII introduction

Select from Book VII

Select book

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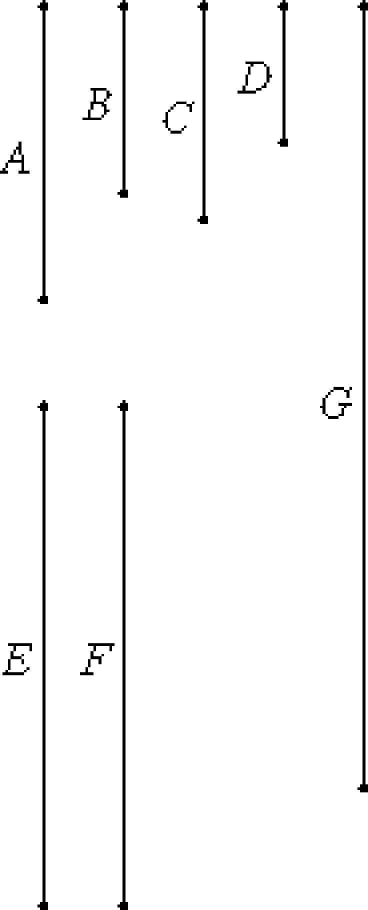
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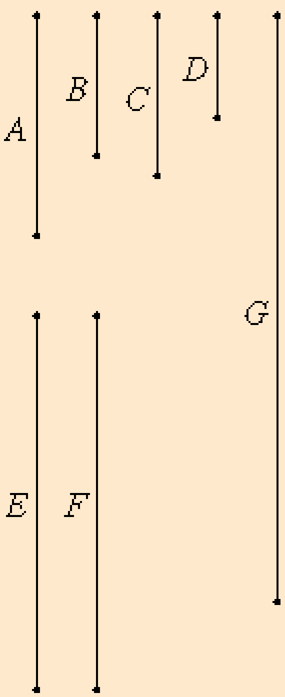
Euclid's Elements
Book VII
Proposition 19

If four numbers are proportional, then the number produced from the first and fourth equals the number produced from the second and third; and, if the number produced from the first and fourth equals that produced from the second and third, then the four numbers are proportional.

Let *A, B, C,* and *D* be four numbers in proportion, so that *A* is to *B* as *C* is to *D,* and let *A* multiplied by *D* make *E,* and let *B* multiplied by *C* make *F.*

I say that *E* equals *F.*



Multiply *A* by *C* to make *G.* Since, then, *A* multiplied by *C* makes *G,* and multiplied by *D* makes *E,* therefore the number *A* multiplied by the two numbers *C* and *D* makes *G* and *E.* Therefore *C* is to *D* as *G* is to *E.* But *C* is to *D* as *A* is to *B,* therefore *A* is to *B* as *G* is to *E.* VII.17 (V.11)

Again, since *A* multiplied by *C* makes *G,* but, further, *B* multiplied by *C* makes *F,* therefore the two numbers *A* and *B* multiplied by a certain number *C* make *G* and *F.* Therefore *A* is to *B* as *G* is to *F.* VII.18

But further *A* is to *B* as *G* is to *E,* therefore *G* is to *E* as *G* is to *F.* Therefore *G* has to each of the numbers *E* and *F* the same ratio. Therefore *E* equals *F.* (V.11) (V.9)

Again, let *E* equal *F.*

I say that *A* is to *B* as *C* is to *D.*

With the same construction, since *E* equals *F,* therefore *G* is to *E* as *G* is to *F.* (V.7)

VII.17

But *G* is to *E* as *C* is to *D,* and *G* is to *F* as *A* is to *B,* therefore *A* is to *B* as *C* is to *D.* VII.18

(V.11)

Therefore, *if four numbers are proportional, then the number produced from the first and fourth equals the number produced from the second and third; and, if the number produced from the first and fourth equals that produced from the second and third, then the four numbers are proportional.*

Q.E.D.

Guide

Algebraically, $a : b = c : d$ if and only if $ad = bc$. These algebraic expressions are meaningful when the variables are all numbers, but not when they are magnitudes in general. They can be interpreted, however, when they are lines, and proposition VI.16 is the analogue in that case.

Twice in this proof Euclid makes conclusions about proportions for numbers that he has neither stated nor proved. These places are indicated by (V.11), (V.9), and (V.7) in the margins, the analogous justifications for magnitudes. Some of the propositions in Book V for magnitudes are stated in proved in Book VII for numbers, in particular, V.16 and VII.13 correspond, and V.22 and VII.14 correspond. But many of the propositions in Book V have no analogue in Book VII, such as V.11, V.9, and V.7.

Now it could be that Euclid considered the missing statements as being obvious, as Heath claims, but being obvious is usually not a reason for Euclid to omit a proposition. Furthermore, other propositions in the next three books assume properties about proportions of numbers without having proofs of those propositions. One explanation is that the books on number theory, including this one, are older, and when the material in Book V was developed by Eudoxus, or when it was incorporated into the *Elements* by Euclid, more careful attention was made to fundamental propositions like V.7, V.9, and V.11.

This proposition is used frequently in Books VII and IX starting with VII.24.

Next proposition: VII.20

Previous: VII.18

Book VII introduction

Select from Book VII

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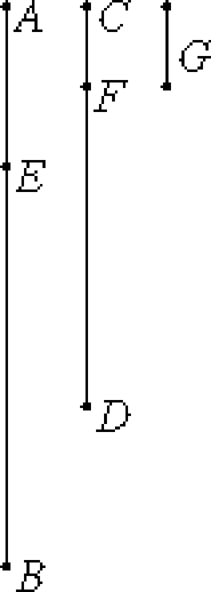
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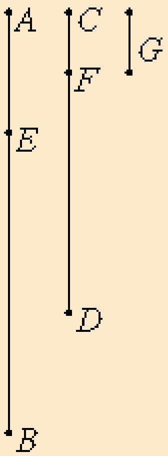
Euclid's Elements
Book VII
Proposition 2

To find the greatest common measure of two given numbers not relatively prime.

Let *AB* and *CD* be the two given numbers not relatively prime.

It is required to find the greatest common measure of *AB* and *CD.*



If now *CD* measures *AB,* since it also measures itself, then *CD* is a common measure of *CD* and *AB.* And it is manifest that it is also the greatest, for no greater number than *CD* measures *CD.*

But, if *CD* does not measure *AB,* then, when the less of the numbers *AB* and *CD* being continually subtracted from the greater, some number is left which measures the one before it.

For a unit is not left, otherwise *AB* and *CD* would be relatively prime, which is contrary to the hypothesis.

VII.Def.12

VII.1

Therefore some number is left which measures the one before it.

Now let *CD,* measuring *BE,* leave *EA* less than itself, let *EA,* measuring *DF*, leave *FC* less than itself, and let *CF* measure *AE.*

Since then, *CF* measures *AE,* and *AE* measures *DF*, therefore *CF* also measures *DF*. But it measures itself, therefore it also measures the whole *CD.*

But *CD* measures *BE,* therefore *CF* also measures *BE.* And it also measures *EA*, therefore it measures the whole *BA.*

But it also measures *CD,* therefore *CF* measures *AB* and *CD.* Therefore *CF* is a common measure of *AB* and *CD.*

I say next that it is also the greatest.

If *CF* is not the greatest common measure of *AB* and *CD,* then some number *G*, which is greater than *CF*, measures the numbers *AB* and *CD.*

Now, since *G* measures *CD,* and *CD* measures *BE,* therefore *G* also measures *BE.* But it also measures the whole *BA,* therefore it measures the remainder *AE.*

But *AE* measures *DF,* therefore *G* also measures *DF.* And it measures the whole *DC,* therefore it also measures the remainder *CF,* that is, the greater measures the less, which is impossible.

Therefore no number which is greater than *CF* measures the numbers *AB* and *CD.* Therefore *CF* is the greatest common measure of *AB* and *CD.*

Corollary

From this it is manifest that, *if a number measures two numbers, then it also measures their greatest common measure.*

Guide

Euclid again uses antenaresis (the Euclidean algorithm) in this proposition, this time to find the greatest common divisor of two numbers that aren't relatively prime. Had Euclid considered the unit (1) to be a number, he could have merged these two propositions into one.

The Euclidean algorithm, antenaresis

The greatest common divisor of two numbers m and n is the largest number which divides both. It's usually denoted GCD(m, n). It can be found by antenaresis by repeatedly subtracting the smaller, whichever it happens to be at the time, from the larger until the smaller divides the larger.

As an illustration consider the problem of computing the greatest common divisor of 884 and 3009. First, repeatedly subtract 884 from 3009 until the remainder is less than 884. An equivalent numerical operation is to divide 884 into 3009; you'll get the same remainder. In this case after subtracting 884 three times, the remainder is 357. The two numbers under our consideration are now 884 and 357. Repeatedly subtract 357 from 884 to get the remainder 170. Repeatedly subtract 170 from 357 to get the remainder 17. Finally, stop since 17 divides 170. We've found GCD(884,3009) equals 17.

The stages of the algorithm are the same as in VII.1 except that the final remainder a_{n+1}, which divides the previous number a_n, is not 1.

$$a_1 = m_1\, a_2 + a_3$$
$$a_2 = m_2\, a_3 + a_4$$
$$\dots$$
$$a_{n-1} = m_{n-1}\, a_n + a_{n+1}.$$

(In Euclid's proof a_1 is *AB*, a_2 is *CD*, a_3 is *AE*, and $a_4 = a_{n+1}$ is *CF*.)

In the first part of the proof, Euclid shows that since a_{n+1} divides a_n, it also divides a_{n-1}, ... , a_2, and a_1. Therefore a_{n+1} is a common divisor of a_2 and a_1. In the last part of the proof, Euclid shows that if any number d divides both a_2 and a_1, then it also divides a_3, ... , a_n, and a_{n+1}. Therefore a_{n+1} is the greatest common divisor. The last part of the proof also shows that every common divisor divides the greatest common divisor as noted in the corollary.

Foundations of number theory

Euclid makes many implicit assumptions about numbers. For instance, he assumes that if $m < n$, then m can be repeatedly subtracted from n until there is eventually a remainder less than or equal to m. He seems to have recognized that magnitudes need not have this property since the property is used as a qualifier in the definition of ratios (V.Def.4), but he didn't recognize its importance for numbers. There are, in fact, nonstandard models of number theory which satisfy the usual properties of numbers, but do not have this property. In such models, there are numbers than can be decreased by 1 infinitely many times but not ever reach 1. The existance of such models implies an axiom is needed to exclude such behavior.

There is a similar assumption that the process of antenaresis eventually reaches an end when applied to numbers. Euclid certainly knew it needn't halt for magnitudes since its halting is used as a criterion for incommensurability (X.2).

There needs to be an explicit axiom to cover these situations. One such axiom is a descending chain condition which states that there is no infinite decreasing sequence of numbers

$$a_1 > a_2 > \ldots > a_n > \ldots$$

Use of this proposition

This proposition and its corollary are used in the next two propositions.

Note how similar this proposition is to X.3, even having the same diagram and the same corollary. The terminology is slightly different and X.3 deals with magnitudes rather than numbers.

Next proposition: VII.3

Previous: VII.1

Book VII introduction

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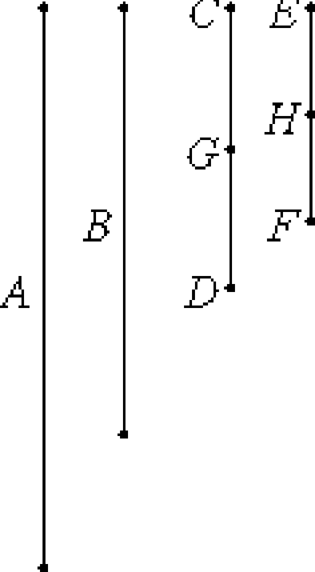
Select topic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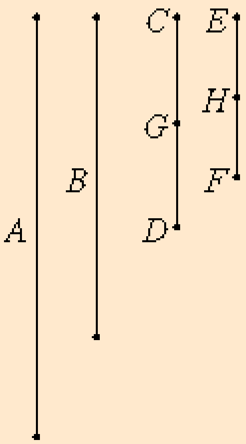
Euclid's Elements
Book VII
Proposition 20

The least numbers of those which have the same ratio with them measure those which have the same ratio with them the same number of times; the greater the greater; and the less the less.

Let *CD* and *EF* be the least numbers of those which have the same ratio with *A* and *B.*

I say that *CD* measures *A* the same number of times that *EF* measures *B.*



Now *CD* is not parts of *A.* If possible, let it be so. Therefore *EF* is also the same parts of *B* that *CD* is of *A.* VII.13 VII.Def.20

Therefore there are as many parts of *B* in *EF* are there are parts of *A* in *CD.*

Divide *CD* into the parts of *A,* namely *CG* and *GD,* and divide *EF* into the parts of *B,* namely *EH* and *HF.* Thus the multitude of *CG* and *GD* equals the multitude of *EH* and *HF.*

Now, since the numbers *CG* and *GD* equal one another, and the numbers *EH* and *HF* also equal one another, while the multitude of *CG* and *GD* equals the multitude of *EH* and *HF,* therefore *CG* is to *EH* as *GD* is to *HF.*

Since one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents, therefore *CG* is to *EH* as *CD* is to *EF.* VII.12

Therefore *CG* and *EH* are in the same ratio with *CD* and *EF,* being less than they, which is impossible, for by hypothesis *CD* and *EF* are the least numbers of those which have the same ratio with them. VII.4
Therefore *CD* is not parts of *A,* therefore it is a part of it.

And *EF* is the same part of *B* that *CD* is of *A,* therefore *CD* measures *A* the same number of times that *EF* measures *B.* VII.13 VII.Def.20

Therefore, *the least numbers of those which have the same ratio with them measure those which have the same ratio with them the same number of times; the greater the greater; and the less the less.*

Q.E.D.

Guide

This proposition says that given a ratio *a*:*b,* if *c*:*d* is the same ratio and the least among all those ratios with the same ratio, then, first of all, *c* divides *a,* and *d* divides *b,* but also, *c* divides *a* the same number of times that *d* divides *b.* For example, the ratio 91:132 is the same ratio as 7:11, which is least among all the ratios equal to 91:132, that is to say 91:132 reduces to 7:11 in lowest terms, therefore 7 divides 91 the same number of times that 11 divides 132, namely, 13 times.

The proof goes along like this. Suppose *a*:*b* reduces to *c*:*e* in lowest terms. In order to show that *c* divides *a,* assume that it doesn't, assume that $c = (m/n)a$. Since *a*:*b* is the same ratio as *c*:*e*, therefore $e = (m/n)d$. But then $c/m = (1/n)a$, and $e/m = (1/n)b$. Therefore $c/m:e/m$ is the same ratio as *a*:*b,* which shows that *c*:*e* is not in lowest terms, a contradiction. Therefore *c* does divide *a*, and *e* divides *b* the same number of times.

Use of this proposition

This proposition is used frequently in Books VII through IX starting with the next proposition.

Next proposition: VII.21

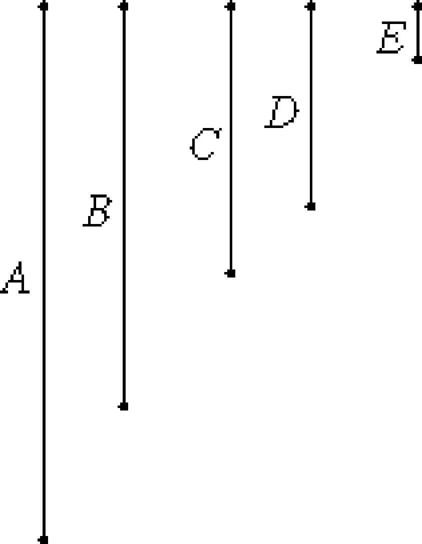
Previous: VII.19

Book VII introduction

Select from Book VII

Select book

Select topic



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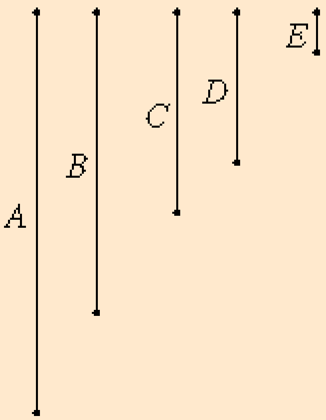
A

Euclid's Elements
Book VII
Proposition 21

Numbers relatively prime are the least of those which have the same ratio with them.

Let *A* and *B* be numbers relatively prime.

I say that *A* and *B* are the least of those which have the same ratio with them.



If not, there are some numbers less than *A* and *B* in the same ratio with *A* and *B*. Let them be *C* and *D*.

Since, then, the least numbers of those which have the same ratio measure those which have the same ratio the same number of times, the greater the greater, and the less the less, that is, the antecedent the antecedent and the consequent the consequent, therefore *C* measures *A* the same number of times that *D* measures *B*. VII.20

Let there be as many units in *E* as the times that *C* measures *A*. Then *D* also measures *B* according to the units in *E*.

And, since *C* measures *A* according to the units in *E*, therefore *E* also measures *A* according to the units in *C*. For the same reason *E* also measures *B* according to the units in *D*. VII.16

Therefore *E* measures *A* and *B* which are relatively prime, which is impossible. VII.Def.12

Therefore there are no numbers less than *A* and *B* which are in the same ratio with *A* and *B*. Therefore *A* and *B* are the least of those which have the same ratio with them.

Therefore, *numbers relatively prime are the least of those which have the same ratio with them.*

Q.E.D.

Guide

The next proposition is the converse of this one. Together they say that a ratio *a*:*b* is reduced to lowest terms if and only if *a* is relatively prime to *b*.

Although it appears that this proposition is pairs of numbers and their ratios, it is used in proposition VII.33 with any quantity of numbers. Stated in terms of three numbers *a*, *b*, and *c*, that proposition says that of all triples with the same ratio as *a*, *b*, and *c*, have, the triple of relatively prime numbers is least.

Use of this proposition

This proposition is used frequently in Books VII through IX starting with VII.24.

Next proposition: VII.22

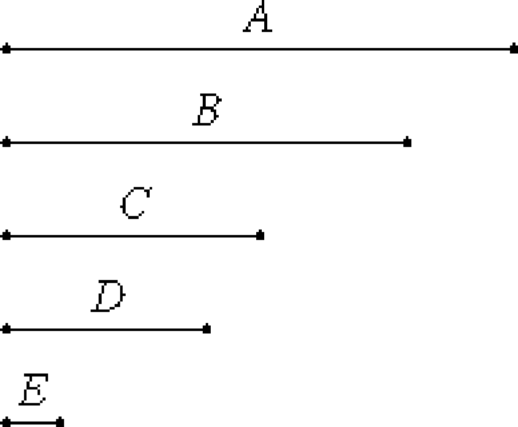
Previous: VII.20

Book VII introduction

Select from Book VII

Select book

Select topic


A
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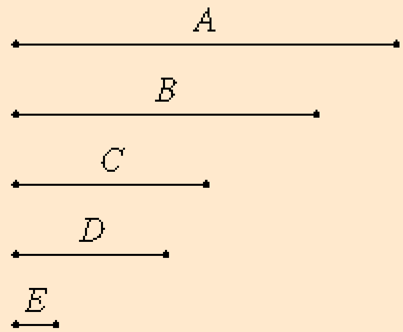
Euclid's Elements
Book VII
Proposition 22

The least numbers of those which have the same ratio with them are relatively prime.

Let *A* and *B* be the least numbers of those which have the same ratio with them.

I say that *A* and *B* are relatively prime.



If they are not relatively prime, then some number *C* measures them.

Let there be as many units in *D* as the times that *C* measures *A*, and as many units in *E* as the times that *C* measures *B*.

Since *C* measures *A* according to the units in *D*, therefore *C* multiplied by *D* makes *A*. For the same reason *C* multiplied by *E* makes *B*. VII.Def.15

Thus the number *C* multiplied by the two numbers *D* and *E* makes *A* and *B*, therefore *D* is to *E* as *A* is to *B*. VII.17

Therefore *D* and *E* are in the same ratio with *A* and *B*, being less than they, which is impossible. Therefore no number measures the numbers *A* and *B*.

Therefore *A* and *B* are relatively prime.

Therefore, *the least numbers of those which have the same ratio with them are relatively prime*.

Q.E.D.

Guide

This proposition is the converse of the last one. Together they say that a ratio *a*:*b* is reduced to lowest terms if and only if *a* is relatively prime to *b*.

Use of this proposition

This proposition is used in propositions VIII.2, VIII.3, and IX.15.

Next proposition: VII.23

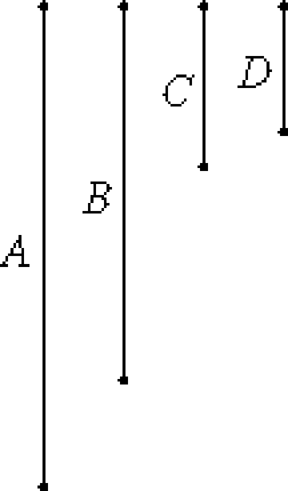
Previous: VII.21

Book VII introduction

Select from Book VII

Select book

Select topic


A
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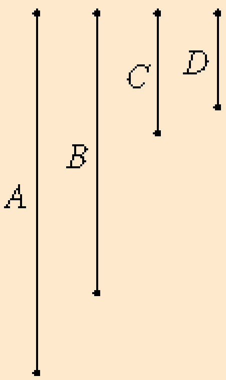
Euclid's Elements
Book VII
Proposition 23

If two numbers are relatively prime, then any number which measures one of them is relatively prime to the remaining number.

Let *A* and *B* be two numbers relatively prime, and let any number *C* measure *A.*

I say that *C* and *B* are also relatively prime.



If *C* and *B* are not relatively prime, then some number *D* measures *C* and *B.*

Since *D* measures *C,* and *C* measures *A,* therefore *D* also measures *A.* But it also measures *B,* therefore *D* measures *A* and *B* which are relatively prime, which is impossible. VII.Def.12

Therefore no number measures the numbers *C* and *B.* Therefore *C* and *B* are relatively prime.

Therefore, *if two numbers are relatively prime, then any number which measures one of them is relatively prime to the remaining number.*

Q.E.D.

Guide

The proof of this proposition is straightforward.

Use of this proposition

This proposition is used in the proof of the next one.

Next proposition: VII.24

Previous: VII.22

Book VII introduction

Select from Book VII

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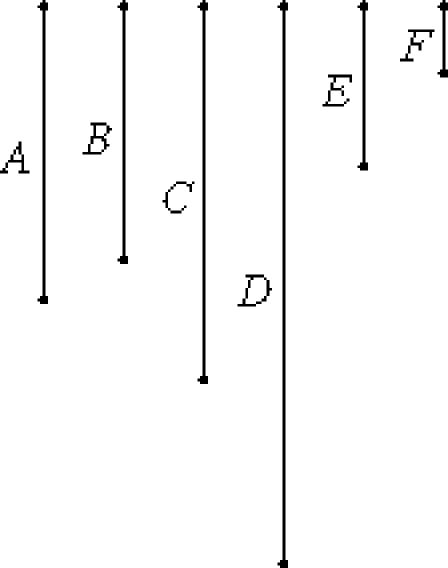
[Clark University](#)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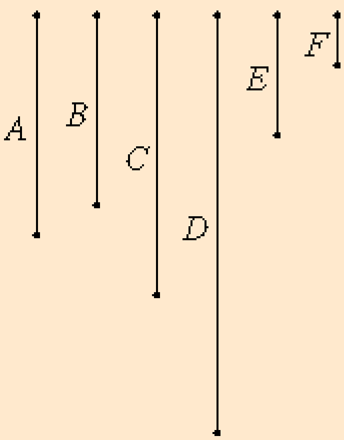
Euclid's Elements
Book VII
Proposition 24

If two numbers are relatively prime to any number, then their product is also relatively prime to the same.

Let the two numbers *A* and *B* [each] be relatively prime to a number *C*, and let *A* multiplied by *B* make *D.*

I say that *C* and *D* are relatively prime.



If *C* and *D* are not relatively prime, then some number *E* measures *C* and *D.*

Now, since *C* and *A* are relatively prime, and a certain number *E* measures *C,* therefore *A* and *E* are relatively prime. VII.23

Let there be as many units in *F* as the times that *E* measures *D.* Then *F* also measures *D* according to the units in *E.* VII.16

Therefore *E* multiplied by *F* makes *D.* Also, *A* multiplied by *B* makes *D*, therefore the product of *E* and *F* equals the product of *A* and *B.* VII.Def.15

But, if the product of the extremes equal that of the means, then the four numbers are proportional. Therefore *E* is to *A* as *B* is to *F*. VII.19

But *A* and *E* are relatively prime, numbers which are relatively prime are also the least of those which have the same ratio, and the least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater, and the less the less, that is, the antecedent the antecedent and the consequent the consequent, therefore *E* measures *B.* VII.21 VII.20

But it also measures *C*, therefore *E* measures *B* and *C* which are relatively prime, which is impossible. VII.Def.12

Therefore no number measures the numbers *C* and *D.* Therefore *C* and *D* are relatively prime.

Therefore, *if two numbers are relatively prime to any number, then their product is also relatively prime to the same.*

Q.E.D.

Guide

Outline of the proof

Assume that two numbers *a* and *b* are each relatively prime to a third number *c.*

Suppose their product *ab* is not relatively prime to *c*. Then there is some number *e* (greater than 1) that divides both *ab* and *c*. Now, since *e* divides *c*, and *c* is relatively prime to *a*, therefore, by VII.23, *e* is also relatively prime to *a*.

Let *f* be the number *ab/e*. Then *e*:*a* = *b*:*f*. Since *e* and *a* are relatively prime, then, by VII.21, *e*:*a* is in lowest terms. Therefore, by VII.20, *e* divides *b*. But then *e* divides both *b* and *c* contradicting the assumption that *b* and *c* are relatively prime.

Therefore, the product *ab* is also relatively prime to *c*.

Use of this proposition

This proposition is used in the next two and in IX.15.

Next proposition: VII.25

Previous: VII.23

Book VII introduction

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Select book

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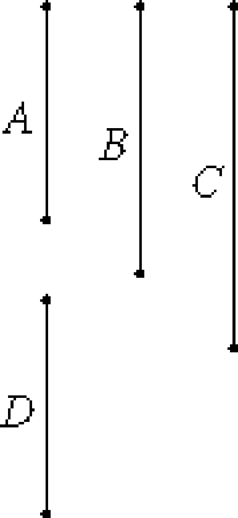
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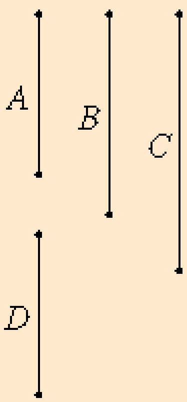
Euclid's Elements
Book VII
Proposition 25

If two numbers are relatively prime, then the product of one of them with itself is relatively prime to the remaining one.

Let *A* and *B* be two numbers relatively prime, and let *A* multiplied by itself make *C*.



I say that *B* and *C* are relatively prime.

Make *D* equal to *A*.

Since *A* and *B* are relatively prime, and *A* equals *D*, therefore *D* and *B* are also relatively prime. Therefore each of the two numbers *D* and *A* is relatively prime to *B*. Therefore the product of *D* and *A* is also relatively prime to *B*. VII.24

But the number which is the product of *D* and *A* is *C*. Therefore *C* and *B* are relatively prime.

Therefore, *if two numbers are relatively prime, then the product of one of them with itself is relatively prime to the remaining one.*

Q.E.D.

Guide

This is a special case of the previous proposition. It is used in VII.27 and IX.15.

Next proposition: VII.26

Previous: VII.24

Book VII introduction

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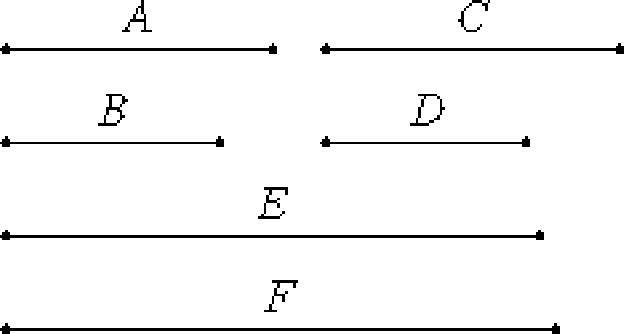
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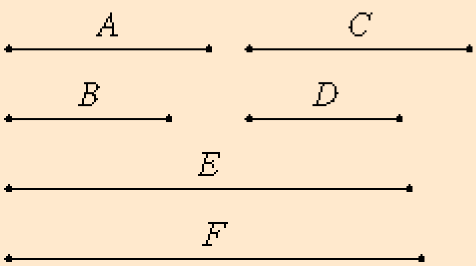
Euclid's Elements
Book VII
Proposition 26

If two numbers are relatively prime to two numbers, both to each, then their products are also relatively prime.

Let the two numbers *A* and *B* be relatively prime to the two numbers *C* and *D*, both to each, and let *A* multiplied by *B* make *E*, and let *C* multiplied by *D* make *F*.

I say that *E* and *F* are relatively prime.



Since each of the numbers *A* and *B* is relatively prime to *C*, therefore the product of *A* and *B* is also relatively prime to *C*. But the product of *A* and *B* is *E*, therefore *E* and *C* are relatively prime. For the same reason *E* and *D* are also relatively prime. Therefore each of the numbers *C* and *D* is relatively prime to *E*. VII.24

Therefore the product of *C* and *D* is also relatively prime to *E*. But the product of *C* and *D* is *F*. Therefore *E* and *F* are relatively prime. VII.24

Therefore, *if two numbers are relatively prime to two numbers, both to each, then their products are also relatively prime.*

Q.E.D.

Guide

The proof of this proposition uses proposition VII.24 twice. If *a* and *b* are both relatively prime to both *c* and *d*, then so is their product *ab*. Now since *c* and *d* are both relatively prime to *ab*, therefore so is their product *cd*.

This proposition is used in the proof of the next one.

Next proposition: VII.27

Previous: VII.25

Book VII introduction

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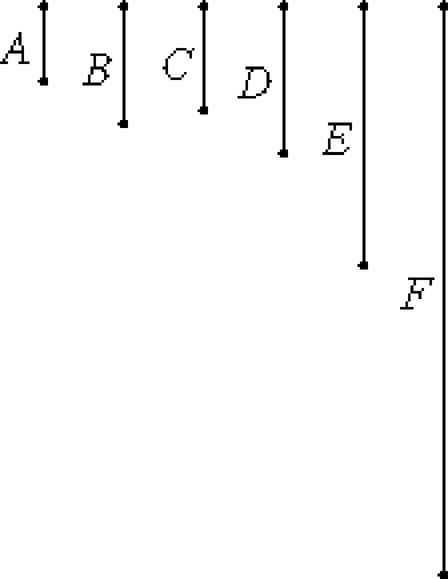
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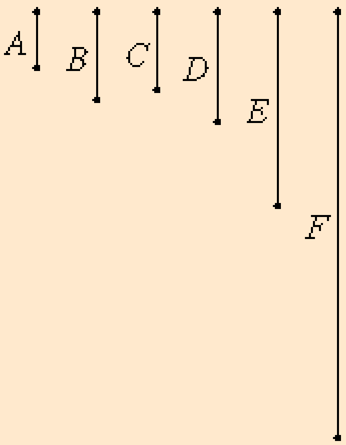
Euclid's Elements
Book VII
Proposition 27

If two numbers are relatively prime, and each multiplied by itself makes a certain number, then the products are relatively prime; and, if the original numbers multiplied by the products make certain numbers, then the latter are also relatively prime.

Let *A* and *B* be two relatively prime numbers, let *A* multiplied by itself make *C*, and multiplied by *C* make *D*, and let *B* multiplied by itself make *E*, and multiplied by *E* make *F*.



I say that *C* and *E* are relatively prime, and that *D* and *F* are relatively prime.

Since *A* and *B* are relatively prime, and *A* multiplied by itself makes *C*, therefore *C* and *B* are relatively prime. VII.25

Since, then, *C* and *B* are relatively prime, and *B* multiplied by itself makes *E*, therefore *C* and *E* are relatively prime.

Again, since *A* and *B* are relatively prime, and *B* multiplied by itself makes *E*, therefore *A* and *E* are relatively prime.

Since, then, the two numbers *A* and *C* are relatively prime to the two numbers *B* and *E*, both to each, therefore the product of *A* and *C* is relatively prime to the product of *B* and *E*. And the product of *A* and *C* is *D*, and the product of *B* and *E* is *F*. VII.26

Therefore *D* and *F* are relatively prime.

Therefore, *if two numbers are relatively prime, and each multiplied by itself makes a certain number, then the products are relatively prime; and, if the original numbers multiplied by the products make certain numbers, then the latter are also relatively prime.*

Q.E.D.

Guide

The proposition states that if two numbers are relatively prime, then their powers are also relatively prime. Explicitly, it only says that their squares are relatively prime, and their cubes are relatively prime, but the way it is used in VIII.2, any powers need to be relatively prime.

The proof of this proposition uses the last two propositions. Assume that a and b are relatively prime. Then applying VII.25 twice, we first get a^2 and b relatively prime, then we get a^2 and b^2 relatively prime.

Again, by VII.25, a and b^2 are relatively prime. Now, a is relatively prime to b^2, and b is relatively prime to a^2, so by VII.26, a^3 is relatively prime to b^3.

Likewise, higher powers of a and b can be shown to be relatively prime.

Use of this proposition

This proposition is used in VIII.2 and VIII.3.

Next proposition: VII.28

Previous: VII.26

Book VII introduction

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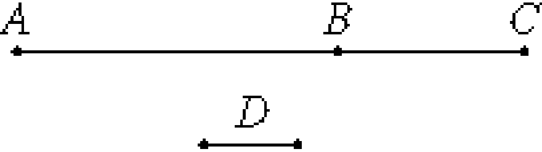
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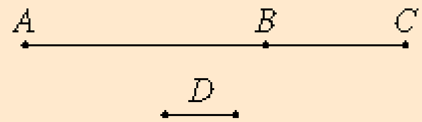
Euclid's Elements
Book VII
Proposition 28

If two numbers are relatively prime, then their sum is also prime to each of them; and, if the sum of two numbers is relatively prime to either of them, then the original numbers are also relatively prime.

Let two relatively prime numbers *AB* and *BC* be added.

I say that their sum *AC* is also relatively prime to each of the numbers *AB* and *BC*.



If *CA* and *AB* are not relatively prime, then some number *D* measures *CA* and *AB*.

Since then *D* measures *CA* and *AB*, therefore it also measures the remainder *BC*. But it also measures *BA*, therefore *D* measures *AB* and *BC* which are relatively prime, which is impossible. VII.Def.12

Therefore no number measures the numbers *CA* and *AB*. Therefore *CA* and *AB* are relatively prime. For the same reason *AC* and *CB* are also relatively prime. Therefore *CA* is relatively prime to each of the numbers *AB* and *BC*.

Next, let *CA* and *AB* be relatively prime.

I say that *AB* and *BC* are also relatively prime.

If *AB* and *BC* are not relatively prime, then some number *D* measures *AB* and *BC*.

Now, since *D* measures each of the numbers *AB* and *BC*, therefore it also measures the whole *CA*. But it measures *AB*, therefore *D* measures *CA* and *AB* which are relatively prime, which is impossible. VII.Def.12

Therefore no number measures the numbers *AB* and *BC*. Therefore *AB* and *BC* are relatively prime.

Therefore, *if two numbers are relatively prime, then their sum is also prime to each of them; and, if the sum of two numbers is relatively prime to either of them, then the original numbers are also relatively prime.*

Q.E.D.

Guide

This proposition is used in IX.15.

Next proposition: VII.29

Previous: VII.27

Book VII introduction

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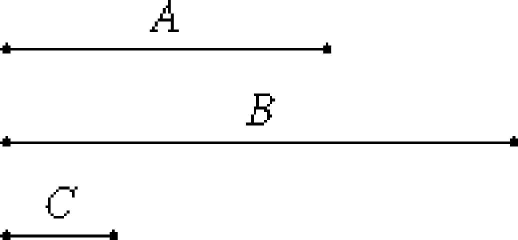
A
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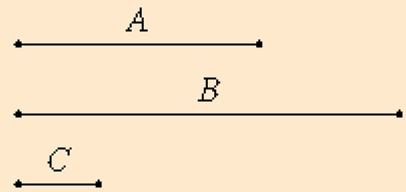
Euclid's Elements
Book VII
Proposition 29

Any prime number is relatively prime to any number which it does not measure.

Let *A* be a prime number, and let it not measure *B*.

I say that *B* and *A* are relatively prime.



If *B* and *A* are not relatively prime, then some number *C* measures them.

Since *C* measures *B*, and *A* does not measure *B*, therefore *C* is not the same as *A*.

Now, since *C* measures *B* and *A*, therefore it also measures *A* which is prime, though it is not the same as it, which is impossible. Therefore no number measures *B* and *A*.

Therefore *A* and *B* are relatively prime.

Therefore, *any prime number is relatively prime to any number which it does not measure.*

Q.E.D.

Guide

This proposition is used in the next one and in propositions IX.12 and IX.36.

Next proposition: VII.30

Previous: VII.28

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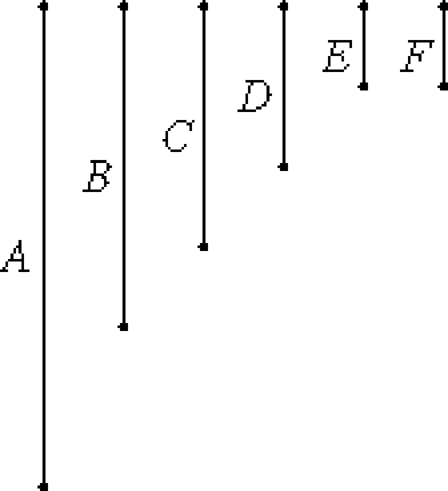
A
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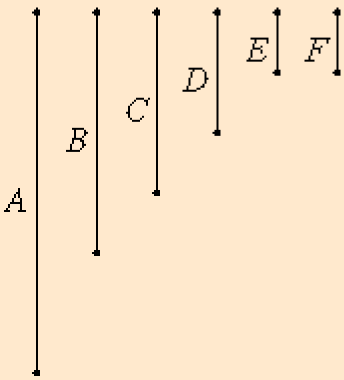
F

Euclid's Elements
Book VII
Proposition 3

To find the greatest common measure of three given numbers not relatively prime.

Let *A, B,* and *C* be the three given numbers not relatively prime.

It is required to find the greatest common measure of *A, B,* and *C.*



Take the greatest common measure, *D,* of the two numbers *A* and *B.* Then either *D* measures, or does not measure, *C.* VII.2

First, let it measure it.

But it measures *A* and *B* also, therefore *D* measures *A, B,* and *C.* Therefore *D* is a common measure of *A, B,* and *C.*

I say that it is also the greatest.

If *D* is not the greatest common measure of *A, B,* and *C,* then some number *E,* greater than *D,* measures the numbers *A, B,* and *C.*

Since then *E* measures *A, B,* and *C,* therefore it measures *A* and *B.* Therefore it also measures the greatest common measure of *A* and *B.* But the greatest common measure of *A* and *B* is *D,* therefore *E* measures *D,* the greater the less, which is impossible. VII.2,Cor.

Therefore no number which is greater than *D* measures the numbers *A, B,* and *C.* Therefore *D* is the greatest common measure of *A, B,* and *C.*

Next, let *D* not measure *C.*

I say first that *C* and *D* are not relatively prime.

Since *A, B,* and *C* are not relatively prime, therefore some number measures them.

Now that which measures *A, B,* and *C* also measures *A* and *B,* and therefore measures *D,* the greatest common measure of *A* and *B.* But it measures *C* also, therefore some number measures the numbers *D* and *C.* Therefore *D* and *C* are not relatively prime. VII.2,Cor.

Take their greatest common measure *E.* VII.2

Then, since *E* measures *D,* and *D* measures *A* and *B,* therefore *E* also measures *A* and *B.* But it measures *C* also, therefore *E* measures *A, B,* and *C.* Therefore *E* is a common measure of *A, B,* and *C.*

I say next that it is also the greatest.

If *E* is not the greatest common measure of *A, B,* and *C,* then some number *F,* greater than *E,* measures the numbers *A, B,* and *C.*

Now, since *F* measures *A, B,* and *C,* it also measures *A* and *B,* therefore it measures the greatest common VII.2,Cor.

measure of *A* and *B*. But the greatest common measure of *A* and *B* is *D*, therefore *F* measures *D*.

And it measures *C* also, therefore *F* measures *D* and *C*. Therefore it also measures the greatest common measure of *D* and *C*. But the greatest common measure of *D* and *C* is *E*, therefore *F* measures *E*, the greater the less, which is impossible. VII.2,Cor.

Therefore no number which is greater than *E* measures the numbers *A*, *B*, and *C*. Therefore *E* is the greatest common measure of *A*, *B*, and *C*.

Q.E.D.

Guide

A common modern notation for the greatest common divisor of two numbers *a* and *b* is GCD(*a*, *b*). Also, the notation *a* | *b* is typically used to indicate that *a* divides *b*.

This proposition constructs the GCD(*a*, *b*, *c*) as GCD(GCD(*a*, *b*), *c*).

The proof that this construction works is simplified if 1 is considered to be a number. Then, two numbers are relatively prime when their GCD is 1, and Euclid's first case in the proof is subsumed in the second.

Let *d* = GCD(*a*, *b*), and let *e* = GCD(*d*, *c*). Since *e* | *d*, *d* | *a*, and *d* | *b*, it follows that *e* | *a* and *e* | *b*, so *e*, in fact, is a common divisor of *a*, *b*, and *c*.

In order to show that *e* is the greatest common divisor, let *f* be any common divisor of *a*, *b*, and *c*. Then as *f* | *a* and *f* | *b*, therefore *f* | GCD(*a*, *b*), that is, *f* | *d*. Also, as *f* | *d* and *f* | *c*, therefore *f* | GCD(*d*, *c*), that is *f* | *e*. Therefore *e* is the greatest common divisor of *a*, *b*, and *c*. Q.E.D.

This is the same proposition as X.4.

This proposition is used in VII.33.

Next proposition: VII.4

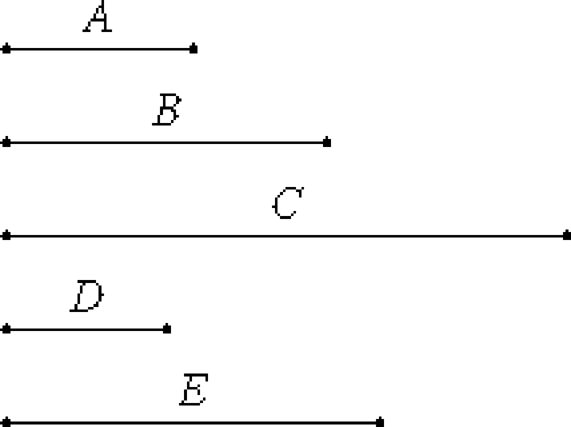
Previous: VII.2

Book VII introduction

Select from Book VII

Select book

Select topic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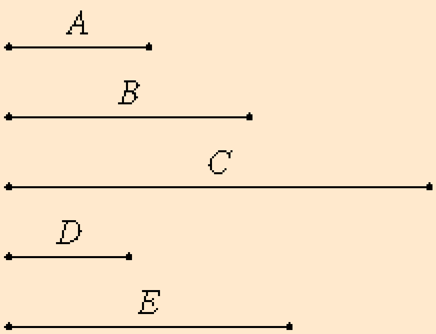
Euclid's Elements
Book VII
Proposition 30

If two numbers, multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers.

Let the two numbers *A* and *B* multiplied by one another make *C*, and let any prime number *D* measure *C*.

I say that *D* measures one of the numbers *A* or *B*.



Let it not measure *A*.

Now *D* is prime, therefore *A* and *D* are relatively prime. VII.29

Let as many units be in *E* as the times that *D* measures *C*.

Since then *D* measures *C* according to the units in *E*, therefore *D* multiplied by *E* makes *C*. VII.Def.15

Further, *A* multiplied by *B* also makes *C*, therefore the product of *D* and *E* equals the product of *A* and *B*.

Therefore *D* is to *A* as *B* is to *E*. VII.19

But *D* and *A* are relatively prime, relatively prime numbers are also least, and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, therefore *D* measures *B*. VII.21 VII.20

Similarly we can also show that, if *D* does not measure *B*, then it measures *A*. Therefore *D* measures one of the numbers *A* or *B*.

Therefore, *if two numbers, multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers.*

Q.E.D.

Guide

This proposition states that if *p* is a prime number, then whenever *p* divides a product of two numbers, then it divides at least one of them. This is actually a property that characterizes prime numbers, that is to say, no composite number has this property. (For if *c* is a composite number, *c ab*, so *c* divides the product but it doesn't divide either factor.)

Outline of the proof

Assume that a prime number *d* divides the product *ab*.

The form of the proof is interesting. Euclid shows that if *d* doesn't divide *a*, then *d* does divide *b*, and similarly, if *d* doesn't divide *b*, then *d* does divide *a*. Therefore, it divides either one or the other.

Suppose *d* does not divide *a*. Then, by VII.29, *d* is relatively prime to *a*. Let *e* be the number *ab/d*. Then *d*:*a* = *b*:*e*. By VII.21, the ratio *d*:*a* is in lowest terms, and so, by VII.20, *d* divides *b*.

Use of this proposition

This proposition is used in IX.14.

Next proposition: VII.31

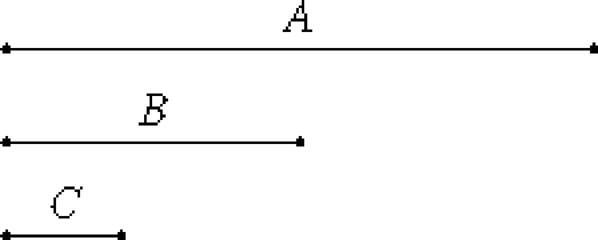
Previous: VII.29

Book VII introduction

Select from Book VII

Select book

Select topic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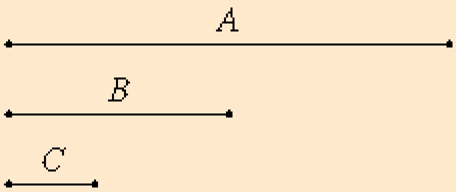
Euclid's Elements
Book VII
Proposition 31

Any composite number is measured by some prime number.

Let A be a composite number.

I say that A is measured by some prime number.



Since A is composite, therefore some number B measures it. [VII.Def.13]

Now, if B is prime, then that which was proposed is done.

But if it is composite, some number measures it. Let a number C measure it. [VII.Def.11,13]

Then, since C measures B, and B measures A, therefore C also measures A. And, if C is prime, then that which was proposed is done. But if it is composite, some number measures it. Thus, if the investigation is continued in this way, then some prime number will be found which measures the number before it, which also measures A. If it is not found, then an infinite sequence of numbers measures the number A, each of which is less than the other, which is impossible in numbers.

Therefore some prime number will be found which measures the one before it, which also measures A.

Therefore any composite number is measured by some prime number.

Therefore, *any composite number is measured by some prime number*.

Q.E.D.

Guide

Euclid does not explain why there can't be an infinite sequence of numbers where each number divides the previous. He simply says that is impossible. Some justification is required such as the principle Euclid uses elsewhere that any decreasing sequence of numbers is finite.

This proposition is used in the next one and in propositions IX.13 and IX.20.

Next proposition: VII.32

Previous: VII.30

Book VII introduction

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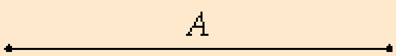
A

Euclid's Elements
Book VII
Proposition 32

Any number is either prime or is measured by some prime number.

Let *A* be a number.

I say that *A* either is prime or is measured by some prime number.



If now *A* is prime, then that which was proposed is done.

But if it is composite, then some prime number measures it. VII.31

Therefore any number either is prime or is measured by some prime number.

Therefore, *any number is either prime or is measured by some prime number.*

Q.E.D.

Guide

After the previous proposition, this one really doesn't need to be stated at all.

Next proposition: VII.33

Previous: VII.31

Book VII introduction

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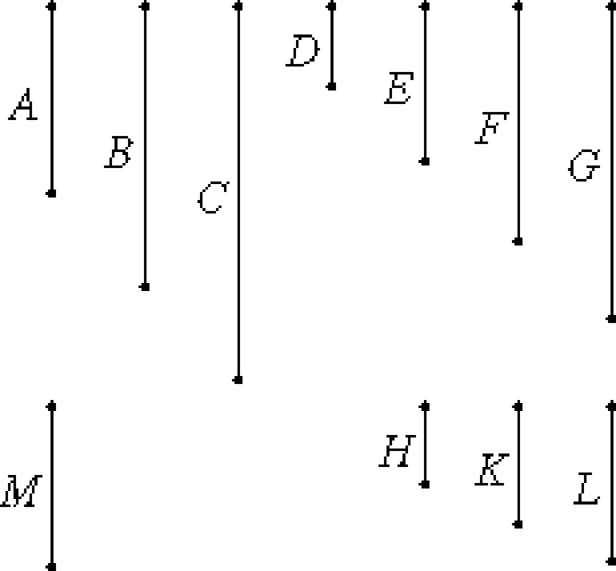
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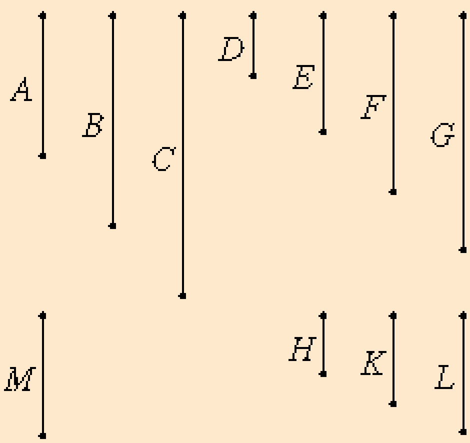
Euclid's Elements
Book VII
Proposition 33

Given as many numbers as we please, to find the least of those which have the same ratio with them.

Let *A, B,* and *C* be the given numbers, as many as we please.

It is required to find the least of those which have the same ratio with *A, B,* and *C.*

Either *A, B,* and *C* are relatively prime or they are not.



Now, if *A, B,* and *C* are relatively prime, then they are the least of those which have the same ratio with them. VII.21

But, if not, take *D* the greatest common measure of *A, B,* and *C.* Let there be as many units in the numbers *E, F,* and *G* as the times that *D* measures the numbers *A, B,* and *C* respectively. VII.3

Therefore the numbers *E, F,* and *G* measure the numbers *A, B,* and *C* respectively according to the units in *D.* Therefore *E, F,* and *G* measure *A, B,* and *C* the same number of times. Therefore *E, F,* and *G* are in the same ratio with *A, B,* and *C.* VII.16 VII.Def.20

I say next that they are the least that are in that ratio.

If *E, F,* and *G* are not the least of those which have the same ratio with *A, B,* and *C,* then there are numbers less than *E, F,* and *G* in the same ratio with *A, B,* and *C.* Let them be *H, K,* and *L.* Therefore *H* measures *A* the same number of times that the numbers *K* and *L* measure the numbers *B* and *C* respectively.

Let there be as many units in *M* as the times that *H* measures *A.* Then the numbers *K* and *L* also measure the numbers *B* and *C* respectively according to the units in *M.*

And, since *H* measures *A* according to the units in *M,* therefore *M* also measures *A* according to the units in *H.* For the same reason *M* also measures the numbers *B* and *C* according to the units in the numbers *K* and *L* respectively. Therefore *M* measures *A, B,* and *C.* VII.16

Now, since *H* measures *A* according to the units in *M,* therefore *H* multiplied by *M* makes *A.* For the same reason also *E* multiplied by *D* makes *A.* VII.Def.15

Therefore the product of *E* and *D* equals the product of *H* and *M.* Therefore *E* is to *H* as *M* is to *D.* VII.19

But *E* is greater than *H,* therefore *M* is also greater than *D.* And it measures *A, B,* and *C,* which is impossible, for by hypothesis, *D* is the greatest common measure of *A, B,* and *C.*

Therefore there cannot be any numbers less than *E*, *F*, and *G* which are in the same ratio with *A*, *B*, and *C*. Therefore *E*, *F*, and *G* are the least of those which have the same ratio with *A*, *B*, and *C*.

Q.E.D.

Guide

This proposition is unusual in that it discusses a ratio $a:b:c$ of three (or more) numbers. It also has the proportion $a:b:c = e:f:g$. These multiterm ratios and proportions may have been left over from an earlier time. Euclid argues that the proportion holds because *e, f,* and *g* measure *a*, *b*, and *c*, respectively, the same number, *d*, times. By the definition of proportion, that observation directly implies

$$a:e = b:f = c:g$$

The desired proportion, $a:b:c = e:f:g$ is an alternate form of that multiple proportion. See V.Def.13 for the definition of alternate ratios.

Use of this proposition

This proposition is used in the next one and several propositions in Book VIII starting with VIII.6.

Next proposition: VII.34

Previous: VII.32

Book VII introduction

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Euclid's Elements

Book VII

Proposition 34

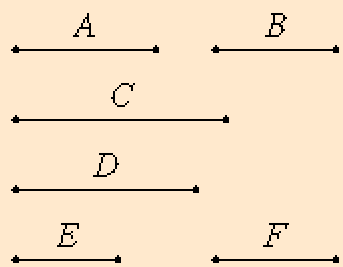
To find the least number which two given numbers measure.

Let *A* and *B* be the two given numbers.

It is required to find the least number which they measure.

Now either *A* and *B* are relatively prime or they are not.

First, let *A* and *B* be relatively prime. Multiply *A* by *B* to make *C*. Then *B* multiplied by *A* makes *C*. Therefore *A* and *B* measure *C*.



I say next that it is also the least number they measure.

If not, then *A* and *B* measure some number *D* less than *C*

Let there be as many units in *E* as the times that *A* measures *D*, and as many units in *F* as the times that *B* measures *D*.

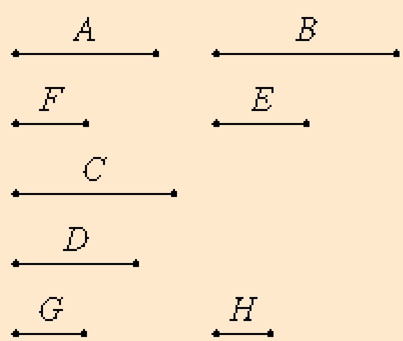
Then *A* multiplied by *E* makes *D*, and *B* multiplied by *F* makes *D*. Therefore the product of *A* and *E* equals the product of *B* and *F*. Therefore *A* is to *B* as *F* is to *E*. VII.Def.15 VII.19

But *A* and *B* are relatively prime, primes are also least, and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, therefore *B* measures *E* as the consequent the consequent. VII.21 VII.20

And, since *A* multiplied by *B* and by *E* makes *C* and *D*, therefore *B* is to *E* as *C* is to *D*. But *B* measures *E*, therefore *C* also measures *D*, the greater the less, which is impossible. VII.17

Therefore *A* and *B* do not measure any number less than *C*. Therefore *C* is the least that is measured by *A* and *B*.

Next, let *A* and *B* not be relatively prime. Take *F* and *E*, the least numbers of those which have the same ratio with *A* and *B*. Therefore the product of *A* and *E* equals the product of *B* and *F*. VII.33 VII.19



Multiply *A* by *E* to make *C*. Then *B* multiplied by *F* makes *C*. Therefore *A* and *B* measure *C*.

I say next that it is also the least number that they measure.

If not, then *A* and *B* measure some number *D* less than *C*.

Let there be as many units in *G* as the times that *A* measures *D*, and as many units in *H* as the times that *B* measures *D*.

Then *A* multiplied by *G* makes *D*, and *B* multiplied by *H* makes *D*. Therefore the product of *A* and *G* equals the product of *B* and *H*. Therefore *A* is to *B* as *H* is to *G*.	VII.19
But *A* is to *B* as *F* is to *E*. Therefore *F* is to *E* as *H* is to *G*.	(V.11)
But *F* and *E* are least, and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, therefore *E* measures *G*.	VII.20
And, since *A* multiplied by *E* and by *G* makes *C* and *D*, therefore *E* is to *G* as *C* is to *D*.	VII.17
But *E* measures *G*, therefore *C* also measures *D*, the greater the less, which is impossible. Therefore *A* and *B* do not measure any number less than *C*. Therefore *C* is the least that is measured by *A* and *B*.	

Q.E.D.

Guide

The least common multiple of two numbers *a* and *b* is the smallest number that they both divide. It is denoted *LCM*(*a*, *b*). This proposition construct it as the product divided by the greatest common divisor:

$$\text{LCM}(a, b) = ab/\text{LCM}(a, b).$$

Summary of the proof

Let *a* and *b* be the two numbers. There are two cases depending on whether they are relatively prime or not.

Case 1. Suppose *a* and *b* are relatively prime. An indirect proof shows that their least common multiple is their product *ab*. If not, then there is a smaller number *d* which both *a* and *b* divide. Since $a:b = (d/b):(d/a)$, and $a:b$ is in lowest terms (since *a* and *b* are relatively prime), therefore *b* divides d/a. Also, $b:(d/a) = ab:d$, so *ab* divides *d*, but *d* is smaller than *ab*, a contradiction. Thus, when *a* and *b* are relatively prime, their least common multiple is their product.

Case 2. Suppose *a* and *b* are not relatively prime. Reduce the ratio $a:b$ to its lowest terms $f:e$ using the previous proposition VII.33. Then $ae = bf$. Let *c* denote this product. (Note that $f = a/\text{GCD}(a, b)$, and $e = b/\text{GCD}(a, b)$, so $c = ab/\text{GCD}(a, b)$.) Both *a* and *b* divide *c*, therefore *c* is a common multiple of *a* and *b*. Suppose that it's not the least common multiple. Then there is a smaller number *d* which both *a* and *b* divide. Now

$$f:e = a:b = (d/b):(d/a),$$

and $f:e$ is in lowest terms, therfore *e* divides d/a. But $e:(d/a) = ae:d$, therefore *ae* also divides *d*. But $c = ae$, and *d* is less than *c*, a contradiction. Thus, $\text{LCM}(a, b) = ab/\text{LCM}(a, b)$.

Use of this proposition

This proposition is used in VII.36 and VIII.4.

Next proposition: VII.35

Previous: VII.33

Book VII introduction

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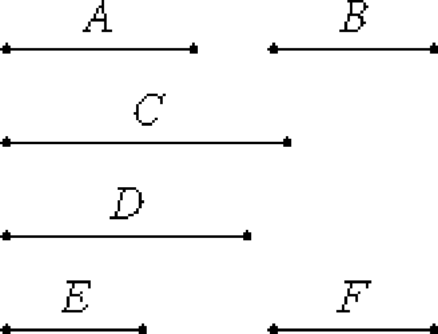
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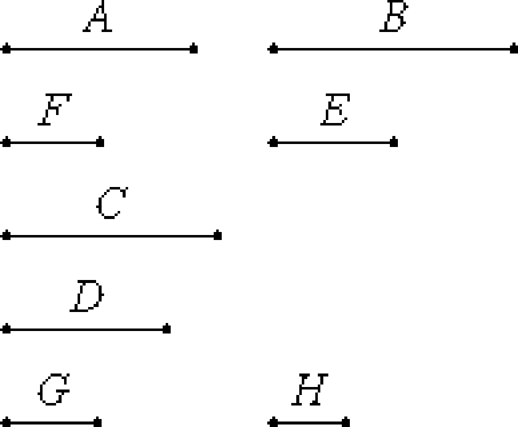
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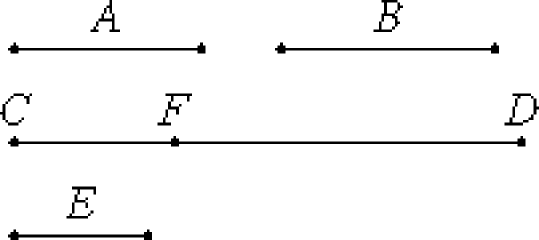
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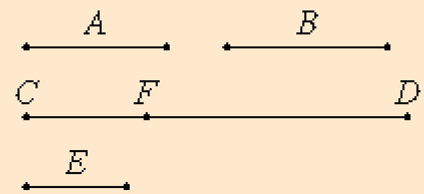
Euclid's Elements
Book VII
Proposition 35

If two numbers measure any number, then the least number measured by them also measures the same.

Let the two numbers *A* and *B* measure any number *CD*, and let *E* be the least that they measure.

I say that *E* also measures *CD.*

If *E* does not measure *CD,* let *E,* measuring *DF,* leave *CF* less than itself.



Now, since *A* and *B* measure *E*, and *E* measures *DF*, therefore *A* and *B* also measure *DF*. But they also measure the whole *CD*, therefore they measure the remainder *CF* which is less than *E,* which is impossible.

Therefore *E* cannot fail to measure *CD.* Therefore it measures it.

Therefore, *if two numbers measure any number, then the least number measured by them also measures the same.*

Q.E.D.

Guide

Outline of the proof

Assume both a and b divide c. Let e be their least common multiple. Suppose that e does not divide c. Then repeatedly subtract e from c to get $c = ke + f$, where the remainder f is less than e and k is some number. Since a and b both divide c and e, they also divide f making f a smaller common multiple than the least common multiple e, a contradiction. Thus the least common multiple also divides c.

Use of this proposition

This proposition is used in the next one and in VIII.4.

Next proposition: VII.36

Previous: VII.34

Book VII introduction

Select from Book VII

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Clark University

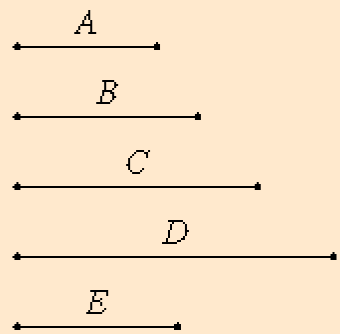
Euclid's Elements
Book VII
Proposition 36

To find the least number which three given numbers measure.

Let *A*, *B*, and *C* be the three given numbers.

It is required to find the least number which they measure.

Take *D* the least number measured by the two numbers *A* and *B*. VII.34



Then *C* either measures, or does not measure, *D*.

First, let it measure it.

But *A* and *B* also measure *D*, therefore *A*, *B*, and *C* measure *D*.

I say next that it is also the least that they measure.

If not, *A*, *B*, and *C* measure some number *E* less than *D*.

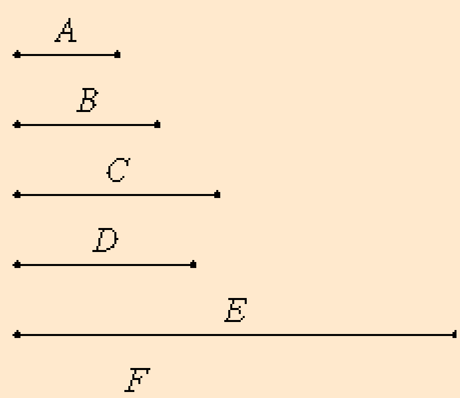
Since *A*, *B*, and *C* measure *E*, therefore *A* and *B* measure *E*. Therefore the least number measured by *A* and *B* also measures *E*. VII.35

But *D* is the least number measured by *A* and *B*, therefore *D* measures *E*, the greater the less, which is impossible.

Therefore *A*, *B*, and *C* do not measure any number less than *D*. Therefore *D* is the least that *A*, *B*, and *C* measure.

Next, let *C* not measure *D*.

Take *E*, the least number measured by *C* and *D*. VII.34



Since *A* and *B* measure *D*, and *D* measures *E*, therefore *A* and *B* also measure *E*. But *C* also measures *E*, therefore *A*, *B*, and *C* also measure *E*.

I say next that it is also the least that they measure.

If not, *A*, *B*, and *C* measure some number *F* less than *E*.

Since *A*, *B*, and *C* measure *F*, therefore *A* and *B* measure *F*. Therefore the least number measured by *A* and *B* also measures *F*. But *D* is the least number measured by *A* and *B*, therefore *D* measures *F*. But *C* also measures *F*, therefore *D* and *C* measure *F*, so that the least number measured by *D* and *C* also measures *F*. VII.35

But *E* is the least number measured by *C* and *D*, therefore *E* measures *F*, the greater the less, which is impossible.

Therefore *A*, *B*, and *C* do not measure any number which is less than *E*. Therefore *E* is the least that is measured by *A*, *B*, and *C*.

Q.E.D.

Guide

The least common multiple of three numbers *a*, *b*, and *c* can be found as

$$\text{LCM}(a, b, c) = \text{LCM}(\text{LCM}(a, b), c)).$$

This proposition is used in the proof of proposition VII.39.

Next proposition: VII.37

Previous: VII.35

Book VII introduction

Select from Book VII

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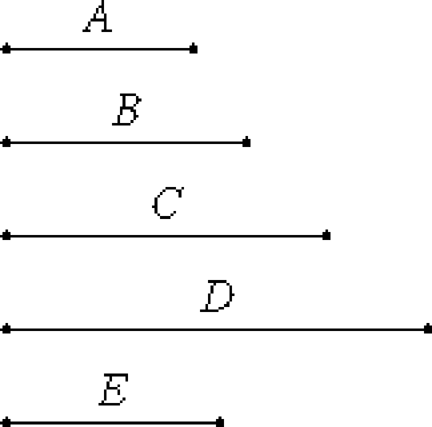
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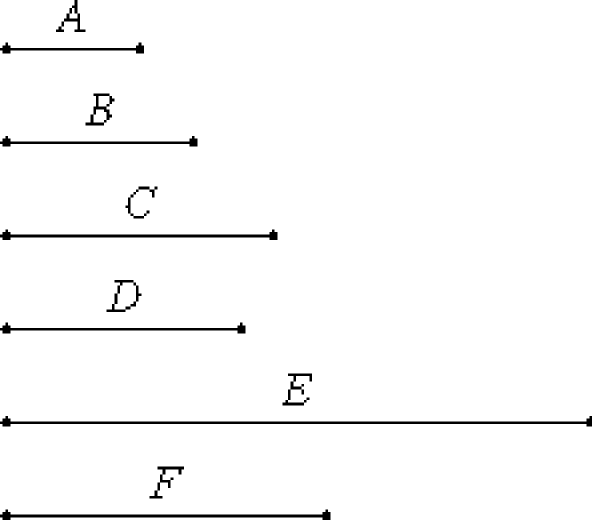
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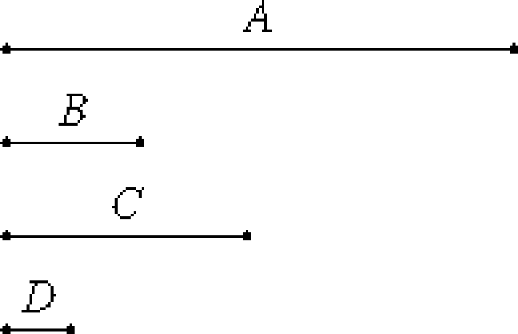
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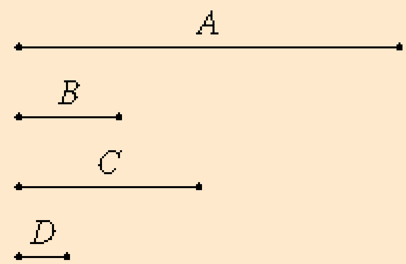
Euclid's Elements
Book VII
Proposition 37

If a number is measured by any number, then the number which is measured has a part called by the same name as the measuring number.

Let the number *A* be measured by any number *B.*

I say that *A* has a part called by the same name as *B.*



Let there be as many units in *C* as the times that *B* measures *A.*

Since *B* measures *A* according to the units in *C,* and the unit *D* also measures the number *C* according to the units in it, therefore the unit *D* measures the number *C* the same number of times as *B* measures *A.*

Therefore, alternately, the unit *D* measures the number *B* the same number of times as *C* measures *A.* Therefore, whatever part the unit *D* is of the number *B*, the same part is *C* of *A* also. But the unit *D* is a part of the number *B* called by the same name as it, therefore *C* is also a part of *A* called by the same name as *B*, so that *A* has a part *C* which is called by the same name as *B.* VII.15

Therefore, *if a number is measured by any number, then the number which is measured has a part called by the same name as the measuring number.*

Q.E.D.

Guide

This proposition says that if *b* divides *a*, then *a* has a one-b^{th} part (namely, *a/b*). For example, 3 divides 12, therefore 12 has a one-third part.

Use of this proposition.

This proposition is used in the proof of proposition VII.39.

Next proposition: VII.38

Previous: VII.36

Book VII introduction

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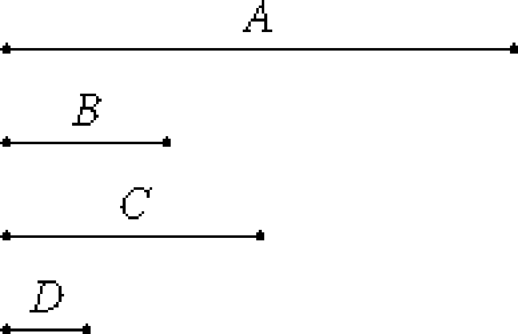
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Euclid's Elements

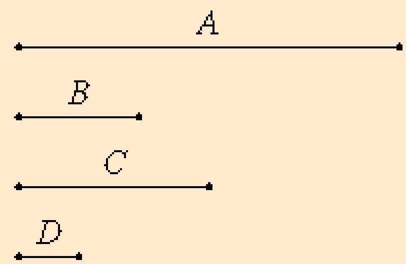
Book VII

Proposition 38

If a number has any part whatever, then it is measured by a number called by the same name as the part.

Let the number *A* have any part whatever, *B*, and let *C* be a number called by the same name as the part *B*.

I say that *C* measures *A*.



Since *B* is a part of *A* called by the same name as *C*, and the unit *D* is also a part of *C* called by the same name as it, therefore the part *B* of *A* is the same part of the unit *D* of the number *C*. Therefore the unit *D* measures the number *C* the same number of times that *B* measures *A*.

Therefore, alternately, the unit *D* measures the number *B* the same number of times that *C* measures *A*. Therefore *C* measures *A*. VII.15

Therefore, *if a number has any part whatever, then it is measured by a number called by the same name as the part.*

Q.E.D.

Guide

This proposition says that if *a* has a one-c^{th} part of *a*, then *c* divides *a*. For example, 12 has a one-third part, 3 divides 12. This is a converse of the last proposition.

Use of this proposition.

This proposition is used in the proof of the next proposition.

Next proposition: VII.39

Previous: VII.37

Book VII introduction

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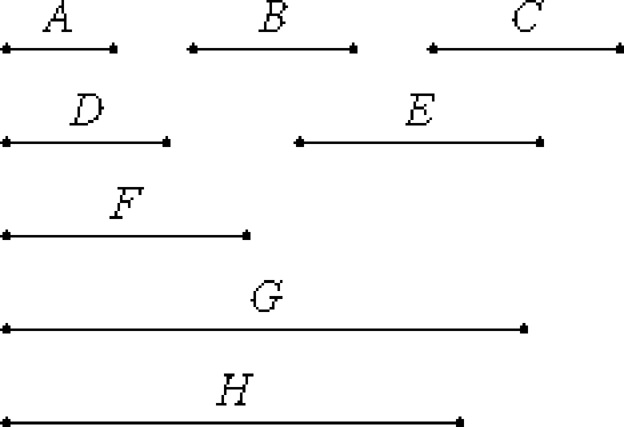
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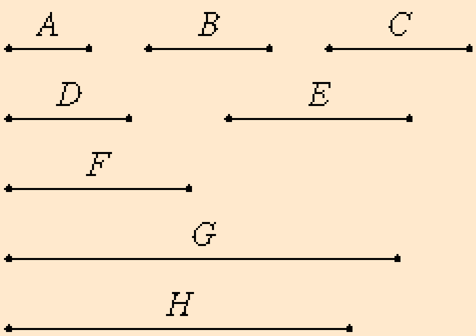
Euclid's Elements
Book VII
Proposition 39

To find the number which is the least that has given parts.

Let *A, B,* and *C* be the given parts.

It is required to find the number which is the least that will have the parts *A, B,* and *C.*



Let *D, E,* and *F* be numbers called by the same name as the parts *A, B,* and *C.* Take *G,* the least number measured by *D, E,* and *F.* VII.36

Therefore *G* has parts called by the same name as *D, E,* and *F.* VII.37

But *A, B,* and *C* are parts called by the same name as *D, E,* and *F,* therefore *G* has the parts *A, B,* and *C.*

I say next that it is also the least number that has.

If not, there is some number *H* less than *G* which has the parts *A, B,* and *C.*

Since *H* has the parts *A, B,* and *C,* therefore *H* is measured by numbers called by the same name as the parts *A, B,* and *C.* But *D, E,* and *F* are numbers called by the same name as the parts *A, B,* and *C,* therefore *H* is measured by *D, E,* and *F.* VII.38

And it is less than *G,* which is impossible. Therefore there is no number less than *G* that has the parts *A, B,* and *C.*

Q.E.D.

Guide

The wording of the proposition is somewhat unclear, but an example will show its intent.

Suppose you want to find the smallest number with given parts, say, a fourth part and a sixth part. Then take the LCM(4,6) which is 12. The number 12 has a 1/4 part, namely 3, and a 1/6 part, namely 2.

Next book: Book VIII

Previous proposition: VII.38

Book VII introduction

Select from Book VII

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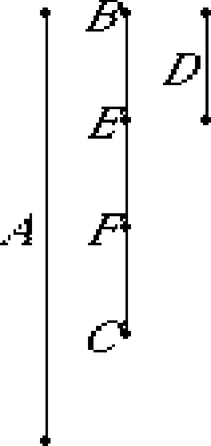
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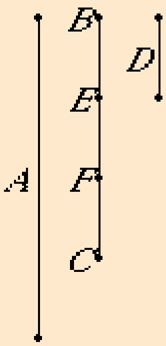
Euclid's Elements
Book VII
Proposition 4

Any number is either a part or parts of any number, the less of the greater.

Let *A* and *BC* be two numbers, and let *BC* be the less.

I say that *BC* is either a part, or parts, of *A*.



Either *A* and *BC* are relatively prime or they are not.

First, let *A* and *BC* be relatively prime.

Then, if *BC* is divided into the units in it, then each unit of those in *BC* is some part of *A*, so that *BC* is parts of *A*. VII.Def.4

Next let *A* and *BC* not be relatively prime, then *BC* either measures, or does not measure, *A*.

Now if *BC* measures *A*, then *BC* is a part of *A*. But, if not, take the greatest common measure *D* of *A* and *BC*, and divide *BC* into the numbers equal to *D*, namely *BE*, *EF*, and *FC*. VII.Def.3 VII.2

Now, since *D* measures *A*, therefore *D* is a part of *A*. But *D* equals each of the numbers *BE*, *EF*, and *FC*, therefore each of the numbers *BE*, *EF*, and *FC* is also a part of *A*, so that *BC* is parts of *A*.

Therefore, *any number is either a part or parts of any number, the less of the greater.*

Q.E.D.

Guide

This proposition says that if *b* is a smaller number than *a*, then *b* is either a part of *a*, that is, *b* is a unit fraction of *a*, or *b* is parts of *a*, that is, a proper fraction, but not a unit fraction, of *a*. For instance, 2 is one part of 6, namely, one third part; but 4 is parts of 6, namely, 2 third parts of 6.

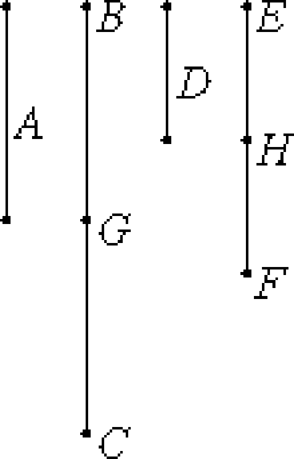
It seems obvious that when one number *b* is less than another *a*, then in all cases *b* would be parts of *a*, namely *b* consists of *b* of the a^{th} parts of *a*. For instance, 4 consists of 4 sixth parts of 6. Yet, the proof of this proposition ignores that possibility, except in the special case when *b* and *a* are relatively prime. In the case of 4 and 6, the proof will find that 4 is 2 third parts of 6. Thus, it appears that a satisfactory answer to the question "How mary parts of *a* is *b*?" requires finding the least number of parts.

The proof has three cases.

1. If *b* and *a* are relatively prime, then *b* consists of *b* of the a^{th} parts of *a*.
2. If *b* divides *a*, then *b* is one part of *a*.
3. Otherwise they're not relatively prime, and *b* does not divide *a*. Let *d* be their greatest common divisor. Then *b* consists of some number, say *c* parts, each part equal to *d*. But these parts also also parts of *a*. Therefore, *b* consists of *c* parts of *a*.

Use of this proposition

This proposition is used in VII.20.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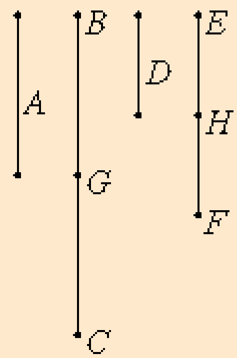
Euclid's Elements
Book VII
Proposition 5

If a number is part of a number, and another is the same part of another, then the sum is also the same part of the sum that the one is of the one.

Let the number *A* be a part of *BC,* and another number *D* be the same part of another number *EF* that *A* is of *BC.*

I say that the sum of *A* and *D* is also the same part of the sum of *BC* and *EF* that *A* is of *BC.*



Since, whatever part *A* is of *BC, D* is also the same part of *EF,* therefore, there are as many numbers equal to *D* in *EF* as there are in *BC* equal to *A.*

Divide *BC* into the numbers equal to *A,* namely *BG* and *GC,* and *EF* into the numbers equal to *D,* namely *EH* and *HF.* Then the multitude of *BG* and *GC* equals the multitude of *EH* and *HF.*

And, since *BG* equals *A,* and *EH* equals *D,* therefore the sum of *BG* and *EH* also equals the sum of *A* and *D.* For the same reason the sum of *GC* and *HF* also equals the sum of *A* and *D.*

Therefore there are as many numbers in *BC* and *EF* equal to *A* and *D* as there are in *BC* equal to *A.* Therefore, the sum of *BC* and *EF* is the same multiple of the sum of *A* and *D* that *BC* is of *A.* Therefore, the sum of *A* and *D* is the same part of the sum of *BC* and *EF* that *A* is of *BC.*

Therefore, *if a number is part of a number, and another is the same part of another, then the sum is also the same part of the sum that the one is of the one.*

Q.E.D.

Guide

This is the first of four propositions that deal with distributivity of division and multiplication over addition and subtraction. This one says division distributes over addition. Algebraically, if $a = b/n$ and $d = e/n$, then $a + d = (b + e)/n$. As a single equation, this says

$$b/n + e/n = (b + e)/n.$$

Use of this proposition

This proposition is used in the proofs of five of the next seven propositions.

Next proposition: VII.6

Previous: VII.4

Book VII introduction

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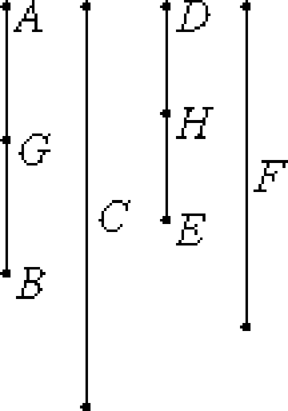
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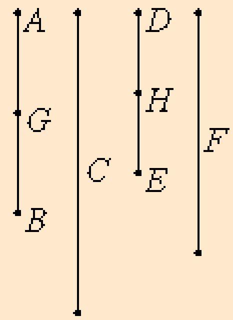
Euclid's Elements
Book VII
Proposition 6

If a number is parts of a number, and another is the same parts of another, then the sum is also the same parts of the sum that the one is of the one.

Let the number *AB* be parts of the number *C*, and another number *DE* be the same parts of another number *F* that *AB* is of *C*.

I say that the sum of *AB* and *DE* is also the same parts of the sum of *C* and *F* that *AB* is of *C*.

Since there are as many parts of *DE* in *F* as there are parts *AB* is of *C*, therefore there are as many parts of *F* in *DE* as there are parts of *C* in *AB*.



Divide *AB* into the parts of *C*, namely *AG* and *GB*, and divide *DE* into the parts of *F*, namely *DH*, and *HE*. Then the multitude of *AG* and *GB* equals the multitude of *DH* and *HE*.

And since *DH* is the same part of *F* that *AG* is of *C*, therefore the sum of *AG* and *DH* is the same part of the sum of *C* and *F* that *AG* is of *C*. For the same reason, the sum of *GB* and *HE* is the same parts of the sum of *C* and *F* that *GB* is of *C*. VII.5

Therefore the sum of *AB* and *DE* is the same parts of the sum of *C* and *F* that *AB* is of *C*.

Therefore, *if a number is parts of a number, and another is the same parts of another, then the sum is also the same parts of the sum that the one is of the one.*

Q.E.D.

Guide

This proposition says multiplication by fractions distributes over addition. Algebraically, if $a = (m/n)b$ and $d = (m/n)e$ then $a + d = (m/n)(b + e)$. As an equation, this says

$$(m/n)b + (m/n)e = (m/n)(b + e).$$

Use of this proposition VII.9.

Next proposition: VII.7

Previous: VII.5

Book VII introduction

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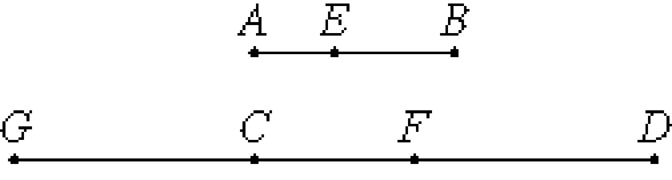
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Euclid's Elements

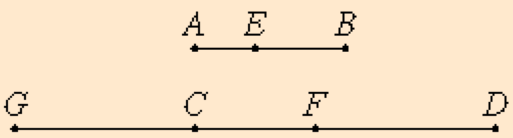
Book VII

Proposition 7

If a number is that part of a number which a subtracted number is of a subtracted number, then the remainder is also the same part of the remainder that the whole is of the whole.

Let the number *AB* be that part of the number *CD* which *AE* subtracted is of *CF* subtracted.

I say that the remainder *EB* is also the same part of the remainder *FD* that the whole *AB* is of the whole *CD*.



Let *EB* be the same part of *CG* that *AE* is of *CF*.

Now since *EB* is the same part of *CG* that *AE* is of *CF*, therefore *AB* is the same part of *GF* that *AE* is of *CF*. VII.5

But, by hypothesis, *AB* is the same part of *CD* that *AE* is of *CF*, therefore *AB* is the same part of *CD* that it is of *GF*. Therefore *GF* equals *CD*.

Subtract *CF* from each. Then the remainder *GC* equals the remainder *FD*.

Now since *EB* is the same part of *GC* that *AE* is of *CF*, and *GC* equals *FD*, therefore *EB* is the same part of *FD* that *AE* is of *CF*.

But *AB* is the same part of *CD* that *AE* is of *CF*, therefore the remainder *EB* is the same part of the remainder *FD* that the whole *AB* is of the whole *CD*.

Therefore, *if a number is that part of a number which a subtracted number is of a subtracted number, then the remainder is also the same part of the remainder that the whole is of the whole.*

Q.E.D.

Guide

This proposition is like VII.5 except it deals with subtraction instead of addition. It says division distributes over subtraction. Algebraically, if $a = b/n$ and $d = e/n$, then $a - d = (b - e)/n$.

This proposition is used in the next proposition and in VII.11.

Next proposition: VII.8

Previous: VII.6

Book VII introduction

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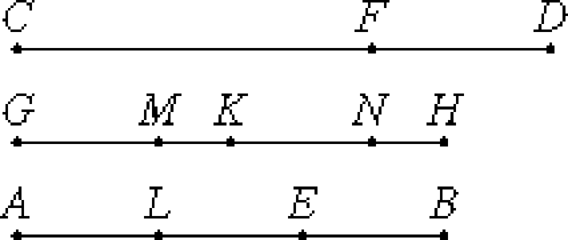
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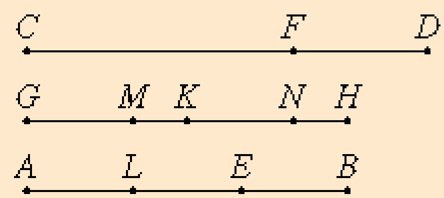
Euclid's Elements
Book VII
Proposition 8

If a number is the same parts of a number that a subtracted number is of a subtracted number, then the remainder is also the same parts of the remainder that the whole is of the whole.

Let the number *AB* be the same parts of the number *CD* that *AE* subtracted is of *CF* subtracted.

I say that the remainder *EB* is also the same parts of the remainder *FD* that the whole *AB* is of the whole *CD.*

Make *GH* equal to *AB.*



Therefore *AE* is the same parts of *CF* that *GH* is of *CD.*

Divide *GH* into the parts of *CD,* namely *GK* and *KH,* and divide *AE* into the parts of *CF,* namely *AL* and *LE.* Then the multitude of *GK* and *KH* equals the multitude of *AL* and *LE.*

Now since *AL* is the same part of *CF* that *GK* is of *CD,* and *CD* is greater than *CF,* therefore *GK* is also greater than *AL.*

Make *GM* equal to *AL.*

Then *GK* is the same part of *CD* that *GM* is of *CF.* Therefore the remainder *MK* is the same part of the remainder *FD* that the whole *GK* is of the whole *CD.* VII.7

Again, since *EL* is the same part of *CF* that *KH* is of *CD,* and *CD* is greater than *CF,* therefore *HK* is also greater than *EL.*

Make *KN* equal to *EL.*

Therefore *KN* is the same part of *CF* that *KH* is of *CD.* Therefore the remainder *NH* is the same part of the remainder *FD* that the whole *KH* is of the whole *CD.* VII.7

But the remainder *MK* was proved to be the same part of the remainder *FD* that the whole *GK* is of the whole *CD,* therefore the sum of *MK* and *NH* is the same parts of *DF* that the whole *HG* is of the whole *CD.*

But the sum of *MK* and *NH* equals *EB,* and *HG* equals *BA,* therefore the remainder *EB* is the same parts of the remainder *FD* that the whole *AB* is of the whole *CD.*

Therefore, *if a number is the same parts of a number that a subtracted number is of a subtracted number, then the remainder is also the same parts of the remainder that the whole is of the whole.*

Q.E.D.

Guide

This proposition says multiplication by fractions distributes over subtraction. Algebraically, if $a = (m/n)b$ and $d = (m/n)e$, then $a + d = (m/n)(b + e)$. The sample value taken for m/n in the proof is 2/3.

This proposition is used in VII.11.

Next proposition: VII.9

Previous: VII.7

Book VII introduction

Select from Book VII

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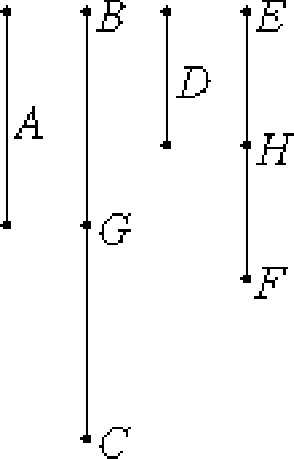
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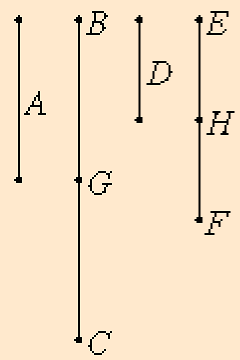
Euclid's Elements
Book VII
Proposition 9

If a number is a part of a number, and another is the same part of another, then alternately, whatever part of parts the first is of the third, the same part, or the same parts, the second is of the fourth.

Let the number *A* be a part of the number *BC,* and and another number *D* be the same part of another number *EF* that *A* is of *BC.*

I say that, alternately, *BC* is the same part or parts of *EF* that *A* is of *D.*

Since *D* is the same part of *EF* that *A* is of *BC*, therefore there are as many numbers *BC* equal to *A* as there are also in *EF* equal to *D.*



Divide *BC* into the numbers equal to *A*, namely *BG* and *GC,* and divide *EF* into those equal to *D*, namely *EH* and *HF.* Then the multitude of *BG* and *GC* equals the multitude of *EH* and *HF*.

Now, since the numbers *BG* and *GC* equal one another, and the numbers *EH* and *HF* also equal one another, while the multitude of *BG* and *GC* equals the multitude of *EH* and *HF*, therefore *GC* is the same part or parts of *HF* that *BG* is of *EH*, so that, in addition, the sum *BC* is the same part or parts of the sum *EF* that *BG* is of *EH.* VII.5 VII.6

But *BG* equals *A*, and *EH* equals *D*, therefore *BC* is the same part or parts of *EF* that *A* is of *D.*

Therefore, *if a number is a part of a number, and another is the same part of another, then alternately, whatever part of parts the first is of the third, the same part, or the same parts, the second is of the fourth.*

Q.E.D.

Guide

In this proposition, Euclid shows that if $a = b/n$, and $d = e/n$, and if $a = (m/n)d$, then $b = (m/n)e$. The sample value taken for $1/n$ in the proof is 1/2.

Proposition VII.15 can be construed as a special case of this one.

This proposition is used in the proof of the next.

Next proposition: VII.10

Previous: VII.8

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propVII6.gif	16-May-1997 10:12	1k	

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propVII20.html	15-Oct-2002 09:49	10k	
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propVII17.html	15-Oct-2002 09:49	8k	
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defVII6.html	15-Oct-2002 09:47	7k
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propVII20.gif	16-May-1997 10:12	1k
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propVII27.gif	16-May-1997 10:12	1k
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propVII5.gif	16-May-1997 10:12	1k
propVII12.gif	16-May-1997 10:12	1k
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propVII37.gif	16-May-1997	10:12	1k
propVII38.gif	16-May-1997	10:12	1k
propVII7.gif	16-May-1997	10:12	1k
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defVII16c.gif	21-Jul-1997	11:56	1k
propVII32.gif	16-May-1997	10:12	1k

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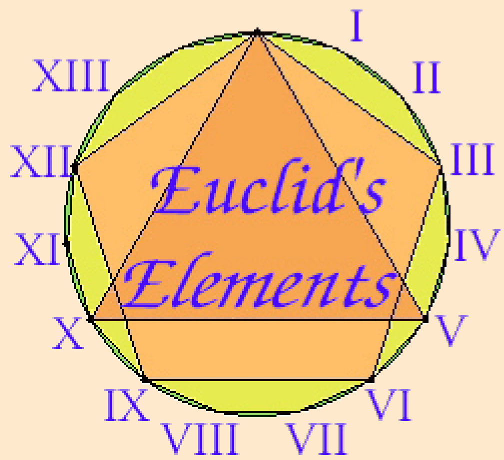
Book VIII



Book VIII

Table of contents

Propositions

Proposition 1.
If there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then the numbers are the least of those which have the same ratio with them.

Proposition 2.
To find as many numbers as are prescribed in continued proportion, and the least that are in a given ratio.

Corollary. If three numbers in continued proportion are the least of those which have the same ratio with them, then the extremes are squares, and, if four numbers, cubes.

Proposition 3.
If as many numbers as we please in continued proportion are the least of those which have the same ratio with them, then the extremes of them are relatively prime.

Proposition 4.
Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Proposition 5.
Plane numbers have to one another the ratio compounded of the ratios of their sides.

Proposition 6.
If there are as many numbers as we please in continued proportion, and the first does not measure the second, then neither does any other measure any other.

Proposition 7.
If there are as many numbers as we please in continued proportion, and the first measures the last, then it also measures the second.

Proposition 8.
If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many also fall in continued proportion between the numbers which have the same ratios with the original numbers.

Proposition 9.
If two numbers are relatively prime, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many also fall between each of them and a unit in continued proportion.

Proposition 10.
If numbers fall between two numbers and a unit in continued proportion, then however many numbers fall between each of them and a unit in continued proportion, so many also fall between the numbers themselves in continued proportion.

Proposition 11.
Between two square numbers there is one mean proportional number, and the square has to the square the duplicate ratio of that which the side has to the side.

Proposition 12.
Between two cubic numbers there are two mean proportional numbers, and the cube has to the cube the triplicate ratio of that which the side has to the side.

Proposition 13.
If there are as many numbers as we please in continued proportion, and each multiplied by itself makes some number, then the products are proportional; and, if the original numbers multiplied by the products make certain numbers, then the latter are also proportional.

Proposition 14.
If a square measures a square, then the side also measures the side; and, if the side measures the side, then the square also measures the square.

Proposition 15.
If a cubic number measures a cubic number, then the side also measures the side; and, if the side measures the side, then the cube also measures the cube.

Proposition 16.
If a square does not measure a square, then neither does the side measure the side; and, if the side does not measure the side, then neither does the square measure the square.

Proposition 17.
If a cubic number does not measure a cubic number, then neither does the side measure the side; and, if the side does not measure the side, then neither does the cube measure the cube.

Proposition 18.
Between two similar plane numbers there is one mean proportional number, and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Proposition 19.
Between two similar solid numbers there fall two mean proportional numbers, and the solid number has to the solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

Proposition 20.
If one mean proportional number falls between two numbers, then the numbers are similar plane numbers.

Proposition 21.
If two mean proportional numbers fall between two numbers, then the numbers are similar solid numbers.

Proposition 22.
If three numbers are in continued proportion, and the first is square, then the third is also square.

Proposition 23.
If four numbers are in continued proportion, and the first is a cube, then the fourth is also a cube.

Proposition 24.
If two numbers have to one another the ratio which a square number has to a square number, and the first is square, then the second is also a square.

Proposition 25.
If two numbers have to one another the ratio which a cubic number has to a cubic number, and the first is a cube, then the second is also a cube.

Proposition 26.
Similar plane numbers have to one another the ratio which a square number has to a square number.

Proposition 27.
Similar solid numbers have to one another the ratio which a cubic number has to a cubic number.

Next book: Book IX

Previous: Book VII

Book VIII introduction

Select from Book VIII

Select book

Select topic

Bad Request

Your browser sent a request that this server could not understand.

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propVIII11.html	16-Oct-2002 09:07	8k	
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propVIII26.html	16-Oct-2002 09:07	7k
propVIII27.html	16-Oct-2002 09:08	7k

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Index of /~djoyce/java/elements/bookVIII

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propVIII2.html	16-Oct-2002 09:06	12k	
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Index of /~djoyce/java/elements/bookVIII

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Index of /~djoyce/java/elements/bookVIII

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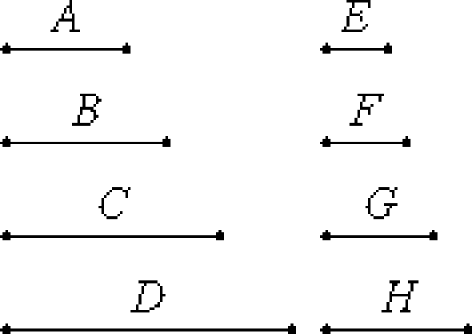
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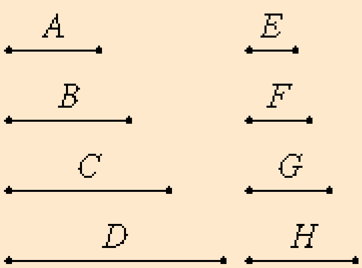
Euclid's Elements
Book VIII
Proposition 1

If there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then the numbers are the least of those which have the same ratio with them.

Let there be as many numbers as we please, *A, B, C,* and *D,* in continued proportion, and let the extremes of them, *A* and *D,* be relatively prime.

I say that *A, B, C,* and *D* are the least of those which have the same ratio with them.

If not, let *E, F, G,* and *H* be less than *A, B, C,* and *D,* and in the same ratio with them.



Now, since *A, B, C,* and *D* are in the same ratio with *E, F, G,* and *H,* and the multitude of the numbers *A, B, C,* and *D* equals the multitude of the numbers *E, F, G,* and *H,* therefore, *ex aequali A* is to *D* as *E* is to *H.* VII.14

But *A* and *D* are relatively prime, numbers which are relatively prime are also least, and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. Therefore *A* measures *E,* the greater the less, which is impossible. VII.21 VII.20

Therefore *E, F, G,* and *H,* which are less than *A, B, C,* and *D,* are not in the same ratio with them. Therefore *A, B, C,* and *D* are the least of those which have the same ratio with them.

Therefore, *if there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then the numbers are the least of those which have the same ratio with them.*

Q.E.D.

Guide

Continued proportions and geometric progressions

Euclid doesn't define a continued proportion. In this proposition we consider only continued proportions with a constant ratio, proportions of the form

$$a_1:a_2 = a_2:a_3 = a_3:a_4 = \ldots = a_n-1:a_n.$$

In proposition VIII.4 continued proportions that don't have a constant ratio will be considered.

An example of continued proportion with constant ratio is

$$1250:750 = 750:450 = 450:270 = 270:162$$

since each of the ratios is the same as the ratio 5:3.

A modern expression for this situation is to say that the numbers a_1, a_2, a_3, ..., a_n-1:a_n are in a *geometric progression* or a *geometric sequence*. The ratio of any consecutive pair in a geometric progression is constant.

Many of the propositions in Books VIII and IX treat geometric progressions. The sum of a geometric progression is found in proposition IX.35.

About this proposition

This proposition says that if the end numbers are relatively prime in a continued proportion with constant ratio, then there is no continued proportion of the same length and same ratio having smaller numbers. We could say that the continued proportion is in *lowest terms*. The proposition generalizes VII.21 when there are only two numbers in continued proportion, that is, a ratio. VII.21 says if the two numbers are relatively prime, then the ratio is in lowest terms.

Notice that the example of a continued proportion given above, 1250:750 = 750:450 = 450:270 = 270:162, is not in lowest terms, since all the numbers may be halved to get continued proportion of the same length and same ratio but with smaller numbers.

Use of this proposition

This proposition is used in the next one and in VIII.9. The converse of this proposition is VIII.3.

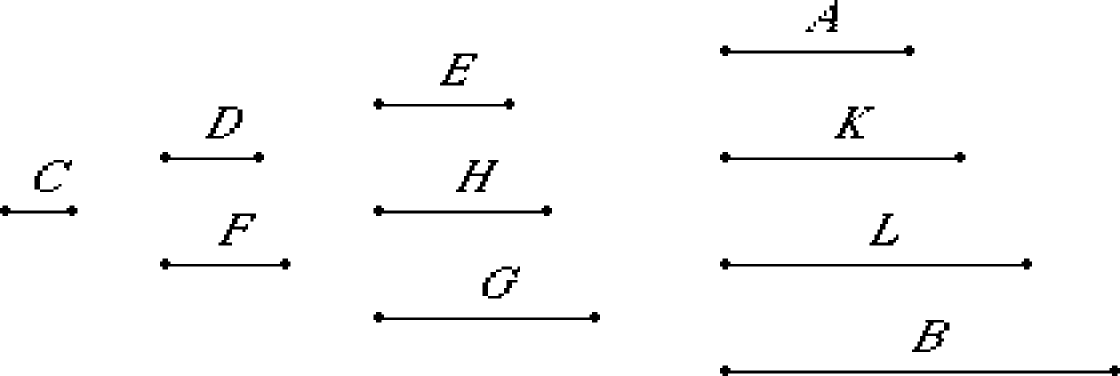
Next proposition: VIII.2

Book VIII introduction

Select from Book VIII

Select book

Select topic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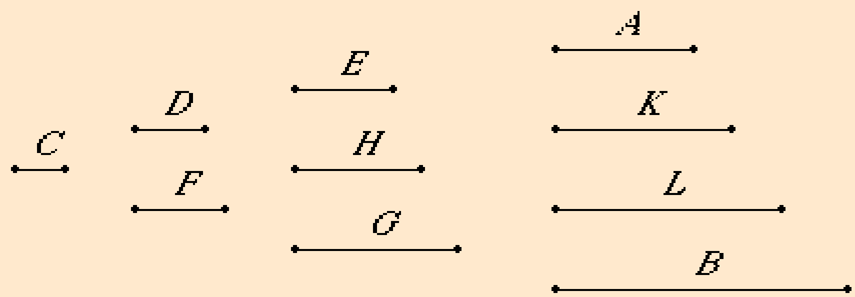
Euclid's Elements
Book VIII
Proposition 10

If numbers fall between two numbers and a unit in continued proportion, then however many numbers fall between each of them and a unit in continued proportion, so many also fall between the numbers themselves in continued proportion.

Let the numbers *D* and *E* and the numbers *F* and *G* respectively fall between the two numbers *A* and *B* and the unit *C* in continued proportion.

I say that, as many numbers have fallen between each of the numbers *A* and *B* and the unit *C* in continued proportion as fall between *A* and *B* in continued proportion.

Multiply *D* by *F* to make *H,* and multiply the numbers *D* and *F* by *H* to make *K* and *L* respectively.



Now, since the unit *C* is to the number *D* as *D* is to *E,* therefore the unit *C* measures the number *D* the same number of times as *D* measures *E.* But the unit *C* measures the number *D* according to the units in *D,* therefore the number *D* also measures *E* according to the units in *D.* Therefore *D* multiplied by itself makes *E.* VII.Def.20

Again, since *C* is to the number *D* as *E* is to *A*, therefore the unit *C* measures the number *D* the same number of times as *E* measures *A.* But the unit *C* measures the number *D* according to the units in *D*, therefore *E* also measures *A* according to the units in *D.* Therefore *D* multiplied by *E* makes *A.*

For the same reason also *F* multiplied by itself makes *G,* and multiplied by *G* makes *B.*

And, since *D* multiplied by itself makes *E* and multiplied by *F* makes *H,* therefore *D* is to *F* as *E* is to *H.* VII.17

For the same reason also *D* is to *F* as *H* is to *G.* Therefore *E* is to *H* as *H* is to *G.* VII.18

Again, since *D* multiplied by the numbers *E* and *H* makes *A* and *K* respectively, therefore *E* is to *H* as *A* is to *K.* But *E* is to *H* as *D* is to *F*, therefore *D* is to *F* as *A* is to *K.* VII.17

Again, since the numbers *D* and *F* multiplied by *H* make *K* and *L* respectively, therefore *D* is to *F* as *K* is to *L*. But *D* is to *F* as *A* is to *K*, therefore *A* is to *K* as *K* is to *L*. Further, since *F* multiplied by the numbers *H* and *G* makes *L* and *B* respectively, therefore *H* is to *G* as *L* is to *B.* VII.18 VII.17

But *H* is to *G* as *D* is to *F*, therefore *D* is to *F* as *L* is to *B.* But it was also proved that *D* is to *F* as *A* is to *K* and as *K* is to *L*, therefore *A* is to *K* as *K* is to *L* and as *L* is to *B*. Therefore *A*, *K*, *L*, and *B* are in continued proportion.

Therefore, as many numbers as fall between each of the numbers *A* and *B* and the unit *C* in continued proportion, so many also fall between *A* and *B* in continued proportion.

Therefore, *if numbers fall between two numbers and a unit in continued proportion, then however many numbers fall between each of them and a unit in continued proportion, so many also fall between the numbers themselves in continued proportion.*

Q.E.D.

Guide

This is a partial converse to the previous proposition; it doesn't require that the generating numbers d and f be relatively prime in order to conclude that the sequence

$$d^{n-1}, d^{n-2}f, d^{n-3}f,^2, \ldots, df^{n-2}, f^{n-1}$$

is in continued proportion.

Next proposition: VIII.11

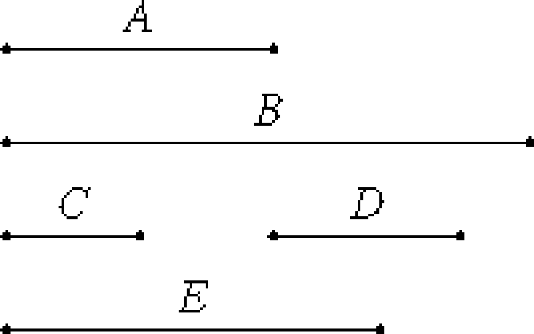
Previous: VIII.9

Book VIII introduction

Select from Book VIII

Select book

Select topic


A
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Euclid's Elements
Book VIII
Proposition 11

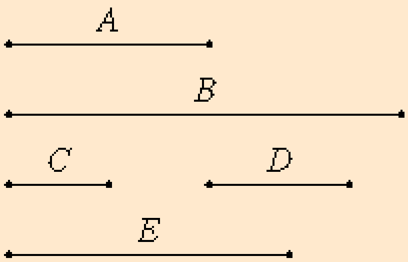
Between two square numbers there is one mean proportional number, and the square has to the square the duplicate ratio of that which the side has to the side.

Let *A* and *B* be square numbers, and let *C* be the side of *A*, and *D* of *B*.

I say that between *A* and *B* there is one mean proportional number, and *A* has to *B* the ratio duplicate of that which *C* has to *D*.

Multiply *C* by *D* to make *E*.

Now, since *A* is a square and *C* is its side, therefore *C* multiplied by itself makes *A*. For the same reason also, *D* multiplied by itself makes *B*.



Since, then, *C* multiplied by the numbers *C* and *D* makes *A* and *E* respectively, therefore *C* is to *D* as *A* is to *E*. VII.17

For the same reason also *C* is to *D* as *E* is to *B*. Therefore *A* is to *E* as *E* is to *B*. Therefore between *A* and *B* there is one mean proportional number. VII.18

I say next that *A* also has to *B* the ratio duplicate of that which *C* has to *D*.

Since *A*, *E*, and *B* are three numbers in proportion, therefore *A* has to *B* the ratio duplicate of that which *A* has to *E*. V.Def.9

But *A* is to *E* as *C* is to *D*, therefore *A* has to *B* the ratio duplicate of that which the side *C* has to *D*.

Therefore, *between two square numbers there is one mean proportional number, and the square has to the square the duplicate ratio of that which the side has to the side.*

Q.E.D.

Guide

Between c^2 and d^2 is the mean proportional cd, and the ratio $c^2:d^2$ is the duplicate ratio of $c:d$. The argument for the latter statement is that $c^2:d^2$ is compounded of the two ratios $c^2:cd$ and $cd:d^2$, but both of those are the same ratio as $c:d$.

Use of this proposition

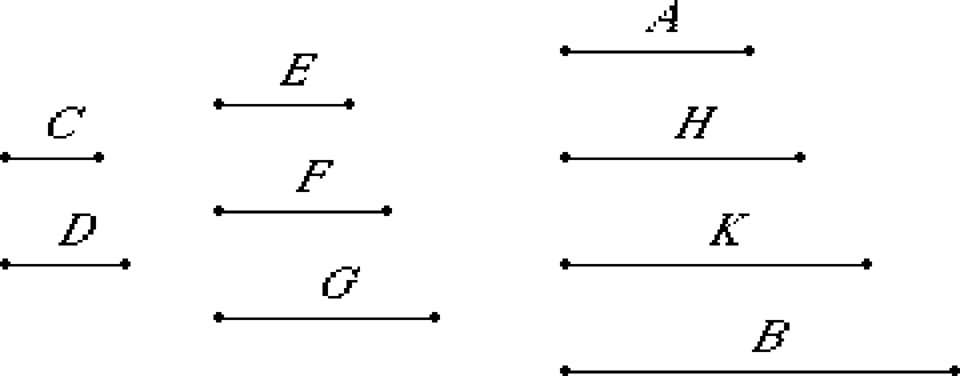
This proposition is used in propositions VIII.14, VIII.15, and X.9.

Previous: VIII.10

Book VIII introduction

Select book

Select topic


A
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Euclid's Elements
Book VIII
Proposition 12

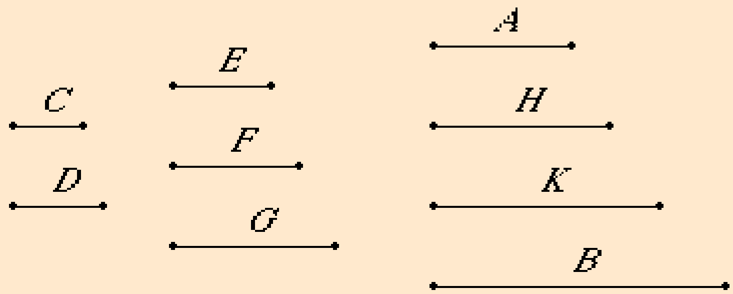
Between two cubic numbers there are two mean proportional numbers, and the cube has to the cube the triplicate ratio of that which the side has to the side.

Let *A* and *B* be cubic numbers, and let *C* be the side of *A*, and *D* of *B*.

I say that between *A* and *B* there are two mean proportional numbers, and *A* has to *K* the ratio triplicate of that which *C* has to *D*.

Multiply *C* by itself to make *E*, and by *D* to make *F*, multiply *D* by itself to make *G*, and multiply the numbers *C* and *D* by *F* to make *H* and *K* respectively.

Now, since *A* is a cube, and *C* its side, and *C* multiplied by itself makes *E*, therefore *C* multiplied by itself makes *E* and multiplied by *E* makes *A*. For the same reason also *D* multiplied by itself makes *G* and multiplied by *G* makes *B*.



And, since *C* multiplied by the numbers *C* and *D* makes *E* and *F* respectively, therefore *C* is to *D* as *E* is to *F*. For the same reason also *C* is to *D* as *F* is to *G*. Again, since *C* multiplied by the numbers *E* and *F* makes *A* and *H* respectively, therefore *E* is to *F* as *A* is to *H*. But *E* is to *F* as *C* is to *D*. Therefore *C* is to *D* as *A* is to *H*. VII.17 VII.18

Again, since the numbers *C* and *D* multiplied by *F* make *H* and *K* respectively, therefore *C* is to *D* as *H* is to *K*. Again, since *D* multiplied by each of the numbers *F* and *G* makes *K* and *B* respectively, therefore *F* is to *G* as *K* is to *B*. VII.18 VII.17

But *F* is to *G* as *C* is to *D*, therefore *C* is to *D* as *A* is to *H*, as *H* is to *K*, and as *K* is to *B*.

Therefore *H* and *K* are two mean proportionals between *A* and *B*.

I say next that *A* also has to *B* the ratio triplicate of that which *C* has to *D*.

Since *A*, *H*, *K*, and *B* are four numbers in proportion, therefore *A* has to *B* the ratio triplicate of that which *A* has to *H*. V.Def.10

But *A* is to *H* as *C* is to *D*, therefore *A* also has to *B* the ratio triplicate of that which *C* has to *D*.

Therefore, *Between two cubic numbers there are two mean proportional numbers, and the cube has to the cube the triplicate ratio of that which the side has to the side.*

Q.E.D.

Guide

This proposition is used in VIII.15.

Next proposition: VIII.13

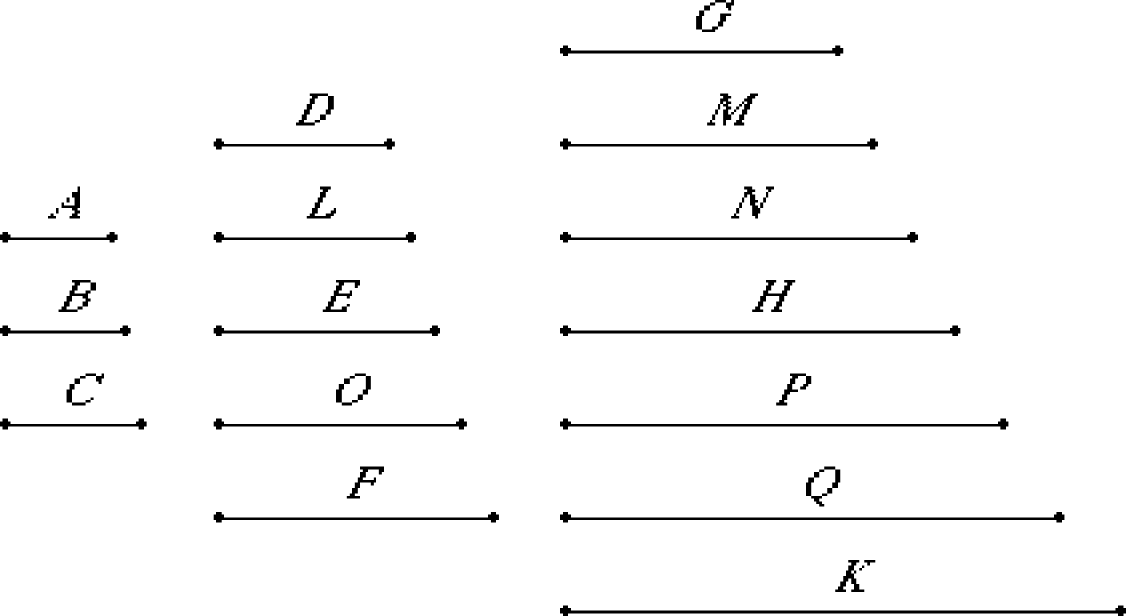
Previous: VIII.11

Book VIII introduction

Select from Book VIII

Select book

Select topic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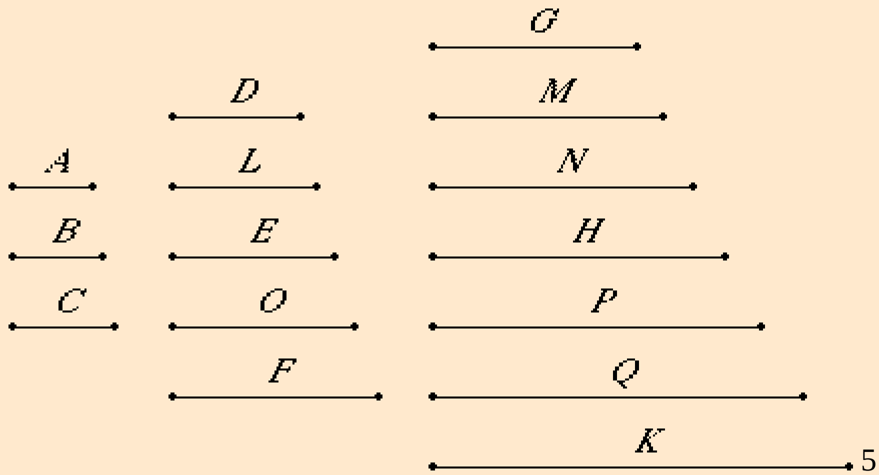
Euclid's Elements
Book VIII
Proposition 13

If there are as many numbers as we please in continued proportion, and each multiplied by itself makes some number, then the products are proportional; and, if the original numbers multiplied by the products make certain numbers, then the latter are also proportional.

Let there be as many numbers as we please, *A, B,* and *C,* in continued proportion, so that *A* is to *B* as *B* is to *C.* Let *A, B,* and *C* multiplied by themselves make *D, E,* and *F,* and multiplied by *D, E,* and *F* let them make *G, H,* and *K.*

I say that *D, E,* and *F* and *G, H,* and *K* are in continued proportion.



Multiply *A* by *B* to make *L,* and multiply the numbers *A* and *B* by *L* to make *M* and *N* respectively. Also multiply *B* by *C* to make *O,* and multiply the numbers *B* and *C* by *O* to make *P* and *Q* respectively.

Then, in manner similar to the foregoing, we can prove that *D, L,* and *E* and *G, M, N,* and *H* are continuously proportional in the ratio of *A* to *B,* and further *E, O,* and *F* and *H, P, Q,* and *K* are continuously proportional in the ratio of *B* to *C.*

Now *A* is to *B* as *B* is to *C,* therefore *D, L,* and *E* are also in the same ratio with *E, O,* and *F,* and further *G, M, N,* and *H* in the same ratio with *H, P, Q,* and *K.* And the multitude of *D, L,* and *E* equals the multitude of *E, O,* and *F* and that of *G, M, N,* and *H* to that of *H, P, Q,* and *K,* therefore, *ex aequali D* is to *E* as *E* is to *F,* and *G* is to *H* as *H* is to *K.* VII.14

Therefore, *if there are as many numbers as we please in continued proportion, and each multiplied by itself makes some number, then the products are proportional; and, if the original numbers multiplied by the products make certain numbers, then the latter are also proportional.*

Q.E.D.

Guide

The proposition says that if the terms of a continued proportion are squared or cubed, then the resulting sequences of numbers are also in continued proportion.

Suppose that the original continued proportion has three terms: a, b, c. Then form two more sequences

$$a^2, ab, b^2, bc, c^2$$

and

$$a^3, a^2b, a\,b^2, b^3, b^2c, b\,c^2, c^3$$

Each of these are in continued proportion with the same ratio as the original sequence. The alternate terms in the second sequence form the continued proportion of the squares of the original sequence where the ratio is duplicate of the original ratio. Likewise, every third term of the third sequence make up a continued proportion of the cubes of the original sequence where the ratio is triplicate of the original ratio.

Next proposition: VIII.14

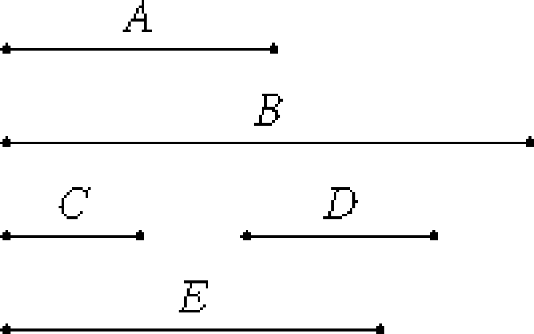
Previous: VIII.12

Book VIII introduction

Select from Book VIII

Select book

Select topic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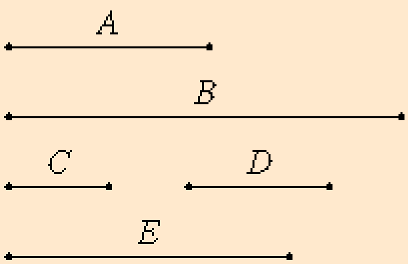
Euclid's Elements
Book VIII
Proposition 14

If a square measures a square, then the side also measures the side; and, if the side measures the side, then the square also measures the square.

Let *A* and *B* be square numbers, let *C* and *D* be their sides, and let *A* measure *B.*

I say that *C* also measures *D.*



Multiply *C* by *D* to make *E.* Then *A, E,* and *B* are continuously proportional in the ratio of *C* to *D.* — as in VIII.11

And, since *A, E,* and *B* are continuously proportional, and *A* measures *B,* therefore *A* also measures *E.* And *A* is to *E* as *C* is to *D,* therefore *C* measures *D.* — VIII.7, VII.Def.20

Next, let *C* measure *D.*

I say that *A* also measures *B.*

With the same construction, we can in a similar manner prove that *A, E,* and *B* are continuously proportional in the ratio of *C* to *D.* And since *C* is to *D* as *A* is to *E,* and *C* measures *D,* therefore *A* also measures *E.* — VII.Def.20

And *A, E,* and *B* are continuously proportional, therefore *A* also measures *B.*

Therefore, *if a square measures a square, then the side also measures the side; and, if the side measures the side, then the square also measures the square.*

Q.E.D.

Guide

This proposition is to prove its contrapositive, VIII.16.

Next proposition: VIII.15

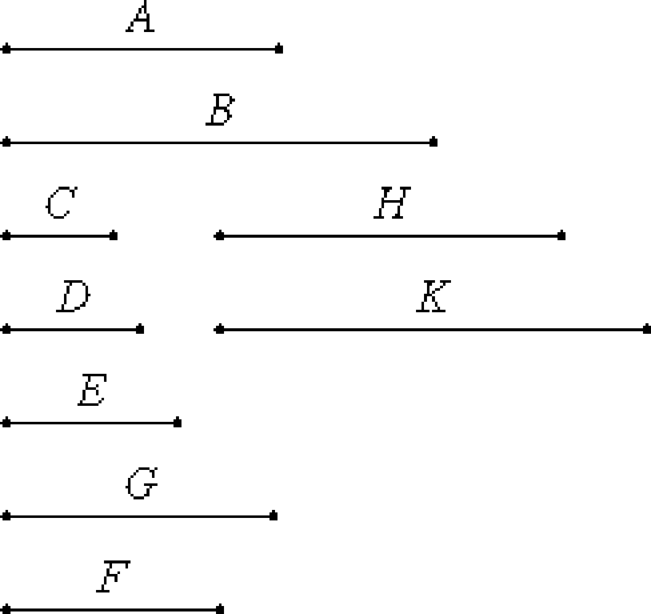
Previous: VIII.13

Book VIII introduction

Select from Book VIII

Select book

Select topic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F

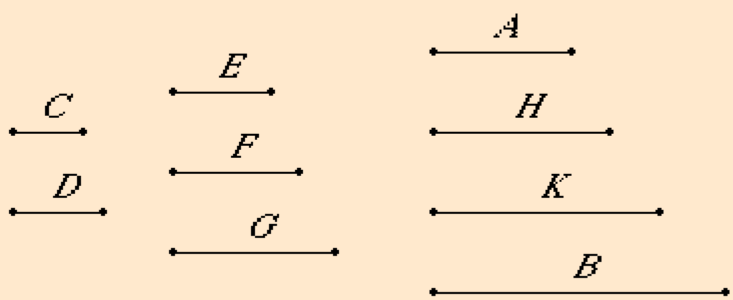
Euclid's Elements
Book VIII
Proposition 15

If a cubic number measures a cubic number, then the side also measures the side; and, if the side measures the side, then the cube also measures the cube.

Let the cubic number *A* measure the cube *B,* and let *C* be the side of *A* and *D* the side of *B.*

I say that *C* measures *D.*

Multiply *C* by itself to make *E,* multiply *D* by itself to make *G,* multiply *C* by *D* to make *F,* and multiply *C* and *D* by *F* to make *H* and *K* respectively.



Now it is manifest that *E, F,* and *G* and *A, H, K,* and *B* are continuously proportional in the ratio of *C* to *D.* And, since *A, H, K,* and *B* are continuously proportional, and *A* measures *B* and *G* therefore it also measures *H.* VIII.11 VIII.12 VIII.7

And *A* is to *H* as *C* is to *D,* therefore *C* also measures *D.* VII.Def.20

Next, let *C* measure *D.*

I say that *A* also measures *B.*

With the same construction, we can prove in a similar manner that *A, H, K,* and *B* are continuously proportional in the ratio of *C* to *D.* And, since *C* measures *D,* and *C* is to *D* as *A* is to *H,* therefore *A* also measures *H,* so that *A* measures *B* also. VII.Def.20

Therefore, *if a cubic number measures a cubic number, then the side also measures the side; and, if the side measures the side, then the cube also measures the cube.*

Q.E.D.

Guide

This proposition is used to prove its contrapositive, VIII.17.

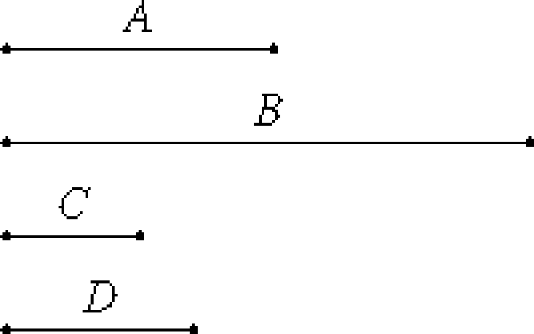
Next proposition: VIII.16 Select from Book VIII

Previous: VIII.14

Book VIII introduction

Select book

Select topic


A
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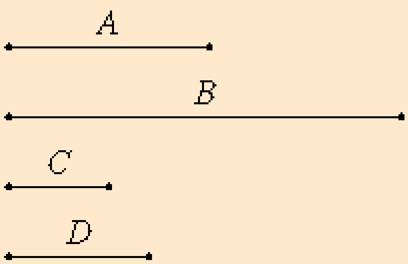
Euclid's Elements
Book VIII
Proposition 16

If a square does not measure a square, then neither does the side measure the side; and, if the side does not measure the side, then neither does the square measure the square.

Let *A* and *B* be square numbers, and let *C* and *D* be their sides, and let *A* not measure *B.*

I say that neither does *C* measure *D.*



If *C* measures *D*, *A* also measures *B.* But *A* does not measure *B*, therefore neither does *C* measure *D.* VIII.14

Next, let *C* not measure *D.*

I say that neither does *A* measure *B.*

If *A* measures *B*, then *C* also measures *D.* But *C* does not measure *D*, therefore neither does *A* measure *B.* VIII.14

Therefore, *if a square does not measure a square, then neither does the side measure the side; and, if the side does not measure the side, then neither does the square measure the square.*

Q.E.D.

Guide

This is simply the contrapositive of VIII.14. It is unclear why papyrus was wasted to state and prove it.

Next proposition: VIII.17

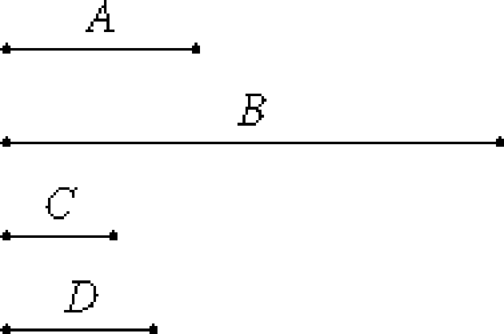
Previous: VIII.15

Book VIII introduction

Select from Book VIII

Select book

Select topic


A
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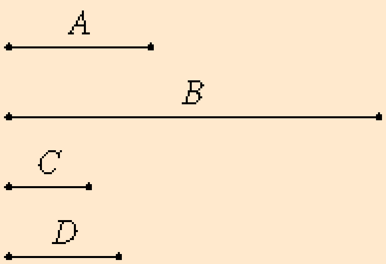
Euclid's Elements
Book VIII
Proposition 17

If a cubic number does not measure a cubic number, then neither does the side measure the side; and, if the side does not measure the side, then neither does the cube measure the cube.

Let the cubic number *A* not measure the cubic number *B*, and let *C* be the side of *A*, and *D* of *B*.

I say that *C* does not measure *D*.



For if *C* measures *D*, then *A* also measures *B*. But *A* does not measure *B*, therefore neither does *C* measure *D*. VIII.15

Next, let *C* not measure *D*.

I say that neither does *A* measure *B*.

If *A* measures *B*, then *C* also measures *D*. But *C* does not measure *D*, therefore neither does *A* measure *B*. VIII.15

Therefore, *if a cubic number does not measure a cubic number, then neither does the side measure the side; and, if the side does not measure the side, then neither does the cube measure the cube.*

Q.E.D.

Guide

This proposition is simply the contrapositive of VIII.15.

"Contrariwise," continued Tweedledee, "if it was so, it would be; but as it isn't, it ain't. That's logic."

Next proposition: VIII.18

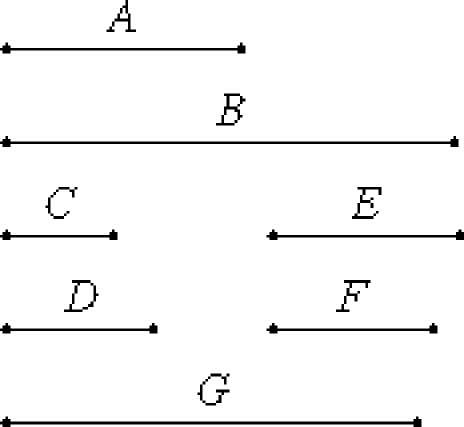
Previous: VIII.16

Book VIII introduction

Select from Book VIII

Select book

Select topic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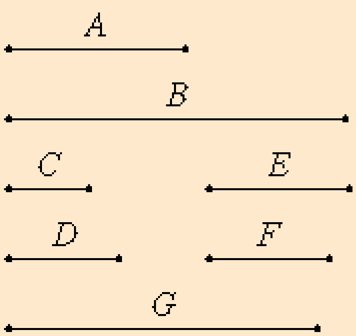
Euclid's Elements
Book VIII
Proposition 18

Between two similar plane numbers there is one mean proportional number, and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Let *A* and *B* be two similar plane numbers, and let the numbers *C* and *D* be the sides of *A*, and *E* and *F* of *B*.

Now, since similar plane numbers are those which have their sides proportional, therefore *C* is to *D* as *E* is to *F*. VII.Def.21

I say then that between *A* and *B* there is one mean proportional number, and *A* has to *B* the ratio duplicate of that which *C* has to *E*, or as *D* has to *F*, that is, of that which the corresponding side has to the corresponding side.



Now since *C* is to *D* as *E* is to *F*, therefore, alternately *C* is to *E* as *D* is to *F*. VII.13

And, since *A* is plane, and *C* and *D* are its sides, therefore *D* multiplied by *C* makes *A*. For the same reason also *E* multiplied by *F* makes *B*.

Multiply *D* by *E* to make *G*. Then, since *D* multiplied by *C* makes *A*, and multiplied by *E* makes *G*, therefore *C* is to *E* as *A* is to *G*. VII.17

But *C* is to *E* as *D* is to *F*, therefore *D* is to *F* as *A* is to *G*. Again, since *E* multiplied by *D* makes *G*, and multiplied by *F* makes *B*, therefore *D* is to *F* as *G* is to *B*. VII.17

But it was also proved that *D* is to *F* as *A* is to *G*, therefore *A* is to *G* as *G* is to *B*. Therefore *A*, *G*, and *B* are in continued proportion.

Therefore between *A* and *B* there is one mean proportional number.

I say next that *A* also has to *B* the ratio duplicate of that which the corresponding side has to the corresponding side, that is, of that which *C* has to *E* or *D* has to *F*.

Since *A*, *G*, and *B* are in continued proportion, *A* has to *B* the ratio duplicate of that which it has to *G*. And *A* is to *G* as *C* is to *E*, and as *D* is to *F*. Therefore *A* also has to *B* the ratio duplicate of that which *C* has to *E* or *D* has to *F*. V.Def.9

Therefore, *between two similar plane numbers there is one mean proportional number, and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.*

Q.E.D.

Guide

This proposition generalizes VIII.11 from squares to similar rectangles.

Outline of the proof

Assume that the similar plane numbers are *cd* and *ef* so that *c*:*d* = *e*:*f*, or, alternately, *c*:*e* = *d*:*f*. Then *c*:*e* = *cd*:*de*, and *d*:*f* = *de*:*ef*, therefore the ratio of the plane numbers *cd*:*ef* is compounded of the ratios of the corresponding sides *c*:*e* and *d*:*f*. Also, since the ratios of the corresponding sides are the same, the ratio of the plane numbers is the duplicate of the ratio of the sides. Furthermore, since *cd*:*de* = *de*:*ef*, the number *de* is a mean proportional between the two plane numbers *cd* and *ef*.

Use of this proposition

This proposition is used in several of the remaining propositions in this book beginning with the next. It is also used in the first two propositions of Book IX. Proposition VIII.20 is a partial converse of this one.

Next proposition: VIII.19

Previous: VIII.17

Book VIII introduction

Select from Book VIII

Select book

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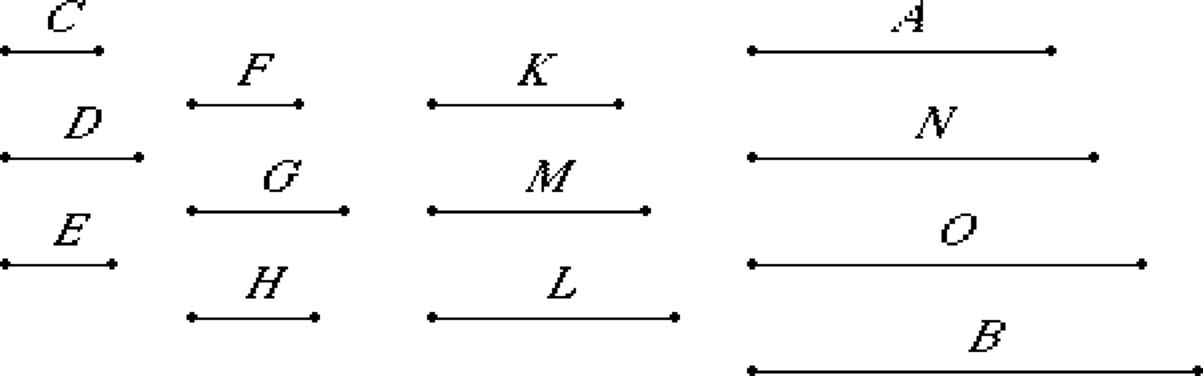
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Euclid's Elements
Book VIII
Proposition 19

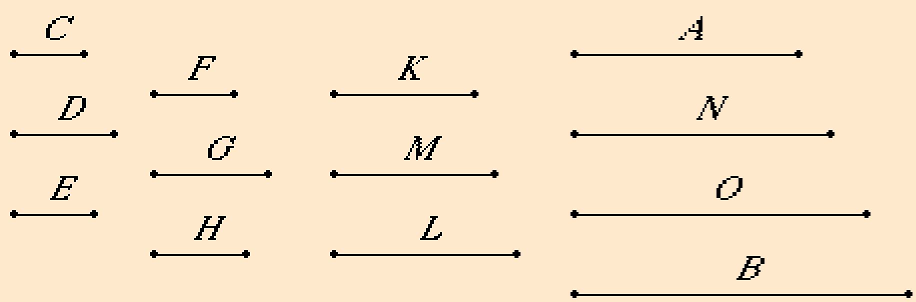
Between two similar solid numbers there fall two mean proportional numbers, and the solid number has to the solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

Let *A* and *B* be two similar solid numbers, and let *C, D,* and *E* be the sides of *A,* and let *F, G,* and *H* be the sides of *B.*

Now, since similar solid numbers are those which have their sides proportional, therefore *C* is to *D* as *F* is to *G,* and *D* is to *E* as *G* is to *H.* VII.Def.21

I say that between *A* and *B* there fall two mean proportional numbers, and *A* has to *B* the ratio triplicate of that which *C* has to *F, D* has to *G,* and *E* has to *H.*

Multiply *C* by *D* to make *K,* and multiply *F* by *G* to make *L.*



Now, since *C* and *D* are in the same ratio with *F* and *G,* and *K* is the product of *C* and *D,* and *L* the product of *F* and *G, K* and *L* are similar plane numbers, therefore between *K* and *L* there is one mean proportional number *M.* VII.Def.21 VIII.18

Therefore *M* is the product of *D* and *F* was proved in the theorem preceding. VIII.18

Now, since *D* multiplied by *C* makes *K,* and multiplied by *F* makes *M,* therefore *C* is to *F* as *K* is to *M.* But *K* is to *M* as *M* is to *L.* Therefore *K, M,* and *L* are continuously proportional in the ratio of *C* to *F.* VII.17

And since *C* is to *D* as *F* is to *G,* alternately therefore *C* is to *F* as *D* is to *G.* For the same reason also *D* is to *G* as *E* is to *H.* VII.13

Therefore *K, M,* and *L* are continuously proportional in the ratio of *C* to *F,* in the ratio of *D* to *G,* and also in the ratio of *E* to *H.*

Next, multiply *E* and *H* by *M* to make *N* and *O* respectively.

Now, since *A* is a solid number, and *C, D,* and *E* are its sides, therefore *E* multiplied by the product of *C* and *D* makes *A.* But the product of *C* and *D* is *K,* therefore *E* multiplied by *K* makes *A.* For the same reason also *H* multiplied by *L* makes *B.*

Now, since *E* multiplied by *K* makes *A,* and further also multiplied by *M* makes *N,* therefore *K* is to *M* as *A* is to *N.* VII.17

But *K* is to *M* as *C* is to *F,* as *D* is to *G,* and as *E* is to *H,* therefore *C* is to *F* as *D* is to *G,* as *E* is to *H,* and as A is to N.

Again, since E and H multiplied by M make N and O respectively, therefore E is to H as N is to O. VII.18

But E is to H as C is to F and as D is to G, therefore C is to F as D is to G, as E is to H, as A is to N, and as N is to O.

Again, since H multiplied by M makes O, and further also multiplied by L makes B, therefore M is to L as O is to B. But M is to L as C is to F, a D is to G, and as E is to H. Therefore C is to F as D is to G, and as E is to H, as are O to B, A to N, and N to O. VII.17

Therefore A, N, O, and B are continuously proportional in the aforesaid ratios of the sides.

I say that A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number C has to F, or D has to G, and also E has to H.

Since A, N, O, and B are four numbers in continued proportion, therefore A has to B the ratio triplicate of that which A has to N. But it was proved that A is to N as C is to F, as D is to G, and as E is to H. V.Def.10

Therefore A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number C has to F, D has to G, and also E has to H.

Therefore, *between two similar solid numbers there fall two mean proportional numbers, and the solid number has to the solid number the ratio triplicate of that which the corresponding side has to the corresponding side.*

Q.E.D.

Guide

Assume cde and fgh are similar solid numbers so that $c:d:e = f:g:h$, or, expressed alternately,

$$c:f = d:g = e:h.$$

Then the numbers

$$cde,\ fde,\ fge, \text{ and } fgh$$

are in continued proportion giving two mean proportionals between cde and fgh. Also, the ratio $cde:fgh$ is compounded of the three equal ratios of the sides $c:f$, $d:g$, and $e:h$, so it is a triplicate ratio of each.

Use of this proposition

This proposition is used in a few propositions in Books VIII and IX starting with VIII.25. Proposition VIII.21 is a partial converse of this one.

Next proposition: VIII.20

Previous: VIII.18

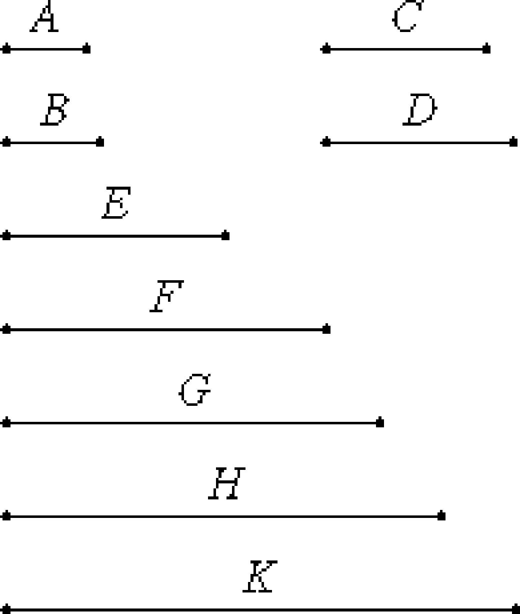
Book VIII introduction

Select from Book VIII

Select book

Select topic

D.E.Joyce
Clark University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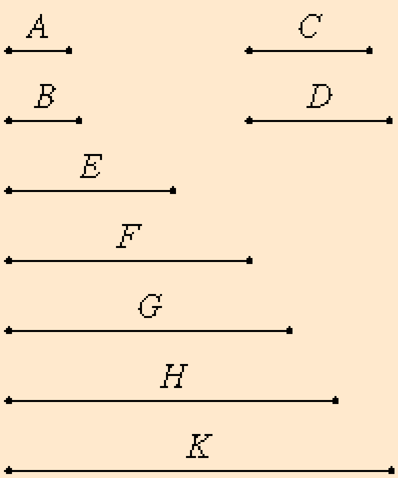
Euclid's Elements
Book VIII
Proposition 2

To find as many numbers as are prescribed in continued proportion, and the least that are in a given ratio.

Let the ratio of *A* to *B* be the given ratio in least numbers.

It is required to find numbers in continued proportion, as many as may be prescribed, the least that are in the ratio of *A* to *B.*

Let four be prescribed. Multiply *A* by itself to make *C*, and by *B* to make *D.* Multiply *B* by itself to make *E.* Also multiply *A* by *C, D,* and *E* to make *F, G,* and *H,* and multiply *B* by *E* to make *K.*



Now, since *A* multiplied by itself makes *C*, and multiplied by *B* makes *D*, therefore *A* is to *B* as *C* is to *D.* VII.17

Again, since *A* multiplied by *B* makes *D*, and *B* multiplied by itself makes *E*, therefore the numbers *A* and *B* multiplied by *B* make the numbers *D* and *E* respectively.

Therefore *A* is to *B* as *D* is to *E.* But *A* is to *B* as *C* is to *D*, therefore *C* is to *D* as *D* is to *E.* VII.18

And, since *A* multiplied by *C* and *D* makes *F* and *G*, therefore *C* is to *D* as *F* is to *G.* VII.17

But *C* is to *D* as *A* is to *B*, therefore *A* is to *B* as *F* is to *G.*

Again, since *A* multiplied by *D* and *E* makes *G* and *H*, therefore *D* is to *E* as *G* is to *H.* But *D* is to *E* as *A* is to *B.* Therefore *A* is to *B* as *G* is to *H.* VII.17

And, since *A* and *B* multiplied by *E* make *H* and *K*, therefore *A* is to *B* as *H* is to *K.* But *A* is to *B* as *F* is to *G*, and as *G* is to *H.* Therefore *F* is to *G* as *G* is to *H*, and as *H* is to *K.* VII.18

Therefore *C, D,* and *E*, and *F, G, H,* and *K* are proportional in the ratio of *A* to *B.*

I say next that they are the least numbers that are so.

Since *A* and *B* are the least of those which have the same ratio with them, and the least of those which have the same ratio are relatively prime, therefore *A* and *B* are relatively prime. VII.22

And the numbers *A* and *B* multiplied by themselves respectively make the numbers *C* and *E*, and multiplied by the numbers *C* and *E* respectively make the numbers *F* and *K*, therefore *C* and *E* and *F* and *K* are VII.27

relatively prime respectively.

But, if there are as many numbers as we please in continued proportion, and the extremes of them are relatively prime, then they are the least of those which have the same ratio with them. Therefore *C, D,* and *E* and *F*, *G*, *H* and *K* are the least of those which have the same ratio with *A* and *B.* VIII.1

Q.E.D.

Corollary.

From this it is manifest that, *if three numbers in continued proportion are the least of those which have the same ratio with them, then the extremes are squares, and, if four numbers, cubes.*

Guide

The problem is to construct *n* numbers in a continued proportion in lowest terms with a given constant ratio. If the ratio is $a:b$ in lowest terms, then the numbers in the continued proportion are

$$a^{n-1}, a^{n-2}b, a^{n-3}b^2, \ldots, a^1b^{n-2}, b^{n-1}$$

For instance, the five numbers in a continued proportion in lowest terms with a ratio of 2:3 form the sequence 2^4, $2^3 \cdot 3$, $2^2 \cdot 3^2$, $2 \cdot 3^3$, and 3^4, that is, the sequence 16, 24, 36, 54, and 81.

The proof is not difficult. First, adjacent terms do have the correct ratio. Also, since *a* and *b* are relatively prime, proposition VII.27 implies that the end terms a^{n-1} and b^{n-1} are relatively prime. The result then follows from the previous proposition VIII.1.

Use of this proposition

This proposition and its corollary are used in several propositions in Book VIII starting with the next and in proposition IX.15 in the next book.

Next proposition: VIII.3

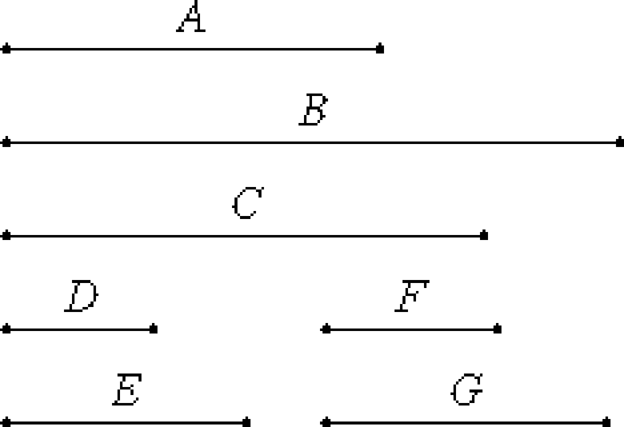
Select from Book VIII

Previous: VIII.1

Select book

Book VIII introduction

Select topic


A
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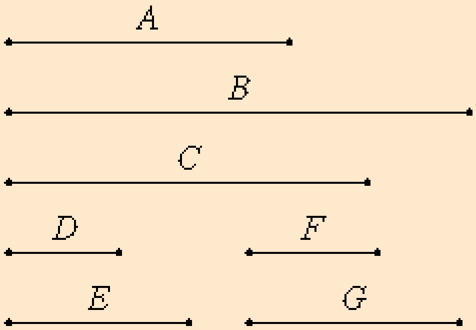
Euclid's Elements
Book VIII
Proposition 20

If one mean proportional number falls between two numbers, then the numbers are similar plane numbers.

Let one mean proportional number *C* fall between the two numbers *A* and *B.*

I say that *A* and *B* are similar plane numbers.

Take *D* and *E,* the least numbers of those which have the same ratio with *A* and *C.* Then *D* measures *A* the same number of times that *E* measures *C.* VII.33 VII.20



Let there be as many units in *F* as times that *D* measures *A.* Then *F* multiplied by *D* makes *A,* so that *A* is plane, and *D* and *F* are its sides.

Again, since *D* and *E* are the least of the numbers which have the same ratio with *C* and *B,* therefore *D* measures *C* the same number of times that *E* measures *B.* VII.20

Let there be as many units in *G* as times that *E* measures *B.* Then *E* measures *B* according to the units in *G.* Therefore *G* multiplied by *E* makes *B.*

Therefore *B* is plane, and *E* and *G* are its sides. Therefore *A* and *B* are plane numbers.

I say next that they are also similar.

Since *F* multiplied by *D* makes *A,* and multiplied by *E* makes *C,* therefore *D* is to *E* as *A* is to *C,* that is, *C* to *B.* VII.17

Again, since *E* multiplied by *F* and *G* makes *C* and *B* respectively, therefore *F* is to *G* as *C* is to *B.* But *C* is to *B* as *D* is to *E,* therefore *D* is to *E* as *F* is to *G.* And alternately *D* is to *F* as *E* is to *G.* VII.17 VII.13

Therefore *A* and *B* are similar plane numbers, for their sides are proportional.

Therefore, *if one mean proportional number falls between two numbers, then the numbers are similar plane numbers.*

Q.E.D.

Guide

This is a partial converse of VIII.18. It says that if two numbers have a mean proportional, then they can be viewed as two similar plane numbers.

An example might clarify the details. The variable refer to the outline of the proof below. The numbers *a* =18 and *b* =50 have a mean proportional *c* = 30. When *a*:*c* is to lowest terms, the result is *d*:*e* = 3:5. Then *f* is 6, and the number *a* = 18 is seen as the plane number *d* = 3 by *f* = 6. Also *g* is 10, and the number *b* = 50 is seen as the plane number *e* = 5 by *g* = 10. The sides of these plane numbers, 3 by 6 and 5 by 10, are proportional.

Outline of the proof

Suppose two numbers *a* and *b* have a mean proportional *c*. Reduce the ratio *a*:*c* to lowest terms *d*:*e*. Then *d* divides *a* the same number of times *e* divides *c*; call that number *f*. Then *a* is a plane number with sides *d* and *f*.

Now since, *c*:*b* is the same ratio as *a*:*c*, it also reduces to the ratio *d*:*e* in lowest terms. Therefore, *d* divides *c* the same number of times that *e* divides *b*; call that number *g*. Then *b* is a plane number with sides *e* and *g*.

Furthermore, the two plane numbers *a* and *b* are similar since we can show their sides are proportional as follows. From the three proportions *d*:*e* = *a*:*c* (which follows from *a* = *fd* and *c* = *fe*), *a*:*c* = *c*:*b* (since *c* is a mean proportional), and *c*:*b* = *f*:*g* (which follows from *g* = *ef* and *b* = *ec*), therefore, *d*:*e* = *f*:*g*, and alternately, *d*:*f* = *e*:*g*. Thus, the two plane numbers have proportional sides.

Use of this proposition

This proposition is used in the next two propositions and also IX.2.

Next proposition: VIII.21

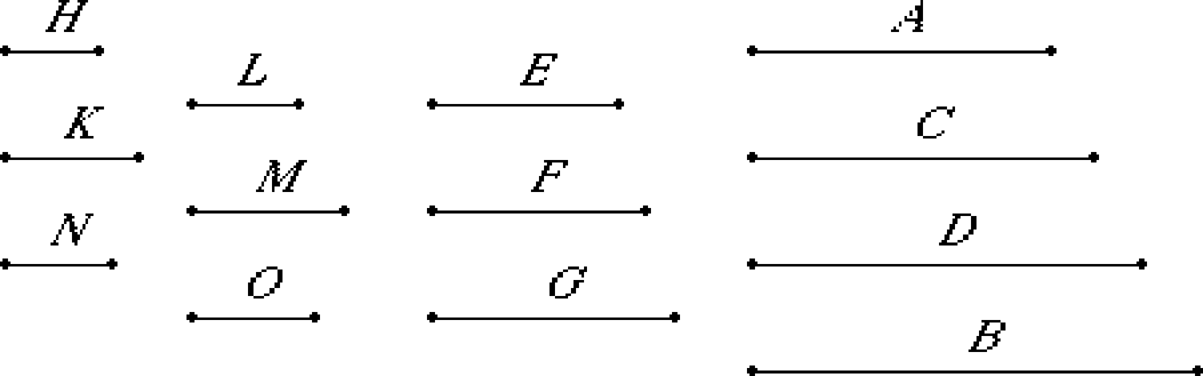
Previous: VIII.19

Book VIII introduction

Select from Book VIII

Select book

Select topic


H
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D
O
G
B

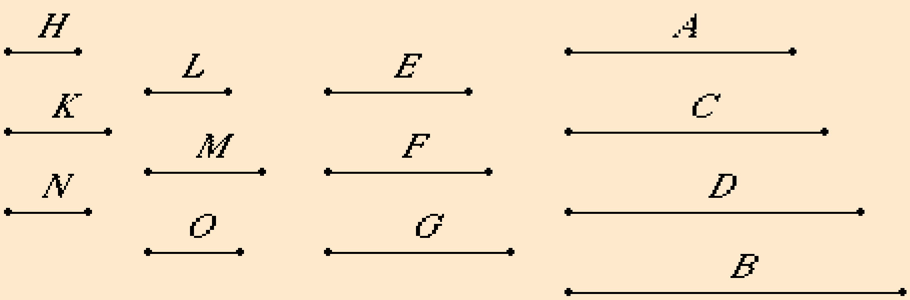
Euclid's Elements
Book VIII
Proposition 21

If two mean proportional numbers fall between two numbers, then the numbers are similar solid numbers.

Let two mean proportional numbers *C* and *D* fall between the two numbers *A* and *B.*

I say that *A* and *B* are similar solid numbers.

Take three numbers *E, F,* and *G,* the least of those which have the same ratio with *A, C,* and *D.* Then the extremes of them *E* and *G* are relatively prime. VII.33 or VIII.2 VIII.3



Now, since one mean proportional number *F* has fallen between *E* and *G,* therefore *E* and *G* are similar plane numbers. VIII.20

Let, then, *H* and *K* be the sides of *E,* and *L* and *M* the sides of *G.*

Therefore it is manifest from the theorem before this that *E, F,* and *G* are continuously proportional in the ratio of *H* to *L,* and that of *K* to *M.*

Now, since *E, F,* and *G* are the least of the numbers which have the same ratio with *A, C,* and *D,* and the multitude of the numbers *E, F,* and *G* equals the multitude of the numbers *A, C,* and *D,* therefore, *ex aequali E* is to *G* as *A* is to *D.* VII.14

But *E* and *G* are relatively prime, primes are also least, and the least measure those which have the same ratio with them the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, therefore *E* measures *A* the same number of times that *G* measures *D.* VII.21 VII.20

Let there be as many units in *N* as times that *E* measures *A.* Then *N* multiplied by *E* makes *A.* But *E* is the product of *H* and *K,* therefore *N* multiplied by the product of *H* and *K* makes *A.*

Therefore *A* is solid, and *H, K,* and *N* are its sides.

Again, since *E, F,* and *G* are the least of the numbers which have the same ratio as *C, D,* and *B,* therefore *E* measures *C* the same number of times that *G* measures *B.*

Let there be as many units in *O* as times that *E* measures *C.* Then *G* measures *B* according to the units in *O,* therefore *O* multiplied by *G* makes *B.*

But *G* is the product of *L* and *M*, therefore *O* multiplied by the product of *L* and *M* makes *B*. Therefore *B* is solid, and *L*, *M*, and *O* are its sides. Therefore *A* and *B* are solid.

I say that they are also similar.

Since *N* and *O* multiplied by *E* make *A* and *C*, therefore *N* is to *O* as *A* is to *C*, that is, *E* to *F*. VII.18

But *E* is to *F* as *H* is to *L*, and as *K* is to *M*, therefore *H* is to *L* as *K* is to *M*, and as *N* is to *O*.

And *H*, *K*, and *N* are the sides of *A*, and *O*, *L*, and *M* the sides of *B*. Therefore *A* and *B* are similar solid numbers.

Therefore, *if two mean proportional numbers fall between two numbers, then the numbers are similar solid numbers.*

Q.E.D.

Guide

This is a partial converse of VIII.19. It says that if two numbers have two mean proportionals, then they can be viewed as two similar solid numbers. It's proof is analogous the previous proposition dealing with plane numbers, but naturally, it is longer and more involved.

Use of this proposition

This proposition is used in VIII.23.

Next proposition: VIII.22

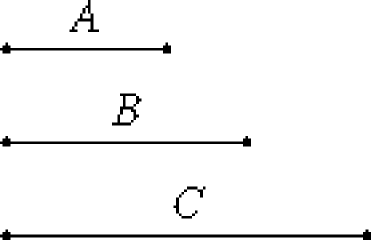
Previous: VIII.20

Book VIII introduction

Select from Book VIII

Select book

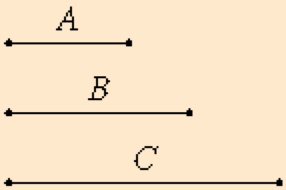
Select topic


A
B
C

Euclid's Elements
Book VIII
Proposition 22

If three numbers are in continued proportion, and the first is square, then the third is also square.

Let *A*, *B*, and *C* be three numbers in continued proportion, and let *A* the first be square.



I say that *C* the third is also square.

Since between *A* and *C* there is one mean proportional number, *B*, therefore *A* and *C* are similar plane numbers. But *A* is square, therefore *C* is also square. VIII.20

Therefore, *if three numbers are in continued proportion, and the first is square, then the third is also square.*

Q.E.D.

Guide

This proposition is used in a few propositions in this and the next book starting with VIII.24.

Next proposition: VIII.23

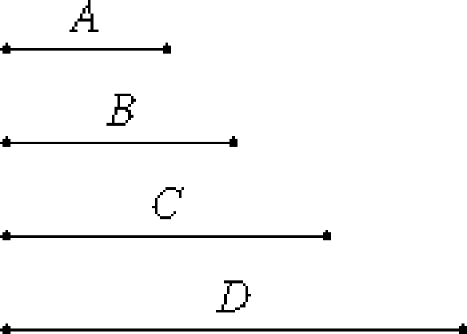
Previous: VIII.21

Book VIII introduction

Select from Book VIII

Select book

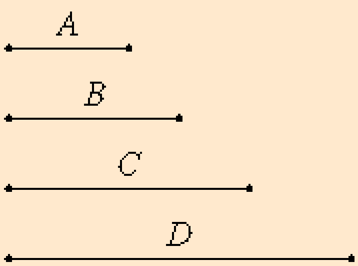
Select topic


A
B
C
D

Euclid's Elements
Book VIII
Proposition 23

If four numbers are in continued proportion, and the first is a cube, then the fourth is also a cube.

Let *A*, *B*, *C*, and *D* be four numbers in continued proportion, and let *A* be a cube.



I say that *D* is also a cube.

Since between *A* and *D* there are two mean proportional numbers *B* and *C*, therefore *A* and *D* are similar solid numbers. But *A* is a cube, therefore *D* is also a cube. VIII.21

Therefore, *if four numbers are in continued proportion, and the first is a cube, then the fourth is also a cube.*

Q.E.D.

Guide

This proposition is used in several propositions in this and the next book starting with VIII.25.

Next proposition: VIII.24

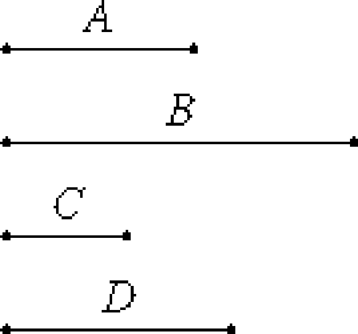
Previous: VIII.22

Book VIII introduction

Select from Book VIII

Select book

Select topic

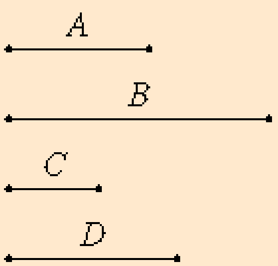

A
B
C
D

Euclid's Elements
Book VIII
Proposition 24

If two numbers have to one another the ratio which a square number has to a square number, and the first is square, then the second is also a square.

Let the two numbers *A* and *B* have to one another the ratio which the square number *C* has to the square number *D,* and let *A* be square.

I say that *B* is also square.



Since *C* and *D* are square, *C* and *D* are similar plane numbers. Therefore one mean proportional number falls between *C* and *D.* VIII.18

And *C* is to *D* as *A* is to *B,* therefore one mean proportional number falls between *A* and *B* also. And *A* is square, therefore *B* is also square. VIII.18 VIII.22

Therefore, *if two numbers have to one another the ratio which a square number has to a square number, and the first is square, then the second is also a square.*

Q.E.D.

Guide

The proof of this proposition is straightforeward.

Next proposition: VIII.25

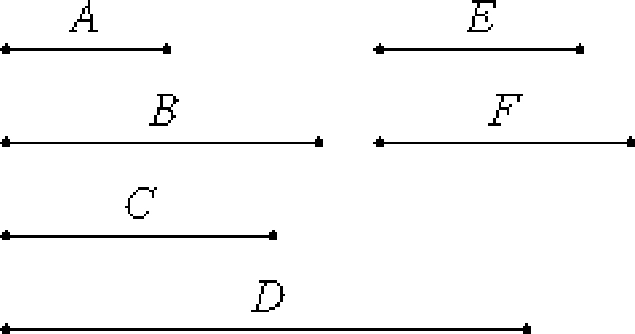
Previous: VIII.23

Book VIII introduction

Select from Book VIII

Select book

Select topic


A
E
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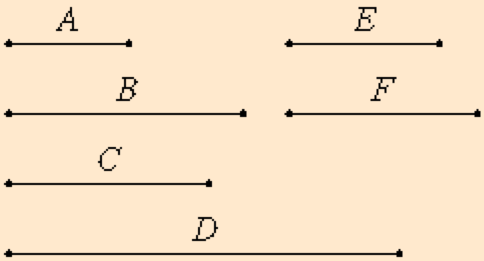
Euclid's Elements
Book VIII
Proposition 25

If two numbers have to one another the ratio which a cubic number has to a cubic number, and the first is a cube, then the second is also a cube.

Let the two numbers *A* and *B* have to one another the ratio which the cubic number *C* has to the cubic number *D,* and let *A* be a cube.



I say that *B* is also a cube.

Since *C* and *D* are cubes, *C* and *D* are similar solid numbers, therefore two mean proportional numbers fall between *C* and *D.* VIII.19

Since as many numbers fall in continued proportion between those which have the same ratio with *C* and *D* as fall between *C* and *D,* therefore two mean proportional numbers *E* and *F* fall between *A* and *B.* VIII.18

Since, then, the four numbers *A, E, F,* and *B* are in continued proportion, and *A* is a cube, therefore *B* is also a cube. VIII.23

Therefore, *if two numbers have to one another the ratio which a cubic number has to a cubic number, and the first is a cube, then the second is also a cube.*

Q.E.D.

Guide

This proposition is analogous to the previous one about squares. Its proof is straightforward.

This proposition is used in IX.10.

Next proposition: VIII.26

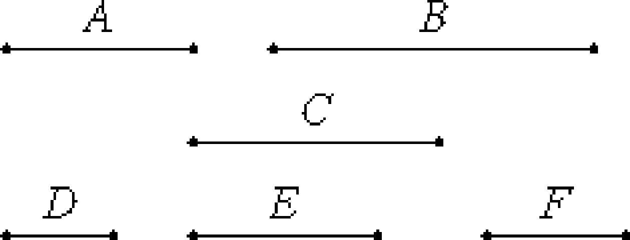
Previous: VIII.24

Book VIII introduction

Select from Book VIII

Select book

Select topic


A
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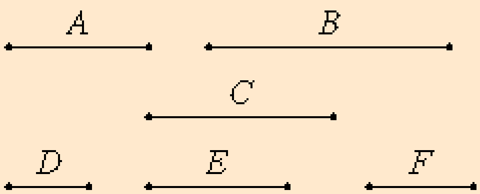
Euclid's Elements
Book VIII
Proposition 26

Similar plane numbers have to one another the ratio which a square number has to a square number.

Let *A* and *B* be similar plane numbers.

I say that *A* has to *B* the ratio which a square number has to a square number.



Since *A* and *B* are similar plane numbers, therefore one mean proportional number *C* falls between *A* and *B*. VIII.18

Take *D*, *E*, and *F*, the least numbers of those which have the same ratio with *A*, *C*, and *B* VII.33 or VIII.2

Then the extremes of them *D* and *F* are square. And since *D* is to *F* as *A* is to *B*, and *D* and *F* are square, therefore *A* has to *B* the ratio which a square number has to a square number. VIII.2,Cor

Therefore, *similar plane numbers have to one another the ratio which a square number has to a square number.*

Q.E.D.

Guide

This proposition is used in propositions IX.10 and X.9.

Next proposition: VIII.27

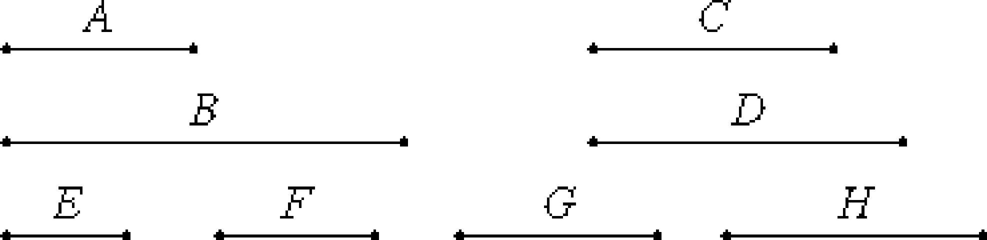
Previous: VIII.25

Book VIII introduction

Select from Book VIII

Select book

Select topic


A
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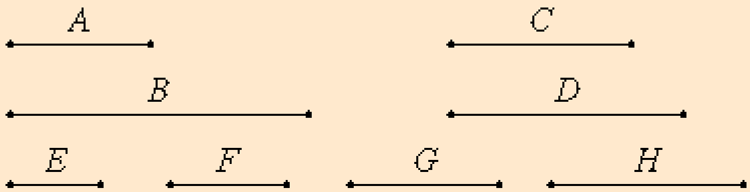
Euclid's Elements
Book VIII
Proposition 27

Similar solid numbers have to one another the ratio which a cubic number has to a cubic number.

Let *A* and *B* be similar solid numbers.

I say that *A* has to *B* the ratio which cubic number has to cubic number.



Since *A* and *B* are similar solid numbers, therefore two mean proportional numbers *C* and *D* fall between *A* and *B.* VIII.19

Take *E*, *F*, *G*, and *H*, the least numbers of those which have the same ratio with *A*, *C*, *D*, and *B*, and equal with them in multitude. VII.33 or VIII.2

Therefore the extremes of them, *E* and *H*, are cubes. And *E* is to *H* as *A* is to *B*, therefore *A* also has to *B* the ratio which a cubic number has to a cubic number. VIII.2,Cor.

Therefore, *similar solid numbers have to one another the ratio which a cubic number has to a cubic number.*

Q.E.D.

Guide

This proposition is analogous to the previous proposition about similar plane numbers.

Next book: Book IX

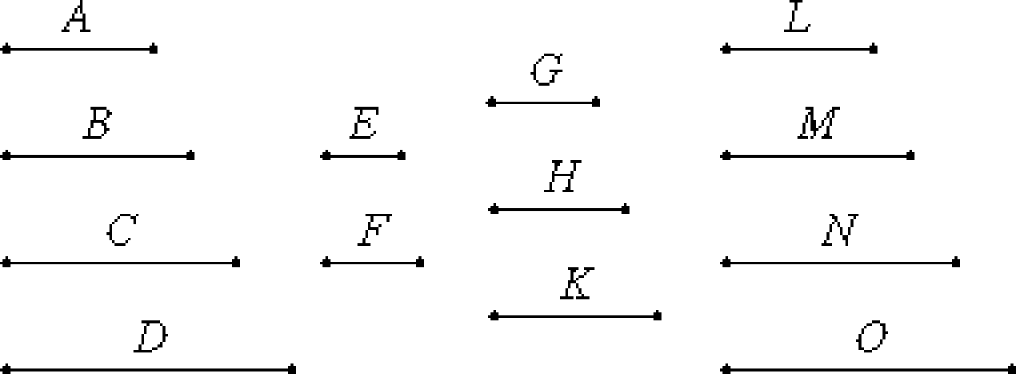
Previous proposition: VIII.26

Book VIII introduction

Select from Book VIII

Select book

Select topic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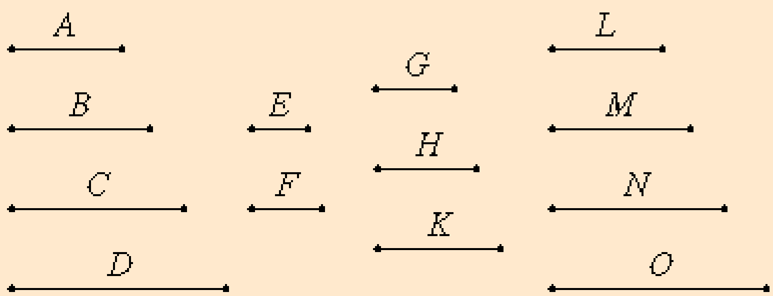
Euclid's Elements
Book VIII
Proposition 3

If as many numbers as we please in continued proportion are the least of those which have the same ratio with them, then the extremes of them are relatively prime.

Let as many numbers as we please, *A*, *B*, *C*, and *D*, in continued proportion be the least of those which have the same ratio with them.

I say that the extremes of them, *A* and *D*, are relatively prime.



Take two numbers *E* and *F*, the least that are in the ratio of *A*, *B*, *C* and *D*, then three others *G*, *H* and *K* with the same property, and others, more by one continually, until the multitude taken becomes equal to the multitude of the numbers *A*, *B*, *C*, and *D*. Let them be *L*, *M*, *N*, and *O*. VII.33 VIII.2

Since *E* and *F* are the least of those which have the same ratio with them, therefore they are relatively prime. And, since the numbers *E* and *F* multiplied by themselves respectively make the numbers *G* and *K*, and multiplied by the numbers *G* and *K* respectively make the numbers *L* and *O*, therefore both *G* and *K* and *L* and *O* are relatively prime. VII.22 VIII.2,Cor VII.27

And, since *A*, *B*, *C*, and *D* are the least of those which have the same ratio with them, while *L*, *M*, *N*, and *O* are the least that are in the same ratio with *A*, *B*, *C*, and *D*, and the multitude of the numbers *A*, *B*, *C*, and *D* equals the multitude of the numbers *L*, *M*, *N*, and *O*, therefore the numbers *A*, *B*, *C*, and *D* equal the numbers *L*, *M*, *N*, and *O* respectively. Therefore *A* equals *L*, and *D* equals *O*.

And *L* and *O* are relatively prime. Therefore *A* and *D* are also relatively prime.

Therefore, *if as many numbers as we please in continued proportion are the least of those which have the same ratio with them, then the extremes of them are relatively prime.*

Q.E.D.

Guide

This proposition, the converse of VIII.1, says if a continued proportion with constant ratio is in lowest terms, then its end numbers are relatively prime.

The proof begins by using the previous proposition VIII.2 to construct the continued proportion in lowest terms, which must be the same as the given continued proportion, and it has relatively prime end numbers.

Use of this proposition

This proposition is used in propositions VIII.6, VIII.8, and VIII.21.

Next proposition: VIII.4

Previous: VIII.2

Book VIII introduction

Select from Book VIII

Select book

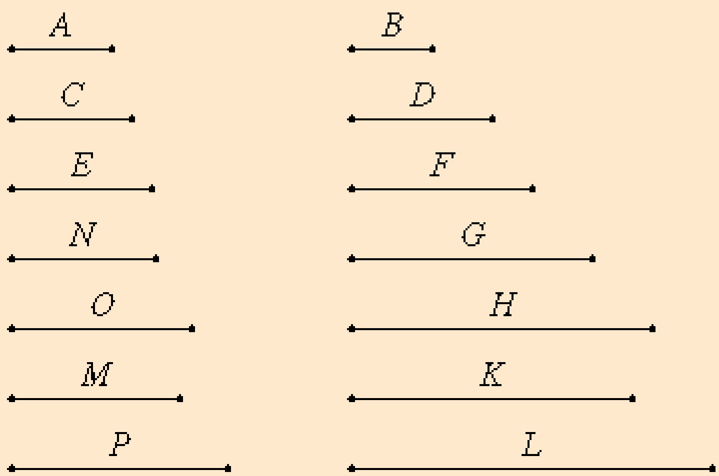
Select topic

Euclid's Elements
Book VIII
Proposition 4

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Let the given ratios in least numbers be that of *A* to *B*, that of *C* to *D*, and that of *E* to *F*.

It is required to find numbers in continued proportion which are the least that are in the ratio of *A* to *B*, in the ratio of *C* to *D*, and in the ratio of *E* to *F*.



Take *G*, the least number measured by *B* and *C*. VII.34

Let *A* measure *H* as many times as *B* measures *G*, and let *D* measure *K* as many times as *C* measures *G*.

Now *E* either measures or does not measure *K*.

First, let it measure it. Let *K* measure *L* as many times as *E* measures *K*.

Now, since *A* measures *H* the same number of times that *B* measures *G*, therefore *A* is to *B* as *H* is to *G*. VII.Def.20 VII.13

For the same reason *C* is to *D* as *G* is to *K*, and *E* is to *F* as *K* is to *L*. Therefore *H*, *G*, *K*, and *L* are continuously proportional in the ratio of *A* to *B*, in the ratio of *C* to *D*, and in the ratio of *E* to *F*.

I say next that they are also the least that have this property.

If *H*, *G*, *K*, and *L* are not the least numbers continuously proportional in the ratios of *A* to *B*, of *C* to *D*, and of *E* to *F*, let them be *N*, *O*, *M*, and *P*.

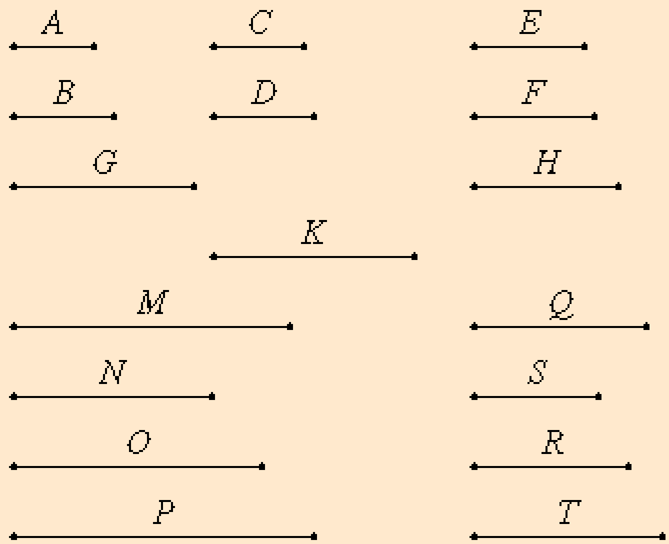
Then since *A* is to *B* as *N* is to *O*, while *A* and *B* are least, and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent, therefore *B* measures *O*. VII.20

For the same reason *C* also measures *O*. Therefore *B* and *C* measure *O*. Therefore the least number measured by *B* and *C* also measures *O*. VII.35

But *G* is the least number measured by *B* and *C*, therefore *G* measures *O*, the greater the less, which is impossible. Therefore there are no numbers less than *H*, *G*, *K*, and *L* which are continuously in the ratio of *A* to *B*, of *C* to *D*, and of *E* to *F*.

Next, let *E* not measure *K*.

Take *M*, the least number measured by *E* and *K*. Let *H* and *G* measure *N* and *O* as many times as *K* measures *M*, respectively, and let *F* measure *P* as many times as *E* measures *M*.



Since *H* measures *N* the same number of times that *G* measures *O*, therefore *H* is to *G* as *N* is to *O*. But *H* is to *G* as *A* is to *B*, therefore *A* is to *B* as is *N* is to *O*. For the same reason *C* is to *D* as is *O* is to *M*. VII.13 VII.Def.20

Again, since *E* measures *M* the same number of times that *F* measures *P*, therefore *E* is to *F* as *M* is to *P*. Therefore *N*, *O*, *M*, and *P* are continuously proportional in the ratios of *A* to *B*, of *C* to *D*, and of *E* to *F*. VII.13 VII.Def.20

I say next that they are also the least that are in the ratios *A*, *B*, *C*, *D*, *E*, and *F*.

If not, there are numbers less than *N*, *O*, *M*, and *P* continuously proportional in the ratios *A*, *B*, *C*, *D*, *E*, and *F*. Let them be *Q*, *R*, *S*, and *T*.

Now since *Q* is to *R* as *A* is to *B*, while *A* and *B* are least, and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent, therefore *B* measures *R*. For the same reason *C* also measures *R*, therefore *B* and *C* measure *R*. VII.20

Therefore the least number measured by *B* and *C* also measures *R*. But *G* is the least number measured by *B* and *C*, therefore *G* measures *R*. VII.35

And *G* is to *R* as *K* is to *S*, therefore *K* also measures *S*. VII.13

But *E* also measures *S*. Therefore *E* and *K* measure *S*.

Therefore the least number measured by *E* and *K* also measures *S*. But *M* is the least number measured by *E* and *K*, therefore *M* measures *S*, the greater the less, which is impossible. VII.35

Therefore there are no numbers less than *N*, *O*, *M*, and *P* continuously proportional in the ratios of *A* to *B*, of *C* to *D*, and of *E* to *F*. Therefore *N*, *O*, *M*, and *P* are the least numbers continuously proportional in the ratios *A*, *B*, *C*, *D*, *E*, and *F*.

Q.E.D.

Guide

This is a generalization of VIII.2 to a more general concept of continued proportion. In this proposition we consider continued proportions having ratios that aren't necessarily constant. These, perhaps, should be called continued ratios. For example, the continued ratio 5:10:20 has a constant ratio of 1:2, but the continued ratio 5:10:30 does not; its first ratio is 1:2 while its second ratio is 1:3. Note that the continued ratio 5:10:30 is not the least with those given ratios since 1:2:6 is smaller with the same ratios of 1:2 and 1:3.

The problem here is to construct the smallest continued ratio having the specified ratios.

An examination of the proof shows that Euclid has a general process to attach two continued proportions into one long one with with the same ratios. Take, for example, the problem of placing the continued ratio 3:7:2:6 in front of the continued ratio 10:4:5 to make a seven-term continued ratio where the first four terms have the ratio 3:7:2:6 and the last three terms have the ratio 10:4:5. The resulting seven-term ratio should be least with the given ratios. The problem is that the last term of the first ratio, 6, does not equal the first term of the second ratio, 10. The solution is to increase the numbers in each ratio to match these numbers. Since 30 = LCM(6, 10), that can be done by multiplying each of the terms of the first ratio by 5 and each of the terms of the second ratio by 3. The resulting ratios, 15:35:10:30 and 30:12:15 can then be merged into the desired ratio 15:35:10:30:12:15.

Outline of the proof

We start with three given ratios $a:b$, $c:d$, and $e:f$, all in lowest terms. First, the two ratios $a:b$ and $c:d$ are merged into a three-term ratio $h:g:k$ so that $h:g = a:b$ and $g:k = c:d$. These are defined equationally as $g = \text{LCM}(b, c)$, $h = (g/b)a$, and $k = (g/c)d$. Then $h:g:k$ has the proper ratios.

Next, the three-term ratio $h:g:k$ is merged with the ratio $e:f$ to get a four-term ratio $n:o:m:p$ so that $n:o = a:b$, $o:m = c:d$, and $m:p = c:d$. These are defined equationally as $m = \text{LCM}(e, k)$, $n = (m/k)h$, and $o = (m/k)g$, and $p = (m/e)f$. Again, $n:o:m:p$ has the proper ratios.

The bulk of the proof consists of showing that the resulting ratio $n:o:m:p$ is the least with the given ratios.

Use of this proposition

This proposition is used in the next one and in proposition X.12.

Next proposition: VIII.5

Previous: VIII.3

Book VIII introduction

Select from Book VIII

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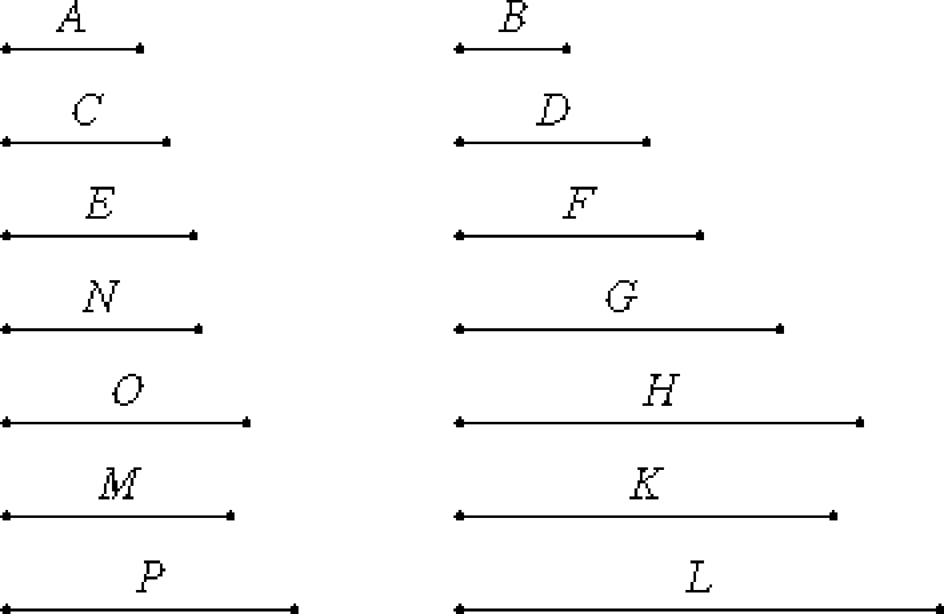
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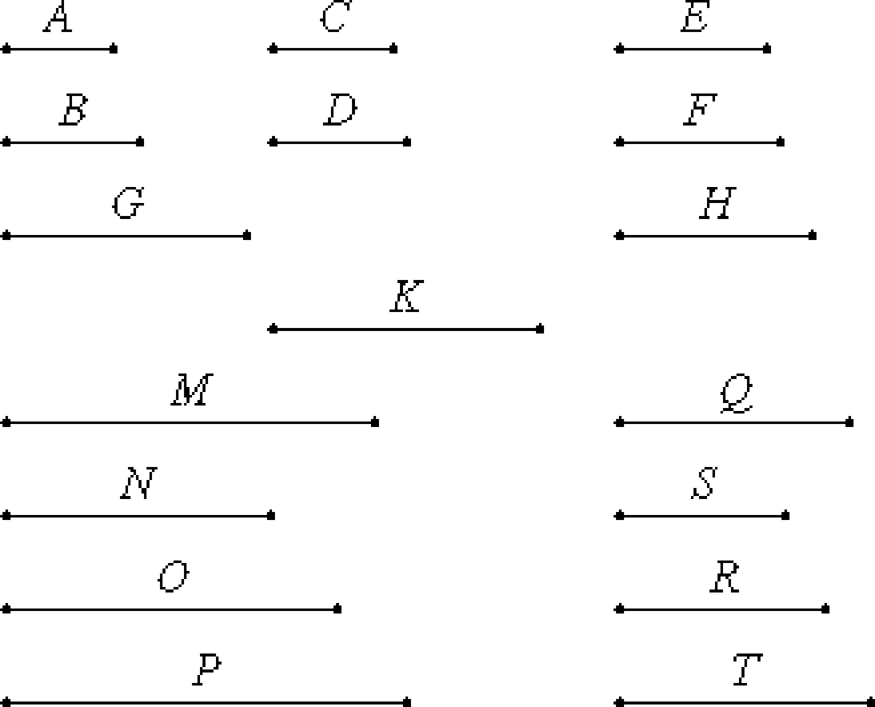
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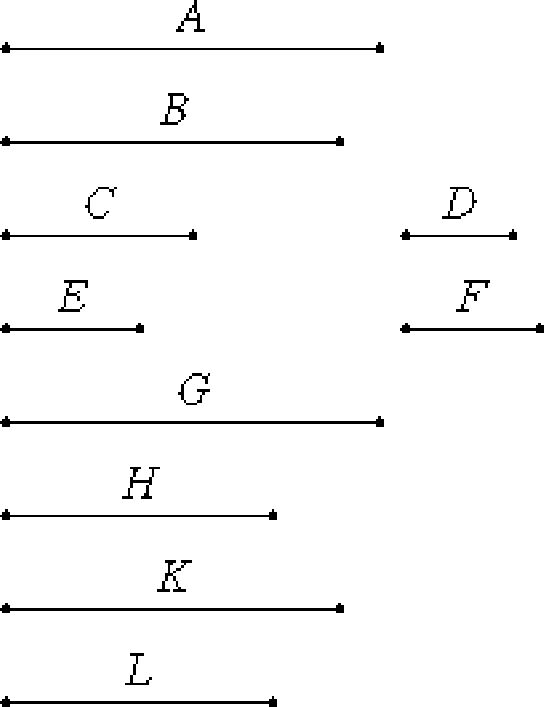
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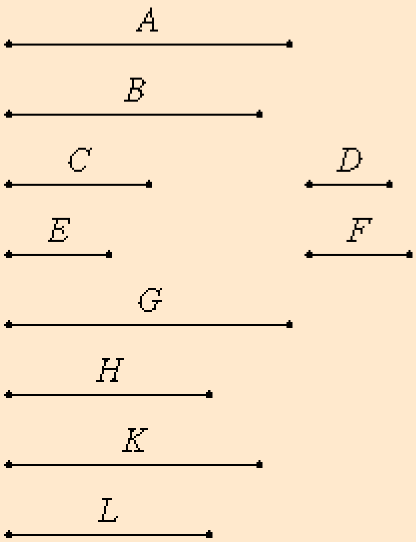
Euclid's Elements
Book VIII
Proposition 5

Plane numbers have to one another the ratio compounded of the ratios of their sides.

Let *A* and *B* be plane numbers, and let the numbers *C* and *D* be the sides of *A,* and *E* and *F* the sides of *B.*

I say that *A* has to *B* the ratio compounded of the ratios of the sides.



The ratios being given which *C* has to *E* and *D* to *F,* take the least numbers *G*, *H*, and *K* that are continuously in the ratios *C*, *E*, *D*, and *F*, so that *C* is to *E* as *G* is to *H,* and *D* is to *F* as *H* is to *K.* VIII.4

Multiply *D* by *E* to make *L.*

Now, since *D* multiplied by *C* makes *A,* and multiplied by *E* makes *L,* therefore *C* is to *E* as *A* is to *L.* But *C* is to *E* as *G* is to *H*, therefore *G* is to *H* as *A* is to *L.* VII.17

Again, since *E* multiplied by *D* makes *L,* and further multiplied by *F* makes *B*, therefore *D* is to *F* as *L* is to *B.* But *D* is to *F* as *H* is to *K,* therefore *H* is to *K* as *L* is to *B.* VII.17

But it was also proved that, *H* as *G* is to *H* as *A* is to *L,* therefore, *ex aequali*, *L* as *G* is to *K* as *A* is to *B.* VII.14

But *G* has to *K* the ratio compounded of the ratios of the sides, therefore *A* also has to *B* the ratio compounded of the ratios of the sides.

Therefore, *plane numbers have to one another the ratio compounded of the ratios of their sides.*

Q.E.D.

Guide

Compounded ratios

Compound ratios as such only appear in a few places in the *Elements.* They appear here in this proposition, and in VI.23, an analogous proposition for rectangles and parallelograms. But duplicate and triplicate ratios are also special kinds of compound ratios, and they are used in Books VI, VIII, IX, X, XI, and XII. Duplicate and triplicate ratios were defined in general in V.9-10, where they are defined as the ratio of the ends of a continued proportion. That is, if *a*:*b* = *b*:c, then the duplicate ratio of *a*:*b* is *a*:*c.* For lines, constructing the third proportional *c* needed to duplicate the ratio *a*:*b* is done in proposition VI.11, but for numbers, the third proportional can be constructed by VIII.2.

The ratio *compounded* from two given ratios *a*:*b* and *b*:*c* is just the ratio *a*:*c.* But if the middle term *b* is not shared by the two given ratios, then equal ratios must be found that do have a shared middle term.

To find the ratio compounded from two given ratios $a:b$ and $c:d$, first find e, f, and g so that $e:f = a:b$ and $f:g = c:d$. Then, the ratio compounded from the ratios $a:b$ and $c:d$ will be the same as the ratio compounded from the ratios $e:f$ and $f:g$, namely $e:g$. For numbers, this construction was done in the previous proposition VIII.4.

Outline of the proof

Let the plane number a be the product cd of its sides, and let the plane number b be the product ef of its sides. Use VIII.4 to construct a continued ratio $g:h:k$ so that $g:h = c:e$ and $h:k = d:f$ so that $g:k$ is the ratio compounded of the ratios $c:e$ and $d:f$ of the sides.

Since $a = cd$, therefore $c:e = a:de$, and so $g:h = a:de$. Since $b = ef$, therefore $d:f = de:b$, and so $h:k = de:b$. From the two proportions $g:h = a:de$ and $h:k = de:b$ therefore, *ex aequali*, $g:k = a:b$. Thus, ratio the plane numbers is the ratio compounded of the ratios of their sides.

The application of VIII.4 to find the least numbers continuously in the ratios $c:d$ and $e:f$ actually makes the proof more difficult. Here's a slightly shorter proof. Since

$$c:e = cd:de = a:de,$$

and

$$d:f = de:ef = de:b,$$

therefore, the ratio compounded from the ratios $c:e$ and $d:f$ of the sides is the ratio of the plane numbers $a:b$.

Next proposition: VIII.6

Previous: VIII.4

Book VIII introduction

Select from Book VIII

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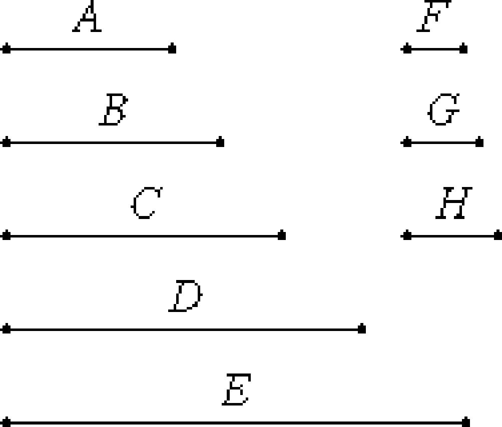
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Euclid's Elements
Book VIII
Proposition 6

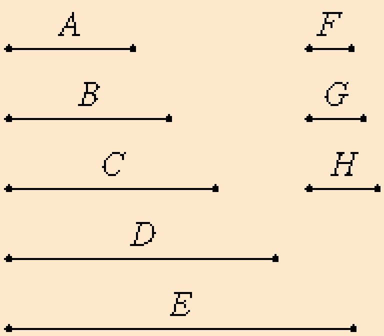
If there are as many numbers as we please in continued proportion, and the first does not measure the second, then neither does any other measure any other.

Let there be as many numbers as we please, *A, B, C, D,* and *E,* in continued proportion, and let *A* not measure *B.*

I say that neither does any other measure any [later] other.

Now it is manifest that *A, B, C, D,* and *E* do not measure one another in order, for *A* does not even measure *B.*

I say, then, that neither does any other measure any [later] other.



If possible, let *A* measure *C.* And, however many *A, B,* and *C* are, take as many numbers *F, G,* and *H,* the least of those which have the same ratio with *A, B,* and *C.* VII.33

Now, since *F, G,* and *H* are in the same ratio with *A, B,* and *C,* and the multitude of the numbers *A, B,* and *C* equals the multitude of the numbers *F, G,* and *H,* therefore, *ex aequali A* is to *C* as *F* is to *H.* VII.14

And since *A* is to *B* as *F* is to *G,* while *A* does not measure *B,* therefore neither does *F* measure *G.* Therefore *F* is not a unit, for the unit measures any number. VII.Def.20

Now *F* and *H* are relatively prime. And *F* is to *H* as *A* is to *C,* therefore neither does *A* measure *C.* VIII.3

Similarly we can prove that neither does any other measure any other.

Therefore, *if there are as many numbers as we please in continued proportion, and the first does not measure the second, then neither does any other measure any other.*

Q.E.D.

Guide

The proposition as stated isn't quite correct. For example, the numbers 24, 12, 6 and 3 are in continued proportion, and 24 does not divide 12, but each of the others does divide others, for instance, 3 divides 6. But none of the others divide others later in the sequence.

Outline of the proof

Consider a sequence of numbers in continued proportion where the first number does not divide the second. Since any number in that sequence has to the next number in the sequence the same ratio as the first has to the second, therefore no number divides the next.

Suppose that some number in the sequence divides a later number. We may call that the former number *a* since it divides the next number in the sequence, and call the number it divides *c*. Take the continued proportion *a*, *b*, ..., *c* and, using VII.33, reduce it a continued proportion *f*, *g*, ..., *h* in lowest terms. Since that's in lowest terms, *f* and *h* are relatively prime. Since *a*:*b* *f*:*g*, and *a* does not divide *b*, therefore *f* does not divide *g*. Since *f* does not divide *g*, in particular *f* does not equal 1, but *f* and *h* are relatively prime by VIII.3, therefore *f* does not divide *h*. Finally, since *a*:*c* *f*:*h*, therefore *a* does not divide *c* either.

Use of this proposition

This proposition is used as a lemma for the following proposition.

Next proposition: VIII.7

Previous: VIII.5

Book VIII introduction

Select from Book VIII

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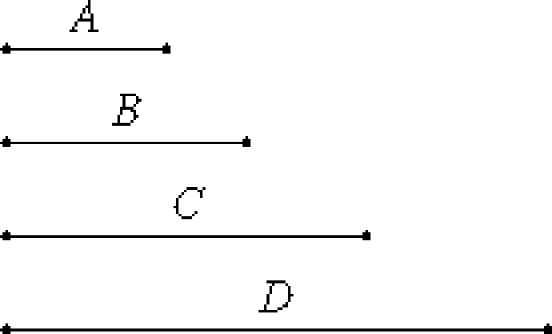
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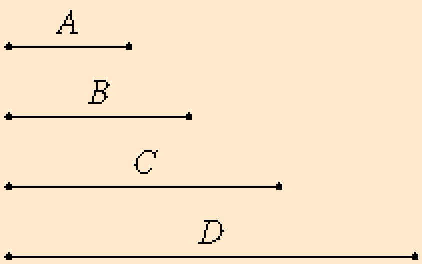
Euclid's Elements
Book VIII
Proposition 7

If there are as many numbers as we please in continued proportion, and the first measures the last, then it also measures the second.

Let there be as many numbers as we please, *A*, *B*, *C*, and *D*, in continued proportion, and let *A* measure *D*.



I say that *A* also measures *B*.

If *A* does not measure *B*, neither does any other of the numbers measure any other. But *A* measures *D*. Therefore *A* also measures *B*. VIII.6

Therefore, *if there are as many numbers as we please in continued proportion, and the first measures the last, then it also measures the second.*

Q.E.D.

Guide

This proposition is the contrapositive of the previous theorem.

Use of this theorem

This proposition is used in propositions VIII.14 and VIII.15.

Next proposition: VIII.8

Previous: VIII.6

Book VIII introduction

Select from Book VIII

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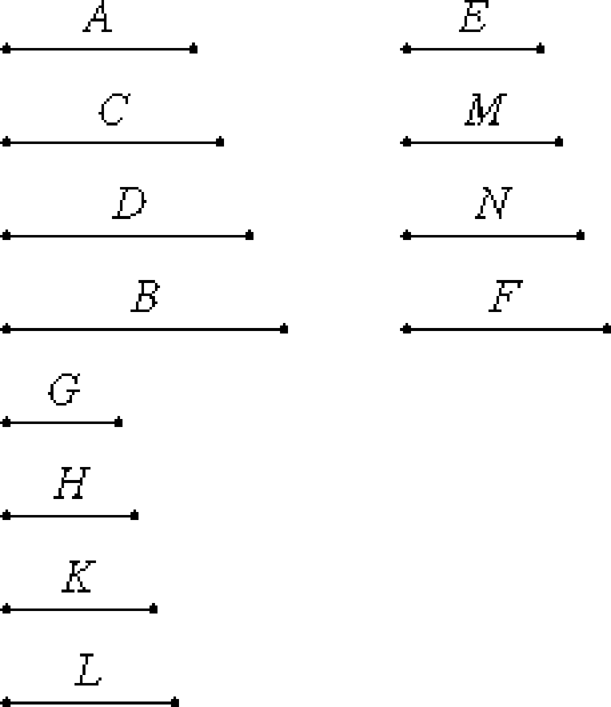
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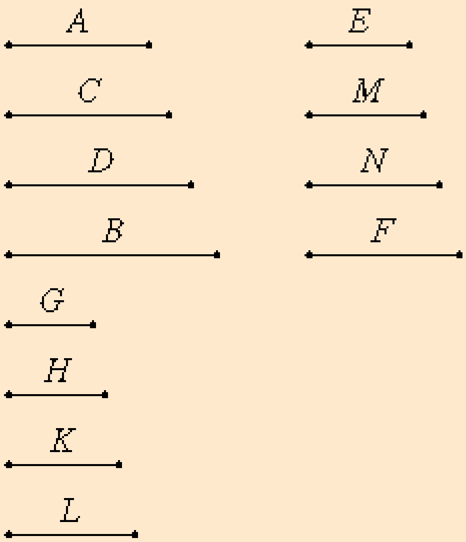
Euclid's Elements
Book VIII
Proposition 8

If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many also fall in continued proportion between the numbers which have the same ratios with the original numbers.

Let the numbers *C* and *D* fall between the two numbers *A* and *B* in continued proportion with them, and make *E* in the same ratio to *F* as *A* is to *B.*

I say that, as many numbers as have fallen between *A* and *B* in continued proportion, so many also fall between *E* and *F* in continued proportion.



As many as *A, B, C,* and *D* are in multitude, take so many numbers *G, H, K,* and *L,* the least of those which have the same ratio with *A, C, D,* and *B.* Then the extremes of them *G* and *L* are relatively prime. VII.33 VIII.3

Now, since *A, C, D,* and *B* are in the same ratio with *G, H, K,* and *L,* and the multitude of the numbers *A, C, D,* and *B* equals the multitude of the numbers *G, H, K,* and *L,* therefore, *ex aequali A* is to *B* as *G* is to *L.* VII.14

But *A* is to *B* as *E* is to *F,* therefore *G* is to *L* as *E* is to *F.* (V.11)

But *G* and *L* are relatively prime, numbers which are relatively prime are also least, and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. VII.21 VII.20

Therefore *G* measures *E* the same number of times as *L* measures *F.*

Next, let *H* and *K* measure *M* and *N,* respectively, as many times as *G* measures *E.* Then *G, H, K,* and *L* measure *E, M, N,* and *F* the same number of times. Therefore *G, H, K,* and *L* are in the same ratio with *E, M, N,* and *F.* VII.Def.20

But *G, H, K,* and *L* are in the same ratio with *A, C, D,* and *B,* therefore *A, C, D,* and *B* are also in the same ratio with *E, M, N,* and *F.*

But *A, C, D,* and *B* are in continued proportion, therefore *E, M, N,* and *F* are also in continued proportion. Therefore, as many numbers as have fallen between *A* and *B* in continued proportion with them, so many numbers have also fallen between *E* and *F* in continued proportion.

Therefore, *if between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many also fall in continued proportion between the numbers which have the same ratios with the original numbers.*

Q.E.D.

Guide

This proposition implies, among other things, that there is no number which forms a mean proportional between a number n and the number $2n$, for if there were, there would be a number m so that 2, m, and 4 would form a continued proportion, but the only number between 2 and 4 is 3, and 2, 3, and 4 do not form a continued proportion. (If 1 is considered to be a number, the argument simplifies.) In modern terminology, this conclusion says the square root of 2 is not a rational number. See proposition X.9 for implications of this conclusion for imcommensurability of line segments.

Outline of the proof

Suppose that $a:b = e:f$ and the sequence $a, c, d, ..., b$ are in continued proportion. Use VII.33 to reduce that sequence to lowest terms $g, h, k, ..., l$. According to VIII.3, the ends of that sequence g and l are relatively prime. Since $a:b = e:f$, we also have $g:l = e:f$. But g is relatively prime to l, so the ratio $g:l$ is in lowest terms (VII.21), and therefore g divides e the same number of times, say n, that l divides f (VII.20). Then the sequence $ng, nh, nk, ..., nl$ is in continued proportion and starts with e and ends with f as required.

Use of this proposition

Although this proposition is not used in Book VIII, it is used in the first six propositions of Book IX.

Next proposition: VIII.9

Previous: VIII.7

Book VIII introduction

Select from Book VIII

Select book

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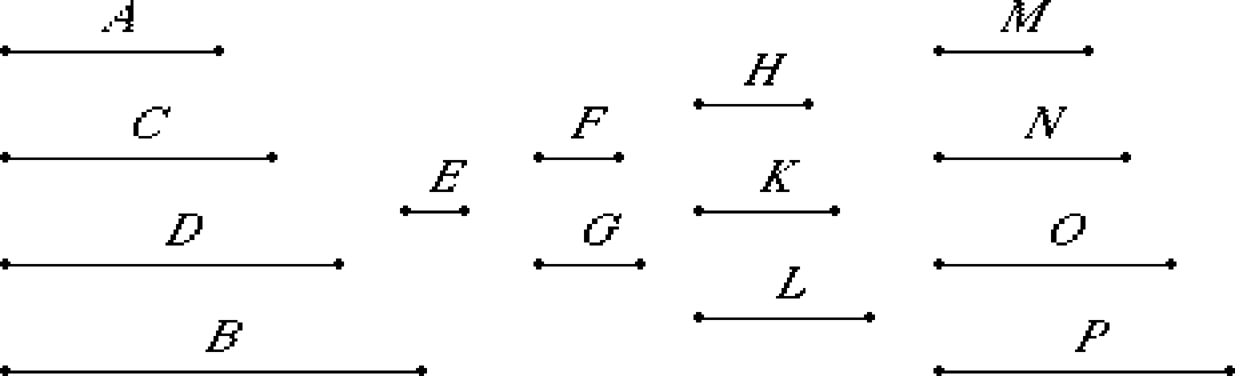
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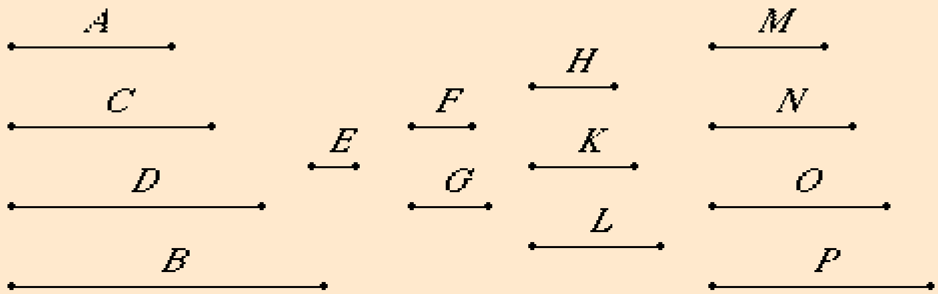
Euclid's Elements
Book VIII
Proposition 9

If two numbers are relatively prime, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many also fall between each of them and a unit in continued proportion.

Let *A* and *B* be two numbers relatively prime, and let *C* and *D* fall between them in continued proportion, and let the unit *E* be set out.

I say that, as many numbers fall between *A* and *B* in continued proportion as fall between either of the numbers *A* or *B* and the unit in continued proportion.



Take two numbers *F* and *G*, the least that are in the ratio of *A*, *C*, *D*, and *B*, three numbers *H*, *K*, and *L* with the same property, and others more by one continually, until their multitude equals the multitude of *A*, *C*, *D*, and *B*. Let them be *M*, *N*, *O*, and *P*. VIII.2

It is now manifest that *F* multiplied by itself makes *H* and multiplied by *H* makes *M*, while *G* multiplied by itself makes *L* and multiplied by *L* makes *P*. VIII.2,Cor

And, since *M*, *N*, *O*, and *P* are the least of those which have the same ratio with *F* and *G*, and *A*, *C*, *D*, and *B* are also the least of those which have the same ratio with *F* and *G*, while the multitude of the numbers *M*, *N*, *O*, and *P* equals the multitude of the numbers *A*, *C*, *D*, and *B*, therefore *M*, *N*, *O*, and *P* equal *A*, *C*, *D*, and *B* respectively. Therefore *M* equals *A*, and *P* equals *B*. VIII.1

Now, since *F* multiplied by itself makes *H*, therefore *F* measures *H* according to the units in *F*. But the unit *E* also measures *F* according to the units in it, therefore the unit *E* measures the number *F* the same number of times as *F* measures *H*.

Therefore the unit *E* is to the number *F* as *F* is to *H*. VII.Def.20

Again, since *F* multiplied by *H* makes *M*, therefore *H* measures *M* according to the units in *F*. But the unit *E* also measures the number *F* according to the units in it, therefore the unit *E* measures the number *F* the same number of times as *H* measures *M*.

Therefore the unit *E* is to the number *F* as *H* is to *M*.

But it was also proved that the unit *E* is to the number *F* as *F* is to *H*, therefore the unit *E* is to the number *F* as *F* is to *H*, and as *H* is to *M*. But *M* equals *A*, therefore the unit *E* is to the number *F* as *F* is to *H*, and as *H* is to *A*. For the same reason also the unit *E* is to the number *G* as *G* is to *L* and as *L* is to *B*.

Therefore as many numbers fall between *A* and *B* in continued proportion as fall between each of the numbers *A* and *B* and the unit *E* in continued proportion.

Therefore, *If two numbers are relatively prime, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many also fall between each of them and a unit in continued proportion.*

Q.E.D.

Guide

Suppose that relatively prime numbers *a* and *b* are the ends of a continued proportion with *n* terms, and that f / g is the ratio for the continued proportion in lowest terms. Then Euclid shows that *a* is the $n-1^{st}$ power of *f*, and *a* is the $n-1^{st}$ power of *g*. The argument is that the sequence

$$f^{n-1}, f^{n-2}g, f^{n-3}g,^2, \ldots, fg^{n-2}, g^{n-1}$$

is in continued proportion with the correct ratio with relatively prime ends, so by VIII.1 they're the same sequence.

Next proposition: VIII.10

Previous: VIII.8

Book VIII introduction

Select from Book VIII

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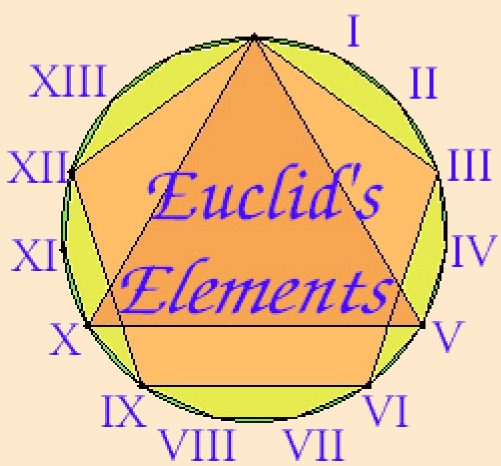
Book X



Book X

Table of contents

Definitions I

Definition 1.

Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.

Definition 2.

Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure.

Definition 3.

With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length and in square, or in square only, *rational*, but those that are incommensurable with it *irrational*.

Definition 4.

And the let the square on the assigned straight line be called *rational*, and those areas which are commensurable with it *rational*, but those which are incommensurable with it *irrational*, and the straight lines which produce them *irrational*, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

Propositions 1-47

Proposition 1.
Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proven even if the parts subtracted are halves.

Proposition 2.
If, when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it, then the two magnitudes are incommensurable.

Proposition 3.
To find the greatest common measure of two given commensurable magnitudes.

Corollary. If a magnitude measures two magnitudes, then it also measures their greatest common measure.

Proposition 4.
To find the greatest common measure of three given commensurable magnitudes.

Corollary. If a magnitude measures three magnitudes, then it also measures their greatest common measure. The greatest common measure can be found similarly for more magnitudes, and the corollary extended.

Proposition 5.
Commensurable magnitudes have to one another the ratio which a number has to a number.

Proposition 6.
If two magnitudes have to one another the ratio which a number has to a number, then the magnitudes are commensurable.

Corollary.

Proposition 7.
Incommensurable magnitudes do not have to one another the ratio which a number has to a number.

Proposition 8.
If two magnitudes do not have to one another the ratio which a number has to a number, then the magnitudes are incommensurable.

Proposition 9.
The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number also have their sides commensurable in length. But the squares on straight lines incommensurable in length do not have to one another the ratio which a square number has to a square number; and squares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either.

Corollary. Straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always also commensurable in length.

Lemma. Similar plane numbers have to one another the ratio which a square number has to a square number, and if two numbers have to one another the ratio which a square number has to a square number, then they are similar plane numbers.

Corollary 2. Numbers which are not similar plane numbers, that is, those which do not have their sides proportional, do not have to one another the ratio which a square number has to a square number

Proposition 10.

To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line.

Proposition 11.

If four magnitudes are proportional, and the first is commensurable with the second, then the third also is commensurable with the fourth; but, if the first is incommensurable with the second, then the third also is incommensurable with the fourth.

Proposition 12.

Magnitudes commensurable with the same magnitude are also commensurable with one another.

Proposition 13.

If two magnitudes are commensurable, and one of them is incommensurable with any magnitude, then the remaining one is also incommensurable with the same.

Proposition 14.

Lemma. Given two unequal straight lines, to find by what square the square on the greater is greater than the square on the less. And, given two straight lines, to find the straight line the square on which equals the sum of the squares on them.

Proposition 14. If four straight lines are proportional, and the square on the first is greater than the square on the second by the square on a straight line commensurable with the first, then the square on the third is also greater than the square on the fourth by the square on a third line commensurable with the third. And, if the square on the first is greater than the square on the second by the square on a straight line incommensurable with the first, then the square on the third is also greater than the square on the fourth by the square on a third line incommensurable with the third.

Proposition 15.

If two commensurable magnitudes are added together, then the whole is also commensurable with each of them; and, if the whole is commensurable with one of them, then the original magnitudes are also commensurable.

Proposition 16.

If two incommensurable magnitudes are added together, the sum is also incommensurable with each of them; but, if the sum is incommensurable with one of them, then the original magnitudes are also incommensurable.

Proposition 17.

Lemma. If to any straight line there is applied a parallelogram but falling short by a square, then the applied parallelogram equals the rectangle contained by the segments of the straight line resulting from the application.

Proposition 17. If there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less but falling short by a square, and if it divides it into parts commensurable in length, then the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater. And if the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater, and if there is applied to the greater a parallelogram equal to the fourth part of the square on the less falling short by a square, then it divides it into parts commensurable in length.

Proposition 18.

If there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less but falling short by a square, and if it divides it into incommensurable parts, then the square on the greater is greater than the square on the less by the square on a straight line incommensurable with the greater. And if the square on the greater is greater than the square on the less by the square on a straight line incommensurable with the greater, and if there is applied to the greater a parallelogram equal to the fourth part of the square on the less but falling short by a square, then it divides it into incommensurable parts.

Proposition 19.

Lemma.

Proposition 19. The rectangle contained by rational straight lines commensurable in length is rational.

Proposition 20.

If a rational area is applied to a rational straight line, then it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.

Proposition 21.

The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called *medial*.

Proposition 22.

Lemma. If there are two straight lines, then the first is to the second as the square on the first is to the rectangle contained by the two straight lines.

Proposition 22. The square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied.

Proposition 23.

A straight line commensurable with a medial straight line is medial.

Corollary. An area commensurable with a medial area is medial.

Proposition 24.

The rectangle contained by medial straight lines commensurable in length is medial.

Proposition 25.

The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

Proposition 26.

A medial area does not exceed a medial area by a rational area.

Proposition 27.

To find medial straight lines commensurable in square only which contain a rational rectangle.

Proposition 28.

To find medial straight lines commensurable in square only which contain a medial rectangle.

Proposition 29.

Lemma 1. To find two square numbers such that their sum is also square.

Lemma 2. To find two square numbers such that their sum is not square.

Proposition 29. To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Proposition 30.

To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater.

Proposition 31.

To find two medial straight lines commensurable in square only, containing a rational rectangle, such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Proposition 32.

To find two medial straight lines commensurable in square only, containing a medial rectangle, such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Proposition 33.

Lemma.

Proposition 33. To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial.

Proposition 34.

To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational.

Proposition 35.

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Proposition 36.

If two rational straight lines commensurable in square only are added together, then the whole is irrational; let it be called *binomial*.

Proposition 37.

If two medial straight lines commensurable in square only and containing a rational rectangle are added together, the whole is irrational; let it be called the *first bimedial* straight line.

Proposition 38.

If two medial straight lines commensurable in square only and containing a medial rectangle are added together, then the whole is irrational; let it be called the *second bimedial* straight line.

Proposition 39.

If two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial are added together, then the whole straight line is irrational; let it be called *major*.

Proposition 40.

If two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational are added together, then the whole straight line is irrational; let it be called the *side of a rational plus a medial area.*

Proposition 41.

If two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and also incommensurable with the sum of the squares on them are added together, then the whole straight line is irrational; let it be called the *side of the sum of two medial areas.*

Lemma.

Proposition 42.

A binomial straight line is divided into its terms at one point only.

Proposition 43.
A first bimedial straight line is divided at one and the same point only.

Proposition 44.
A second bimedial straight line is divided at one point only.

Proposition 45.
A major straight line is divided at one point only.

Proposition 46.
The side of a rational plus a medial area is divided at one point only.

Proposition 47.
The side of the sum of two medial areas is divided at one point only.

Definitions II

Definition 1.
Given a rational straight line and a binomial, divided into its terms, such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then, if the greater term is commensurable in length with the rational straight line set out, let the whole be called a *first binomial* straight line;

Definition 2.
But if the lesser term is commensurable in length with the rational straight line set out, let the whole be called a *second binomial*;

Definition 3.
And if neither of the terms is commensurable in length with the rational straight line set out, let the whole be called a *third binomial.*

Definition 4.
Again, if the square on the greater term is greater than the square on the lesser by the square on a straight line incommensurable in length with the greater, then, if the greater term is commensurable in length with the rational straight line set out, let the whole be called a *fourth binomial*;

Definition 5.
If the lesser, a *fifth binomial*;

Definition 6.
And, if neither, a *sixth binomial*.

Propositions 48-84

Proposition 48.
To find the first binomial line.

Proposition 49.
To find the second binomial line.

Proposition 50.
To find the third binomial line.

fifth binomial.

Proposition 65.
The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.

Proposition 66.
A straight line commensurable with a binomial straight line is itself also binomial and the same in order.

Proposition 67.
A straight line commensurable with a bimedial straight line is itself also bimedial and the same in order.

Proposition 68.
A straight line commensurable with a major straight line is itself also major.

Proposition 69.
A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area.

Proposition 70.
A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas.

Proposition 71.
If a rational and a medial are added together, then four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area.

Proposition 72.
If two medial areas incommensurable with one another are added together, then the remaining two irrational straight lines arise, namely either a second bimedial or a side of the sum of two medial areas.

Proposition. The binomial straight line and the irrational straight lines after it are neither the same with the medial nor with one another.

Proposition 73.
If from a rational straight line there is subtracted a rational straight line commensurable with the whole in square only, then the remainder is irrational; let it be called an *apotome.*

Proposition 74.
If from a medial straight line there is subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle, then the remainder is irrational; let it be called *first apotome of a medial* straight line.

Proposition 75.
If from a medial straight line there is subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a medial rectangle, then the remainder is irrational; let it be called *second apotome of a medial* straight line.

Proposition 76.
If from a straight line there is subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them added together rational, but the rectangle contained by them medial, then the remainder is irrational; let it be called *minor*.

Proposition 77.
If from a straight line there is subtracted a straight line which is incommensurable in square with the whole, and

which with the whole makes the sum of the squares on them medial but twice the rectangle contained by them rational, then the remainder is irrational; let it be called *that which produces with a rational area a medial whole.*

Proposition 78.

If from a straight line there is subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them, then the remainder is irrational; let it be called *that which produces with a medial area a medial whole.*

Proposition 79.

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only.

Proposition 80.

To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle.

Proposition 81.

To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

Proposition 82.

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of squares on them rational but twice the rectangle contained by them medial.

Proposition 83.

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of squares on them medial but twice the rectangle contained by them rational.

Proposition 84.

To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of squares on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them.

Definitions III

Definition 1.

Given a rational straight line and an apotome, if the square on the whole is greater than the square on the annex by the square on a straight line commensurable in length with the whole, and the whole is commensurable in length with the rational line set out, let the apotome be called a *first apotome.*

Definition 2.

But if the annex is commensurable with the rational straight line set out, and the square on the whole is greater than that on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *second apotome.*

Definition 3.

But if neither is commensurable in length with the rational straight line set out, and the square on the whole is greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *third apotome.*

Definition 4.

Again, if the square on the whole is greater than the square on the annex by the square on a straight line incommensurable with the whole, then, if the whole is commensurable in length with the rational straight line set out, let the apotome be called a *fourth apotome*;

Definition 5.

If the annex be so commensurable, a *fifth*;

Definition 6.

And, if neither, a *sixth*.

Propositions 85-115

Proposition 85.

To find the first apotome.

Proposition 86.

To find the second apotome.

Proposition 87.

To find the third apotome.

Proposition 88.

To find the fourth apotome.

Proposition 89.

To find the fifth apotome.

Proposition 90.

To find the sixth apotome.

Proposition 91.

If an area is contained by a rational straight line and a first apotome, then the side of the area is an apotome.

Proposition 92.

If an area is contained by a rational straight line and a second apotome, then the side of the area is a first apotome of a medial straight line.

Proposition 93.

If an area is contained by a rational straight line and a third apotome, then the side of the area is a second apotome of a medial straight line.

Proposition 94.

If an area is contained by a rational straight line and a fourth apotome, then the side of the area is minor.

Proposition 95.

If an area is contained by a rational straight line and a fifth apotome, then the side of the area is a straight line which produces with a rational area a medial whole.

Proposition 96.

If an area is contained by a rational straight line and a sixth apotome, then the side of the area is a straight line which produces with a medial area a medial whole.

Proposition 97.

The square on an apotome of a medial straight line applied to a rational straight line produces as breadth a first apotome.

Proposition 98.
The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.

Proposition 99.
The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.

Proposition 100.
The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome.

Proposition 101.
The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome.

Proposition 102.
The square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome.

Proposition 103.
A straight line commensurable in length with an apotome is an apotome and the same in order.

Proposition 104.
A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order.

Proposition 105.
A straight line commensurable with a minor straight line is minor.

Proposition 106.
A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole.

Proposition 107.
A straight line commensurable with that which produces a medial area and a medial whole is itself also a straight line which produces with a medial area a medial whole.

Proposition 108.
If from a rational area a medial area is subtracted, the side of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line.

Proposition 109.
If from a medial area a rational area is subtracted, then there arise two other irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.

Proposition 110.
If from a medial area there is subtracted a medial area incommensurable with the whole, then the two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produce with a medial area a medial whole.

Proposition 111.
The apotome is not the same with the binomial straight line.

Proposition. The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another. There are, in order, thirteen irrational straight lines in all:

Medial
Binomial
First bimedial
Second bimedial
Major
Side of a rational plus a medial area
Side of the sum of two medial areas
Apotome
First apotome of a medial straight line
Second apotome of a medial straight line
Minor
Producing with a rational area a medial whole
Producing with a medial area a medial whole

Proposition 112.
The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial straight line and moreover in the same ratio; and further the apotome so arising has the same order as the binomial straight line.

Proposition 113.
The square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.

Proposition 114.
If an area is contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, then the side of the area is rational.

Corollary. It is possible for a rational area to be contained by irrational straight lines.

Proposition 115.
From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any preceding.

Elements Introduction - Book IX - Book XI.

Bad Request

Your browser sent a request that this server could not understand.

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Index of /~djoyce/java/elements/bookX

Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
bkX.gif	19-May-1997 20:47	1k	
bookX.html	26-Oct-1998 10:32	37k	
bookX.html%	26-Oct-1998 10:32	37k	
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defX.II.html	21-Jul-1997 17:22	2k	
defX.III.html	21-Jul-1997 17:22	2k	
favicon.ico	07-May-2001 14:54	1k	
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propX1.html	21-Jul-1997 17:22	6k	
propX10.gif	19-May-1997 20:47	1k	
propX10.html	21-Jul-1997 17:22	6k	
propX100.html	21-Jul-1997 17:22	8k	
propX101.html	21-Jul-1997 17:22	7k	
propX102.html	21-Jul-1997 17:22	7k	
propX103.gif	19-May-1997 20:47	1k	
propX103.html	21-Jul-1997 17:22	5k	
propX104.html	21-Jul-1997 17:22	5k	
propX105.html	21-Jul-1997 17:22	5k	
propX106.html	21-Jul-1997 17:22	4k	
propX107.html	21-Jul-1997 17:22	4k	
propX108.gif	19-May-1997 20:48	1k	
propX108.html	21-Jul-1997 17:23	5k	
propX109.gif	19-May-1997 20:48	2k	
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propX11.html	21-Jul-1997 17:23	4k	
propX110.html	21-Jul-1997 17:23	6k	
propX111.gif	19-May-1997 20:48	1k	
propX111.html	21-Jul-1997 17:23	7k	
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propX114.gif	19-May-1997 20:48	1k	
propX114.html	21-Jul-1997 17:23	6k	
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propX16.gif	19-May-1997	20:49	1k
propX16.html	21-Jul-1997	17:23	4k
propX17.gif	26-Jun-1997	22:25	1k
propX17.html	21-Jul-1997	17:23	10k
propX17l.gif	19-May-1997	20:49	1k
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propX28.html	21-Jul-1997	17:24	5k
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propX29l2.gif	19-May-1997	20:51	1k
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propX37.html	21-Jul-1997	17:25	3k
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propX4.gif	19-May-1997	20:52	1k
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propX41.html	21-Jul-1997	17:25	8k
propX41l.gif	19-May-1997	20:52	1k
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propX46.html	21-Jul-1997	17:26	3k
propX47.html	21-Jul-1997	17:26	5k
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?	propX88.html	21-Jul-1997 17:30	5k
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propX91.gif	19-May-1997 20:58	3k	
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propX88.html	21-Jul-1997 17:30	5k	
propX88.gif	19-May-1997 20:57	1k	
propX87.html	21-Jul-1997 17:30	8k	
propX87.gif	19-May-1997 20:57	1k	
propX86.html	21-Jul-1997 17:30	5k	
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	propX73.gif	19-May-1997 20:56	1k
	propX72.html	21-Jul-1997 17:28	8k
	propX71.html	21-Jul-1997 17:28	8k
	propX71.gif	19-May-1997 20:56	2k
	propX70.html	21-Jul-1997 17:28	3k
	propX7.html	21-Jul-1997 17:28	3k
	propX7.gif	19-May-1997 20:55	1k
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	propX64.html	21-Jul-1997 17:28	5k
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	propX62.html	21-Jul-1997 17:27	6k
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	propX60.gif	19-May-1997 20:55	1k
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propX4.gif	19-May-1997	20:52	1k
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propX36.gif	19-May-1997	20:52	1k
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propX28.gif	19-May-1997	20:51	1k
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propX113.gif	19-May-1997	20:48	1k
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propX112.gif	19-May-1997	20:48	1k
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propX111.gif	19-May-1997	20:48	1k
propX110.html	21-Jul-1997	17:23	6k

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bookX.html	26-Oct-1998	10:32	37k
bkX.gif	19-May-1997	20:47	1k

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Euclid's Elements
Book X
Definitions I

Definition 1

Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.

Guide

Two magnitudes *A* and *B* of the same kind are *commensurable* if there is another magnitude *C* of the same kind such that both are multiples of *C,* that is, there are numbers *m* and *n* such that $nC = A$ and $mC = B$. See definition V.Def.5 for the defintion of equality of ratios (also known as a *proportion*). If the two magnitudes are not commensurable, then they're called *incommensurable.*

Propositions X.2 through X.8 and several later ones deal with commensurable and incommensurable magnitudes. In particular X.5 and X.6 state that two magnitudes are commensurable if and only if their ratio is the ratio of a number to a number. For example, if $nC = A$ and $mC = B$, then the ratio of magnitudes $A:B$ is the same as the ratio of numbers $m:n$. And conversely, if $A:B = m:n$, then the $1/n^{th}$ part of *A* equals the $1/m^{th}$ part of *B.*

Ratios of numbers are known to modern mathematicians as *rational numbers* while other ratios are known as *irrational numbers.* Unfortunately, Euclid used the words "rational" and "irrational" in a different way in Definition 3, see below.

Definition 2

Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure.

Guide

Note that this definition only applies to lines, that is, only lines are ever said to be "commensurable in square." Certainly, commensurable lines are also commensurable in square, but lines can be commensurable in square but not commensurable, in other words, "commensurable is square only." The most famous example of this phenomenon consists of the side *A* and the diagonal *B* of a square. They are commensurable in square since the square on *B* is twice the square on *A*, by I.47. But they are not commensurable lines. In modern terms we would say that the square root of 2 is not a rational number.

Definition 3

With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length and in square, or in square only, *rational*, but those that are incommensurable with it *irrational*.

Guide

The proof referred to at the beginning of this definition is that of X.10 which finds lines commensurable in square only, and lines incommensurable in square.

Euclid uses the words "rational" and "irrational" differently than mathematicians both before and after him. The usual uses of these words correspond to commensurable and incommensurable, respectively. But when applied to lines Euclid makes them correspond to commensurable in square and incommensurable in square. First, one line is chosen as a standard, then another line is called *rational* if it is commensurable in square, and *irrational* if not. Thus, the diagonal on the square on the standard line is rational, even though it's incommensurable with the standard line, since it's commensurable in square with it.

Definition 4

And the let the square on the assigned straight line be called *rational*, and those areas which are commensurable with it *rational*, but those which are incommensurable with it *irrational*, and the straight lines which produce them *irrational*, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

Guide

Although Euclid uses "rational" in an unusual way for lines, he uses it in the usual way for areas, so that an area is *rational*, according to Euclid, if it's commensurable with the standard square, and *irrational* otherwise.

Book X Introduction - Proposition X.1.

Euclid's Elements

Book X

Definitions II

Definition 1.

Given a rational straight line and a binomial, divided into its terms, such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then, if the greater term is commensurable in length with the rational straight line set out, let the whole be called a *first binomial* straight line;

Definition 2.

But if the lesser term is commensurable in length with the rational straight line set out, let the whole be called a *second binomial*;

Definition 3.

And if neither of the terms is commensurable in length with the rational straight line set out, let the whole be called a *third binomial.*

Definition 4.

Again, if the square on the greater term is greater than the square on the lesser by the square on a straight line incommensurable in length with the greater, then, if the greater term is commensurable in length with the rational straight line set out, let the whole be called a *fourth binomial*;

Definition 5.

If the lesser, a *fifth binomial*;

Definition 6.

And, if neither, a *sixth binomial*.

Guide

(Forthcoming)

Book X Introduction - Proposition X.47 - Proposition X.48.

Euclid's Elements

Book X

Definitions III

Definition 1.

Given a rational straight line and an apotome, if the square on the whole is greater than the square on the annex by the square on a straight line commensurable in length with the whole, and the whole is commensurable in length with the rational line set out, let the apotome be called a *first apotome.*

Definition 2.

But if the annex is commensurable with the rational straight line set out, and the square on the whole is greater than that on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *second apotome.*

Definition 3.

But if neither is commensurable in length with the rational straight line set out, and the square on the whole is greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *third apotome.*

Definition 4.

Again, if the square on the whole is greater than the square on the annex by the square on a straight line incommensurable with the whole, then, if the whole is commensurable in length with the rational straight line set out, let the apotome be called a *fourth apotome*;

Definition 5.

If the annex be so commensurable, a *fifth*;

Definition 6.

And, if neither, a *sixth*.

Guide

(Forthcoming)

Book X Introduction - Proposition X.84 - Proposition X.85.

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propX106.html	21-Jul-1997	17:22	4k
propX105.html	21-Jul-1997	17:22	5k
propX104.html	21-Jul-1997	17:22	5k
propX103.html	21-Jul-1997	17:22	5k
propX102.html	21-Jul-1997	17:22	7k
propX101.html	21-Jul-1997	17:22	7k
propX100.html	21-Jul-1997	17:22	8k
propX10.html	21-Jul-1997	17:22	6k
propX1.html	21-Jul-1997	17:22	6k
defX.III.html	21-Jul-1997	17:22	2k
defX.II.html	21-Jul-1997	17:22	2k
propX17.gif	26-Jun-1997	22:25	1k
propX97.gif	19-May-1997	20:58	2k
propX92.gif	19-May-1997	20:58	3k

propX91.gif	19-May-1997	20:58	3k
propX9.gif	19-May-1997	20:57	1k
propX89.gif	19-May-1997	20:57	1k
propX88.gif	19-May-1997	20:57	1k
propX87.gif	19-May-1997	20:57	1k
propX86.gif	19-May-1997	20:57	1k
propX85.gif	19-May-1997	20:57	1k
propX81.gif	19-May-1997	20:56	1k
propX79.gif	19-May-1997	20:56	1k
propX75.gif	19-May-1997	20:56	1k
propX73.gif	19-May-1997	20:56	1k
propX71.gif	19-May-1997	20:56	2k
propX7.gif	19-May-1997	20:55	1k
propX66.gif	19-May-1997	20:55	1k
propX60l.gif	19-May-1997	20:55	1k
propX60.gif	19-May-1997	20:55	1k
propX6.gif	19-May-1997	20:54	1k
propX54l.gif	19-May-1997	20:54	2k
propX54.gif	19-May-1997	20:54	2k
propX53.gif	19-May-1997	20:54	1k
propX52.gif	19-May-1997	20:53	1k
propX51.gif	19-May-1997	20:53	1k
propX50.gif	19-May-1997	20:53	1k
propX5.gif	19-May-1997	20:53	1k
propX49.gif	19-May-1997	20:53	1k
propX48.gif	19-May-1997	20:53	1k
propX44.gif	19-May-1997	20:52	1k
propX42.gif	19-May-1997	20:52	1k
propX41l.gif	19-May-1997	20:52	1k
propX41.gif	19-May-1997	20:52	1k
propX4.gif	19-May-1997	20:52	1k
propX38.gif	19-May-1997	20:52	1k
propX36.gif	19-May-1997	20:52	1k
propX34.gif	19-May-1997	20:51	1k
propX33.gif	19-May-1997	20:51	1k
propX31.gif	19-May-1997	20:51	1k
propX3.gif	19-May-1997	20:51	1k
propX29l2.gif	19-May-1997	20:51	1k
propX29l1.gif	19-May-1997	20:51	1k
propX29.gif	19-May-1997	20:51	1k
propX28.gif	19-May-1997	20:51	1k
propX27.gif	19-May-1997	20:50	1k
propX26.gif	19-May-1997	20:50	1k
propX25.gif	19-May-1997	20:50	2k

propX24.gif	19-May-1997 20:50	1k
propX23.gif	19-May-1997 20:50	1k
propX22l.gif	19-May-1997 20:50	1k
propX22.gif	19-May-1997 20:50	2k
propX20.gif	19-May-1997 20:50	1k
propX2.gif	19-May-1997 20:50	1k
propX19.gif	19-May-1997 20:50	1k
propX17l.gif	19-May-1997 20:49	1k
propX16.gif	19-May-1997 20:49	1k
propX15.gif	19-May-1997 20:49	1k
propX14l.gif	19-May-1997 20:49	1k
propX14.gif	19-May-1997 20:49	1k
propX13.gif	19-May-1997 20:49	1k
propX12.gif	19-May-1997 20:49	1k
propX115.gif	19-May-1997 20:49	1k
propX114.gif	19-May-1997 20:48	1k
propX113.gif	19-May-1997 20:48	1k
propX112.gif	19-May-1997 20:48	1k
propX111.gif	19-May-1997 20:48	1k
propX11.gif	19-May-1997 20:48	1k
propX109.gif	19-May-1997 20:48	2k
propX108.gif	19-May-1997 20:48	1k
propX103.gif	19-May-1997 20:47	1k
propX10.gif	19-May-1997 20:47	1k
propX1.gif	19-May-1997 20:47	1k
bkX.gif	19-May-1997 20:47	1k

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Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
bkX.gif	19-May-1997 20:47	1k	
bookX.html	26-Oct-1998 10:32	37k	
bookX.html%	26-Oct-1998 10:32	37k	
defX.I.html	10-Oct-2002 09:21	6k	
defX.II.html	21-Jul-1997 17:22	2k	
defX.III.html	21-Jul-1997 17:22	2k	
favicon.ico	07-May-2001 14:54	1k	
propX1.gif	19-May-1997 20:47	1k	
propX1.html	21-Jul-1997 17:22	6k	
propX10.gif	19-May-1997 20:47	1k	
propX10.html	21-Jul-1997 17:22	6k	
propX100.html	21-Jul-1997 17:22	8k	
propX101.html	21-Jul-1997 17:22	7k	
propX102.html	21-Jul-1997 17:22	7k	
propX103.gif	19-May-1997 20:47	1k	
propX103.html	21-Jul-1997 17:22	5k	
propX104.html	21-Jul-1997 17:22	5k	
propX105.html	21-Jul-1997 17:22	5k	
propX106.html	21-Jul-1997 17:22	4k	
propX107.html	21-Jul-1997 17:22	4k	
propX108.gif	19-May-1997 20:48	1k	
propX108.html	21-Jul-1997 17:23	5k	
propX109.gif	19-May-1997 20:48	2k	
propX109.html	21-Jul-1997 17:23	5k	
propX11.gif	19-May-1997 20:48	1k	
propX11.html	21-Jul-1997 17:23	4k	
propX110.html	21-Jul-1997 17:23	6k	
propX111.gif	19-May-1997 20:48	1k	
propX111.html	21-Jul-1997 17:23	7k	
propX112.gif	19-May-1997 20:48	1k	
propX112.html	21-Jul-1997 17:31	8k	
propX113.gif	19-May-1997 20:48	1k	
propX113.html	21-Jul-1997 17:31	9k	
propX114.gif	19-May-1997 20:48	1k	
propX114.html	21-Jul-1997 17:23	6k	
propX115.gif	19-May-1997 20:49	1k	
propX115.html	21-Jul-1997 17:23	4k	
propX12.gif	19-May-1997 20:49	1k	

	propX12.html	21-Jul-1997 17:23	6k
	propX13.gif	19-May-1997 20:49	1k
	propX13.html	21-Jul-1997 17:31	3k
	propX14.gif	19-May-1997 20:49	1k
	propX14.html	23-Jul-1997 11:25	10k
	propX14l.gif	19-May-1997 20:49	1k
	propX15.gif	19-May-1997 20:49	1k
	propX15.html	21-Jul-1997 17:23	4k
	propX16.gif	19-May-1997 20:49	1k
	propX16.html	21-Jul-1997 17:23	4k
	propX17.gif	26-Jun-1997 22:25	1k
	propX17.html	21-Jul-1997 17:23	10k
	propX17l.gif	19-May-1997 20:49	1k
	propX18.html	21-Jul-1997 17:23	7k
	propX19.gif	19-May-1997 20:50	1k
	propX19.html	21-Jul-1997 17:24	5k
	propX2.gif	19-May-1997 20:50	1k
	propX2.html	10-Oct-2002 09:19	8k
	propX20.gif	19-May-1997 20:50	1k
	propX20.html	21-Jul-1997 17:24	3k
	propX21.html	21-Jul-1997 17:24	3k
	propX22.gif	19-May-1997 20:50	2k
	propX22.html	21-Jul-1997 17:24	8k
	propX22l.gif	19-May-1997 20:50	1k
	propX23.gif	19-May-1997 20:50	1k
	propX23.html	21-Jul-1997 17:24	5k
	propX24.gif	19-May-1997 20:50	1k
	propX24.html	21-Jul-1997 17:24	3k
	propX25.gif	19-May-1997 20:50	2k
	propX25.html	21-Jul-1997 17:24	6k
	propX26.gif	19-May-1997 20:50	1k
	propX26.html	21-Jul-1997 17:24	5k
	propX27.gif	19-May-1997 20:50	1k
	propX27.html	21-Jul-1997 17:24	4k
	propX28.gif	19-May-1997 20:51	1k
	propX28.html	21-Jul-1997 17:24	5k
	propX29.gif	19-May-1997 20:51	1k
	propX29.html	21-Jul-1997 17:24	12k
	propX29l1.gif	19-May-1997 20:51	1k
	propX29l2.gif	19-May-1997 20:51	1k
	propX2a.gif	10-Oct-2002 09:20	3k
	propX3.gif	19-May-1997 20:51	1k
	propX3.html	21-Jul-1997 17:24	5k
	propX30.html	21-Jul-1997 17:25	4k

propX31.gif	19-May-1997 20:51	1k
propX31.html	21-Jul-1997 17:25	5k
propX32.html	21-Jul-1997 17:25	6k
propX33.gif	19-May-1997 20:51	1k
propX33.html	21-Jul-1997 17:25	9k
propX34.gif	19-May-1997 20:51	1k
propX34.html	21-Jul-1997 17:25	4k
propX35.html	21-Jul-1997 17:25	5k
propX36.gif	19-May-1997 20:52	1k
propX36.html	21-Jul-1997 17:25	3k
propX37.html	21-Jul-1997 17:25	3k
propX38.gif	19-May-1997 20:52	1k
propX38.html	21-Jul-1997 17:25	6k
propX39.html	21-Jul-1997 17:25	3k
propX4.gif	19-May-1997 20:52	1k
propX4.html	21-Jul-1997 17:25	6k
propX40.html	21-Jul-1997 17:25	3k
propX41.gif	19-May-1997 20:52	1k
propX41.html	21-Jul-1997 17:25	8k
propX41l.gif	19-May-1997 20:52	1k
propX42.gif	19-May-1997 20:52	1k
propX42.html	21-Jul-1997 17:25	4k
propX43.html	21-Jul-1997 17:26	3k
propX44.gif	19-May-1997 20:52	1k
propX44.html	21-Jul-1997 17:26	7k
propX45.html	21-Jul-1997 17:26	3k
propX46.html	21-Jul-1997 17:26	3k
propX47.html	21-Jul-1997 17:26	5k
propX48.gif	19-May-1997 20:53	1k
propX48.html	21-Jul-1997 17:26	6k
propX49.gif	19-May-1997 20:53	1k
propX49.html	23-Jul-1997 11:26	6k
propX5.gif	19-May-1997 20:53	1k
propX5.html	23-Jul-1997 11:25	5k
propX50.gif	19-May-1997 20:53	1k
propX50.html	21-Jul-1997 17:26	7k
propX51.gif	19-May-1997 20:53	1k
propX51.html	21-Jul-1997 17:26	5k
propX52.gif	19-May-1997 20:53	1k
propX52.html	23-Jul-1997 11:26	5k
propX53.gif	19-May-1997 20:54	1k
propX53.html	21-Jul-1997 17:26	7k
propX54.gif	19-May-1997 20:54	2k
propX54.html	21-Jul-1997 17:26	13k

[?]	propX54l.gif	19-May-1997 20:54	2k
[?]	propX55.html	21-Jul-1997 17:27	8k
[?]	propX56.html	21-Jul-1997 17:27	5k
[?]	propX57.html	21-Jul-1997 17:27	7k
[?]	propX58.html	21-Jul-1997 17:27	6k
[?]	propX59.html	21-Jul-1997 17:27	6k
[?]	propX6.gif	19-May-1997 20:54	1k
[?]	propX6.html	21-Jul-1997 17:27	7k
[?]	propX60.gif	19-May-1997 20:55	1k
[?]	propX60.html	21-Jul-1997 17:27	10k
[?]	propX60l.gif	19-May-1997 20:55	1k
[?]	propX61.html	21-Jul-1997 17:27	6k
[?]	propX62.html	21-Jul-1997 17:27	6k
[?]	propX63.html	21-Jul-1997 17:27	6k
[?]	propX64.html	21-Jul-1997 17:28	5k
[?]	propX65.html	21-Jul-1997 17:28	5k
[?]	propX66.gif	19-May-1997 20:55	1k
[?]	propX66.html	21-Jul-1997 17:28	7k
[?]	propX67.html	21-Jul-1997 17:28	5k
[?]	propX68.html	21-Jul-1997 17:28	5k
[?]	propX69.html	21-Jul-1997 17:28	3k
[?]	propX7.gif	19-May-1997 20:55	1k
[?]	propX7.html	21-Jul-1997 17:28	3k
[?]	propX70.html	21-Jul-1997 17:28	3k
[?]	propX71.gif	19-May-1997 20:56	2k
[?]	propX71.html	21-Jul-1997 17:28	8k
[?]	propX72.html	21-Jul-1997 17:28	8k
[?]	propX73.gif	19-May-1997 20:56	1k
[?]	propX73.html	21-Jul-1997 17:29	3k
[?]	propX74.html	21-Jul-1997 17:29	3k
[?]	propX75.gif	19-May-1997 20:56	1k
[?]	propX75.html	21-Jul-1997 17:29	6k
[?]	propX76.html	21-Jul-1997 17:29	3k
[?]	propX77.html	21-Jul-1997 17:29	3k
[?]	propX78.html	21-Jul-1997 17:29	5k
[?]	propX79.gif	19-May-1997 20:56	1k
[?]	propX79.html	21-Jul-1997 17:29	4k
[?]	propX8.html	21-Jul-1997 17:29	3k
[?]	propX80.html	21-Jul-1997 17:29	4k
[?]	propX81.gif	19-May-1997 20:56	1k
[?]	propX81.html	21-Jul-1997 17:29	8k
[?]	propX82.html	21-Jul-1997 17:29	3k
[?]	propX83.html	21-Jul-1997 17:29	3k
[?]	propX84.html	21-Jul-1997 17:29	7k

propX85.gif	19-May-1997 20:57	1k	
propX85.html	21-Jul-1997 17:29	5k	
propX86.gif	19-May-1997 20:57	1k	
propX86.html	21-Jul-1997 17:30	5k	
propX87.gif	19-May-1997 20:57	1k	
propX87.html	21-Jul-1997 17:30	8k	
propX88.gif	19-May-1997 20:57	1k	
propX88.html	21-Jul-1997 17:30	5k	
propX89.gif	19-May-1997 20:57	1k	
propX89.html	21-Jul-1997 17:30	5k	
propX9.gif	19-May-1997 20:57	1k	
propX9.html	15-Dec-1998 14:26	9k	
propX9.html%	15-Dec-1998 14:26	8k	
propX90.html	21-Jul-1997 17:30	8k	
propX91.gif	19-May-1997 20:58	3k	
propX91.html	21-Jul-1997 17:30	9k	
propX92.gif	19-May-1997 20:58	3k	
propX92.html	21-Jul-1997 17:30	9k	
propX93.html	21-Jul-1997 17:30	11k	
propX94.html	21-Jul-1997 17:30	9k	
propX95.html	21-Jul-1997 17:31	7k	
propX96.html	21-Jul-1997 17:31	8k	
propX97.gif	19-May-1997 20:58	2k	
propX97.html	21-Jul-1997 17:31	8k	
propX98.html	21-Jul-1997 17:31	7k	
propX99.html	21-Jul-1997 17:31	8k	

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Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
propX99.html	21-Jul-1997 17:31	8k	
propX98.html	21-Jul-1997 17:31	7k	
propX97.html	21-Jul-1997 17:31	8k	
propX97.gif	19-May-1997 20:58	2k	
propX96.html	21-Jul-1997 17:31	8k	
propX95.html	21-Jul-1997 17:31	7k	
propX94.html	21-Jul-1997 17:30	9k	
propX93.html	21-Jul-1997 17:30	11k	
propX92.html	21-Jul-1997 17:30	9k	
propX92.gif	19-May-1997 20:58	3k	
propX91.html	21-Jul-1997 17:30	9k	
propX91.gif	19-May-1997 20:58	3k	
propX90.html	21-Jul-1997 17:30	8k	
propX9.html%	15-Dec-1998 14:26	8k	
propX9.html	15-Dec-1998 14:26	9k	
propX9.gif	19-May-1997 20:57	1k	
propX89.html	21-Jul-1997 17:30	5k	
propX89.gif	19-May-1997 20:57	1k	
propX88.html	21-Jul-1997 17:30	5k	
propX88.gif	19-May-1997 20:57	1k	
propX87.html	21-Jul-1997 17:30	8k	
propX87.gif	19-May-1997 20:57	1k	
propX86.html	21-Jul-1997 17:30	5k	
propX86.gif	19-May-1997 20:57	1k	
propX85.html	21-Jul-1997 17:29	5k	
propX85.gif	19-May-1997 20:57	1k	
propX84.html	21-Jul-1997 17:29	7k	
propX83.html	21-Jul-1997 17:29	3k	
propX82.html	21-Jul-1997 17:29	3k	
propX81.html	21-Jul-1997 17:29	8k	
propX81.gif	19-May-1997 20:56	1k	
propX80.html	21-Jul-1997 17:29	4k	
propX8.html	21-Jul-1997 17:29	3k	
propX79.html	21-Jul-1997 17:29	4k	
propX79.gif	19-May-1997 20:56	1k	
propX78.html	21-Jul-1997 17:29	5k	
propX77.html	21-Jul-1997 17:29	3k	
propX76.html	21-Jul-1997 17:29	3k	

propX75.html	21-Jul-1997	17:29	6k
propX75.gif	19-May-1997	20:56	1k
propX74.html	21-Jul-1997	17:29	3k
propX73.html	21-Jul-1997	17:29	3k
propX73.gif	19-May-1997	20:56	1k
propX72.html	21-Jul-1997	17:28	8k
propX71.html	21-Jul-1997	17:28	8k
propX71.gif	19-May-1997	20:56	2k
propX70.html	21-Jul-1997	17:28	3k
propX7.html	21-Jul-1997	17:28	3k
propX7.gif	19-May-1997	20:55	1k
propX69.html	21-Jul-1997	17:28	3k
propX68.html	21-Jul-1997	17:28	5k
propX67.html	21-Jul-1997	17:28	5k
propX66.html	21-Jul-1997	17:28	7k
propX66.gif	19-May-1997	20:55	1k
propX65.html	21-Jul-1997	17:28	5k
propX64.html	21-Jul-1997	17:28	5k
propX63.html	21-Jul-1997	17:27	6k
propX62.html	21-Jul-1997	17:27	6k
propX61.html	21-Jul-1997	17:27	6k
propX60l.gif	19-May-1997	20:55	1k
propX60.html	21-Jul-1997	17:27	10k
propX60.gif	19-May-1997	20:55	1k
propX6.html	21-Jul-1997	17:27	7k
propX6.gif	19-May-1997	20:54	1k
propX59.html	21-Jul-1997	17:27	6k
propX58.html	21-Jul-1997	17:27	6k
propX57.html	21-Jul-1997	17:27	7k
propX56.html	21-Jul-1997	17:27	5k
propX55.html	21-Jul-1997	17:27	8k
propX54l.gif	19-May-1997	20:54	2k
propX54.html	21-Jul-1997	17:26	13k
propX54.gif	19-May-1997	20:54	2k
propX53.html	21-Jul-1997	17:26	7k
propX53.gif	19-May-1997	20:54	1k
propX52.html	23-Jul-1997	11:26	5k
propX52.gif	19-May-1997	20:53	1k
propX51.html	21-Jul-1997	17:26	5k
propX51.gif	19-May-1997	20:53	1k
propX50.html	21-Jul-1997	17:26	7k
propX50.gif	19-May-1997	20:53	1k
propX5.html	23-Jul-1997	11:25	5k
propX5.gif	19-May-1997	20:53	1k

propX49.html	23-Jul-1997 11:26	6k
propX49.gif	19-May-1997 20:53	1k
propX48.html	21-Jul-1997 17:26	6k
propX48.gif	19-May-1997 20:53	1k
propX47.html	21-Jul-1997 17:26	5k
propX46.html	21-Jul-1997 17:26	3k
propX45.html	21-Jul-1997 17:26	3k
propX44.html	21-Jul-1997 17:26	7k
propX44.gif	19-May-1997 20:52	1k
propX43.html	21-Jul-1997 17:26	3k
propX42.html	21-Jul-1997 17:25	4k
propX42.gif	19-May-1997 20:52	1k
propX41l.gif	19-May-1997 20:52	1k
propX41.html	21-Jul-1997 17:25	8k
propX41.gif	19-May-1997 20:52	1k
propX40.html	21-Jul-1997 17:25	3k
propX4.html	21-Jul-1997 17:25	6k
propX4.gif	19-May-1997 20:52	1k
propX39.html	21-Jul-1997 17:25	3k
propX38.html	21-Jul-1997 17:25	6k
propX38.gif	19-May-1997 20:52	1k
propX37.html	21-Jul-1997 17:25	3k
propX36.html	21-Jul-1997 17:25	3k
propX36.gif	19-May-1997 20:52	1k
propX35.html	21-Jul-1997 17:25	5k
propX34.html	21-Jul-1997 17:25	4k
propX34.gif	19-May-1997 20:51	1k
propX33.html	21-Jul-1997 17:25	9k
propX33.gif	19-May-1997 20:51	1k
propX32.html	21-Jul-1997 17:25	6k
propX31.html	21-Jul-1997 17:25	5k
propX31.gif	19-May-1997 20:51	1k
propX30.html	21-Jul-1997 17:25	4k
propX3.html	21-Jul-1997 17:24	5k
propX3.gif	19-May-1997 20:51	1k
propX2a.gif	10-Oct-2002 09:20	3k
propX29l2.gif	19-May-1997 20:51	1k
propX29l1.gif	19-May-1997 20:51	1k
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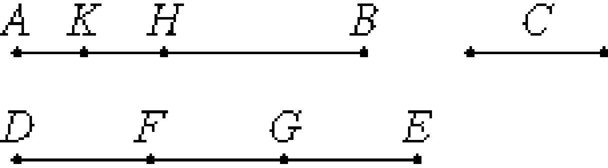
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Euclid's Elements
Book X
Proposition 1

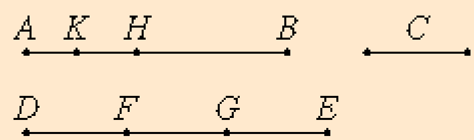
Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out.

Let *AB* and *C* be two unequal magnitudes of which *AB* is the greater.

I say that, if from *AB* there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude which is less than the magnitude *C*.

Some multiple *DE* of *C* is greater than *AB*. cf. V.Def.4

Divide *DE* into the parts *DF*, *FG*, and *GE* equal to *C*. From *AB* subtract *BH* greater than its half, and from *AH* subtract *HK* greater than its half, and repeat this process continually until the divisions in *AB* are equal in multitude with the divisions in *DE*.



Let, then, *AK*, *KH*, and *HB* be divisions equal in multitude with *DF*, *FG*, and *GE*.

Now, since *DE* is greater than *AB*, and from *DE* there has been subtracted *EG* less than its half, and, from *AB*, *BH* greater than its half, therefore the remainder *GD* is greater than the remainder *HA*.

And, since *GD* is greater than *HA*, and there has been subtracted from *GD* the half *GF*, and from *HA*, *HK* greater than its half, therefore the remainder *DF* is greater than the remainder *AK*.

But *DF* equals *C*, therefore *C* is also greater than *AK*. Therefore *AK* is less than *C*.

Therefore there is left of the magnitude *AB* the magnitude *AK* which is less than the lesser magnitude set out, namely *C*.

Therefore, *two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out.*

Q.E.D.

And the theorem can similarly be proven even if the parts subtracted are halves.

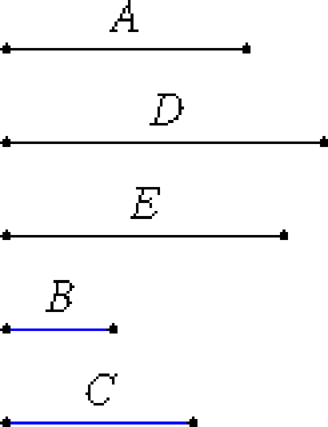
Guide

The proof begins with two magnitudes *C* and *AB* and claims that some multiple of *C* is greater then *AB*. Definition V.Def.4 is not a justification for this statement. Euclid himself proved that a horn angle is less than any rectilinear angle in proposition III.16 and must have recognized that if the magnitude *C* is a horn angle, and the magnitude *AB* is a rectilinear angle, then no multiple of *C* is greater than *AB*. Nonetheless, he did not qualify this proposition to say that it only holds for certain kinds of magnitudes.

Use of this proposition

This proposition is the foundation of the method of exhaustion of Book XII. It is not used in the rest of Book X and would, perhaps, be better placed at the beginning of Book XII. This method is used in the propositions concerning areas of circles and volumes of solids. It is specifically used in propositions XII.2, XII.5, XII.10, XII.11, XII.12, and XII.16.

Book X Introduction - Definitions X.I - Proposition X.2.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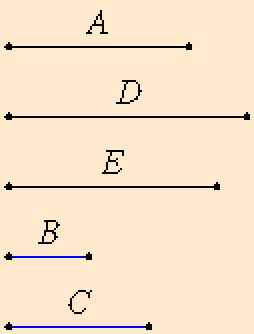
Euclid's Elements
Book X
Proposition 10

To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line.

Let *A* be the assigned straight line.

It is required to find two straight lines incommensurable, the one in length only, and the other in square also, with *A*.

Set out two numbers *B* and *C* which do not have to one another the ratio which a square number has to a square number, that is, which are not similar plane numbers, and let it be contrived that *B* is to *C* as the square on *A* is to the square on *D*, for we have learned how to do this. X.6,Cor.



Therefore the square on *A* is commensurable with the square on *D*. X.6

And, since *B* does not have to *C* the ratio which a square number has to a square number, therefore neither has the square on *A* to the square on *D* the ratio which a square number has to a square number, therefore *A* is incommensurable in length with *D*. X.9

Take a mean proportional *E* between *A* and *D*. Then *A* is to *D* as the square on *A* is to the square on *E*. V.Def.9

But *A* is incommensurable in length with *D*, therefore the square on *A* is also incommensurable with the square on *E*. Therefore *A* is incommensurable in square with *E*. X.11

Therefore two straight lines *D* and *E* have been found incommensurable, *D* in length only, and *E* in square and of course in length also, with the assigned straight line *A*.

Q.E.D.

Guide

This proposition exhibits the lines promised in X.Def.I.3. Just take a line *D* so that the square on *A* to the square on *D* is the ratio of two numbers which are not a square number to a square number. For instance, if *A* is the side of a square and *D* the diagonal of that square, then the square on *A* to the square on *D* is in the ratio 1:2, which is not the ratio of square number to a square number. Therefore *D* is commensurable in square only with *A*. (The ratio *D*:*A* is the square root of 2.)

Next, if *E* is the mean proportional between *A* and *D*, then *E* is incommensurable in square with *A*. (The ratio *E*:*A* is the fourth root of 2.)

It is certain that this proposition is not genuine. For one thing, its proof uses the next proposition. Also, the phrase "for we have learned how to do this" is the sort of thing a student would write. Finally, in the manuscript P (the primary one used by Peyrard and Heiberg) this proposition is not numbered and the next one is numbered 10.

Although not genuine, this proposition ought to be, since it is used in proposition X.27 and others.

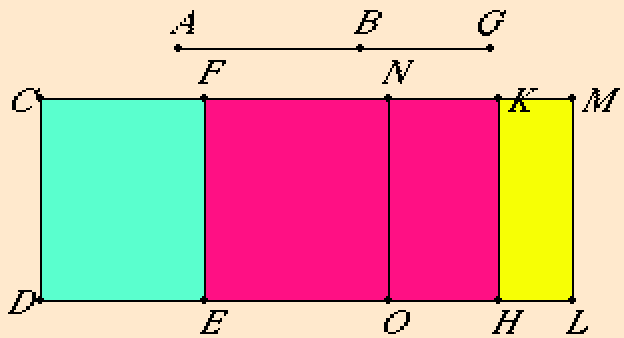
Book X Introduction - Proposition X.9 - Proposition X.11.

Euclid's Elements
Book X
Proposition 100

The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome.

Let *AB* be a minor and *CD* a rational straight line, and to the rational straight line *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth.

I say that *CF* is a fourth apotome.



Let *BG* be the annex to *AB*. Then *AG* and *GB* are straight lines incommensurable in square which make the sum of the squares on *AG* and *GB* rational, but twice the rectangle *AG* by *GB* medial. X.76

To *CD* apply *CH*, equal to the square on *AG*, producing *CK* as breadth, and *KL*, equal to the square on *BG*, producing *KM* as breadth. Then the whole *CL* equals the sum of the squares on *AG* and *GB*.

And the sum of the squares on *AG* and *GB* is rational, therefore *CL* is also rational.

And it is applied to the rational straight line *CD* producing *CM* as breadth, therefore *CM* is also rational and commensurable in length with *CD*. X.20

Since the whole *CL* equals the sum of the squares on *AG* and *GB*, and, in these, *CE* equals the square on *AB*, therefore the remainder *FL* equals twice the rectangle *AG* by *GB*. II.7

Bisect *FM* at the point *N*, and draw *NO* through *N* parallel to either of the straight lines *CD* or *ML*. Then each of the rectangles *FO* and *NL* equals the rectangle *AG* by *GB*.

And, since twice the rectangle *AG* by *GB* is medial and equals *FL*, therefore *FL* is also medial.

And it is applied to the rational straight line *FE* producing *FM* as breadth, therefore *FM* is rational and incommensurable in length with *CD*. X.22

Since the sum of the squares on *AG* and *GB* is rational, while twice the rectangle *AG* by *GB* is medial, therefore the sum of the squares on *AG* and *GB* is incommensurable with twice the rectangle *AG* by *GB*.

But *CL* equals the sum of the squares on *AG* and *GB*, and *FL* equals twice the rectangle *AG* by *GB*, therefore *CL* is incommensurable with *FL*.

But *CL* is to *FL* as *CM* is to *MF*, therefore *CM* is incommensurable in length with *MF*. VI.1 X.11

And both are rational, therefore *CM* and *MF* are rational straight lines commensurable in square only. Therefore *CF* is an apotome. X.73

I say that it is also a fourth apotome.

Since *AG* and *GB* are incommensurable in square, therefore the square on *AG* is also incommensurable with the square on *GB*. And *CH* equals the square on *AG*, and *KL* equal to the square on *GB*, therefore *CH* is incommensurable with *KL*.

But *CH* is to *KL* as *CK* is to *KM*, therefore *CK* is incommensurable in length with *KM*. VI.1 X.11

Since the rectangle *AG* by *GB* is a mean proportional between the squares on *AG* and *GB* the square on *AG* equals *CH*, the square on *GB* equals *KL*, and the rectangle *AG* by *GB* equals *NL*, therefore *NL* is a mean proportional between *CH* and *KL*. Therefore *CH* is to *NL* as *NL* is to *KL*.

But *CH* is to *NL* as *CK* is to *NM*, and *NL* is to *KL* as *NM* is to *KM*, therefore *CK* is to *MN* as *MN* is to *KM*. VI.1 V.11

Therefore the rectangle *CK* by *KM* equals the square on *MN*, that is, to the fourth part of the square on *FM*. VI.17

Since *CM* and *MF* are two unequal straight lines, and the rectangle *CK* by *KM*, equal to the fourth part of the square on *MF* and deficient by a square figure, has been applied to *CM* and divides it into incommensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. X.18

And the whole *CM* is commensurable in length with the rational straight line *CD* set out, therefore *CF* is a fourth apotome. X.Def.III.4

Therefore, *the square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome.*

Q.E.D.

Guide

This proposition is used in X.111.

Book X Introduction - Proposition X.99 - Proposition X.101.

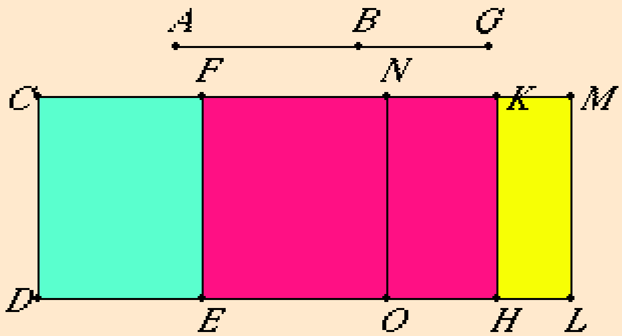
Euclid's Elements
Book X
Proposition 101

The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome.

Let *AB* be the straight line which produces with a rational area a medial whole, and *CD* a rational straight line, and to *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth.

I say that *CF* is a fifth apotome.

Let *BG* be the annex to *AB*. Then *AG* and *GB* are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational. X.77



To *CD* apply *CH* equal to the square on *AG*, and *KL* equal to the square on *GB*. Then the whole *CL* equals the sum of the squares on *AG* and *GB*.

But the sum of the squares on *AG* and *GB* together is medial, therefore *CL* is medial.

And it is applied to the rational straight line *CD* producing *CM* as breadth, therefore *CM* is rational and incommensurable with *CD*. X.22

Since the whole *CL* equals the sum of the squares on *AG* and *GB*, and, in these, *CE* equals the square on *AB*, therefore the remainder *FL* equals twice the rectangle *AG* by *GB*. II.7

Bisect *FM* at *N*, and draw *NO* through *N* parallel to either of the straight lines *CD* or *ML*. Then each of the rectangles *FO* and *NL* equals the rectangle *AG* by *GB*.

And, since twice the rectangle *AG* by *GB* is rational and equal to *FL*, therefore *FL* is rational.

And it is applied to the rational straight line *EF* producing *FM* as breadth, therefore *FM* is rational and commensurable in length with *CD*. X.20

Now, since *CL* is medial, and *FL* rational, therefore *CL* is incommensurable with *FL*.

But *CL* is to *FL* as *CM* is to *MF*, therefore *CM* is incommensurable in length with *MF*. VI.1 X.11

And both are rational, therefore *CM* and *MF* are rational straight lines commensurable in square only. Therefore *CF* is an apotome. X.73

I say next that it is also a fifth apotome.

We can prove similarly that the rectangle *CK* by *KM* equals the square on *NM*, that is, the fourth part of the square on *FM*.

And, since the square on *AG* is incommensurable with the square on *GB*, while the square on *AG* equals *CH*, and the square on *GB* equals *KL*, therefore *CH* is incommensurable with *KL*.

But *CH* is to *KL* as *CK* is to *KM*, therefore *CK* is incommensurable in length with *KM*. VI.1 X.11

Since *CM* and *MF* are two unequal straight lines, and a parallelogram equal to the fourth part of the square on *FM* and deficient by a square figure has been applied to *CM*, and divides it into incommensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. X.18

And the annex *FM* is commensurable with the rational straight line *CD* set out, therefore *CF* is a fifth apotome. X.Def.III.5

Therefore, *the square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome.*

Q.E.D.

Guide

This proposition is used in X.111.

Book X Introduction - Proposition X.100 - Proposition X.102.

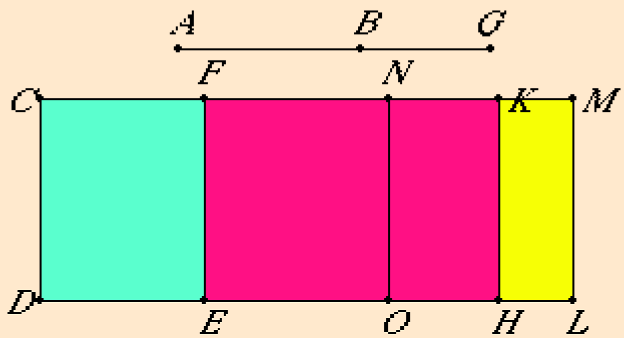
Euclid's Elements
Book X
Proposition 102

The square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome.

Let *AB* be the straight line which produces with a medial area a medial whole, and *CD* a rational straight line, and to *CD* let *CE* be applied equal to the square on *AB* and producing *CF* as breadth.

I say that *CF* is a sixth apotome.

Let *BG* be the annex to *AB*. Then *AG* and *GB* are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle *AG* by *GB* medial, and the sum of the squares on *AG* and *GB* incommensurable with twice the rectangle *AG* by *GB*. X.78



Now to *CD* apply *CH* equal to the square on *AG* and producing *CK* as breadth, and *KL* equal to the square on *BG*. Then the whole *CL* equals the sum of the squares on *AG* and *GB*. Therefore *CL* is also medial.

And it is applied to the rational straight line *CD* producing *CM* as breadth, therefore *CM* is rational and incommensurable in length with *CD*. X.22

Since *CL* equals the sum of the squares on *AG* and *GB*, and, in these, *CE* equals the square on *AB*, therefore the remainder *FL* equals twice the rectangle *AG* by *GB*. And twice the rectangle *AG* by *GB* is medial, therefore *FL* is also medial. II.7

And it is applied to the rational straight line *FE* producing *FM* as breadth, therefore *FM* is rational and incommensurable in length with *CD*. X.22

Since the sum of the squares on *AG* and *GB* is incommensurable with twice the rectangle *AG* by *GB*, *CL* equals the sum of the squares on *AG* and *GB*, and *FL* equals twice the rectangle *AG* by *GB*, therefore *CL* is incommensurable with *FL*.

But *CL* is to *FL* as *CM* is to *MF*, therefore *CM* is incommensurable in length with *MF*. And both are rational. VI.1 X.11

Therefore *CM* and *MF* are rational straight lines commensurable in square only, therefore *CF* is an apotome. X.73

I say next that it is also a sixth apotome.

Since *FL* equals twice the rectangle *AG* by *GB*, bisect *FM* at *N*, and draw *NO* through *N* parallel to *CD*, therefore each of the rectangles *FO* and *NL* equals the rectangle *AG* by *GB*.

And, since *AG* and *GB* are incommensurable in square, therefore the square on *AG* is incommensurable with the square on *GB*.

But *CH* equals the square on *AG*, and *KL* equals the square on *GB*, therefore *CH* is incommensurable with *KL*.

But *CH* is to *KL* as *CK* is to *KM*, therefore *CK* is incommensurable with *KM*. VI.1 X.11

Since the rectangle *AG* by *GB* is a mean proportional between the squares on *AG* and *GB*, *CH* equals the square on *AG*, *KL* equals the square on *GB*, and *NL* equals the rectangle *AG* by *GB*, therefore *NL* is also a mean proportional between *CH* and *KL*. Therefore *CH* is to *NL* as *NL* is to *KL*.

And for the same reason as before the square on *CM* is greater than the square on *MF* by the square on a straight line incommensurable with *CM*. X.18

And neither of them is commensurable with the rational straight line *CD* set out, therefore *CF* is a sixth apotome. X.Def.III.6

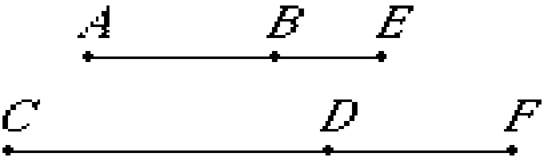
Therefore, *the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome.*

Q.E.D.

Guide

This proposition is used in X.111.

Book X Introduction - Proposition X.101 - Proposition X.103.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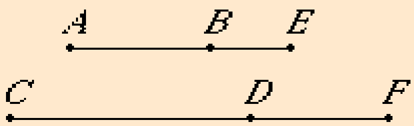
Euclid's Elements
Book X
Proposition 103

A straight line commensurable in length with an apotome is an apotome and the same in order.

Let *AB* be an apotome, *B* and let *CD* be commensurable in length with *AB*.

I say that *CD* is also an apotome and the same in order with *AB*.

Since *AB* is an apotome, let *BE* be the annex to it, therefore *AE* and *EB* are rational straight lines commensurable in square only. X.73



Let it be contrived that the ratio of *BE* to *DF* is the same as the ratio of *AB* to *CD*. Then one is to one as are all to all. Therefore the whole *AE* is to the whole *CF* as *AB* is to *CD*. VI.12 V.12

But *AB* is commensurable in length with *CD*, therefore *AE* is also commensurable with *CF*, and *BE* with *DF*. X.11

And *AE* and *EB* are rational straight lines commensurable in square only, therefore *CF* and *FD* are also rational straight lines commensurable in square only. X.13

Now since *AE* is to *CF* as *BE* is to *DF*, therefore, alternately, *AE* is to *EB* as *CF* is to *FD*. And the square on *AE* is greater than the square on *EB* either by the square on a straight line commensurable with *AE* or by the square on a straight line incommensurable with it. V.16

If then the square on *AE* is greater than the square on *EB* by the square on a straight line commensurable with *AE*, then the square on *CF* is also greater than the square on *FD* by the square on a straight line commensurable with *CF*. X.14

And, if *AE* is commensurable in length with the rational straight line set out, then *CF* is also; if *BE*, then *DF* also; and, if neither of the straight lines *AE* nor *EB*, then neither of the straight lines *CF* nor *FD*. X.12 X.13

But, if the square on *AE* is greater than the square on *EB* by the square on a straight line incommensurable with *AE*, then the square on *CF* is also greater than the square on *FD* by the square on a straight line incommensurable with *CF*. X.14

And, if *AE* is commensurable in length with the rational straight line set out, then *CF* is also; if *BE*, then *DF* also; and, if neither of the straight lines *AE* nor *EB*, then neither of the straight lines *CF* nor *FD*. Therefore *CD* is an apotome and the same in order with *AB*. X.12 X.13

Therefore, *a straight line commensurable in length with an apotome is an apotome and the same in order*.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.102 - Proposition X.104.

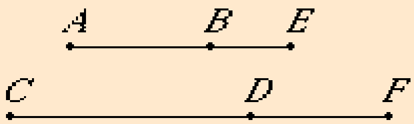
Euclid's Elements
Book X
Proposition 104

A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order.

Let *AB* be an apotome of a medial straight line, and let *CD* be commensurable in length with *AB*.

I say that *CD* is also an apotome of a medial straight line and the same in order with *AB*.

Since *AB* is an apotome of a medial straight line, let *EB* be the annex to it.



Then *AE* and *EB* are medial straight lines commensurable in square only. X.74 X.75

Let it be contrived that *AB* is to *CD* as *BE* is to *DF*. Then *AE* is also commensurable with *CF*, and *BE* with *DF*. VI.12 V.12 X.11

But *AE* and *EB* are medial straight lines commensurable in square only, therefore *CF* and *FD* are also medial straight lines commensurable in square only. X.23 X.13

Therefore *CD* is an apotome of a medial straight line. X.74 X.75

I say next that it is also the same in order with *AB*.

Since *AE* is to *EB* as *CF* is to *FD*, therefore the square on *AE* is to the rectangle *AE* by *EB* as the square on *CF* is to the rectangle *CF* by *FD*.

But the square on *AE* is commensurable with the square on *CF*, therefore the rectangle *AE* by *EB* is also commensurable with the rectangle *CF* by *FD*. V.16 X.11

Therefore, if the rectangle *AE* by *EB* is rational, then the rectangle *CF* by *FD* is also rational, and if the rectangle *AE* by *EB* is medial, the rectangle *CF* by *FD* is also medial. X.Def.4 X.23,Cor.

Therefore *CD* is an apotome of a medial straight line and the same in order with *AB*. X.74 X.75

Therefore, *a straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order.*

Q.E.D.

Guide

(Forthcoming)

Euclid's Elements

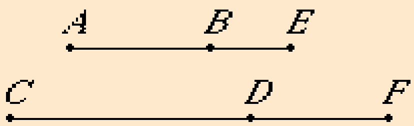
Book X

Proposition 105

A straight line commensurable with a minor straight line is minor.

Let *AB* be a minor straight line, and *CD* commensurable with *AB*.

I say that *CD* is also minor.



Make the same construction as before. Then, since *AE* and *EB* are incommensurable in square, therefore *CF* and *FD* are also incommensurable in square. X.76 X.13

Now since *AE* is to *EB* as *CF* is to *FD*, therefore the square on *AE* is to the square on *EB* as the square on *CF* is to the square on *FD*. V.12 V.16 VI.22

Therefore, taken jointly, the sum of the squares on *AE* and *EB* is to the square on *EB* as the sum of the squares on *CF* and *FD* is to the square on *FD*. V.18

But the square on *BE* is commensurable with the square on *DF*, therefore the sum of the squares on *AE* and *EB* is also commensurable with the sum of the squares on *CF* and *FD*. V.16 X.11

But the sum of the squares on *AE* and *EB* is rational, therefore the sum of the squares on *CF* and *FD* is also rational. X.76 X.Def.4

Again, since the square on *AE* is to the rectangle *AE* by *EB* as the square on *CF* is to the rectangle *CF* by *FD*, while the square on *AE* is commensurable with the square on *CF*, therefore the rectangle *AE* by *EB* is also commensurable with the rectangle *CF* by *FD*.

But the rectangle *AE* by *EB* is medial, therefore the rectangle *CF* by *FD* is also medial. X.76 X23,Cor.

Therefore *CF* and *FD* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial.

Therefore *CD* is minor. X.76

Therefore, *a straight line commensurable with a minor straight line is minor.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.104 - Proposition X.106.

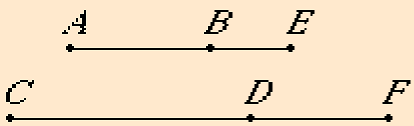
[Clark University]

Euclid's Elements
Book X
Proposition 106

A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole.

Let *AB* be a straight line which produces with a rational area a medial whole, and *CD* commensurable with *AB*.

I say that *CD* is also a straight line which produces with a rational area a medial whole.



Let *BE* be the annex to *AB*, therefore *AE* and *EB* are straight lines incommensurable in square which make the sum of the squares on *AE* and *EB* medial but the rectangle contained by them rational. X.77

Make the same construction.

Then we can prove, in manner similar to the foregoing, that *CF* and *FD* are in the same ratio as *AE* and *EB*, the sum of the squares on *AE* and *EB* is commensurable with the sum of the squares on *CF* and *FD*, and the rectangle *AE* by *EB* is commensurable with the rectangle *CF* by *FD*, so that *CF* and *FD* are also straight lines incommensurable in square which make the sum of the squares on *CF* and *FD* medial but the rectangle contained by them rational.

Therefore *CD* is a straight line which produces with a rational area a medial whole. X.77

Therefore, *a straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole.*

Q.E.D.

Guide

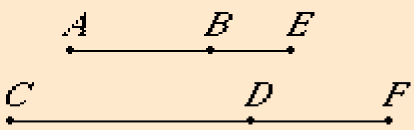
(Forthcoming)

Book X Introduction - Proposition X.105 - Proposition X.107.

Euclid's Elements
Book X
Proposition 107

A straight line commensurable with that which produces a medial area and a medial whole is itself also a straight line which produces with a medial area a medial whole.

Let *AB* be a straight line which produces with a medial area a medial whole, and let *CD* be commensurable with *AB*.



I say that *CD* is also a straight line which produces with a medial area a medial whole.

Let *BE* be the annex to *AB*, and make the same construction.

Then *AE* and *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further, the sum of the squares on them incommensurable with the rectangle contained by them. X.78

Now as was proved, *AE* and *EB* are commensurable with *CF* and *FD*, the sum of the squares on *AE* and *EB* with the sum of the squares on *CF* and *FD*, and the rectangle *AE* by *EB* with the rectangle *CF* by *FD*, therefore *CF* and *FD* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further, the sum of the squares on them incommensurable with the rectangle contained by them.

Therefore *CD* is a straight line which produces with a medial area a medial whole. X.78

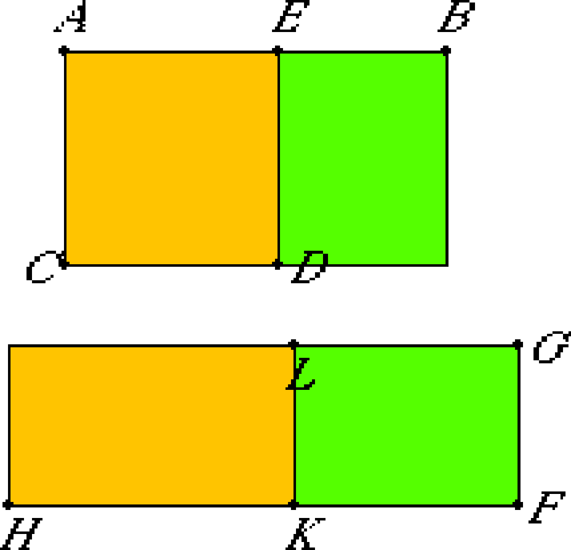
Therefore, *a straight line commensurable with that which produces a medial area and a medial whole is itself also a straight line which produces with a medial area a medial whole.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.106 - Proposition X.108.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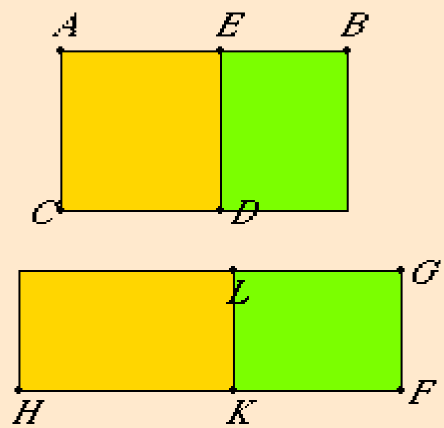
Euclid's Elements
Book X
Proposition 108

If a medial area is subtracted from a rational area, then the side of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line.

Let the medial area *BD* be subtracted from the rational area *BC*.

I say that the side of the remainder *EC* becomes one of two irrational straight lines, either an apotome or a minor straight line.



Set out a rational straight line *FG*, to *FG* apply the rectangular parallelogram *GH* equal to *BC*, and subtract *GK* equal to *DB*. Then the remainder *EC* equals *LH*.

Since, then, *BC* is rational, and *BD* medial, while *BC* equals *GH*, and *BD* equals *GK*, therefore *GH* is rational, and *GK* is medial.

And they are applied to the rational straight line *FG*, therefore *FH* is rational and commensurable in length with *FG*, while *FK* is rational and incommensurable in length with *FG*. Therefore *FH* is incommensurable in length with *FK*. X.20 X.22 X.13

Therefore *FH* and *FK* are rational straight lines commensurable in square only. Therefore *KH* is an apotome, and *KF* the annex to it. X.73

Now the square on *HF* is greater than the square on *FK* by the square on a straight line either commensurable with *HF* or not commensurable.

First, let the square on it be greater by the square on a straight line commensurable with it.

Now the whole *HF* is commensurable in length with the rational straight line *FG* set out, therefore *KH* is a first apotome. X.Def.III.2

But the side of the rectangle contained by a rational straight line and a first apotome is an apotome. Therefore the side of *LH*, that is, of *EC*, is an apotome. X.91

But, if the square on *HF* is greater than the square on *FK* by the square on a straight line incommensurable with *HF*, while the whole *FH* is commensurable in length with the rational straight line *FG* set out, then *KH* is a fourth apotome. X.Def.III.4

But the side of the rectangle contained by a rational straight line and a fourth apotome is minor. X.94

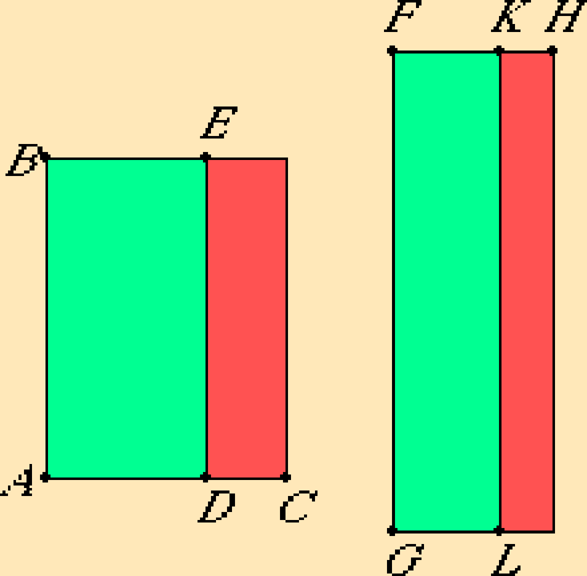
Therefore, *if a medial area is subtracted from a rational area, then the side of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.107 - Proposition X.109.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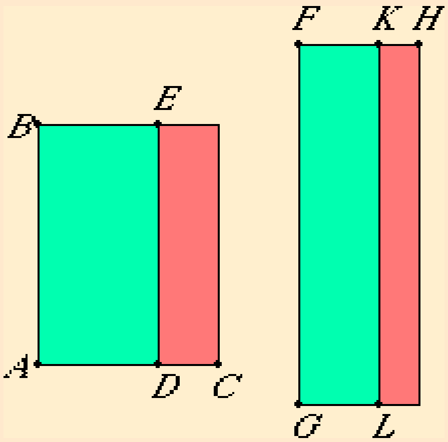
Euclid's Elements
Book X
Proposition 109

If a rational area is subtracted from a medial area, then there arise two other irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.

Let the rational area *BD* be subtracted from the medial area *BC*.

I say that the side of the remainder *EC* becomes one of two irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.



Set out a rational straight line *FG*, and apply the areas similarly. Then *FH* is rational and incommensurable in length with *FG*, while *KF* is rational and commensurable in length with *FG*, therefore *FH* and *FK* are rational straight lines commensurable in square only. X.13

Therefore *KH* is an apotome, and *FK* the annex to it. X.73

Now the square on *HF* is greater than the square on *FK* either by the square on a straight line commensurable with *HF* or by the square on a straight line incommensurable with it.

If the square on *HF* is greater than the square on *FK* by the square on a straight line commensurable with *HF*, while the annex *FK* is commensurable in length with the rational straight line *FG* set out, then *KH* is a second apotome. X.Def.III.2

But *FG* is rational, so that the side of *LH*, that is, of *EC*, is a first apotome of a medial straight line. X.92

But, if the square on *HF* is greater than the square on *FK* by the square on a straight line incommensurable with *HF*, while the annex *FK* is commensurable in length with the rational straight line *FG* set out, then *KH* is a fifth apotome, so that the side of *EC* is a straight line which produces with a rational area a medial whole. X.Def.III.5 X.95

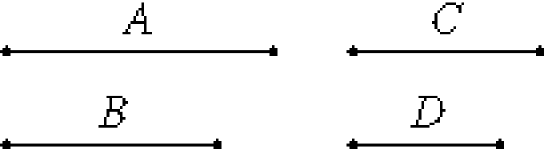
Therefore, *if a rational area is subtracted from a medial area, then there arise two other irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.108 - Proposition X.110.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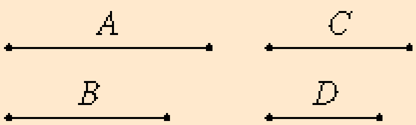
Euclid's Elements
Book X
Proposition 11

If four magnitudes are proportional, and the first is commensurable with the second, then the third also is commensurable with the fourth; but, if the first is incommensurable with the second, then the third also is incommensurable with the fourth.

Let *A*, *B*, *C*, and *D* be four magnitudes in proportion, so that *A* is to *B* as *C* is to *D*, and let *A* be commensurable with *B*.

I say that *C* is also commensurable with *D*.



Since *A* is commensurable with *B*, therefore *A* has to *B* the ratio which a number has to a number. X.5

And *A* is to *B* as *C* is to *D*, therefore *C* also has to *D* the ratio which a number has to a number. Therefore *C* is commensurable with *D*. V.11 X.6

Next, let *A* be incommensurable with *B*.

I say that *C* is also incommensurable with *D*.

Since *A* is incommensurable with *B*, therefore *A* does not have to *B* the ratio which a number has to a number. X.7

And *A* is to *B* as *C* is to *D*, therefore neither has *C* to *D* the ratio which a number has to a number. Therefore *C* is incommensurable with *D*. V.11 X.8

Therefore, *if four magnitudes are proportional, and the first is commensurable with the second, then the third also is commensurable with the fourth; but, if the first is incommensurable with the second, then the third also is incommensurable with the fourth.*

Q.E.D.

Guide

The proof if very direct. If *A*:*B* = *C*:*D*, and the first ratio equals a numeric ratio, then the second equals that, too, but if the first is not a numeric ratio, then neither is the second.

This proposition is used in repeatedly in Book X starting with X.14. It is also used in the previous proposition which was, no doubt, not in the original *Elements*.

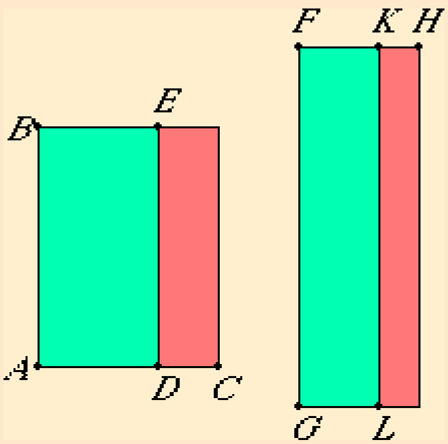
Book X Introduction - Proposition X.10 - Proposition X.12.

Euclid's Elements
Book X
Proposition 110

If a medial area incommensurable with the whole is subtracted from a medial area, then two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

As in the foregoing figures, let there be subtracted the medial area *BD* incommensurable with the whole from the medial area *BC*.

I say that the side of *EC* is one of two irrational straight lines, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.



Since each of the rectangles *BC* and *BD* is medial, and *BC* is incommensurable with *BD*, therefore each of the straight lines *FH* and *FK* is rational and incommensurable in length with *FG*. X.22

Since *BC* is incommensurable with *BD*, that is, *GH* with *GK*, therefore *HF* is also incommensurable with *FK*. VI.1 X.11

Therefore *FH* and *FK* are rational straight lines commensurable in square only. Therefore *KH* is an apotome. X.73

If then the square on *FH* is greater than the square on *FK* by the square on a straight line commensurable with *FH*, while neither of the straight lines *FH* nor *FK* is commensurable in length with the rational straight line *FG* set out, then *KH* is a third apotome. X.Def.III.3

But *KL* is rational, and the rectangle contained by a rational straight line and a third apotome is irrational, and the side of it is irrational, and is called a second apotome of a medial straight line, so that the side of *LH*, that is, of *EC*, is a second apotome of a medial straight line. X.93

But, if the square on *FH* is greater than the square on *FK* by the square on a straight line incommensurable with *FH*, while neither of the straight lines *HF* nor *FK* is commensurable in length with *FG*, then *KH* is a sixth apotome. X.Def.III.6

But the side of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial whole. X.96

Therefore the side of *LH*, that is, of *EC*, is a straight line which produces with a medial area a medial whole.

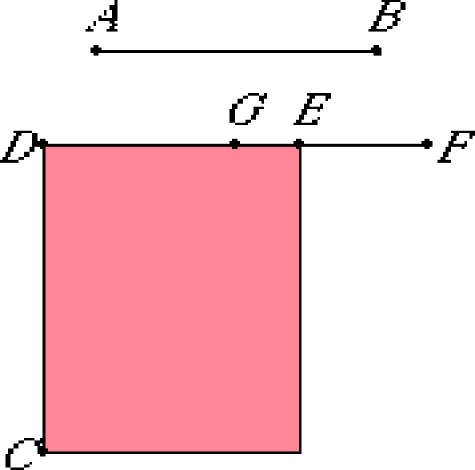
Therefore, *if a medial area incommensurable with the whole is subtracted from a medial area, then two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.109 - Proposition X.111.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Euclid's Elements
Book X
Proposition 111

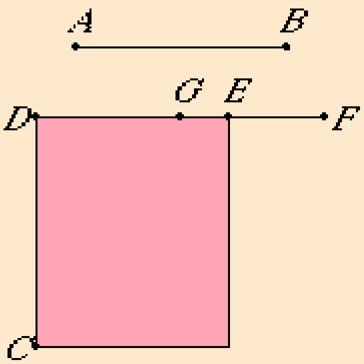
The apotome is not the same with the binomial straight line.

Let *AB* be an apotome.

I say that *AB* is not the same with the binomial straight line.

If possible, let it be so. Set out a rational straight line *DC*, and to *CD* apply the rectangle *CE* equal to the square on *AB* and producing *DE* as breadth.

Then, since *AB* is an apotome, *DE* is a first apotome. X.97



Let *EF* be the annex to it. Then *DF* and *FE* are rational straight lines commensurable in square only, the square on *DF* is greater than the square on *FE* by the square on a straight line commensurable with *DF*, and *DF* is commensurable in length with the rational straight line *DC* set out. X.Def.III.2

Again, since *AB* is binomial, therefore *DE* is a first binomial straight line. X.60

Divide it into its terms at *G*, and let *DG* be the greater term. Then *DG* and *GE* are rational straight lines commensurable in square only, the square on *DG* is greater than the square on *GE* by the square on a straight line commensurable with *DG*, and the greater term *DG* is commensurable in length with the rational straight line *DC* set out. X.Def.II.1

Therefore *DF* is also commensurable in length with *DG*. Therefore the remainder *GF* is also commensurable in length with *DF*. X.12 X.15

But *DF* is incommensurable in length with *EF*, therefore *FG* is also incommensurable in length with *EF*. X.13

Therefore *GF* and *FE* are rational straight lines commensurable in square only, therefore *EG* is an apotome. But it is also rational: which is impossible. X.73

Therefore, *the apotome is not the same with the binomial straight line.*

Q.E.D.

Remark

The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another.

For the square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied, X.22

while the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome, X.97

the square on a first apotome of a medial straight line, if applied to a rational straight line, produces as breadth a second apotome, X.98

the square on a second apotome of a medial straight line, if applied to a rational straight line, produces as breadth a third apotome, X.99

the square on a minor straight line, if applied to a rational straight line, produces as breadth a fourth apotome, X.100

the square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome, X.101

and the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome. X.102

Since the said breadths differ from the first and from one another, from the first because it is rational, and from one another since they are not the same in order, it is clear that the irrational straight lines themselves also differ from one an other.

Since the apotome has been proved not to be the same as the binomial straight line, X.111

but, if applied to a rational straight line, the straight lines following the apotome produce breadths, each according to its own order, apotomes, and those following the binomial straight line themselves also, according to their order, produce the binomials as breadths, therefore those following the apotome are different, and those following the binomial straight line are different, so that there are, in order, thirteen irrational straight lines in all:

Medial

Binomial

First bimedial

Second bimedial

Major

Side of a rational plus a medial area

Side of the sum of two medial areas

Apotome

First apotome of a medial straight line

Second apotome of a medial straight line

Minor

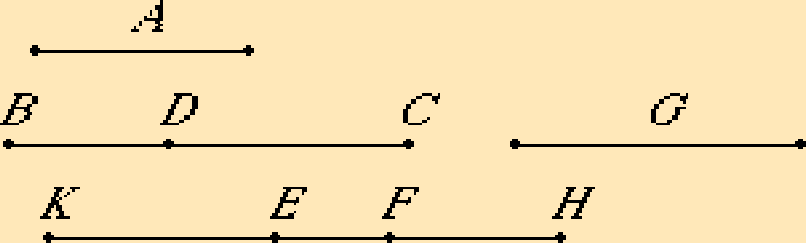
Producing with a rational area a medial whole

Producing with a medial area a medial whole

Guide

(Forthcoming)

Book X Introduction - Proposition X.110 - Proposition X.112.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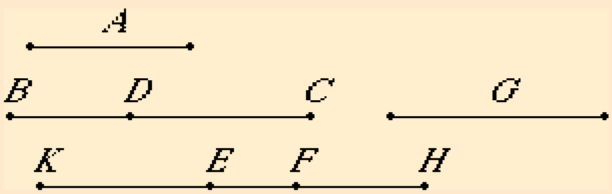
Euclid's Elements
Book X
Proposition 112

The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial straight line and moreover in the same ratio; and further the apotome so arising has the same order as the binomial straight line.

Let *A* be a rational straight line, let *BC* be a binomial, let *DC* be its greater term, and let the rectangle *BC* by *EF* equal the square on *A*.

I say that *EF* is an apotome the terms of which are commensurable with *CD* and *DB*, and in the same ratio, and further, *EF* has the same order as *BC*.

Again let the rectangle *BD* by *G* equal the square on *A*.



Since, then, the rectangle *BC* by *EF* equals the rectangle *BD* by *G*, therefore *CB* is to *BD* as *G* is to *EF*. But *CB* is greater than *BD*, therefore *G* is also greater than *EF*. VI.16 (V.14)

Let *EH* equal *G*. Then *CB* is to *BD* as *HE* is to *EF*, therefore, taken separately, *CD* is to *BD* as *HF* is to *FE*. V.17

Let it be contrived that *HF* is to *FE* as *FK* is to *KE*. Then the whole *HK* is to the whole *KF* as *FK* is to *KE*, for one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents. V.12

But *FK* is to *KE* as *CD* is to *DB*, therefore *HK* is to *KF* as *CD* is to *DB*. V.11

But the square on *CD* is commensurable with the square on *DB*, therefore the square on *HK* is commensurable with the square on *KF*. X.36 VI.22 X.11

And the square on *HK* is to the square on *KF* as *HK* is to *KE*, since the three straight lines *HK*, *KF*, and *KE* are proportional. Therefore *HK* is commensurable in length with *KE*, so that *HE* is also commensurable in length with *EK*. V.Def.9 X.15

Now, since the square on *A* equals the rectangle *EH* by *BD*, while the square on *A* is rational, therefore the rectangle *EH* by *BD* is also rational.

And it is applied to the rational straight line *BD*, therefore *EH* is rational and commensurable in length with *BD*, so that *EK*, being commensurable with it, is also rational and commensurable in length with *BD*. X.20

Since, then *CD* is to *DB* as *FK* is to *KE*, while *CD* and *DB* are straight lines commensurable in square only, therefore *FK* and *KE* are also commensurable in square only. But *KE* is rational, therefore *FK* is also rational. X.11

Therefore *FK* and *KE* are rational straight lines commensurable in square only, therefore *EF* is an apotome. X.73

Now the square on *CD* is greater than the square on *DB* either by the square on a straight line commensurable with *CD* or by the square on a straight line incommensurable with it.

If the square on *CD* is greater than the square on *DB* by the square on a straight line commensurable with *CD*, then the square on *FK* is also greater than the square on *KE* by the square on a straight line commensurable with *FK*. X.14

And, if *CD* is commensurable in length with the rational straight line set out, then *FK* is also; if *BD* is so commensurable, then *KE* is also; but, if neither of the straight lines *CD* nor *DB* is so commensurable, then neither of the straight lines *FK* nor *KE* is so. X.11 X.12

But, if the square on *CD* is greater than the square on *DB* by the square on a straight line incommensurable with *CD*, then the square on *FK* is also greater than the square on *KE* by the square on a straight line incommensurable with *FK*. X.14

And, if *CD* is commensurable with the rational straight line set out, then *FK* is also; if *BD* is so commensurable, then *KE* is also; but, if neither of the straight lines *CD* nor *DB* is so commensurable, then neither of the straight lines *FK* nor *KE* is so, so that *FE* is an apotome, the terms of which, *FK* and *KE* are commensurable with the terms *CD* and *DB* of the binomial straight line and in the same ratio, and it has the same order as *BC*.

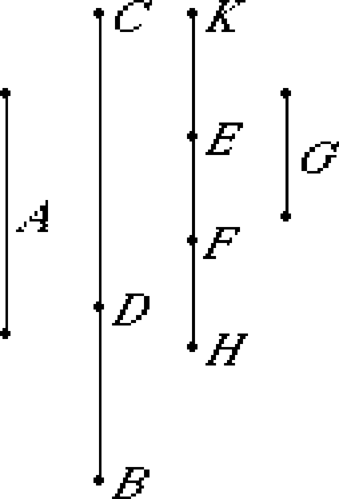
Therefore, *the square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial straight line and moreover in the same ratio; and further the apotome so arising has the same order as the binomial straight line.*

Q.E.D.

Guide

Note that it isn't proposition V.14 being invoked near the beginning of the proof, but an alternate form of it. See the Guide to V.14.

Book X Introduction - Proposition X.111 - Proposition X.113.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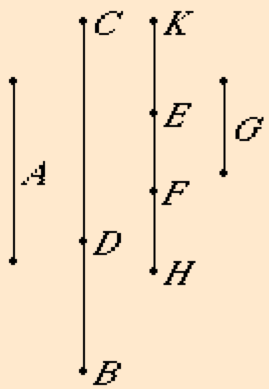
Euclid's Elements
Book X
Proposition 113

But *BC* is greater than *BD*, therefore *KH* is also greater than *G*.

The square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.

Let *A* be a rational straight line and *BD* an apotome, and let the rectangle *BD* by *KH* equal the square on *A*, so that the square on the rational straight line *A* when applied to the apotome *BD* produces *KH* as breadth.

I say that *KH* is a binomial straight line the terms of which are commensurable with the terms of *BD* and in the same ratio, and further, *KH* has the same order as *BD*.



Let *DC* be the annex to *BD*. Then *BC* and *CD* are rational straight lines commensurable in square only. Let the rectangle *BC* by *G* also equal the square on *A*. X.73

But the square on *A* is rational, therefore the rectangle *BC* by *G* is also rational. And it has been applied to the rational straight line *BC*, therefore *G* is rational and commensurable in length with *BC*. X.20

Since now the rectangle *BC* by *G* equals the rectangle *BD* by *KH*, therefore, *CB* is to *BD* as *KH* is to *G*. But *BC* is greater than *BD*, therefore *KH* is also greater than *G*. VI.16 (V.14)

Make *KE* equal to *G*. Then *KE* is commensurable in length with *BC*.

Since *CB* is to *BD* as *HK* is to *KE*, therefore, in conversion, *BC* is to *CD* as *KH* is to *HE*. V.19,Cor.

Let it be contrived that *KH* is to *HE* as *HF* is to *FE*. Then the remainder *KF* is to *FH* as *KH* is to *HE*, that is *BC* is to *CD*. V.19

But *BC* and *CD* are commensurable in square only, therefore *KF* and *FH* are also commensurable in square only. V.11

Since *KH* is to *HE* as *KF* is to *FH*, while *KH* is to *HE* as *HF* is to *FE*, therefore *KF* is to *FH* as *HF* is to *FE*, so that also the first is to the third as the square on the first to the square on the second. Therefore *KF* is to *FE* as the square on *KF* is to the square on *FH*. V.11 V.Def.9

But the square on *KF* is commensurable with the square on *FH*, for *KF* and *FH* are commensurable in square, therefore *KF* is also commensurable in length with *FE*, so that *KF* is also commensurable in length with *KE*. X.11 X.15

But *KE* is rational and commensurable in length with *BC*, therefore *KF* is also rational and commensurable in length with *BC*. X.12

Since *BC* is to *CD* as *KF* is to *FH*, alternately, *BC* is to *KF* as *DC* is to *FH*. V.16

But *BC* is commensurable with *KF*, therefore *FH* is also commensurable in length with *CD*. X.11

But *BC* and *CD* are rational straight lines commensurable in square only, therefore *KF* and *FH* are also rational straight lines commensurable in square only. Therefore *KH* is binomial. X.Def.3 X.36

If now the square on *BC* is greater than the square on *CD* by the square on a straight line commensurable with *BC*, then the square on *KF* is also greater than the square on *FH* by the square on a straight line commensurable with *KF*. X.14

And, if *BC* is commensurable in length with the rational straight line set out, then *KF* is also; if *CD* is commensurable in length with the rational straight line set out, then *FH* is also; but, if neither of the straight lines *BC* nor *CD*, then neither of the straight lines *KF* nor *FH*.

But, if the square on *BC* is greater than the square on *CD* by the square on a straight line incommensurable with *BC*, then the square on *KF* is also greater than the square on *FH* by the square on a straight line incommensurable with *KF*. X.14

And, if *BC* is commensurable with the rational straight line set out, then *KF* is also; if *CD* is so commensurable, then *FH* is also; but, if neither of the straight lines *BC* nor *CD*, then neither of the straight lines *KF* nor *FH*.

Therefore *KH* is a binomial straight line, the terms of which *KF* and *FH* are commensurable with the terms *BC* and *CD* of the apotome and in the same ratio, and further, *KH* has the same order as *BD*.

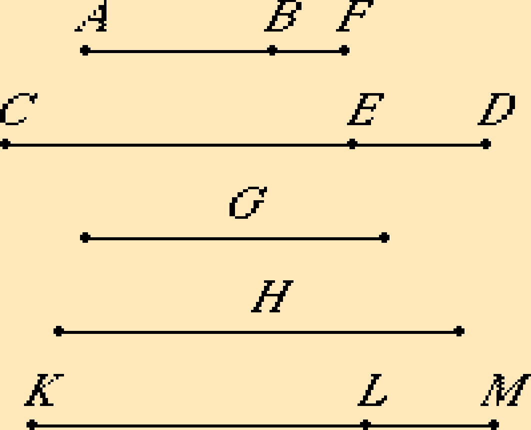
Therefore, *the square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.112 - Proposition X.114.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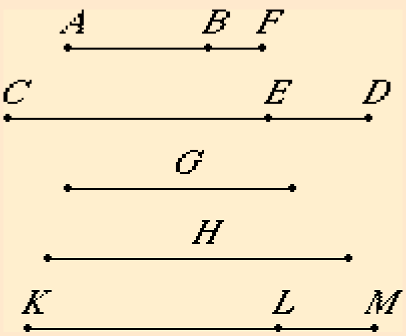
Euclid's Elements
Book X
Proposition 114

If an area is contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, then the side of the area is rational.

Let an area, the rectangle *AB* by *CD*, be contained by the apotome *AB* and the binomial straight line *CD*, and let *CE* be the greater term of the latter, let the terms *CE* and *ED* of the binomial straight line be commensurable with the terms *AF* and *FB* of the apotome and in the same ratio, and let the side of the rectangle *AB* by *CD* be *G*.

I say that *G* is rational.

Set out a rational straight line *H*, and to *CD* apply a rectangle equal to the square on *H* and producing *KL* as breadth. Then *KL* is an apotome.



Let its terms be *KM* and *ML* commensurable with the terms *CE* and *ED* of the binomial straight line and in the same ratio. X.112

But *CE* and *ED* are also commensurable with *AF* and *FB* and in the same ratio, therefore *AF* is to *FB* as *KM* is to *ML*.

Therefore, alternately, *AF* is to *KM* as *BF* is to *LM*. Therefore the remainder *AB* is to the remainder *KL* as *AF* is to *KM*. V.19

But *AF* is commensurable with *KM*, therefore *AB* is also commensurable with *KL*. X.12 X.11

And *AB* is to *KL* as the rectangle *CD* by *AB* is to the rectangle *CD* by *KL*, therefore the rectangle *CD* by *AB* is also commensurable with the rectangle *CD* by *KL*. VI.1 X.11

But the rectangle *CD* by *KL* equals the square on *H*, therefore the rectangle *CD* by *AB* is commensurable with the square on *H*.

But the square on *G* equals the rectangle *CD* by *AB*, therefore the square on *G* is commensurable with the square on *H*.

But the square on *H* is rational, therefore the square on *G* is also rational.

Therefore *G* is rational. And it is the side of the rectangle *CD* by *AB*.

Therefore, *if an area is contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, then the side of the area is rational.*

Corollary.

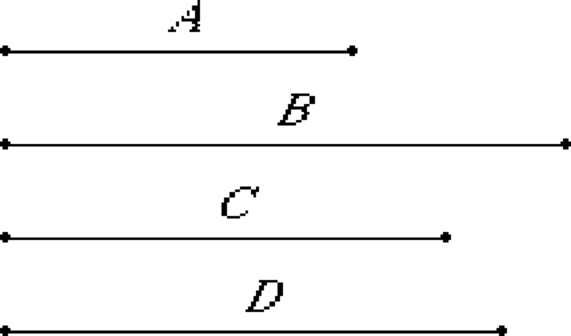
And it is made manifest to us by this also that *it is possible for a rational area to be contained by irrational straight lines*.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.113 - Proposition X.115.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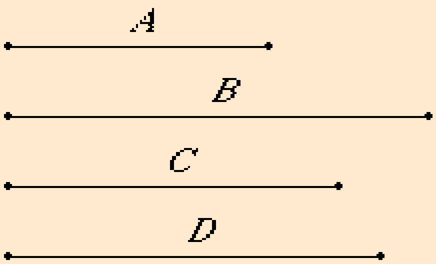
Euclid's Elements
Book X
Proposition 115

From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any preceding.

Let *A* be a medial straight line.

I say that from *A* there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.



Set out a rational straight line *B*, and let the square on *C* equal the rectangle *B* by *A*. Then *C* is irrational, for that which is contained by an irrational and a rational straight line is irrational. X.Def.4 X.20

And it is not the same with any of the preceding, for the square on none of the preceding, if applied to a rational straight line will produce as breadth a medial straight line.

Again, let the square on *D* equal the rectangle *B* by *C*. Then the square on *D* is irrational. X.20

Therefore *D* is irrational, and it is not the same with any of the preceding, for the square on none of the preceding, if applied to a rational straight line, will produce *C* as breadth. X.Def.4

Similarly, if this arrangement proceeds *ad infinitum*, it is manifest that from the medial straight line there arise irrational straight lines infinite in number, and none is the same with any of the preceding.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.114 - Book XI Introduction.

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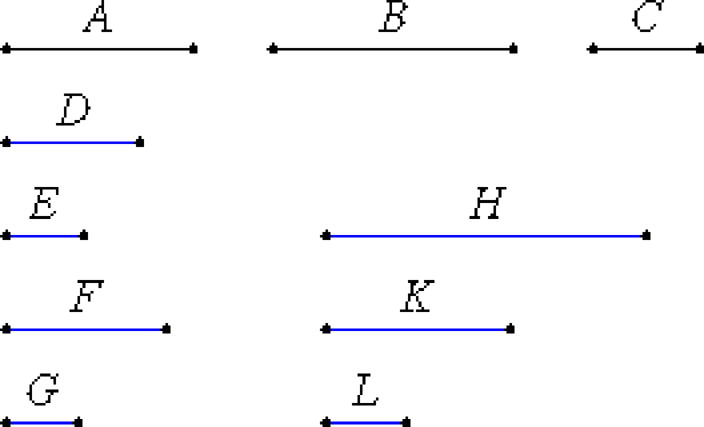
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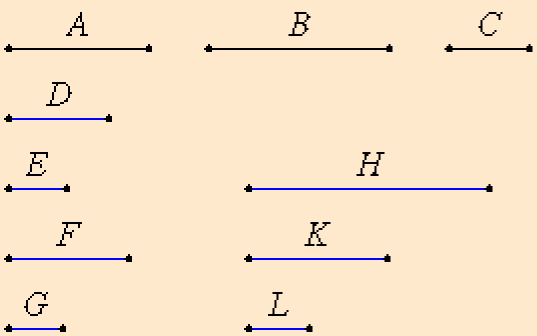
Euclid's Elements
Book X
Proposition 12

Magnitudes commensurable with the same magnitude are also commensurable with one another.

Let each of the magnitudes *A* and *B* be commensurable with *C*.

I say that *A* is also commensurable with *B*.



Since *A* is commensurable with *C*, therefore *A* has to *C* the ratio which a number has to a number. Let it have the ratio which *D* has to *E*. Again, since *C* is commensurable with *B*, therefore *C* has to *B* the ratio which a number has to a number. Let it have the ratio which *F* has to *G*. X.5

And, given any number of ratios we please, namely the ratio which *D* has to *E* and that which *F* has to *G*, take the numbers *H*, *K*, and *L* continuously in the given ratios, so that *D* is to *E* as *H* is to *K*, and *F* is to *G* as *K* is to *L*. VIII.4

Since *A* is to *C* as *D* is to *E*, while *D* is to *E* as *H* is to *K*, therefore *A* is to *C* as *H* is to *K*. Again, since *C* is to *B* as *F* is to *G*, while *F* is to *G* as *K* is to *L*, therefore *C* is to *B* as *K* is to *L*. V.11

But *A* is to *C* as *H* is to *K*, therefore, *ex aequali*, *A* is to *B* as *H* is to *L*. V.22

Therefore *A* has to *B* the ratio which a number has to a number. Therefore *A* is commensurable with *B*. X.6

Therefore, *magnitudes commensurable with the same magnitude are also commensurable with one another*.

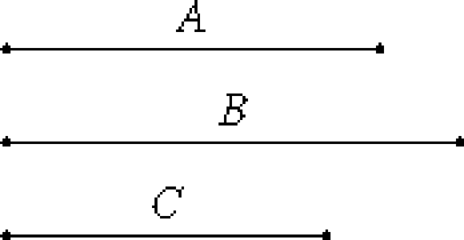
Q.E.D.

Guide

The proof is primarily an application of VIII.4.

This proposition is used in frequently in Book X starting with the next proposition. It is also used in XIII.11.

Book X Introduction - Proposition X.11 - Proposition X.13.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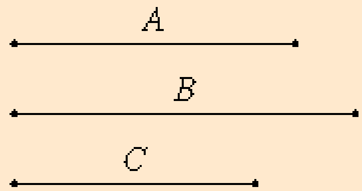
Euclid's Elements
Book X
Proposition 13

If two magnitudes are commensurable, and one of them is incommensurable with any magnitude, then the remaining one is also incommensurable with the same.



Let *A* and *B* be two commensurable magnitudes, and let one of them, *A*, be incommensurable with some other magnitude *C*.

I say that the remaining one, *B*, is also incommensurable with *C*.

If *B* is commensurable with *C*, while *A* is also commensurable with *B*, then *A* is also commensurable with *C*. X.12

But it is also incommensurable with it, which is impossible. Therefore *B* is not commensurable with *C*. Therefore it is incommensurable with it.

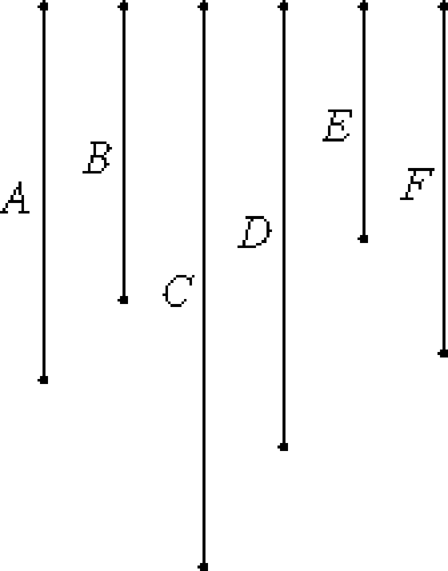
Therefore, *if two magnitudes are commensurable, and one of them is incommensurable with any magnitude, then the remaining one is also incommensurable with the same.*

Q.E.D.

Guide

The proposition is a logical variant of the previous. It is used in very frequently in Book X starting with X.18.

Book X Introduction - Proposition X.12 - Proposition X.14.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Euclid's Elements

Book X

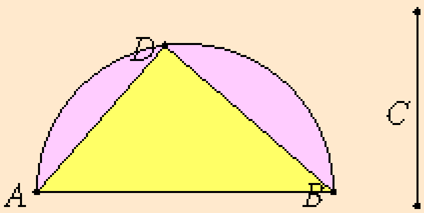
Proposition 14

Lemma.

Given two unequal straight lines, to find by what square the square on the greater is greater than the square on the less.

Let *AB* and *C* be the given two unequal straight lines, and let *AB* be the greater of them.

It is required to find by what square the square on *AB* is greater than the square on *C*.



Describe the semicircle *ADK* on *AB*, fit *AD* into it equal to *C*, and join *DB*. IV.1

It is then manifest that the angle *ADB* is right, and that the square on *AB* is greater than the square on *AD*, that is, *C*, by the square on *DB*. III.31 I.47

Similarly also, *if two straight lines are given, then the straight line the square on which equals the sum of the squares on them is found in this manner.*

Let *AD* and *DB* be the given two straight lines, and let it be required to find the straight line the square on which equals the sum of the squares on them.

Place them so as to contain a right angle *ADB*, and join *AB*.

It is again manifest that the straight line the square on which equals the sum of the squares on *AD* and *DB* is *AB*. I.47

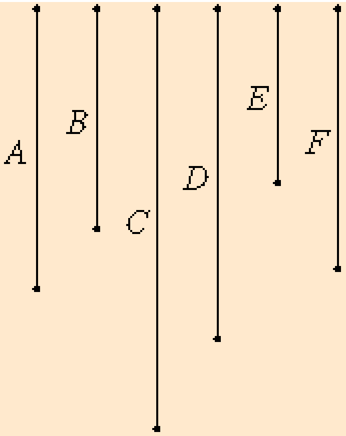
Proposition 14

If four straight lines are proportional, and the square on the first is greater than the square on the second by the square on a straight line commensurable with the first, then the square on the third is also greater than the square on the fourth by the square on a third line commensurable with the third. And, if the square on the first is greater than the square on the second by the square on a straight line incommensurable with the first, then the square on the third is also greater than the square on the fourth by the square on a third line incommensurable with the third.

Let *A*, *B*, *C*, and *D* be four straight lines in proportion, so that *A* is to *B* as *C* is to *D*, and let the square on *A* be greater than the square on *B* by the square on *E*, and let the square on *C* be greater than the square on *D* by the square on *F*. Lemma

I say that, if *A* is commensurable with *E*, then *C* is also commensurable with *F*, and, if *A* is incommensurable with *E*, then *C* is also incommensurable with *F*.

Since *A* is to *B* as *C* is to *D*, therefore the square on *A* is to the square on *B* as the square on *C* is to the square on *D*. VI.22



But the sum of the squares on *E* and *B* equals the square on *A*, and the sum of the squares on *D* and *F* equals the square on *C*. Therefore the sum of the squares on *E* and *B* is to the square on *B* as the sum of the squares on *D* and *F* is to the square on *D*.

Therefore, taken separately, the square on *E* is to the square on *B* as the square on *F* is to the square on *D*. Therefore *E* is to *B* as *F* is to *D*. Therefore, inversely, *B* is to *E* as *D* is to *F*. V.17 VI.22 V.7.Cor

But *A* is to *B* as *C* is to *D*, therefore, *ex aequali*, *A* is to *E* as *C* is to *F*. V.22

Therefore, if *A* is commensurable with *E*, then *C* is also commensurable with *F*, but if *A* is incommensurable with *E*, then *C* is also incommensurable with *F*. X.11

Therefore, *if four straight lines are proportional, and the square on the first is greater than the square on the second by the square on a straight line commensurable with the first, then the square on the third is also greater than the square on the fourth by the square on a third line commensurable with the third. And, if the square on the first is greater than the square on the second by the square on a straight line incommensurable with the first, then the square on the third is also greater than the square on the fourth by the square on a third line incommensurable with the third.*

Q.E.D.

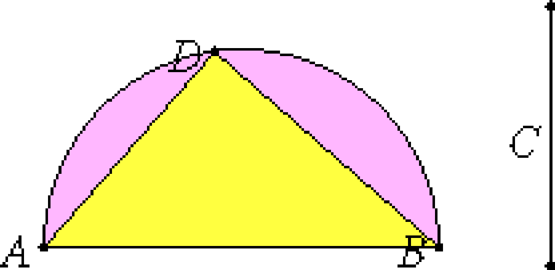
Guide

A little modern algebra clarifies the situation. We assume $A:B = C:D$. Then if $\sqrt{(A^2 - B^2)} : A$ is a numeric ratio, then so is $\sqrt{(C^2 - D^2)} : C$. It's simply because $\sqrt{(A^2 - B^2)} : A = \sqrt{(C^2 - D^2)} : C$.

The lemma is the same as the lemma for proposition XI.23.

The proposition is used in several propositions in Book X starting with X.31.

Book X Introduction - Proposition X.13 - Proposition X.15.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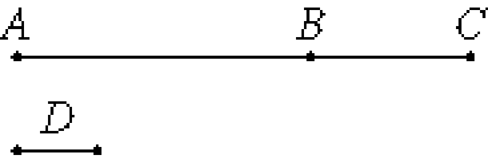
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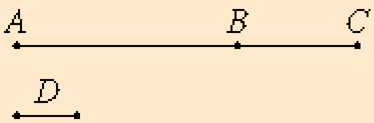
Euclid's Elements
Book X
Proposition 15

If two commensurable magnitudes are added together, then the whole is also commensurable with each of them; and, if the whole is commensurable with one of them, then the original magnitudes are also commensurable.

Let the two commensurable magnitudes *AB* and *BC* be added together.

I say that the whole *AC* is also commensurable with each of the magnitudes *AB* and *BC*.

Since *AB* and *BC* are commensurable, some magnitude *D* measures them.



Since then *D* measures *AB* and *BC*, therefore it also measures the whole *AC*. But it measures *AB* and *BC* also, therefore *D* measures *AB*, *BC*, and *AC*. X.Def.1
Therefore *AC* is commensurable with each of the magnitudes *AB* and *BC*.

Next, let *AC* be commensurable with *AB*.

I say that *AB* and *BC* are also commensurable.

Since *AC* and *AB* are commensurable, some magnitude *D* measures them.

Since then *D* measures *CA* and *AB*, therefore it also measures the remainder *BC*.

But it measures *AB* also, therefore *D* measures *AB* and *BC*. Therefore *AB* and *BC* are commensurable. X.Def.1

Therefore, *if two commensurable magnitudes are added together, then the whole is also commensurable with each of them; and, if the whole is commensurable with one of them, then the original magnitudes are also commensurable.*

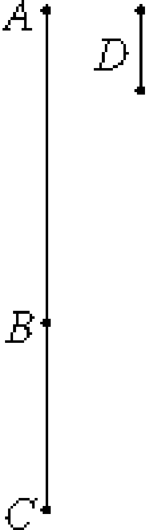
Q.E.D.

Guide

This fundamental proposition on commensurability of sums and differences is used in very frequently in Book X starting with X.17. It is also used in XIII.11.

Book X Introduction - Proposition X.14 - Proposition X.16.

D.E.Joyce
Clark University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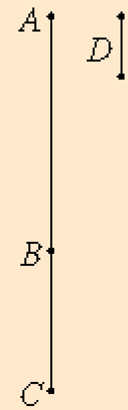
Euclid's Elements
Book X
Proposition 16

If two incommensurable magnitudes are added together, the sum is also incommensurable with each of them; but, if the sum is incommensurable with one of them, then the original magnitudes are also incommensurable.

Let the two incommensurable magnitudes *AB* and *BC* be added together.



I say that the whole *AC* is also incommensurable with each of the magnitudes *AB* and *BC*.

For, if *CA* and *AB* are not incommensurable, then some magnitude *D* measures them.

Since then *D* measures *CA* and *AB*, therefore it also measures the remainder *BC*. But it also measures *AB*, therefore *D* measures *AB* and *BC*. Therefore *AB* and *BC* are commensurable, but they were also, by hypothesis, incommensurable, which is impossible.

Therefore no magnitude measures *CA* and *AB*. Therefore *CA* and *AB* are incommensurable. X.Def.1

Similarly we can prove that *AC* and *CB* are also incommensurable. Therefore *AC* is incommensurable with each of the magnitudes *AB* and *BC*.

Next, let *AC* be incommensurable with one of the magnitudes *AB* or *BC*.

First, let it be incommensurable with *AB*.

I say that *AB* and *BC* are also incommensurable.

For, if they are commensurable, then some magnitude *D* measures them.

Since, then, *D* measures *AB* and *BC*, therefore it also measures the whole *AC*. But it also measures *AB*, therefore *D* measures *CA* and *AB*. Therefore *CA* and *AB* are commensurable, but they were also, by hypothesis, incommensurable, which is impossible.

Therefore no magnitude measures *AB* and *BC*. Therefore *AB* and *BC* are incommensurable. X.Def.1

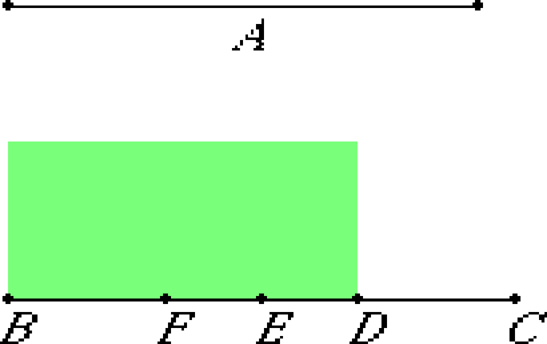
Therefore, *if two incommensurable magnitudes are added together, the sum is also incommensurable with each of them; but, if the sum is incommensurable with one of them, then the original magnitudes are also incommensurable.*

Q.E.D.

Guide

This proposition is a logical variant of the previous one, but it is proved afresh. It is used in several others in Book X starting with X.18.

Book X Introduction - Proposition X.15 - Proposition X.17.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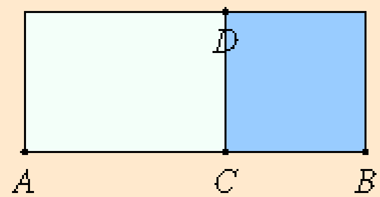
Euclid's Elements
Book X
Proposition 17

Lemma.

If to any straight line there is applied a parallelogram but falling short by a square, then the applied parallelogram equals the rectangle contained by the segments of the straight line resulting from the application.



Apply to the straight line *AB* the parallelogram *AD* but falling short by the square *DB*.

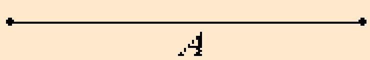
I say that *AD* equals the rectangle *AC* by *CB*.

This is indeed at once manifest, for, since *DB* is a square, *DC* equals *CB*, and *AD* is the rectangle *AC* by *CD*, that is, the rectangle *AC* by *CB*.

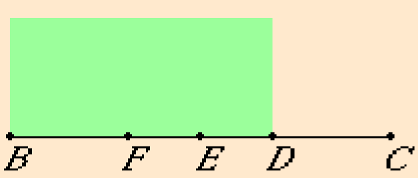
Proposition 17

If there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less minus a square figure, and if it divides it into parts commensurable in length, then the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater. And if the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater, and if there is applied to the greater a parallelogram equal to the fourth part of the square on the less minus a square figure, then it divides it into parts commensurable in length.

Let *A* and *BC* be two unequal straight lines, of which *BC* is the greater, and let there be applied to *BC* a parallelogram equal to the fourth part of the square on the less, *A*, that is, equal to the square on the half of *A* but falling short by a square figure. Let this be the rectangle *BD* by *DC*, and let *BD* be commensurable in length with *DC*. Lemma



I say that the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.



Bisect *BC* at the point *E*, and make *EF* equal to *DE*. I.10 I.3

Therefore the remainder *DC* equals *BF*. And, since the straight line *BC* was cut into equal parts at *E*, and into unequal parts at *D*, therefore the rectangle *BD* by *DC*, together with the square on *ED*, equals the square on *EC*. II.5

And the same is true of their quadruples, therefore four times the rectangle *BD* by *DC*, together with four times the square on *DE*, equals four times the square on *EC*.

But the square on *A* equals four times the rectangle *BD* by *DC*, and the square on *DF* equals four times the square on *DE*, for *DF* is double *DE*. And the square on *BC* equals four times the square on *EC*, for again *BC* is double *CE*.

Therefore the sum of the squares on *A* and *DF* equals the square on *BC*, so that the square on *BC* is greater than the square on *A* by the square on *DF*.

It is to be proved that *BC* is also commensurable with *DF*.

Since *BD* is commensurable in length with *DC*, therefore *BC* is also commensurable in length with *CD*. X.15

But *CD* is commensurable in length with *CD* and *BF*, for *CD* equals *BF*. X.6

Therefore *BC* is also commensurable in length with *BF* and *CD*, so that *BC* is also commensurable in length with the remainder *FD*. Therefore the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*. X.12 X.15

Next, let the square on *BC* be greater than the square on *A* by the square on a straight line commensurable with *BC*. Apply to *BC* a parallelogram equal to the fourth part of the square on *A* but falling short by a square figure, and let it be the rectangle *BD* by *DC*.

It is to be proved that *BD* is commensurable in length with *DC*.

With the same construction, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*. X.15

But the square on *BC* is greater than the square on *A* by the square on a straight line commensurable with *BC*.

Therefore *BC* is commensurable in length with *FD*, so that *BC* is also commensurable in length with the remainder, the sum of *BF* and *DC*.

But the sum of *BF* and *DC* is commensurable with *DC*, so that *BC* is also commensurable in length with *CD*, and therefore, taken separately, *BD* is commensurable in length with *DC*. X.6 X.12 X.15

Therefore, *if there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less minus a square figure, and if it divides it into parts commensurable in length, then the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater. And if the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater, and if there is applied to the greater a parallelogram equal to the fourth part of the square on the less minus a square figure, then it divides it into parts commensurable in length.*

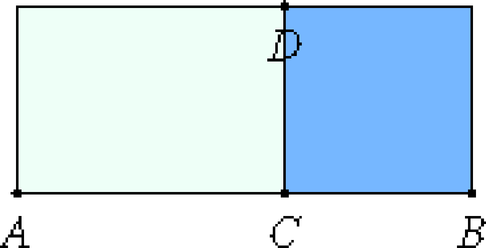
Q.E.D.

Guide

Here is an algebraic description. Let b denote *BC*. Then *DC* is $(b - \sqrt{(b^2 - A^2)})/2$. Then the proposition asserts that the ratio $b : (b - \sqrt{(b^2 - A^2)})/2$ is a numeric ratio if and only if the ratio $\sqrt{(b^2 - A^2)} : A$ is a numeric ratio.

The lemma is also used in the next proposition. The proposition is used in several times in Book X starting with X.54.

Book X Introduction - Proposition X.16 - Proposition X.18.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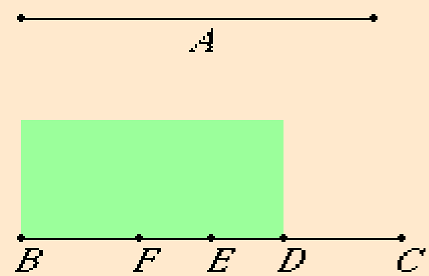
Euclid's Elements
Book X
Proposition 18

If there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less but falling short by a square, and if it divides it into incommensurable parts, then the square on the greater is greater than the square on the less by the square on a straight line incommensurable with the greater. And if the square on the greater is greater than the square on the less by the square on a straight line incommensurable with the greater, and if there is applied to the greater a parallelogram equal to the fourth part of the square on the less but falling short by a square, then it divides it into incommensurable parts.

Let *A* and *BC* be two unequal straight lines, of which *BC* is the greater, and to *BC* let there be applied a parallelogram equal to the fourth part of the square on the less, *A*, but falling short by a square. Let this be the rectangle *BD* by *DC*, and let *BD* be incommensurable in length with *DC*. X.17,Lemma



I say that the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*.

With the same construction as before, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

It is to be proved that *BC* is incommensurable in length with *DF*.

Since *BD* is incommensurable in length with *DC*, therefore *BC* is also incommensurable in length with *CD*. X.16

But *DC* is commensurable with the sum of *BF* and *DC*, therefore *BC* is incommensurable with the sum of *BF* and *DC*, so that *BC* is also incommensurable in length with the remainder *FD*. X.6 X.13 X.16

And the square on *BC* is greater than the square on *A* by the square on *FD*, therefore the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*.

Next, let the square on *BC* be greater than the square on *A* by the square on a straight line incommensurable with *BC*. Apply to *BC* a parallelogram equal to the fourth part of the square on *A* but falling short by a square. Let this be the rectangle *BD* by *DC*.

It is to be proved that *BD* is incommensurable in length with *DC*.

With the same construction, we can prove similarly that the square on *BC* is greater than the square on *A* by the square on *FD*.

But the square on *BC* is greater than the square on *A* by the square on a straight line incommensurable with *BC*, therefore *BC* is incommensurable in length with *FD*, so that *BC* is also incommensurable with the remainder, the sum of *BF* and *DC*. X.16

But the sum of *BF* and *DC* is commensurable in length with *DC*, therefore *BC* is also incommensurable in length with *DC*, so that, taken separately, *BD* is also incommensurable in length with *DC*. X.6 X.13 X.16

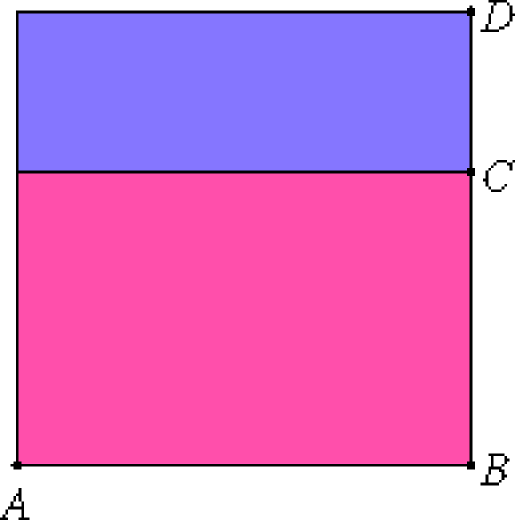
Therefore, *if there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less but falling short by a square, and if it divides it into incommensurable parts, then the square on the greater is greater than the square on the less by the square on a straight line incommensurable with the greater. And if the square on the greater is greater than the square on the less by the square on a straight line incommensurable with the greater, and if there is applied to the greater a parallelogram equal to the fourth part of the square on the less but falling short by a square, then it divides it into incommensurable parts.*

Q.E.D.

Guide

This proposition is a logical variant of the last. It is used in frequently in Book X starting with X.33.

Book X Introduction - Proposition X.17 - Proposition X.19.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Euclid's Elements
Book X
Proposition 19

Lemma.

Since it has been proved that straight lines commensurable in length are always commensurable in square also, while those commensurable in square are not always commensurable in length also, but can of course be either commensurable or incommensurable in length, it is manifest that, if any straight line is commensurable in length with a given rational straight line, it is called rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also.

But, if any straight line is commensurable in square with a given rational straight line, then, if it is also commensurable in length with it, in this case it is also called rational and commensurable with it both in length and in square, but, if again any straight line, being commensurable in square with a given rational straight line, is incommensurable in length with it, in this case it is also called rational but commensurable in square only.

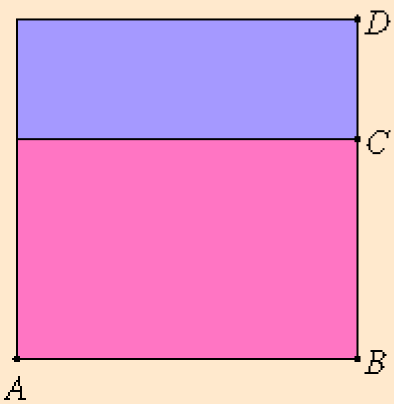
Proposition 19

The rectangle contained by rational straight lines commensurable in length is rational.

Let the rectangle *AC* be contained by the rational straight lines *AB* and *BC* commensurable in length.

I say that *AC* is rational.

Describe the square *AD* on *AB*. Then *AD* is rational. I.46 X.Def.4



And, since *AB* is commensurable in length with *BC*, while *AB* equals *BD*, therefore *BD* is commensurable in length with *BC*.

And *BD* is to *BC* as *DA* is to *AC*. VI.1

Therefore *DA* is commensurable with *AC*. X.11

But *DA* is rational, therefore *AC* is also rational. X.Def.4

Therefore, *the rectangle contained by rational straight lines commensurable in length is rational.*

Q.E.D.

Guide

This is the first proposition that deals with rational lines and rational squares. As required by definitions X.Def.I.3 and

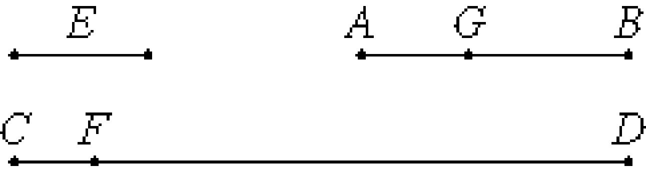
X.Def.I.3, there is some assigned straight line to act as a standard to which other lines and squares are compared for rationality. That line is usually not mentioned in the propositions.

In this proposition, it is assumed that both sides of the rectangle *AB* and *BC* are rational lines. That means these lines are commensurable in square to the standard line, that is, their squares are commensurable with the standard square. It is also assumed that *AB* and *BC* are commensurable with each other. Therefore the rectangle *AC* is commensurable with the square on *AB*, but that's commensurable with the standard square, so the rectangle *AC* is too.

The proposition is used several times starting with X.25. The lemma is used in X.23.

The next proposition is a converse of this one, but the language obscures that from notice.

Book X Introduction - Proposition X.18 - Proposition X.20.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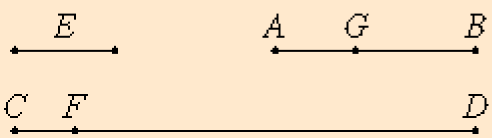
Euclid's Elements
Book X
Proposition 2

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it, then the two magnitudes are incommensurable.

There being two unequal magnitudes *AB* and *CD,* with *AB* being the less, when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it.

I say that the magnitudes *AB* and *CD* are incommensurable.

If they are commensurable, then some magnitude *E* measures them.



Let *AB,* measuring *FD,* leave *CF* less than itself, let *CF* measuring *BG,* leave *AG* less than itself, and let this process be repeated continually, until there is left some magnitude which is less than *E.*

Suppose this done, and let there be left *AG* less than *E.*

Then, since *E* measures *AB,* while *AB* measures *DF,* therefore *E* also measures *FD.* But it measures the whole *CD* also, therefore it also measures the remainder *CF.* But *CF* measures *BG,* therefore *E* also measures *BG.* But it measures the whole *AB* also, therefore it also measures the remainder *AG,* the greater the less, which is impossible.

Therefore no magnitude measures the magnitudes *AB* and *CD.* Therefore the magnitudes *AB* and *CD* are incommensurable. X.Def.1

Therefore, *if, when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it, then the two magnitudes are incommensurable.*

Q.E.D.

Guide

Antenaresis (also called the Euclidean algorithm), first used in proposition VII.1, is again used in this proposition. Beginning with two magnitudes, the smaller, whichever it is, is repeated subtracted from the larger. Proposition VII.1 concerns relatively prime numbers. It is similar to this proposition, but its conclusion is different.

Heath claims that Euclid uses X.1 to prove this proposition, in particular, to show that antenaresis eventually leaves some magnitude which is less than *E*. It is hard to tell what Euclid thought his justification was. Since both magnitudes are multiples of *E,* whatever justification Euclid intended back in proposition VII.2 works just as well here. Euclid did, however, put X.1 just before this proposition, perhaps for an intended logical connection. If so, there is a missing statement to the effect that *GB* is greater than half of *AB,* and so forth, so that X.1 might be invoked.

An example of incommensurable magnitudes

Consider the 36°-72°-72° triangle constructed *ABC* in proposition IV.10. This triangle was used in the following proposition IV.11 to construct regular pentagons. When its base *BC* is subtracted from a side *AC* then the remainder *CD* is the base of a similar triangle *BCD*. Likewise, when the base *CD* of this new triangle is subtracted from its side *BD* then the remainder *DE* is the base of yet another smaller similar triangle *CDE*. And so forth.

Thus, when we begin with the two lines *AB* and *BC* and apply the algorithm of antenaresis to them, we get a series of lines which never ends *AB*, *BC*, *CD*, *DE*, *EF*, and so forth, and these lines form a never-ending continued proportion.

$$AB:BC = BC:CD = CD:DE = DE:EF = \ldots$$

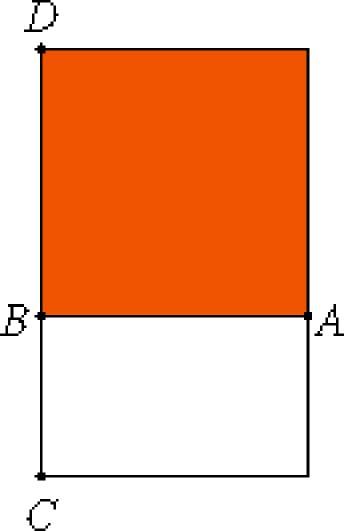
Thus, according to this proposition, the two quantities *AB* and *BC* are incommensurable.

Cutting the line *AB* at *C* to make this ratio *AB*:*BC* is called in VI.Def.3 cutting *AB* into extreme and mean ratio. A more modern name for this ratio is the "golden ratio."

Use of this proposition

This proposition is used in the next one.

Book X Introduction - Proposition X.1 - Proposition X.3.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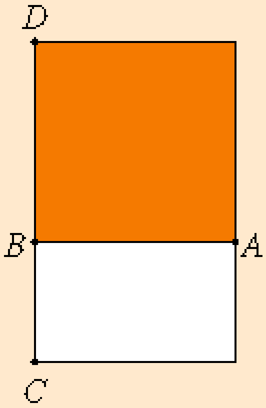
Euclid's Elements
Book X
Proposition 20

If a rational area is applied to a rational straight line, then it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.

Let the rational area *AC* be applied to *AB*, a straight line once more rational in any of the aforesaid ways, producing *BC* as breadth.

I say that *BC* is rational and commensurable in length with *BA*.

Describe the square *AD* on *AB*. Then *AD* is rational. I.46 X.Def.4



But *AC* is also rational, therefore *DA* is commensurable with *AC*. And *DA* is to *AC* as *DB* is to *BC*. Therefore *DB* is also commensurable with *BC*, and *DB* equals *BA*. Therefore *AB* is also commensurable with *BC*. VI.1 X.11

But *AB* is rational, *C* therefore *BC* is also rational and commensurable in length with *AB*.

Therefore, *if a rational area is applied to a rational straight line, then it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.*

Q.E.D.

Guide

This proposition is a converse of the last, except that it's preceded by applying an area to a straight line to get the rectangle. That would be more evident if it read "If one side of a rational rectangle is rational, then the other side is rational and commensurable with the first.

This proposition is used frequently in Book X starting with X.26.

Book X Introduction - Proposition X.19 - Proposition X.21.

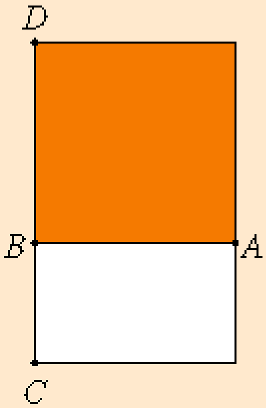
Euclid's Elements
Book X
Proposition 21

The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial.

Let the rectangle *AC* be contained by the rational straight lines *AB* and *BC* commensurable in square only.

I say that *AC* is irrational, and the side of the square equal to it is irrational, and let the latter be called medial.

Describe the square *AD* on *AB*. Then *AD* is rational. X.Def.4



And, since *AB* is incommensurable in length with *BC*, for by hypothesis they are commensurable in square only, while *AB* equals *BD*, therefore *DB* is also incommensurable in length with *BC*.

And *DB* is to *BC* as *AD* is to *AC*, therefore *DA* is incommensurable with *AC*. VI.1 X.11

But *DA* is rational, therefore *AC* is irrational, so that the side of the square *AC* is also irrational. X.Def.4

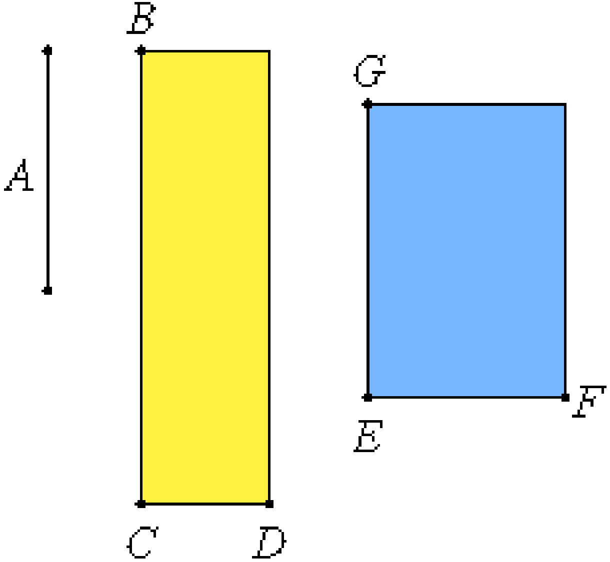
Let the latter be called *medial*.

Q.E.D.

Guide

This proposition is used frequently in Book X starting with the next propostition.

Book X Introduction - Proposition X.20 - Proposition X.22.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Euclid's Elements

Book X

Proposition 22

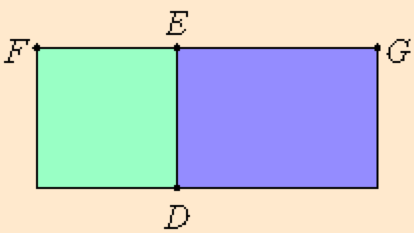
Lemma.

If there are two straight lines, then the first is to the second as the square on the first is to the rectangle contained by the two straight lines.

Let *FE* and *EG* be two straight lines.

I say that *FE* is to *EG* as the square on *FE* is to the rectangle *FE* by *EG*.

Describe the square *DF* on *FE*, and complete *GD*.



Since then *FE* is to *EG* as *FD* is to *DG*, and *FD* is the square on *FE*, and *DG* the rectangle *DE* by *EG*, that is, the rectangle *FE* by *EG*, therefore *FE* is to *EG* as the square on *FE* is to the rectangle *FE* by *EG*. Similarly the rectangle *GE* by *EF* is to the square on *EF*, that is *GD* is to *FD*, as *GE* is to *EF*. VI.1

Q.E.D.

Proposition 22

The square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied.

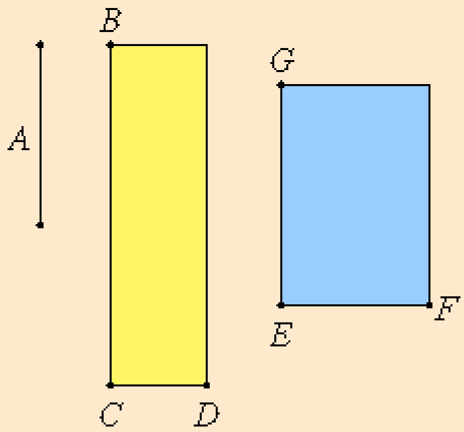
Let *A* be medial and *CB* rational, and let a rectangular area *BD* equal to the square on *A* be applied to *BC*, producing *CD* as breadth.

I say that *CD* is rational and incommensurable in length with *CB*.

Since *A* is medial, the square on it equals a rectangular area contained by rational straight lines commensurable in square only. X.21

Let the square on it equal *GF*. But the square on it also equals *BD*, therefore *BD* equals *GF*.

But it is also equiangular with it, and in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional, therefore, *BC* is to *EG* as *EF* is to *CD*. VI.14



Therefore the square on *BC* is to the square on *EG* as the square on *EF* is to the square on *CD*. VI.22

But the square on *CB* is commensurable with the square on *EG*, for each of these straight lines is rational, therefore the square on *EF* is also commensurable with the square on *CD*. X.11

But the square on *EF* is rational, therefore the square on *CD* is also rational. Therefore *CD* is rational. X.Def.4

And since *EF* is incommensurable in length with *EG*, for they are commensurable in square only, while *EF* is to *EG* as the square on *EF* is to the rectangle *FE* by *EG*, therefore the square on *EF* is incommensurable with the rectangle *FE* by *EG*. Lemma X.11

But the square on *CD* is commensurable with the square on *EF*, for the straight lines are rational in square, and the rectangle *DC* by *CB* is commensurable with the rectangle *FE* by *EG*, for they equal the square on *A*, therefore the square on *CD* is incommensurable with the rectangle *DC* by *CB*. X.13

But the square on *CD* is to the rectangle *DC* by *CB* as *DC* is to *CB*, therefore *DC* is incommensurable in length with *CB*. Lemma X.11

Therefore *CD* is rational and incommensurable in length with *CB*.

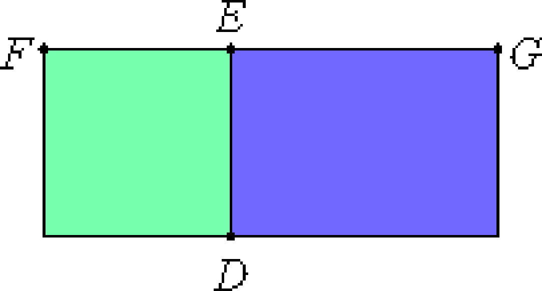
Therefore, *the square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied.*

Q.E.D.

Guide

This proposition is used frequently in Book X starting with the next propostition.

Book X Introduction - Proposition X.21 - Proposition X.23.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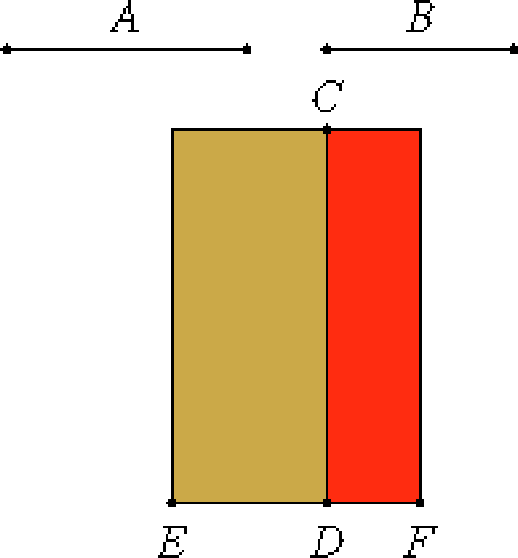

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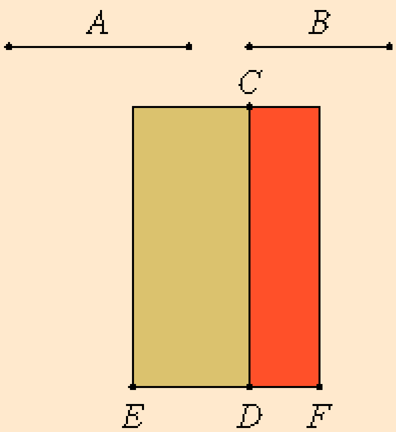
Euclid's Elements
Book X
Proposition 23

A straight line commensurable with a medial straight line is medial.

Let *A* be medial, and let *B* be commensurable with *A*.

I say that *B* is also medial.

Set out a rational straight line *CD*. Apply the rectangular area *CE* to *CD* equal to the square on *A*, producing *ED* as breadth. Then *ED* is rational and incommensurable in length with *CD*. And apply the rectangular area *CF* to *CD* equal to the square on *B*, producing *DF* as breadth. X.22



Since *A* is commensurable with *B*, therefore the square on *A* is also commensurable with the square on *B*. But *EC* equals the square on *A*, and *CF* equals the square on *B*, therefore *EC* is commensurable with *CF*.

And *EC* is to *CF* as *ED* is to *DF*, therefore *ED* is commensurable in length with *DF*. VI.1 X.11

But *ED* is rational and incommensurable in length with *DC*, therefore *DF* is also rational and incommensurable in length with *DC*. X.13 X.Def.3

Therefore *CD* and *DF* are rational and commensurable in square only.

But the straight line is medial on which the square equals the rectangle contained by rational straight lines commensurable in square only, therefore the side of the square equals the rectangle *CD* by *DF* is medial. X.21

And *B* is the side of the square equal the rectangle *CD* by *DF*, therefore *B* is medial.

Therefore, *a straight line commensurable with a medial straight line is medial.*

Q.E.D.

Corollary

From this it is manifest that *an area commensurable with a medial area is medial*.

Note

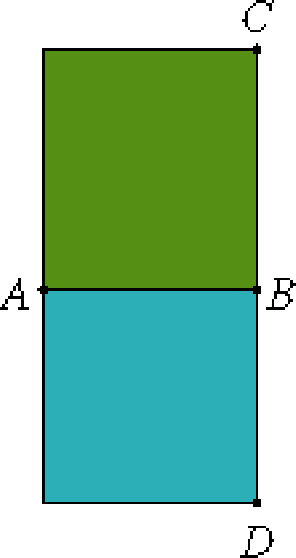
And in the same way as was explained in the case of rationals it follows regards medials, that a straight line commensurable in length with a medial straight line is called medial and commensurable with it not X.18,Lemma only in length but in square also, since, in general, straight lines commensurable in length are always commensurable in square also.

But, if any straight line is commensurable in square with a medial straight line, then if it is also commensurable in length with it, the straight lines are called, in this case too, medial and commensurable in length and in square, but, if in square only, they are called medial straight lines commensurable in square only.

Guide

The proposition is used in X.67 and X.104, the corollary in X.33 and many others, and the note in X.27 and a few others.

Book X Introduction - Proposition X.22 - Proposition X.24.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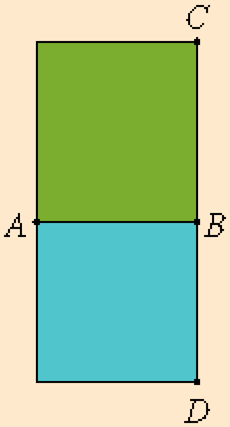
Euclid's Elements
Book X
Proposition 24

The rectangle contained by medial straight lines commensurable in length is medial.

Let the rectangle *AC* be contained by the medial straight lines *AB* and *BC* which are commensurable in length.

I say that *AC* is medial.



Describe the square *AD* on *AB*. Then *AD* is medial.

And, since *AB* is commensurable in length with *BC*, while *AB* equals *BD*, therefore *DB* is commensurable in length with *BC*, so that *DA* is commensurable with *AC*. VI.1 X.11

But *DA* is medial, therefore *AC* is also medial. X.23,Cor.

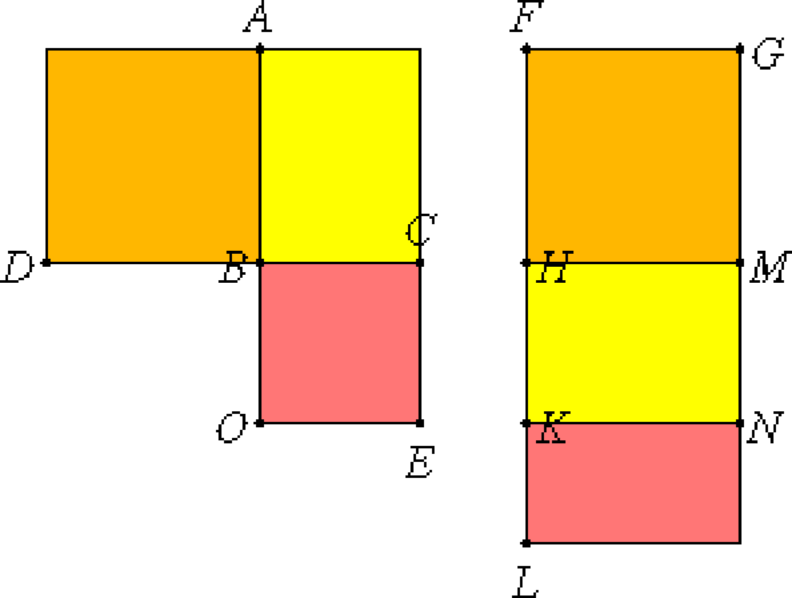
Therefore, *the rectangle contained by medial straight lines commensurable in length is medial.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.23 - Proposition X.25.



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Euclid's Elements
Book X
Proposition 25

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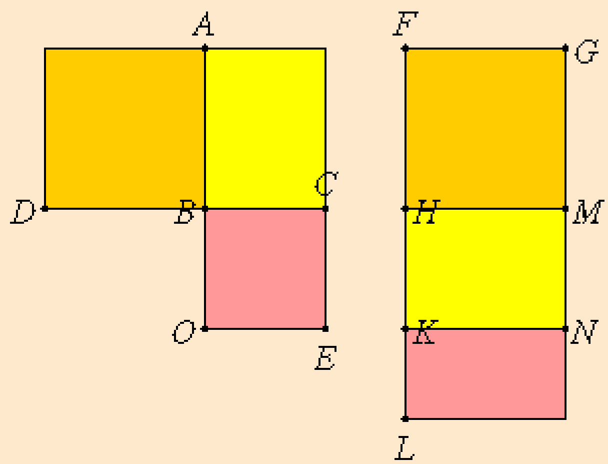
The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

Let the rectangle *AC* be contained by the medial straight lines *AB* and *BC* which are commensurable in square only.

I say that *AC* is either rational or medial.

Describe the squares *AD* and *BE* on *AB* and *BC*. Then each of the squares *AD* and *BE* is medial.



Set out a rational straight line *FG*. Apply the rectangular parallelogram *GH* to *FG* equal to *AD*, producing *FH* as breadth, apply the rectangular parallelogram *MK* to *HM* equal to *AC*, producing *HK* as breadth, and further apply similarly *NL* to *KN* equal to *BE*, producing *KL* as breadth. Then *FH*, *HK*, and *KL* are in a straight line.

Since each of the squares *AD* and *BE* is medial, and *AD* equals *GH* while *BE* equals *NL*, therefore each of the rectangles *GH* and *NL* is also medial.

And they are applied to the rational straight line *FG*, therefore each of the straight lines *FH* and *KL* is rational and incommensurable in length with *FG*. X.22

And, since *AD* is commensurable with *BE*, therefore *GH* is commensurable with *NL*. And *GH* is to *NL* as *FH* is to *KL*, therefore *FH* is commensurable in length with *KL*. VI.1 X.11

Therefore *FH* and *KL* are rational straight lines commensurable in length, therefore the rectangle *FH* by *KL* is rational. X.19

And, since *DB* equals *BA* while *OB* equals *BC*, therefore *DB* is to *BC* as *AB* is to *BO*.

But *DB* is to *BC* as *DA* is to *AC*, and *AB* is to *BO* as *AC* is to *CO*, therefore *DA* is to *AC* as *AC* is to *CO*. VI.1

But *AD* equals *GH*, *AC* equals *MK*, and *CO* equals *NL*, therefore *GH* is to *MK* as *MK* is to *NL*. Therefore *FH* is to *HK* as *HK* is to *KL*. Therefore the rectangle *FH* by *KL* equals the square on *HK*. VI.1 V.11 VI.17

But the rectangle *FH* by *KL* is rational, therefore the square on *HK* is also rational.

Therefore *HK* is rational. And, if it is commensurable in length with *FG*, then *HN* is rational, but, if it is incommensurable in length with *FG*, then *KH* and *HM* are rational straight lines commensurable in square only, and therefore *HN* is medial. X.19 X.21

Therefore *HN* is either rational or medial. But *HN* equals *AC*, therefore *AC* is either rational or medial.

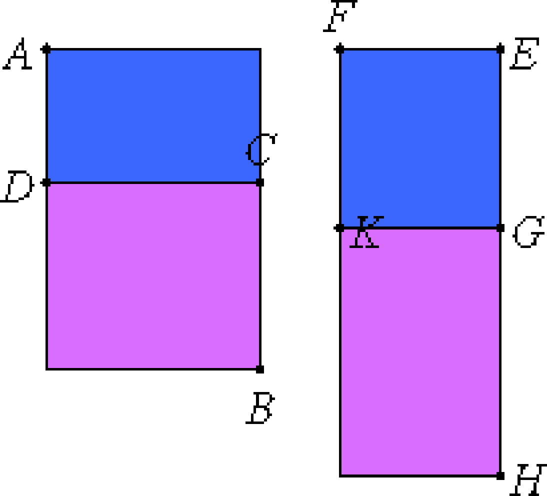
Therefore, *the rectangle contained by medial straight lines commensurable in square only is either rational or medial.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.24 - Proposition X.26.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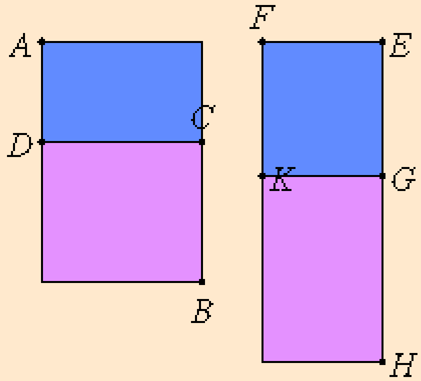
Euclid's Elements
Book X
Proposition 26

A medial area does not exceed a medial area by a rational area.

If possible, let the medial area *AB* exceed the medial area *AC* by the rational area *DB*.

Set out a rational straight line *EF*. Apply to *EF* the rectangular parallelogram *FH* equal to *AB* producing *EH* as breadth. Subtract the rectangle *FG* equal to *AC*. Then the remainder *BD* equals the remainder *KH*.



But *DB* is rational, therefore *KH* is also rational.

Since each of the rectangles *AB* and *AC* is medial, and *AB* equals *FH* while *AC* equals *FG*, therefore each of the rectangles *FH* and *FG* is also medial.

They are applied to the rational straight line *EF*, therefore each of the straight lines *HE* and *EG* is rational and incommensurable in length with *EF*. X.22

Since *DB* is rational and equals *KH*, therefore *KH* is rational. And it is applied to the rational straight line *EF*, therefore *GH* is rational and commensurable in length with *EF*. X.20

But *EG* is also rational, and is incommensurable in length with *EF*, therefore *EG* is incommensurable in length with *GH*. X.13

And *EG* is to *GH* as the square on *EG* is to the rectangle *EG* by *GH*, therefore the square on *EG* is incommensurable with the rectangle *EG* by *GH*. X.11

But the squares on *EG* and *GH* are commensurable with the square on *EG*, for both are rational, and twice the rectangle *EG* by *GH* is commensurable with the rectangle *EG* by *GH*, for it is double it, therefore the sum of the squares on *EG* and *GH* is incommensurable with twice the rectangle *EG* by *GH*. X.6 X.13

Therefore the sum of the squares on *EG* and *GH* plus twice the rectangle *EG* by *GH*, that is, the square on *EH* is incommensurable with the sum of the squares on *EG* and *GH*. II.4 X.16

But the squares on *EG* and *GH* are rational, therefore the square on *EH* is irrational. X.Def.4

Therefore *EH* is irrational. But it is also rational, which is impossible.

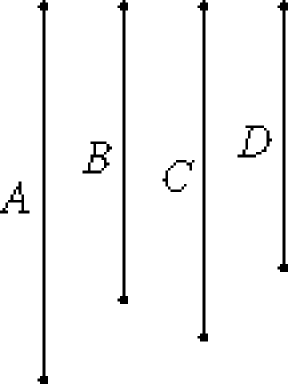
Therefore, *a medial area does not exceed a medial area by a rational area.*

Q.E.D.

Guide

This proposition is used in several others in Book X starting with X.42.

Book X Introduction - Proposition X.25 - Proposition X.27.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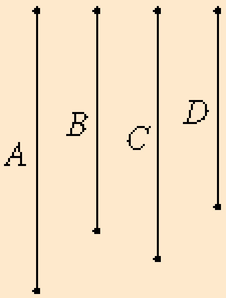
Euclid's Elements

Book X

Proposition 27

To find medial straight lines commensurable in square only which contain a rational rectangle.

Set out two rational straight lines *A* and *B* commensurable in square only. Take a mean proportional *C* between *A* and *B*. Let it be contrived that *A* is to *B* as *C* is to *D*. X.10 VI.13 VI.12



Then, since *A* and *B* are rational and commensurable in square only, therefore the rectangle *A* by *B*, that is, the square on *C*, is medial. Therefore *C* is medial. VI.17 X.21

And since *A* is to *B* as *C* is to *D*, and *A* and *B* are commensurable in square only, therefore *C* and *D* are also commensurable in square only. X.11

And *C* is medial, therefore *D* is also medial. X.23.Note

Therefore *C* and *D* are medial and commensurable in square only.

I say that they also contain a rational rectangle.

Since *A* is to *B* as *C* is to *D*, therefore, alternately, *A* is to *C* as *B* is to *D*. V.16

But *A* is to *C* as *C* is to *B*, therefore *C* is to *B* as *B* is to *D*. Therefore the rectangle *C* by *D* equals the square on *B*. But the square on *B* is rational, therefore the rectangle *C* by *D* is also rational.

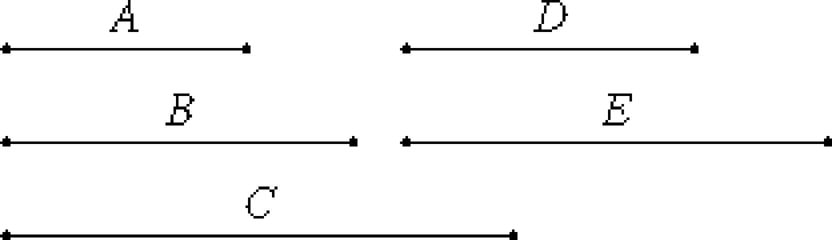
Therefore medial straight lines commensurable in square only have been found which contain a rational rectangle.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.26 - Proposition X.28.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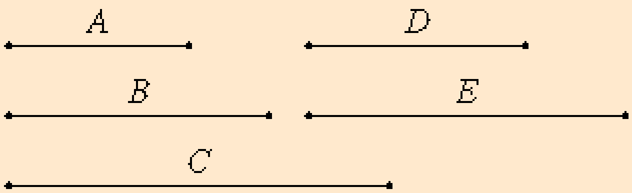
Euclid's Elements
Book X
Proposition 28

To find medial straight lines commensurable in square only which contain a medial rectangle.

Set out the rational straight lines *A*, *B*, and *C* commensurable in square only. Take a mean proportional *D* between *A* and *B*. Let it be contrived that *B* is to *C* as *D* is to *E*. X.10 VI.13 VI.12



Since *A* and *B* are rational straight lines commensurable in square only, therefore the rectangle *A* by *B*, that is, the square on *D*, is medial. Therefore *D* is medial. VI.17 X.21

And since *B* and *C* are commensurable in square only, and *B* is to *C* as *D* is to *E*, therefore *D* and *E* are also commensurable in square only. X.11

But *D* is medial, therefore *E* is also medial. X.23,Note

Therefore *D* and *E* are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

Since *B* is to *C* as *D* is to *E*, therefore, alternately, *B* is to *D* as *C* is to *E*. V.16

But *B* is to *D* as *D* is to *A*, therefore *D* is to *A* as *C* is to *E*. Therefore the rectangle *A* by *C* equals the rectangle *D* by *E*. VI.16

But the rectangle *A* by *C* is medial, therefore the rectangle *D* by *E* is also medial. X.21

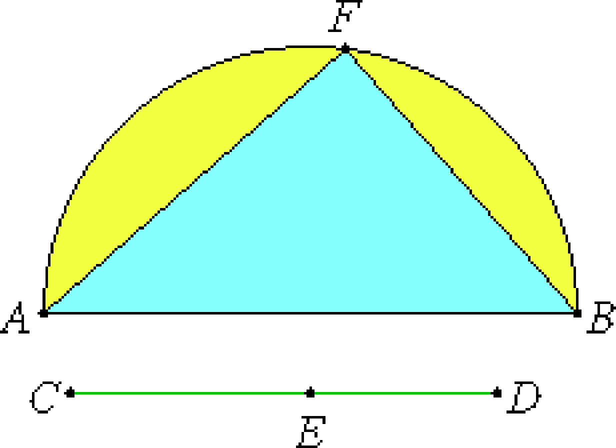
Therefore medial straight lines commensurable in square only have been found which contain a medial rectangle.

Q.E.D.

Guide

Lemma 1 is used in X.48, and the proposition itself is used in X.75.

Book X Introduction - Proposition X.27 - Proposition X.29.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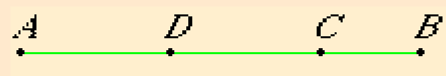
Euclid's Elements
Book X
Proposition 29

Lemma 1.

To find two square numbers such that their sum is also square.

Set out two numbers *AB* and *BC*, and let them be either both even or both odd. Then since, whether an even number is subtracted from an even number, or an odd number from an odd number, the remainder is even, therefore the remainder *AC* is even. IX.24 IX.26

Bisect *AC* at *D*. Let *AB* and *BC* also be either similar plane numbers, or square numbers, which are themselves also similar plane numbers.



Now the product of *AB* and *BC* together with the square on *CD* equals the square on *BD*. And the product of *AB* and *BC* is square, inasmuch as it was proved that, if two similar plane numbers multiplied by one another make some number, the product is square. Therefore two square numbers, the product of *AB* and *BC*, and the square on *CD*, have been found which, when added together, make the square on *BD*. II.6 IX.1

And it is manifest that two square numbers, the square on *BD* and the square on *CD*, have again been found such that their difference, the product of *AB* and *BC*, is a square, whenever *AB* and *BC* are similar plane numbers. But when they are not similar plane numbers, two square numbers, the square on *BD* and the square on *DC*, have been found such that their difference, the product of *AB* and *BC*, is not square.

Q.E.D.

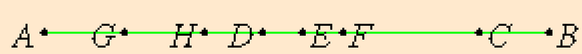
Lemma 2.

To find two square numbers such that their sum is not square.

Let the product of *AB* and *BC* we said, be square, and *CA* even. Bisect *CA* by *D*.

It is then manifest that the square product of *AB* and *BC* together with the square on *CD* equals the square on *BD*. Lemma 1

Subtract the unit *DE*. Therefore the product of *AB* and *BC* together with the square on *CE* is less than the square on *BD*.



I say then that the product of *AB* and *BC* together with the square on *CE* is not square.

If it is square, then it either equals the square on *BE*, or is less than the square on *BE*, but cannot be greater lest the unit be divided.

First, if possible, let the product of *AB* and *BC* together with the square on *CE* equal the square on *BE*, and let *GA* be double the unit *DE*.

Since the whole *AC* is double the whole *CD*, and in them *AG* is double *DE*, therefore the remainder *GC* is also double the remainder *EC*. Therefore *GC* is bisected by *E*.

Therefore the product of *GB* and *BC* together with the square on *CE* equals the square on *BE* II.6

But the product of *AB* and *BC* together with the square on *CE* also, by hypothesis, equals the square on *BE*, therefore the product of *GB* and *BC* together with the square on *CE* equals the product of *AB* and *BC* together with the square on *CE*.

And, if the common square on *CE* is subtracted, then it follows that *AB* equals *GB*, which is absurd.

Therefore the product of *AB* and *BC* together with the square on *CE* does not equal the square on *BE*.

I say next that neither is it less than the square on *BE*.

For, if possible, let it equal the square on *BF*, and let *HA* be double *DF*.

Now it will again follow that *HC* is double *CF*, so that *CH* is bisected at *F*, and for this reason the product of *HB* and *BC* together with the square on *FC* equals the square on *BF*. II.6

But, by hypothesis, the product of *AB* and *BC* together with the square on *CE* also equals the square on *BF*.

Thus the product of *HB* and *BC* together with the square on *CF* also equals the product of *AB* and *BC* together with the square on *CE*, which is absurd.

Therefore the product of *AB* and *BC* together with the square on *CE* is not less than the square on *BE*.

And it was proved that neither does it equal the square on *BE*.

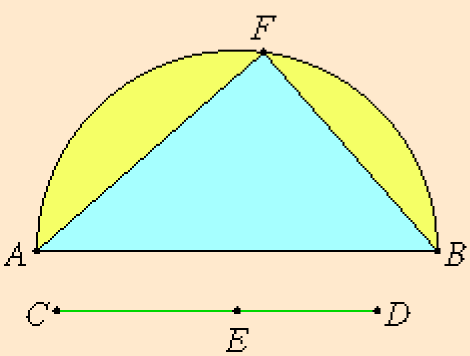
Therefore the product of *AB* and *BC* together with the square on *CE* is not square.

Q.E.D.

Proposition 29

To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Set out any rational straight line *AB*, and two square numbers *CD* and *DE* such that their difference *CE* is not square. Lemma 1



Describe the semicircle *AFB* on *AB*, and let it be contrived that *DC* is to *CE* as the square on *BA* is to the square on *AF*. Join *FB*. X.6,Cor.

Since the square on *BA* is to the square on *AF* as *DC* is to *CE*, therefore the square on *BA* has to the square on *AF* the ratio which the number *DC* has to the number *CE*. Therefore the square on *BA* is commensurable with the square on *AF*. X.6

But the square on *AB* is rational, therefore the square on *AF* is also rational. Therefore *AF* is also rational. X.Def.4

And, since *DC* does not have to *CE* the ratio which a square number has to a square number, neither does the square on *BA* have to the square on *AF* the ratio which a square number has to a square number, X.9

therefore *AB* is incommensurable in length with *AF*.

Therefore *BA* and *AF* are rational straight lines commensurable in square only.

And since *DC* is to *CE* as the square on *BA* is to the square on *AF*, therefore, in conversion, *CD* is to *DE* as the square on *AB* is to the square on *BF*. V.19,Cor. III.31 I.47

But *CD* has to *DE* the ratio which a square number has to a square number, therefore the square on *AB* has to the square on *BF* the ratio which a square number has to a square number. Therefore *AB* is commensurable in length with *BF*. X.9

And the square on *AB* equals the sum of the squares on *AF* and *FB*, therefore the square on *AB* is greater than the square on *AF* by the square on *BF* commensurable with *AB*.

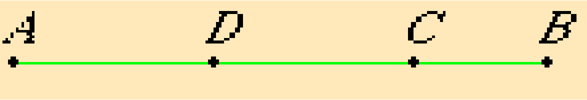
Therefore there have been found two rational straight lines *BA* and *AF* commensurable in square only such that the square on the greater *AB* is greater than the square on the less *AF* by the square on *BF* commensurable in length with *AB*.

Q.E.D.

Guide

This proposition is used in X.31 and X.32.

Book X Introduction - Proposition X.28 - Proposition X.30.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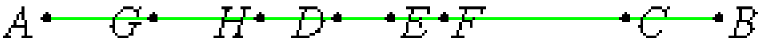
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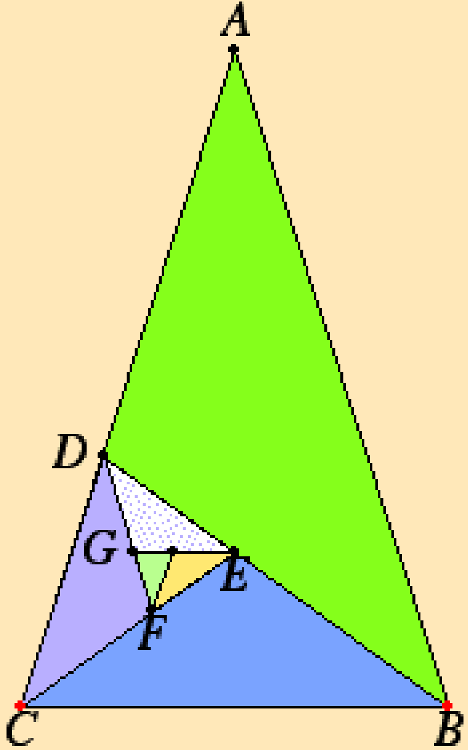
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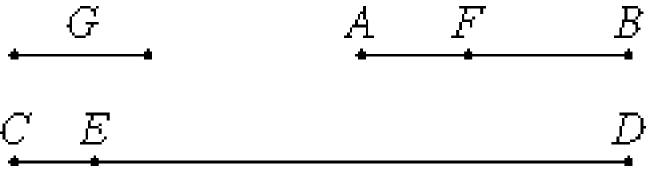
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Euclid's Elements
Book X
Proposition 3

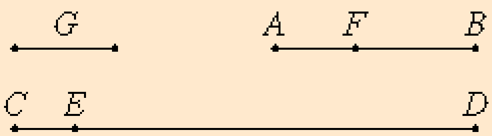
To find the greatest common measure of two given commensurable magnitudes.

Let the two given commensurable magnitudes be *AB* and *CD* with *AB* the less.

It is required to find the greatest common measure of *AB* and *CD*.

Now the magnitude *AB* either measures *CD* or it does not.

If it measures it, and it does measures itself, then *AB* is a common measure of *AB* and *CD*. And it is manifest that it is also the greatest, for a greater magnitude than the magnitude *AB* does not measure *AB*.



Next, let *AB* not measure *CD*.

Then, if the less is continually subtracted in turn from the greater, then that which is left over will sometime measure the one before it, because *AB* and *CD* are not incommensurable. X.2

Let *AB*, measuring *ED*, leave *EC* less than itself, let *EC*, measuring *FB*, leave *AF* less than itself, and let *AF* measure *CE*.

Since, then, *AF* measures *CE*, while *CE* measures *FB*, therefore *AF* also measures *FB*. But it measures itself also, therefore *AF* also measures the whole *AB*. But *AB* measures *DE*, therefore *AF* also measures *ED*. But it measures *CE* also, therefore it also measures the whole *CD*.

Therefore *AF* is a common measure of *AB* and *CD*.

I say next that it is also the greatest.

If not, then there there is some magnitude *G* greater than *AF* which measures *AB* and *CD*. Since then *G* measures *AB*, while *AB* measures *ED*, therefore *G* also measures *ED*.

But it measures the whole *CD* also, therefore *G* measures the remainder *CE*. But *CE* measures *FB*, therefore *G* also measures *FB*.

But it measures the whole *AB* also, and it therefore measures the remainder *AF*, the greater the less, which is impossible.

Therefore no magnitude greater than *AF* measures *AB* and *CD*. Therefore *AF* is the greatest common measure of *AB* and *CD*.

Therefore the greatest common measure of the two given commensurable magnitudes *AB* and *CD* has been found.

Q.E.D.

Corollary.

From this it is manifest that, *if a magnitude measures two magnitudes, then it also measures their greatest common* *measure*.

Guide

This is the same proposition as VII.3 with the same diagram and the same corollary, only the terminology is slightly different.

This proposition and its corollary are used in the next proposition.

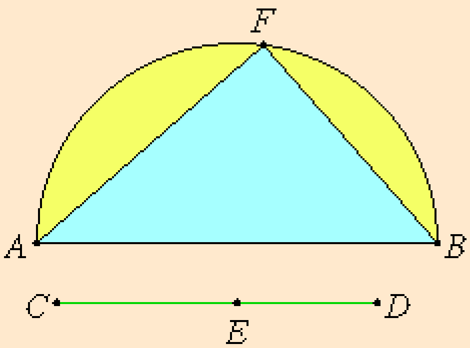
Book X Introduction - Proposition X.2 - Proposition X.4.

Euclid's Elements
Book X
Proposition 30

To find two rational straight lines commensurable in square only such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater.

Set out a rational straight line *AB*, and two square numbers *CE* and *ED* such that their sum *CD* is not square. Describe the semicircle *AFB* on *AB*. Let it be contrived that *DC* is to *CE* as the square on *BA* is to the square on *AF*, and join *FB*. X.29.Lemma 2 X.6,Cor.

Then, in a similar manner to the preceding, we can prove that *BA* and *AF* are rational straight lines commensurable in square only.



Since *DC* is to *CE* as the square on *BA* is to the square on *AF*, therefore, in conversion, *CD* is to *DE* as the square on *AB* is to the square on *BF*. V.19,Cor. III.31 I.47

But *CD* does not have to *DE* the ratio which a square number has to a square number, therefore neither has the square on *AB* to the square on *BF* the ratio which a square number has to a square number. Therefore *AB* is incommensurable in length with *BF*. X.9

And the square on *AB* is greater than the square on *AF* by the square on *FB* incommensurable with *AB*.

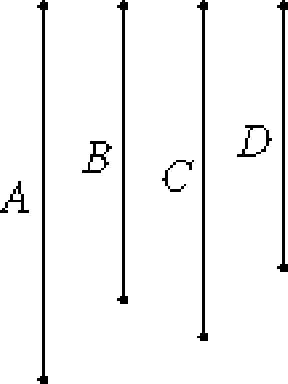
Therefore *AB* and *AF* are rational straight lines commensurable in square only, and the square on *AB* is greater than the square on *AF* by the square on *FB* incommensurable in length with *AB*.

Q.E.D.

Guide

This proposition is used in the next three propositions.

Book X Introduction - Proposition X.29 - Proposition X.31.



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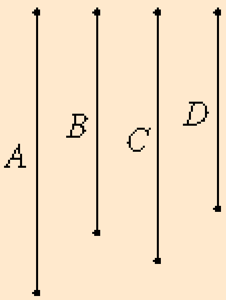
Euclid's Elements
Book X
Proposition 31

To find two medial straight lines commensurable in square only, containing a rational rectangle, such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Set out two rational straight lines *A* and *B* commensurable in square only such that the square on *A*, being the greater, is greater than the square on *B* the less by the square on a straight line commensurable in length with *A*. X.29



Let the square on *C* equal the rectangle *A* by *B*.

Now the rectangle *A* by *B* is medial, therefore the square on *C* is also medial. Therefore *C* is also medial. X.21

Let the rectangle *C* by *D* equal the square on *B*.

Now the square on *B* is rational, therefore the rectangle *C* by *D* is also rational. And since *A* is to *B* as the rectangle *A* by *B* is to the square on *B*, while the square on *C* equals the rectangle *A* by *B*, and the rectangle *C* by *D* equals the square on *B*, therefore *A* is to *B* as the square on *C* is to the rectangle *C* by *D*.

But the square on *C* is to the rectangle *C* by *D* as *C* is to *D*, therefore *A* is to *B* as *C* is to *D*.

But *A* is commensurable with *B* in square only, therefore *C* is also commensurable with *D* in square only. X.11

And *C* is medial, therefore *D* is also medial. X.23,Note

Since *A* is to *B* as *C* is to *D*, and the square on *A* is greater than the square on *B* by the square on a straight line commensurable with *A*, therefore the square on *C* is greater than the square on *D* by the square on a straight line commensurable with *C*. X.14

Therefore two medial straight lines *C* and *D*, commensurable in square only and containing a rational rectangle, have been found, and the square on *C* is greater than the square on *D* by the square on a straight line commensurable in length with *C*.

Similarly also it can be proved that the square on *C* exceeds the square on *D* by the square on a straight line incommensurable with *C*, when the square on *A* is greater than the square on *B* by the square on a straight line incommensurable with *A*. X.30

Q.E.D.

Guide

This proposition is used in X.34 and X.35.

Book X Introduction - Proposition X.30 - Proposition X.32.

Euclid's Elements
Book X
Proposition 32

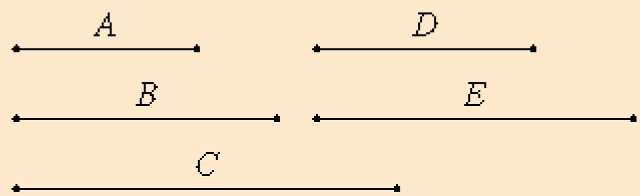
To find two medial straight lines commensurable in square only, containing a medial rectangle, such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Set out three rational straight lines *A*, *B*, and *C* commensurable in square only, such that the square on *A* is greater than the square on *C* by the square on a straight line commensurable with *A*. Let the square on *D* equal the rectangle *A* by *B*. X.29

Then the square on *D* is medial. Therefore *D* is also medial. X.21

Let the rectangle *D* by *E* equal the rectangle *B* by *C*.

Then since as the rectangle *A* by is is to the rectangle *B* by *C* as *A* is to *C*, while the square on *D* equals the rectangle *A* by *B*, and the rectangle *D* by *E* equals the rectangle *B* by *C*, therefore *A* is to *C* as the square on *D* is to the rectangle *D* by *E*.



But the square on *D* is to the rectangle *D* by *E* as *D* is to *E*, therefore *A* is to *C* as *D* is to *E*. But *A* is commensurable with *C* in square only, therefore *D* is also commensurable with *E* in square only. X.11

But *D* is medial, therefore *E* is also medial. X.23,Note

And, since *A* is to *C* as *D* is to *E*, while the square on *A* is greater than the square on *C* by the square on a straight line commensurable with *A*, therefore the square on *D* is greater than the square on *E* by the square on a straight line commensurable with *D*. X.14

I say next that the rectangle *D* by *E* is also medial.

Since the rectangle *B* by *C* equals the rectangle *D* by *E*, while the rectangle *B* by *C* is medial, therefore the rectangle *D* by *E* is also medial. X.21

Therefore two medial straight lines *D* and *E*, commensurable in square only, and containing a medial rectangle, have been found such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

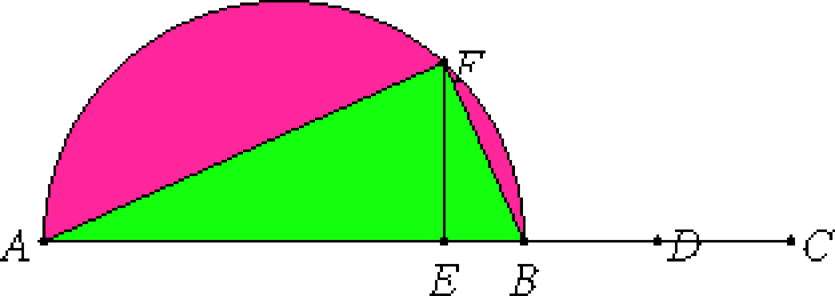
Similarly again it can be proved that the square on *D* is greater than the square on *E* by the square on a straight line incommensurable with *D* when the square on *A* is greater than the square on *C* by the square on a straight line incommensurable with *A*. X.30

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.31 - Proposition X.33.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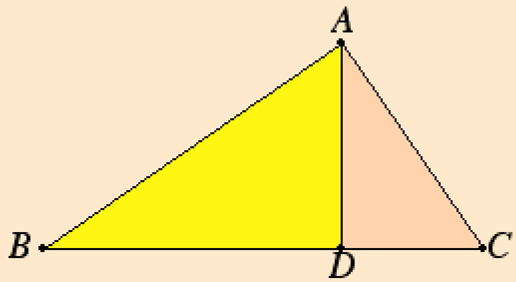
Euclid's Elements
Book X
Proposition 33

Lemma.

Let *ABC* be a right-angled triangle having the angle *A* right, and let the perpendicular *AD* be drawn.

I say that the rectangle *CB* by *BD* equals the square on *BA*, the rectangle *BC* by *CD* equals the square on *CA*, the rectangle *BD* by *DC* equals the square on *AD*, and the rectangle *BC* by *AD* equals the rectangle *BA* by *AC*, and first that the rectangle *CB* by *BD* equals the square on *BA*.



Since in a right-angled triangle *AD* has been drawn from the right angle perpendicular to the base, therefore the triangles *ABD* and *ADC* are similar both to the whole *ABC* and to one another. VI.8

Since the triangle *ABC* is similar to the triangle *ABD*, therefore *CB* is to *BA* as *BA* is to *BD*. Therefore the rectangle *CB* by *BD* equals the square on *AB*. For the same reason the rectangle *BC* by *CD* also equals the square on *AC*. VI.4 VI.17

Since, if in a right-angled triangle a perpendicular is drawn from the right angle to the base, then the perpendicular so drawn is a mean proportional between the segments of the base, therefore *BD* is to *DA* as *AD* is to *DC*. Therefore the rectangle *BD* by *DC* equals the square on *AD*. VI.8,Cor. VI.17

I say that the rectangle *BC* by *AD* also equals the rectangle *BA* by *AC*.

Since we said, *ABC* is similar to *ABD*, therefore *BC* is to *CA* as *BA* is to *AD*. VI.4

Therefore the rectangle *BC* by *AD* equals the rectangle *BA* by *AC*. VI.16

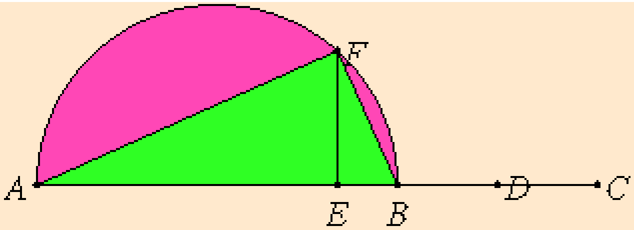
Q.E.D.

Proposition 33

To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial.

Set out two rational straight lines *AB* and *BC* commensurable in square only such that the square on the greater *AB* is greater than the square on the less *BC* by the square on a straight line incommensurable with *AB*. X.30

Bisect *BC* at *D*. Apply to *AB* a parallelogram equal to the square on either of the straight lines *BD* or *DC* and deficient by a square figure, and let it be the rectangle *AE* by *EB*. VI.28



Describe the semicircle *AFB* on *AB*, draw *EF* at right angles to *AB*, and join *AF* and *FB*.

Since *AB* and *BC* are unequal straight lines, and the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*, while there was applied to *AB* a parallelogram equal to the fourth part of the square on *BC*, that is, to the square on half of it, and deficient by a square figure, making the rectangle *AE* by *EB*, therefore *AE* is incommensurable with *EB*. X.18

And *AE* is to *EB* as the rectangle *BA* by *AE* is to the rectangle *AB* by *BE*, while the rectangle *BA* by *AE* equals the square on *AF*, and the rectangle *AB* by *BE* is to the square on *BF*, therefore the square on *AF* is incommensurable with the square on *FB*. Therefore *AF* and *FB* are incommensurable in square.

Since *AB* is rational, therefore the square on *AB* is also rational, so that the sum of the squares on *AF* and *FB* is also rational. I.47

Since, again, the rectangle *AE* by *EB* equals the square on *EF*, and, by hypothesis, the rectangle *AE* by *EB* also equals the square on *BD*, therefore *FE* equals *BD*. Therefore *BC* is double *FE*, so that the rectangle *AB* by *BC* is also commensurable with the rectangle *ABEF*.

But the rectangle *AB* by *BC* is medial, therefore the rectangle *AB* by *EF* is also medial. X.21 X.23,Cor.

But the rectangle *AB* by *EF* equals the rectangle *AF* by *FB*, therefore the rectangle *AF* by *FB* is also medial. Lemma

But it was also proved that the sum of the squares on these straight lines is rational.

Therefore two straight lines *AF* and *FB* incommensurable in square have been found which make the sum of the squares on them rational, but the rectangle contained by them medial.

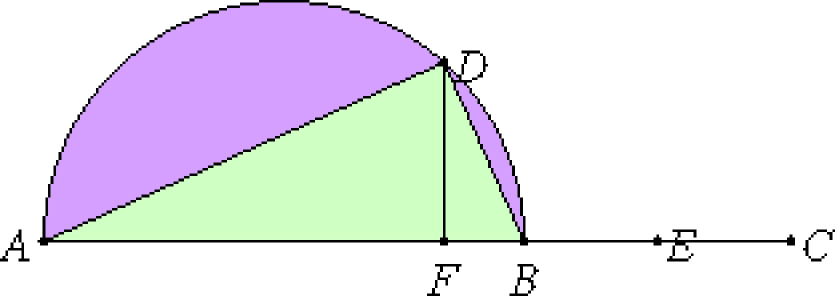
Q.E.D.

Guide

The first part of the lemma encompasses proposition I.47, but the proof of it depends on the theory of similar triangles developed in Book VI, unlike Euclid's proof of I.47.

This proposition is used in propositions X.39 and X.76.

Book X Introduction - Proposition X.32 - Proposition X.34.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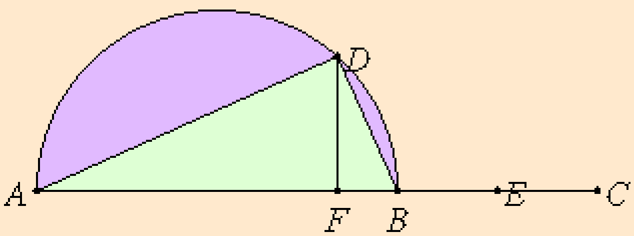
Euclid's Elements
Book X
Proposition 34

To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational.

Set out two medial straight lines *AB* and *BC*, commensurable in square only, such that the rectangle which they contain is rational and the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*. X.31 ad fin.

Describe the semicircle *ADB* on *AB*. Apply a parallelogram to *AB* equal to the square on *BE* and deficient by a square figure, namely the rectangle *AF* by *FB*. Then *AF* is incommensurable in length with *FB*. VI.28 X.18



Draw *FD* from *F* at right angles to *AB*, and join *AD* and *DB*.

Since *AF* is incommensurable in length with *FB*, therefore the rectangle *BA* by *AF* is also incommensurable with the rectangle *AB* by *BF*. X.11

But the rectangle *BA* by *AF* equals the square on *AD*, and the rectangle *AB* by *BF* equals the square on *DB*, therefore the square on *AD* is also incommensurable with the square on *DB*.

And, since the square on *AB* is medial, therefore the sum of the squares on *AD* and *DB* is also medial. III.31 I.47

And, since *BC* is double *DF*, therefore the rectangle *AB* by *BC* is also double the rectangle *AB* by *FD*.

But the rectangle *AB* by *BC* is rational, therefore the rectangle *AB* by *FD* is also rational. X.6

But the rectangle *AB* by *FD* equals the rectangle *AD* by *DB*, so that the rectangle *AD* by *DB* is also rational. Lemma

Therefore two straight lines *AD* and *DB* incommensurable in square have been found which make the sum of the squares on them medial, but the rectangle contained by them rational.

Q.E.D.

Guide

This proposition is used in proposition X.40.

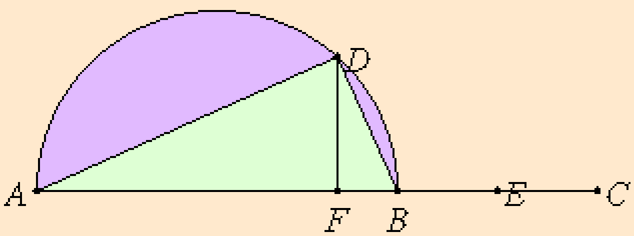
Book X Introduction - Proposition X.33 - Proposition X.35.

Euclid's Elements
Book X
Proposition 35

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Set out two medial straight lines *AB* and *BC* commensurable in square only, containing a medial rectangle, such that the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*. Describe the semicircle *ADB* on *AK*, and make the rest of the construction as above. X.31 ad fin.

Since *AF* is incommensurable in length with *FB*, therefore *AD* is also incommensurable in square with *DB*. X.18 X.11



Since the square on *AB* is medial, therefore the sum of the squares on *AD* and *DB* is also medial. III.31 I.47

Since the rectangle *AF* by *FB* equals the square on each of the straight lines *BE* and *DF*, therefore *BE* equals *DF*. Therefore *BC* is double *FD*, so that the rectangle *AB* by *BC* is also double the rectangle *AB* by *FD*. But the rectangle *AB* by *BC* is medial, therefore the rectangle *AB* by *FD* is also medial. X.32,Cor.

And it equals the rectangle *AD* by *DB*, therefore the rectangle *AD* by *DB* is also medial. X.33,Lemma

Since *AB* is incommensurable in length with *BC*, while *CB* is commensurable with *BE*, therefore *AB* is also incommensurable in length with *BE*, so that the square on *AB* is also incommensurable with the rectangle *AB* by *BE*. X.13 X.11

But the sum of the squares on *AD* and *DB* equals the square on *AB*, and the rectangle *AB* by *FD*, that is, the rectangle *AD* by *DB*, equals the rectangle *AB* by *BE*, therefore the sum of the squares on *AD* and *DB* is incommensurable with the rectangle *AD* by *DB*. I.47

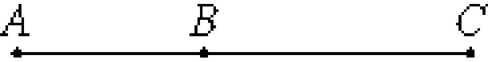
Therefore two straight lines *AD* and *DB* incommensurable in square have been found which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Q.E.D.

Guide

This proposition is used in propositions X.41 and X.78.

Book X Introduction - Proposition X.34 - Proposition X.36.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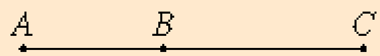
Euclid's Elements
Book X
Proposition 36

If two rational straight lines commensurable in square only are added together, then the whole is irrational; let it be called binomial.

Let two rational straight lines *AB* and *BC* commensurable in square only be added together.

I say that the whole *AC* is irrational.



Since *AB* is incommensurable in length with *BC*, for they are commensurable in square only, and *AB* is to *BC* as the rectangle *AB* by *BC* is to the square on *BC*, therefore the rectangle *AB* by *BE* is incommensurable with the square on *BC*. X.11

But twice the rectangle *AB* by *BC* is commensurable with the rectangle *AB* by *BC*, and the sum of the squares on *AB* and *BC* is commensurable with the square on *BC*, for *AB* and *BC* are rational straight lines commensurable in square only, therefore twice the rectangle *AB* by *BC* is incommensurable with the sum of the squares on *AB* and *BC*. X.6 X.15 X.13

And, taken jointly, twice the rectangle *AB* by *BC* together with the squares on *AB* and *BC*, that is, the square on *AC*, is incommensurable with the sum of the squares on *AB* and *BC*. II.4 X.16

But the sum of the squares on *AB* and *BC* is rational, therefore the square on *AC* is irrational, so that *AC* is also irrational. Let it be called *binomial*. X.Def.4

Q.E.D.

Guide

This proposition is used very frequently in Book X starting with the next proposition.

Book X Introduction - Proposition X.35 - Proposition X.37.

Euclid's Elements

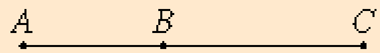
Book X

Proposition 37

If two medial straight lines commensurable in square only and containing a rational rectangle are added together, the whole is irrational; let it be called the first bimedial straight line.

Let two medial straight lines *AB* and *BC* commensurable in square only and containing a rational rectangle be added together.

I say that the whole *AC* is irrational.



Since *AB* is incommensurable in length with *BC*, therefore the sum of the squares on *AB* and *BC* is also incommensurable with twice the rectangle *AB* by *BC*, and, taken jointly, the sum of the squares on *AB* and *BC* together with twice the rectangle *AB* by *BC*, that is, the square on *AC*, is incommensurable with the rectangle *AB* by *BC*. X.36 II.4 X.16

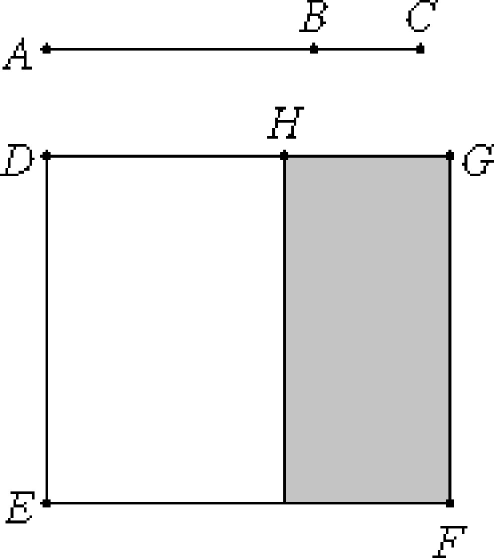
But the rectangle *AB* by *BC* is rational, for, by hypothesis, *AB* and *BC* are straight lines containing a rational rectangle, therefore the square on *AC* is irrational. Therefore *AC* is irrational. And let it be called a *first bimedial* straight line. X.Def.4

Q.E.D.

Guide

This proposition is used in X.43 and a few others in Book X.

Book X Introduction - Proposition X.36 - Proposition X.38.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Euclid's Elements

Book X

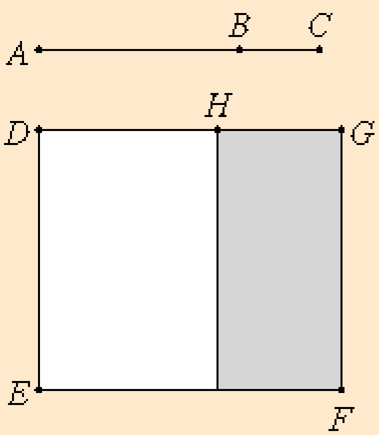
Proposition 38

If two medial straight lines commensurable in square only and containing a medial rectangle are added together, then the whole is irrational; let it be called the second bimedial straight line.

Let two medial straight lines *AB* and *BC* commensurable in square only and containing a medial rectangle be added together.

I say that *AC* is irrational.

Set out a rational straight line *DE*, and apply parallelogram *DF* to *DE* equal to the square on *AC*, producing *DG* as breadth. I.41



Since the square on *AC* equals the sum of the squares on *AB* and *BC* and twice the rectangle *AB* by *BC*, apply *EH*, equal to the sum of the squares on *AB* and *BC*, to *DE*. Then the remainder *HF* equals twice the rectangle *AB* by *BC*. II.4

Since each of the straight lines *AB* and *BC* is medial, therefore the squares on *AB* and *BC* are also medial. But, by hypothesis, twice the rectangle *AB* by *BC* is also medial. And *EH* equals the sum of the squares on *AB* and *BC*, while *FH* equals twice the rectangle *AB* by *BC*, therefore each of the rectangles *EH* and *HF* is medial.

And they are applied to the rational straight line *DE*, therefore each of the straight lines *DH* and *HG* is rational and incommensurable in length with *DE*. X.22

Since *AB* is incommensurable in length with *BC*, and *AB* is to *BC* as the square on *AB* is to the rectangle *AB* by *BC*, therefore the square on *AB* is incommensurable with the rectangle *AB* by *BC*. X.11

But the sum of the squares on *AB* and *BC* is commensurable with the square on *AB*, and twice the rectangle *AB* by *BC* is commensurable with the rectangle *AB* by *BC*. X.15 X.6

Therefore the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*. X.13

But *EH* equals the sum of the squares on *AB* and *BC*, and *HF* equals twice the rectangle *AB* by *BC*.

Therefore *EH* is incommensurable with *HF*, so that *DH* is also incommensurable in length with *HG*. VI.1 X.11

Therefore *DH* and *HG* are rational straight lines commensurable in square only, so that *DG* is irrational. X.36

But *DE* is rational, and the rectangle contained by an irrational and a rational straight line is irrational, therefore the area *DF* is irrational, and the side of the square equal to it is irrational. cf. X.20 X.Def.4

But *AC* is the side of the square equal to *DF*, therefore *AC* is irrational. Let it be called a *second bimedial* straight line.

Q.E.D.

Guide

This proposition is used in X.44 and a few others in Book X.

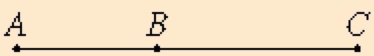
Book X Introduction - Proposition X.37 - Proposition X.39.

Euclid's Elements
Book X
Proposition 39

If two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial are added together, then the whole straight line is irrational; let it be called major.

Let two straight lines *AB* and *BC* incommensurable in square, and fulfilling the given conditions, be added together. X.33

I say that *AC* is irrational.



Since the rectangle *AB* by *BC* is medial, therefore twice the rectangle *AB* by *BC* is also medial. X.6 X.23,Cor.

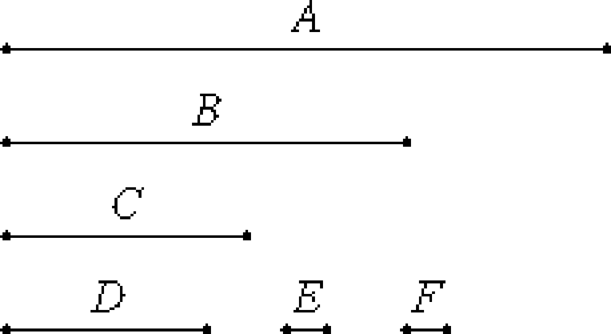
But the sum of the squares on *AB* and *BC* is rational, therefore twice the rectangle *AB* by *BC* is incommensurable with the sum of the squares on *AB* and *BC*, so that the sum of the squares on *AB* and *BC* together with twice the rectangle *AB* by *BC*, that is, the square on *AC*, is also incommensurable with the sum of the squares on *AB* and *BC*. Therefore the square on *AC* is irrational, so that *AC* is also irrational. Let it be called *major*. X.16 X.Def.4

Q.E.D.

Guide

This proposition is used in X.57 and a few others in Book X.

Book X Introduction - Proposition X.38 - Proposition X.40.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Euclid's Elements
Book X
Proposition 4

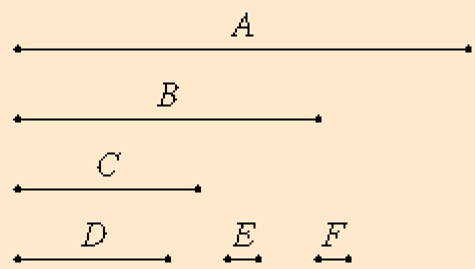
To find the greatest common measure of three given commensurable magnitudes.

Let *A*, *B*, and *C* be the three given commensurable magnitudes.

It is required to find the greatest common measure of *A*, *B*, and *C*.

Take the greatest common measure *D* of the two magnitudes *A* and *B*. X.3

Either *D* measures *C*, or it does not measure it.



First, let it measure it.

Since then *D* measures *C*, while it also measures *A* and *B*, therefore *D* is a common measure of *A*, *B*, and *C*. And it is manifest that it is also the greatest, for a greater magnitude than the magnitude *D* does not measure *A* and *B*.

Next, let *D* not measure *C*.

I say first that *C* and *D* are commensurable.

Since *A*, *B*, and *C* are commensurable, some magnitude measures them, and this of course measures *A* and *B* also, so that it also measures the greatest common measure of *A* and *B*, namely *D*. X.3.Cor.

But it also measures *C*, so that the said magnitude measures *C* and *D*, therefore *C* and *D* are commensurable.

Now take their greatest common measure *E*. X.3

Since *E* measures *D*, while *D* measures *A* and *B*, therefore *E* also measures *A* and *B*. But it measures *C* also, therefore *E* measures *A*, *B*, and *C*. Therefore *E* is a common measure of *A*, *B*, and *C*.

I say next that it is also the greatest.

For, if possible, let there be some magnitude *F* greater than *E*, and let it measure *A*, *B*, and *C*.

Now, since *F* measures *A*, *B*, and *C*, it also measures *A* and *B*, and therefore measures the greatest common measure of *A* and *B*. X.3,Cor.

But the greatest common measure of *A* and *B* is *D*, therefore *F* measures *D*.

But it measures *C* also, therefore *F* measures *C* and *D*. Therefore *F* also measures the greatest common measure of *C* and *D*. But that is *E*, therefore *F* measures *E*, the greater the less, which is impossible. X.3,Cor.

Therefore no magnitude greater than the magnitude *E* measures *A*, *B*, and *C*. Therefore *E* is the greatest common measure of *A*, *B*, and *C* if *D* does not measure *C*, but if it measures it, then *D* is itself the greatest common measure.

Therefore the greatest common measure of the three given commensurable magnitudes has been found.

Corollary.

From this it is manifest that, *if a magnitude measures three magnitudes, then it also measures their greatest common measure. The greatest common measure can be found similarly for more magnitudes, and the corollary extended.*

Q.E.D.

Guide

This is the same proposition as VII.3. This proposition and the last explain how to find the common measure of commensurable magnitudes. Although not explicitly invoked, they bear on the succeeding propositions which use common measures of commensurable magnitudes.

Book X Introduction - Proposition X.3 - Proposition X.5.

Euclid's Elements

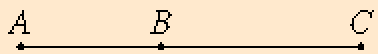
Book X

Proposition 40

If two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational are added together, then the whole straight line is irrational; let it be called the side of a rational plus a medial area.

Let two straight lines *AB* and *BC* incommensurable in square, and fulfilling the given conditions, be added together. X.34

I say that *AC* is irrational.



Since the sum of the squares on *AB* and *BC* is medial, while twice the rectangle *AB* by *BC* is rational, therefore the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*, so that the square on *AC* is also incommensurable with twice the rectangle *AB* by *BC*. X.16

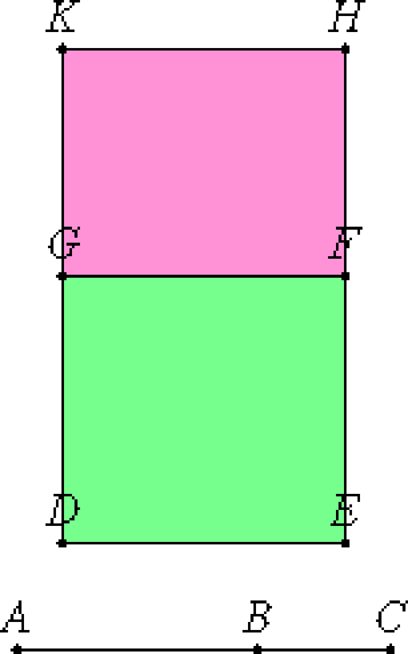
But twice the rectangle *AB* by *BC* is rational, therefore the square on *AC* is irrational. Therefore *AC* is irrational. Let it be called the *side of a rational plus a medial area.* X.Def.4

Q.E.D.

Guide

This proposition is used in X.46 and a few others in Book X.

Book X Introduction - Proposition X.39 - Proposition X.41.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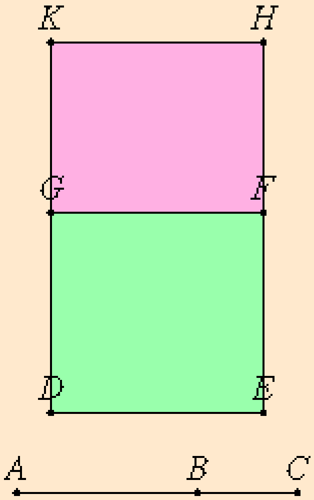
Euclid's Elements
Book X
Proposition 41

If two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and also incommensurable with the sum of the squares on them are added together, then the whole straight line is irrational; let it be called the side of the sum of two medial areas.

Let two straight lines *AB* and *BC* incommensurable in square and satisfying the given conditions be added together. X.35

I say that *AC* is irrational.



Set out a rational straight line *DE*. Apply to *DE* the rectangle *DF* equal to the sum of the squares on *AB* and *BC*, and apply to *DE* the rectangle *GH* equal to twice the rectangle *AB* by *BC*. Then the whole *DH* equals the square on *AC*. II.4

Now, since the sum of the squares on *AB* and *BC* is medial, and equals *DF*, therefore *DF* is also medial. And it is applied to the rational straight line *DE*, therefore *DG* is rational and incommensurable in length with *DE*. For the same reason *GK* is also rational and incommensurable in length with *GF*, that is, *DE*. X.22

Since the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*, therefore *DF* is incommensurable with *GH*, so that *DG* is also incommensurable with *GK*. VI.1 X.11

And they are rational, therefore *DG* and *GK* are rational straight lines commensurable in square only. Therefore *DK* is irrational and what is called binomial. X.36

But *DE* is rational, therefore *DH* is irrational, and the side of the square which equals it is irrational. X.Def.4

But *AC* is the side of the square equal to *HD*, therefore *AC* is irrational. Let it be called the side of the sum two medial areas.

Q.E.D.

Lemma.

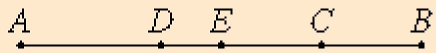
And that *the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question* we will now prove after premising the following lemma.

Set out the straight line *AB*, cut the whole into unequal parts at each of the points *C* and *D*, and let *AC* be supposed greater than *DB*.

I say that the sum of the squares on *AC* and *CB* is greater than the sum of the squares on *AD* and *DB*.

Bisect *AB* at *E*.

Since *AC* is greater than *DB*, subtract *DC* from each, therefore the remainder *AD* is greater than the remainder *CB*.



But *AE* equals *EB*, therefore *DE* is less than *EC*. Therefore the points *C* and *D* are not equidistant from the point of bisection.

Since the rectangle *AC* by *CB* together with the square on *EC* equals the square on *EB*, and, further, the rectangle *AD* by *DB* together with the square on *DE* equals the square on *EB*, therefore the rectangle *AC* by *CB* together with the square on *EC* equals the rectangle *AD* by *DB* together with the square on *DE*. II.5

And of these the square on *DE* is less than the square on *EC*, therefore the remainder, the rectangle *AC* by *CB*, is also less than the rectangle *AD* by *DB* so that twice the rectangle *AC* by *CB* is also less than twice the rectangle *AD* by *DB*.

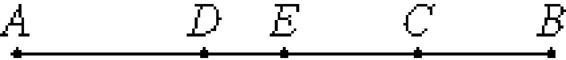
Therefore the remainder, the sum of the squares on *AC* and *CB*, is greater than the sum of the squares on *AD* and *DB*.

Q.E.D.

Guide

This proposition is used in X.65 and a couple of others in Book X.

Book X Introduction - Proposition X.40 - Proposition X.42.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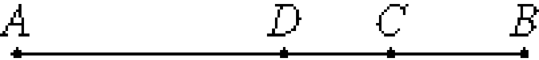
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Euclid's Elements

Book X

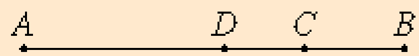
Proposition 42

A binomial straight line is divided into its terms at one point only.

Let *AB* be a binomial straight line divided into its terms at *C*. Then *AC* and *CB* are rational straight lines commensurable in square only.

I say that *AB* is not divided at another point into two rational straight lines commensurable in square only.

For, if possible, let it be divided at *D* also, so that *AD* and *DB* are also rational straight lines commensurable in square only.



It is then manifest that *AC* is not the same as *DB*.

If possible, let it be so. Then *AB* is also the same as *CB*, and *AC* is to *CB* as *BD* is to *DA*. Thus *AB* is divided at *D* also in the same way as by the division at *C*, which is contrary to the hypothesis.

Therefore *AC* is not the same with *DB*.

For this reason also the points *C* and *D* are not equidistant from the point of bisection.

Therefore that by which the sum of the squares on *AC* and *CB* differs from the sum of the squares on *AD* and *DB* is also that by which twice the rectangle *AD* by *DB* differs from twice the rectangle *AC* by *CB*, because both the squares on *AC* and *CB* together with twice the rectangle *AC* by *CB*, and the squares on *AD* and *DB* together with twice the rectangle *AD* by *DB*, equal the square on *AB*. II.4

But the sum of the squares on *AC* and *CB* differs from the sum of the squares on *AD* and *DB* by a rational area, for both are rational, therefore twice the rectangle *AD* by *DB* also differs from twice the rectangle *AC* by *CB* by a rational area, though they are medial, which is absurd, for a medial area does not exceed a medial by a rational area. X.21 X.26

Therefore a binomial straight line is not divided at different points. Therefore it is divided at one point only.

Therefore, *a binomial straight line is divided into its terms at one point only.*

Q.E.D.

Guide

This proposition is used in X.47.

Book X Introduction - Proposition X.41 - Proposition X.43.

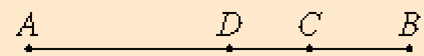
Euclid's Elements
Book X
Proposition 43

A first bimedial straight line is divided at one and the same point only.

Let *AB* be a first bimedial straight line divided at *C*, so that *AC* and *CB* are medial straight lines commensurable in square only and containing a rational rectangle. X.37

I say that *AB* is not so divided at another point.

If possible, let it also be divided at *D*, so that *AD* and *DB* are also medial straight lines commensurable in square only and containing a rational rectangle.



Since, then, that by which twice the rectangle *AD* by *DB* differs from twice the rectangle *AC* by *CB* is that by which the sum of the squares on *AC* and *CB* differs from the sum of the squares on *AD* and *DB*, while twice the rectangle *AD* by *DB* differs from twice the rectangle *AC* by *CB* by a rational area, for both are rational, therefore the sum of the squares on *AC* and *CB* also differs from the sum of the squares on *AD* and *DB* by a rational area, though they are medial, which is absurd. X.26

Therefore a first bimedial straight line is not divided into its terms at different points. Therefore it is so divided at one point only.

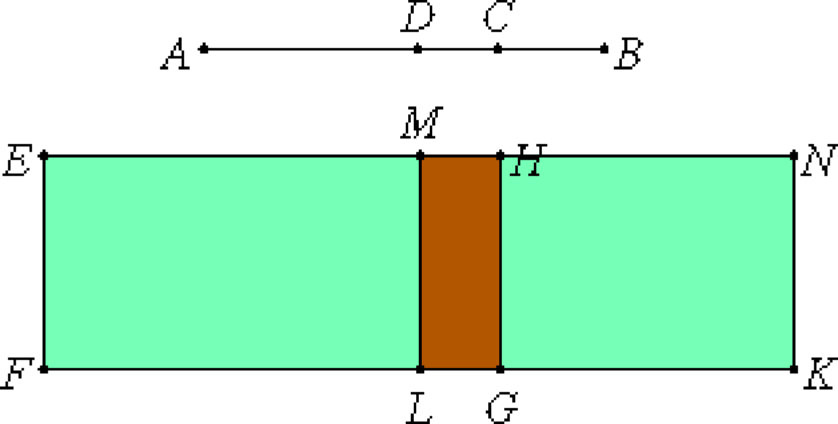
Therefore, *a first bimedial straight line is divided at one and the same point only.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.42 - Proposition X.44.



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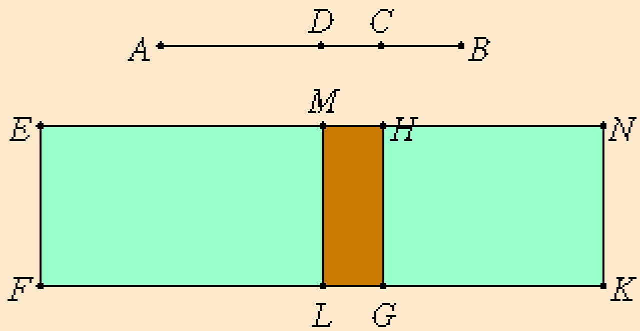
Euclid's Elements
Book X
Proposition 44

A second bimedial straight line is divided at one point only.

Let *AB* be a second bimedial straight line divided at *C*, so that *AC* and *CB* are medial straight lines commensurable in square only and containing a medial rectangle. It is then manifest that *C* is not at the point of bisection, because the segments are not commensurable in length. X.38

I say that *AB* is not so divided at another point.



If possible, let it also be divided at *D*, so that *AC* is not the same with *DB*, but *AC* is supposed greater. It is then clear that the sum of the squares on *AD* and *DB* is also, as we proved above, less than the sum of the squares on *AC* and *CB*. Suppose that *AD* and *DB* are medial straight lines commensurable in square only and containing a medial rectangle. Lemma

Set out a rational straight line *EF*, apply to *EF* the rectangular parallelogram *EK* equal to the square on *AB*, and subtract *EG*, equal to the sum of the squares on *AC* and *CB*. Then the remainder *HK* equals twice the rectangle *AC* by *CB*. II.4

Again, subtract *EL*, equal to the sum of the squares on *AD* and *DB*, which were proved less than the sum of the squares on *AC* and *CB*. Then the remainder *MK* also equals twice the rectangle *AD* by *DB*. Lemma

Now, since the squares on *AC* and *CB* are medial, therefore *EG* is medial.

And it is applied to the rational straight line *EF*, therefore *EH* is rational and incommensurable in length with *EF*. X.22

For the same reason *HN* is also rational and incommensurable in length with *EF*.

And, since *AC* and *CB* are medial straight lines commensurable in square only, therefore *AC* is incommensurable in length with *CB*.

But *AC* is to *CB* as the square on *AC* is to the rectangle *AC* by *CB*, therefore the square on *AC* is incommensurable with the rectangle *AC* by *CB*. X.11

But the sum of the squares on *AC* and *CB* is commensurable with the square on *AC*, for *AC* and *CB* are commensurable in square. X.15

And twice the rectangle *AC* by *CB* is commensurable with the rectangle *AC* by *CB*. X.6

Therefore the sum of the squares on *AC* and *CB* is also incommensurable with twice the rectangle *AC* by *CB*. X.13

But *EG* equals the sum of the squares on *AC* and *CB*, and *HK* equals twice the rectangle *AC* by *CB*, VI.1
therefore *EG* is incommensurable with *HK*, so that *EH* is also incommensurable in length with *HN*. X.11

And they are rational, therefore *EH* and *HN* are rational straight lines commensurable in square only.

But, if two rational straight lines commensurable in square only are added together, then the whole is the irrational which is called binomial. X.36

Therefore *EN* is a binomial straight line divided at *H*.

In the same way *EM* and *MN* is also proved to be rational straight lines commensurable in square only, and *EN* is a binomial straight line divided at different points, *H* and *M*.

And *EH* is not the same with *MN*, for the sum of the squares on *AC* and *CB* is greater than the sum of the squares on *AD* and *DB*.

But the sum of the squares on *AD* and *DB* is greater than twice the rectangle *AD* by *DB*, therefore the sum of the squares on *AC* and *CB*, that is, *EG*, is much greater than twice the rectangle *AD* by *DB*, that is, *MK*, so that *EH* is also greater than *MN*.

Therefore *EH* is not the same with *MN*.

Therefore, *a second bimedial straight line is divided at one point only*.

Q.E.D.

Guide

(Forthcoming)

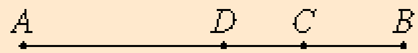
Book X Introduction - Proposition X.43 - Proposition X.45.

Euclid's Elements
Book X
Proposition 45

A major straight line is divided at one point only.

Let *AB* be a major straight line divided at *C*, so that *AC* and *CB* are incommensurable in square, and let the sum of the squares on *AC* and *CB* be rational, but the rectangle *AC* by *CB* medial.

I say that *AB* is not so divided at another point.



If possible, let it also be divided at *D*, so that *AD* and *DB* are incommensurable in square and the sum of the squares on *AD* and *DB* is rational, but the rectangle contained by them medial.

Then, since that by which the sum of the squares on *AC* and *CB* differs from the sum of the squares on *AD* and *DB* is also that by which twice the rectangle *AD* by *DB* differs from twice the rectangle *AC* by *CB*, while the sum of the squares on *AC* and *CB* exceeds the sum of the squares on *AD* and *DB* by a rational area, for both are rational, therefore twice the rectangle *AD* by *DB* also exceeds twice the rectangle *AC* by *CB* by a rational area, though they are medial, which is impossible. X.26

Therefore a major straight line is not divided at different points. Therefore it is only divided at one and the same point.

Therefore, *a major straight line is divided at one point only.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.44 - Proposition X.46.

Euclid's Elements

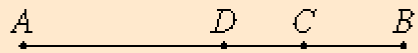
Book X

Proposition 46

The side of a rational plus a medial area is divided at one point only.

Let *AB* be the side of a rational plus a medial area divided at *C*, so that *AC* and *CB* are incommensurable in square and let the sum of the squares on *AC* and *CB* be medial, but twice the rectangle *AC* by *CB* rational. X.40

I say that *AB* is not so divided at another point.



If possible, let it be divided at *D* also, so that *AD* and *DB* are also incommensurable in square and the sum of the squares on *AD* and *DB* is medial, but twice the rectangle *AD* by *DB* rational.

Since, then, that by which twice the rectangle *AC* by *CB* differs from twice the rectangle *AD* by *DB* is also that by which the sum of the squares on *AD* and *DB* differs from the sum of the squares on *AC* and *CB*, while twice the rectangle *AC* by *CB* exceeds twice the rectangle *AD* by *DB* by a rational area, therefore the sum of the squares on *AD* and *DB* also exceeds the sum of the squares on *AC* and *CB* by a rational area, though they are medial, which is impossible. X.26

Therefore the side of a rational plus a medial area is not divided at different points, therefore it is divided at one point only.

Therefore, *the side of a rational plus a medial area is divided at one point only.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.45 - Proposition X.47.

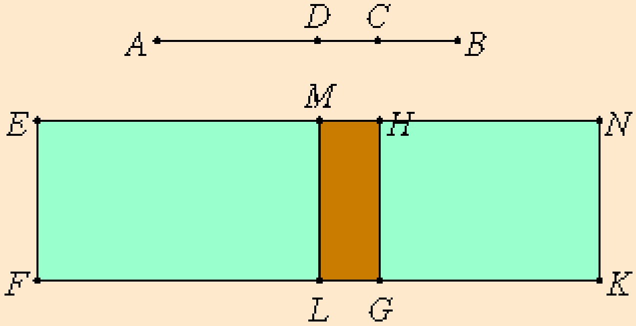
Euclid's Elements
Book X
Proposition 47

The side of the sum of two medial areas is divided at one point only.

Let *AB* be divided at *C*, so that *AC* and *CB* are incommensurable in square and let the sum of the squares on *AC* and *CB* be medial, and the rectangle *AC* by *CB* medial and also incommensurable with the sum of the squares on them.

I say that *AB* is not divided at another point so as to fulfill the given conditions.

If possible, let it be divided at *D*, so that again *AC* is of course not the same as *BD*, but *AC* is supposed greater.



Set out a rational straight line *EF*, and apply to *EF* the rectangle *EG* equal to the sum of the squares on *AC* and *CB*, and the rectangle *HK* equal to twice the rectangle *AC* by *CB*. Then the whole *EK* equals the square on *AB*. II.4

Again, to *EP* apply *EL*, equal to the sum of the squares on *AD* and *DB*. Then the remainder, twice the rectangle *AD* by *DB*, equals the remainder *MK*.

And since, by hypothesis, the sum of the squares on *AC* and *CB* is medial, therefore *EG* is also medial.

And it is applied to the rational straight line *EF*, therefore *HE* is rational and incommensurable in length with *EF*. X.22

For the same reason *HN* is also rational and incommensurable in length with *EF*. And, since the sum of the squares on *AC* and *CB* is incommensurable with twice the rectangle *AC* by *CB*, therefore *EG* is also incommensurable with *GN*, so that *EH* is also incommensurable with *HN*. VI.1 X.11

And they are rational, therefore *EH* and *HN* are rational straight lines commensurable in square only. Therefore *EN* is a binomial straight line divided at *H*. X.36

Similarly we can prove that it is also divided at *M*. And *EH* is not the same with *MN*, therefore a binomial has been divided at different points, which is absurd. X.42

Therefore a side of the sum of two medial areas is not divided at different points. Therefore it is divided at one point only.

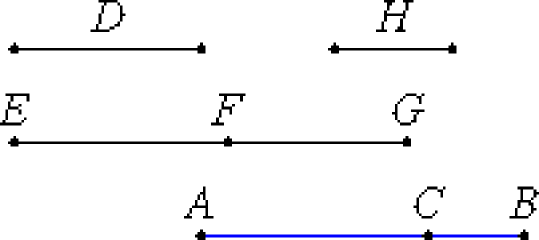
Therefore, *the side of the sum of two medial areas is divided at one point only.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.46 - Definitions II of Book X.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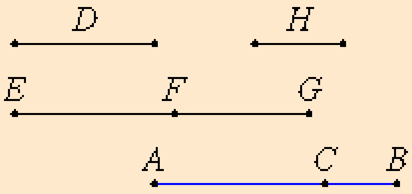
Euclid's Elements
Book X
Proposition 48

To find the first binomial line.

Set out two numbers *AC* and *CB* such that the sum of them *AB* has to *BC* the ratio which a square number has to a square number, but does not have to *CA* the ratio which a square number has to a square number. X.28,Lemma1

Set out any rational straight line *D*, and let *EF* be commensurable in length with *D*. Therefore *EF* is also rational.



Let it be contrived that the number *BA* is to *AC* as the square on *EF* is to the square on *FG*. X.6,Cor.

But *AB* has to *AC* the ratio which a number has to a number, therefore the square on *EF* also has to the square on *FG* the ratio which a number has to a number, so that the square on *EF* is commensurable with the square on *FG*. X.6

And *EF* is rational, therefore *FG* is also rational. And, since *BA* does not have to *AC* the ratio which a square number has to a square number, neither, therefore, has the square on *EF* to the square on *FG* the ratio which a square number has to a square number. Therefore *EF* is incommensurable in length with *FG*. X.9

Therefore *EF* and *FG* are rational straight lines commensurable in square only. Therefore *EG* is binomial. X.36

I say that it is also a first binomial straight line.

Since the number *BA* is to *AC* as the square on *EF* is to the square on *FG*, while *BA* is greater than *AC*, therefore the square on *EF* is also greater than the square on *FG*.

Let then the sum of the squares on *FG* and *H* equal the square on *EF*.

Now since *BA* is to *AC* as the square on *EF* is to the square on *FG*, therefore, in conversion, *AB* is to *BC* as the square on *EF* is to the square on *H*. V.19,Cor.

But *AB* has to *BC* the ratio which a square number has to a square number, therefore the square on *EF* also has to the square on *H* the ratio which a square number has to a square number.

Therefore *EF* is commensurable in length with *H*. Therefore the square on *EF* is greater than the square on *FG* by the square on a straight line commensurable with *EF*. X.9

And *EF* and *FG* are rational, and *EF* is commensurable in length with *D*.

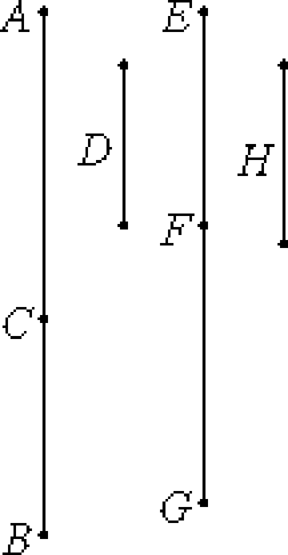
Therefore *EF* is a first binomial straight line.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Definitions II of Book X - Proposition X.49.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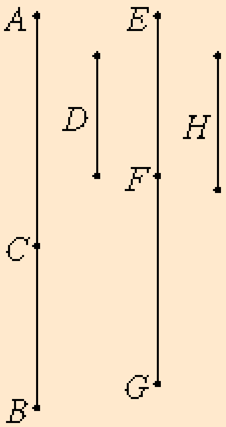
Euclid's Elements
Book X
Proposition 49

To find the second binomial line.

Set out two numbers *AC* and *CB* such that the sum of them *AB* has to *BC* the ratio which a square number has to a square number, but does not have to *AC* the ratio which a square number has to a square number. Set out a rational straight line *D*, and let *EF* be commensurable in length with *D*, therefore *EF* is rational.

Let it be contrived then that as the number *CA* is to *AB*, so is the square on *EF*to the square on *FG*, therefore the square on *EF* is commensurable with the square on *FG*. Therefore *FG* is also rational. X.6,Cor. X.6



Now, since the number *CA* does not have to *AB* the ratio which a square number has to a square number, neither does the square on *EF* have to the square on *FG* the ratio which a square number has to a square number.

Therefore *EF* is incommensurable in length with *FG*. Therefore *EF* and *FG* are rational straight lines commensurable in square only. Therefore *EG* is binomial. X.9 X.36

It is next to be proved that it is also a second binomial straight line.

Since, inversely, the number *BA* is to *AC* as the square on *GF* is to the square on *FE*, while *BA* is greater than *AC*, therefore the square on *GF* is greater than the square on *FE*. V.7.Cor

Let the sum of the squares on *EF* and *H* equal the square on *GF*. Then, in conversion, *AB* is to *BC* as the square on *FG* is to the square on *H*. V.19,Cor.

But *AB* has to *BC* the ratio which a square number has to a square number, therefore the square on *FG* also has to the square on *H* the ratio which a square number has to a square number.

Therefore *FG* is commensurable in length with *H*, so that the square on *FG* is greater than the square on *FE* by the square on a straight line commensurable with *FG*. X.9

And *FG* and *FE* are rational straight lines commensurable in square only, and *EF*, the lesser term, is commensurable in length with the rational straight line *D* set out.

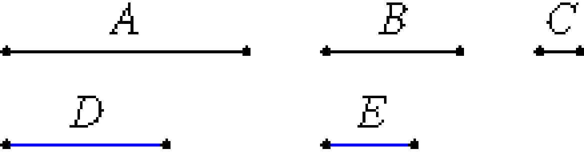
Therefore *EG* is a second binomial straight line.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.48 - Proposition X.50.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Euclid's Elements
Book X
Proposition 5

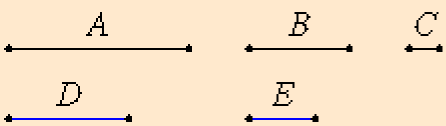
Commensurable magnitudes have to one another the ratio which a number has to a number.

Let *A* and *B* be commensurable magnitudes.

I say that *A* has to *B* the ratio which a number has to a number.

Since *A* and *B* are commensurable, some magnitude *C* measures them.

As many times as *C* measures *A*, let so many units be in *D*, and, as many times as *C* measures *B*, let so many units be in *E*.



Since *C* measures *A* according to the units in *D*, while the unit also measures *D* according to the units in it, therefore the unit measures the number *D* the same number of times as the magnitude *C* measures *A*. VII.Def.20

Therefore *C* is to *A* as the unit is to *D*. Therefore, inversely, *A* is to *C* as *D* is to the unit. V.7.Cor

Again, since *C* measures *B* according to the units in *E*, while the unit also measures *E* according to the units in it, therefore the unit measures *E* the same number of times as *C* measures *B*. Therefore *C* is to *B* as the unit is to *E*.

But it was also proved that *A* is to *C* as *D* is to the unit, therefore, *ex aequali*, *A* is to *B* as the number *D* is to *E*. V.22

Therefore the commensurable magnitudes *A* and *B* have to one another the ratio which the number *D* has to the number *E*.

Therefore, *commensurable magnitudes have to one another the ratio which a number has to a number.*

Q.E.D.

Guide

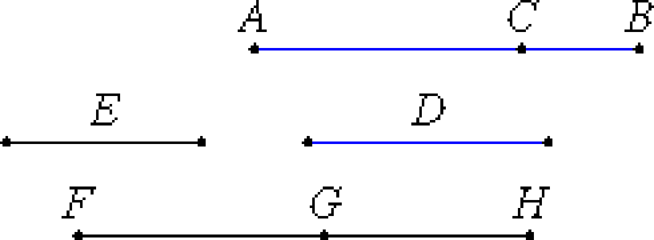
If $A = mC$ and $B = nC$, then $A:B = m:n$.

The proof here assumes that numbers are magnitudes, that is to say, the two definitions of proportion V.Def.5 and VII.Def.20 are compatible.

This proposition is used in X.8, its contrapositive, and a few propositions after that.

The next proposition is the converse of this one, and the two following that are its contrapositive, and the contrapositive of this one. It is not clear why these four statements are separated into four propositions, but the following four statements are bundled together into one proposition X.9 instead of being separate. Perhaps originally each group of four was bundled together, but later the first group was separated.

Book X Introduction - Proposition X.4 - Proposition X.6.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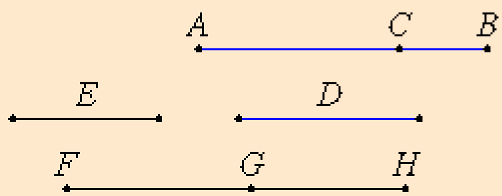
Euclid's Elements
Book X
Proposition 50

To find the third binomial line.

Set out two numbers *AC* and *CB* such that the sum of them *AB* has to *BC* the ratio which a square number has to a square number, but does not have to *AC* the ratio which a square number has to a square number.

Also set out any other number *D*, not square, and let it not have to either of the numbers *BA* and *AC* the ratio which a square number has to a square number.

Set out any rational straight line *E*, and let it be contrived that *D* is to *AB* as the square on *E* is to the square on *FG*. Then the square on *E* is commensurable with the square on *FG*. X.6,Cor. X.6



And *E* is rational, therefore *FG* is also rational. And, since *D* does not have to *AB* the ratio which a square number has to a square number, neither does the square on *E* have to the square on *FG* the ratio which a square number has to a square number, therefore *E* is incommensurable in length with *FG*. X.9

Next let it be contrived that the number *BA* is to *AC* as the square on *FG* is to the square on *GH*. Then the square on *FG* is commensurable with the square on *GH*. X.6,Cor. X.6

But *FG* is rational, therefore *GH* is also rational. And, since *BA* does not have to *AC* the ratio which a square number has to a square number, neither does the square on *FG* have to the square on *HG* the ratio which a square number has to a square number, therefore *FG* is incommensurable in length with *GH*. X.9

Therefore *FG* and *GH* are rational straight lines commensurable in square only. Therefore *FH* is binomial. X.36

I say next that it is also a third binomial straight line.

Since *D* is to *AB* as the square on *E* is to the square on *FG*, and *BA* is to *AC* as the square on *FG* is to the square on *GH*, therefore, *ex aequali*, *D* is to *AC* as the square on *E* is to the square on *GH*. V.22

But *D* does not have to *AC* the ratio which a square number has to a square number, therefore neither does the square on *E* have to the square on *GH* the ratio which a square number has to a square number. Therefore *E* is incommensurable in length with *GH*. X.9

Since *BA* is to *AC* as the square on *FG* is to the square on *GH*, therefore the square on *FG* is greater than the square on *GH*.

Let then the sum of the squares on *GH* and *K* equal the square on *FG*. Then, in conversion, *AB* is to *BC* as the square on *FG* is to the square on *K*. V.19,Cor.

But *AB* has to *BC* the ratio which a square number has to a square number, therefore the square on *FG* also has to the square on *K* the ratio which a square number has to a square number. Therefore *FG* is commensurable in length with *K*. X.9

Therefore the square on *FG* is greater than the square on *GH* by the square on a straight line commensurable with *FG*.

And *FG* and *GH* are rational straight lines commensurable in square only, and neither of them is commensurable in length with *E*.

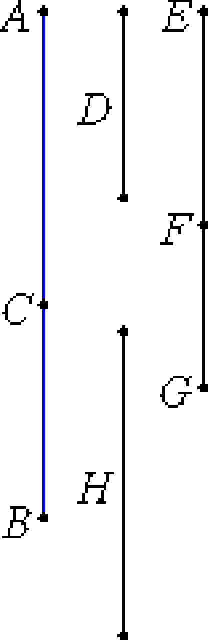
Therefore *FH* is a third binomial straight line.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.49 - Proposition X.51.



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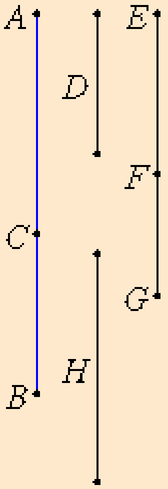
Euclid's Elements
Book X
Proposition 51

To find the fourth binomial straight line.

Set out two numbers *AC* and *CB* such that *AB* has neither to *BC* nor to *AC* the ratio which a square number has to a square number.

Set out a rational straight line *D*, and let *EF* be commensurable in length with *D*. Then *EF* is also rational.



Let it be contrived that the number *BA* is to *AC* as the square on *EF* is to the square on *FG*. Then the square on *EF* is commensurable with the square on *FG*. Therefore *FG* is also rational. X.6,Cor. X.6

Now, since *BA* does not have to *AC* the ratio which a square number has to a square number, neither does the square on *EF* have to the square on *FG* the ratio which a square number has to a square number, therefore *EF* is incommensurable in length with *FG*. X.9

Therefore *EF* and *FG* are rational straight lines commensurable in square only, so that *EG* is binomial.

I say next that it is also a fourth binomial straight line.

Since *BA* is to *AC* as the square on *EF* is to the square on *FG*, therefore the square on *EF* is greater than the square on *FG*.

Let then the sum of the squares on *FG* and *H* equal the square on *EF*. Then, in conversion, the number *AB* is to *BC* as the square on *EF* is to the square on *H*. V.19,Cor.

But *AB* does not have to *BC* the ratio which a square number has to a square number, therefore neither does the square on *EF* have to the square on *H* the ratio which a square number has to a square number.

Therefore *EF* is incommensurable in length with *H*. Therefore the square on *EF* is greater than the square on *GF* by the square on a straight line incommensurable with *EF*. X.9

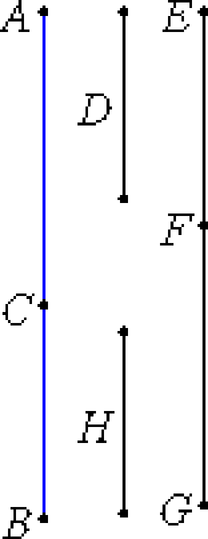
And *EF* and *FG* are rational straight lines commensurable in square only, and *EF* is commensurable in length with *D*. Therefore *EG* is a fourth binomial straight line.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.50 - Proposition X.52.



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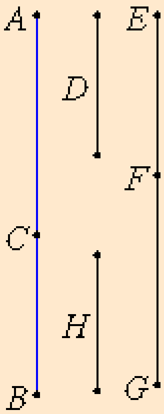
Euclid's Elements
Book X
Proposition 52

To find the fifth binomial line.

Set out two numbers *AC* and *CB* such that *AB* does not have to either of them the ratio which a square number has to a square number. Set out any rational straight line *D*, and let *EF* be commensurable with *D*. Then *EF* is rational.



Let it be contrived that *CA* is to *AB* as the square on *EF* is to the square on *FG*. X.6,Cor.

But *CA* does not have to *AB* the ratio which a square number has to a square number, therefore neither does the square on *EF* have to the square on *FG* the ratio which a square number has to a square number.

Therefore *EF* and *FG* are rational straight lines commensurable in square only, therefore *EG* is binomial. X.36 X.9

I say next that it is also a fifth binomial straight line.

Since *CA* is to *AB* as the square on *EF* is to the square on *FG*, therefore, inversely, *BA* is to *AC* as the square on *FG* is to the square on *FE*. Therefore the square on *GF* is greater than the square on *FE*. V.7.Cor

Let then the sum of the squares on *EF* and *H* equal the square on *GF*. Then, in conversion, the number *AB* is to *BC* as the square on *GF* is to the square on *H*. V.19,Cor.

But *AB* does not have to *BC* the ratio which a square number has to a square number, therefore neither does the square on *FG* have to the square on *H* the ratio which a square number has to a square number.

Therefore *FG* is incommensurable in length with *H*, so that the square on *FG* is greater than the square on *FE* by the square on a straight line incommensurable with *FG*. X.9

And *GF* and *FE* are rational straight lines commensurable in square only, and the lesser term *EF* is commensurable in length with the rational straight line *D* set out.

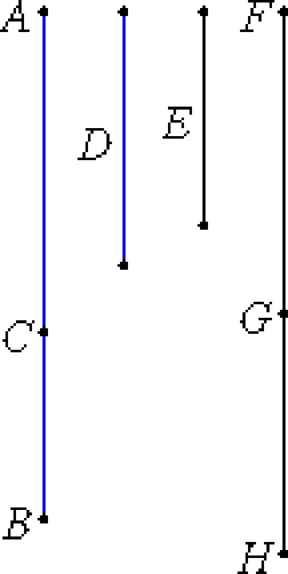
Therefore *EG* is a fifth binomial straight line.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.51 - Proposition X.53.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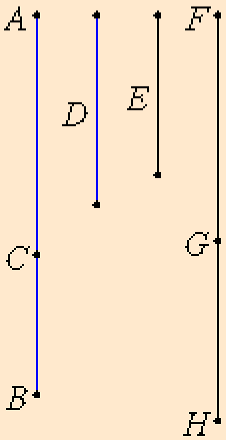
Euclid's Elements

Book X

Proposition 53

To find the sixth binomial line.

Set out two numbers *AC* and *CB* such that *AB* does not have to either of them the ratio which a square number has to a square number, and let there also be another number *D* which is not square and which does not have to either of the numbers *BA* or *AC* the ratio which a square number has to a square number.



Set out any rational straight line *E*, and let it be contrived that *D* is to *AB* as the square on *E* is to the square on *FG*. Then the square on *E* is commensurable with the square on *FG*. And *E* is rational, therefore *FG* is also rational. X.6,Cor. X.6

Now, since *D* does not have to *AB* the ratio which a square number has to a square number, neither does the square on *E* have to the square on *FG* the ratio which a square number has to a square number, therefore *E* is incommensurable in length with *FG*. X.9

Again, let it be contrived that *BA* is to *AC* as the square on *FG* is to the square on *GH*. Then the square on *FG* is commensurable with the square on *HG*. X.6,Cor. X.6

Therefore the square on *HG* is rational. Therefore *HG* is rational. And, since *BA* does not have to *AC* the ratio which a square number has to a square number, neither does the square on *FG* have to the square on *GH* the ratio which a square number has to a square number, therefore *FG* is incommensurable in length with *GH*. X.9

Therefore *FG* and *GH* are rational straight lines commensurable in square only. Therefore *FH* is binomial. X.36

It is next to be proved that it is also a sixth binomial straight line.

Since *D* is to *AB* as the square on *E* is to the square on *FG*, and also *BA* is to *AC* as the square on *FG* is to the square on *GH*, therefore, *ex aequali*, *D* is to *AC* as the square on *E* is to the square on *GH*. V.22

But *D* does not have to *AC* the ratio which a square number has to a square number, therefore neither does the square on *E* have to the square on *GH* the ratio which a square number has to a square number, therefore *E* is incommensurable in length with *GH*. X.9

But it was also proved incommensurable with *FG*, therefore each of the straight lines *FG* and *GH* is incommensurable in length with *E*.

And, since *BA* is to *AC* as the square on *FG* is to the square on *GH*, therefore the square on *FG* is greater than the square on *GH*.

Let then the sum of the squares on *GH* and *K* equal the square on *FG*. Then, in conversion, *AB* is to *BC* as the square on *FG* is to the square on *K*. V.19,Cor.

But *AB* does not have to *BC* the ratio which a square number has to a square number, so that neither does the square on *FG* have to the square on *K* the ratio which a square number has to a square number.

Therefore *FG* is incommensurable in length with *A*. Therefore the square on *FG* is greater than the square on *GH* by the square on a straight line incommensurable with *FG*. X.9

And *FG* and *GH* are rational straight lines commensurable in square only, and neither of them is commensurable in length with the rational straight line *E* set out.

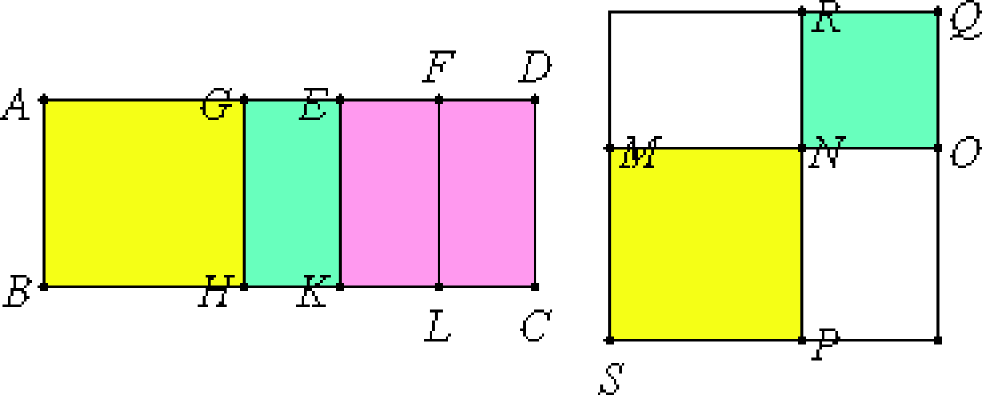
Therefore *FH* is a sixth binomial straight line.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.52 - Proposition X.54.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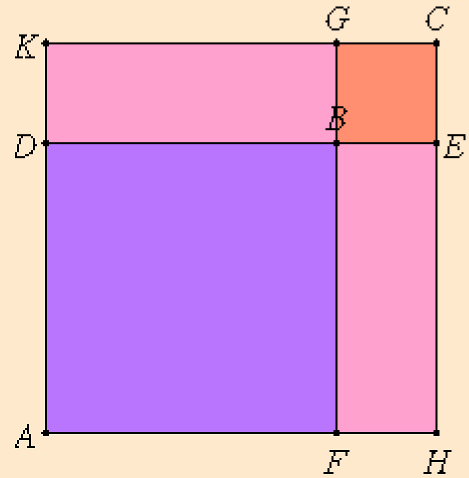
Euclid's Elements
Book X
Proposition 54

Lemma.

Let there be two squares *AB* and *BC*, and let them be placed so that *DB* is in a straight line with *BE*. Then *FB* is also in a straight line with *BG*. Complete the parallelogram *AC*.



I say that *AC* is a square, that *DG* is a mean proportional between *AB* and *BC*, and further that *DC* is a mean proportional between *AC* and *CB*.

Since *DB* equals *BF*, and *BE* is to *BG*, therefore the whole *DE* equals the whole *FG*.

But *DE* equals each of the straight lines *AH* and *KC*, and *FG* equals each of the straight lines *AK* and *HC*, therefore each of the straight lines *AH* and *KC* also equals each of the straight lines *AK* and *HC*. I.34

Therefore the parallelogram *AC* is equilateral. And it is also rectangular, therefore *AC* is a square.

Since *FB* is to *BG* as *DB* is to *BE*, while *FB* is to *BG* as *AB* is to *DG*, and *DB* is to *BE* as *DG* is to *BC*, therefore *AB* is to *DG* as *DG* is to *BC*. VI.1 VI.11

Therefore *DG* is a mean proportional between *AB* and *BC*.

I say next that *DC* is also a mean proportional between *AC* and *CB*.

Since *AD* is to *DK* as *KG* is to *GC*, for they are equal respectively, and, taken jointly, *AK* is to *KD* as *KC* is to *CG*, while *AK* is to *KD* as *AC* is to *CD*, and *AC* is to *CG* as *DC* is to *CB*, therefore *AC* is to *DC* as *DC* is to *BC*. V.18 VI.1 VI.11

Therefore *DC* is a mean proportional between *AC* and *CB*.

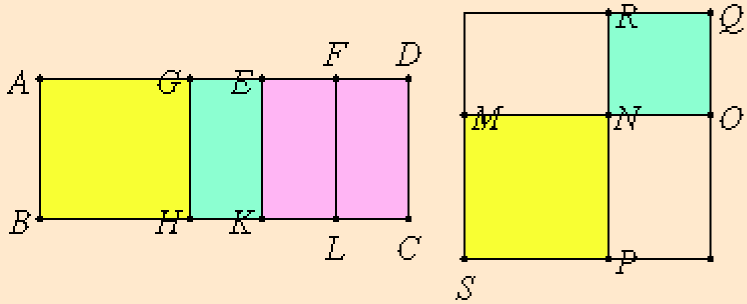
Proposition 54

If an area is contained by a rational straight line and the first binomial, then the side of the area is the irrational straight line which is called binomial.

Let the area *AC* be contained by the rational straight line *AB* and the first binomial *AD*.

I say that the side of the area *AC* is the irrational straight line which is called binomial.

Since *AD* is a first binomial straight line, divide it into its terms at *E*, and let *AE* be the greater term.



It is then manifest that *AE* and *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, and *AE* is commensurable in length with the rational straight line *AB* set out. X.Def.II.1

Bisect *ED* at the point *F*.

Then, since the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, therefore, if there is applied to the greater *AE* a parallelogram equal to the fourth part of the square on the less, that is, to the square on *EF*, and deficient by a square figure, then it divides it into commensurable parts. X.17

Apply to *AE* the rectangle *AG* by *GE* equal to the square on *EF*. Then *AG* is commensurable in length with *EG*.

Draw *GH*, *EK*, and *FL* from *G*, *E*, and *F* parallel to either of the straight lines *AB* and *CD*. Construct the square *SN* equal to the parallelogram *AH*, and the square *NQ* equal to *GK*, and place them so that *MN* is in a straight line with *NO*. Then *RN* is also in a straight line with *NP*. Complete the parallelogram *SQ*. Then *SQ* is a square. II.4 Lemma

Now, since the rectangle *AG* by *GE* equals the square on *EF*, therefore *AG* is to *EF* as *FE* is to *EG*. VI.17

Therefore *AH* is to *EL* as *EL* is to *KG*. Therefore *EL* is a mean proportional between *AH* and *GK*. VI.1

But *AH* equals *SN*, and *GK* equals *NQ*, therefore *EL* is a mean proportional between *SN* and *NQ*. But *MR* is also a mean proportional between the same *SN* and *NQ*, therefore *EL* equals *MR*, so that it also equals *PO*. Lemma

But *AH* and *GK* also equal *SN* and *NQ*, therefore the whole *AC* equals the whole *SQ*, that is, it equals the square on *MO*, Therefore *MO* is the side of *AC*.

I say next that *MO* is binomial.

Since *AG* is commensurable with *GE*, therefore *AE* is also commensurable with each of the straight lines *AG* and *GE*. X.15

But *AE* is also, by hypothesis, commensurable with *AB*, therefore *AG* and *GE* are also commensurable with *AB*. X.12

And *AB* is rational, therefore each of the straight lines *AG* and *GE* is also rational. Therefore each of the rectangles *AH* and *GK* is rational, and *AH* is commensurable with *GK*. X.19

But *AH* equals *SN*, and *GK* equals *NQ*, therefore the sum of *SN* and *NQ*, that is the squares on *MN* and *NO*, are rational and commensurable.

Since *AE* is incommensurable in length with *ED*, while *AE* is commensurable with *AG*, and *DE* is commensurable with *EF*, therefore *AG* is also incommensurable with *EF*, so that *AH* is also incommensurable with *EL*. X.13 VI.1 X.11

But *AH* equals *SN*, and *EL* equals *MR*, therefore *SN* is also incommensurable with *MR*. But *SN* is to *MR* as VI.1

PN is to *NR*, therefore *PN* is incommensurable with *NR*. X.11

But *PN* equals *MN*, and *NR* equals *NO*, therefore *MN* is incommensurable with *NO*. And the square on *MN* is commensurable with the square on *NO*, and each is rational, therefore *MN* and *NO* are rational straight lines commensurable in square only.

Therefore *MO* is binomial and the side of *AC*. X.36

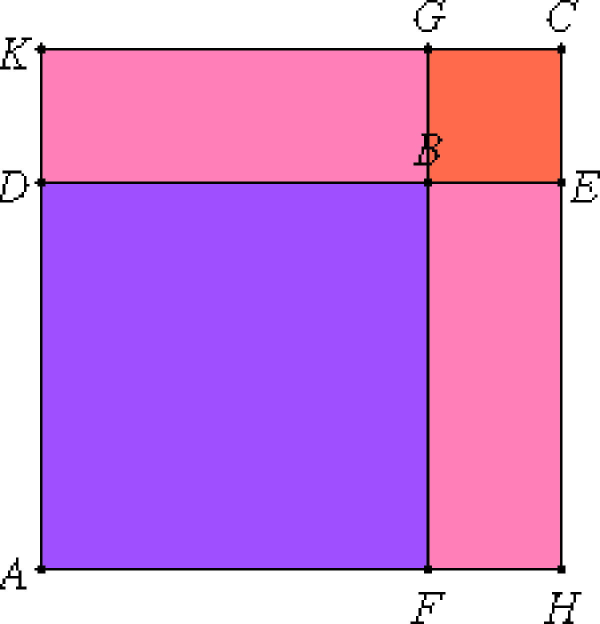
Therefore, *if an area is contained by a rational straight line and the first binomial, then the side of the area is the irrational straight line which is called binomial.*

Q.E.D.

Guide

The lemma before the proposition is used in this proposition, X.60, and X.11. The proposition itself is used in X.71.

Book X Introduction - Proposition X.53 - Proposition X.55.



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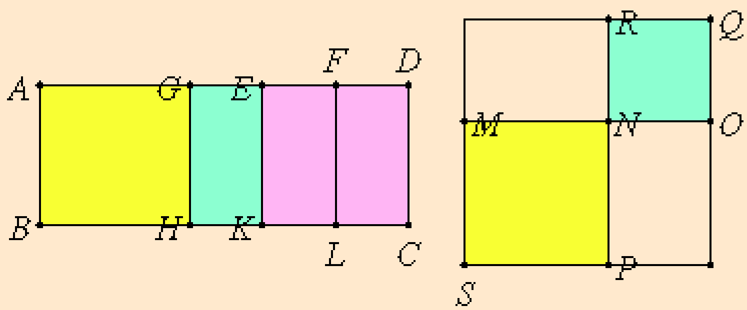
Euclid's Elements
Book X
Proposition 55

If an area is contained by a rational straight line and the second binomial, then the side of the area is the irrational straight line which is called a first bimedial.

Let the area *ABCD* be contained by the rational straight line *AB* and the second binomial *AD*.

I say that the side of the area *AC* is a first bimedial straight line.

Since *AD* is a second binomial straight line, divide it into its terms at *E*, so that *AE* is the greater term. Then *AE* and *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, and the lesser term *ED* is commensurable in length with *AB*. X.Def.II.2



Bisect *ED* at *F*, and apply to *AE* the rectangle *AG* by *GE* equal to the square on *EF* and deficient by a square figure. Then *AG* is commensurable in length with *GE*. X.17

Draw *GH*, *EK*, and *FL* through *G*, *E*, and *F* parallel to *AB* and *CD*. Construct the square *SN* equal to the parallelogram *AH*, and the square *NQ* equal to *GK*, and place them so that *MN* is in a straight line with *NO*. Then *RN* is also in a straight line with *NP*. Complete the square *SQ*.

It is then manifest from what was proved before that *MR* is a mean proportional between *SN* and *NQ* and equals *EL*, and that is the side of the area *AC*.

It is now to be proved that *MO* is a first bimedial straight line.

Since *AE* is commensurable in length with *ED*, while *ED* is commensurable with *AB*, therefore *AE* is incommensurable with *AB*. X.13

Since *AG* is commensurable with *EG*, therefore *AE* is also commensurable with each of the straight lines *AG* and *GE*. X.15

But *AE* is incommensurable in length with *AB*, therefore *AG* and *GE* are also incommensurable with *AB*. X.13

Therefore *BA* and *AG*, and *BA* and *GE*, are pairs of rational straight lines commensurable in square only, so that each of the rectangles *AH* and *GK* is medial. X.21

Hence, each of the squares *SN* and *NQ* is medial. Therefore *MN* and *NO* are also medial.

Since *AG* is commensurable in length with *GE*, therefore *AH* is also commensurable with *GK*, that is, *SN* is commensurable with *NQ*, that is, the square on *MN* with the square on *NO*. VI.1 X.11

Since *AE* is incommensurable in length with *ED*, while *AE* is commensurable with *AG*, and *ED* is commensurable with *EF*, therefore *AG* is incommensurable with *EF*, so that *AH* is also incommensurable with *EL*, that is, *SN* is incommensurable with *MR*, that is, *PN* with *NR*, that is, *MN* is incommensurable in length with *NO*. X.13 VI.1 X.11

But *MN* and *NO* were proved to be both medial and commensurable in square, therefore *MN* and *NO* are medial straight lines commensurable in square only.

I say next that they also contain a rational rectangle.

Since *DE* is, by hypothesis, commensurable with each of the straight lines *AB* and *EF*, therefore *EF* is also commensurable with *EK*. X.12

And each of them is rational, therefore *EL*, that is, *MR* is rational, and *MR* is the rectangle *MN* by *NO*. X.19

But, if two medial straight lines commensurable in square only and containing a rational rectangle are added together, then the whole is irrational and is called a first bimedial straight line. Therefore *MO* is a first bimedial straight line. X.37

Therefore, *if an area is contained by a rational straight line and the second binomial, then the side of the area is the irrational straight line which is called a first bimedial.*

Q.E.D.

Guide

This proposition is used in X.71.

Book X Introduction - Proposition X.54 - Proposition X.56.

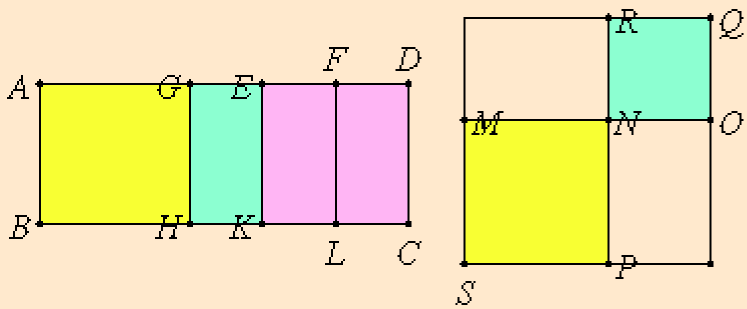
Euclid's Elements
Book X
Proposition 56

If an area is contained by a rational straight line and the third binomial, then the side of the area is the irrational straight line called a second bimedial.

Let the area *ABCD* be contained by the rational straight line *AB* and the third binomial *AD* divided into its terms at *E*, of which terms *AE* is the greater.

I say that the side of the area *AC* is the irrational straight line called a second bimedial.

Make the same construction as before.



Now, since *AD* is a third binomial straight line, therefore *AE* and *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line commensurable with *AE*, and neither of the terms *AE* and *ED* is commensurable in length with *AB*. X.Def.II.3

Then, in manner similar to the foregoing, we shall prove that *MO* is the side of the area *AC*, and *MN* and *NO* are medial straight lines commensurable in square only, so that *MO* is bimedial.

It is next to be proved that it is also a second bimedial straight line.

Since *DE* is incommensurable in length with *AB*, that is, with *EK*, and *DE* is commensurable with *EF*, therefore *EF* is incommensurable in length with *EK*. X.13

And they are rational, therefore *FE* and *EK* are rational straight lines commensurable in square only. Therefore *EL*, that is, *MR*, is medial. X.21

And it is contained by *MN* and *NO*, therefore the rectangle *MN* by *NO* is medial. Therefore *MO* is a second bimedial straight line. X.38

Therefore, *if an area is contained by a rational straight line and the third binomial, then the side of the area is the irrational straight line called a second bimedial.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.55 - Proposition X.57.

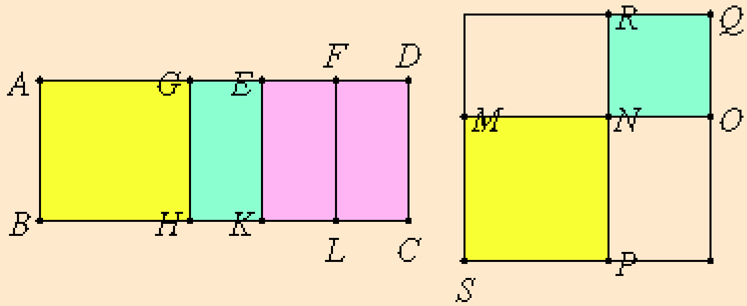
Euclid's Elements
Book X
Proposition 57

If an area is contained by a rational straight line and the fourth binomial, then the side of the area is the irrational straight line called major.

Let the area *AC* be contained by the rational straight line *AB* and the fourth binomial *AD* divided into its terms at *E*, of which terms let *AE* be the greater.

I say that the side of the area *AC* is the irrational straight line called major.

Since *AD* is a fourth binomial straight line, therefore *AE* and *ED* are rational straight lines commensurable in square only, the square on *AE* is greater than the square on *ED* by the square on a straight line incommensurable with *AE*, and *AE* is commensurable in length with *AB*. X.Def.II.4



Bisect *DE* at *F*, and apply to *AE* a parallelogram, the rectangle *AG* by *GE*, equal to the square on *EF*. Then *AG* is incommensurable in length with *GE*. X.18

Draw *GH*, *EK*, and *FL* parallel to *AB*, and make the rest of the construction as before.

It is then manifest that *MO* is the side of the area *AC*.

It is next to be proved that *MO* is the irrational straight line called major.

Since *AG* is incommensurable with *EG*, therefore *AH* is also incommensurable with *GK*, that is, *SN* with *NQ*. Therefore *MN* and *NO* are incommensurable in square. VI.1 X.11

Since *AE* is commensurable with *AB*, therefore *AK* is rational, and it equals the sum of the squares on *MN* and *NO*. Therefore the sum of the squares on *MN* and *NO* is also rational. X.19

Since *DE* is incommensurable in length with *AB*, that is, with *EK*, while *DE* is commensurable with *EF*, therefore *EF* is incommensurable in length with *EK*. X.13

Therefore *EK* and *EF* are rational straight lines commensurable in square only. Therefore *LE*, that is, *MR*, is medial. X.21

And it is contained by *MN* and *NO*, therefore the rectangle *MN* by *NO* is medial. And the sum of the squares on *MN* and *NO* is rational, and *MN* and *NO* are incommensurable in square.

But, if two straight lines incommensurable in square and making the sum of the squares on them rational, but the rectangle contained by them medial, are added together, then the whole is irrational and is called major. Therefore *MO* is the irrational straight line called major and is the side of the area *AC*. X.39

Therefore, *if an area is contained by a rational straight line and the fourth binomial, then the side of the area is the irrational straight line called major.*

Q.E.D.

Guide

This proposition is used in X.70.

Book X Introduction - Proposition X.56 - Proposition X.58.

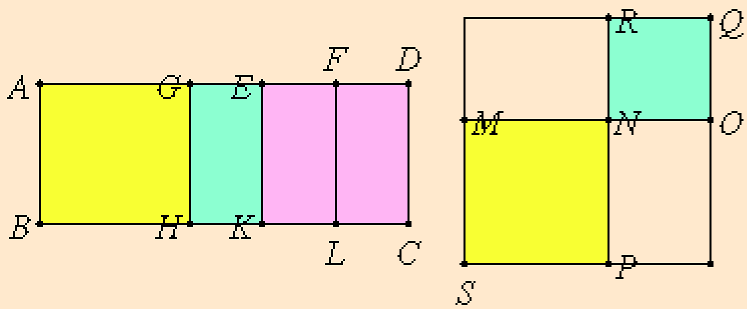
Euclid's Elements
Book X
Proposition 58

If an area is contained by a rational straight line and the fifth binomial, then the side of the area is the irrational straight line called the side of a rational plus a medial area.

Let the area *AC* be contained by the rational straight line *AB* and the fifth binomial *AD* divided into its terms at *E*, so that *AE* is the greater term.

I say that the side of the area *AC* is the irrational straight line called the side of a rational plus a medial area.

Make the same construction shown before. It is then manifest that *MO* is the side of the area *AC*. It is then to be proved that *MO* is the side of a rational plus a medial area.



Since *AG* is incommensurable with *GE*, therefore *AH* is also commensurable with *HE*, that is, the square on *MN* with the square on *NO*. Therefore *MN* and *NO* are incommensurable in square. X.18 VI.1 X.11

Since *AD* is a fifth binomial straight line, and *ED* the lesser segment, therefore *ED* is commensurable in length with *AB*. X.Def.II.5

But *AE* is incommensurable with *ED*, therefore *AB* is also incommensurable in length with *AE*. Therefore *AK*, that is, the sum of the squares on *MN* and *NO*, is medial. X.13 X.21

Since *DE* is commensurable in length with *AB*, that is, with *EK*, while *DE* is commensurable with *EF*, therefore *EF* is also commensurable with *EK*. X.12

And *EK* is rational, therefore *EL*, that is, *MR*, that is, the rectangle *MN* by *NO*, is also rational. X.19

Therefore *MN* and *NO* are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational.

Therefore *MO* is the side of a rational plus a medial area and is the side of the area *AC*. X.40

Therefore, *if an area is contained by a rational straight line and the fifth binomial, then the side of the area is the irrational straight line called the side of a rational plus a medial area.*

Q.E.D.

Guide

This proposition is used in X.71.

Book X Introduction - Proposition X.57 - Proposition X.59.

Euclid's Elements
Book X
Proposition 59

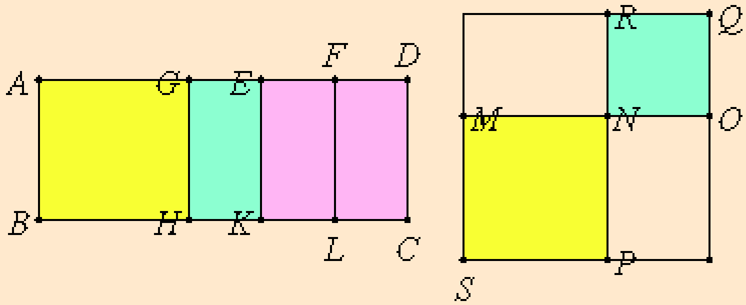
If an area is contained by a rational straight line and the sixth binomial, then the side of the area is the irrational straight line called the side of the sum of two medial areas.

Let the area *ABCD* be contained by the rational straight line *AB* and the sixth binomial *AD*, divided into its terms at *E*, so that *AE* is the greater term.

I say that the side of *AC* is the side of the sum of two medial areas.

Make the same construction as shown before.

It is then manifest that *MO* is the side of *AC*, and that l! *IN* is incommensurable in square with *NO*.



Now, since *EA* is incommensurable in length with *AB*, therefore *EA* and *AB* are rational straight lines commensurable in square only, therefore *AK*, that is, the sum of the squares on *MN* and *NO*, is medial. X.21

Again, since *ED* is incommensurable in length with *AB*, therefore *FE* is also incommensurable with *EK*. Therefore *FE* and *EK* are rational straight lines commensurable in square only. Therefore *EL*, that is, *MR*, that is, the rectangle *MN* by *NO*, is medial. X.13 X.21

Since *AE* is incommensurable with *EF*, therefore *AK* is also incommensurable with *EL*. VI.1 X.11

But *AK* is the sum of the squares on *MN* and *NO*, and *EL* is the rectangle *MN* by *NO*, therefore the sum of the squares on *MN* and *NO* is incommensurable with the rectangle *MN* by *NO*. And each of them is medial, and *MN* and *NO* are incommensurable in square.

Therefore *MO* is the side of the sum of two medial areas, and is the side of *AC*. X.41

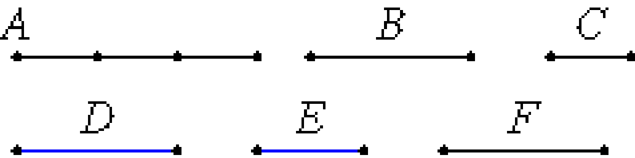
Therefore, *if an area is contained by a rational straight line and the sixth binomial, then the side of the area is the irrational straight line called the side of the sum of two medial areas.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.58 - Proposition X.60.



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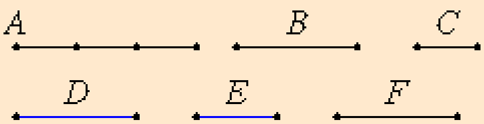
Euclid's Elements
Book X
Proposition 6

If two magnitudes have to one another the ratio which a number has to a number, then the magnitudes are commensurable.

Let the two magnitudes *A* and *B* have to one another the ratio which the number *D* has to the number *E*.

I say that the magnitudes *A* and *B* are commensurable.

Divide *A* into as many equal parts as there are units in *D*, and let *C* equal one of them, and let *F* be made up of as many magnitudes equal to *C* as there are units in *E*.



Since then there are in *A* as many magnitudes equal to *C* as there are units in *D*, whatever part the unit is of *D*, the same part is *C* of *A* also. Therefore *C* is to *A* as the unit is to *D*. VII.Def.20

But the unit measures the number *D*, therefore *C* also measures *A*. And since *C* is to *A* as the unit is to *D*, therefore, inversely, *A* is to *C* as the number *D* is to the unit. V.7,Cor.

Again, since there are in *F* as many magnitudes equal to *C* as there are units in *E*, therefore *C* is to *F* as the unit is to *E*. VII.Def.20

But it was also proved that *A* is to *C* as *D* is to the unit, therefore, *ex aequali*, *A* is to *F* as *D* is to *E*. V.22

But *D* is to *E* as *A* is to *B*, therefore *A* is to *B* as it is to *F* also. V.11

Therefore *A* has the same ratio to each of the magnitudes *B* and *F*. Therefore *B* equals *F*. V.9

But *C* measures *F*, therefore it measures *B* also. Further it measures *A* also, therefore *C* measures *A* and *B*.

Therefore *A* is commensurable with *B*.

Therefore, *if two magnitudes have to one another the ratio which a number has to a number, then the magnitudes are commensurable.*

Q.E.D.

Corollary.

From this it is manifest that, *if there are two numbers as D and E, and a straight line as A, then it is possible to make a straight line F such that the given straight line is to it as the number D is to the number E.*

And *if a mean proportional is also taken between A and F, as B, then A is to F as the square on A is to the square on B, that is, the first is to the third as the figure on the first is to that which is similar and similarly described on the second.* V.19,Cor.

But *A* is to *F* as the number *D* is to the number *E*, therefore the number *D* is to the number *E* as the figure on the straight line *A* is to the figure on the straight line *B*.

Guide

If A:B = m:n, then, with C equal to A/m, it follows that $A = mC$ and $B = nC$.

The proof assumes that magnitudes are divisible. Not all magnitudes, however, are constructively divisible. For instance, a 60° angle cannot be trisected by a Euclidean construction. An alternate proof which does not depend on divisibility of magnitudes can be based on antenaresis.

Use of this propostion

The proposition is used in very frequently in Book X starting with the next proposition, its contrapositive. It is also used in proposition XIII.6. The corollary is also used frequently in Book X starting with X.10.

Book X Introduction - Proposition X.5 - Proposition X.7.

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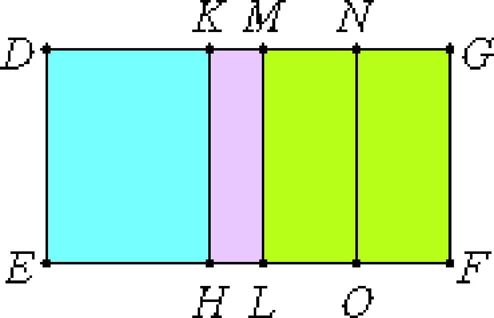
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Euclid's Elements
Book X
Proposition 60

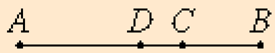
Lemma.

If a straight line is cut into unequal parts, then the sum of the squares on the unequal parts is greater than twice the rectangle contained by the unequal parts.

Let *AB* be a straight line, and let it be cut into unequal parts at *C*, and let *AC* be the greater.

I say that the sum of the squares on *AC* and *CB* is greater than twice the rectangle *AC* by *CB*.

Bisect *AB* at *D*.



Since a straight line is cut into equal parts at *D* and into unequal parts at *C*, therefore the rectangle *AC* by *CB* together with the square on *CD* equals the square on *AD*, so that the rectangle *AC* by *CB* is less than the square on *AD*. Therefore twice the rectangle *AC* by *CB* is less than double the square on *AD*. II.5

But the sum of the squares on *AC* and *CB* is double the sum of the squares on *AD* and *DC*, therefore the sum of the squares on *AC* and *CB* is greater than twice the rectangle *AC* by *CB*. II.9

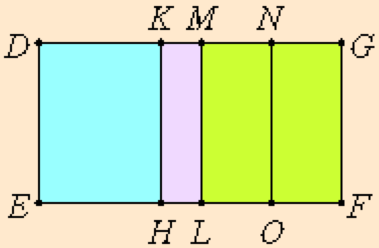
Q.E.D.

Proposition 60

The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial.

Let *AB* be a binomial straight line divided into its terms at *C*, so that *AC* is the greater term, let a rational straight line *DE* be set out, and let *DEFG* equal the square on *AB* be applied to *DE* producing *DG* as its breadth.

I say that *DG* is a first binomial straight line.



Apply to *DE* the rectangle *DH* equal to the square on *AC*, and *KL* equal to the square on *BC*. Then the remainder, twice the rectangle *AC* by *CB*, equals *MF*.

Bisect *MG* at *N*, and draw *NO* parallel to *ML* or *GF*. Then each of the rectangles *MO* and *NF* equals once the rectangle *AC* by *CB*.

Now, since *AB* is a binomial divided into its terms at *C*, therefore *AC* and *CB* are rational straight lines commensurable in square only. X.36

Therefore the squares on *AC* and *CB* are rational and commensurable with one an other, so that the sum X.15

of the squares on *AC* and *CB* is also rational. And it equals *DL*, therefore *DL* is rational.

And it is applied to the rational straight line *DE*, therefore *DM* is rational and commensurable in length with *DE*. X.20

Again, since *AC* and *CB* are rational straight lines commensurable in square only, therefore twice the rectangle *AC* by *CB*, that is *MF*, is medial. X.21

And it is applied to the rational straight line *ML*, therefore *MG* is also rational and incommensurable in length with *ML*, that is, *DE*. X.22

But *MD* is also rational and is commensurable in length with *DE*, therefore *DM* is incommensurable in length with *MG*. X.13

And they are rational, therefore *DM* and *MG* are rational straight lines commensurable in square only. Therefore *DG* is binomial. X.36

It is next to be proved that it is also a first binomial straight line.

Since the rectangle *AC* by *CB* is a mean proportional between the squares on *AC* and *CB*, therefore *MO* is also a mean proportional between *DH* and *KL*. X.54,Lemma

Therefore *DH* is to *MO* as *MO* is to *KL*, that is *DK* is to *MN* as *MN* is to *MK*. Therefore the rectangle *DK* by *KM* equals the square on *MN*. VI.1 VI.17

Since the square on *AC* is commensurable with the square on *CB*, therefore *DH* is also commensurable with *KL*, so that *DK* is also commensurable with *KM*. VI.1 X.11

Since the sum of the squares on *AC* and *CB* is greater than twice the rectangle *AC* by *CB*, therefore *DL* is also greater than *MF*, so that *DM* is also greater than *MG*. VI.1 Lemma

And the rectangle *DK* by *KM* equals the square on *MN*, that is, to the fourth part of the square on *MG*, and *DK* is commensurable with *KM*.

But, if there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divides it into commensurable parts, then the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater. Therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*. X.17

And *DM* and *MG* are rational, and *DM*, which is the greater term, is commensurable in length with the rational straight line *DE* set out.

Therefore *DG* is a first binomial straight line. X.Def.II.1

Therefore, *the square on the binomial straight line applied to a rational straight line produces as breadth the first binomial.*

Q.E.D.

Guide

This proposition is used in X.72 and X.111.

Book X Introduction - Proposition X.59 - Proposition X.61.

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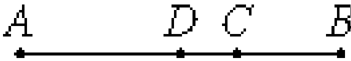
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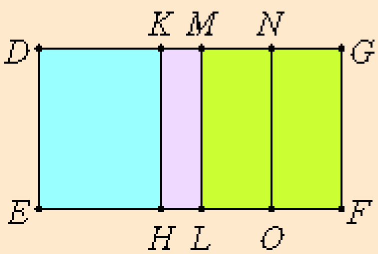
Euclid's Elements
Book X
Proposition 61

The square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial.

Let *AB* be a first bimedial straight line divided into its medials at *C*, of which medials *AC* is the greater.
Let a rational straight line *DE* be set out, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth.

I say that *DG* is a second binomial straight line.

Make the same construction as before.



Then, since *AB* is a first bimedial divided at *C*, therefore *AC* and *CB* are medial straight lines commensurable in square only, and containing a rational rectangle, so that the squares on *AC* and *CB* are also medial. X.37 X.21

Therefore *DL* is medial. And it was applied to the rational straight line *DE*, therefore *MD* is rational and incommensurable in length with *DE*. X.15 X.23,Cor. X.22

Again, since twice the rectangle *AC* by *CB* is rational, therefore *MF* is also rational.

And it is applied to the rational straight line *ML*, therefore *MG* is also rational and commensurable in length with *ML*, that is, *DE*. Therefore *DM* is incommensurable in length with *MG*. X.20 X.13

And they are rational, therefore *DM* and *MG* are rational straight lines commensurable in square only. Therefore *DG* is binomial. X.36

It is next to be proved that it is a second binomial straight line.

Since the sum of the squares on *AC* and *CB* is greater than twice the rectangle *AC* by *CB*, therefore *DL* is also greater than *MF*, so that *DM* is also greater than *MG*. VI.1

Since the square on *AC* is commensurable with the square on *CB*, therefore *DH* is also commensurable with *KL*, so that *DK* is also commensurable with *KM*. VI.1 X.11

And the rectangle *DK* by *KM* equals the square on *MN*, therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*. And *MG* is commensurable in length with *DE*. X.17

Therefore *DG* is a second binomial straight line. X.Def.II.2

Therefore, *the square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.60 - Proposition X.62.

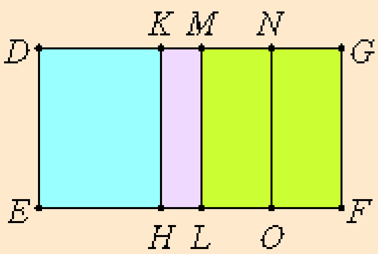
Euclid's Elements
Book X
Proposition 62

The square on the second bimedial straight line applied to a rational straight line produces as breadth the third binomial.

Let *AB* be a second bimedial straight line divided into its medials at *C*, so that *AC* is the greater segment, let *DE* be any rational straight line, and to *DE* let there be applied the parallelogram *DF* equal to the square on *AB* and producing *DG* as its breadth.

I say that *DG* is a third binomial straight line.

Make the same construction as shown before.



Then, since *AB* is a second bimedial divided at *C*, therefore *AC* and *CB* are medial straight lines commensurable in square only and containing a medial rectangle, so that the sum of the squares on *AC* and *CB* is also medial. X.38 X.15 X.23,Cor.

And it equals *DL*, therefore *DL* is also medial. And it is applied to the rational straight line *DE*, therefore *MD* is also rational and incommensurable in length with *DE*. X.22

For the same reason, *MG* is also rational and incommensurable in length with *ML*, that is, with *DE*, therefore each of the straight lines *DM* and *MG* is rational and incommensurable in length with *DE*.

Since *AC* is incommensurable in length with *CB*, and *AC* is to *CB* as the square on *AC* is to the rectangle *AC* by *CB*, therefore the square on *AC* is also incommensurable with the rectangle *AC* by *CB*. X.11

Hence the sum of the squares on *AC* and *CB* is incommensurable with twice the rectangle *AC* by *CB*, that is, *DL* is incommensurable with *MF*, so that *DM* is also incommensurable with *MG*. X.12 X.13 VI.1 X.11

And they are rational, therefore *DG* is binomial. X.36

It is to be proved that it is a third binomial straight line.

In manner similar to the foregoing we may conclude that *DM* is greater than *MG*, and that *DK* is commensurable with *KM*.

And the rectangle *DK* by *KM* equals the square on *MN*, therefore the square on *DM* is greater than the square on *MG* by the square on a straight line commensurable with *DM*. And neither of the straight lines *DM* nor *MG* is commensurable in length with *DE*.

Therefore *DG* is a third binomial straight line. X.Def.II.3

Therefore, *the square on the second bimedial straight line applied to a rational straight line produces as breadth the third binomial.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.61 - Proposition X.63.

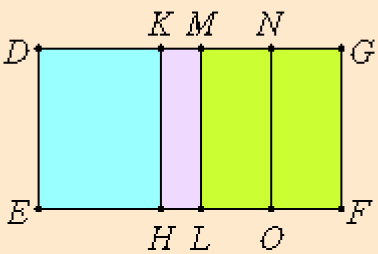
Euclid's Elements
Book X
Proposition 63

The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial.

Let *AB* be a major straight line divided at *C*, so that *AC* is greater than *CB*, let *DE* be a rational straight line, and to *DE* let there be applied the parallelogram *DF* equal to the square on *AB* and producing *DG* as its breadth.

I say that *DG* is a fourth binomial straight line.

Make the same construction as shown before.



Since *AB* is a major straight line divided at *C*, therefore *AC* and *CB* are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial. X.39

Since the sum of the squares on *AC* and *CB* is rational, therefore *DL* is rational. Therefore *DM* is also rational and commensurable in length with *DE*. X.20

Again, since twice the rectangle *AC* by *CB*, that is, *MF*, is medial, and it is applied to the rational straight line *ML*, therefore *MG* is also rational and incommensurable in length with *DE*. X.22

Therefore *DM* is also incommensurable in length with *MG*. Therefore *DM* and *MG* are rational straight lines commensurable in square only. Therefore *DG* is binomial. X.13 X.36

It is to be proved that it is a fourth binomial straight line.

In manner similar to the foregoing we can prove that *DM* is greater than *MG*, and that the rectangle *DK* by *KM* equals the square on *MN*.

Since the square on *AC* is incommensurable with the square on *CB*, therefore *DH* is also incommensurable with *KL*, so that *DK* is also incommensurable with *KM*. VI.1 X.11

But, if there are two unequal straight lines, and to the greater there is applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divides it into incommensurable parts, then the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater, therefore the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable with *DM*. X.18

And *DM* and *MG* are rational straight lines commensurable in square only, and *DM* is commensurable with the rational straight line *DE* set out.

Therefore *DG* is a fourth binomial straight line. X.Def.II.4

Therefore, *the square on the major straight line applied to a rational straight line produces as breadth the fourth* *binomial.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.62 - Proposition X.64.

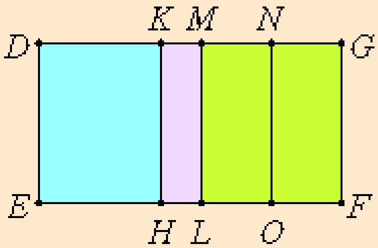
Euclid's Elements
Book X
Proposition 64

The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial.

Let *AB* be the side of a rational plus a medial area, divided into its straight lines at *C*, so that *AC* is the greater, let a rational straight line *DE* be set out, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth.

I say that *DG* is a fifth binomial straight line.

Make the same construction as before.



Since *AB* is the side of a rational plus a medial area, divided at *C*, therefore *AC* and *CB* are straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational. X.40

Since, then, the sum of the squares on *AC* and *CB* is medial, therefore *DL* is medial, so that *DM* is rational and incommensurable in length with *DE*. X.22

Again, since twice the rectangle *AC* by *CB*, that is *MF*, is rational, therefore *MG* is rational and commensurable with *DE*. X.20

Therefore *DM* is incommensurable with *MG*. Therefore *DM* and *MG* are rational straight lines commensurable in square only. Therefore *DG* is binomial. X.13 X.36

I say next that it is also a fifth binomial straight line.

For it can be proved similarly that the rectangle *DK* by *KM* equals the square on *MN*, and that *DK* is incommensurable in length with *KM*. Therefore the square on *DM* is greater than the square on *MG* by the square on a straight line incommensurable with *DM*. X.18

And *DM* and *MG* are commensurable in square only, and the less, *MG*, is commensurable in length with *DE*.

Therefore *DG* is a fifth binomial.

Therefore, *the square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.63 - Proposition X.65.

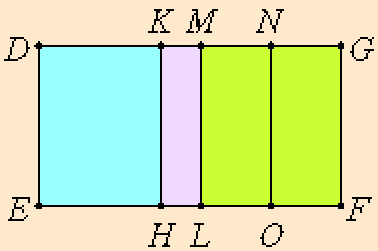
Euclid's Elements
Book X
Proposition 65

The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.

Let *AB* be the side of the sum of two medial areas, divided at *C*, let *DE* be a rational straight line, and let there be applied to *DE* the parallelogram *DF* equal to the square on *AB*, producing *DG* as its breadth.

I say that *DG* is a sixth binomial straight line.

Make the same construction as before.



Since *AB* is the side of the sum of two medial areas divided at *C*, therefore *AC* and *CB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and moreover the sum of the squares on them incommensurable with the rectangle contained by them. X.41

So that, in accordance with what was before proved, each of the rectangles *DL* and *MF* is medial. And they are applied to the rational straight line *DE*, therefore each of the straight lines *DM* and *MG* is rational and incommensurable in length with *DE*. X.22

Since the sum of the squares on *AC* and *CB* is incommensurable with twice the rectangle *AC* by *CB*, therefore *DL* is incommensurable with *MF*.

Therefore *DM* is also incommensurable with *MG*. VI.1 X.11

Therefore *DM* and *MG* are rational straight lines commensurable in square only. Therefore *DG* is binomial. X.36

I say next that it is a sixth binomial straight line.

Similarly again we can prove that the rectangle *DK* by *KM* equals the square on *MN*, and that *DK* is incommensurable in length with *KM*, and, for the same reason, the square on *DM* is greater than the square *MG* by the square on a straight line incommensurable in length with *DM*.

And neither of the straight lines *DM* nor *MG* is commensurable in length with the rational straight line *DE* set out.

Therefore *DG* is a sixth binomial straight line.

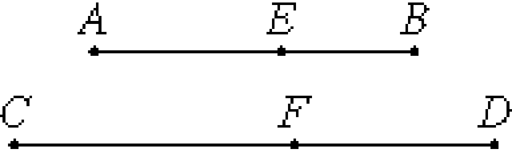
Therefore, *the square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.*

Q.E.D.

Guide

This proposition is used in X.72.

Book X Introduction - Proposition X.64 - Proposition X.66.



A
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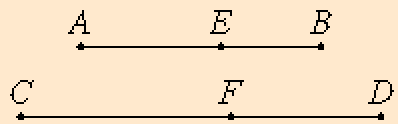
Euclid's Elements
Book X
Proposition 66

A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order.

Let *AB* be binomial, and let *CD* be commensurable in length with *AB*.

I say that *CD* is binomial and the same in order with *AB*.



Since *AB* is binomial, divide it into its terms at *E*, and let *AE* be the greater term, therefore *AE* and *EB* are rational straight lines commensurable in square only. X.36

Let it be contrived that *AB* is to *CD* as *AE* is to *CF*. Then the remainder *EB* is to the remainder *FD* as *AB* is to *CD*. VI.12 V.19

But *AB* is commensurable in length with *CD*, therefore *AE* is also commensurable with *CF*, and *EB* with *FD*. X.11

And *AE* and *EB* are rational, therefore *CF* and *FD* are also rational.

And *AE* is to *CF* as *EB* is to *FD*. Therefore, alternately, *AE* is to *EB* as *CF* is to *FD*. V.11 V.16

But *AE* and *EB* are commensurable in square only, therefore *CF* and *FD* are also commensurable in square only. X.11

And they are rational, therefore *CD* is binomial. X.36

I say next that it is the same in order with *AB*.

For the square on *AE* is greater than the square on *EB* either by the square on a straight line commensurable with *AE* or by the square on a straight line incommensurable with it.

If then the square on *AE* is greater than the square on *EB* by the square on a straight line commensurable with *AE*, then the square on *CF* is also greater than the square on *FD* by the square on a straight line commensurable with *CF*. X.14

And, if *AE* is commensurable with the rational straight line set out, then *CF* is also commensurable with it, and for this reason each of the straight lines *AB* and *CD* is a first binomial, that is, the same in order. X.12 X.Def.II.1

But, if *EB* is commensurable with the rational straight line set out, then *FD* is also commensurable with it, and for this reason again *CD* is the same in order with *AB*, for each of them is a second binomial. X.12 X.Def.II.2

But, if neither of the straight lines *AE* nor *EB* is commensurable with the rational straight line set out, then neither of the straight lines *CF* nor *FD* is commensurable with it, and each of the straight lines *AB* and *CD* is a third binomial. X.13 X.Def.II.3

But, if the square on *AE* is greater than the square on *EB* by the square on a straight line incommensurable with *AE*, then the square on *CF* is also greater than the square on *FD* by the square on a straight line X.14

incommensurable with *CF*.

And, if *AE* is commensurable with the rational straight line set out, then *CF* is also commensurable with it, and each of the straight lines *AB* and *CD* is a fourth binomial. X.Def.II.4

But, if *EB* is so commensurable, then *FD* is also, and each of the straight lines *AB* and *CD* is a fifth binomial. X.Def.II.5

But, if neither of the straight lines *AE* or *EB* is so commensurable, then neither of the straight lines *CF* or *FD* is commensurable with the rational straight line set out, and each of the straight lines *AB* and *CD* is a sixth binomial. X.Def.II.6

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order.

Therefore, *a straight line commensurable in length with a binomial straight line is itself also binomial and the same in order*.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.65 - Proposition X.67.

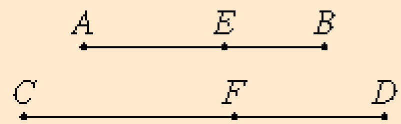
Euclid's Elements
Book X
Proposition 67

A straight line commensurable with a bimedial straight line is itself also bimedial and the same in order.

Let *AB* be bimedial, and let *CD* be commensurable in length with *AB*.

I say that *CD* is bimedial and the same in order with *AB*.

Since *AB* is bimedial, divide it into its medials at *E*.



Then *AE* and *EB* are medial straight lines commensurable in square only. X.37 X.38

Let it be contrived that *AB* is to *CD* as *AE* is to *CF*. Then the remainder *EB* is to the remainder *FD* as *AB* is to *CD*. V.19

But *AB* is commensurable in length with *CD*, therefore *AE* and *EB* are commensurable with *CF* and *FD* respectively. X.11

But *AE* and *EB* are medial, therefore *CF* and *FD* are also medial. X.23

Since *AE* is to *EB* as *CF* is to *FD*, and *AE* and *EB* are commensurable in square only, therefore *CF* and *FD* are also commensurable in square only. V.11 X.11

But they were also proved medial, therefore *CD* is bimedial.

I say next that it is also the same in order with *AB*.

Since *AE* is to *EB* as *CF* is to *FD*, therefore the square on *AE* is to the rectangle *AE* by *EB* as the square on *CF* is to the rectangle *CF* by *FD*. Therefore, alternately, the square on *AE* is to the square on *CF* as the rectangle *AE* by *EB* is to the rectangle *CF* by *FD*. V.16

But the square on *AE* is commensurable with the square on *CF*, therefore the rectangle *AE* by *EB* is commensurable with the rectangle *CF* by *FD*.

Therefore if the rectangle *AE* by *EB* is rational, then the rectangle *CF* by *FD* is also rational, and for this reason *CD* is a first bimedial, but if medial, medial, and each of the straight lines *AB* and *CD* is a second bimedial. And for this reason *CD* is the same in order with *AB*. X.37 X.23.Cor. X.38

Therefore, *a straight line commensurable with a bimedial straight line is itself also bimedial and the same in order*.

Q.E.D.

Guide

(Forthcoming)

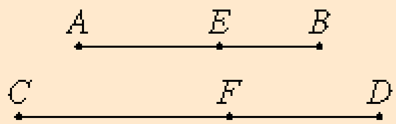
Book X Introduction - Proposition X.66 - Proposition X.68.

Euclid's Elements
Book X
Proposition 68

A straight line commensurable with a major straight line is itself also major.

Let *AB* be major, and let *CD* be commensurable with *AB*.

I say that *CD* is major.



Divide *AB* at *E*. Then *AE* and *EB* are straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial. X.39

Make the same construction as before.

Since *AB* is to *CD* as *AE* is to *CF*, and *EB* is to *FD*, therefore *AE* is to *CF* as *EB* is to *FD*. V.11

But *AB* is commensurable with *CD*, therefore *AE* and *EB* are commensurable with *CF* and *FD* respectively. X.11

Since *AE* is to *CF* as *EB* is to *FD*, alternately, also *AE* is to *EB* as *CF* is to *FD*, therefore, taken jointly, *AB* is to *BE* as *CD* is to *DF*. V.16 V.18

Therefore the square on *AB* is to the square on *BE* as the square on *CD* is to the square on *DF*. VI.20

Similarly we can prove that the square on *AB* is to the square on *AE* as the square on *CD* is to the square on *CF*. Therefore the square on *AB* is to the squares on *AE* and *EB* as the square on *CD* is to the squares on *CF* and *FD*, therefore, alternately, the square on *AB* is to the square on *CD*, so are the squares on *AE* and *EB* to the squares on *CF* and *FD*. V.16

But the square on *AB* is commensurable with the square on *CD*, therefore the squares on *AE* and *EB* are also commensurable with the squares on *CF* and *FD*.

And the squares on *AE* and *EB* together are rational, therefore the squares on *CF* and *FD* together are rational.

Similarly also twice the rectangle *AE* by *EB* is commensurable with twice the rectangle *CF* by *FD*. And twice the rectangle *AE* by *EB* is medial, therefore twice the rectangle *CF* by *FD* is also medial. X.23,Cor.

Therefore *CF* and *FD* are straight lines incommensurable in square which make, at the same time, the sum of the squares on them rational, but the rectangle contained by them medial, therefore the whole *CD* is the irrational straight line called major. X.39

Therefore a straight line commensurable with the major straight line is major.

Therefore, *a straight line commensurable with a major straight line is itself also major.*

Q.E.D.

Guide

(Forthcoming)

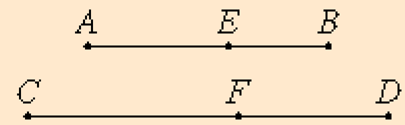
Book X Introduction - Proposition X.67 - Proposition X.69.

Euclid's Elements
Book X
Proposition 69

A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area.

Let *AB* be the side of a rational plus a medial area, and let *CD* be commensurable with *AB*.

It is to be proved that *CD* is also the side of a rational plus a medial area.



Divide *AB* into its straight lines at *E*. Then *AE* and *EB* are straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational. X.40

Make the same construction as before.

We can then prove similarly that *CF* and *FD* are incommensurable in square, and the sum of the squares on *AE* and *EB* is commensurable with the sum of the squares on *CF* and *FD*, and the rectangle *AE* by *EB* with the rectangle *CF* by *FD*, so that the sum of the squares on *CF* and *FD* is also medial, and the rectangle *CF* by *FD* rational. Therefore *CD* is the side of a rational plus a medial area.

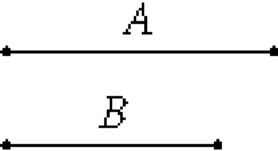
Therefore, *a straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.68 - Proposition X.70.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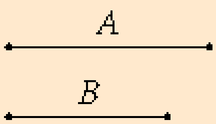
Euclid's Elements
Book X
Proposition 7

Incommensurable magnitudes do not have to one another the ratio which a number has to a number.

Let *A* and *B* be incommensurable magnitudes.

I say that *A* does not have to *B* the ratio which a number has to a number.



If *A* does have to *B* the ratio which a number has to a number, then *A* is commensurable with *B*. X.6

But it is not, therefore *A* does not have to *B* the ratio which a number has to a number.

Therefore, *incommensurable magnitudes do not have to one another the ratio which a number has to a number.*

Q.E.D.

Guide

This proposition is the contrapositive of the last one. It is used in X.11.

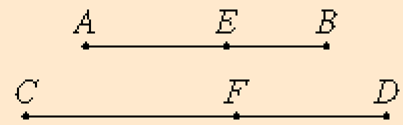
Book X Introduction - Proposition X.6 - Proposition X.8.

Euclid's Elements
Book X
Proposition 70

A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas.

Let *AB* be the side of the sum of two medial areas, and *CD* commensurable with *AB*.

It is to be proved that *CD* is also the side of the sum of two medial areas.



Since *AB* is the side of the sum of two medial areas, divide it into its straight lines at *E*, therefore *AE* and *EB* are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and furthermore the sum of the squares on *AE* and *EB* incommensurable with the rectangle *AE* by *EB*. X.41

Make the same construction as before.

We can then prove similarly that *CF* and *FD* are also incommensurable in square, the sum of the squares on *AE* and *EB* is commensurable with the sum of the squares on *CF* and *FD*, and the rectangle *AE* by *EB* with the rectangle *CF* by *FD*, so that the sum of the squares on *CF* and *FD* is also medial, the rectangle *CF* by *FD* is medial, and moreover the sum of the squares on *CF* and *FD* is incommensurable with the rectangle *CF* by *FD*.

Therefore *CD* is the side of the sum of two medial areas.

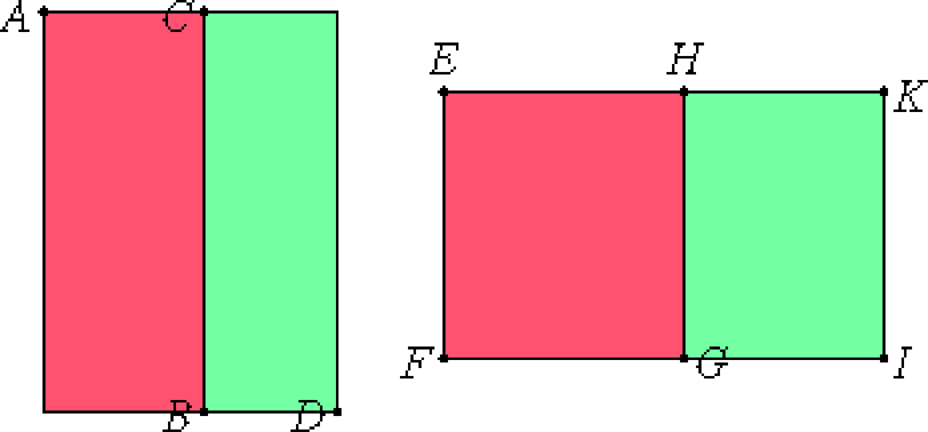
Therefore, *a straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.69 - Proposition X.71.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Euclid's Elements
Book X
Proposition 71

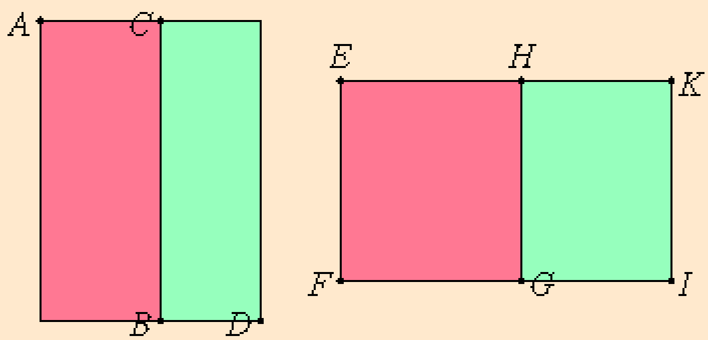
If a rational and a medial are added together, then four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area.

Let *AB* be rational, and *CD* medial.

I say that the side of the area *AD* is a binomial or a first bimedial or a major or a side of a rational plus a medial area.

For *AB* is either greater or less than *CD*.

First, let it be greater. Set out a rational straight line *EF* , apply to *EF* the rectangle *EG* equal to *AB*, producing *EH* as breadth, and apply to *EF HI*, equal to *DC*, producing *HK* as breadth.



Then, since *AB* is rational and equals *EG*, therefore *EG* is also rational. And it is applied to *EF*, producing *EH* as breadth, therefore *EH* is rational and commensurable in length with *EF*. X.20

Again, since *CD* is medial and equals *HI*, therefore *HI* is also medial. And it is applied to the rational straight line *EF*, producing *HK* as breadth, therefore *HK* is rational and incommensurable in length with *EF*. X.22

Since *CD* is medial, while *AB* is rational, therefore *AB* is incommensurable with *CD*, so that *EG* is also incommensurable with *HI*.

But *EG* is to *HI* as *EH* is to *HK*, therefore *EH* is also incommensurable in length with *HK*. VI.1 X.11

And both are rational, therefore *EH* and *HK* are rational straight lines commensurable in square only. Therefore *EK* is a binomial straight line, divided at *H*. X.36

Since *AB* is greater than *CD*, while *AB* equals *EG* and *CD* equals *HI*, therefore *EG* is also greater than *HI*. Therefore *EH* is also greater than *HK*. The square, then, on *EH* is greater than the square on *HK* either by the square on a straight line commensurable in length with *EH* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable with itself.

Now the greater straight line *HE* is commensurable in length with the rational straight line *EF* set out, therefore *EK* is a first binomial. X.Def.II.1

But *EF* is rational, and, if an area is contained by a rational straight line and the first binomial, then the side of the square equal to the area is binomial. Therefore the side of *EI* is binomial, so that the side of *AD* is also binomial. X.54

Next, let the square on *EH* be greater than the square on *HK* by the square on a straight line incommensurable with *EH*.

Now the greater straight line *EH* is commensurable in length with the rational straight line *EF* set out, therefore *EK* is a fourth binomial. X.Def.II.4

But *EF* is rational, and, if an area be contained by a rational straight line and the fourth binomial, then the side of the area is the irrational straight line called major. Therefore the side of the area *EI* is major, so that the side of the area *AD* is also major. X.57

Next, let *AB* be less than *CD*. Then *EG* is also less than *HI*, so that *EH* is also less than *HK*.

Now the square on *HK* is greater than the square on *EH* either by the square on a straight line commensurable with *HK* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now the lesser straight line *EH* is commensurable in length with the rational straight line *EF* set out, therefore *EK* is a second binomial. X.Def.II.2

But *EF* is rational, and, if an area is contained by a rational straight line and the second binomial, then the side of the square it is a first bimedial, therefore the side of the area *EI* is a first bimedial, so that the side of *AD* is also a first bimedial. X.55

Next, let the square on *HK* be greater than the square on *HE* by the square on a straight line incommensurable with *HK*.

Now the lesser straight line *EH* is commensurable with the rational straight line *EF* set out, therefore *EK* is a fifth binomial. X.Def.II.5

But *EF* is rational, and, if an area is contained by a rational straight line and the fifth binomial, then the side of the square equal to the area is a side of a rational plus a medial area. X.58

Therefore the side of the area *EI* is a side of a rational plus a medial area, so that the side of the area *AD* is also a side of a rational plus a medial area.

Therefore, *if a rational and a medial are added together, then four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.70 - Proposition X.72.

Euclid's Elements
Book X
Proposition 72

If two medial areas incommensurable with one another are added together, then the remaining two irrational straight lines arise, namely either a second bimedial or a side of the sum of two medial areas.

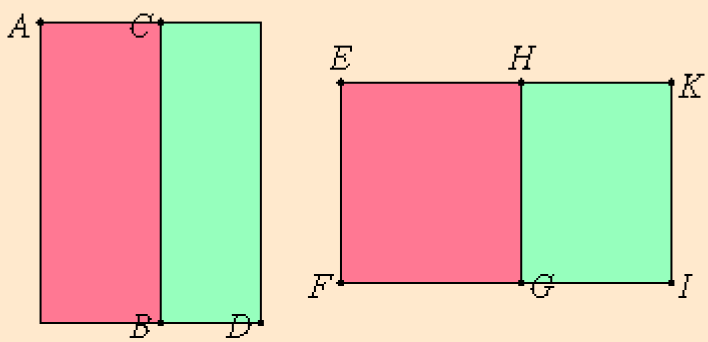
Let two medial areas *AB* and *CD* incommensurable with one another be added together.

I say that the side of the area *AD* is either a second bimedial or a side of the sum of two medial areas.

For *AB* is either greater or less than *CD*.

First, let *AB* be greater than *CD*.

Set out the rational straight line *EF*, and apply to *EF* the rectangle *EG* equal to *AB* and producing *EH* as breadth, and the rectangle *HI* equal to *CD* and producing *HK* as breadth.



Now, since each of the areas *AB* and *CD* is medial, therefore each of the areas *EG* and *HI* is also medial.

And they are applied to the rational straight line *FE* producing *EH* and *HK* as breadth, therefore each of the straight lines *EH* and *HK* is rational and incommensurable in length with *EF*. X.22

Since *AB* is incommensurable with *CD*, and *AB* equals *EG*, and *CD* equals *HI*, therefore *EG* is also incommensurable with *HI*.

But *EG* is to *HI* as *EH* is to *HK*, therefore *EH* is incommensurable in length with *HK*. VI.1 X.11

Therefore *EH* and *HK* are rational straight lines commensurable in square only, therefore *EK* is binomial. X.36

But the square on *EH* is greater than the square on *HK* either by the square on a straight line commensurable with *EH* or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now neither of the straight lines *EH* nor *HK* is commensurable in length with the rational straight line *EF* set out, therefore *EK* is a third binomial. X.Def.II.3

But *EF* is rational, and, if an area is contained by a rational straight line and the third binomial, then the side of the area is a second bimedial, therefore the side of *EI*, that is, of *AD*, is a second bimedial. X.56

Next, let the square on *EH* be greater than the square on *HK* by the square on a straight line incommensurable in length with *EH*.

Now each of the straight lines *EH* and *HK* is incommensurable in length with *EF*, therefore *EK* is a sixth binomial. X.Def.II.6

But, if an area is contained by a rational straight line and the sixth binomial, then the side of the area is the side of the sum of two medial areas, so that the side of the area *AD* is also the side of the sum of two medial areas. X.59

Therefore, *if two medial areas incommensurable with one another are added together, then the remaining two irrational straight lines arise, namely either a second bimedial or a side of the sum of two medial areas.*

Q.E.D.

Proposition

The binomial straight line and the irrational straight lines after it are neither the same with the medial nor with one another.

For the square on a medial, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied. But the square on the binomial, if applied to a rational straight line, produces as breadth the first binomial. X.22 X.60

The square on the first bimedial, if applied to a rational straight line, produces as breadth the second binomial. X.61

The square on the second bimedial, if applied to a rational straight line, produces as breadth the third binomial. X.62

The square on the major, if applied to a rational straight line, produces as breadth the fourth binomial. X.63

The square on the side of a rational plus a medial area, if applied to a rational straight line, produces as breadth the fifth binomial. X.64

The square on the side of the sum of two medial areas, if applied to a rational straight line, produces as breadth the sixth binomial. X.65

And the said breadths differ both from the first and from one another, from the first because it is rational, and from one another because they are not the same in order, so that the irrational straight lines themselves also differ from one another.

Guide

(Forthcoming)

Book X Introduction - Proposition X.71 - Proposition X.73.

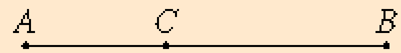

A
C
B

Euclid's Elements
Book X
Proposition 73

If from a rational straight line there is subtracted a rational straight line commensurable with the whole in square only, then the remainder is irrational; let it be called an apotome.

From the rational straight line *AB* let the rational straight line *BC*, commensurable with the whole in square only, be subtracted.

I say that the remainder *AC* is the irrational straight line called apotome.



Since *AB* is incommensurable in length with *BC*, and *AB* is to *BC* as the square on *AB* is to the rectangle *AB* by *BC*, therefore the square on *AB* is incommensurable with the rectangle *AB* by *BC*. X.11

But the sum of the squares on *AB* and *BC* is commensurable with the square on *AB*, and twice the rectangle *AB* by *BC* is commensurable with the rectangle *AB* by *BC*. X.15 X.6

And, inasmuch as the sum of the squares on *AB* and *BC* equal twice the rectangle *AB* by *BC* together with the square on *CA*, therefore the sum of the squares on *AB* and *BC* is also incommensurable with the remainder, the square on *AC*. II.7 X.13 X.16

But the sum of the squares on *AB* and *BC* is rational, therefore *AC* is irrational. Let it be called an apotome. X.Def.4

Q.E.D.

Guide

This proposition is used very frequently in the rest of Book X starting with X.75. It is also used in propositions XIII.6 and XIII.11.

Book X Introduction - Proposition X.72 - Proposition X.74.

Euclid's Elements

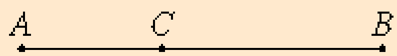
Book X

Proposition 74

If from a medial straight line there is subtracted a medial straight line which is commensurable with the whole in square only and which contains with the whole a rational rectangle, then the remainder is irrational; let it be called first apotome of a medial straight line.

From the medial straight line *AB* let there be subtracted the medial straight line *BC* which is commensurable with *AB* in square only and with *AB* makes the rectangle *AB* by *BC* rational.

I say that the remainder *AC* is irrational, and let it be called an apotome of a medial straight line.



Since *AB* and *BC* are medial, the squares on *AB* and *BC* are also medial. But twice the rectangle *AB* by *BC* is rational, therefore the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*.

Therefore twice the rectangle *AB* by *BC* is also incommensurable with the remainder, the square on *AC*, since, if the whole is incommensurable with one of the magnitudes, then the original magnitudes are also incommensurable. cf. II.7 X.16

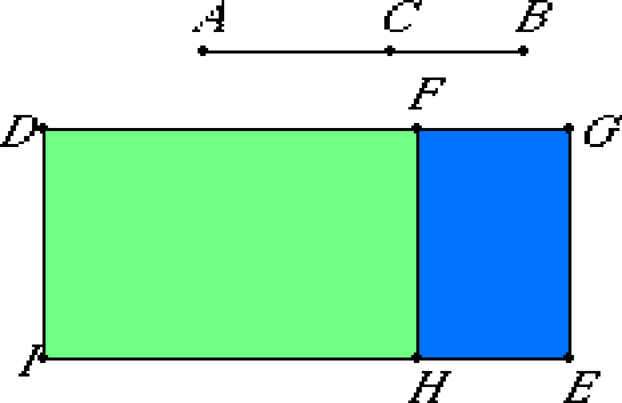
But twice the rectangle *AB* by *BC* is rational, therefore the square on *AC* is irrational, therefore *AC* is irrational. Let it be called a first apotome of a medial straight line. X.Def.4

Q.E.D.

Guide

This proposition is used for a few later propositions in Book X starting with X.80.

Book X Introduction - Proposition X.73 - Proposition X.75.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Euclid's Elements

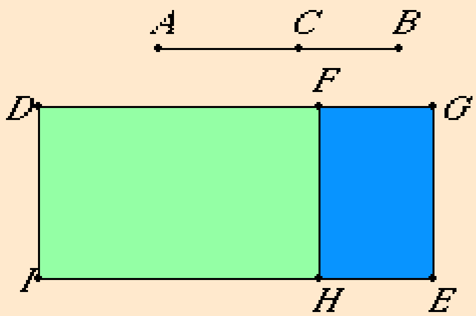
Book X

Proposition 75

If from a medial straight line there is subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a medial rectangle, then the remainder is irrational; let it be called second apotome of a medial straight line.

From the medial straight line *AB* let there be subtracted the medial straight line *CB* which is commensurable with the whole *AB* in square only such that the rectangle *AB* by *BC* which it contains with the whole *AB*, is medial. X.28

I say that the remainder *AC* is irrational, and let it be called a second apotome of a medial straight line.



Set out a rational straight line *DI*. Apply *DE*, equal to the sum of the squares on *AB* and *BC*, to *DI* producing *DG* as breadth. Apply *DH*, equal to twice the rectangle *AB* by *BC*, to *DI* producing *DF* as breadth. Then the remainder *FE* equals the square on *AC*. II.7

Now, since the squares on *AB* and *BC* are medial and commensurable, therefore *DE* is also medial. X.15 X.23,Cor.

And it is applied to the rational straight line *DI*, producing *DG* as breadth, therefore *DG* is rational and incommensurable in length with *DI*. X.22

Again, since the rectangle *AB* by *BC* is medial, therefore twice the rectangle *AB* by *BC* is also medial. X.23,Cor.

And it equals *DH*, therefore *DH* is also medial.

And it is applied to the rational straight line *DI*, producing *DF* as breadth, therefore *DF* is rational and incommensurable in length with *DI*. X.22

Since *AB* and *BC* are commensurable in square only, therefore *AB* is incommensurable in length with *BC*. Therefore the square on *AB* is also incommensurable with the rectangle *AB* by *BC*. X.11

But the sum of the squares on *AB* and *BC* is commensurable with the square on *AB*, and twice the rectangle *AB* by *BC* is commensurable with the rectangle *AB* by *BC*, therefore twice the rectangle *AB* by *BC* is incommensurable with the sum of the squares on *AB* and *BC*. X.15 X.6 X.13

But *DE* equals the sum of the squares on *AB* and *BC*, and *DH* equals twice the rectangle *AB* by *BC*, therefore *DE* is incommensurable with *DH*. But *DE* is to *DH* as *GD* is to *DF*, therefore *GD* is incommensurable with *DF*. VI.1 X.11

And both are rational, therefore *GD* and *DF* are rational straight lines commensurable in square only. Therefore *FG* is an apotome. X.73

But *DI* is rational, and the rectangle contained by a rational and an irrational straight line is irrational, and its side is irrational. X.20

And *AC* is the side of *FE*, therefore *AC* is irrational.

Let it be called a second apotome of a medial straight line.

Q.E.D.

Guide

This proposition is used for a few later propositions in Book X starting with X.81.

Book X Introduction - Proposition X.74 - Proposition X.76.

Euclid's Elements

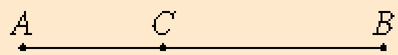
Book X

Proposition 76

If from a straight line there is subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them added together rational, but the rectangle contained by them medial, then the remainder is irrational; let it be called minor.

From the straight line *AB* let there be subtracted the straight line *BC* which is incommensurable in square with the whole and fulfills the given conditions. X.33

I say that the remainder *AC* is the irrational straight line called minor.



Since the sum of the squares on *AB* and *BC* is rational, while twice the rectangle *AB* by *BC* is medial, therefore the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*, and, in conversion, the sum of the squares on *AB* and *BC* is incommensurable with the remainder, the square on *AC*. II.7 X.16

But the sum of the squares on *AB* and *BC* is rational, therefore the square on *AC* is irrational. Therefore *AC* is irrational.

Let it be called minor.

Q.E.D.

Guide

This proposition is used for a few later propositions in Book X starting with X.82.

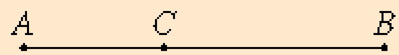
Book X Introduction - Proposition X.75 - Proposition X.77.

Euclid's Elements
Book X
Proposition 77

If from a straight line there is subtracted a straight line which is incommensurable in square with the whole, and which with the whole makes the sum of the squares on them medial but twice the rectangle contained by them rational, then the remainder is irrational; let it be called that which produces with a rational area a medial whole.

From the straight line *AB* let there be subtracted the straight line *BC* which is incommensurable in square with *AB* and fulfills the given conditions.

I say that the remainder *AC* is the irrational straight line aforesaid.



Since the sum of the squares on *AB* and *BC* is medial, while twice the rectangle *AB* by *BC* is rational, therefore the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*. Therefore the remainder, the square on *AC*, is also incommensurable with twice the rectangle *AB* by *BC*. II.7 X.16

And twice the rectangle *AB* by *BC* is rational, therefore the square on *AC* is irrational. Therefore *AC* is irrational. Let it be called that which produces with a rational area a medial whole.

Q.E.D.

Guide

This proposition is used for a few later propositions in Book X starting with X.83.

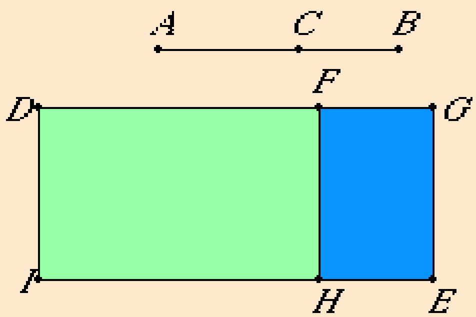
Book X Introduction - Proposition X.76 - Proposition X.78.

Euclid's Elements
Book X
Proposition 78

If from a straight line there is subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, twice the rectangle contained by them medial, and further the sum of the squares on them incommensurable with twice the rectangle contained by them, then the remainder is irrational; let it be called that which produces with a medial area a medial whole.

From the straight line *AB* let there be subtracted the straight line *BC* incommensurable in square with *AB* and fulfilling the given conditions. X.35

I say that the remainder *AC* is the irrational straight line called that which produces with a medial area a medial whole.



Set out a rational straight line *DI*. Apply *DE*, equal to the sum of the squares on *AB* and *BC*, to *DI* producing *DG* as breadth. Subtract *DH* equal twice the rectangle *AB* by *BC*. Then the remainder *FE* equals the square on *AC*, so that *AC* is the side of *FE*. II.7

Now, since the sum of the squares on *AB* and *BC* is medial and equals *DE*, therefore *DE* is medial.

And it is applied to the rational straight line *DI* producing *DG* as breadth, therefore *DG* is rational and incommensurable in length with *DI*. X.22

Again, since twice the rectangle *AB* by *BC* is medial and equals *DH*, therefore *DH* is medial. And it is applied to the rational straight line *DI* producing *DF* as breadth, therefore *DF* is also rational and incommensurable in length with *DI*. X.22

Since the sum of the squares on *AB* and *BC* is incommensurable with twice the rectangle *AB* by *BC*, therefore *DE* is also incommensurable with *DH*.

But *DE* is to *DH* as *DG* is to *DF*, therefore *DG* is incommensurable with *DF*. X.11 VI.1

And both are rational, therefore *GD* and *DF* are rational straight lines commensurable in square only. Therefore *FG* is an apotome. X.73

And *FH* is rational, but the rectangle contained by a rational straight line and an apotome is irrational, and its side is irrational. X.20

And *AC* is the side of *FE*, therefore *AC* is irrational.

Let it be called *that which produces with a medial area a medial whole*.

Q.E.D.

Guide

This proposition is used for a few later propositions in Book X starting with X.84.

Book X Introduction - Proposition X.77 - Proposition X.79.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Euclid's Elements
Book X
Proposition 79

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only.

Let *AB* be an apotome, and *BC* an annex to it. Then *AC* and *CB* are rational straight lines commensurable in square only. X.73

I say that no other rational straight line can be annexed to *AB* which is commensurable with the whole in square only.

If possible, let *BD* be so annexed. Then *AD* and *DB* are also rational straight lines commensurable in square only. X.73

Now, since the excess of the sum of the squares on *AD* and *DB* over twice the rectangle *AD* by *DB* is also the excess of the sum of the squares on *AC* and *CB* over twice the rectangle *AC* by *CB*, for both exceed by the same, the square on *AB*, therefore, alternately, the excess of the sum of the squares on *AD* and *DB* over the sum of the squares on *AC* and *CB* is the excess of twice the rectangle *AD* by *DB* over twice the rectangle *AC* by *CB*. II.7

But the sum of the squares on *AD* and *DB* exceeds the sum of the squares on *AC* and *CB* by a rational area, for both are rational, therefore twice the rectangle *AD* by *DB* also exceeds twice the rectangle *AC* by *CB* by a rational area, which is impossible, for both are medial, and a medial area does not exceeded a medial by a rational area. X.21 X.26

Therefore no other rational straight line can be annexed to *AB* which is commensurable with the whole in square only.

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only.

Therefore, *to an apotome only one rational straight line can be annexed which is commensurable with the whole in square only.*

Q.E.D.

Guide

This proposition is used in X.81 and X.84.

Book X Introduction - Proposition X.78 - Proposition X.80.

Euclid's Elements

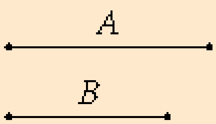
Book X

Proposition 8

If two magnitudes do not have to one another the ratio which a number has to a number, then the magnitudes are incommensurable.

Let the two magnitudes *A* and *B* not have to one another the ratio which a number has to a number.

I say that the magnitudes *A* and *B* are incommensurable.



For, if they are commensurable, then *A* has to *B* the ratio which a number has to a number. X.5

But it does not, therefore the magnitudes *A* and *B* are incommensurable.

Therefore, *if two magnitudes do not have to one another the ratio which a number has to a number, then the magnitudes are incommensurable.*

Q.E.D.

Guide

This proposition is the contrapositive of X.5. It is used in frequently in X.11.

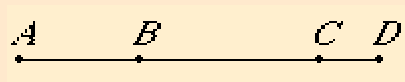
Book X Introduction - Proposition X.7 - Proposition X.9.

Euclid's Elements
Book X
Proposition 80

To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle.

Let *AB* be a first apotome of a medial straight line, and let *KC* be an annex to *AB*. Then *AC* and *CB* are medial straight lines commensurable in square only such that the rectangle *AC* by *CB* which they contain is rational. X.74

I say that no other medial straight line can be annexed to *AB* which is commensurable with the whole in square only and which contains with the whole a rational area.



If possible, let *DB* also be so annexed. Then *AD* and *DB* are medial straight lines commensurable in square only such that the rectangle *AD* by *DB* which they contain is rational. X.74

Now, since the excess of the sum of the squares on *AD* and *DB* over twice the rectangle *AD* by *DB* is also the excess of the sum of the squares on *AC* and *CB* over twice the rectangle *AC* by *CB*, for they exceed by the same, the square on *AB*, therefore, alternately, the excess of the sum of the squares on *AD* and *DB* over the sum of the squares on *AC* and *CB* is also the excess of twice the rectangle *AD* by *DB* over twice the rectangle *AC* by *CB*. II.7

But twice the rectangle *AD* by *DB* exceeds twice the rectangle *AC* by *CB* by a rational area, for both are rational. X.15

Therefore the sum of the squares on *AD* and *DB* also exceeds the sum of the squares on *AC* and *CB* by a rational area, which is impossible, for both are medial, and a medial area does not exceed a medial by a rational area. X.23,Cor. X.26

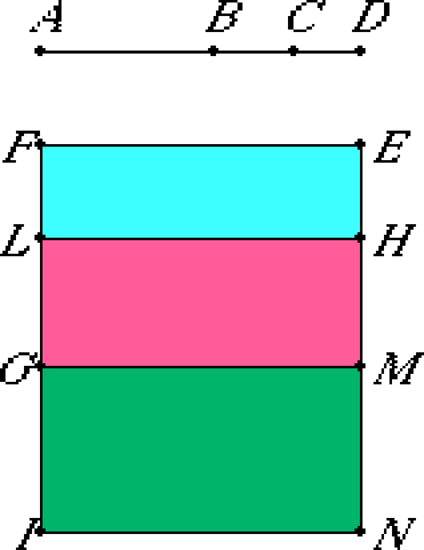
Therefore, *to a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.79 - Proposition X.81.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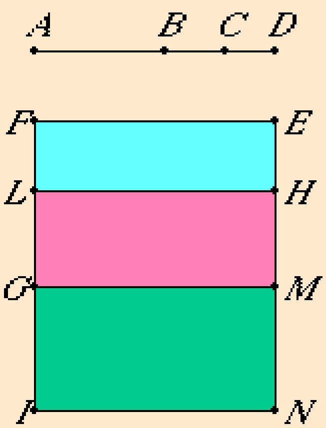
Euclid's Elements
Book X
Proposition 81

To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

Let *AB* be a second apotome of a medial straight line and *BC* an annex to *AB*. Then *AC* and *CB* are medial straight lines commensurable in square only such that the rectangle *AC* by *CB* which they contain is medial. X.75

I say that no other medial straight line can be annexed to *AB* which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

If possible, let *BD* also be so annexed. Then *AD* and *DB* are also medial straight lines commensurable in square only such that the rectangle *AD* by *DB* which they contain is medial. X.75



Set out a rational straight line *EF*. Apply *EG*, equal to the sum of the squares on *AC* and *CB*, to *EF* producing *EM* as breadth. Subtract *HG*, equal to twice the rectangle *AC* by *CB*, producing *HM* as breadth. Then the remainder *EL* equals the square on *AB*, so that *AB* is the side of *EL*. II.7

Again, apply *EI*, equal to the sum of the squares on *AD* and *DB*, to *EF* producing *EN* as breadth.

But *EL* also equals the square on *AB*, therefore the remainder *HI* equals twice the rectangle *AD* by *DB*. II.7

Now, since *AC* and *CB* are medial straight lines, therefore the squares on *AC* and *CB* are also medial. And they equal *EG*, therefore *EG* is also medial. X.15 X.23,Cor.

And it is applied to the rational straight line *EF*, producing *EM* as breadth, therefore *EM* is rational and incommensurable in length with *EF*. X.22

Again, since the rectangle *AC* by *CB* is medial, twice the rectangle *AC* by *CB* is also medial. And it equals *HG*, therefore *HG* is also medial. X.23,Cor.

And it is applied to the rational straight line *EF*, producing *HM* as breadth, therefore *HM* is also rational and incommensurable in length with *EF*. X.22

Since *AC* and *CB* are commensurable in square only, therefore *AC* is incommensurable in length with *CB*.

But *AC* is to *CB* as the square on *AC* is to the rectangle *AC* by *CB*, therefore the square on *AC* is incommensurable with the rectangle *AC* by *CB*. X.11

But the sum of the squares on *AC* and *CB* is commensurable with the square on *AC*, while twice the rectangle *AC* by *CB* is commensurable with the rectangle *AC* by *CB*, therefore the sum of the squares on *AC* and *CB* is incommensurable with twice the rectangle *AC* by *CB*. X.6 X.13

And *EG* equals the sum of the squares on *AC* and *CB*, while *GH* equals twice the rectangle *AC* by *CB*, therefore *EG* is incommensurable with *HG*.

But *EG* is to *HG* as *EM* is to *HM*, therefore *EM* is incommensurable in length with *MH*. VI.1 X.11

And both are rational, therefore *EM* and *MH* are rational straight lines commensurable in square only, therefore *EH* is an apotome, and *HM* an annex to it. X.73

Similarly we can prove that *HN* is also an annex to it. Therefore to an apotome different straight lines are annexed which are commensurable with the wholes in square only, which is impossible. X.79

Therefore, *to a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.80 - Proposition X.82.

Euclid's Elements
Book X
Proposition 82

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of squares on them rational but twice the rectangle contained by them medial.

Let *AB* be the minor straight line, and let *BC* be an annex to *AB*. Then *AC* and *CB* are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial. X.76

I say that no other straight line can be annexed to *AB* fulfilling the same conditions.

If possible, let *BD* be so annexed. Then *AD* and *DB* are both straight lines incommensurable in square which fulfill the aforesaid conditions. X.76

Now, since the excess of the sum of the squares on *AD* and *DB* over the sum of the squares on *AC* and *CB* is also the excess of twice the rectangle *AD* by *DB* over twice the rectangle *AC* by *CB*, while the sum of the squares on *AD* and *DB* exceed the sum of the squares on *AC* and *CB* by a rational area, for both are rational, therefore twice the rectangle *AD* by *DB* also exceeds twice the rectangle *AC* by *CB* by a rational area, which is impossible, for both are medial. X.26

Therefore, *to a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of squares on them rational but twice the rectangle contained by them medial.*

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.81 - Proposition X.83.

Euclid's Elements
Book X
Proposition 83

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of squares on them medial but twice the rectangle contained by them rational.

Let *AB* be the straight line which produces with a rational area a medial whole, and let *BC* be an annex to *AB*. Then *AC* and *CB* are straight lines incommensurable in square which fulfill the given conditions. X.77

I say that no other straight line can be annexed to *AB* which fulfills the same conditions.

If possible, let *BD* be so annexed. Then *AD* and *DB* are both straight lines incommensurable in square which fulfill the given conditions. X.77

As in the preceding cases, the excess of the sum of the squares on *AD* and *DB* over the sum of the squares on *AC* and *CB* is also the excess of twice the rectangle *AD* by *DB* over twice the rectangle *AC* by *CB*, while twice the rectangle *AD* by *DB* exceeds twice the rectangle *AC* by *CB* by a rational area, for both are rational, therefore the sum of the squares on *AD* and *DB* also exceeds the sum of the squares on *AC* and *CB* by a rational area, which is impossible, for both are medial. X.26

Therefore no other straight line can be annexed to *AB* which is incommensurable in square with the whole and which with the whole fulfills the aforesaid conditions, therefore only one straight line can be so annexed.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Proposition X.82 - Proposition X.84.

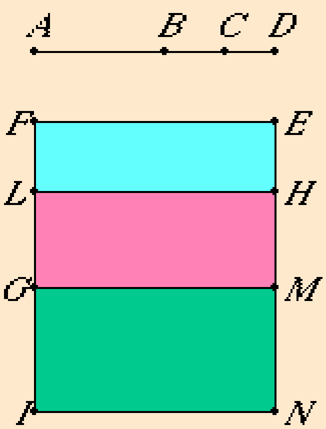
Euclid's Elements
Book X
Proposition 84

To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of squares on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them.

Let *AB* be the straight line which produces with a medial area a medial whole, and *BC* an annex to it. Then *AC* and *CB* are straight lines incommensurable in square which fulfill the aforesaid conditions. X.78

I say that no other straight line can be annexed to *AB* which fulfills the aforesaid conditions.

If possible, let *BD* be so annexed, so that *AD* and *DB* are also straight lines incommensurable in square which make the squares on *AD* and *DB* added together medial, twice the rectangle *AD* by *DB* medial, and also the sum of the squares on *AD* and *DB* incommensurable with twice the rectangle *AD* by *DB*. X.78



Set out a rational straight line *EF*. Apply *EG*, equal to the sum of the squares on *AC* and *CB*, to *EF* producing *EM* as breadth. Apply *HG*, equal to twice the rectangle *AC* by *CB*, to *EF* producing *HM* as breadth. Then the remainder, the square on *AB*, equals *EL*. Therefore *AB* is the side of *EL*. II.7

Again, apply *EI*, equal to the sum of the squares on *AD* and *DB*, to *EF* producing *EN* as breadth.

But the square on *AB* also equals *EL*, therefore the remainder, twice the rectangle *AD* by *DB*, equals *HI*. II.7

Now, since the sum of the squares on *AC* and *CB* is medial and equals *EG*, therefore *EG* is also medial. And it is applied to the rational straight line *EF* producing *EM* as breadth, therefore *EM* is rational and incommensurable in length with *EF*. X.22

Again, since twice the rectangle *AC* by *CB* is medial and equals *HG*, therefore *HG* is also medial. And it is applied to the rational straight line *EF* producing *HM* as breadth, therefore *HM* is rational and incommensurable in length with *EF*. X.22

Since the sum of the squares on *AC* and *CB* is incommensurable with twice the rectangle *AC* by *CB*, therefore *EG* is also incommensurable with *HG*. Therefore *EM* is also incommensurable in length with *MH*. VI.1 X.11

And both are rational, therefore *EM* and *MH* are rational straight lines commensurable in square only. Therefore *EH* is an apotome, and *HM* an annex to it. X.73

Similarly we can prove that *EH* is again an apotome and *HN* an annex to it. Therefore to an apotome different rational straight lines are annexed which are commensurable with the wholes in square only, which was proved impossible. X.79

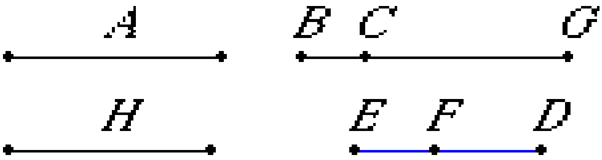
Therefore no other straight line can be so annexed to *AB*. Therefore to *AB* only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial, twice the rectangle contained by them medial, and also the sum of the squares on them incommensurable with twice the rectangle contained by them.

Q.E.D.

Guide

(Forthcoming)

Book X Introduction - Definitions III of Book X - Proposition X.85.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Euclid's Elements
Book X
Proposition 85

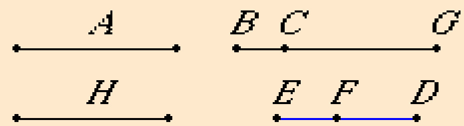
To find the first apotome.

Set out a rational straight line, and let *BG* be commensurable in length with *A*. Then *BG* is also rational.

Set out two square numbers *DE* and *EF*, and let their difference *FD* not be square. Then *ED* does not have to *DF* the ratio which a square number has to a square number.

Let it be contrived that *ED* is to *DF* as the square on *BG* is to the square on *GC*. Then the square on *BG* is commensurable with the square on *GC*. X.6,Cor. X.6

But the square on *BG* is rational, therefore the square on *GC* is also rational. Therefore *GC* is also rational.



Since *ED* does not have to *DF* the ratio which a square number has to a square number, therefore neither has the square on *BG* to the square on *GC* the ratio which a square number has to a square number. Therefore *BG* is incommensurable in length with *GC*. X.9

And both are rational, therefore *BG* and *GC* are rational straight lines commensurable in square only. Therefore *BC* is an apotome. X.73

I say next that it is also a first apotome.

Let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Now since *ED* is to *FD* as the square on *BG* is to the square on *GC*, therefore, in conversion, as *DE* is to *EF* as the square on *GB* is to the square on *H*. V.19,Cor.

But *DE* has to *EF* the ratio which a square number has to a square number, for each is square, therefore the square on *GB* also has to the square on *H* the ratio which a square number has to a square number. Therefore *BG* is commensurable in length with *H*. X.9

And the square on *BG* is greater than the square on *GC* by the square on *H*, therefore the square on *BG* is greater than the square on *GC* by the square on a straight line commensurable in length with *BG*.

And the whole *BG* is commensurable in length with the rational straight line *A* set out.

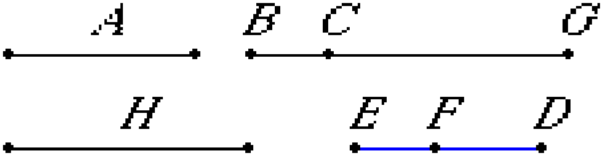
Therefore *BC* is a first apotome. Therefore the first apotome *BC* has been found. X.Def.III.2

Q.E.F.

Guide

(Forthcoming)

Book X Introduction - Definitions III of Book X - Proposition X.86.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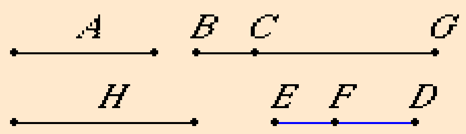
Euclid's Elements
Book X
Proposition 86

To find the second apotome.

Set out a rational straight line *A*, and let *GC* be commensurable in length with *A*. Then *GC* is rational. Set out two square numbers *DE* and *EF*, and let their difference *DF* not be square.

Now let it be contrived that *FD* is to *DE* as the square on *CG* is to the square on *GB*. X.6,Cor.



Then the square on *CG* is commensurable with the square on *GB*. X.6

But the square on *CG* is rational, therefore the square on *GB* is also rational. Therefore *BG* is rational. And, since the square on *GC* does not have to the square on *GB* the ratio which a square number has to a square number, therefore *CG* is incommensurable in length with *GB*. X.9

And both are rational, therefore *CG* and *GB* are rational straight lines commensurable in square only. Therefore *BC* is an apotome. X.73

I say next that it is also a second apotome.

Let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since the square on *BG* is to the square on *GC* as the number *ED* is to the number *DF*, therefore, in conversion, the square on *BG* is to the square on *H* as *DE* is to *EF*. V.19,Cor.

And each of the numbers *DE* and *EF* is square, therefore the square on *BG* has to the square on *H* the ratio which a square number has to a square number. Therefore *BG* is commensurable in length with *H*. X.9

And the square on *BG* is greater than the square on *GC* by the square on *H*, therefore the square on *BG* is greater than the square on *GC* by the square on a straight line commensurable in length with *BG*.

And *CG*, the annex, is commensurable with the rational straight line *A* set out, therefore *BC* is a second apotome. X.Def.III.2

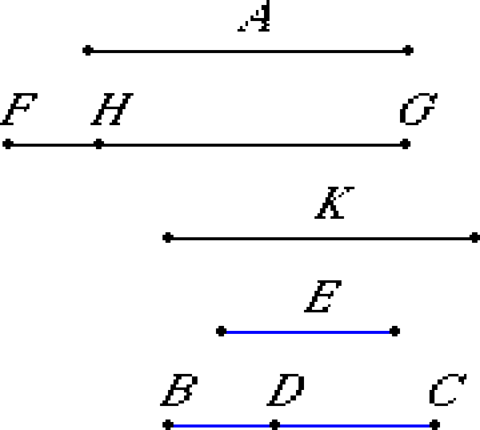
Therefore the second apotome *BC* has been found.

Q.E.F.

Guide

(Forthcoming)

Book X Introduction - Proposition X.85 - Proposition X.87.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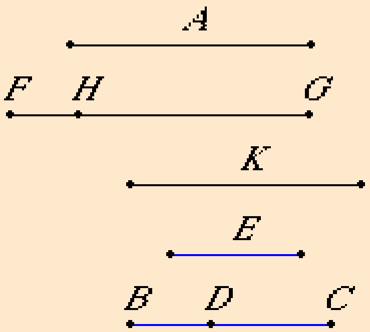
Euclid's Elements
Book X
Proposition 87

To find the third apotome.

Set out a rational straight line *A*. Set out three numbers *E*, *BC*, and *CD* which do not have to one another the ratio which a square number has to a square number, but let *CB* have to *BD* the ratio which a square number has to a square number.



Let it be contrived that *E* is to *BC* as the square on *A* is to the square on *FG*, and *BC* is to *CD* as the square on *FG* is to the square on *GH*. X.6,Cor.

Since *E* is to *BC* as the square on *A* is to the square on *FG*, therefore the square on *A* is commensurable with the square on *FG*. X.6

But the square on *A* is rational, therefore the square on *FG* is also rational, therefore *FG* is rational.

Since *E* does not have to *BC* the ratio which a square number has to a square number, therefore neither has the square on *A* to the square on *FG* the ratio which 3 square number has to a square number. Therefore *A* is incommensurable in length with *FG*. X.9

Again, since *BC* is to *CD* as the square on *FG* is to the square on *GH*, therefore the square on *FG* is commensurable with the square on *GH*. X.6

But the square on *FG* is rational, therefore the square on *GH* is also rational, therefore *GH* is rational.

Since *BC* does not have to *CD* the ratio which a square number has to a square number, therefore neither has the square on *FG* to the square on *GH* the ratio which a square number has to a square number. Therefore *FG* is incommensurable in length with *GH*. X.9

And both are rational, therefore *FG* and *GH* are rational straight lines commensurable in square only. Therefore *FH* is an apotome. X.73

I say next that it is also a third apotome.

Since *E* is to *BC* as the square on *A* is to the square on *FG*, and *BC* is to *CD* as the square on *FG* is to the square on *HG*, therefore, *ex aequali*, *E* is to *CD* as the square on *A* is to the square on *HG*. V.22

But *E* does not have to *CD* the ratio which a square number has to a square number, therefore neither has the square on *A* to the square on *GH* the ratio which a square number has to a square number. Therefore *A* is incommensurable in length with *GH*. X.9

Therefore neither of the straight lines *FG* nor *GH* is commensurable in length with the rational straight line *A* set out.

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*.

Since *BC* is to *CD* as the square on *FG* is to the square on *GH*, therefore, in conversion, *BC* is to *BD* as the square on *FG* is to the square on *K*. V.19,Cor.

But *BC* has to *BD* the ratio which a square number has to a square number, therefore the square on *FG* also has to the square on *K* the ratio which a square number has to a square number.

Therefore *FG* is commensurable in length with *K*, and the square on *FG* is greater than the square on *GH* by the square on a straight line commensurable with *FG*. X.9

And neither of the straight lines *FG* nor *GH* is commensurable in length with the rational straight line *A* set out, therefore *FH* is a third apotome. X.Def.III.3

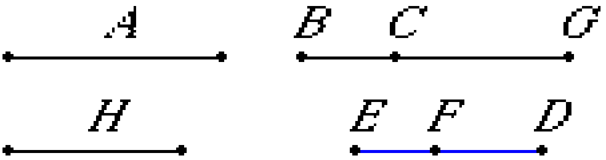
Therefore the third apotome *FH* has been found.

Q.E.F.

Guide

(Forthcoming)

Book X Introduction - Proposition X.86 - Proposition X.88.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Euclid's Elements

Book X

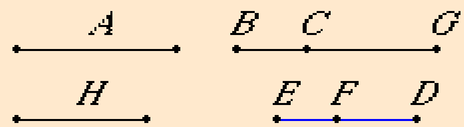
Proposition 88

To find the fourth apotome.

Set out a rational straight line *A*, and let *BG* be commensurable in length with it. Set out two numbers *DF* and *FE* such that the whole *DE* has to neither of the numbers *DE* nor *EF* the ratio which a square number has to a square number.

Let it be contrived that *DE* is to *EF* as the square on *BG* is to the square on *GC*. Then the square on *BG* is commensurable with the square on *GC*. X.6,Cor. X.6

But the square on *BG* is rational, therefore the square on *GC* is also rational. Therefore *GC* is rational.



Now, since *DE* does not have to *EF* the ratio which a square number has to a square number, therefore neither has the square on *BG* to the square on *GC* the ratio which a square number has to a square number. Therefore *BG* is incommensurable in length with *GC*. X.9

And both are rational, therefore *BG* and *GC* are rational straight lines commensurable in square only. Therefore *BC* is an apotome. X.73

Now let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since *DE* is to *EF* as the square on *BG* is to the square on *GC*, therefore, in conversion, *ED* is to *DF* as the square on *GB* is to the square on *H*. V.19,Cor.

But *ED* does not have to *DF* the ratio which a square number has to a square number, therefore neither has the square on *GB* to the square on *H* the ratio which a square number has to a square number. Therefore *BG* is incommensurable in length with *H*. X.9

And the square on *BG* is greater than the square on *GC* by the square on *H*, therefore the square on *BG* is greater than the square on *GC* by the square on a straight line incommensurable with *BG*.

And the whole *BG* is commensurable in length with the rational straight line *A* set out. Therefore *BC* is a fourth apotome.

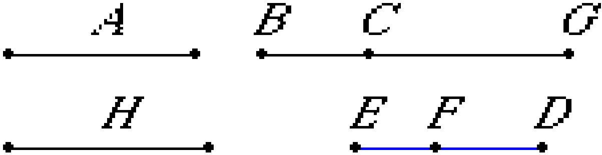
Therefore the fourth apotome has been found. X.Def.III.4

Q.E.F.

Guide

(Forthcoming)

Book X Introduction - Proposition X.87 - Proposition X.89.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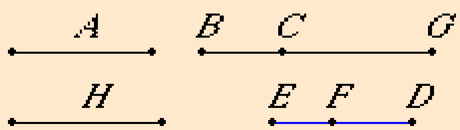
Euclid's Elements
Book X
Proposition 89

To find the fifth apotome.

Set out a rational straight line *A*, and let *CG* be commensurable in length with *A*. Then *CG* is rational.

Set out two numbers *DF* and *FE* such that *DE* again has to neither of the numbers *DF* nor *FE* the ratio which a square number has to a square number, and let it be contrived that *FE* is to *ED* as the square on *CG* is to the square on *GB*.



Then the square on *GB* is also rational. Therefore *BG* is also rational. X.6

Now since *DE* is to *EF* as the square on *BG* is to the square on *GC*, while *DE* does not have to *EF* the ratio which a square number has to a square number, therefore neither does the square on *BG* have to the square on *GC* the ratio which a square number has to a square number. Therefore *BG* is incommensurable in length with *GC*. X.9

And both are rational, therefore *BG* and *GC* are rational straight lines commensurable in square only. Therefore *BC* is an apotome. X.73

I say next that it is also a fifth apotome.

Let the square on *H* be that by which the square on *BG* is greater than the square on *GC*.

Since the square on *BG* is to the square on *GC* as *DE* is to *EF*, therefore, in conversion, *ED* is to *DF* as the square on *BG* is to the square on *H*. V.19,Cor.

But *ED* does not have to *DF* the ratio which a square number has to a square number, therefore neither has the square on *BG* to the square on *H* the ratio which a square number has to a square number. Therefore *BG* is incommensurable in length with *H*. X.9

And the square on *BG* is greater than the square on *GC* by the square on *H*, therefore the square on *GB* is greater than the square on *GC* by the square on a straight line incommensurable in length with *GB*.

And the annex *CG* is commensurable in length with the rational straight line *A* set out, therefore *BC* is a fifth apotome. X.Def.III.5

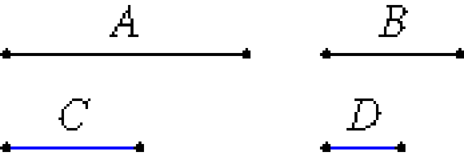
Therefore the fifth apotome *BC* has been found.

Q.E.F.

Guide

(Forthcoming)

Book X Introduction - Proposition X.88 - Proposition X.90.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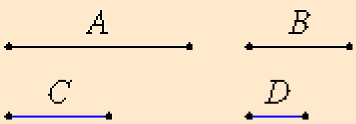
Euclid's Elements
Book X
Proposition 9

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number also have their sides commensurable in length. But the squares on straight lines incommensurable in length do not have to one another the ratio which a square number has to a square number; and squares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either.

Let *A* and *B* be commensurable in length.

I say that the square on *A* has to the square on *B* the ratio which a square number has to a square number.



Since *A* is commensurable in length with *B*, therefore *A* has to *B* the ratio which a number has to a number. Let it have to it the ratio which *C* has to *D*. X.5

Since then *A* is to *B* as *C* is to *D*, while the ratio of the square on *A* to the square on *B* is duplicate of the ratio of *A* to *B*, for similar figures are in the duplicate ratio of their corresponding sides, and the ratio of the square on *C* to the square on *D* is duplicate of the ratio of *C* to *D*, for between two square numbers there is one mean proportional number, and the square number has to the square number the ratio duplicate of that which the side has to the side, therefore the square on *A* is to the square on *B* as the square on *C* is to the square on *D*. VI.20,Cor. VIII.11

Next, as the square on *A* is to the square on *B*, so let the square on *C* be to the square on *D*.

I say that *A* is commensurable in length with *B*.

Since the square on *A* is to the square on *B* as the square on *C* is to the square on *D*, while the ratio of the square on *A* to the square on *B* is duplicate of the ratio of *A* to *B*, and the ratio of the square on *C* to the square on *D* is duplicate of the ratio of *C* to *D*, therefore *A* is to *B* as *C* is to *D*.

Therefore *A* has to *B* the ratio which the number *C* has to the number *D*. Therefore *A* is commensurable in length with *B*. X.6

Next, let *A* be incommensurable in length with *B*.

I say that the square on *A* does not have to the square on *B* the ratio which a square number has to a square number.

If the square on *A* does have to the square on *B* the ratio which a square number has to a square number, then *A* is commensurable with *B*. Above

But it is not, therefore the square on *A* does not have to the square on *B* the ratio which a square number has to a square number.

Finally, let the square on *A* not have to the square on *B* the ratio which a square number has to a square number.

I say that *A* is incommensurable in length with *B*.

For, if *A* is commensurable with *B*, then the square on *A* has to the square on *B* the ratio which a square number has to a square number. Above

But it does not, therefore *A* is not commensurable in length with *B*.

Therefore, *the squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number also have their sides commensurable in length. But the squares on straight lines incommensurable in length do not have to one another the ratio which a square number has to a square number; and squares which do not have to one another the ratio which a square number has to a square number also do not have their sides commensurable in length either.*

Q.E.D.

Corollary.

And it is manifest from what has been proved that *straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always also commensurable in length.*

Lemma.

It has been proved in the arithmetical books that *similar plane numbers have to one another the ratio which a square number has to a square number, and that, if two numbers have to one another the ratio which a square number has to a square number, then they are similar plane numbers*. VIII.26 and converse

Corollary 2.

And it is manifest from these propositions that *numbers which are not similar plane numbers, that is, those which do not have their sides proportional, do not have to one another the ratio which a square number has to a square number*.

For, if they have, then they are similar plane numbers, which is contrary to the hypothesis. Therefore numbers which are not similar plane numbers do not have to one another the ratio which a square number has to a square number.

Guide

This proposition has a statement, its converse, and its and its converse's contrapositives. It says lines are commensurable if and only if the squares on them are in the ratio of a square number to a square number.

For example, the diagonal of a square and the side of the square are not commensurable since the squares on them are in the ratio 2:1, and 2:1 is not the ratio of a square number to a square number, see the guide to proposition VIII.8.

The proposition is used repeatedly in Book X starting with the next. It is also used in Book XIII in propositions XIII.6 and XIII.11.

Book X Introduction - Proposition X.8 - Proposition X.10.

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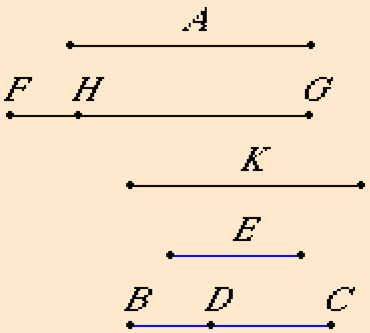
Euclid's Elements
Book X
Proposition 90

To find the sixth apotome.

Set out a rational straight line *A*, and set out three numbers *E*, *BC*, and *CD* not having to one another the ratio which a square number has to a square number, and further let *CB* also not have to *BD* the ratio which a square number has to a square number.

Let it be contrived that *E* is to *BC* as the square on *A* is to the square on *FG*, and *BC* is to *CD* as the square on *FG* is to the square on *GH*. X.6,Cor.



Now since *E* is to *BC* as the square on *A* is to the square on *FG*, therefore the square on *A* is commensurable with the square on *FG*. X.6

But the square on *A* is rational, therefore the square on *FG* is also rational. Therefore *FG* is also rational.

Since *E* does not have to *BC* the ratio which a square number has to a square number, therefore neither does the square on *A* have to the square on *FG* the ratio which a square number has to a square number, therefore *A* is incommensurable in length with *FG*. X.9

Again, since *BC* is to *CD* as the square on *FG* is to the square on *GH*, therefore the square on *FG* is commensurable with the square on *GH*. X.6

But the square on *FG* is rational, therefore the square on *GH* is also rational. Therefore *GH* is also rational.

Since *BC* does not have to *CD* the ratio which a square number has to a square number, therefore neither does the square on *FG* have to the square on *GH* the ratio which a square number has to a square number. Therefore *FG* is incommensurable in length with *GH* X.9

And both are rational, therefore *FG* and *GH* are rational straight lines commensurable in square only. Therefore *FH* is an apotome. X.73

I say next that it is also a sixth apotome.

Since *E* is to *BC* as the square on *A* is to the square on *FG*, and *BC* is to *CD* as the square on *FG* is to the square on *GH*, therefore, *ex aequali*, *E* is to *CD* as the square on *A* is to the square on *GH*. V.22

But *E* does not have to *CD* the ratio which a square number has to a square number, therefore neither does the square on *A* have to the square on *GH* the ratio which a square number has to a square number. Therefore *A* is incommensurable in length with *GH*. Therefore neither of the straight lines *FG* nor *GH* is commensurable in length with the rational straight line *A*. X.9

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*.

Since *BC* is to *CD* as the square on *FG* is to the square on *GH*, therefore, in conversion, *CB* is to *BD* as V.19,Cor.

the square on *FG* is to the square on *K*.

But *CB* does not have to *BD* the ratio which a square number has to a square number, therefore neither does the square on *FG* have to the square on *K* the ratio which a square number has to a square number, therefore *FG* is incommensurable in length with *K*. X.9

And the square on *FG* is greater than the square on *GH* by the square on *K*, therefore the square on *FG* is greater than the square on *GH* by the square on a straight line incommensurable in length with *FG*.

X.Def.III.6

And neither of the straight lines *FG* nor *GH* is commensurable with the rational straight line *A* set out. Therefore *FH* is a sixth apotome.

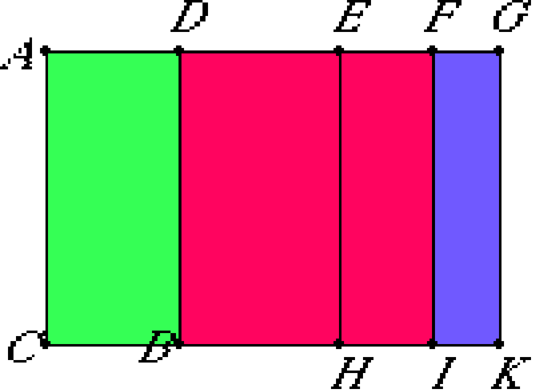
Therefore the sixth apotome *FH* has been found.

Q.E.F.

Guide

(Forthcoming)

Book X Introduction - Proposition X.89 - Proposition X.91.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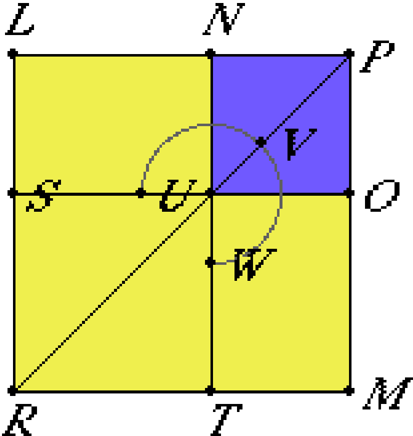
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Euclid's Elements

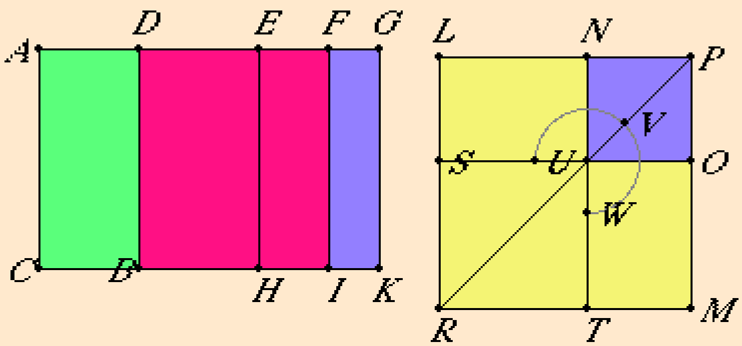
Book X

Proposition 91

If an area is contained by a rational straight line and a first apotome, then the side of the area is an apotome.

Let the area *AB* be contained by the rational straight line *AC* and the first apotome *AD*.

I say that the side of the area *AB* is an apotome.



Since *AD* is a first apotome, let *DG* be its annex, therefore *AG* and *GD* are rational straight lines commensurable in square only. Also, the whole *AG* is commensurable with the rational straight line *AC* set out, and the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable in length with *AG*. X.73 X.Def.III.2

Therefore if there is applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, then it divides it into commensurable parts. X.17

Bisect *DG* at *E*, apply to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF* by *FG*. Then *AF* is commensurable with *FG*.

Draw *EH*, *FI*, and *GK* through the points *E*, *F*, and *G* parallel to *AC*.

Now, since *AF* is commensurable in length with *FG*, therefore *AG* is also commensurable in length with each of the straight lines *AF* and *FG*. X.15

But *AG* is commensurable with *AC*, therefore each of the straight lines *AF* and *FG* is commensurable in length with *AC*. X.12

And *AC* is rational, therefore each of the straight lines *AF* and *FG* is also rational, so that each of the rectangles *AI* and *FK* is also rational. X.19

Now, since *DE* is commensurable in length with *EG*, therefore *DG* is also commensurable in length with each of the straight lines *DE* and *EG*. X.15

But *DG* is rational and incommensurable in length with *AC*, therefore each of the straight lines *DE* and *EG* is also rational and incommensurable in length with *AC*. Therefore each of the rectangles *DH* and *EK* is medial. X.13 X.21

Now make the square *LM* equal to *AI*, and subtract the square *NO* having a common angle with it, the angle *LPM*, and equal to *FK*. Then the squares *LM* and *NO* are about the same diameter. VI.26

Let *PR* be their diameter, and draw the figure.

Since the rectangle *AF* by *FG* equals the square on *EG*, therefore *AF* is to *EG* as *EG* is to *FG*. VI.17

But *AF* is to *EG* as *AI* is to *EK*, and *EG* is to *FG* as *EK* is to *KF*, therefore *EK* is a mean proportional between *AI* and *KF*. VI.1 V.11

But it was proved before that *MN* is also a mean proportional between *LM* and *NO*, and *AI* equals the square *LM*, and *KF* equals *NO*, therefore *MN* also equals *EK*. X.54's Lemma

But *EK* equals *DH*, and *MN* equals *LO*, therefore *DK* equals the gnomon *UVW* and *NO*.

But *AK* also equals the sum of the squares *LM* and *NO*, therefore the remainder *AB* equals *ST*.

But *ST* is the square on *LN*, therefore the square on *LN* equals *AB*. Therefore *LN* is the side of *AB*.

I say next that *LN* is an apotome.

Since each of the rectangles *AI* and *FK* is rational, and they equal *LM* and *NO*, therefore each of the squares *LM* and *NO*, that is, the squares on *LP* and *PN* respectively, is also rational. Therefore each of the straight lines *LP* and *PN* is also rational.

Again, since *DH* is medial and equals *LO*, therefore *LO* is also medial.

Since, then, *LO* is medial, while *NO* is rational, therefore *LO* is incommensurable with *NO*.

But *LO* is to *NO* as *LP* is to *PN*, therefore *LP* is incommensurable in length with *PN*. VI.1 X.11

And both are rational, therefore *LP* and *PN* are rational straight lines commensurable in square only. Therefore *LN* is an apotome. X.73

And it is the side of the area *AB*, therefore the side of the area *AB* is an apotome.

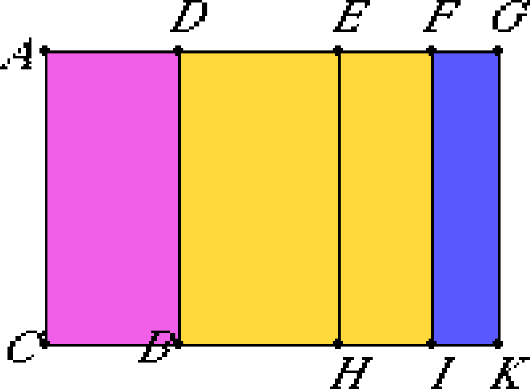
Therefore, *if an area is contained by a rational straight line and a first apotome, then the side of the area is an apotome.*

Q.E.D.

Guide

This proposition is used in X.108.

Book X Introduction - Proposition X.90 - Proposition X.92.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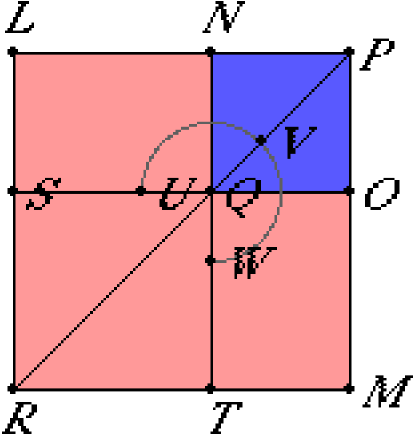
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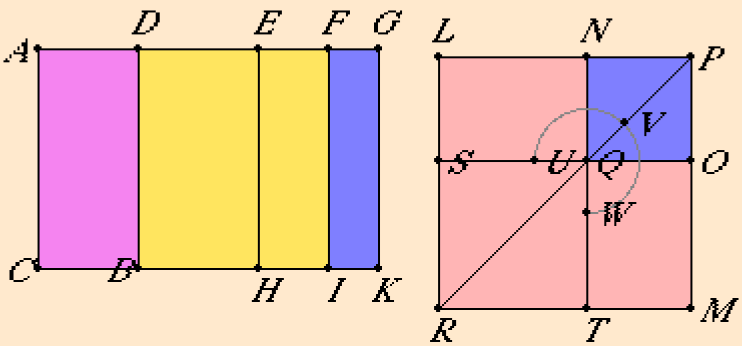
Euclid's Elements
Book X
Proposition 92

If an area is contained by a rational straight line and a second apotome, then the side of the area is a first apotome of a medial straight line.

Let the area *AB* be contained by the rational straight line *AC* and the second apotome *AD*.

I say that the side of the area *AB* is a first apotome of a medial straight line.



Let *DG* be the annex to *AD*. Then *AG* and *GD* are rational straight lines commensurable in square only, and the annex *DG* is commensurable with the rational straight line *AC* set out, while the square on the whole *AG* is greater than the square on the annex *GD* by the square on a straight line commensurable in length with *AG*. X.73 X.Def.III.2

Since the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable with *AG*, therefore, if there is applied to *AG* a parallelogram equal to the fourth part of the square on *GD* and deficient by a square figure, then it divides it into commensurable parts. X.17

Bisect, then, *DG* at *E*, apply to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF* by *FG*. Then *AF* is commensurable in length with *FG*.

Therefore *AG* is also commensurable in length with each of the straight lines *AF* and *FG*. X.15

But *AG* is rational and incommensurable in length with *AC*, therefore each of the straight lines *AF* and *FG* is also rational and incommensurable in length with *AC*. Therefore each of the rectangle *AI* by *FK* is medial. X.13 X.21

Again, since *DE* is commensurable with *EG*, therefore *DG* is also commensurable with each of the straight lines *DE* and *EG*. X.15

But *DG* is commensurable in length with *AC*.

Therefore each of the rectangles *DH* and *EK* is rational. X.19

Construct the square *LM* equal to *AI*, and subtract *NO*, equal to *FK*, about the same angle with *LM*, namely the angle *LPM*. Then the squares *LM* and *NO* are about the same diameter. VI.26

Let *PR* be their diameter, and draw the figure.

Since *AI* and *FK* are medial and equal the squares on *LP* and *PN*, the squares on *LP* and *PN* are also medial, therefore *LP* and *PN* are also medial straight lines commensurable in square only.

Since the rectangle *AF* by *FG* equals the square on *EG*, therefore *AF* is to *EG* as *EG* is to *FG*, while *AF* is to *EG* as *AI* is to *EK*, and *EG* is to *FG* as *EK* is to *FK*. Therefore *EK* is a mean proportional between *AI* and *FK*. VI.17 VI.1 V.11

But *MN* is also a mean proportional between the squares *LM* and *NO*, and *AI* equals *LM* while *FK* equals *NO*, therefore *MN* also equals *EK*.

But *DH* equals *EK*, and *LO* equals *MN*, therefore the whole *DK* equals the gnomon *UVW* and *NO*.

Since, then, the whole *AK* equals *LM* and *NO*, and, in these, *DK* equals the gnomon *UVW* and *NO*, therefore the remainder *AB* equals *TS*.

But *TS* is the square on *LN*, therefore the square on *LN* equals the area *AB*. Therefore *LN* is the side of the area *AB*.

I say that *LN* is a first apotome of a medial straight line.

Since *EK* is rational and equals *LO*, therefore *LO*, that is, the rectangle *LP* by *PN*, is rational.

But *NO* was proved medial, therefore *LO* is incommensurable with *NO*.

But *LO* is to *NO* as *LP* is to *PN*, therefore *LP* and *PN* are incommensurable in length. VI.1 X.11

Therefore *LP* and *PN* are medial straight lines commensurable in square only, which contain a rational rectangle. Therefore *LN* is a first apotome of a medial straight line. X.74

And it is the side of the area *AB*.

Therefore the side of the area *AB* is a first apotome of a medial straight line.

Therefore, *if an area is contained by a rational straight line and a second apotome, then the side of the area is a first apotome of a medial straight line.*

Q.E.D.

Guide

This proposition is used in X.109.

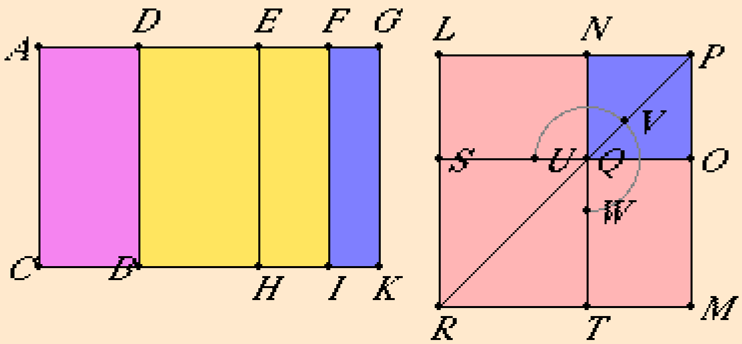
Book X Introduction - Proposition X.91 - Proposition X.93.

Euclid's Elements
Book X
Proposition 93

If an area is contained by a rational straight line and a third apotome, then the side of the area is a second apotome of a medial straight line.

Let the area *AB* be contained by the rational straight line *AC* and the third apotome *AD*.

I say that the side of the area *AB* is a second apotome of a medial straight line.



Let *DG* be the annex to *AD*. Then *AG* and *GD* are rational straight lines commensurable in square only, and neither of the straight lines *AG* and *GD* is commensurable in length with the rational straight line *AC* set out, while the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line commensurable with *AG*. X.Def.III.3

Since, then, the square on *AG* is greater than the square on *GD* by the square on a straight line commensurable with *AG*, therefore, if there is applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, then it divides it into commensurable parts. X.17

Bisect *DG* at *E*, apply to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF* by *FG*. Draw *EH*, *FI*, and *GK* through the points *E*, *F*, and *G* parallel to *AC*.

Then *AF* and *FG* are commensurable. Therefore *AI* is also commensurable with *FK*. VI.1 X.11

Since *AF* and *FG* are commensurable in length, therefore *AG* is also commensurable in length with each of the straight lines *AF* and *FG*. X.15

But *AG* is rational and incommensurable in length with *AC*, so that *AF* and *FG* are so also. X.13

Therefore each of the rectangles *AI* and *FK* is medial. X.21

Again, since *DE* is commensurable in length with *EG*, therefore *DG* is also commensurable in length with each of the straight lines *DE* and *EG*. X.15

But *GD* is rational and incommensurable in length with *AC*, therefore each of the straight lines *DE* and *EG* is also rational and incommensurable in length with *AC*. Therefore each of the rectangles *DH* and *EK* is medial. X.13 X.21

Since *AG* and *GD* are commensurable in square only, therefore *AG* is incommensurable in length with *GD*.

But AG is commensurable in length with AF, and DG with EG, therefore AF is incommensurable in length with EG. X.13

But AF is to EG as AI is to EK, therefore AI is incommensurable with EK. VI.1 X.11

Now construct the square LM equal to AI, and subtract NO, equal to FK, about the same angle with LM. Then LM and NO are about the same diameter. VI.26

Let PR be their diameter, and draw the figure.

Now, since the rectangle AF by FG equals the square on EG, therefore AF is to EG as EG is to FG. VI.17

But AF is to EG as AI is to EK, and EG is to FG as EK is to FK, therefore AI is to EK as EK is to FK. Therefore EK is a mean proportional between AI and FK. VI.1 VI.11

But MN is also a mean proportional between the squares LM and NO, and AI equals LM, and FK equals NO, therefore EK also equals MN.

But MN equals LO, and EK equals DH, therefore the whole DK also equals the gnomon UVW and NO.

But AK equals the sum of LM and NO, therefore the remainder AB equals ST, that is, to the square on LN. Therefore LN is the side of the area AB.

I say that LN is a second apotome of a medial straight line.

Since AI and FK were proved medial, and equal the squares on LP, therefore each of the squares on LP and PN is also medial. Therefore each of the straight lines LP and PN is medial.

Since AI is commensurable with FK, therefore the square on LP is also commensurable with the square on PN. VI.1 X.11

Again, since AI was proved incommensurable with EK, therefore LM is also incommensurable with MN, that is, the square on LP with the rectangle LP by PN, so that LP is also incommensurable in length with PN. VI.1 X.11

Therefore LP and PN are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

Since EK was proved medial, and equals the rectangle LP by PN, therefore the rectangle LP by PN is also medial, so that LP and PN are medial straight lines commensurable in square only which contain a medial rectangle.

Therefore LN is a second apotome of a medial straight line, and it is the side of the area AB. X.75

Therefore the side of the area AB is a second apotome of a medial straight line.

Therefore, *if an area is contained by a rational straight line and a third apotome, then the side of the area is a second apotome of a medial straight line.*

Q.E.D.

Guide

This proposition is used in X.110.

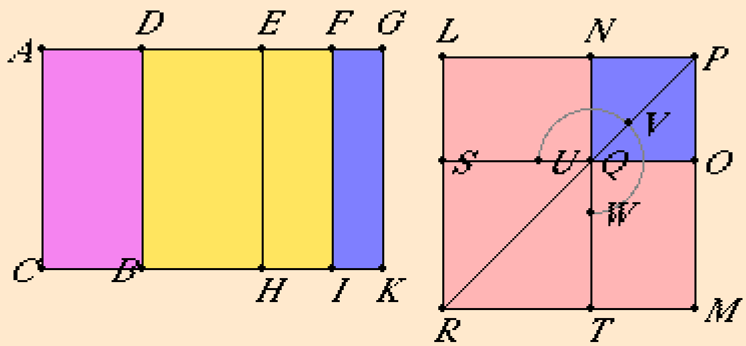
Book X Introduction - Proposition X.92 - Proposition X.94.

Euclid's Elements
Book X
Proposition 94

If an area is contained by a rational straight line and a fourth apotome, then the side of the area is minor.

Let the area be contained by the rational straight line *AC* and the fourth apotome *AD*.

I say that the side of the area *AB* is minor.



Let *DG* be the annex to *AD*, therefore *AG* and *GD* are rational straight lines commensurable in square only, *AG* is commensurable in length with the rational straight line *AC* set out, and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable in length with *AG*. X.Def.III.4

Since the square on *AG* is greater than the square on *GD* by the square on a straight line incommensurable in length with *AG*, therefore, if there is applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, then it divides it into incommensurable parts. X.18

Bisect *DG* at *E*, apply to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF* by *FG*. Then *AF* is incommensurable in length with *FG*.

Draw *EH*, *FI*, and *GK* through *E*, *F*, and *G* parallel to *AC* and *BD*.

Since *AG* is rational and commensurable in length with *AC*, therefore the whole *AK* is rational. X.19

Again, since *DG* is incommensurable in length with *AC*, and both are rational, therefore *DK* is medial. X.21

Again, since *AF* is incommensurable in length with *FG*, therefore *AI* is incommensurable with *FK*. VI.1 X.11

Now construct the square *LM* equal to *AI*, and subtract *NO*, equal to *FK*, about the same angle, the angle *LPM*.

Therefore the squares *LM* and *NO* are about the same diameter. Let *PR* be their diameter, and draw the figure. VI.26

Since the rectangle *AF* by *FG* equals the square on *EG*, therefore, *AF* is to *EG* as *EG* is to *FG*. VI.17

But *AF* is to *EG* as *AI* is to *EK*, and *EG* is to *FG* as *EK* is to *FK*, therefore *EK* is a mean proportional between *AI* and *FK*. VI.1 V.11

But *MN* is also a mean proportional between the squares *LM* and *NO*, and *AI* equals *LM*, and *FK* equals *NO* therefore *EK* also equals *MN*.

But *DH* equals *EK*, and *LO* equals *MN*, therefore the whole *DK* equals the gnomon *UVW* and *NO*.

Since, then, the whole *AK* equals the sum of the squares *LM* and *NO*, and, in these, *DK* equals the gnomon *UVW* and the square *NO*, therefore the remainder *AB* equals *ST*, that is, to the square on *LN*. Therefore *LN* is the side of the area *AB*.

I say that *LN* is the irrational straight line called minor.

Since *AK* is rational and equals the sum of the squares on *LP* and *PN*, therefore the sum of the squares on *LP* and *PN* is rational.

Again, since *DK* is medial, and *DK* equals twice the rectangle *LP* by *PN*, therefore twice the rectangle *LP* by *PN* is medial.

And, since *AI* was proved incommensurable with *FK*, therefore the square on *LP* is also incommensurable with the square on *PN*.

Therefore *LP* and *PN* are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial.

Therefore *LN* is the irrational straight line called minor, and it is the side of the area *AB*. X.76

Therefore the side of the area *AB* is minor.

Therefore, *if an area is contained by a rational straight line and a fourth apotome, then the side of the area is minor.*

Q.E.D.

Guide

This proposition is used in X.108. It is also used in proposition XIII.11.

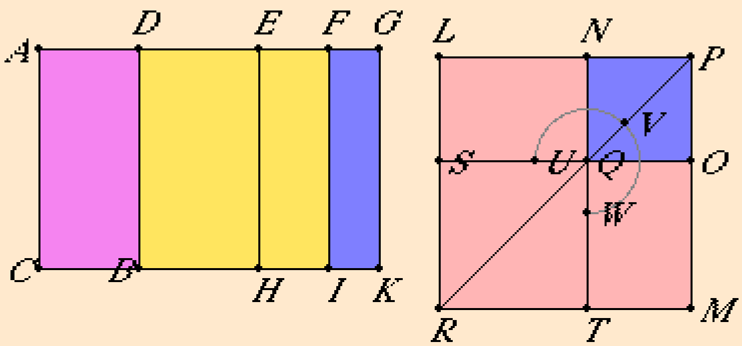
Book X Introduction - Proposition X.93 - Proposition X.95.

Euclid's Elements
Book X
Proposition 95

If an area is contained by a rational straight line and a fifth apotome, then the side of the area is a straight line which produces with a rational area a medial whole.

Let the area *AB* be contained by the rational straight line *AC* and the fifth apotome *AD*.

I say that the side of the area *AB* is a straight line which produces with a rational area a medial whole.



Let *DG* be the annex to *AD*. Then *AG* and *GD* are rational straight lines commensurable in square only, the annex *GD* is commensurable in length with the rational straight line *AC* set out, and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable with *AG*. X.Def.III.5

Therefore, if there is applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, then it divides it into in commensurable parts. X.18

Bisect *DG* at the point *E*, apply to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF* by *FG*. Then *AF* is incommensurable in length with *FG*.

Now, since *AG* is incommensurable in length with *CA*, and both are rational, therefore *AK* is medial. X.21

Again, since *DG* is rational and commensurable in length with *AC*, therefore *DK* is rational. X.19

Now construct the square *LM* equal to *AI*, and subtract the square *NO*, equal to *FK* and about the same angle, the angle *LPM*. Then the squares *LM* and *NO* are about the same diameter. Let *PR* be their diameter, and draw the figure. VI.26

Similarly then we can prove that *LN* is the side of the area *AB*.

I say that *LN* is the straight line which produces with a rational area a medial whole.

Since *AK* was proved medial and equals the sum of the squares on *LP* and *PN*, therefore the sum of the squares on *LP* and *PN* is medial.

Again, since *DK* is rational and equals twice the rectangle *LP* by *PN*, therefore the latter is itself also rational.

And, since *AI* is incommensurable with *FK*, therefore the square on *LP* is also incommensurable with the square on *PN*. Therefore *LP* and *PN* are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational.

Therefore the remainder *LN* is the irrational straight line called that which produces with a rational area a medial whole, and it is the side of the area *AB*. X.77

Therefore the side of the area *AB* is a straight line which produces with a rational area a medial whole.

Therefore, *if an area is contained by a rational straight line and a fifth apotome, then the side of the area is a straight line which produces with a rational area a medial whole.*

Q.E.D.

Guide

This proposition is used in X.109.

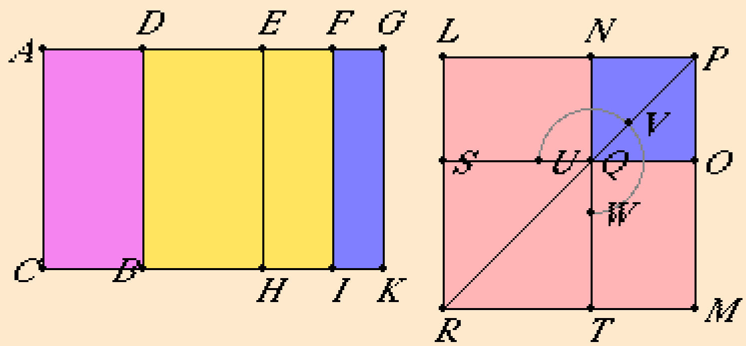
Book X Introduction - Proposition X.94 - Proposition X.96.

Euclid's Elements
Book X
Proposition 96

If an area is contained by a rational straight line and a sixth apotome, then the side of the area is a straight line which produces with a medial area a medial whole.

Let the area *AB* be contained by the rational straight line *AC* and the sixth apotome *AD*.

I say that the side of the area *AB* is a straight line which produces with a medial area a medial whole.



Let *DG* be the annex to *AD*. Then *AG* and *GD* are rational straight lines commensurable in square only, neither of them is commensurable in length with the rational straight line *AC* set out, and the square on the whole *AG* is greater than the square on the annex *DG* by the square on a straight line incommensurable in length with *AG*. X.Def.III.6

Since the square on *AG* is greater than the square on *GD* by the square on a straight line incommensurable in length with *AG*, therefore, if there is applied to *AG* a parallelogram equal to the fourth part of the square on *DG* and deficient by a square figure, then it divides it into incommensurable parts. X.18

Bisect *DG* at *E*, apply to *AG* a parallelogram equal to the square on *EG* and deficient by a square figure, and let it be the rectangle *AF* by *FG*. Then *AF* is incommensurable in length with *FG*.

But *AF* is to *FG* as *AI* is to *FK*, therefore *AI* is incommensurable with *FK*. X.11

Since *AG* and *AC* are rational straight lines commensurable in square only, therefore *AK* is medial. Again, since *AC* and *DG* are rational straight lines and incommensurable in length, *DK* is also medial. X.21

Now, since *AG* and *GD* are commensurable in square only, therefore *AG* is incommensurable in length with *GD*.

But *AG* is to *GD* as *AK* is to *KD*, therefore *AK* is incommensurable with *KD*. VI.1 X.11

Now construct the square *LM* equal to *AI*, and subtract *NO*, equal to *FK*, about the same angle. Then the squares *LM* and *NO* are about the same diameter. VI.26

Let *PR* be their diameter, and draw the figure. Then in manner similar to the above we can prove that *LN* is the side of the area *AB*.

I say that *LN* is a straight line which produces with a medial area a medial whole.

Since *AK* was proved medial and equals the sum of the squares on *LP* and *PN*, therefore the sum of the squares on *LP* and *PN* is medial. Again, since *DK* was proved medial and equals twice the rectangle *LP* by *PN*, therefore twice the rectangle *LP* by *PN* is also medial.

Since *AK* was proved incommensurable with *DK*, therefore the sum of the squares on *LP* and *PN* is also incommensurable with twice the rectangle *LP* by *PN*. And, since *AI* is incommensurable with *FK*, therefore the square on *LP* is also incommensurable with the square on *PN*.

Therefore *LP* and *PN* are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle contained by them medial, and further, the sum of the squares on them incommensurable with twice the rectangle contained by them.

Therefore *LN* is the irrational straight line called that which produces with a medial area a medial whole, and it is the side of the area *AB*. Therefore the side of the area is a straight line which produces with a medial area a medial whole. X.78

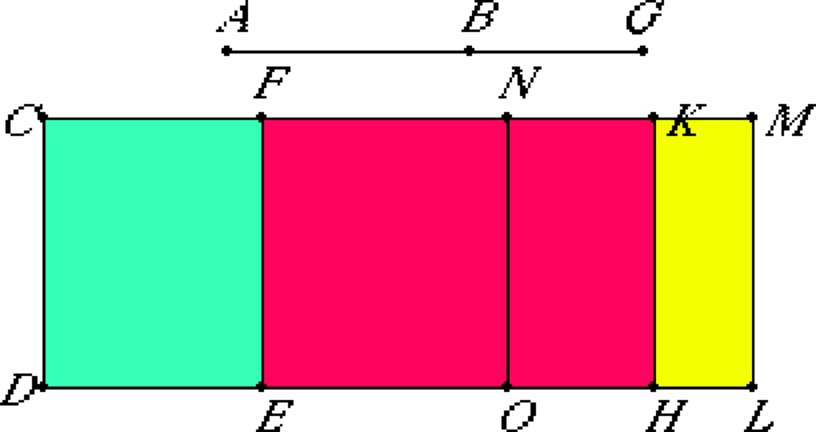
Therefore, *if an area is contained by a rational straight line and a sixth apotome, then the side of the area is a straight line which produces with a medial area a medial whole.*

Q.E.D.

Guide

This proposition is used in X.110.

Book X Introduction - Proposition X.95 - Proposition X.97.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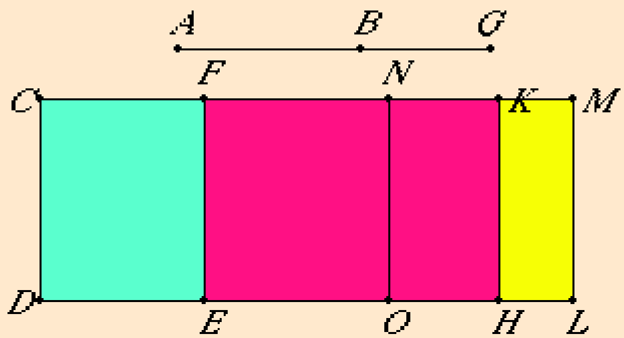
Euclid's Elements
Book X
Proposition 97

The square on an apotome of a medial straight line applied to a rational straight line produces as breadth a first apotome.

Let *AB* be an apotome, and *CD* rational, and to *CD* let there be applied *CE* equal to the square on *AB* and producing *CF* as breadth.

I say that *CF* is a first apotome.

Let *BG* be the annex to *AB*. Then *AG* and *GB* are rational straight lines commensurable in square only. X.73



To *CD* apply *CH*, equal to the square on *AG*, and *KL*, equal to the square on *BG*.

Then the whole *CL* equals the sum of the squares on *AG* and *GB*, and, in these, *CE* equals the square on *AB*, therefore the remainder *FL* equals twice the rectangle *AG* by *GB*. II.7

Bisect *FM* at the point *N*, and draw *NO* through *N* parallel to *CD*. Then each of the rectangles *FO* and *LN* equals the rectangle *AG* by *GB*.

Now, since the sum of the squares on *AG* and *GB* is rational, and *DM* equals the sum of the squares on *AG* and *GB*, therefore *DM* is rational.

And it is applied to the rational straight line *CD* producing *CM* as breadth, therefore *CM* is rational and commensurable in length with *CD*. X.20

Again, since twice the rectangle *AG* by *GB* is medial, and *FL* equals twice the rectangle *AG* by *GB*, therefore *FL* is medial. And it is applied to the rational straight line *CD* producing *FM* as breadth, therefore *FM* is rational and incommensurable in length with *CD*. X.22

Since the squares on *AG* and *GB* are rational, while twice the rectangle *AG* by *GB* is medial, therefore the sum of the squares on *AG* and *GB* is incommensurable with twice the rectangle *AG* by *GB*.

And *CL* equals the sum of the squares on *AG* and *GB*, and *FL* equals twice the rectangle *AG* by *GB*, therefore *DM* is incommensurable with *FL*.

But *DM* is to *FL* as *CM* is to *FM*, therefore *CM* is incommensurable in length with *FM*. VI.1 X.11

And both are rational, therefore *CM* and *MF* are rational straight lines commensurable in square only. Therefore *CF* is an apotome. X.73

I say next that it is also a first apotome.

Since the rectangle *AG* by *GB* is a mean proportional between the squares on *AG* and *GB*, *CH* equals the square on *AG*, *KL* equals the square on *BG*, and *NL* equals the rectangle *AG* by *GB*, therefore *NL* is also a mean proportional between *CH* and *KL*. Therefore *CH* is to *NL* as *NL* is to *KL*.

But *CH* is to *NL* as *CK* is to *NM*, and *NL* is to *KL* as *NM* is to *KM*, therefore the rectangle *CK* by *KM* equals the square on *NM*, that is, the fourth part of the square on *FM*. VI.1 VI.17

Since the square on *AG* is commensurable with the square on *GB*, therefore *CH* is also commensurable with *KL*.

But *CH* is to *KL* as *CK* is to *KM*, therefore *CK* is commensurable with *KM*. VI.1 X.11

Since *CM* and *MF* are two unequal straight lines, and to *CM* there has been applied the rectangle *CK* by *KM* equal to the fourth part of the square on *FM* and deficient by a square figure, while *CK* is commensurable with *KM*, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable in length with *CM*. X.17

And *CM* is commensurable in length with the rational straight line *CD* set out, therefore *CF* is a first apotome. X.Def.III.2

Therefore, *the square on an apotome of a medial straight line applied to a rational straight line produces as breadth a first apotome.*

Q.E.D.

Guide

This proposition is used in X.111. It is also used in proposition XIII.6.

Book X Introduction - Proposition X.96 - Proposition X.98.

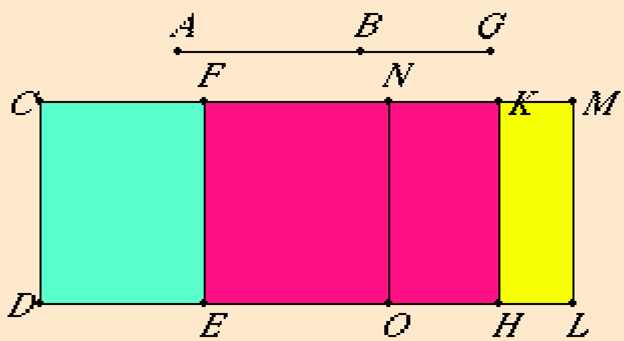
Euclid's Elements
Book X
Proposition 98

The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.

Let *AB* be a first apotome of a medial straight line and *CD* a rational straight line, and to *CD* let there be applied *CE* equal to the square on *AB* producing *CF* as breadth.

I say that *CF* is a second apotome.

Let *BG* be the annex to *AB*. Then *AG* and *GB* are medial straight lines commensurable in square only which contain a rational rectangle. X.74



To *CD* apply *CH*, equal to the square on *AG*, producing *CK* as breadth, and *KL*, equal to the square on *GB*, producing *KM* as breadth.

Therefore the whole *CL* equals the sum of the squares on *AG*. Therefore *CL* is also medial. X.15 X.23,Cor.

And it is applied to the rational straight line *CD* producing *CM* as breadth, therefore *CM* is rational and incommensurable in length with *CD*. X.22

Now, since *CL* equals the sum of the squares on *AG* and *GB*, and, in these, the square on *AB* equals *CE*, therefore the remainder, twice the rectangle *AG* by *GB*, equals *FL*. II.7

But twice the rectangle *AG* by *GB* is rational, therefore *FL* is rational.

And it is applied to the rational straight line *FE* producing *FM* as breadth, therefore *FM* is also rational and commensurable in length with *CD*. X.20

Now, since the sum of the squares on *AG* and *GB*, that is, *CL*, is medial, while twice the rectangle *AG* by *GB*, that is, *FL*, is rational, therefore *CL* is incommensurable with *FL*.

But *CL* is to *FL* as *CM* is to *FM*, therefore *CM* is incommensurable in length with *FM*. VI.1 X.11

And both are rational, therefore *CM* and *MF* are rational straight lines commensurable in square only. Therefore *CF* is an apotome. X.73

I say next that it is also a second apotome.

Bisect *FM* at *N*, and draw *NO* through *N* parallel to *CD*. Then each of the rectangles *FO* and *NL* equals the rectangle *AG* by *GB*.

Now, since the rectangle *AG* by *GB* is a mean proportional between the squares on *AG* and *GB*, the square on *AG* equals *CH*, the rectangle *AG* by *GB* equals *NL*, and the square on *BG* equals *KL*, therefore *NL* is also a mean proportional between *CH* and *KL*. Therefore *CH* is to *NL* as *NL* is to *KL*.

But *CH* is to *NL* as *CK* is to *NM*, and *NL* is to *KL* as *NM* is to *MK*, therefore *CK* is to *NM* as *NM* is to *KM*. Therefore the rectangle *CK* by *KM* equals the square on *NM*, that is, the fourth part of the square on *FM*. VI.1 V.11 VI.17

Since *CM* and *MF* are two unequal straight lines, and the rectangle *CK* by *KM*, equal to the fourth part of the square on *MF* and deficient by a square figure, has been applied to the greater, *CM*, and divides it into commensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable in length with *CM*. X.17

And the annex *FM* is commensurable in length with the rational straight line *CD* set out, therefore *CF* is a second apotome. X.Def.III.2

Therefore, *the square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.*

Q.E.D.

Guide

This proposition is used in X.111.

Book X Introduction - Proposition X.97 - Proposition X.99.

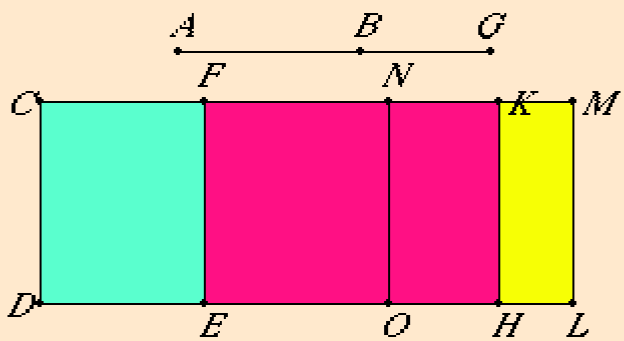
Euclid's Elements
Book X
Proposition 99

The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.

Let *AB* be a second apotome of a medial straight line, and *CD* rational, and to *CD* let there be applied *CE* equal to the square on *AB* producing *CF* as breadth.

I say that *CF* is a third apotome.

Let *BG* be the annex to *AB*, therefore *AG* and *GB* are medial straight lines commensurable in square only which contains a medial rectangle. X.75



Apply *CH*, equal to the square on *AG*, to *CD* producing *CK* as breadth, and apply *KL*, equal to the square on *BG*, to *KH* producing *KM* as breadth. Then the whole *CL* equals the sum of the squares on *AG* and *GB*. Therefore *CL* is also medial. X.15 X.23,Cor.

And it is applied to the rational straight line *CD* producing *CM* as breadth, therefore *CM* is rational and incommensurable in length with *CD*. X.22

Now, since the whole *CL* equals the sum of the squares on *AG* and *GB*, and, in these, *CE* equals the square on *AB*, therefore the remainder *LF* equals twice the rectangle *AG* by *GB*. II.7

Bisect *FM* at the point *N*, and draw *NO* parallel to *CD*. Then each of the rectangles *FO* and *NL* equals the rectangle *AG* by *GB*.

But the rectangle *AG* by *GB* is medial, therefore *FL* is also medial.

And it is applied to the rational straight line *EF* producing *FM* as breadth, therefore *FM* is also rational and incommensurable in length with *CD*. X.22

Since *AG* and *GB* are commensurable in square only, therefore *AG* is incommensurable in length with *GB*. Therefore the square on *AG* is also incommensurable with the rectangle *AG* by *GB*. VI.1 X.11

But the sum of the squares on *AG* and *GB* is commensurable with the square on *AG*, and twice the rectangle *AG* by *GB* with the rectangle *AG* by *GB*, therefore the sum of the squares on *AG* and *GB* is incommensurable with twice the rectangle *AG* by *GB*. X.13

But *CL* equals the sum of the squares on *AG* and *GB*, and *FL* equals twice the rectangle *AG* by *GB*, therefore *CL* is also incommensurable with *FL*.

But *CL* is to *FL* as *CM* is to *FM*, therefore *CM* is incommensurable in length with *FM*. VI.1

And both are rational, therefore *CM* and *MF* are rational straight lines commensurable in square only, therefore *CF* is an apotome. X.73

I say next that it is also a third apotome.

Since the square on *AG* is commensurable with the square on *GB*, therefore *CH* is also commensurable with *KL*, so that *CK* is also commensurable with *KM*. VI.1 X.11

Since the rectangle *AG* by *GB* is a mean proportional between the squares on *AG* and *GB*, *CH* equals the square on *AG*, *KL* equals the square on *GB*, and *NL* equals the rectangle *AG* by *GB*, therefore *NL* is also a mean proportional between *CH* and *KL*. Therefore *CH* is to *NL* as *NL* is to *KL*.

But *CH* is to *NL* as *CK* is to *NM*, and *NL* is to *KL* as *NM* is to *KM*, therefore *CK* is to *MN* as *MN* is to *KM*. Therefore the rectangle *CK* by *KM* equals the square on *MN*, that is, to the fourth part of the square on *FM*. VI.1 V.11

Since, then, *CM* and *MF* are two unequal straight lines, and a parallelogram equal to the fourth part of the square on *FM* and deficient by a square figure has been applied to *CM*, and divides it into commensurable parts, therefore the square on *CM* is greater than the square on *MF* by the square on a straight line commensurable with *CM*. X.17

And neither of the straight lines *CM* nor *MF* is commensurable in length with the rational straight line *CD* set out, therefore *CF* is a third apotome. X.Def.III.3

Therefore, *the square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.*

Q.E.D.

Guide

This proposition is used in X.111.

Book X Introduction - Proposition X.98 - Proposition X.100.

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Book XI

Euclid's Elements

Book XI

Table of contents

Definitions

Definition 1.
A *solid* is that which has length, breadth, and depth.

Definition 2.
A face of a solid is a surface.

Definition 3.
A straight line is *at right angles* to a plane when it makes right angles with all the straight lines which meet it and are in the plane.

Definition 4.
A plane is *at right angles* to a plane when the straight lines drawn in one of the planes at right angles to the intersection of the planes are at right angles to the remaining plane.

Definition 5.
The *inclination* of a straight line to a plane is, assuming a perpendicular drawn from the end of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the end of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up.

Definition 6.
The *inclination* of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to the intersection at the same point, one in each of the planes.

Definition 7.

A plane is said to be *similarly inclined* to a plane as another is to another when the said angles of the inclinations equal one another.

Definition 8.
Parallel planes are those which do not meet.

Definition 9.
Similar solid figures are those contained by similar planes equal in multitude.

Definition 10.
Equal and similar solid figures are those contained by similar planes equal in multitude and magnitude.

Definition 11.
A *solid angle* is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines, that is, a *solid angle* is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

Definition 12.
A *pyramid* is a solid figure contained by planes which is constructed from one plane to one point.

Definition 13.
A *prism* is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar, and parallel, while the rest are parallelograms.

Definition 14.
When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *sphere*.

Definition 15.
The *axis of the sphere* is the straight line which remains fixed and about which the semicircle is turned.

Definition 16.
The *center of the sphere* is the same as that of the semicircle.

Definition 17.
A *diameter of the sphere* is any straight line drawn through the center and terminated in both directions by the surface of the sphere.

Definition 18.
When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *cone*. And, if the straight line which remains fixed equals the remaining side about the right angle which is carried round, the cone will be *right-angled*; if less, *obtuse-angled*; and if greater, *acute-angled*.

Definition 19.
The *axis of the cone* is the straight line which remains fixed and about which the triangle is turned.

Definition 20.
And the *base* is the circle described by the straight in which is carried round.

Definition 21.
When a rectangular parallelogram with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *cylinder*.

Definition 22.

The *axis of the cylinder* is the straight line which remains fixed and about which the parallelogram is turned.

Definition 23.
And the *bases* are the circles described by the two sides opposite to one another which are carried round.

Definition 24.
Similar cones and cylinders are those in which the axes and the diameters of the bases are proportional.

Definition 25.
A *cube* is a solid figure contained by six equal squares.

Definition 26.
An *octahedron* is a solid figure contained by eight equal and equilateral triangles.

Definition 27.
An *icosahedron* is a solid figure contained by twenty equal and equilateral triangles.

Definition 28.
A *dodecahedron* is a solid figure contained by twelve equal, equilateral and equiangular pentagons.

Propositions

Proposition 1.
A part of a straight line cannot be in the plane of reference and a part in plane more elevated.

Proposition 2.
If two straight lines cut one another, then they lie in one plane; and every triangle lies in one plane.

Proposition 3.
If two planes cut one another, then their intersection is a straight line.

Proposition 4.
If a straight line is set up at right angles to two straight lines which cut one another at their common point of section, then it is also at right angles to the plane passing through them.

Proposition 5.
If a straight line is set up at right angles to three straight lines which meet one another at their common point of section, then the three straight lines lie in one plane.

Proposition 6.
If two straight lines are at right angles to the same plane, then the straight lines are parallel.

Proposition 7.
If two straight lines are parallel and points are taken at random on each of them, then the straight line joining the points is in the same plane with the parallel straight lines.

Proposition 8.
If two straight lines are parallel, and one of them is at right angles to any plane, then the remaining one is also at right angles to the same plane.

Proposition 9
Straight lines which are parallel to the same straight line but do not lie in the same plane with it are also parallel to each other.

Proposition 10.

is to the solid.

Proposition 26.
To construct a solid angle equal to a given solid angle on a given straight line at a given point on it.

Proposition 27.
To describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid on a given straight line.

Proposition 28.
If a parallelepipedal solid is cut by a plane through the diagonals of the opposite planes, then the solid is bisected by the plane.

Proposition 29.
Parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are on the same straight lines, equal one another.

Proposition 30.
Parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are not on the same straight lines, equal one another.

Proposition 31.
Parallelepipedal solids which are on equal bases and of the same height equal one another.

Proposition 32.
Parallelepipedal solids which are of the same height are to one another as their bases.

Proposition 33.
Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

Corollary. If four straight lines are continuously proportional, then the first is to the fourth as a parallelepipedal solid on the first is to the similar and similarly situated parallelepipedal solid on the second, in as much as the first has to the fourth the ratio triplicate of that which it has to the second.

Proposition 34.
In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

Proposition 35.
If there are two equal plane angles, and on their vertices there are set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points are taken at random and perpendiculars are drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines are joined to the vertices of the original angles, then they contain with the elevated straight lines equal angles.

Proposition 36.
If three straight lines are proportional, then the parallelepipedal solid formed out of the three equals the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.

Proposition 37.
If four straight lines are proportional, then parallelepipedal solids on them which are similar and similarly described are also proportional; and, if the parallelepipedal solids on them which are similar and similarly described are proportional, then the straight lines themselves are also proportional.

Proposition 38.

If the sides of the opposite planes of a cube are bisected, and the planes are carried through the points of section, then the intersection of the planes and the diameter of the cube bisect one another.

Proposition 39.

If there are two prisms of equal height, and one has a parallelogram as base and the other a triangle, and if the parallelogram is double the triangle, then the prisms are equal.

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Euclid's Elements
Book XI
Definitions 1 and 2

Def. 1. *A* solid *is that which has length, breadth, and depth.*

Def. 2. *A face of a solid is a surface.*

Guide

The 28 definitions at the beginning of Book XI serve Books XII and XIII as well.

The first two definitions correspond to definitions I.Def.2 and I.Def.3 for a line and its ends, and definitions I.Def.5 and I.Def.6 for a surface and its edges.

Some examples of solids that appear in Books XI through XII are parallelepipedal solids (see proposition XI.24 and the following propositions), prisms (XI.Def.13 and proposition XI.39), pyramids (XI.Def.12, XII.3 and the following propositions), cones and cylinders (XI.Def.18 through XI.Def.24, XII.10 and the following propositions), spheres (XI.Def.14 through XI.Def.17, and propositions XII.17 and XII.18), octahedra (XI.Def.26, XIII.14), cubes (XI.Def.25, XIII.15), icosahedra (XI.Def.27, XIII.16), and dodecahedra (XI.Def.28, XIII.17).

Next definition: XI.Def.3-5

Book XI introduction

Select from Book XI

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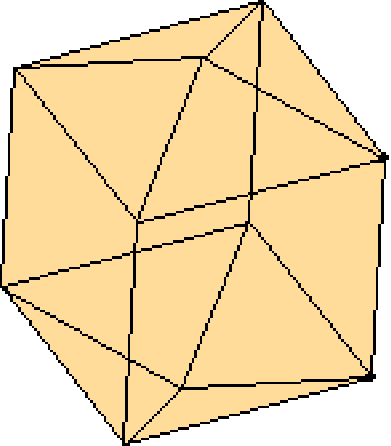
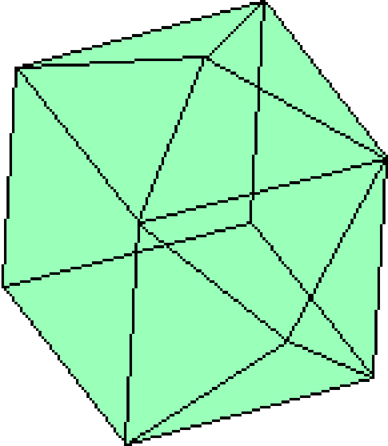

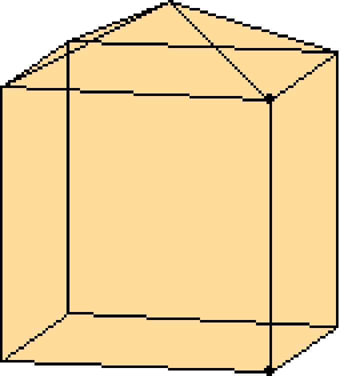
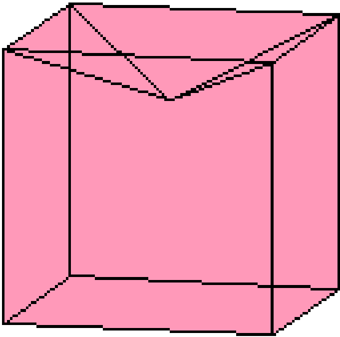

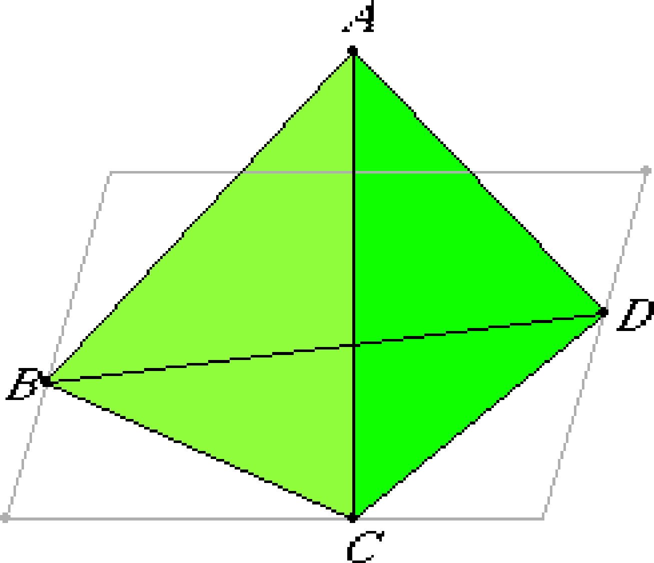
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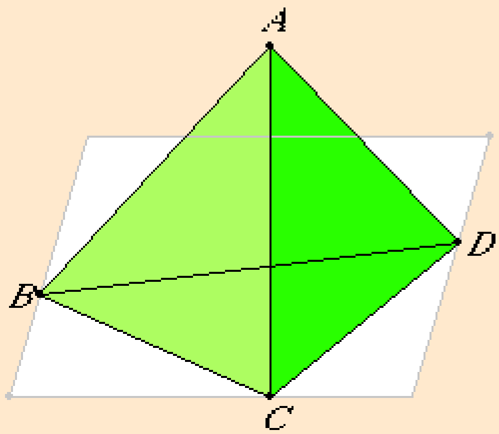
Euclid's Elements
Book XI
Definition 11

A solid angle *is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines, that is, a* solid angle *is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.*

Guide



A solid angle is intended to be bounded by three or more planes meeting at a point. The solid angle at *A* is bounded by the three planes *ABC*, *ACD*, and *ADB.* The figure *ABCD* is a triangular pyramid. Pyramids are defined in definition XI.Def.12 coming next.

The two definitions given here for solid angle are not strictly equivalent. In the first the lines mentioned are not specified as being straight, and the surfaces are not specified as being planes. In the second the surfaces are specified as being planes, and since planes meet in straight lines (XI.3), the lines must be straight. The difference, however, may well be an oversight.

Next definition: XI.Def.12-13

Previous: XI0.Def.9-10

Book XI introduction

Select from Book XI

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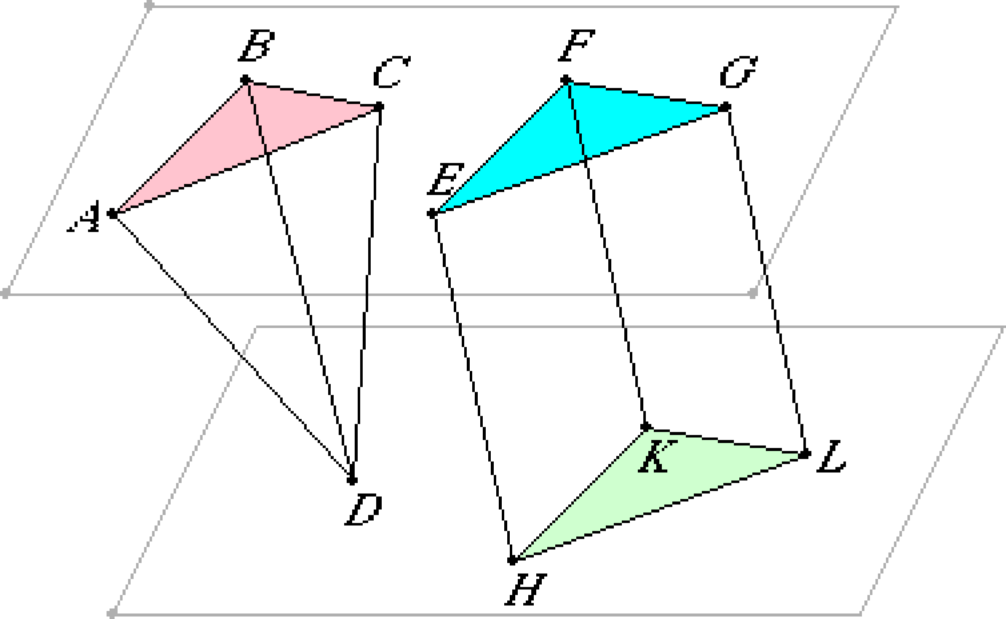
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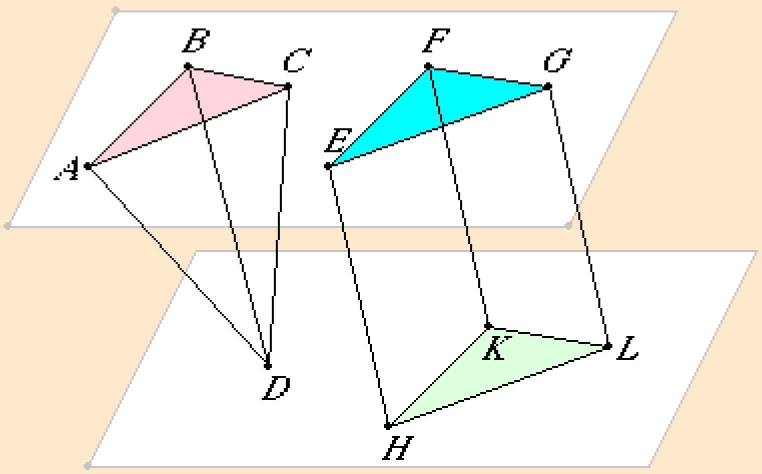
Euclid's Elements
Book XI
Definitions12 and 13

Def. 12. *A* pyramid *is a solid figure contained by planes which is constructed from one plane to one point.*

Def. 13. *A* prism *is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar, and parallel, while the rest are parallelograms.*

Guide

In the diagram below, *ABCD* is a pyramid with vertex *D* and triangular base *ABC.* Since it's a tetrahedron, it's still a triangular pyramid when any of the other three sides is considered the base. Also, *EFGHKL* is a prism with opposite triangular sides *EFG* and *HKL*.



Definition 12 for pyramids is rather abbreviated, but the intention is clear. Note that in definition 13 the term "equal and similar" is used for congruence of plane figures.

Prisms, by that name, are first discussed in proposition XI.39. Pyramids are treated in propositions XII.3 through XII.9 in Book XII.

Next definition: XI.Def.14-17

Previous: XI.Def.11

Book XI introduction

Select from Book XI

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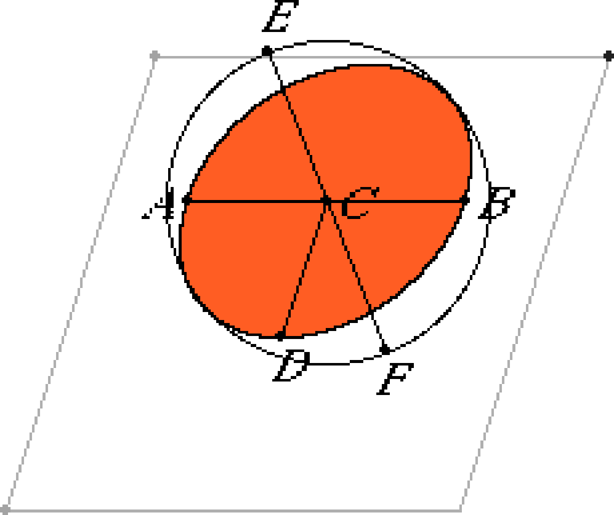
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Euclid's Elements
Book XI
Definitions 14 through 17

Def. 14. *When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a* sphere.

Def. 15. *The* axis of the sphere *is the straight line which remains fixed and about which the semicircle is turned.*

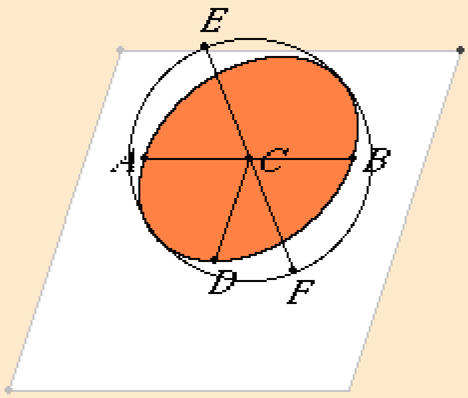
Def. 16. *The* center of the sphere *is the same as that of the semicircle.*

Def. 17. *A* diameter of the sphere *is any straight line drawn through the center and terminated in both directions by the surface of the sphere.*

Guide

There are alternative definitions for a sphere, but Euclid chose this one, perhaps, to be analogous to the definitions of cone in XI.Def.18 and cylinder in XI.Def.22. These are all defined as *solids of revolution*, that is, solids generated by rotating a plane figure around a straight line called the *axis of revolution.*

Another book would probably be required to develop the theory of spheres to the degree that Euclid developed the theory of circles in Book III, but that, apparently, was not his goal. The lack of propositions is so severe that it is not even shown that any two points on the surface of a sphere are equidistant from the center. (Any point on the surface of the sphere is a point on the circumference of one of the rotated semicircles, and all the points on any of these semicircles are equidistant from the center of the semicircles.)



In the illustration at the right there is a semicircle *ADB* with center *C* and diameter *AB* in a plane. When the semicircle is revolved around *AB,* a sphere results. The sphere's axis is *AB*, and its center is *C*. If *E* is any point on the sphere and *F* the antipodal point, then the line *EF* is a diameter of the sphere.

There are very few propositions about spheres in the *Elements.* Proposition XII.17 allows a kind of approximation of spheres by polyhedra preliminary to proposition XII.18 on the ratio of volumes of spheres. Also, regular polyhedra are inscribed in spheres in Book XIII

With so few propositions there are gaps in the proofs. For instance, in XII.17 it is claimed that the the intersection of a plane and a sphere is a circle, but a justification is lacking.

Next definition: XI.Def.18-20

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Previous: XI.Def.12-13

Book XI introduction

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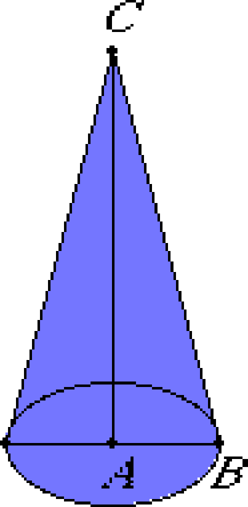
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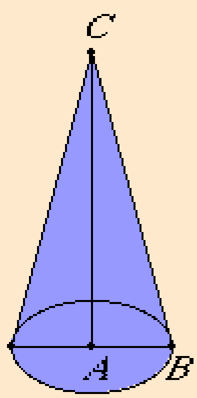
Euclid's Elements
Book XI
Definitions 18 through 20

Def. 18. *When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a* cone. *And, if the straight line which remains fixed equals the remaining side about the right angle which is carried round, the cone will be* right-angled; *if less,* obtuse-angled; *and if greater,* acute-angled.

Def. 19. *The* axis of the cone *is the straight line which remains fixed and about which the triangle is turned.*

Def. 20. *And the* base *is the circle described by the straight line which is carried round.*

Guide



The right triangle *ABC* with right angle at *A* is rotated about the side *AC* to produce a cone. The axis of the cone is *AC,* and its base is the circle with center at *A* and radius *AB.*

The three different kinds of cone are not used by Euclid in the *Elements,* but they were important in the theory of conic sections until Apollonius' work *Conics.* In Euclid's time conic sections were taken as the intersections of a plane at right angles to an edge (straight line from the vertex) of a cone. When the cone is acute-angled, the section is an ellipse; when right-angled, a parabola; and when obtuse-angle, a hyperbola. Even the names of these three curves were given by the kind of angle, so, for instance, Euclid knew a parabola as a "section of a right-angled cone." It was Apollonius who named them ellipse, parabola, and hyperbola.

Next definition: XI.Def.21-23

Previous: XI.Def.14-17

Book XI introduction

Select from Book XI

Select book

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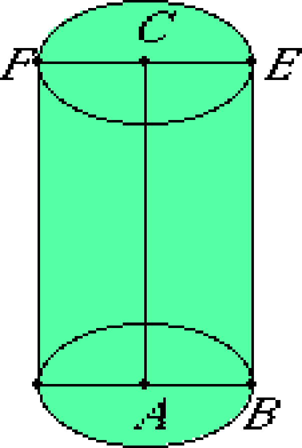
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Euclid's Elements
Book XI
Definitions 21 through 23

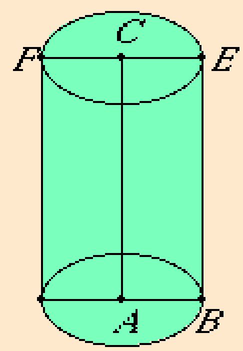
Def. 21. *When a rectangular parallelogram with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a* cylinder.

Def. 22. *The* axis of the cylinder *is the straight line which remains fixed and about which the parallelogram is turned.*

Def. 23. *And the* bases *are the circles described by the two sides opposite to one another which are carried round.*

Guide

Rectangle *ABEC* is rotated around the side *AC* to produce a cylinder. Its axis is *AC* and it has two circles for bases.



The concept of cylinder has been generalized since Euclid's time as have so many ancient mathematical concepts. Euclid's cylinders are right, circular cylinders since their axes are at right angles to their bases and their bases are circles.

Next definition: XI.Def.24

Previous: XI.Def.18-20

Book XI introduction

Select from Book XI

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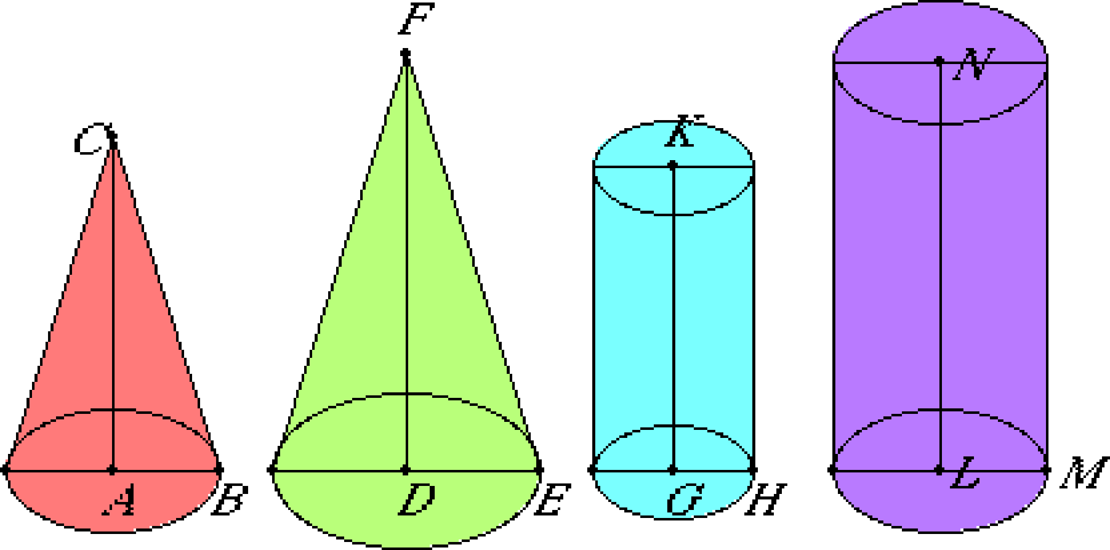
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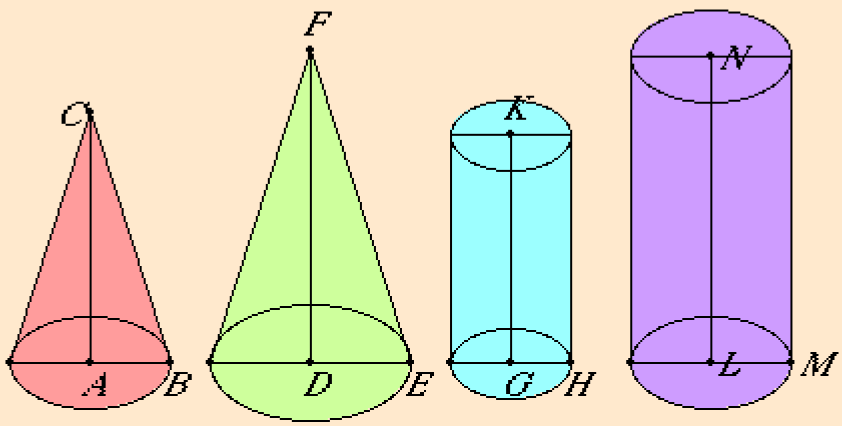
Euclid's Elements

Book XI

Definition 24

Similar cones and cylinders *are those in which the axes and the diameters of the bases are proportional.*

Guide



Two cones are similar if the axis of the first is to the axis of the second as the base diameter of the first is to the base diameter of the second. Likewise, two cylinders are similar if the axis of the first is to the axis of the second as the base diameter of the first is to the base diameter of the second.

It can be shown that an equivalent condition is that the vertex angles of the cones are equal. Thus, all right-angled cones are similar.

Next defintion: XI.Def.25-28

Previous: XI.Def.21-23

Book XI introduction

Select from Book XI

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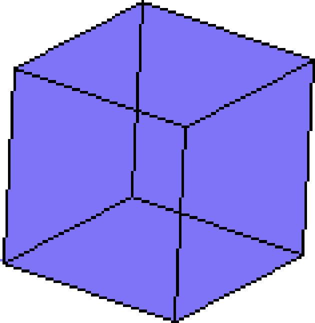


Euclid's Elements
Book XI
Definitions 25 through 28

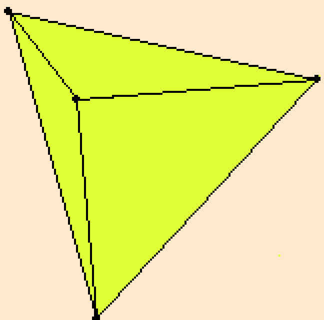
Def. 25. *A* cube *is a solid figure contained by six equal squares.*

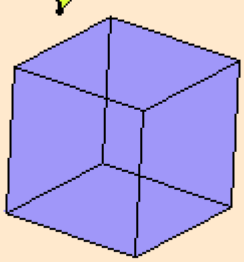
Def. 26. *An* octahedron *is a solid figure contained by eight equal and equilateral triangles.*

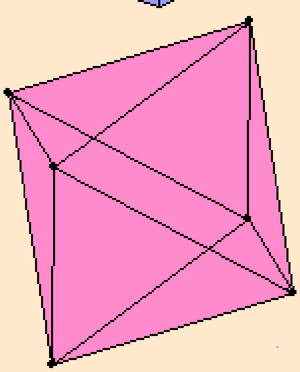
Def. 27. *An* icosahedron *is a solid figure contained by twenty equal and equilateral triangles.*

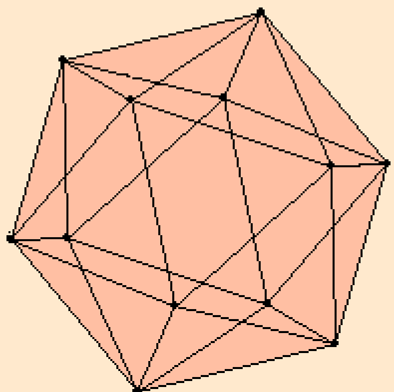
Def. 28. *A* dodecahedron *is a solid figure contained by twelve equal, equilateral and equiangular pentagons.*

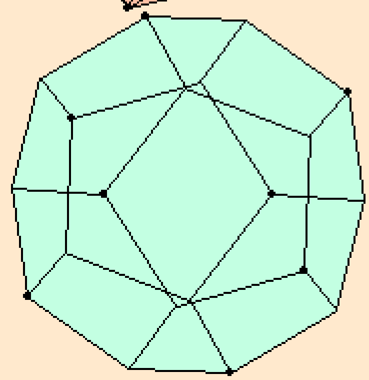
Guide











These are four of the five regular solids. The tetrahedron is not mentioned here since it is a certain triangular pyramid. It's called simply the "pyramid" in Book XIII.

The regular tetrahedron is constructed in proposition XIII.13, the cube in XIII.15, the octahedron in XIII.14, the icosahedron in XIII.16, and the dodecahedron in XIII.17. These five are shown to be the only regular solids in proposition XIII.18.

Next proposition: XI.1

Previous: XI.Def.24

Select from Book XI

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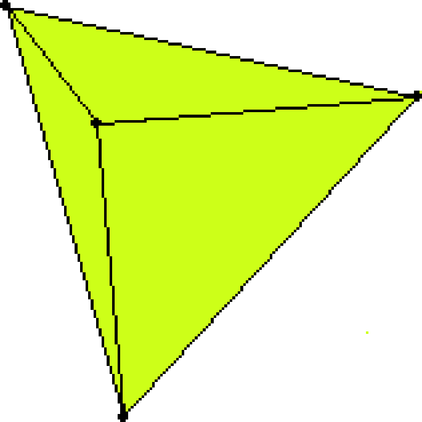

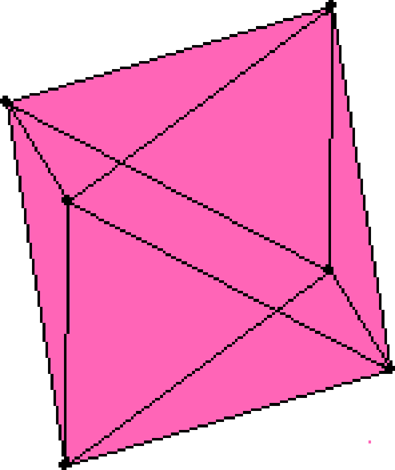

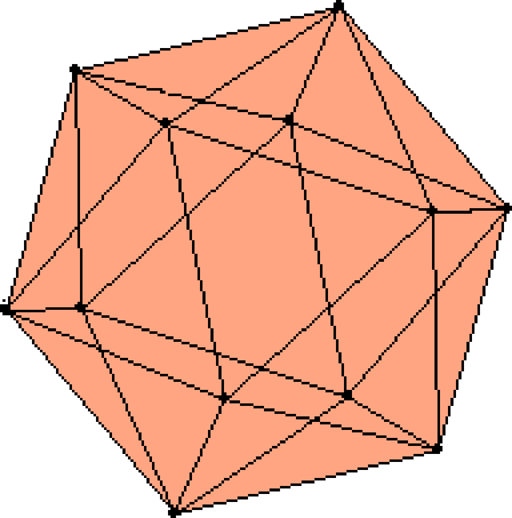

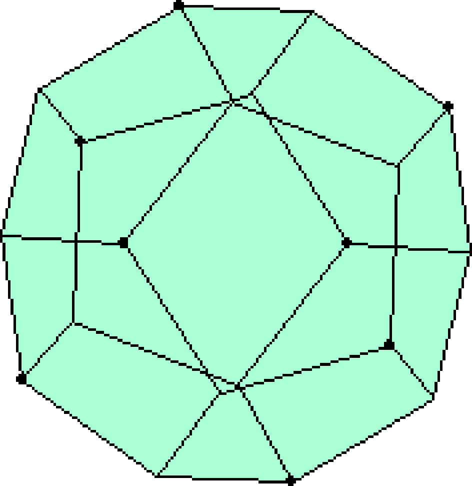

Euclid's Elements
Book XI
Definitions 3 through 5

Def. 3. *A straight line is* at right angles *to a plane when it makes right angles with all the straight lines which meet it and are in the plane.*

Def. 4. *A plane is* at right angles *to a plane when the straight lines drawn in one of the planes at right angles to the intersection of the planes are at right angles to the remaining plane.*

Def. 5. *The* inclination *of a straight line to a plane is, assuming a perpendicular drawn from the end of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the end of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up.*

Guide

Although definition 3 states that a line needs to be at right angles with all of the straight lines which meet it and lie in the plane, proposition XI.4 states that it is only necessary that a straight line be at right angles to two lines in the plane in order that it be at right angles to all the rest.

There is an implicit assumption in definition 3 as it speaks of a straight line making right angles with straight lines which meet it and are in the plane. The concept of two lines making a right angle assumes that the two sides of the angles lie in one plane, that is, that two intersecting lines lie in a plane, a statement that is supposedly verified in proposition XI.2.

The concept of a line being perpendicular to a plane is central to solid geometry. It is developed and used in many propositions in Book XI, starting with XI.4.

There is also an implicit assumption in definition 4, namely that the intersection of the two planes is a straight line, a statement that is supposedly verified in proposition XI.3. The concept of planes perpendicular to planes first appears inproposition XI.18 which states that if *one* straight line drawn in one of the planes is at right angles to the other plane, then the two planes are at perpendicular.

Definition 5 is meant to define the inclination (angle) between a line and a plane as the angle between that line and the projection of it in the plane. This requires that there is a line at right angles to a plane from a point not on the plane which is assured by proposition XI.11. It also requires that the angle constructed in the definition is independent of the construction.

Next defintion: XI.Def.6-8

Previous: XI.Def.1-2

Book XI introduction

Select from Book XI

Select book

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D.E.Joyce
Clark University

Euclid's Elements
Book XI
Definitions 6 through 8

Def. 6. *The* inclination *of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to the intersection at the same point, one in each of the planes.*

Def. 7. *A plane is said to be* similarly inclined *to a plane as another is to another when the said angles of the inclinations equal one another.*

Def. 8. Parallel planes *are those which do not meet.*

Guide

As the previous definition requires certain assumptions, so does definition 6. It assumes that any two such acute angles are equal, something Euclid does not prove but could have in the course of Book XI.

Definition 8 is analogous to definition I.23 for parallel lines in a plane. There is no proposition in Book XI which states that parallelism of planes is a transitive relation, but that is not difficult to prove given the rest of the propositions in the book. The first appearance of parallel planes is in proposition XI.14.

When two planes are not parallel, then, by this definition, they intersect. Proposition XI.3 proclaims that this intersection is a straight line.

Note that it is not defined when a line is parallel to a plane, but that would be when they don't meet.

Next defintion: XI.Def.9-10

Previous: XI.Def.3-5

Book XI introduction

Select from Book XI

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Euclid's Elements
Book XI
Definitions 9 and 10

Def. 9. Similar solid figures *are those contained by similar planes equal in multitude.*

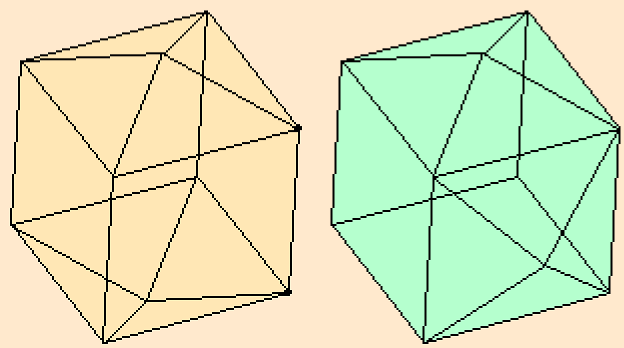
Def. 10. Equal and similar solid figures *are those contained by similar planes equal in multitude and magnitude.*

Guide

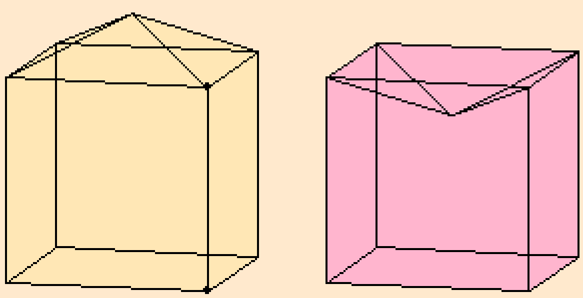
While definition 9 defines similar solid figures, definition 10 describes what is commonly called "congruent" solid figures. Euclid uses "equal and similar" plane figures for "congruent" plane figures in these later books, but that could not be done for plane figures before Book VI where similarity of plane figures was defined.

These definitions are incomplete as was VI.Def.1 for similar rectilinear figures. The notion of similarity for plane figures implicitly assumed a correspondence of consecutive vertices and sides. This notion of similar solid figures assumes a correspondence of of adjacent edges and faces.

Different solid figures can sometimes be constructed with the same faces but with different adjacencies. For example, there are two distinct ways to attach two pyramids to two of the square faces of a cube. They could be attached to opposite faces of the cube or to adjacent faces of the cube. The resulting solids both have four remaining square faces and eight new triangular faces, but the positioning of the squares and triangles is different.



Heron's definition of similar solid figures, "those which are contained by equal and similarly situated planes, equal in number and magnitude," is a bit more explicit than Euclid's.



It is also apparent that Euclid did not consider the possibility of concave solids and the problems they cause his definition, problems that Simson noticed. Take a cube and first add a pyramid on the outside of one square face, and second subtract the same pyramid from the inside of a square face. The resulting two solids both have five square faces and four triangular faces, and the adjacencies are the same, but they are very different solids.

To eliminate this problem, further conditions must be made on the definition of similar solids. For instance, conditions on dihedral angles between faces, or conditions on distances between all corresponding pairs of vertices.

Next defintion: XI.Def.11 Select from Book XI

Previous: XI.Def.6-8

Book XI introduction

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Index of /~djoyce/java/elements/bookXI

Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
propXI1.gif	16-May-1997 09:58	1k	
propXI10.gif	16-May-1997 09:58	2k	
propXI11.gif	16-May-1997 09:58	2k	
propXI12.gif	16-May-1997 09:58	1k	
propXI13.gif	16-May-1997 09:58	1k	
propXI14.gif	16-May-1997 09:58	1k	
propXI15.gif	16-May-1997 09:58	2k	
propXI16.gif	16-May-1997 09:58	1k	
propXI17.gif	16-May-1997 09:58	3k	
propXI18.gif	16-May-1997 09:58	2k	
propXI19.gif	16-May-1997 09:58	3k	
propXI2.gif	16-May-1997 09:58	1k	
propXI20.gif	16-May-1997 09:58	1k	
propXI21.gif	16-May-1997 09:58	1k	
propXI22.gif	16-May-1997 09:58	2k	
propXI23l.gif	16-May-1997 09:58	1k	
propXI24.gif	16-May-1997 09:58	2k	
propXI3.gif	16-May-1997 09:58	2k	
propXI4.gif	16-May-1997 09:58	3k	
propXI5.gif	16-May-1997 09:58	1k	
propXI6.gif	16-May-1997 09:58	2k	
propXI7.gif	16-May-1997 09:58	1k	
propXI8.gif	16-May-1997 09:58	2k	
propXI9.gif	16-May-1997 09:58	1k	
defXI11.gif	23-May-1997 01:08	2k	
defXI12.gif	23-May-1997 01:09	3k	
propXI25.gif	23-May-1997 01:09	4k	
propXI26.gif	23-May-1997 01:09	3k	
propXI27.gif	23-May-1997 01:09	2k	
propXI28.gif	23-May-1997 01:09	2k	
propXI29.gif	23-May-1997 01:09	3k	
propXI30.gif	23-May-1997 01:09	4k	
propXI31a.gif	23-May-1997 01:09	5k	
propXI31b.gif	23-May-1997 01:09	5k	
propXI33.gif	24-May-1997 09:24	5k	
propXI34b.gif	24-May-1997 09:24	5k	
propXI34a.gif	24-May-1997 09:25	3k	
propXI35.gif	24-May-1997 09:25	4k	

propXI36.gif	24-May-1997	09:25	3k
propXI37.gif	24-May-1997	09:25	5k
propXI38.gif	24-May-1997	09:25	4k
defXI28.gif	12-Jun-1997	21:16	2k
defXI27.gif	12-Jun-1997	21:16	2k
defXI25a.gif	12-Jun-1997	21:16	1k
defXI26.gif	12-Jun-1997	21:16	1k
defXI25.gif	12-Jun-1997	21:16	1k
defXI24.gif	12-Jun-1997	21:20	4k
defXI21.gif	12-Jun-1997	21:20	1k
defXI18.gif	12-Jun-1997	21:20	1k
defXI14.gif	12-Jun-1997	21:20	2k
bkXI.gif	20-Jun-1997	15:43	1k
propXI39.gif	20-Jun-1997	15:44	3k
defXI10a.gif	20-Jun-1997	15:45	3k
defXI10b.gif	20-Jun-1997	15:45	2k
propXI32.gif	23-Jun-1997	11:18	4k
propXI23b.gif	29-Jun-1997	21:07	4k
propXI23a.gif	29-Jun-1997	21:07	3k
propXI9a.gif	03-Mar-1998	11:08	1k
bookXI.html	26-Oct-1998	10:32	16k
favicon.ico	07-May-2001	14:54	1k
defXI1.html	29-Oct-2002	12:22	6k
defXI3.html	29-Oct-2002	12:22	7k
defXI6.html	29-Oct-2002	12:22	6k
defXI9.html	29-Oct-2002	12:22	13k
defXI11.html	29-Oct-2002	12:22	7k
defXI12.html	29-Oct-2002	12:22	8k
defXI14.html	29-Oct-2002	12:23	8k
defXI18.html	29-Oct-2002	12:23	7k
defXI21.html	29-Oct-2002	12:23	7k
defXI24.html	29-Oct-2002	12:23	8k
defXI25.html	29-Oct-2002	12:23	6k
propXI1.html	29-Oct-2002	12:23	8k
propXI2.html	29-Oct-2002	12:23	8k
propXI3.html	29-Oct-2002	12:23	9k
propXI4.html	29-Oct-2002	12:23	11k
propXI5.html	29-Oct-2002	12:23	8k
propXI6.html	29-Oct-2002	12:23	10k
propXI7.html	29-Oct-2002	12:23	8k
propXI8.html	29-Oct-2002	12:23	10k
propXI9.html	29-Oct-2002	12:23	10k
propXI10.html	29-Oct-2002	12:23	9k
propXI11.html	29-Oct-2002	12:24	10k

propXI12.html	29-Oct-2002 12:24	7k
propXI13.html	29-Oct-2002 12:24	7k
propXI14.html	29-Oct-2002 12:24	8k
propXI15.html	29-Oct-2002 12:24	10k
propXI16.html	29-Oct-2002 12:24	8k
propXI17.html	29-Oct-2002 12:24	9k
propXI18.html	29-Oct-2002 12:24	8k
propXI19.html	29-Oct-2002 12:24	8k
propXI20.html	29-Oct-2002 12:24	10k
propXI21.html	29-Oct-2002 12:24	9k
propXI22.html	29-Oct-2002 12:24	10k
propXI23.html	29-Oct-2002 12:24	25k
propXI24.html	29-Oct-2002 12:24	10k
propXI25.html	29-Oct-2002 12:24	14k
propXI26.html	29-Oct-2002 12:24	10k
propXI27.html	29-Oct-2002 12:24	9k
propXI28.html	29-Oct-2002 12:24	9k
propXI29.html	29-Oct-2002 12:25	9k
propXI30.html	29-Oct-2002 12:25	9k
propXI31.html	29-Oct-2002 12:25	19k
propXI32.html	29-Oct-2002 12:25	10k
propXI33.html	29-Oct-2002 12:25	13k
propXI34.html	29-Oct-2002 12:25	21k
propXI35.html	29-Oct-2002 12:25	15k
propXI36.html	29-Oct-2002 12:25	11k
propXI37.html	29-Oct-2002 12:25	9k
propXI38.html	29-Oct-2002 12:25	11k
propXI39.html	29-Oct-2002 12:25	10k
propXI23d.gif	29-Oct-2002 12:28	5k

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Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
propXI23d.gif	29-Oct-2002 12:28	5k	
propXI39.html	29-Oct-2002 12:25	10k	
propXI38.html	29-Oct-2002 12:25	11k	
propXI37.html	29-Oct-2002 12:25	9k	
propXI36.html	29-Oct-2002 12:25	11k	
propXI35.html	29-Oct-2002 12:25	15k	
propXI34.html	29-Oct-2002 12:25	21k	
propXI33.html	29-Oct-2002 12:25	13k	
propXI32.html	29-Oct-2002 12:25	10k	
propXI31.html	29-Oct-2002 12:25	19k	
propXI30.html	29-Oct-2002 12:25	9k	
propXI29.html	29-Oct-2002 12:25	9k	
propXI28.html	29-Oct-2002 12:24	9k	
propXI27.html	29-Oct-2002 12:24	9k	
propXI26.html	29-Oct-2002 12:24	10k	
propXI25.html	29-Oct-2002 12:24	14k	
propXI24.html	29-Oct-2002 12:24	10k	
propXI23.html	29-Oct-2002 12:24	25k	
propXI22.html	29-Oct-2002 12:24	10k	
propXI21.html	29-Oct-2002 12:24	9k	
propXI20.html	29-Oct-2002 12:24	10k	
propXI19.html	29-Oct-2002 12:24	8k	
propXI18.html	29-Oct-2002 12:24	8k	
propXI17.html	29-Oct-2002 12:24	9k	
propXI16.html	29-Oct-2002 12:24	8k	
propXI15.html	29-Oct-2002 12:24	10k	
propXI14.html	29-Oct-2002 12:24	8k	
propXI13.html	29-Oct-2002 12:24	7k	
propXI12.html	29-Oct-2002 12:24	7k	
propXI11.html	29-Oct-2002 12:24	10k	
propXI10.html	29-Oct-2002 12:23	9k	
propXI9.html	29-Oct-2002 12:23	10k	
propXI8.html	29-Oct-2002 12:23	10k	
propXI7.html	29-Oct-2002 12:23	8k	
propXI6.html	29-Oct-2002 12:23	10k	
propXI5.html	29-Oct-2002 12:23	8k	
propXI4.html	29-Oct-2002 12:23	11k	
propXI3.html	29-Oct-2002 12:23	9k	

propXI2.html	29-Oct-2002	12:23	8k
propXI1.html	29-Oct-2002	12:23	8k
defXI25.html	29-Oct-2002	12:23	6k
defXI24.html	29-Oct-2002	12:23	8k
defXI21.html	29-Oct-2002	12:23	7k
defXI18.html	29-Oct-2002	12:23	7k
defXI14.html	29-Oct-2002	12:23	8k
defXI12.html	29-Oct-2002	12:22	8k
defXI11.html	29-Oct-2002	12:22	7k
defXI9.html	29-Oct-2002	12:22	13k
defXI6.html	29-Oct-2002	12:22	6k
defXI3.html	29-Oct-2002	12:22	7k
defXI1.html	29-Oct-2002	12:22	6k
favicon.ico	07-May-2001	14:54	1k
bookXI.html	26-Oct-1998	10:32	16k
propXI9a.gif	03-Mar-1998	11:08	1k
propXI23a.gif	29-Jun-1997	21:07	3k
propXI23b.gif	29-Jun-1997	21:07	4k
propXI32.gif	23-Jun-1997	11:18	4k
defXI10b.gif	20-Jun-1997	15:45	2k
defXI10a.gif	20-Jun-1997	15:45	3k
propXI39.gif	20-Jun-1997	15:44	3k
bkXI.gif	20-Jun-1997	15:43	1k
defXI14.gif	12-Jun-1997	21:20	2k
defXI18.gif	12-Jun-1997	21:20	1k
defXI21.gif	12-Jun-1997	21:20	1k
defXI24.gif	12-Jun-1997	21:20	4k
defXI25.gif	12-Jun-1997	21:16	1k
defXI26.gif	12-Jun-1997	21:16	1k
defXI25a.gif	12-Jun-1997	21:16	1k
defXI27.gif	12-Jun-1997	21:16	2k
defXI28.gif	12-Jun-1997	21:16	2k
propXI38.gif	24-May-1997	09:25	4k
propXI37.gif	24-May-1997	09:25	5k
propXI36.gif	24-May-1997	09:25	3k
propXI35.gif	24-May-1997	09:25	4k
propXI34a.gif	24-May-1997	09:25	3k
propXI34b.gif	24-May-1997	09:24	5k
propXI33.gif	24-May-1997	09:24	5k
propXI31b.gif	23-May-1997	01:09	5k
propXI31a.gif	23-May-1997	01:09	5k
propXI30.gif	23-May-1997	01:09	4k
propXI29.gif	23-May-1997	01:09	3k
propXI28.gif	23-May-1997	01:09	2k

?	propXI27.gif	23-May-1997 01:09	2k
?	propXI26.gif	23-May-1997 01:09	3k
?	propXI25.gif	23-May-1997 01:09	4k
?	defXI12.gif	23-May-1997 01:09	3k
?	defXI11.gif	23-May-1997 01:08	2k
?	propXI9.gif	16-May-1997 09:58	1k
?	propXI8.gif	16-May-1997 09:58	2k
?	propXI7.gif	16-May-1997 09:58	1k
?	propXI6.gif	16-May-1997 09:58	2k
?	propXI5.gif	16-May-1997 09:58	1k
?	propXI4.gif	16-May-1997 09:58	3k
?	propXI3.gif	16-May-1997 09:58	2k
?	propXI24.gif	16-May-1997 09:58	2k
?	propXI23l.gif	16-May-1997 09:58	1k
?	propXI22.gif	16-May-1997 09:58	2k
?	propXI21.gif	16-May-1997 09:58	1k
?	propXI20.gif	16-May-1997 09:58	1k
?	propXI2.gif	16-May-1997 09:58	1k
?	propXI19.gif	16-May-1997 09:58	3k
?	propXI18.gif	16-May-1997 09:58	2k
?	propXI17.gif	16-May-1997 09:58	3k
?	propXI16.gif	16-May-1997 09:58	1k
?	propXI15.gif	16-May-1997 09:58	2k
?	propXI14.gif	16-May-1997 09:58	1k
?	propXI13.gif	16-May-1997 09:58	1k
?	propXI12.gif	16-May-1997 09:58	1k
?	propXI11.gif	16-May-1997 09:58	2k
?	propXI10.gif	16-May-1997 09:58	2k
?	propXI1.gif	16-May-1997 09:58	1k

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Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
bkXI.gif	20-Jun-1997 15:43	1k	
bookXI.html	26-Oct-1998 10:32	16k	
defXI1.html	29-Oct-2002 12:22	6k	
defXI10a.gif	20-Jun-1997 15:45	3k	
defXI10b.gif	20-Jun-1997 15:45	2k	
defXI11.gif	23-May-1997 01:08	2k	
defXI11.html	29-Oct-2002 12:22	7k	
defXI12.gif	23-May-1997 01:09	3k	
defXI12.html	29-Oct-2002 12:22	8k	
defXI14.gif	12-Jun-1997 21:20	2k	
defXI14.html	29-Oct-2002 12:23	8k	
defXI18.gif	12-Jun-1997 21:20	1k	
defXI18.html	29-Oct-2002 12:23	7k	
defXI21.gif	12-Jun-1997 21:20	1k	
defXI21.html	29-Oct-2002 12:23	7k	
defXI24.gif	12-Jun-1997 21:20	4k	
defXI24.html	29-Oct-2002 12:23	8k	
defXI25.gif	12-Jun-1997 21:16	1k	
defXI25.html	29-Oct-2002 12:23	6k	
defXI25a.gif	12-Jun-1997 21:16	1k	
defXI26.gif	12-Jun-1997 21:16	1k	
defXI27.gif	12-Jun-1997 21:16	2k	
defXI28.gif	12-Jun-1997 21:16	2k	
defXI3.html	29-Oct-2002 12:22	7k	
defXI6.html	29-Oct-2002 12:22	6k	
defXI9.html	29-Oct-2002 12:22	13k	
favicon.ico	07-May-2001 14:54	1k	
propXI1.gif	16-May-1997 09:58	1k	
propXI1.html	29-Oct-2002 12:23	8k	
propXI10.gif	16-May-1997 09:58	2k	
propXI10.html	29-Oct-2002 12:23	9k	
propXI11.gif	16-May-1997 09:58	2k	
propXI11.html	29-Oct-2002 12:24	10k	
propXI12.gif	16-May-1997 09:58	1k	
propXI12.html	29-Oct-2002 12:24	7k	
propXI13.gif	16-May-1997 09:58	1k	
propXI13.html	29-Oct-2002 12:24	7k	
propXI14.gif	16-May-1997 09:58	1k	

propXI14.html	29-Oct-2002	12:24	8k
propXI15.gif	16-May-1997	09:58	2k
propXI15.html	29-Oct-2002	12:24	10k
propXI16.gif	16-May-1997	09:58	1k
propXI16.html	29-Oct-2002	12:24	8k
propXI17.gif	16-May-1997	09:58	3k
propXI17.html	29-Oct-2002	12:24	9k
propXI18.gif	16-May-1997	09:58	2k
propXI18.html	29-Oct-2002	12:24	8k
propXI19.gif	16-May-1997	09:58	3k
propXI19.html	29-Oct-2002	12:24	8k
propXI2.gif	16-May-1997	09:58	1k
propXI2.html	29-Oct-2002	12:23	8k
propXI20.gif	16-May-1997	09:58	1k
propXI20.html	29-Oct-2002	12:24	10k
propXI21.gif	16-May-1997	09:58	1k
propXI21.html	29-Oct-2002	12:24	9k
propXI22.gif	16-May-1997	09:58	2k
propXI22.html	29-Oct-2002	12:24	10k
propXI23.html	29-Oct-2002	12:24	25k
propXI23a.gif	29-Jun-1997	21:07	3k
propXI23b.gif	29-Jun-1997	21:07	4k
propXI23d.gif	29-Oct-2002	12:28	5k
propXI23l.gif	16-May-1997	09:58	1k
propXI24.gif	16-May-1997	09:58	2k
propXI24.html	29-Oct-2002	12:24	10k
propXI25.gif	23-May-1997	01:09	4k
propXI25.html	29-Oct-2002	12:24	14k
propXI26.gif	23-May-1997	01:09	3k
propXI26.html	29-Oct-2002	12:24	10k
propXI27.gif	23-May-1997	01:09	2k
propXI27.html	29-Oct-2002	12:24	9k
propXI28.gif	23-May-1997	01:09	2k
propXI28.html	29-Oct-2002	12:24	9k
propXI29.gif	23-May-1997	01:09	3k
propXI29.html	29-Oct-2002	12:25	9k
propXI3.gif	16-May-1997	09:58	2k
propXI3.html	29-Oct-2002	12:23	9k
propXI30.gif	23-May-1997	01:09	4k
propXI30.html	29-Oct-2002	12:25	9k
propXI31.html	29-Oct-2002	12:25	19k
propXI31a.gif	23-May-1997	01:09	5k
propXI31b.gif	23-May-1997	01:09	5k
propXI32.gif	23-Jun-1997	11:18	4k

propXI32.html	29-Oct-2002 12:25	10k
propXI33.gif	24-May-1997 09:24	5k
propXI33.html	29-Oct-2002 12:25	13k
propXI34.html	29-Oct-2002 12:25	21k
propXI34a.gif	24-May-1997 09:25	3k
propXI34b.gif	24-May-1997 09:24	5k
propXI35.gif	24-May-1997 09:25	4k
propXI35.html	29-Oct-2002 12:25	15k
propXI36.gif	24-May-1997 09:25	3k
propXI36.html	29-Oct-2002 12:25	11k
propXI37.gif	24-May-1997 09:25	5k
propXI37.html	29-Oct-2002 12:25	9k
propXI38.gif	24-May-1997 09:25	4k
propXI38.html	29-Oct-2002 12:25	11k
propXI39.gif	20-Jun-1997 15:44	3k
propXI39.html	29-Oct-2002 12:25	10k
propXI4.gif	16-May-1997 09:58	3k
propXI4.html	29-Oct-2002 12:23	11k
propXI5.gif	16-May-1997 09:58	1k
propXI5.html	29-Oct-2002 12:23	8k
propXI6.gif	16-May-1997 09:58	2k
propXI6.html	29-Oct-2002 12:23	10k
propXI7.gif	16-May-1997 09:58	1k
propXI7.html	29-Oct-2002 12:23	8k
propXI8.gif	16-May-1997 09:58	2k
propXI8.html	29-Oct-2002 12:23	10k
propXI9.gif	16-May-1997 09:58	1k
propXI9.html	29-Oct-2002 12:23	10k
propXI9a.gif	03-Mar-1998 11:08	1k

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Name	Last modified	Size	Description
Parent Directory	22-Oct-2002 08:54	-	
propXI9a.gif	03-Mar-1998 11:08	1k	
propXI9.html	29-Oct-2002 12:23	10k	
propXI9.gif	16-May-1997 09:58	1k	
propXI8.html	29-Oct-2002 12:23	10k	
propXI8.gif	16-May-1997 09:58	2k	
propXI7.html	29-Oct-2002 12:23	8k	
propXI7.gif	16-May-1997 09:58	1k	
propXI6.html	29-Oct-2002 12:23	10k	
propXI6.gif	16-May-1997 09:58	2k	
propXI5.html	29-Oct-2002 12:23	8k	
propXI5.gif	16-May-1997 09:58	1k	
propXI4.html	29-Oct-2002 12:23	11k	
propXI4.gif	16-May-1997 09:58	3k	
propXI39.html	29-Oct-2002 12:25	10k	
propXI39.gif	20-Jun-1997 15:44	3k	
propXI38.html	29-Oct-2002 12:25	11k	
propXI38.gif	24-May-1997 09:25	4k	
propXI37.html	29-Oct-2002 12:25	9k	
propXI37.gif	24-May-1997 09:25	5k	
propXI36.html	29-Oct-2002 12:25	11k	
propXI36.gif	24-May-1997 09:25	3k	
propXI35.html	29-Oct-2002 12:25	15k	
propXI35.gif	24-May-1997 09:25	4k	
propXI34b.gif	24-May-1997 09:24	5k	
propXI34a.gif	24-May-1997 09:25	3k	
propXI34.html	29-Oct-2002 12:25	21k	
propXI33.html	29-Oct-2002 12:25	13k	
propXI33.gif	24-May-1997 09:24	5k	
propXI32.html	29-Oct-2002 12:25	10k	
propXI32.gif	23-Jun-1997 11:18	4k	
propXI31b.gif	23-May-1997 01:09	5k	
propXI31a.gif	23-May-1997 01:09	5k	
propXI31.html	29-Oct-2002 12:25	19k	
propXI30.html	29-Oct-2002 12:25	9k	
propXI30.gif	23-May-1997 01:09	4k	
propXI3.html	29-Oct-2002 12:23	9k	
propXI3.gif	16-May-1997 09:58	2k	
propXI29.html	29-Oct-2002 12:25	9k	

propXI29.gif	23-May-1997	01:09	3k
propXI28.html	29-Oct-2002	12:24	9k
propXI28.gif	23-May-1997	01:09	2k
propXI27.html	29-Oct-2002	12:24	9k
propXI27.gif	23-May-1997	01:09	2k
propXI26.html	29-Oct-2002	12:24	10k
propXI26.gif	23-May-1997	01:09	3k
propXI25.html	29-Oct-2002	12:24	14k
propXI25.gif	23-May-1997	01:09	4k
propXI24.html	29-Oct-2002	12:24	10k
propXI24.gif	16-May-1997	09:58	2k
propXI23l.gif	16-May-1997	09:58	1k
propXI23d.gif	29-Oct-2002	12:28	5k
propXI23b.gif	29-Jun-1997	21:07	4k
propXI23a.gif	29-Jun-1997	21:07	3k
propXI23.html	29-Oct-2002	12:24	25k
propXI22.html	29-Oct-2002	12:24	10k
propXI22.gif	16-May-1997	09:58	2k
propXI21.html	29-Oct-2002	12:24	9k
propXI21.gif	16-May-1997	09:58	1k
propXI20.html	29-Oct-2002	12:24	10k
propXI20.gif	16-May-1997	09:58	1k
propXI2.html	29-Oct-2002	12:23	8k
propXI2.gif	16-May-1997	09:58	1k
propXI19.html	29-Oct-2002	12:24	8k
propXI19.gif	16-May-1997	09:58	3k
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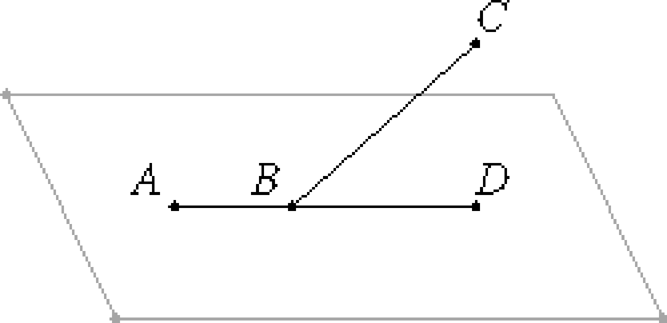
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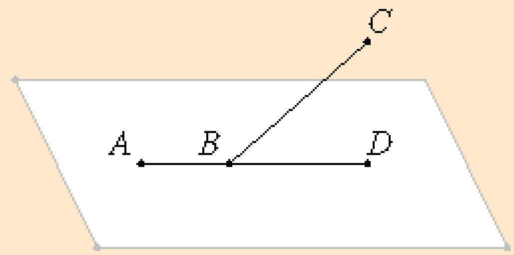
Euclid's Elements
Book XI
Proposition 1

A part of a straight line cannot be in the plane of reference and a part in plane more elevated.

For, if possible, let a part *AB* of the straight line *ABC* be in the plane of reference, and a part *BC* be in a plane more elevated.



Then there is in the plane of reference some straight line continuous with *AB* in a straight line. Let it be *BD*. Therefore *AB* is a common segment of the two straight lines *ABC* and *ABD*, which is impossible, since, if we describe a circle with center *B* and radius *AB*, then the diameters cut off unequal circumferences of the circle.

Therefore, *a part of a straight line cannot be in the plane of reference and a part in plane more elevated.*

Q. E. D.

Guide

Not only is the proof of this proposition unclear, so is the statement of it. The meaning of the "plane of reference" and the role it is to play in solid geometry are unclear. Is the intent of the statement that if part of a line lies in a plane, then all of it does? At least that would be a meaningful statement.

The proof of this proposition is unclear for more than one reason. Before a circle with center *B* and radius *AB* can be described, a plane has to be specified in which to describe the circle. In space there are infinitely many circles that have the same center *B*, the same radius *AB*, and even contain the point *A*. Indeed, this possibility of many circles with with same diameter was used to define a sphere in definition XI.Def.14. The last statement about unequal circumferences is incomprehensible.

The problem is that there are no postulates for solid geometry. The postulates in Book I apparently refer to an ambient plane. Certainly Post.3, "to describe a circle with any center and radius," and Post.5 (which refers to interior angles when one line crosses two others) do. Without any postulates for nonplanar geometry it is impossible for solid geometry to get off the ground.

Use of this proposition

This proposition is used in the proof of the next one as well as others in the last three books of the *Elements*.

Next proposition: XI.2

Select from Book XI

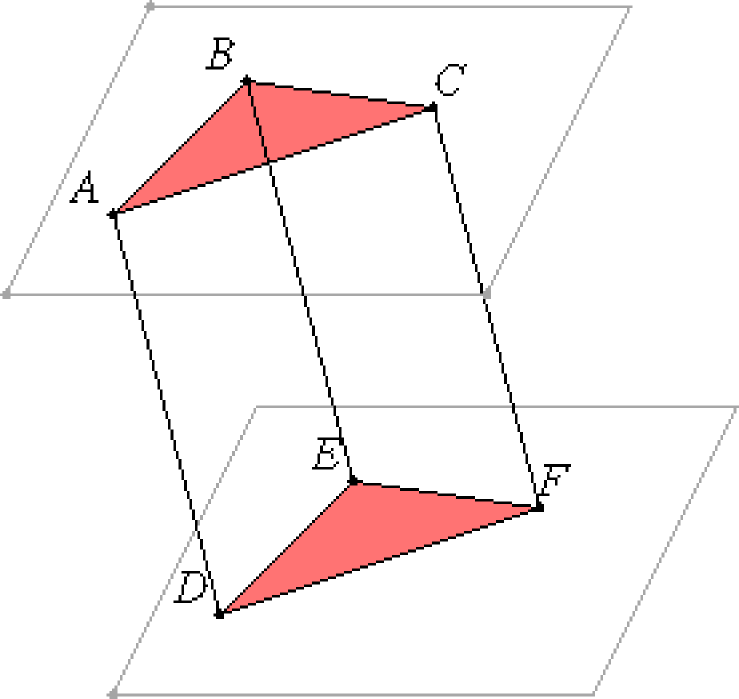
Previous: XI.Def.25-28

Select book

Book XI introduction

Select topic

Select topic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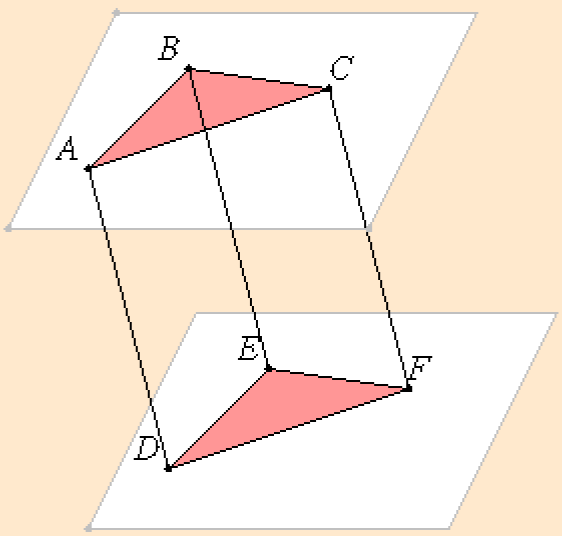
Euclid's Elements
Book XI
Proposition 10

If two straight lines meeting one another are parallel to two straight lines meeting one another not in the same plane, then they contain equal angles.

Let the two straight lines *AB* and *BC* meeting one another be parallel to the two straight lines *DE* and *EF* meeting one another not in the same plane.

I say that the angle *ABC* equals the angle *DEF*.



Cut *BA*, *BC*, *ED*, and *EF* off equal to one another, and join *AD*, *CF*, *BE*, *AC*, and *DF*. I.3

Now, since *BA* equals and is parallel to *ED*, therefore *AD* also equals and is parallel to *BE*. For the same reason *CF* also equals and is parallel to *BE*. I.33

Therefore each of the straight lines *AD* and *CF* equals and is parallel to *BE*. But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another, therefore *AD* is parallel and equal to *CF*. XI.9

And *AC* and *DF* join them, therefore *AC* also equals and is parallel to *DF*. I.33

Now, since the two sides *AB* and *BC* equal the two sides *DE* and *EF*, and the base *AC* equals the base *DF*, therefore the angle *ABC* equals the angle *DEF*. I.8

Therefore, *if two straight lines meeting one another are parallel to two straight lines meeting one another not in the same plane, then they contain equal angles.*

Q. E. D.

Guide

Of course it is necessary to be careful about in which directions the lines head. If one is changed to head into the opposite direction, then the angles won't be equal but supplementary instead.

Use of this proposition

This proposition is used in the proofs of propositions XI.24 and XII.3.

Next proposition: XI.11

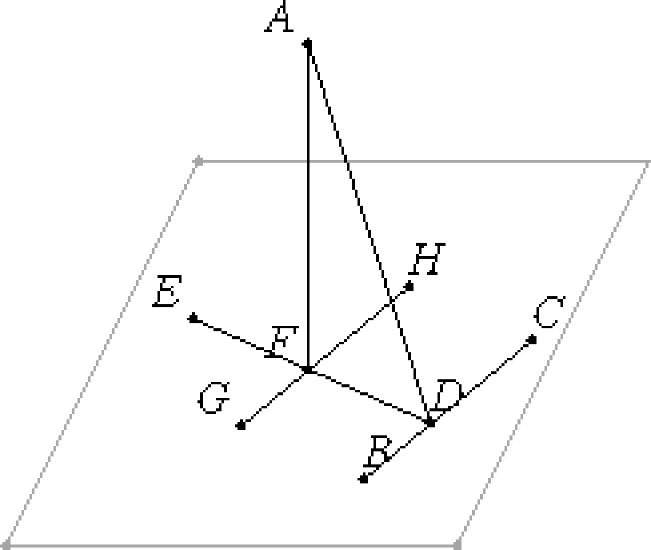
Previous: XI.9

Book XI introduction

Select from Book XI

Select book

Select topic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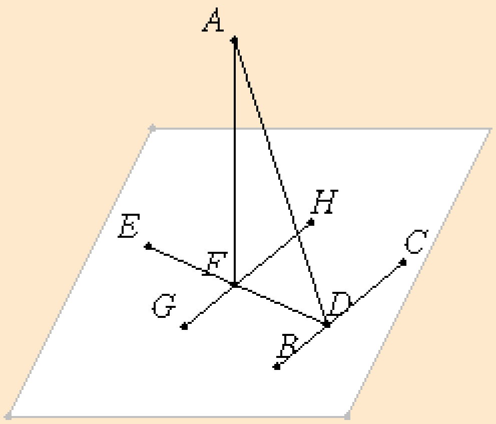
Euclid's Elements
Book XI
Proposition 11

To draw a straight line perpendicular to a given plane from a given elevated point.

Let *A* be the given elevated point, and the plane of reference the given plane.

It is required to draw from the point *A* a straight line perpendicular to the plane of reference.



Draw any straight line *BC* at random in the plane of reference, and draw *AD* from the point *A* perpendicular to *BC*. I.12

Then if *AD* is also perpendicular to the plane of reference, then that which was proposed is done.

But if not, draw *DE* from the point *D* at right angles to *BC* and in the plane of reference, draw *AF* from *A* perpendicular to *DE*, and draw *GH* through the point *F* parallel to *BC*. I.11 I.12 I.31

Now, since *BC* is at right angles to each of the straight lines *DA* and *DE*, therefore *BC* is also at right angles to the plane through *ED* and *DA*. XI.4

And *GH* is parallel to it, but if two straight lines are parallel, and one of them is at right angles to any plane, then the remaining one is also at right angles to the same plane, therefore *GH* is also at right angles to the plane through *ED* and *DA*. XI.8

And *GH* is parallel to it, but if two straight lines are parallel, and one of them is at right angles to any plane, then the remaining one is also at right angles to the same plane, therefore *GH* is also at right angles to the plane through *ED* and *DA*. XI.8

Therefore *GH* is also at right angles to all the straight lines which meet it and are in the plane through *ED* and *DA*. XI.Def.3

But *AF* meets it and lies in the plane through *ED* and *DA*, therefore *GH* is at right angles to *FA*, so that *FA* is also at right angles to *GH*. But *AF* is also at right angles to *DE*, therefore *AF* is at right angles to each of the straight lines *GH* and *DE*.

But if a straight is set up at right angles to two straight lines which cut one another at their intersection point, then it also is at right angles to the plane through them. Therefore *FA* is at right angles to the plane through *ED* and *GH*. XI.4

But the plane through *ED* and *GH* is the plane of reference, therefore *AF* is at right angles to the plane of reference.

Therefore from the given elevated point *A* the straight line *AF* has been drawn perpendicular to the plane of reference.

Q. E. F.

Guide

In the proof, before the line *AD* can be drawn from the point *A* perpendicular to the line *BC*, it is necesary to know that the point and line belong to the same plane. Such a plane can be specified by taking the line *BC* and a line from *A* to any point on *BC* since two intersecting lines determine a plane (XI.2).

Use of this proposition

The construction in this proposition is used frequently in the last three books of the *Elements*.

Next proposition: XI.12

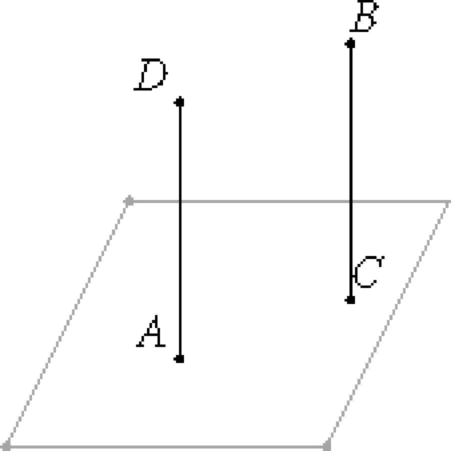
Previous: XI.10

Book XI introduction

Select from Book XI

Select book

Select topic



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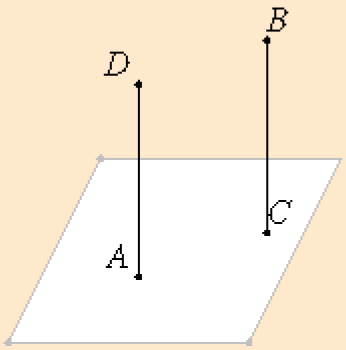
Euclid's Elements
Book XI
Proposition 12

To set up a straight line at right angles to a given plane from a given point in it.

Let the plane of reference be the given plane and *A* the point in it.

It is required to set up from the point *A* a straight line at right angles to the plane of reference.



From an elevated point *B* draw *BC* perpendicular to the plane of reference, and draw *AD* parallel to to *BC* through the point *A*. XI.11 I.31

Then since *AD* and *BC* are two parallel straight lines, and one of them, *BC*, is at right angles to the plane of reference, therefore the remaining one, *AD*, is also at right angles to the plane of reference. XI.8

Therefore *AD* is set up at right angles to the given plane from the point *A* in it.

Q. E. F.

Guide

This proposition, like the last, is used frequently in the rest of the *Elements* to construct lines perpendicular to planes.

Next proposition: XI.13

Previous: XI.11

Book XI introduction

Select from Book XI

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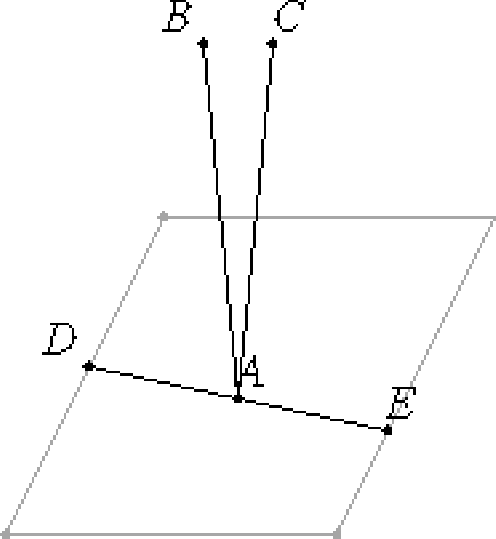
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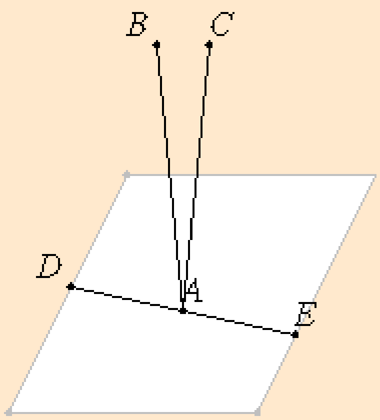
Euclid's Elements
Book XI
Proposition 13

From the same point two straight lines cannot be set up at right angles to the same plane on the same side.

For, if possible, from the same point *A* let the two straight lines *AB* and *AC* be set up at right angles to the plane of reference and on the same side.



Draw a plane through *BA* and *AC*. It intersects the plane of reference in a straight line through *A*. Let the line be *DAE*. XI.3

Therefore the straight lines *AB*, *AC*, and *DAE* lie in one plane. And, since *CA* is at right angles to the plane of reference, it also makes right angles with all the straight lines which meet it and lie in the plane of reference. XI.Def.3

But *DAE* meets it and lies in the plane of reference, therefore the angle *CAE* is right. For the same reason the angle *BAE* is also right. Therefore the angle *CAE* equals the angle *BAE*.

And they lie in one plane, which is impossible.

Therefore, *from the same point two straight lines cannot be set up at right angles to the same plane on the same side.*

Q. E. D.

Guide

This proposition is used in the proof of proposition XI.19. Also, the result of this proposition is implicitly used in the proof of XI.6.

Next proposition: XI.14

Previous: XI.12

Book XI introduction

Select from Book XI

Select book

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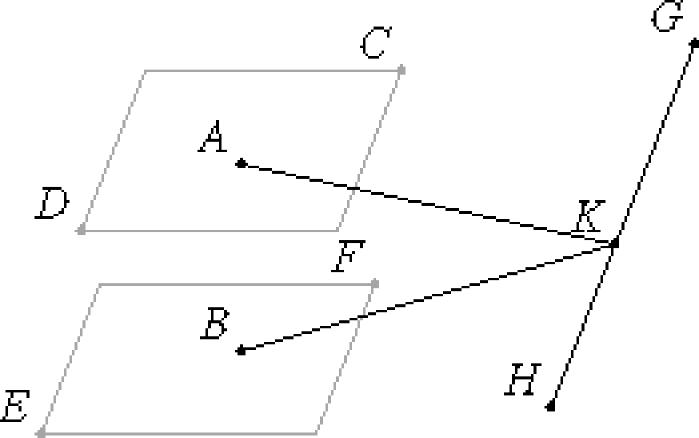
D.E.Joyce
Clark University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Euclid's Elements

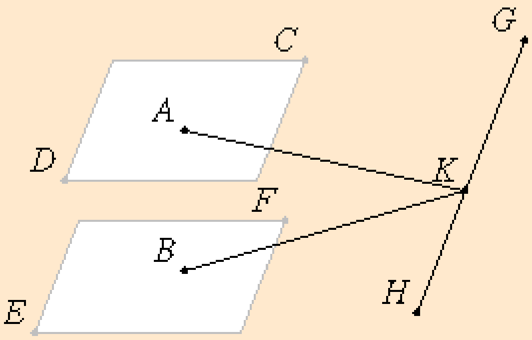
Book XI

Proposition 14

Planes at right angles to the same straight line are parallel.

Let any straight line *AB* be at right angles to each of the planes *CD* and *EF*.

I say that the planes are parallel.



For, if not, then they meet when produced. Let them meet. Then they intersect as a straight line. Let it be *GH*. XI.Def.8 XI.3

Take a point *K* at random on *GH*, and join *AK* and *BK*.

Now, since *AB* is at right angles to the plane *EF*, therefore *AB* is also at right angle to *BK* which is a straight line in the plane *EF* produced. Therefore the angle *ABK* is right. For the same reason the angle *BAK* is also right. XI.Def.3

Thus, in the triangle *ABK* the sum of the two angles *ABK* and *BAK* equals two right angles, which is impossible. I.17

Therefore the planes *CD* and *EF* do not meet when produced. Therefore the planes *CD* and *EF* are parallel. XI.Def.8

Therefore, *planes at right angles to the same straight line are parallel.*

Q. E. D.

Guide

Part of this proof is unnecessary. The line *GH* is irrelevant. If the two points meet, then they meet at some point, so the point *K* might just as well be taken as a point where they meet. The proof should, however, include another case, and that is where the given line meets both given planes at a point common to both planes, so that the three points *A*, *B*, and *K* are all the same point.

Use of this proposition

This proposition is used in the proof of the next one.

Next proposition: XI.15

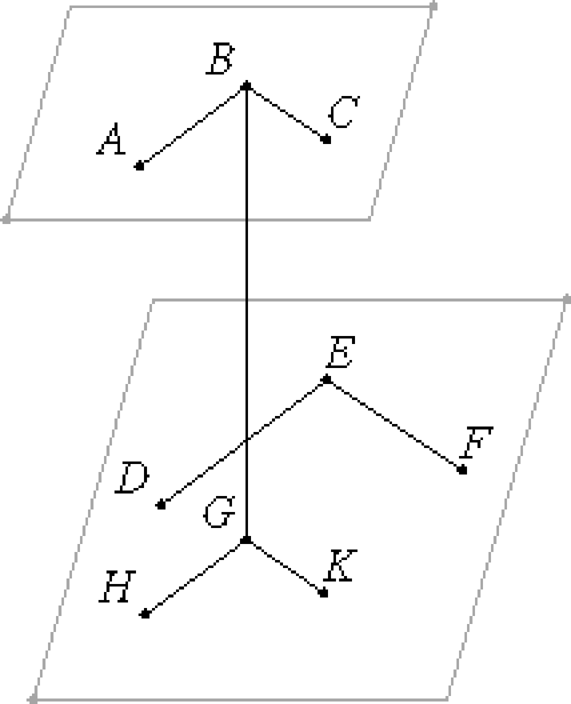
Select from Book XI

Previous: XI.13

Select book

Book XI introduction

Select topic



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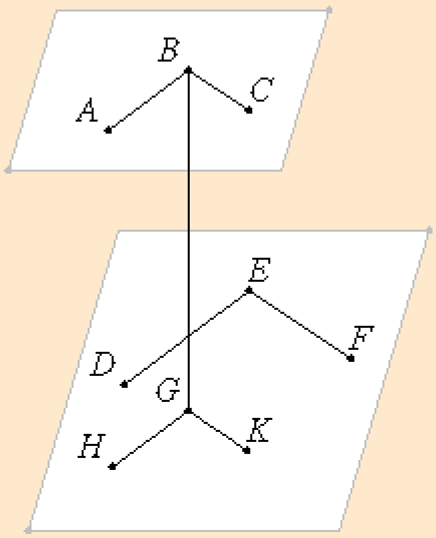
Euclid's Elements
Book XI
Proposition 15

If two straight lines meeting one another are parallel to two straight lines meeting one another not in the same plane, then the planes through them are parallel.

Let the two straight lines *AB* and *BC* meeting one another be parallel to the two straight lines *DE* and *EF* meeting one another not in the same plane.

I say that the plane produced through *AB* and *BC* and the plane produced through *DE* and *EF* do not meet one another.



Draw *BG* from the point *B* perpendicular to the plane through *DE* and *EF* to where it meets the plane at the point *G*. XI.11

Draw *GH* through *G* parallel to *ED*, and *GK* parallel to *EF*. I.31

Now, since *BG* is at right angles to the plane through *DE* and *EF*, therefore it makes right angles with all the straight lines which meet it and lie in the plane through *DE* and *EF*. XI.Def.3

But each of the straight lines *GH* and *GK* meets it and lies in the plane through *DE* and *EF*, therefore each of the angles *BGH* and *BGK* is right.

And, since *BA* is parallel to *GH*, therefore the sum of the angles *GBA* and *BGH* equals two right angles. XI.9 I.29

But the angle *BGH* is right, therefore the angle *GBA* is also right. Therefore *GB* is at right angles to *BA*. For the same reason *GB* is also at right angles to *BC*.

Since then the straight line *GB* is set up at right angles to the two straight lines *BA* and *BC* which cut one another, therefore *GB* is also at right angles to the plane through *BA* and *BC*. XI.4

But planes to which the same straight line is at right angles are parallel, therefore the plane through *AB* and *BC* is parallel to the plane through *DE* and *EF*. XI.14

Therefore, *if two straight lines meeting one another are parallel to two straight lines meeting one another not in the same plane, then the planes through them are parallel.*

Q. E. D.

Guide

This proposition is not used in the rest of the *Elements*.

Next proposition: XI.16

Previous: XI.14

Book XI introduction

Select from Book XI

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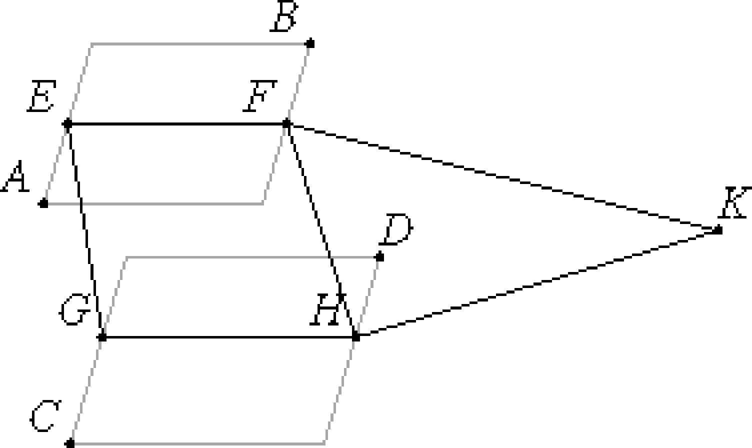
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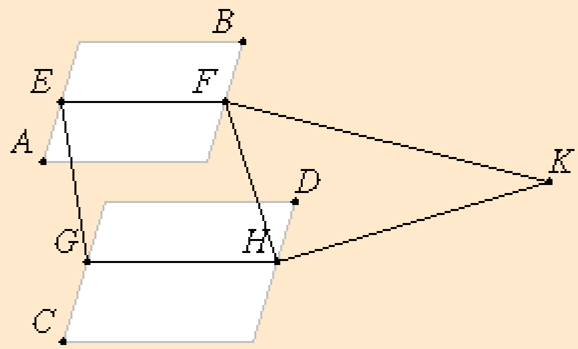
Euclid's Elements
Book XI
Proposition 16

If two parallel planes are cut by any plane, then their intersections are parallel.

Let the two parallel planes *AB* and *CD* be cut by the plane *EFGH,* and let *EF* and *GH* be their intersections. XI.3

I say that *EF* is parallel to *GH.*



If not, then *EF* and *GH* will, when produced, meet either in the direction of *F* and *H* or in the direction of *E* and *G.*

First, let them meet when produced in the direction of *F* and *H* at *K.*

Now, since *EFK* lies in the plane *AB,* therefore all the points on *EFK* also lie in the plane *AB.* But *K* is one of the points on the straight line *EFK,* therefore *K* lies in the plane *AB.* For the same reason *K* also lies in the plane *CD.* Therefore the planes *AB* and *CD* will meet when produced. XI.1

But they do not meet, because, by hypothesis, they are parallel. Therefore the straight lines *EF* and *GH* do not meet when produced in the direction of *F* and *H.*

Similarly we can prove that neither do the straight lines *EF* and *GH* meet when produced in the direction of *E* and *G.*

But straight lines which do not meet in either direction are parallel. Therefore *EF* is parallel to *GH.*

Therefore, *if two parallel planes are cut by any plane, then their intersections are parallel.*

Q. E. D.

Guide

This proposition is used in the proof of the next proposition as well as proposition XI.24.

Next proposition: XI.17

Previous: XI.15

Book XI introduction

Select from Book XI

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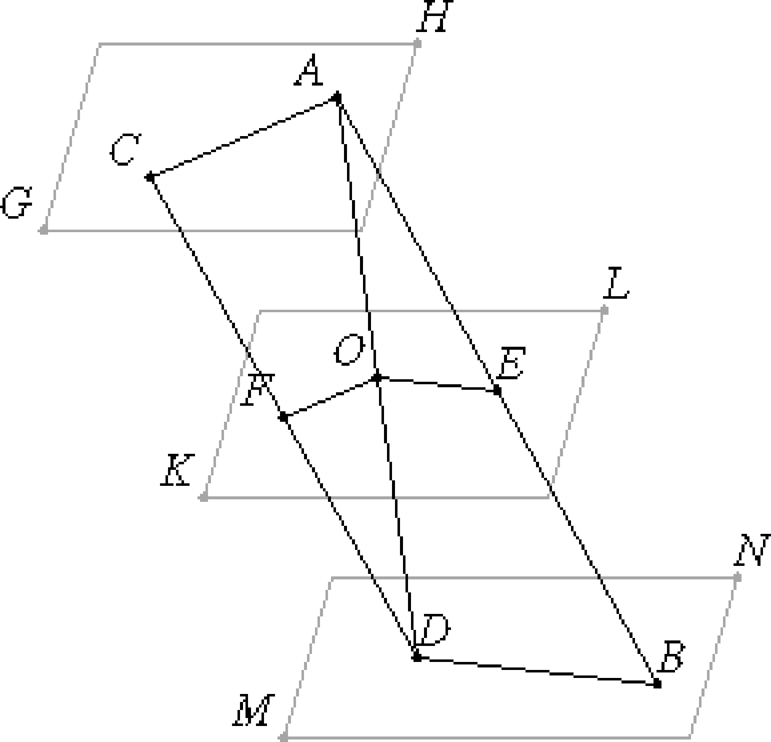
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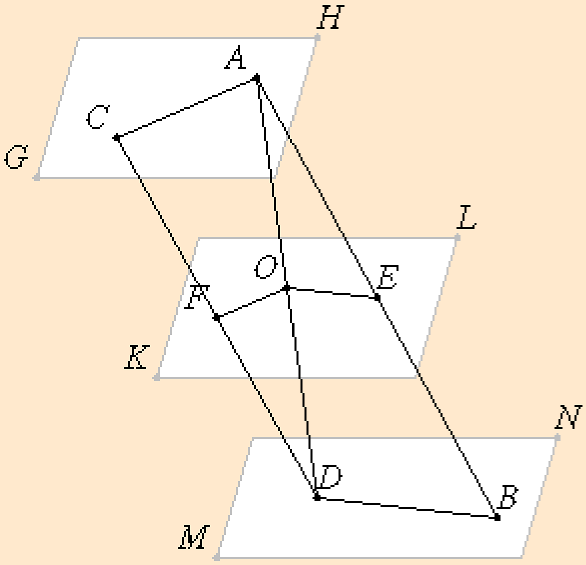
Euclid's Elements
Book XI
Proposition 17

If two straight lines are cut by parallel planes, then they are cut in the same ratios.

Let the two straight lines *AB* and *CD* be cut by the parallel planes *GH, KL,* and *MN* at the points *A, E,* and *B,* and at the points *C, F,* and *D,* respectively.

I say that the straight line *AE* is to *EB* as *CF* is to *FD.*



Join *AC*, *BD*, and *AD.* Let *AD* meet the plane *KL* at the point *O.* Join *EO* and *FO.*

Now, since the two parallel planes *KL* and *MN* are cut by the plane *EBDO,* therefore their intersections *EO* and *BD* are parallel. For the same reason, since the two parallel planes *GH* and *KL* are cut by the plane *AOFC,* their intersections *AC* and *OF* are parallel. XI.16

And, since the straight line *EO* is parallel to *BC,* one of the sides of the triangle *ABD,* therefore proportionally *AE* is to *EB* as *AO* is to *OD.* Again, since the straight line *FO* is parallel to *CA,* one of the sides of the triangle *ADC,* therefore proportionally *AO* is to *OD* as *CF* is to *FD.* VI.2

But it was prove that *AO* is to *OD* as *AE* is to *EB,* therefore *AE* is to *EB* as *CF* is to *FD.* V.11

Therefore, *if two straight lines are cut by parallel planes, then they are cut in the same ratios.*

Q. E. D.

Guide

This proposition is used in the proof of proposition XII.4.

Next proposition: XI.18

Previous: XI.16

Book XI introduction

Select from Book XI

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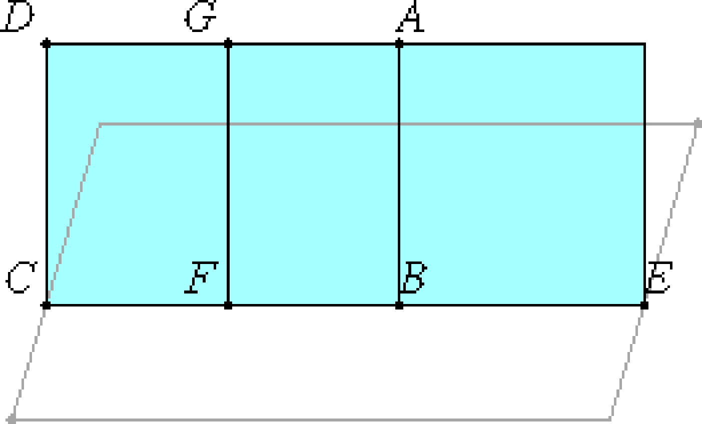
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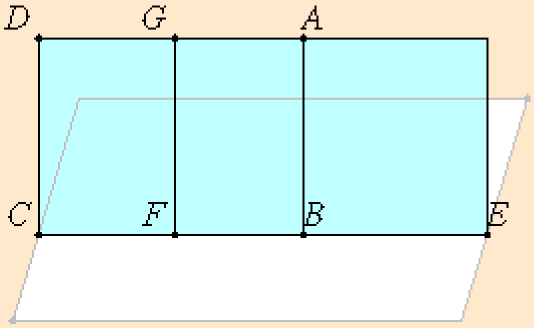
Euclid's Elements
Book XI
Proposition 18

If a straight line is at right angles to any plane, then all the planes through it are also at right angles to the same plane.

Let any straight line *AB* be at right angles to the plane of reference.

I say that all the planes through *AB* are also at right angles to the plane of reference.



Let the plane *DE* be drawn through *AB*. Let *CE* be the intersection of the plane *DE* and the plane of reference.

Take a point *F* at random on *CE*, and draw *FG* from *F* at right angles to *CE* in the plane *DE*. I.11

Now, since *AB* is at right angles to the plane of reference, therefore *AB* is also at right angles to all the straight lines which meet it and lie in the plane of reference, so that it is also at right angles to *CE*. Therefore the angle *ABF* is right. XI.Def.3

But the angle *GFB* is also right, therefore *AB* is parallel to *FG*. I.28

But *AB* is at right angles to the plane of reference, therefore *FG* is also at right angles to the plane of reference. XI.8

Now a plane is at right angles to a plane when the straight lines drawn in one of the planes at right angles to the intersection of the planes are at right angles to the remaining plane. And *FG*, drawn in one of the planes *DE* at right angles to *CE*, the intersection of the planes, was proved to be at right angles to the plane of reference. Therefore the plane *DE* is at right angles to the plane of reference. XI.Def.4

Similarly it can also be proved that all the planes through *AB* are at right angles to the plane of reference.

Therefore, *if a straight line is at right angles to any plane, then all the planes through it are also at right angles to the same plane.*

Q. E. D.

Guide

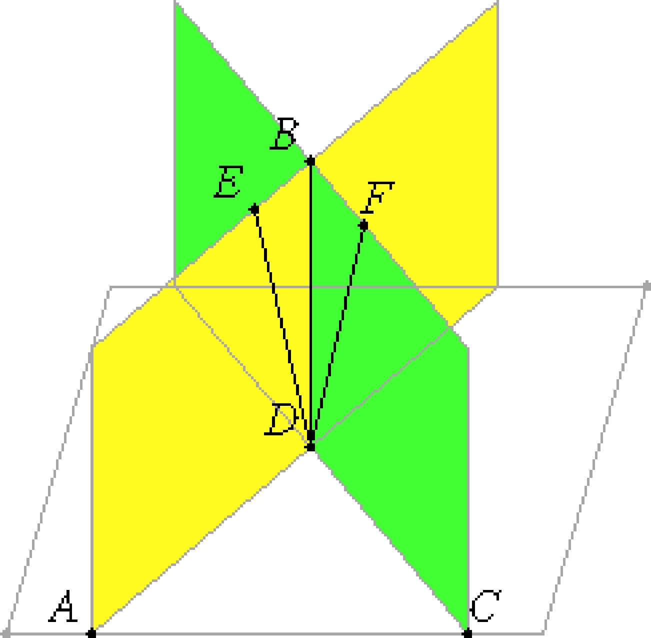
This proposition is used in the proof of proposition XII.17.

Next proposition: XI.19

Select from Book XI

Previous: XI.17

Select book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Euclid's Elements

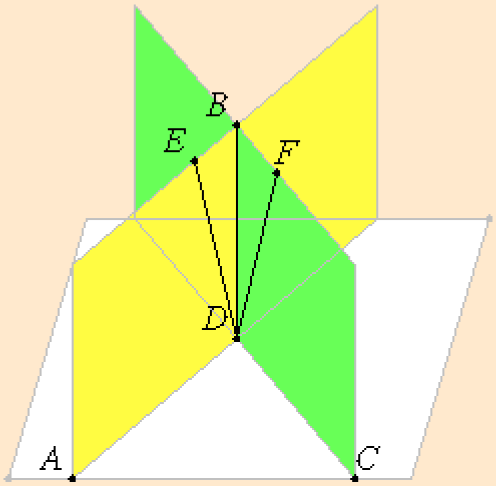
Book XI

Proposition 19

If two planes which cut one another are at right angles to any plane, then their intersection is also at right angles to the same plane.

Let the two planes *AB* and *BC* be at right angles to the plane of reference, and let *BC* be their intersection.

I say that *BD* is at right angles to the plane of reference.



Suppose it is not. From the point *D* draw *DE* at right angles to the straight line *AD* in the plane *AB*, and draw *DF* at right angles to *CD* in the plane *BC*. I.11

Now, since the plane *AB* is at right angles to the plane of reference, and *DE* is at right angles in the plane *AB* to *AD*, their intersection, therefore *DE* is at right angles to the plane of reference. XI.Def.4

Similarly we can prove that *DF* is also at right angles to the plane of reference. Therefore from the same point *D* two straight lines have been set up at right angles to the plane of reference on the same side, which is impossible. XI.13

Therefore no straight line except the intersection *DB* of the planes *AB* and *BC* can be set up from the point *D* at right angles to the plane of reference.

Therefore, *if two planes which cut one another are at right angles to any plane, then their intersection is also at right angles to the same plane.*

Q. E. D.

Guide

This proposition is not used in the rest of the *Elements.*

Next proposition: XI.20

Previous: XI.18

Book XI introduction

Select from Book XI

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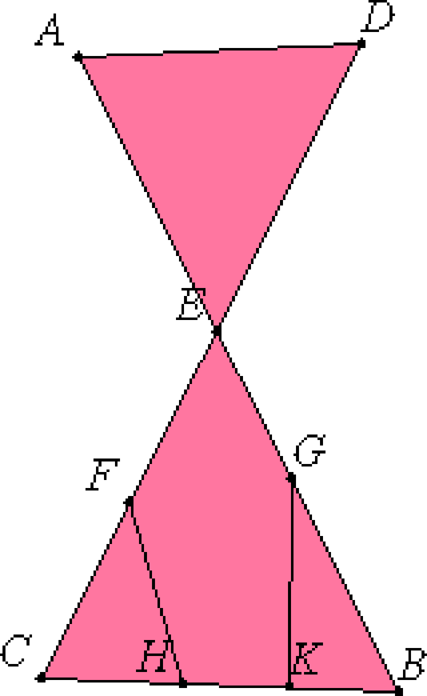
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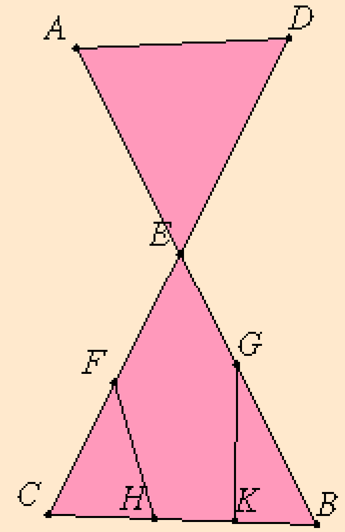
Euclid's Elements
Book XI
Proposition 2

If two straight lines cut one another, then they lie in one plane; and every triangle lies in one plane.

For let the two straight lines *AB* and *CD* cut one another at the point *E.*

I say that *AB* and *CD* lie in one plane, and that every triangle lies in one plane.



Take the points *F* and *G* at random on *EC* and *EB,* join *CB* and *FG,* and draw *FH* and *GK* across.

I say first that the triangle *ECB* lies in one plane.

For, if part of the triangle *ECB,* either *FHC* or *GBK,* is in the plane of reference, and the rest in another, then a part also of one of the straight lines *EC* or *EB* is in the plane of reference, and a part in another.

But, if the part *FCBG* of the triangle *ECB* is in the plane of reference, and the rest in another, then a part also of both the straight lines *EC* and *EB* is in the plane of reference and a part in another, which was proved absurd. XI.1

Therefore the triangle *ECB* lies in one plane.

But, in whatever plane the triangle *ECB* lies, each of the straight lines *EC* and *EB* also lies, and in whatever plane each of the straight lines *EC* and *EB* lies, *AB* and *CD* also lie. XI.1

Therefore the straight lines *AB* and *CD* lie in one plane; and every triangle lies in one plane.

Therefore, *if two straight lines cut one another, then they lie in one plane; and every triangle lies in one plane.*

Q. E. D.

Guide

The goal of the proof in this proposition is to produce a plane for the two lines *AB* and *CD* to lie in. Yet the proof fails to produce any plane at all. Near the beginning is the phrase "the plane of reference" occurs, but there is no reference as no planes have been mentioned. As the two lines *AB* and *CD* could be placed anywhere in space, any previously conceived plane would be irrelevant to them.

Postulates of some sort are needed to justify the existance of planes. One could state that three noncollinear points determine a plane. Another might be that there are four noncoplanar points.

Use of this proposition

This proposition is used in the proofs of propositions XI.4, XI.6, and XII.17.

Next proposition: XI.3

Previous: XI.1

Book XI introduction

Select from Book XI

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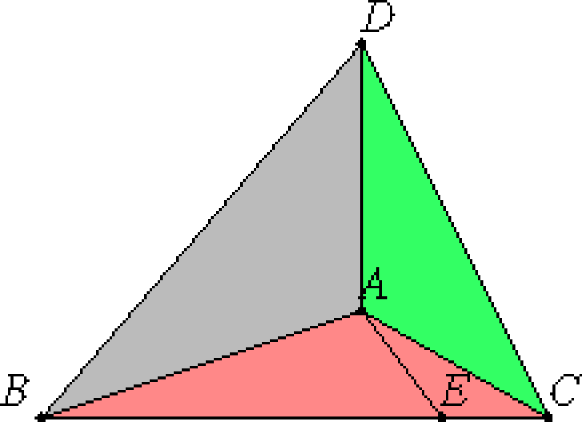

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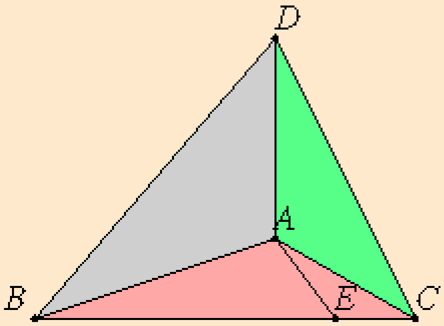
Euclid's Elements
Book XI
Proposition 20

If a solid angle is contained by three plane angles, then the sum of any two is greater than the remaining one.

Let the solid angle at *A* be contained by the three plane angles *BAC*, *CAD*, and *DAB*.

I say that the sum of any two of the angles *BAC*, *CAD*, and *DAB* is greater than the remaining one.



If the angles *BAC*, *CAD*, and *DAB* are equal to one another, then it is clear that the sum of any two is greater than the remaining one.

But, if not, let *BAC* be greater. In the plane through *BA* and *AC*, construct the angle *BAE* equal to the angle *DAB* at the point *A* on the straight line *AB*. Make *AE* equal to *AD*, draw *BEC* across through the point *E* cutting the straight lines *AB* and *AC* at the points *B* and *C*, and join *DB* and *DC*. I.23 I.3

Now, since *DA* equals *AE*, and *AB* is common, therefore two sides are equal to two sides. And the angle *DAB* equals the angle *BAE*, therefore the base *DB* equals the base *BE*. I.4

And, since the sum of the two sides *BD* and *DC* is greater than *BC*, and of these *DB* was proved equal to *BE*, therefore the remainder *DC* is greater than the remainder *EC*. I.20

Now, since *DA* equals *AE*, and *AC* is common, and the base *DC* is greater than the base *EC*, therefore the angle *DAC* is greater than the angle *EAC*. I.25

But the angle *BAE* equals the angle *DAB*, therefore the sum of the angles *DAB* and *DAC* is greater than the angle *BAC*.

Similarly we can prove that the sum of any two of the remaining angles is greater than the remaining one.

Therefore, *if a solid angle is contained by three plane angles, then the sum of any two is greater than the remaining one.*

Q. E. D.

Guide

This is one of two necessary conditions for constructing a solid angle out of three plane angles. The next necessary condition is stated in the next proposition, and the two conditions together are shown to be sufficient in XI.23.

About the proof

The structure of the proof is not entirely clear. The goal is to show that the sum of any two of the angles is greater than the third. Notice is make that if they are all equal, then the goal is clearly satisfied. Then, under the assumption that if one is greater than a second, then the sum of the second and third is greater than the first. Then the other cases are declared to be similarly provable.

Various interpretations have been made of the intent of the form of proof, but all require minor changes to clarify the stucture.

About three-dimensional analogues of two-dimensional constructions

Up until this proposition, each construction in Book XI takes place within a plane, although different constructions in the same proposition may occur in different planes. One of the constructions here, however, takes place in two different planes. The angle *BAE* is constructed in one plane to equal a given angle *BAE* in a different plane. The construction in I.23 to construct one angle equal to a given angle, strictly speaking, takes place in only one plane. Tracing that construction back through Book I leads through proposition I.22 to I.3. Proposition I.3 cuts one line off equal to another line. That basic construction can easily be modified so that the two lines are in different planes. Once that's done, the rest of the constructions in Book I also apply when their components lie in different planes. Still, the details should be verified before applying the results as done in the proof of this proposition.

Next proposition: XI.21

Previous: XI.19

Book XI introduction

Select from Book XI

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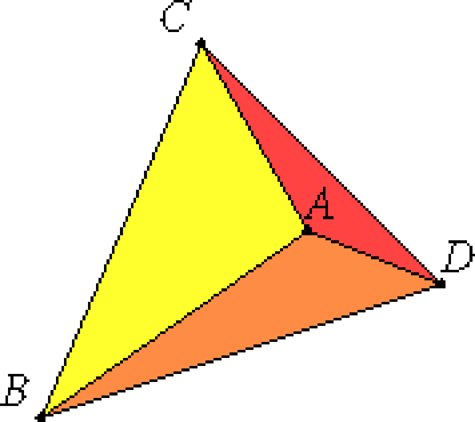
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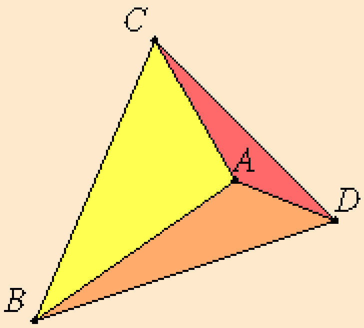
Euclid's Elements
Book XI
Proposition 21

Any solid angle is contained by plane angles whose sum is less than four right angles.

Let the angle at *A* be a solid angle contained by the plane angles *BAC, CAD,* and *DAB.*

I say that the sum of the angles *BAC, CAD,* and *DAB* is less than four right angles.



Take points *B, C,* and *D* at random on the straight lines *AB, AC,* and *AD* respectively, and join *BC, CD,* and *DB.*

Now, since the solid angle at *B* is contained by the three plane angles *CBA, ABD,* and *CBD,* and the sum of any two is greater than the remaining one, therefore the sum of the angles *CBA* and *ABD* is greater than the angle *CBD.* XI.20

For the same reason the sum of the angles *BCA* and *ACD* is greater than the angle *BCD,* and the sum of the angles *CDA* and *ADB* is greater than the angle *CDB.* Therefore the sum of the six angles *CBA, ABD, BCA, ACD, CDA,* and *ADB* is greater than the sum of the three angles *CBD, BCD,* and *CDB.*

But the sum of the three angles *CBD, BDC,* and *BCD* equals two right angles, therefore the sum of the six angles *CBA, ABD, BCA, ACD, CDA,* and *ADB* is greater than two right angles. I.32

And, since the each sum of the three angles of the triangles *ABC, ACD,* and *ADB* equals two right angles, therefore the sum of the nine angles of the three triangles, the angles *CBA, ACB, BAC, ACD, CDA, CAD, ADB, DBA,* and *BAD* equals six right angles. Of them the sum of the six angles *ABC, BCA, ACD, CDA, ADB,* and *DBA* are greater than two right angles, therefore the sum of the remaining three angles *BAC, CAD,* and *DAB* containing the solid angle is less than four right angles.

Therefore, *any solid angle is contained by plane angles whose sum is less than four right angles.*

Q.E.D.

Guide

In proposition XI.23 the condition stated here and the condition in XI.20 (the sum of any two plane angles is less than the third) together are shown to be sufficient to construct a solid angle.

About the proof

The proof only shows that the sum of the plane angles in all cases is less than four right angles when there are three plane angles, not when there are more than three. When there are four ore more plane angles, the proof is analogous, but it is necessary to invoke Proclus' first corollary to I.32, which states that "the sum of the interior angles of a convex rectilinear figure equals twice as many angles as the figure has sides, less four."

Use of this proposition

This proposition is used in the proof of remark after proposition XIII.18 to show that the five regular polyhedra constructed in Book XIII are the only five possible.

Next proposition: XI.22

Previous: XI.20

Book XI introduction

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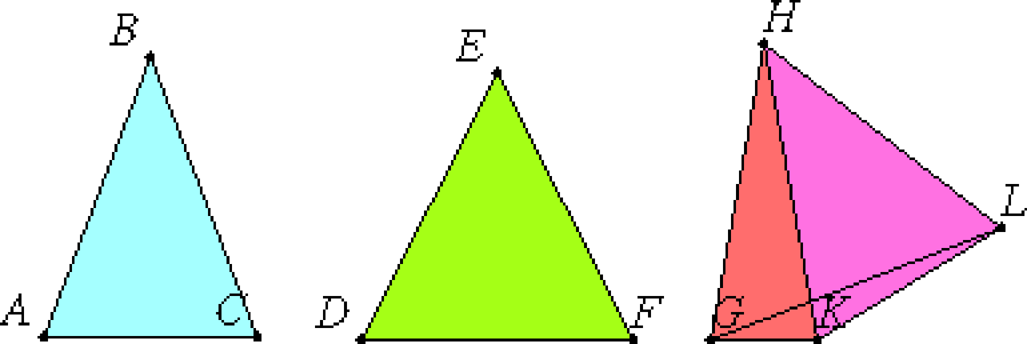
D.E.Joyce
Clark University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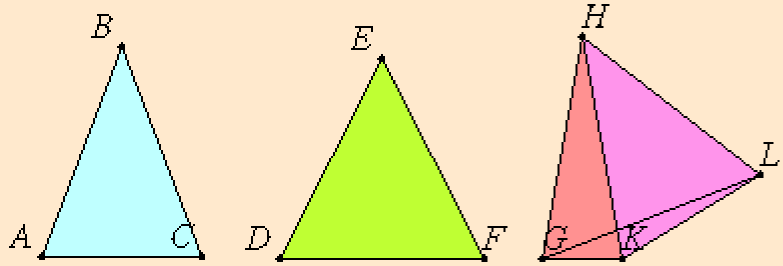
Euclid's Elements
Book XI
Proposition 22

If there are three plane angles such that the sum of any two is greater than the remaining one, and they are contained by equal straight lines, then it is possible to construct a triangle out of the straight lines joining the ends of the equal straight lines.

Let there be three plane angles *ABC, DEF,* and *GHK,* of which the sum of any two is greater than the remaining one, so that the sum of the angles *ABC* and *DEF* is greater than the angle *GHK,* the sum of the angles *DEF* and *GHK* is greater than the angle *ABC,* and, further, the sum of the angles *GHK* and *ABC* is greater than the angle *DEF.* Also let the straight lines *AB, BC, DE, EF, GH,* and *HK* be equal.

Join *AC, DF,* and *GK.*

I say that it is possible to construct a triangle out of straight lines equal to *AC, DF,* and *GK,* that is, that the sum of any two of the straight lines *AC, DF,* and *GK* is greater than the remaining one.



Now, if the angles *ABC, DEF,* and *GHK* equal one another, then it is clear that, *AC, DF,* and *GK* also being equal, it is possible to construct a triangle out of straight lines equal to *AC, DF,* and *GK.* I.4 I.1

But, if not, let them be unequal. Construct the angle *KHL* equal to the angle *ABC* at the point *H* on the straight line *HK.* Make *HL* equal to any one of the straight lines *AB, BC, DE, EF, GH,* or *HK.* Join *KL* and *GL.* I.23 I.3

Now, since the two sides *AB* and *BC* equal the two sides *KH* and *HL,* and the angle at *B* equals the angle *KHL,* therefore the base *AC* equals the base *KL.* I.4

And, since the sum of the angles *ABC* and *GHK* is greater than the angle *DEF,* while the angle *ABC* equals the angle *KHL,* therefore the angle *GHL* is greater than the angle *DEF.*

And, since the two sides *GH* and *HL* equal the two sides *DE* and *EF,* and the angle *GHL* is greater than the angle *DEF,* therefore the base *GL* is greater than the base *DF.* I.24

But the sum of *GK* and *KL* is greater than *GL.* Therefore the sum of *GK* and *KL* is much greater than *DF.*

But *KL* equals *AC,* therefore the sum of *AC* and *GK* is greater than the remaining straight line *DF.*

Similarly we can prove that the sum of *AC* and *DF* is greater than *GK,* and further, the sum of *DF* and *GK* is greater than *AC.*

Therefore it is possible to construct a triangle out of straight lines equal to *AC, DF,* and *GK.* (I.22)

Therefore, *if there are three plane angles such that the sum of any two is greater than the remaining one, and they are contained by equal straight lines, then it is possible to construct a triangle out of the straight lines joining the ends of the equal straight lines.*

Q. E. D.

Guide

This construction is the first stage of the construction in the next proposition to make a solid angle given three plane angles.

The proof succeeds in showing that if each of the three plane angles is less than the sum of the other two, then each of the three lines *AC*, *DF*, and *DK* is less than the sum of the other two. The latter is a necessary condition for a triangle to be made with its three sides equal to those three lines according to I.20. But it was never shown to be sufficient to make such a triangle in I.22, and it is that sufficiency which is being invoked in this proof. Thus, there is a serious flaw in the proof.

Next proposition: XI.23

Previous: XI.21

Book XI introduction

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Euclid's Elements
Book XI
Proposition 23

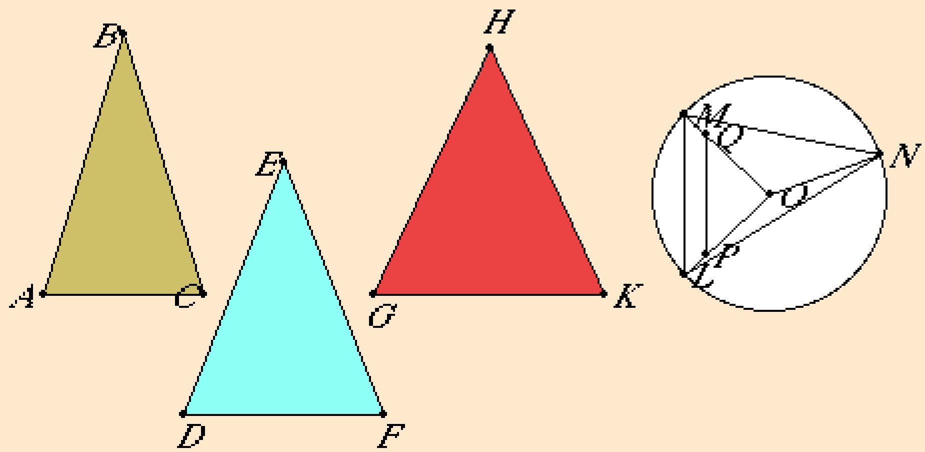
To construct a solid angles out of three plane angles such that the sum of any two is greater than the remaining one: thus the sum of the three angles must be less than four right angles. XI.20 XI.21

Let the angles *ABC, DEF,* and *GHK* be the three given plane angles, and let the sum of any two of them be greater than the remaining one, and further, let the sum of all three be less than four right angles.

It is required to construct a solid angle out of angles equal to the angles *ABC*, *DEF,* and *GHK.*

Cut off *AB, BC, DE, EF, GH,* and *HK* equal to one another, and join *AC, DF,* and *GK.* I.3

It is therefore possible to construct a triangle out of straight lines equal to *AC*, *DF*, and *GK.* Construct *LMN* so that *AC* equals *LM, DF* equals *MN*, and *GK* equals *NL.* XI.22



Describe the circle *LMN* about the triangle *LMN*, and take its center *O.* Join *LO, MO,* and *NO.* IV.5 III.1

I say that *AB* is greater than *LO.*

For, if not, *AB* either equals *LO*, or is less.

First, let it be equal. Then, since *AB* equals *LO,* while *AB* equals *BC*, and *LO* equals *OM,* therefore the two sides *AB* and *BC* equal the two sides *LO* and *OM* respectively. And, by hypothesis, the base *AC* equals the base *LM,* therefore the angle *ABC* equals the angle *LOM.* I.8

For the same reason the angle *DEF* also equals the angle *MON*, and the angle *GHK* equals the angle *NOL.* Therefore the sum of the three angles *ABC, DEF,* and *GHK* equals the sum of the three angles *LOM, MON,* and *NOL.*

But the sum of the three angles *LOM, MON,* and *NOL* equals four right angles, therefore the sum of the three angles *ABC, DEF,* and *GHK* equals four right angles.

But the sum is also, by hypothesis, less than four right angles, which is absurd. Therefore *AB* is not equal to *LO.*

I say next that neither is *AB* less than *LO.*

For, if possible, let it be so. Make *OP* equal to *AB,* and *OQ* equal to *BC,* and join *PQ.* I.3

Then, since *AB* equals *BC,* therefore *OP* also equals *OQ,* so that the remainder *LP* equals *QM.*

Therefore *LM* is parallel to *PQ,* and *LMO* is equiangular with *PQO.* VI.2 I.29

Therefore, *OL* is to *LM* as *OP* is to *PQ,* and alternately, *LO* is to *OP* as *LM* is to *PQ.* VI.4 V.16

But *LO* is greater than *OP,* therefore *LM* is greater than *PQ.* And *LM* equals *AC,* therefore *AC* is greater than *PQ.*

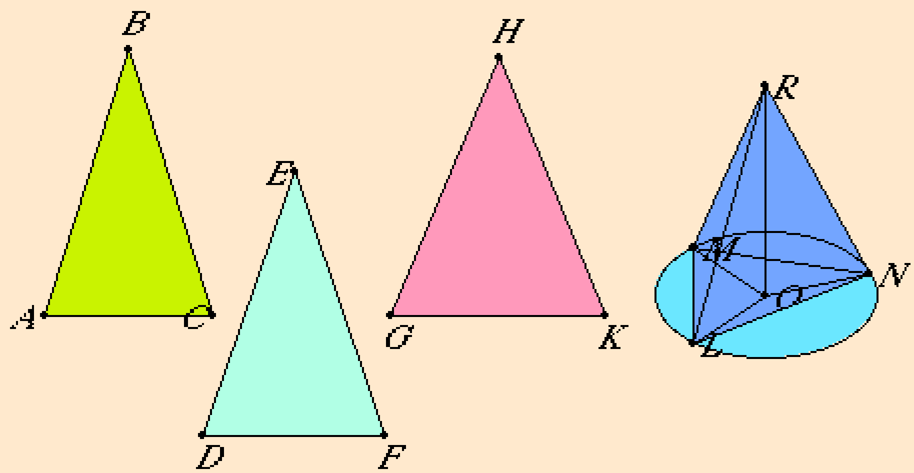
Since, then, the two sides *AB* and *BC* equal the two sides *PO* and *OQ,* and the base *AC* is greater than the base *PQ,* therefore the angle *ABC* is greater than the angle *POQ.* I.25

Similarly we can prove that the angle *DEF* is also greater than the angle *MON,* and the angle *GHK* is greater than the angle *NOL.*

Therefore the sum of the three angles *ABC, DEF,* and *GHK* is greater than the sum of the three angles *LOM, MON,* and *NOL.* But, by hypothesis, the sum of the angles *ABC, DEF,* and *GHK* is less than four right angles, therefore the sum of the angles *LOM, MON,* and *NOL* is much less than four right angles. But the sum also equals four right angles, which is absurd. Therefore *AB* is not less than *LO.*

And it was proved that neither is it equal, therefore *AB* is greater than *LO.*

Next set up *OR* from the point *O* at right angles to the plane of the circle *LMN* so that the square on *OR* equals the square on *AB* minus the square on *LO.* Join *RL, RM,* and *RN.* XI.12 Lemma below



Then, since *RO* is at right angles to the plane of the circle *LMN,* therefore *RO* is also at right angles to each of the straight lines *LO, MO,* and *NO.* And, since *LO* equals *OM,* and *OR* is common and at right angles, therefore the base *RL* equals the base *RM.* XI.Def.3 I.4

For the same reason *RN* also equals each of the straight lines *RL* and *RM.* Therefore the three straight lines *RL, RM,* and *RN* equal one another.

Next, since by hypothesis the square on *OR* equals equals the square on *AB* minus the square on *LO,* therefore the square on *AB* equals the sum of the squares on *LO* and *OR.*

But the square on *LR* equals the sum of the squares on *LO* and *OR,* for the angle *LOR* is right, therefore the square on *AB* equals the square on *RL.* Therefore *AB* equals *RL.* I.47

But each of the straight lines *BC, DE, EF, GH,* and *HK* equals *AB,* while each of the straight lines *RM* and *RN* equals *RL,* therefore each of the straight lines *AB, BC, DE, EF, GH,* and *HK* equals each of the straight lines *RL, RM,* and *RN.*

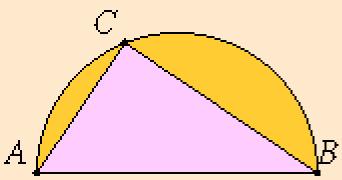
Since the two sides *LR* and *RM* equal the two sides *AB* and *BC*, and, by hypothesis, the base *LM* equals the base *AC*, therefore the angle *LRM* equals the angle *ABC.* For the same reason the angle *MRN* equals the angle *DEF*, and the angle *LRN* equals the angle *GHK.* I.8

Therefore, out of the three plane angles *LRM*, *MRN*, and *LRN*, which equal the three given angles *ABC, DEF*, and *GHK*, the solid angle at *R* has been constructed, which is contained by the angles *LRM*, *MRN*, and *LRN.*

Q.E.F.

Lemma

But how it is possible to take the square on *OR* equal to the square on *AB* minus the square on *LO* we can show as follows.



Set out the straight lines *AB* and *LO*, and let *AB* be the greater. Describe the semicircle *ABC* on *AB.* Fit *AC* into the semicircle *ABC* equal to the straight line *LO*, not being greater than the diameter *AB.* Join *CB.* IV.1

Since the angle *ACB* is an angle in the semicircle *ACB*, therefore the angle *ACB* is right. III.31

Therefore the square on *AB* equals the sum of the squares on *AC* and *CB.* I.47

Hence the square on *AB* equals the square on *AC* minus the square on *CB.* But *AC* equals *LO.* Therefore the square on *AB* equals the square on *LO* minus the square on *CB.* Therefore if we cut off *OR* equal to *BC*, then the square on *AB* will equal the square on *LO* minus the square on *OR.*

Q. E. F.

Guide

This proposition shows that the necessary conditions for constructing a solid angle found in XI.20 (the sum any two angles must be less than the third) and XI.21 (the sum of the three angles must be less than four right angles) are, in fact, sufficient. It is interesting to see how parts of the construction disappear as the angles *B*, *E*, and *H* grow so that these conditions fail.

This proposition completes the introductory portion of Book XI. Most of the remainder deals with parallelepipedal solids and their properties.

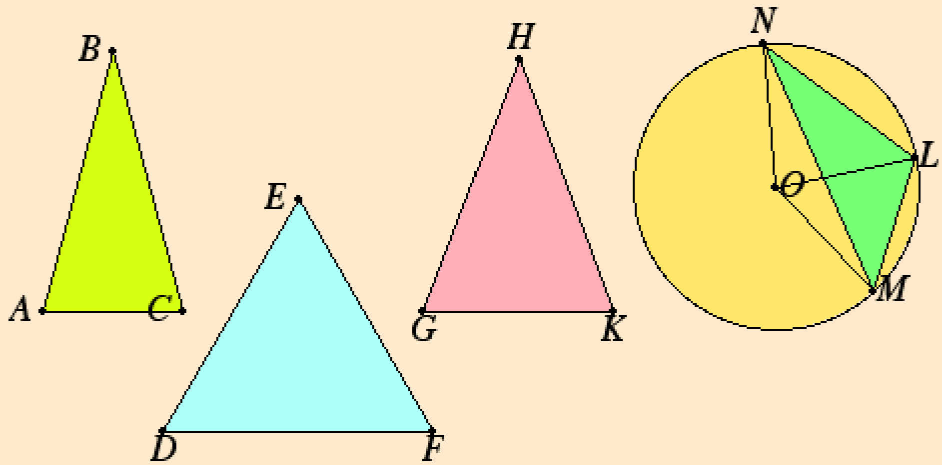
About the proof

This is a rather long proof that has several stages. First, the base *LMN* for the proposed solid angle is constructed. This first stage has been set off as the provious proposition XI.24.

After the circumcircle for this base is constructed, it is shown that the proposed edges for the solid angle, which are all equal, are greater than the radius of the circle. That part of the demonstration takes some time, and it is separated into two parts to show, first, that the edges can't equal the radius, and, second, that the edges can't be less than the radius.

The next stage is to place the proposed vertex *R* for the solid angle. It is placed above the center *O* of the circumcircle so that OR^2 is the difference of the square of the edge and the square of the radius. A separate lemma appears after the proposition to construct a line of this particular length. This lemma is the same as the lemma for proposition X.14 in Book X.

The remainder of the proof is the verification that the proposed solid angle satisfies the requirements of the construction.



The proof only covers the case when the circumcenter *O* of the triangle *LMN* lies within that triangle. Two other cases need to be considered as well—when *O* lies outside the triangle and when *O* lies on the boundary of the triangle. The three different cases need only be considered in the stage which shows that the proposed edges are greater than the radius of the circumcircle; the proof doesn't have to be split into three cases for the other stages of the proof.

Next proposition: XI.24

Previous: XI.22

Book XI introduction

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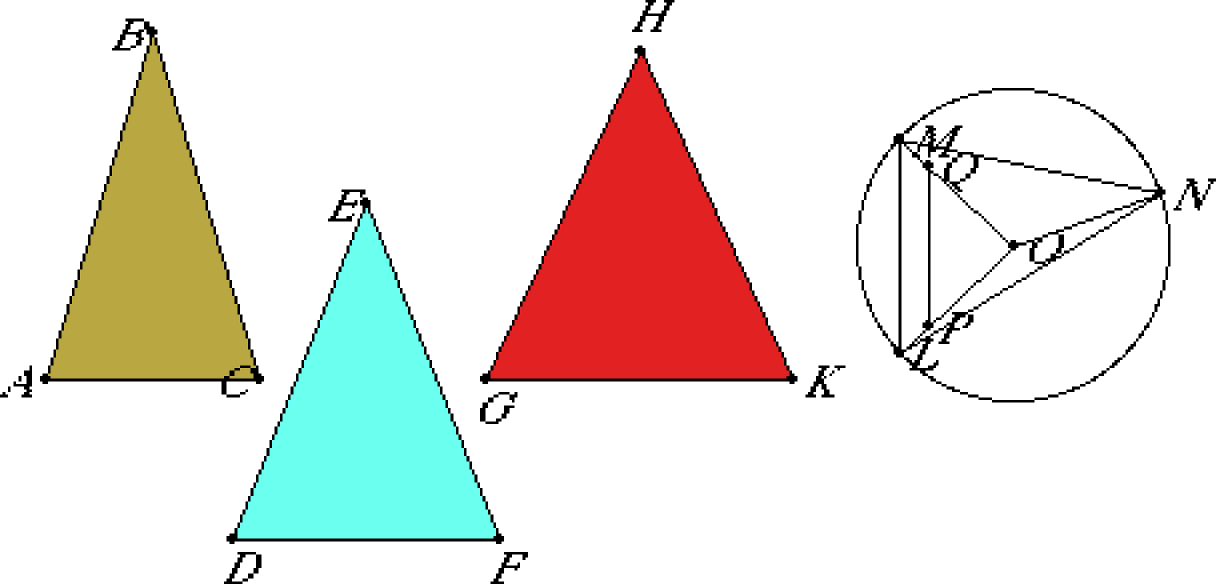

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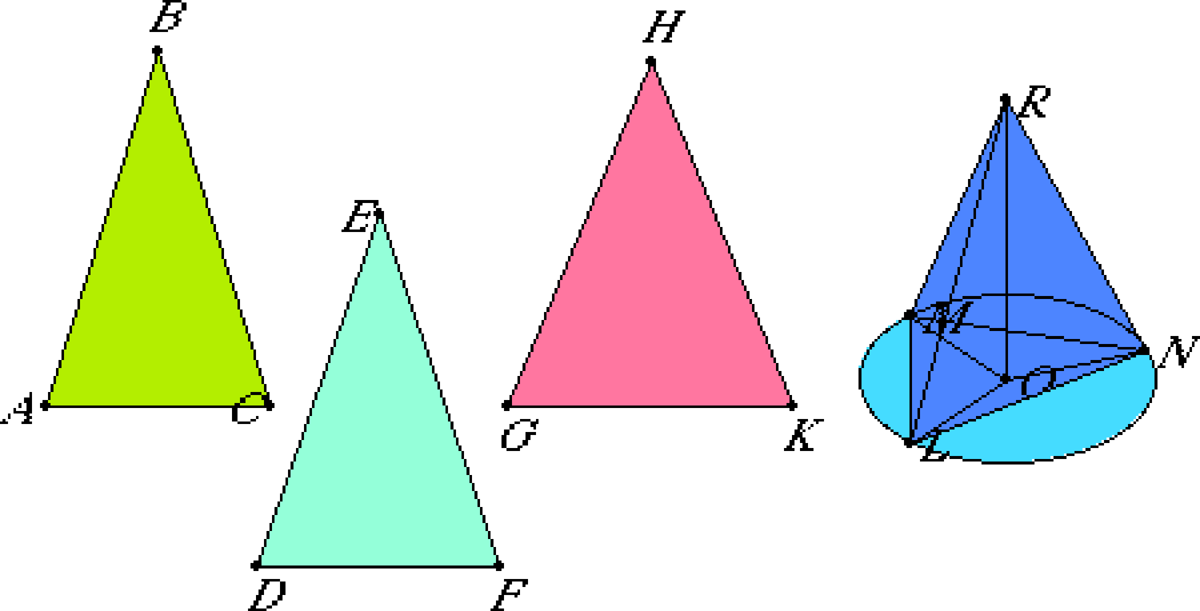
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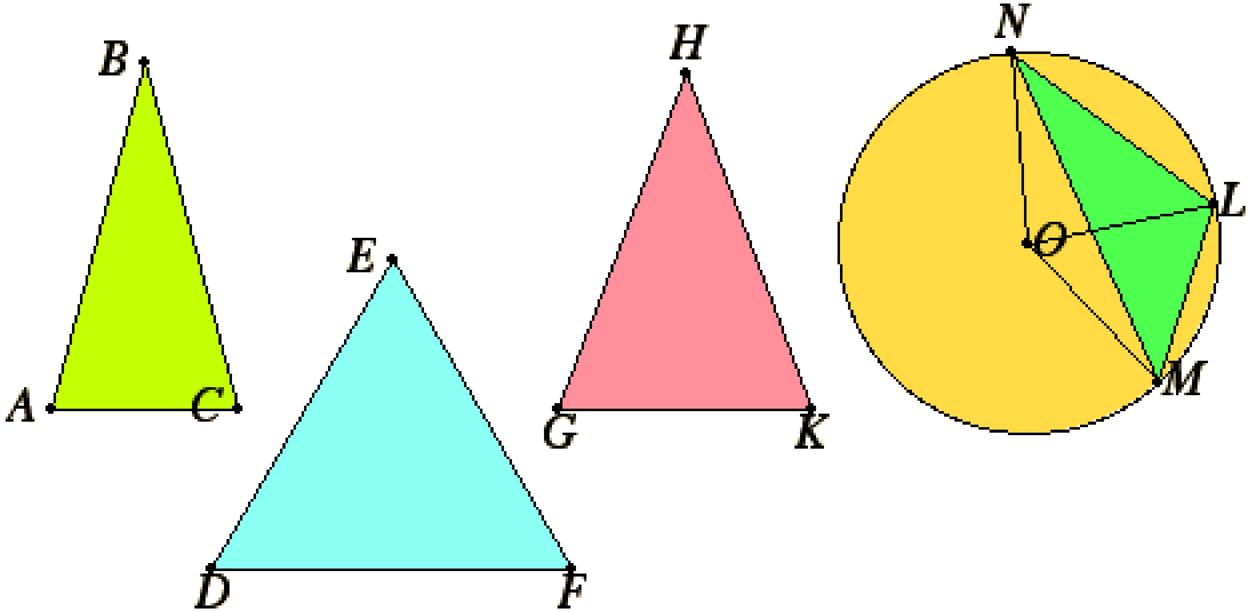
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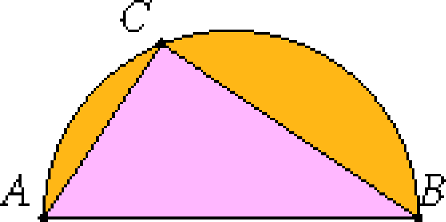
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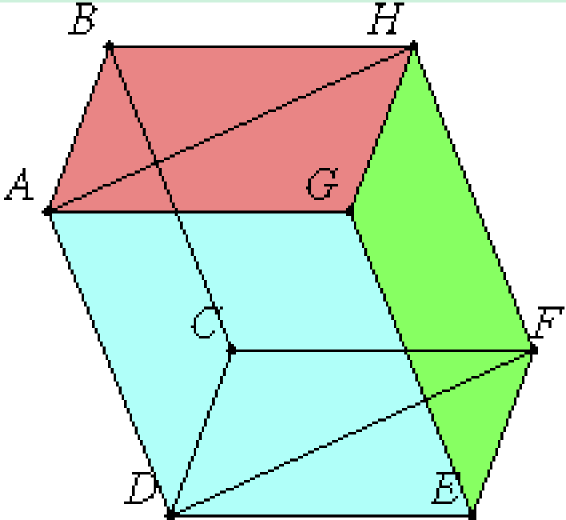

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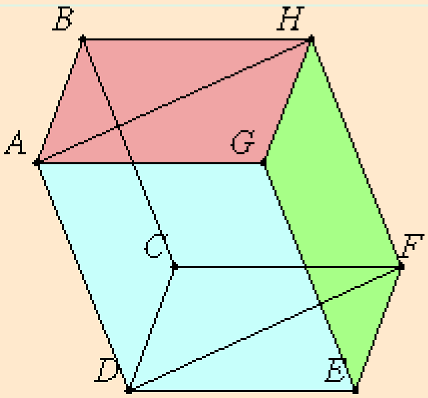
Euclid's Elements
Book XI
Proposition 24

If a solid is contained by parallel planes, then the opposite planes in it are equal and parallelogrammic.

Let the solid *CDHG* be contained by the parallel planes *AC*, *GF*, *AH*, *DF*, *BF*, and *AE*.

I say that the opposite planes in it are equal and parallelogrammic.



Since the two parallel planes *BG* and *CE* are cut by the plane *AC*, therefore their common sections are parallel. Therefore *AB* is parallel to *DC*. Again, since the two parallel planes *BF* and *AE* are cut by the plane *AC*, therefore their intersections are parallel. Therefore *BC* is parallel to *AD*. XI.16

But *AB* was proved parallel to *DC*, therefore *AC* is a parallelogram. Similarly we can prove that each of the planes *DF*, *FG*, *GB*, *BF*, and *AE* is a parallelogram.

Join *AH* and *DF*.

Then, since *AB* is parallel to *DC*, and *BH* is parallel to *CF*, therefore the two straight lines *AB* and *BH*, which meet one another, are parallel to the two straight lines *DC* and *CF*, which meet one another, not in the same plane. Therefore they contain equal angles. Therefore the angle *ABH* equals the angle *DCF*. XI.10

And, since the two sides *AB* and *BH* equal the two sides *DC* and *CF*, and the angle *ABH* equals the angle *DCF*, therefore the base *AH* equals the base *DF*, and the triangle *ABH* equals the triangle *DCF*. I.34 I.4

And the parallelogram *BG* is double the triangle *ABH*, and the parallelogram *CE* is double the triangle *DCF*, therefore the parallelogram *BG* equals the parallelogram *CE*. I.34

Similarly we can prove that *AC* equals *GF*, and *AE* equals *BF*.

Therefore, *if a solid is contained by parallel planes, then the opposite planes in it are equal and parallelogrammic.*

Q. E. D.

Guide

The statement of the theorem is not sufficiently detailed. All three of the octahedron, icosahedron, and dodecahedron (see XI.Def.26-28) are contained by parallel planes, but their faces are triangles or pentagons, not parallelograms. They do not have six faces, however, but eight, twenty, or twelve.

The correct hypothesis for this proposition is that the solid is contained by three pairs of parallel planes. Then the intersection of each plane with the other four nonparallel planes can be shown to be sides of a parallelgram, and the parallelograms on opposite planes can be shown to be congruent, what Euclid would call similar and equal parallelograms. That the opposite parallelgrams are not just equal but also similar should stated in the conclusion of the proposition.

Parallelepipeds

The term "parallelepipedal solid," abbreviated as "parallelepiped," is used for the solid treated by this proposition. It can be defined as a solid bounded by three pairs of parallel faces. Then this proposition shows that a parallelepiped has the further properties that each face is a parallelogram, and opposite parallelograms have parallel and equal corresponding sides, and equal corresponding angles.

Parallelepipeds are to solid geometry what parallelograms are to plane geometry. This proposition is the analogue of proposition I.34 which introduces parallelograms just as this proposition introduces parallelepipeds. It is likely that both are the product of Euclid's own research.

Use of this proposition

This proposition is used in the next as well as others in this book and the next.

Next proposition: XI.25

Previous: XI.23

Book XI introduction

Select from Book XI

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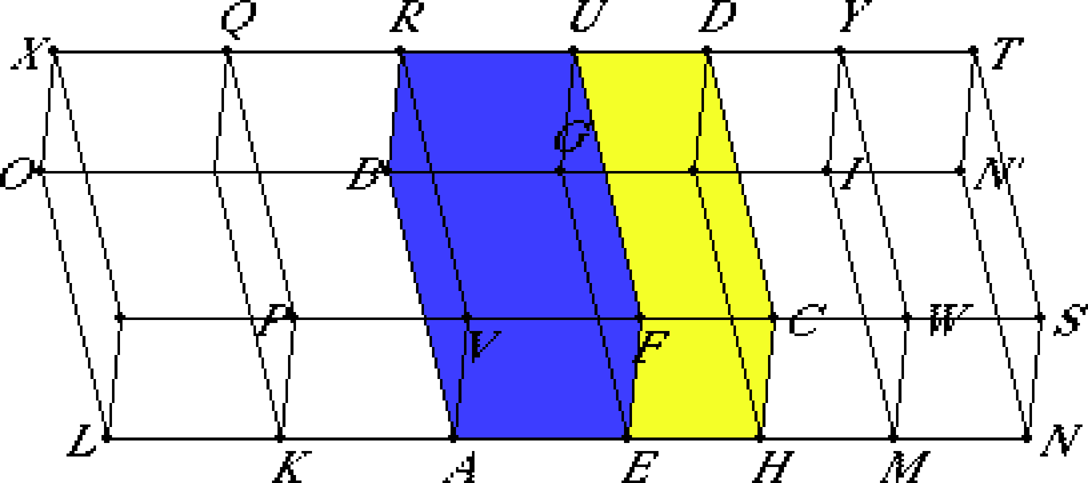
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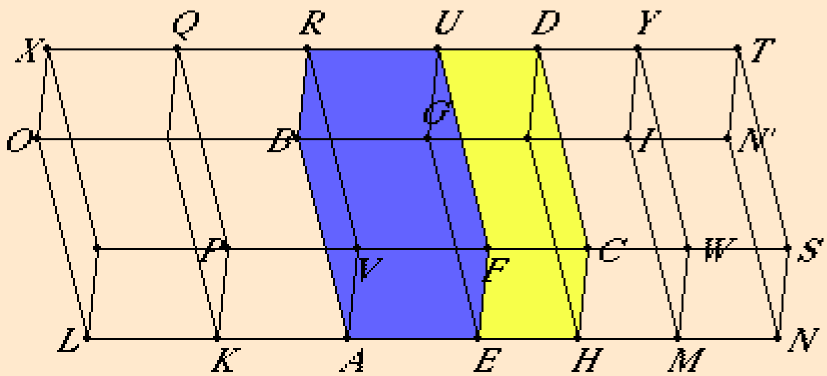
Euclid's Elements
Book XI
Proposition 25

If a parallelepipedal solid is cut by a plane parallel to the opposite planes, then the base is to the base as the solid is to the solid.

Let the parallelepipedal solid *ABCD* be cut by the plane *FG* which is parallel to the opposite planes *RA* and *DH.*

I say that the base *AEFV* is to the base *FHCF* as the solid *ABFU* is to the solid *EGCD.*

Produce *AH* in each direction. Make any number of straight lines *AK* and *KL* equal to *AE,* and any number *HM* and *MN* equal to *EH.* Complete the parallelograms *LP*, *KV*, *HW*, and *MS* and the solids *LQ*, *KR*, *DM*, and *MT*. I.3 I.31



Then, since the straight lines *LK*, *KA*, and *AE* equal one another, therefore the parallelograms *LP*, *KV*, and *AF* equal one another, *KO*, *KB*, and *AG* equal one another, and further *LX*, *KQ*, and *AR* equal one another, for they are opposite. For the same reason the parallelograms *EC*, *HW*, and *MS* are equal one another, *HG*, *HI*, and *IN* equal one another, and further, *DH*, *MY*, *NT* and equal one another. XI.24

Therefore in the solids *LQ*, *KR*, and *AU* three planes equal three planes. But the three planes equal the three opposite, therefore the three solids *LQ*, *KR*, and *AU* equal one another. For the same reason the three solids *ED*, *DM*, and *MT* also equal one another. Therefore, the solid *LU* is the same multiple of the solid *AU* that the base *LF* is of the base *AF*. For the same reason, the solid *NU* is the same multiple of the solid *HU* that the base *NF* is of the base *FH.* XI.Def.10

And, if the base *LF* equals the base *NF,* then the solid *LU* also equals the solid *NU*; if the base *LF* exceeds the base *NF,* then the solid *LU* also exceeds the solid *NU*; and, if one falls short, then the other falls short.

Therefore, there being four magnitudes, the two bases *AF* and *FH,* and the two solids *AU* and *UH,* equimultiples have been taken of the base *AF* and the solid *AU,* namely the base *LF* and the solid *LU,* and equimultiples of the base *HF* and the solid *HU,* namely the base *NF* and the solid *NU,* and it has been proved that, if the base *LF* exceeds the base *FN*, then the solid *LU* also exceeds the solid *NU*; if the bases are equal, then the solids are equal; and if the base falls short, then the solid falls short. Therefore, the base *AF* is to the base *FH* as the solid *AU* is to the solid *UH.* V.Def.5

Therefore *If a parallelepipedal solid is cut by a plane parallel to the opposite planes, then the base is to the base as the solid is to the solid.*

Q. E. D.

Guide

This is the first of the propositions on volumes of solids. Most of the rest of this book deals with volumes of parallelepipeds, and Book XII develops the theory of volumes for pyramids, prisms, cones, cylinders, and spheres.

Euclid's foundations for volume are (1) his definition XI.Def.10 which says that if two solid figures have congruent faces, then the solids are equal, and (2) solids are magnitudes for which cut and paste principles hold. See the comments on XI.Def.10 for details.

Outline of the proof

The analogous proposition for two dimensions is proposition VI.1. In both propositions the ratio of two figures in one dimension is shown to be equal to the ratio of two figures in another dimension. In this proposition, the dimensions are 2 and 3, while in VI.1, the dimensions are 1 and 2. Eudoxus' definition of proportion, V.Def.5, allows these ratios of different kinds to be compared.

The goal of this proof is to show that the ratio of the bases of the the two parallelepipeds is the same as the ratio of the two parallelepipeds themselves. The parallelpiped *AU* has the parallelogram *AF* as its base, while the parallelepiped *HU* has the parallelogram *HF* as its base. Thus, the goal is to derive the proportion

$$AU:HU = AF:HF.$$

By the definition of proportion, V.Def.5, that means for any number *m* and any number *n* that

$$m\ AF >=< n\ HF \text{ when } m\ AU >=< n\ HU.$$

Note that Euclid takes both *m* and *n* to be 3 in his proof, just as he did in VI.1. Now *m AU* equals the parallelepiped *LU*, *n HU* equals the parallelepiped *NU*, *m AF* equals the parallelogram *LF*, and *n HF* equals the parallelogram *NF*. So what has to be shown is that

$$LF >=< NF \text{ when } LU >=< NU.$$

Euclid makes no attempt to show that; he just states it as fact:

> And, if the base *LF* equals the base *NF*, then the solid *LU* also equals the solid *NU*; if the base *LF* exceeds the base *NF*, then the solid *LU* also exceeds the solid *NU*; and, if one falls short, then the other falls short.

The case of equality is based directly on defintion XI.Def.10, for if the bases are equal, then all the planes bounding the solids are equal, which is the defintion of the solids being equal. The two cases when one base exceeds or falls short of the other implicitly depend on finding a part of one solid equal to the whole of the other solid, equality again using XI.Def.10, then concluding that one whole solid is greater than the other whole solid (C.N.5).

Use of this proposition

This proposition is used for the proofs of propositions XI.31, XI.32, and XI.34.

Next proposition: XI.26

Select from Book XI

Previous: XI.24

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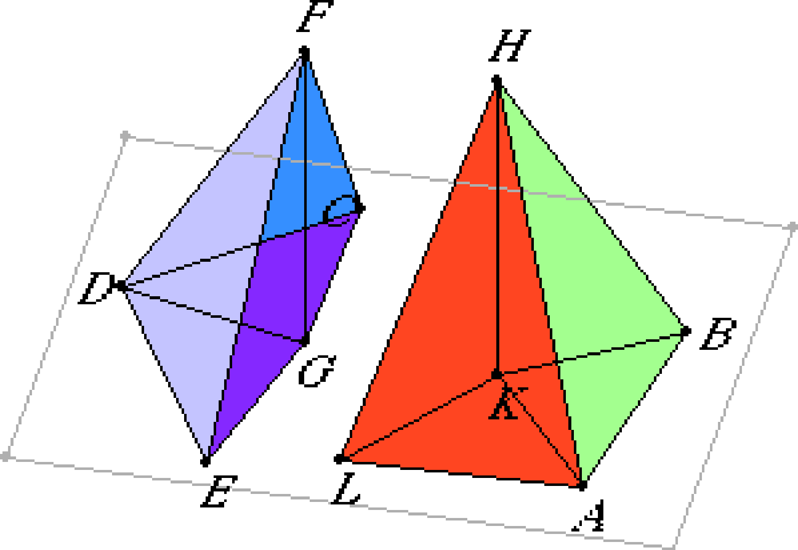
Book XI introduction

Select topic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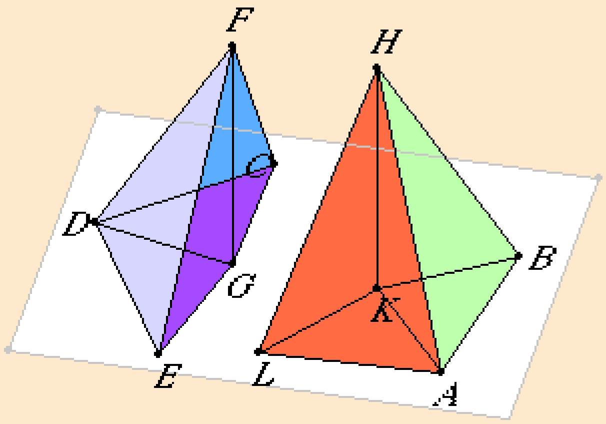
Euclid's Elements
Book XI
Proposition 26

To construct a solid angle equal to a given solid angle on a given straight line at a given point on it.

Let *A* be the given point on the given straight line *AB*, and let the angle at *D* be the given solid angle contained by the angles *EDC*, *EDF*, and *FDC*.

It is required to construct at the point *A* on the straight line *AB* a solid angle equal to the solid angle at *D*.

Take a point *F* at random on *DF*, draw *FG* from *F* perpendicular to the plane through *ED* and *DC*, and let it meet the plane at *G*. Join *DG*. XI.11



At the point *A* on the straight line *AB* construct the angle *BAL* equal to the angle *EDC*, and construct the angle *BAK* equal to the angle *EDG*. I.23

Make *AK* equal to *DG*. Set *KH* up from the point *K* at right angles to the plane through *BA* and *AL*. Make *KH* equal to *GF*, and join *HA*. XI.12

I say that the solid angle at *A* contained by the angles *BAL*, *BAH*, and *HAL* equals the solid angle at *D* contained by the angles *EDC*, *EDF*, and *FDC*.

Cut *AB* and *DE* off equal to one another, and join *HB*, *KB*, *FE*, and *GE*.

Then, since *FG* is at right angles to the plane of reference, therefore it is also at right angles with all the straight lines which meet it and are in the plane of reference. Therefore each of the angles *FGD* and *FGE* is right. For the same reason each of the angles *HKA* and *HKB* is also right. XI.Def.3

And, since the two sides *KA* and *AB* equal the two sides *GD* and *DE* respectively, and they contain equal angles, therefore the base *KB* equals the base *GE*. But *KH* also equals *GF*, and they contain right angles, therefore *HB* also equals *FE*. Again, since the two sides *AK* and *KH* equal the two sides *DG* and *GF*, and they contain right angles, therefore the base *AH* equals the base *FD*. I.4

But *AB* also equals *DE*, therefore the two sides *HA* and *AB* are equal to the two sides *DF* and *DE*. And the base *HB* is equal to the base *FE*, therefore the angle *BAH* equals the angle *EDF*. For the same reason the angle *HAL* also equals the angle *FDC*. I.8

And the angle *BAL* also equals the angle *EDC*. Therefore at the point *A* on the straight line *AB* a solid angle has been constructed equal to the given solid angle at *D*.

Q. E. F.

Guide

This construction is used in the next one to construct similar parallelepipeds.

Next proposition: XI.27

Previous: XI.25

Book XI introduction

Select from Book XI

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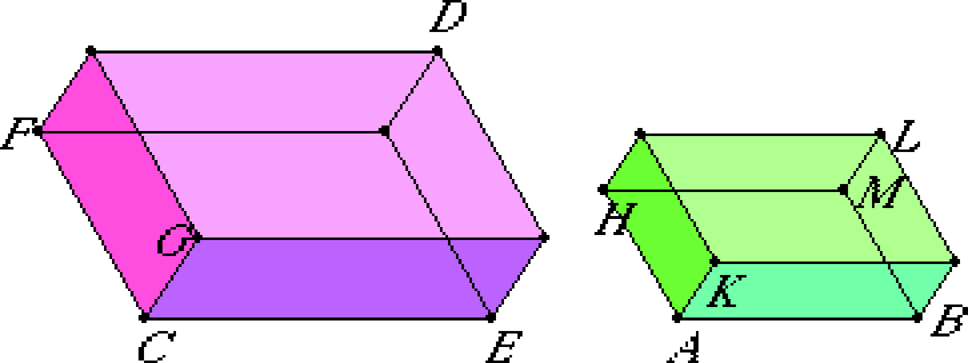
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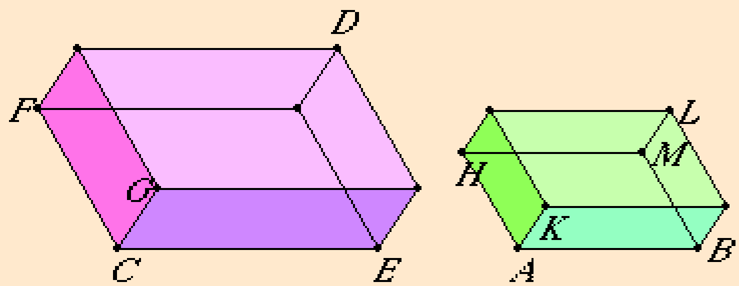
Euclid's Elements
Book XI
Proposition 27

To describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid on a given straight line.

Let *AB* be the given straight line and *CD* the given parallelepipedal solid.

It is required to describe on the given straight line *AB* a parallelepipedal solid similar and similarly situated to the given parallelepipedal solid *CD.*



Construct the solid angle contained by the angles *BAH, HAK,* and *KAB* at the point *A* on the straight line *AB* equal to the solid angle *C* so that the angle *BAH* equals the angle *ECF*, the angle *BAK* equals the angle *ECG*, and the angle *KAH* equals the angle *GCF*,so that *EC* is to *CG* as *BA* is to *AK,* and *GC* is to *CF* as *KA* is to *AH.* XI.26 VI.12

Therefore, *ex aequali*, *EC* is to *CF* as *BA* is to *AH.* V.22

Complete the parallelogram *HB* and the solid *AL.*

Now since *EC* is to *CG* as *BA* is to *AK,* and the sides about the equal angles *ECG* and *BAK* are thus proportional, therefore the parallelogram *GE* is similar to the parallelogram *KB.* For the same reason the parallelogram *KH* is similar to the parallelogram *GF*, and also *FE* is similar to *HB.*

Therefore three parallelograms of the solid *CD* are similar to three parallelograms of the solid *AL.* But the former three are both equal and similar to the three opposite parallelograms, and the latter three are both equal and similar to the three opposite parallelograms, therefore the whole solid *CD* is similar to the whole solid *AL.* XI.Def.9

Therefore on the given straight line *AB* there has been described *AL* similar and similarly situated to the given parallelepipedal solid *CD.*

Q. E. F.

Guide

This proposition is analogous to proposition VI.18 which constructs a similar plane figure on a line, but it is not as general since it applies only to parallelepipeds and not all polyhedra. It is not used later in the *Elements*.

Next proposition: XI.28

Select from Book XI

Previous: XI.26

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Book XI introduction

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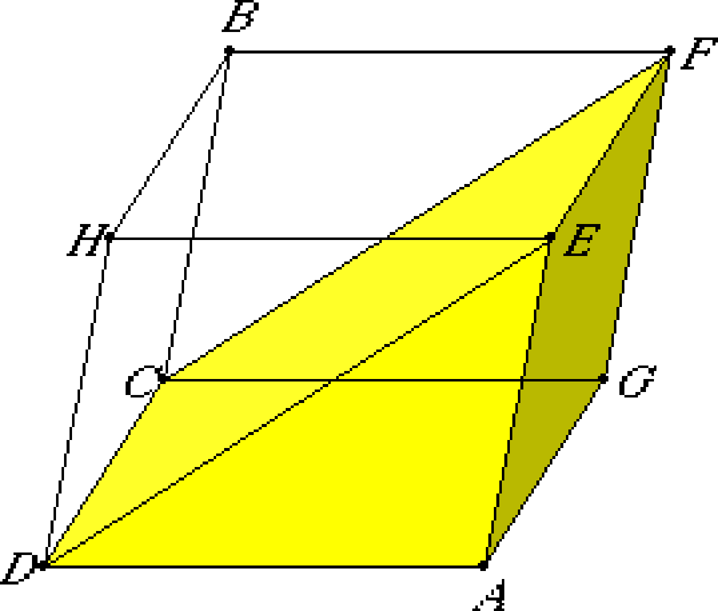
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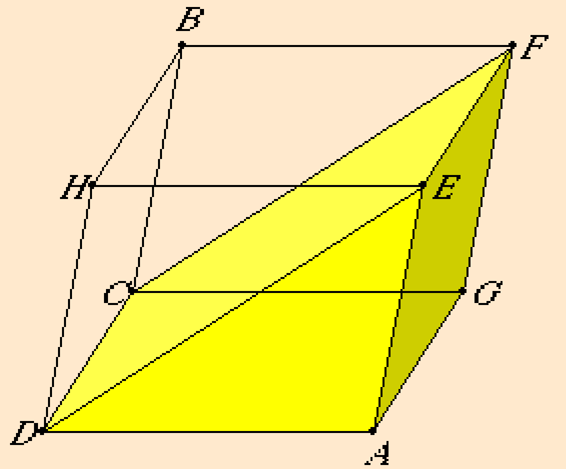
Euclid's Elements
Book XI
Proposition 28

If a parallelepipedal solid is cut by a plane through the diagonals of the opposite planes, then the solid is bisected by the plane.



Let the parallelepipedal solid *AB* be cut by the plane *CDEF* through the diagonals *CF* and *DE* of opposite planes.

I say that the solid *AB* is bisected by the plane *CDEF.*

Since the triangle *CGF* equals the triangle *CFB*, and *ADE* equals *DEH,* while the parallelogram *CA* equals the parallelogram *EB*, for they are opposite, and *GE* equals *CH,* therefore the prism contained by the two triangles *CGF* and *ADE* and the three parallelograms *GE*, *AC*, and *CE* equals the prism contained by the two triangles *CFB* and *DEH* and the three parallelograms *CH*, *BE*, and *CE*, for they are contained by planes equal both in multitude and in magnitude. I.34 XI.Def.10

Hence the whole solid *AB* is bisected by the plane *CDEF*.

Therefore, *if a parallelepipedal solid is cut by a plane through the diagonals of the opposite planes, then the solid is bisected by the plane.*

Q. E. D.

Guide

A minor point missing from the beginning of the proof of is that the two diagonals *CF* and *DE* lie in one plane, but it is easy to show that they lie in the lines *CD* and *EF* are parallel, and therefore, by XI.7, *CF* and *DE* lie in the plane spanned by *CD* and *EF*.

This is the second proposition concerning volumes. (The first was XI.25.) The final conclusion of the proof here is justified by XI.Def.10: since the faces of the two prisms are congruent, therefore the prisms are equal and similar (that is, congruent). Several authors have criticized this conclusion because the two prisms are mirror images of each other and cannot be applied to each other in the sense of moving one in space to coincide with the other.

From some points of view this criticism is valid. But the method of superposition is subject to even greater criticism. In modern geometry, depending on the style of geometry, superposition is either eliminated entirely or else completely formalized using the theory of group transformations.

Use of this proposition

Although this proposition is not used in the rest of this book, it is used for several propositions in the next book that deal with triangular prisms.

Next proposition: XI.29

Previous: XI.27

Book XI introduction

Select from Book XI

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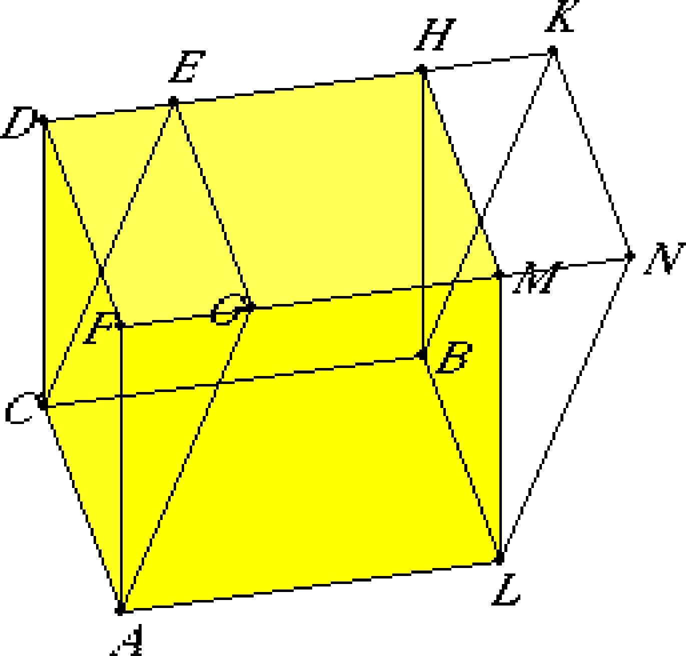

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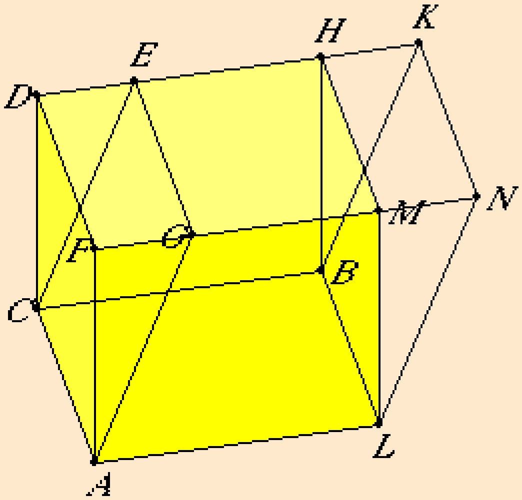
Euclid's Elements

Book XI

Proposition 29

Parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are on the same straight lines, equal one another.

Let *CM* and *CN* be parallelepipedal solids on the same base *AB* and of the same height, and let the ends of their edges which stand up, namely *AG, AF, LM, LN, CD, CE, BH,* and *BK*, be on the same straight lines *FN* and *DK*.



I say that the solid *CM* equals the solid *CN*.

Since each of the figures *CH* and *CK* is a parallelogram, therefore *CB* equals each of the straight lines *DH* and *EK*. Therefore *DH* also equals *EK*. I.34

Subtract *EH* from each, therefore the remainder *DE* equals the remainder *HK*. Therefore the triangle *DCE* also equals the triangle *HBK*, and the parallelogram *DG* equals the parallelogram *HN*. For the same reason the triangle *AFG* equals the triangle *MLN*. I.8 I.4 I.36

But the parallelogram *CF* equals the parallelogram *BM*, and *CG* equals *BN*, for they are opposite, therefore the prism contained by the two triangles *AFG* and *DCE* and the three parallelograms *AD*, *DG*, and *CG* equals the prism contained by the two triangles *MLN* and *HBK* and the three parallelograms *BM*, *HN*, and *BN*. XI.Def.10

Add to each the solid of which the parallelogram *AB* is the base and *GEHM* its opposite, therefore the whole parallelepipedal solid *CM* equals the whole parallelepipedal solid *CN*.

Therefore, *parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are on the same straight lines, equal one another.*

Q. E. D.

Guide

This proposition is the first step in the theory of volume for parallelepipeds. It is used in the proof of three of the next five propositions.

Next proposition: XI.30 Select from Book XI

Select from Book XI

Previous: XI.28

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Book XI introduction

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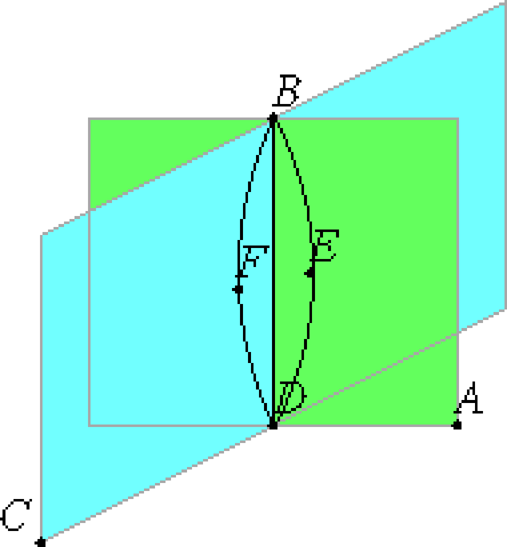
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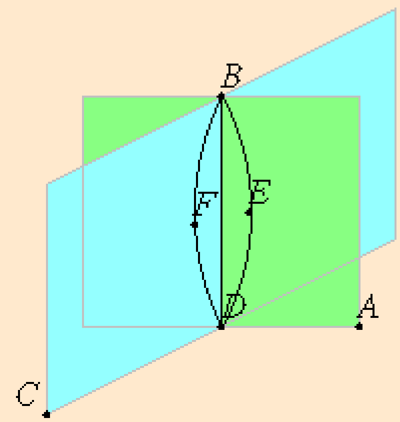
Euclid's Elements
Book XI
Proposition 3

If two planes cut one another, then their intersection is a straight line.

Let two planes *AB* and *BC* cut one another, and let the line *DB* be their intersection.

I say that the line *DB* is a straight line.



For, if not, join the straight line *DEB* from *D* to *B* in the plane *AB*, and the straight line *DFB* in the plane *BC*.

Then the two straight lines *DEB* and *DFB* have the same ends and clearly enclose an area, which is absurd.

Therefore *DEB* and *DFB* are not straight lines.

Similarly we can prove that neither is there any other straight line joined from *D* to *B* except *DB*, the intersection of the planes *AB* and *BC*.

Therefore, *if two planes cut one another, then their intersection is a straight line.*

Q. E. D.

Guide

The proof of this proposition has some flaws. Postulate I (from Book I) states that a straight line can be drawn from any point to any point. It seems to be interpreted as saying that for any plane from any point in that plane to any point in that plane a straight line in that plane can be drawn. Next it is stated that the lines in those two planes "clearly enclose an area, which is absurd." But the two lines do not lie in the same plane, so it is unclear that they enclose an area. Furthermore, the statement that two straight lines cannot enclose an area did not appear in the original elements, although it was later appended to Post.1.

A more serious criticism of the proof is that it fails to prove the statement of the proposition. At most it shows that if two planes intersect at more than one point, then the line that joins them also lies in their intersection. But the possibility that their intersection consists of only one point is ignored. This is important as this proposition is used in XI.5 from two planes known to intersect at one point, a line of intersection is generated.

The real problem is that there is no postulate limiting space to three dimensions. In four or or more dimensions two planes may intersect in only one point. Neither Euclid nor anyone else before the nineteenth century recognized the possibility of higher dimensional geometry, but the flaws in this proof are apparent nonetheless. There are alternative postulates to limit the geometry to three dimensions. For instance, one is based on the idea that a plane divides space into two sides.

Use of this proposition

This proposition is used frequently, first in the proof of XI.5.

Next proposition: XI.4

Previous: XI.2

Book XI introduction

Select from Book XI

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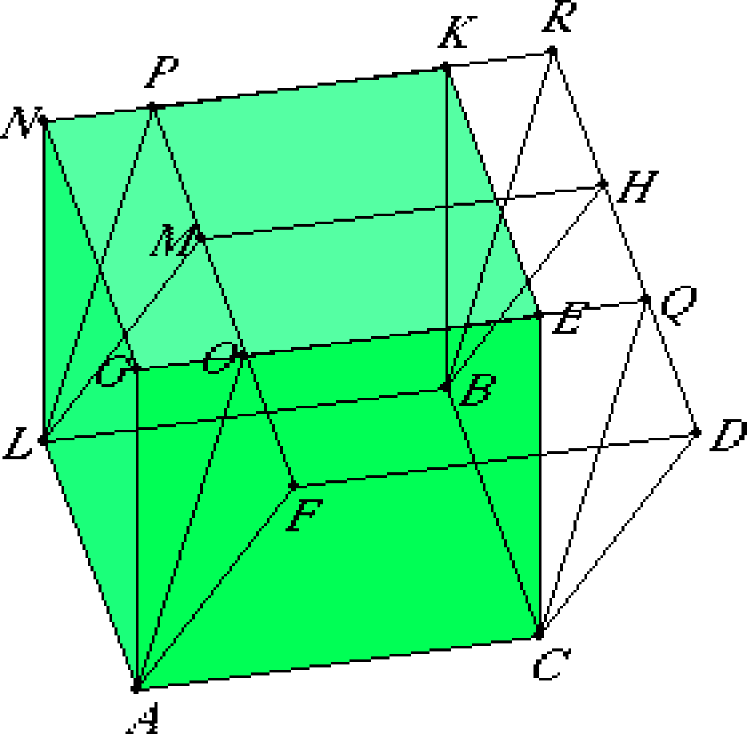

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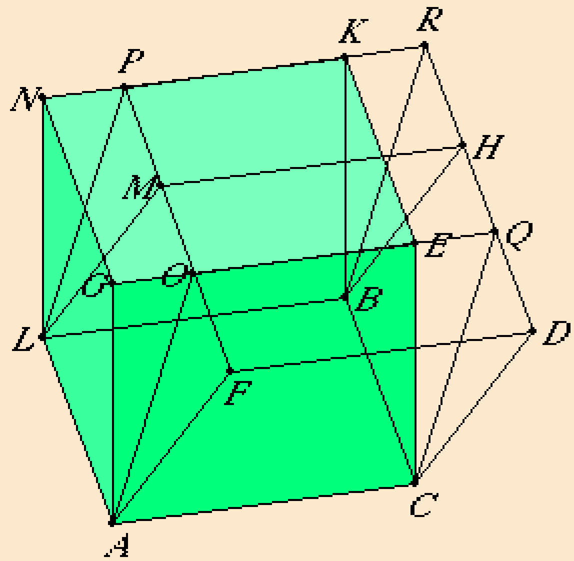
Euclid's Elements
Book XI
Proposition 30

Parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are not on the same straight lines, equal one another.

Let *CM* and *CN* be parallelepipedal solids on the same base *AB* and of the same height, and let the ends of their edges which stand up, namely *AF, AG, LM, LN, CD, CE, BH,* and *BK*, not be on the same straight lines.

I say that the solid *CM* equals the solid *CN.*



Produce *NK* and *DH* to meet one another at *R*, and produce *FM* and *GE* to *P* and *Q*. Join *AO, LP, CQ,* and *BR.*

Then the solid *CM,* of which the parallelogram *ACBL* is the base and *FDHM* its opposite, equals the solid *CP*, of which the parallelogram *ACBL* is the base and *OQRP* its opposite, for they are on the same base *ACBL* and of the same height, and the ends of their edges which stand up, namely *AF, AO, LM, LP, CD, CQ, BH,* and *BR*, are on the same straight lines *FP* and *DR.* XI.29

But the solid *CP,* of which the parallelogram *ACBL* is the base and *OQRP* its opposite, equals the solid *CN,* of which the parallelogram *ACBL* is the base and *GEKN* its opposite, for they are again on the same base *ACBL* and of the same height, and the ends of their edges which stand up, namely *AG, AO, CE, CQ, LN, LP, BK,* and *BR,* are on the same straight lines *GQ* and *NR.* XI.29

Hence the solid *CM* also equals the solid *CN.*

Therefore, *parallelepipedal solids which are on the same base and of the same height, and in which the ends of their edges which stand up are not on the same straight lines, equal one another.*

Q. E. D.

Guide

Two applications of the previous proposition allow its generalization to the present proposition. It is generalized one step further in the next proposition.

Next proposition: XI.31

Previous: XI.29

Book XI introduction

Select from Book XI

Select book

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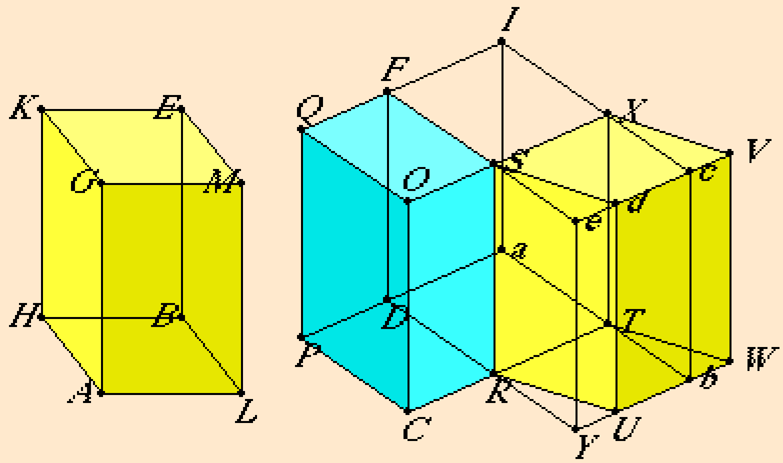
Euclid's Elements
Book XI
Proposition 31

Parallelepipedal solids which are on equal bases and of the same height equal one another.

Let the parallelepipedal solids *AE* and *CF* of the same height be on equal bases *AB* and *CD.*

I say that the solid *AE* equals the solid *CF.*

First, let the sides which stand up, *HK, BE, AG, LM, PQ, DF, CO,* and *RS,* be at right angles to the bases *AB* and *CD.* Produce the straight line *RT* in a straight line with *CR.* Construct the angle *TRU* equal to the angle *ALB* at the point *R* on the straight line *RT.* Make *RT* equal to *A,* and *RU* equal to *LB.* Complete the base *RW* and the solid *XU.* I.23 I.3 I.31



Now, since the two sides *TR* and *RU* equal the two sides *AL* and *LB,* and they contain equal angles, therefore the parallelogram *RW* equals and is similar to the parallelogram *HL.* Since again *AL* equals *RT,* and *LM* equals *RS,* and they contain right angles, therefore the parallelogram *RX* equals and is similar to the parallelogram *AM.* For the same reason *LE* also equals and is similar to *SU.*

Therefore three parallelograms of the solid *AE* equal and are similar to three parallelograms of the solid *XU.* But the former three equal and are similar to the three opposite, and the latter three equal and are similar the three opposite, therefore the whole parallelepipedal solid *AE* equals the whole parallelepipedal solid *XU.* XI.24 XI.Def.10

Draw *DR* and *WU* through to meet one another at *Y,* draw *aTb* through *T* parallel to *DY,* produce *PD* to *a,* and complete the solids *YX* and *RI.* I.31

Then the solid *XY,* of which the parallelogram *RX* is the base and *Yc* its opposite, equals the solid *XU,* of which the parallelogram *RX* is the base and *UV* its opposite, for they are on the same base *RX* and of the same height, and the ends of their edges which stand up, namely *RY, RU, Tb, TW, Se, Sd, Xc,* and *XV,* are on the same straight lines *YW* and *eV.* But the solid *XU* equals *AE,* therefore the solid *XY* also equals the solid *AE.* XI.29

And, since the parallelogram *RUWT* equals the parallelogram *YT,* for they are on the same base *RT* and in the same parallels *RT* and *YW,* and *RUWT* equals *CD,* since it also equals *AB,* therefore the parallelogram *YT* also equals *CD.* I.35

But *DT* is another parallelogram, therefore the base *CD* is to *DT* as *YT* is to *DT.* V.7

And, since the parallelepipedal solid *CI* is cut by the plane *RF* which is parallel to opposite planes, therefore the base *CD* is to the base *DT* as the solid *CF* is to the solid *RI*. For the same reason, since the parallelepipedal solid *YI* is cut by the plane *RX* which is parallel to opposite planes, therefore the base *YT* is to the base *TD* as the solid *YX* is to the solid *RI*. XI.25

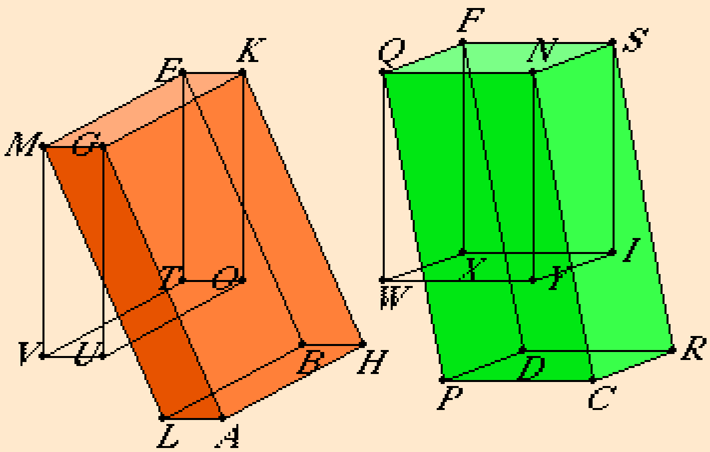
But the base *CD* is to *DT* as *YT* is to *DT*, therefore the solid *CF* is to the solid *RI* as the solid *YX* is to *RI*. V.11

Therefore each of the solids *CF* and *YX* has to *RI* the same ratio. Therefore the solid *CF* equals the solid *YX*. But *YX* was proved equal to *AE*, therefore *AE* also equals *CF*. V.9

Next, let the sides standing up, *AG, HK, BE, LM, CN, PQ, DF*, and *RS*, not be at right angles to the bases *AB* and *CD.*

I say again that the solid *AE* equals the solid *CF*.

Draw *KO, ET, GU, MV, QW, FX, NY* and *SI* from the points *K, E, G, M, Q, F, N*, and *S* perpendicular to the plane of reference, and let them meet the plane at the points *O, T, U, V, W, X, Y*, and *I*. XI.11



Then the solid *KV* equals the solid *QI*, for they are on the equal bases *KM* and *QS* and of the same height, and their sides which stand up are at right angles to their bases. Above

But the solid *KV* equals the solid *AE*, and *QI* equals *CF*, for they are on the same base and of the same height, while the ends of their edges which stand up are not on the same straight lines. XI.30

Therefore the solid *AE* also equals the solid *CF*.

Therefore, *parallelepipedal solids which are on equal bases and of the same height equal one another*.

Q.E.D.

Guide

The statement that started with XI.29 has now been generalized two steps. In the next proposition the heights of the two parallelepipeds remain equal, but the bases vary. The present proposition is used not only in the proof of the next, but also in three more of the remaining propositions of Book XI.

Next proposition: XI.32

Previous: XI.30

Select from Book XI

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Book XI introduction

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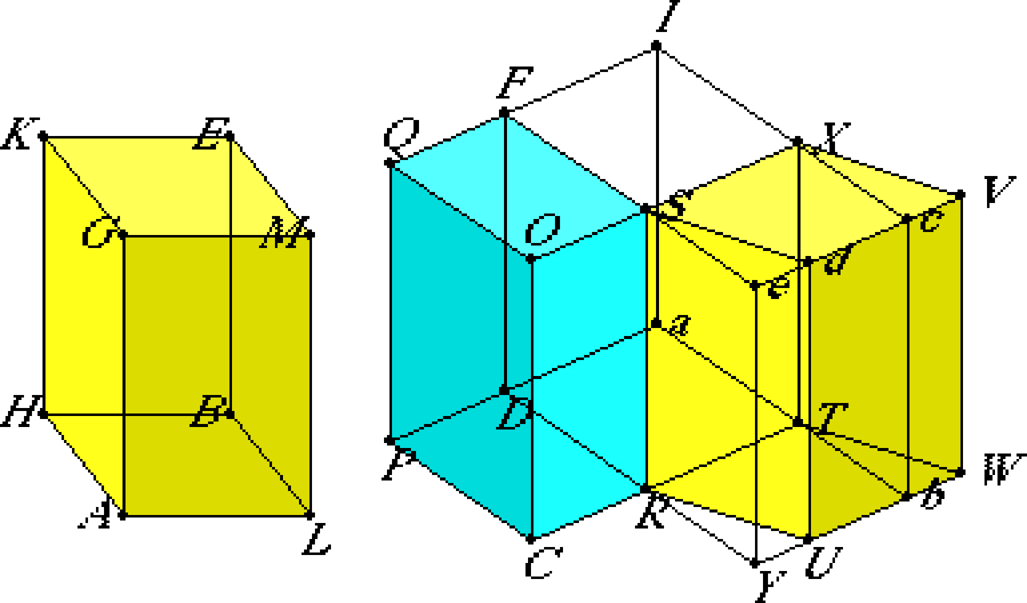
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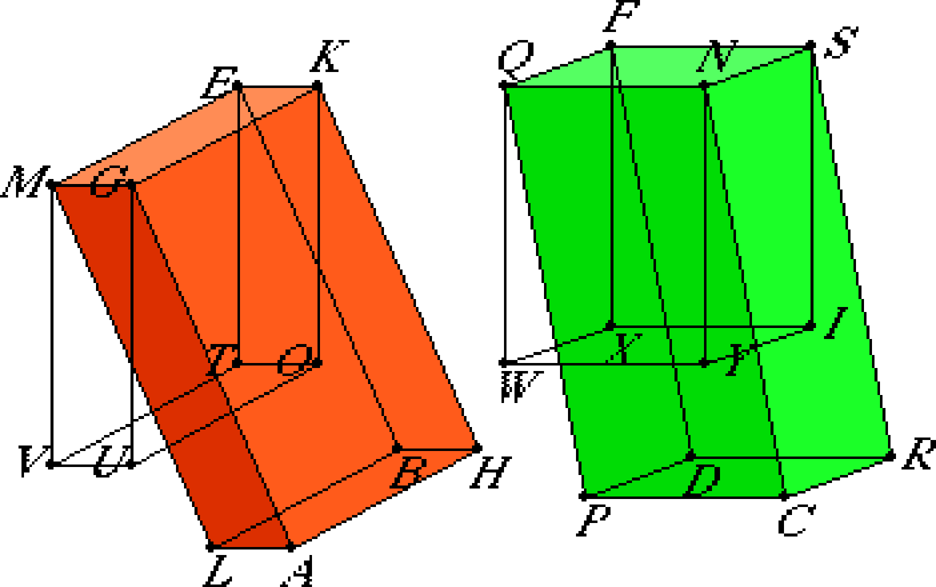

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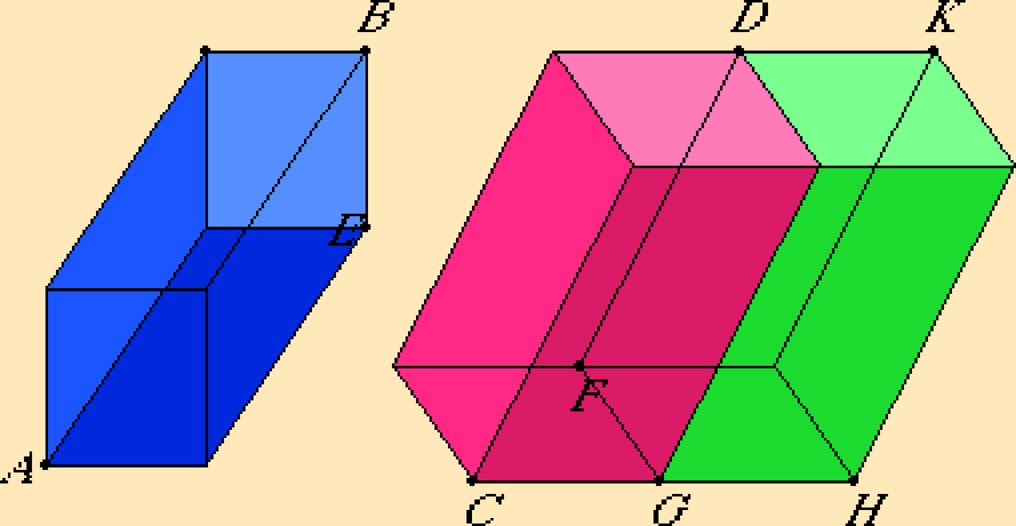
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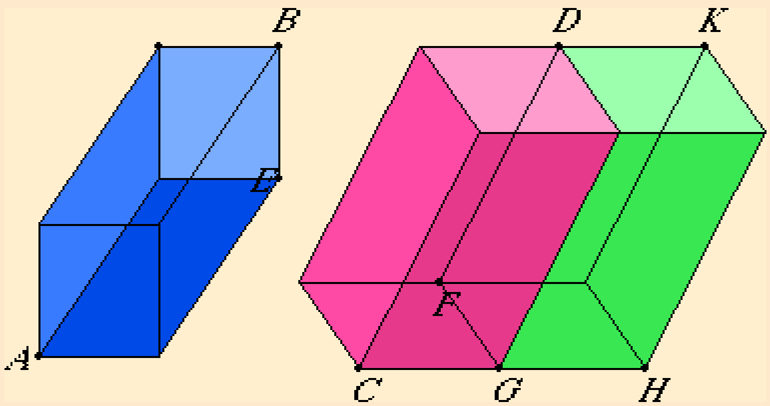
Euclid's Elements
Book XI
Proposition 32

Parallelepipedal solids which are of the same height are to one another as their bases.

Let *AB* and *CD* be parallelepipedal solids of the same height.

I say that the parallelepipedal solids *AB* and *CD* are to one another as their bases, that is, that the solid *AB* to the solid *CD* as the base *AE* is to the base *CF* .

Apply *FH* equal to *AE* to *FG*. Complete the parallelepipedal solid *GK* with the same height as that of *CD* on *FH* as base. I.45 I.31



Then the solid *AB* equals the solid *GK* for they are on equal bases *AE* and *FH* and of the same height. XI.31

And, since the parallelepipedal solid *CK* is cut by the plane *DG* which is parallel to opposite planes, therefore the solid *CD* is to the solid *DH* as the base *CF* is to the base *FH*. XI.25

But the base *FH* equals the base *AE*, and the solid *GK* equals the solid *AB*, therefore the solid *AB* to the solid *CD* as the base *AE* is to the base *CF*.

Therefore, *parallelepipedal solids which are of the same height are to one another as their bases.*

Q.E.D.

Guide

This completes the sequence of generalizations of proposition XI.29. Euclid does not specifically have a corresponding proposition "parallelepipedal solids with equal bases are to one another as their heights," but in the next two propositions, which depend on this one, he investigates other aspects of volumes of parallelepipeds.

Next proposition: XI.33

Previous: XI.31

Book XI introduction

Select from Book XI

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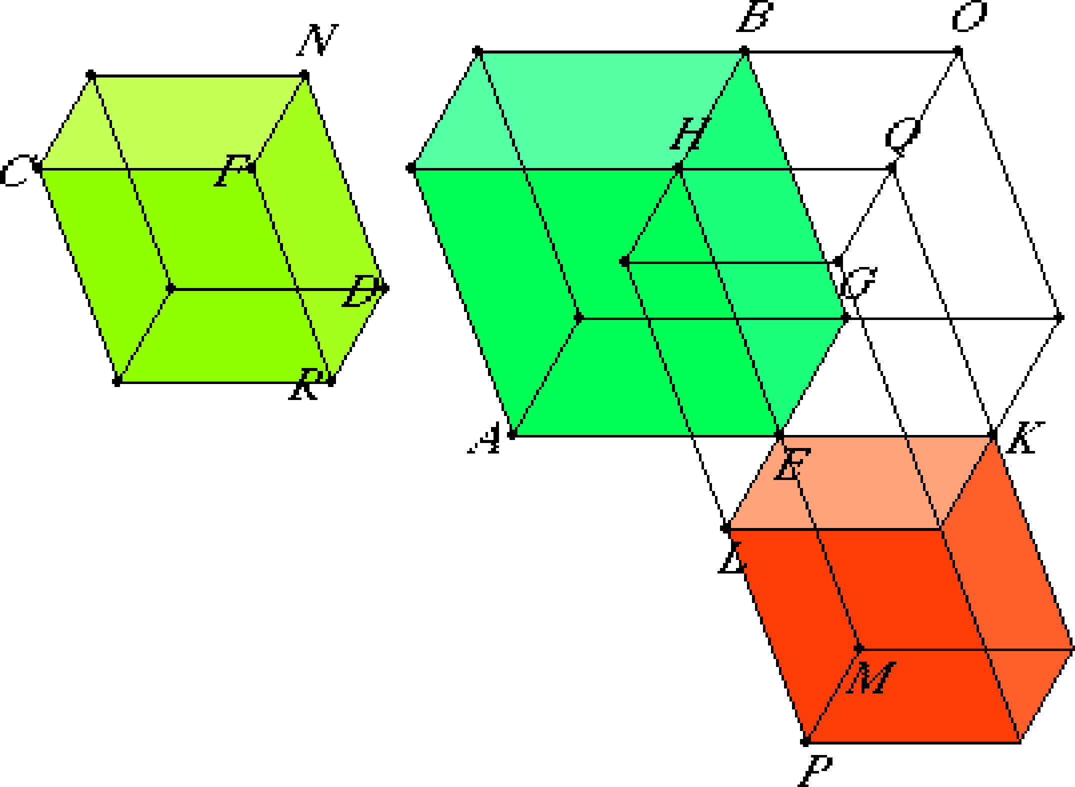
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Euclid's Elements

Book XI

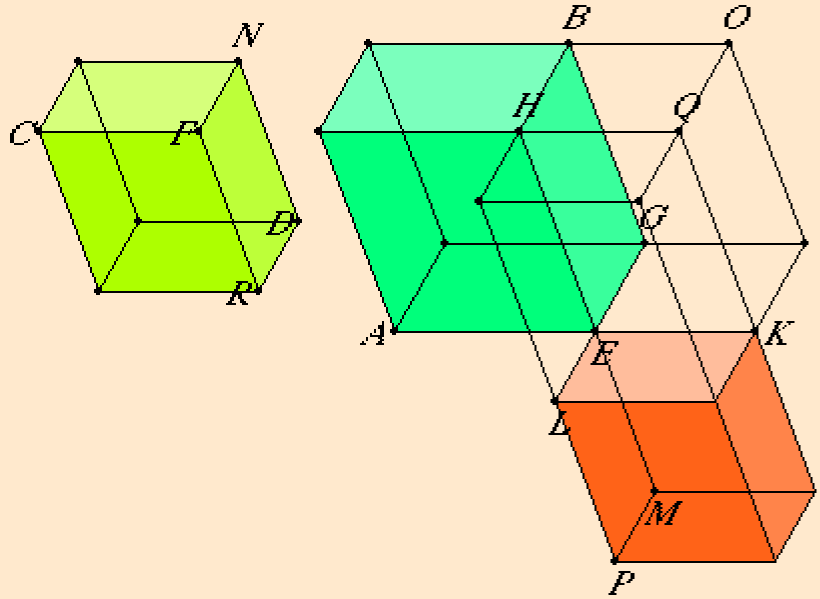
Proposition 33

Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

Let *AB* and *CD* be similar parallelepipedal solids, and let *AE* be the side corresponding to *CF*.

I say that the solid *AB* has to the solid *CD* the ratio triplicate of that which *AE* has to *CF*.

Produce *EK*, *EL*, and *EM* in a straight line with *AE*, *GE*, and *HE*. Make *EK* equal to *CF*, *EL* equal to *FN*, and *EM* equal to *FR*. Complete the parallelogram *KL* and the solid *KP*. I.3 I.31



Now, since the two sides *KE* and *EL* equal the two sides *CF* and *FN*, while the angle *KEL* equals the angle *CFN*, for the angle *AEG* also equals the angle *CFN* because *AB* and *CD* are similar solids, therefore the parallelogram *KL* equals and is similar to the parallelogram *CN*. For the same reason the parallelogram *KM* equals and is similar to *CR*, and *EP* equals and is similar to *DF*.

Therefore three parallelograms of the solid *KP* equal and are similar to three parallelograms of the solid *CD*. But the former three parallelograms equal and are similar to their opposites, and the latter three equal and are similar to their opposites, therefore the whole solid *KP* equals and is similar to the whole solid *CD*. XI.24 XI.Def.10

Complete the parallelogram *GK*, and complete the solids *EO* and *LQ* on the parallelograms *GK* and *KL* as bases with the same height as that of *AB*. I.31

Then since the solids *AB* and *CD* are similar, therefore *AE* is to *CF* as *EG* is to *FN*, and as *EH* is to *FR*. And *CF* equals *EK*, *FN* equals *EL*, and *FR* equals *EM*, therefore *AE* is to *EK* as *GE* is to *EL*, and as *HE* is to *EM*. XI.Def.9

But *AE* is to *EK* as *AG* is to the parallelogram *GK*, therefore *GE* is to *EL* as *GK* is to *KL*, and *HE* is to *EM* as *QE* is to *KM*. Therefore the parallelogram *AG* is to *GK* as *GK* to is *KL*, and as *QE* is to *KM*. VI.1

But *AG* is to *GK* as the solid *AB* is to the solid *EO*, *GK* is to *KL* as the solid *OE* is to the solid *QL*, and

QE is to *KM* as the solid *QL* is to the solid *KP*, therefore the solid *AB* is to *EO* as *EO* is *to QL*, and as *QL* is to *KP*. XI.32

But, if four magnitudes are continuously proportional, then the first has to the fourth the ratio triplicate of that which it has to the second, therefore the solid *AB* has to *KP* the ratio triplicate of that which *AB* has to *EO*. V.Def.10

But *AB* is to *EO* as the parallelogram *AG* is to *GK*, and as the straight line *AE* is to *EK*, hence the solid *AB* also has to *KP* the ratio triplicate of that which *AE* has to *EK*. VI.1

But the solid *KP* equals the solid *CD*, and the straight line *EK* equals *CF*, therefore the solid *AB* has also to the solid *CD* the ratio triplicate of that which the corresponding side of it, *AE*, has to the corresponding side *CF*.

Therefore, *Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.*

Q. E. D.

Corollary.

If four straight lines are continuously proportional, then the first is to the fourth as a parallelepipedal solid on the first is to the similar and similarly situated parallelepipedal solid on the second, in as much as the first has to the fourth the ratio triplicate of that which it has to the second.

Guide

This proposition is used in the proof of proposition XI.37 and later in XII.8, an analogous proposition about similar pyramids.

Next proposition: XI.34

Previous: XI.32

Book XI introduction

Select from Book XI

Select book

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Euclid's Elements
Book XI
Proposition 34

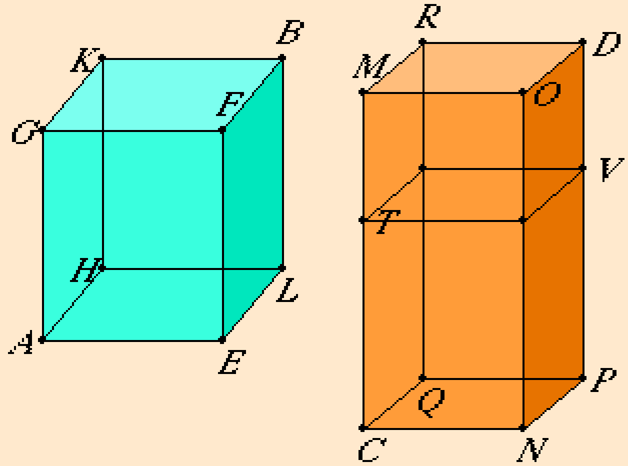
In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

Let *AB* and *CD* be equal parallelepipedal solids.

I say that in the parallelepipedal solids *AB* and *CD* the bases are reciprocally proportional to the heights, that is the base *EH* is to the base *NQ* as the height of the solid *CD* is to the height of the solid *AB.*

First, let the sides which stand up, namely *AG, EF, LB, HK, CM, NO, PD,* and *QR*, be at right angles to their bases.

I say that the base *EH* is to the base *NQ* as *CM* is to *AG.*



If now the base *EH* equals the base *NQ*, and the solid *AB* equals the solid *CD*, then *CM* equal *AG*. For parallelepipedal solids of the same height are to one another as the bases, and the base *EH* is to *NQ* as *CM* is to *AG*, and it is clear that in the parallelepipedal solids *AB* and *CD* the bases are reciprocally proportional to the heights. XI.32

Next, let the base *EH* not be equal to the base *NQ*, but let *EH* be greater.

Now the solid *AB* equals the solid *CD*, therefore *CM* is also greater than *AG.*

Make *CT* equal to *AG*, complete the parallelepipedal solid *VC* on *NQ* as base with *CT* as height. I.3 I.31

Now, since the solid *AB* equals the solid *CD*, and *CV* is outside them, and equals have to the same the same ratio, therefore the solid *AB* is to the solid *CV* as the solid *CD* is to the solid *CV*. V.7

But the solid *AB* is to the solid *CY* as the base *EH* is to the base *NQ*, for the solids *AB* and *CV* are of equal height, and the solid *CD* is to the solid *CV* as the base *MQ* is to the base *TQ* and *CM* is to *CT*, therefore the base *EH* is to the base *NQ* as *MC* is to *CT*. XI.32 XI.25 VI.1

But *CT* equals *AG*, therefore the base *EH* is to the base *NQ* as *MC* is to *AG.*

Therefore in the parallelepipedal solids *AB* and *CD* the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids *AB* and *CD* let the bases be reciprocally proportional to the heights, that is the base *EH* is to the base *NQ,* so let the height of the solid *CD* be to the height of the solid *AB.*

I say that the solid *AB* equals the solid *CD.*

Let the sides which stand up be at right angles to the bases.

Now, if the base *EH* equals the base *NQ,* and the base *EH* is to the base *NQ* as the height of the solid *CD* is to the height of the solid *AB,* therefore the height of the solid *CD* also equals the height of the solid *AB.*

But parallelepipedal solids on equal bases and of the same height equal one another, therefore the solid *AB* equals the solid *CD.* XI.31

Next, let the base *EH* not be equal to the base *NQ,* but let *EH* be greater.

Therefore the height of the solid *CD* is also greater than the height of the solid *AB,* that is, *CM* is greater than *AG.*

Make *CT* equal to *AG* again, and complete the solid *CV.* I.3 I.31

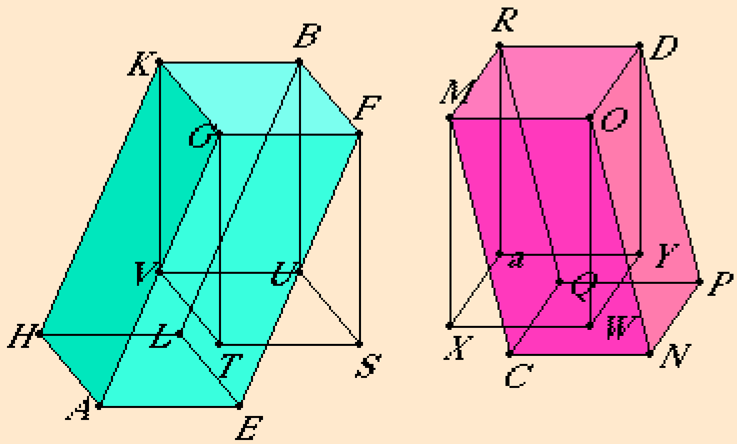
Since the base *EH* is to the base *NQ* as *MC* is to *AG,* and *AG* equals *CT,* therefore the base *EH* is to the base *NQ* as *CM* is to *CT.*

But the base *EH* is to the base *NQ* as the solid *AB* is to the solid *CV,* for the solids *AB* and *CV* are of equal height, and *CM* is to *CT* as the base *MQ* is to the base *QT* and as the solid *CD* is to the solid *CV.* XI.32 VI.1 XI.25

Therefore the solid *AB* is to the solid *CV* as the solid *CD* to the solid *CV.* Therefore each of the solids *AB* and *CD* has to *CV* the same ratio. Therefore the solid *AB* equals the solid *CD.* V.9

Now let the sides which stand up, *FE, BL, GA, HK, ON, DP, MC,* and *RQ,* not be at right angles to their bases. Draw perpendiculars from the points *F, G, B, K, O, M, D,* and *R* to the planes through *EH* and *NQ,* and let them meet the planes at *S, T, U, V, W, X, Y,* and *a*. Complete the solids *FV* and *Oa.* X.11

I say that, in this case too, if the solids *AB* and *CD* are equal, then the bases are reciprocally proportional to the heights, that is, the base *EH* is to the base *NQ* as the height of the solid *CD* to the height of the solid *AB.*



Since the solid *AB* equals the solid *CD,* and *AB* equals *BT,* for they are on the same base *FK* and of the same height, and the solid *CD* equals *DX,* for they are again on the same base *RO* and of the same height, therefore the solid *BT* also equals the solid *DX.* XI.29 XI.30

Therefore the base *FK* is to the base *OR* as the height of the solid *DX* is to the height of the solid *BT.* But the base *FK* equals the base *EH,* and the base *OR* equals the base *NQ,* therefore the base *EH* is to the base *NQ* as the height of the solid *DX* is to the height of the solid *BT.* Above

But the solids *DX* and *BT* and the solids *DC* and *BA* have the same heights respectively, therefore the base *EH* is to the base *NQ* as the height of the solid *DC* is to the height of the solid *AB.*

Therefore in the parallelepipedal solids *AB* and *CD* the bases are reciprocally proportional to the heights.

Next, in the parallelepipedal solids *AB* and *CD* let the bases be reciprocally proportional to the heights, that is, as the base *EH* is to the base *NQ,* so let the height of the solid *CD* be to the height of the solid *AB.*

I say that the solid *AB* equals the solid *CD.*

With the same construction, since the base *EH* is to the base *NQ* as the height of the solid *CD* is to the height of the solid *AB,* and the base *EH* equals the base *FK*, and *NQ* equals *OR*, therefore the base *FK* is to the base *OR* as the height of the solid *CD* is to the height of the solid *AB.*

But the solids *AB* and *CD* and the solids *BT* and *DX* have the same heights respectively, therefore the base *FK* is to the base *OR* as the height of the solid *DX* is to the height of the solid *BT*.

Therefore in the parallelepipedal solids *BT* and *DX* the bases are reciprocally proportional to the heights. Therefore the solid *BT* equals the solid *DX*. Above

But *BT* equals *BA*, for they are on the same base *FK* and of the same height, and the solid *DX* equals the solid *DC.* Therefore the solid *AB* also equals the solid *CD.* XI.29 XI.30

Therefore, *in equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.*

Q. E. D.

Guide

A proof of this proposition could be made with very little regard to geometry but almost entirely in terms of abstract proportions. The volume of a parallelepiped is proportional to its base for equal heights (XI.32), and proportional to its height for equal bases (not actually stated by Euclid), therefore the base and height are inversely proportional for equal volumes. Such a proof, although simpler, is not in Euclid's style.

This proposition is used in the proof of proposition XII.9, an analogous statement about pyramids.

Next proposition: XI.35

Previous: XI.33

Book XI introduction

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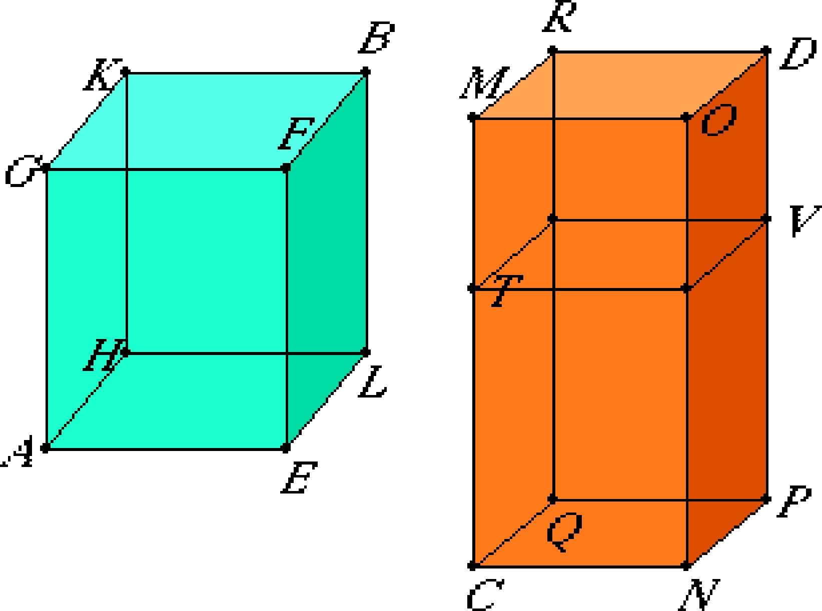

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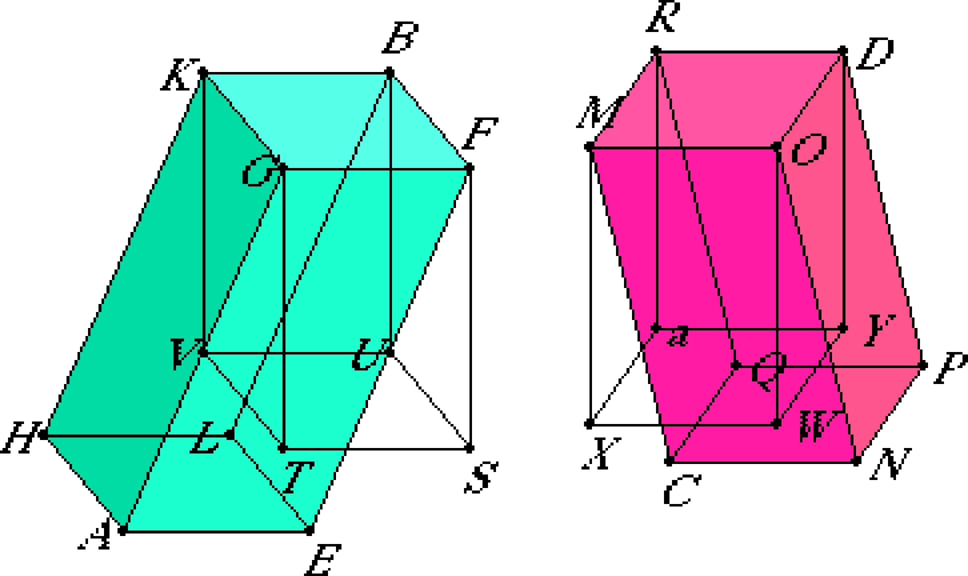

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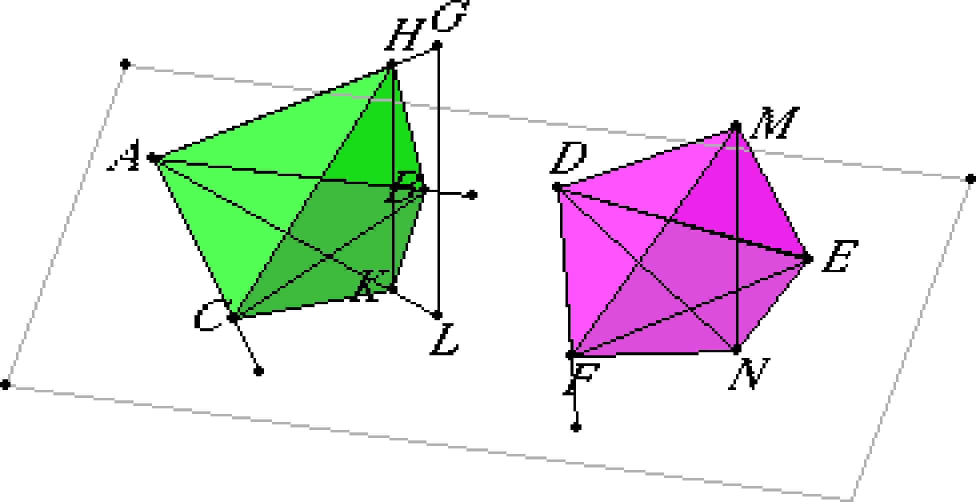
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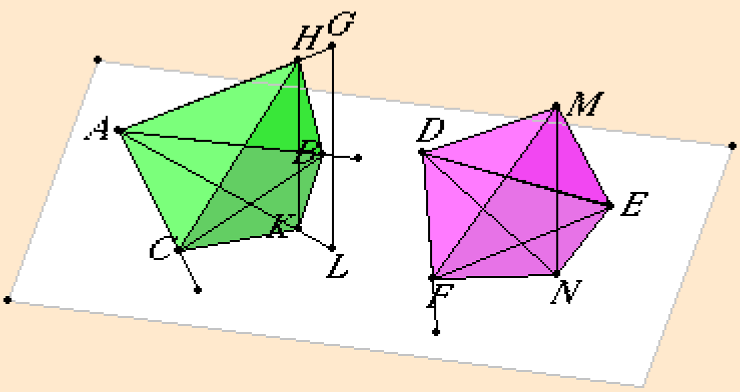
Euclid's Elements
Book XI
Proposition 35

If there are two equal plane angles, and on their vertices there are set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points are taken at random and perpendiculars are drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines are joined to the vertices of the original angles, then they contain with the elevated straight lines equal angles.

Let the angles *BAC* and *EDF* be two equal rectilinear angles, and from the points *A* and *D* let the elevated straight lines *AG* and *DM* be set up containing, with the original straight lines, equal angles respectively, namely, the angle *MDE* equal to the angle *GAB* and the angle *MDF* equal to the angle *GAC*.

Take the points *G* and *M* at random on *AG* and *DM*. Draw *GL* and *MN* from the points *G* and *M* perpendicular to the plane through *BA* and *AC* and the plane through *ED* and *DF*, and let them meet the planes at *L* and *N*. Join *LA* and *ND*. XI.11

I say that the angle *GAL* equals the angle *MDN*.



Make *AH* equal to *DM*, and draw *HK* through the point *H* parallel to *GL*. I.3 I.31

Since *GL* is perpendicular to the plane through *BA* and *AC*, therefore *HK* is also perpendicular to the plane through *BA* and *AC*. XI.8

Draw *KC*, *NF*, *KB* and *NE* from the points *K* and *N* perpendicular to the straight lines *AC*, *DF*, *AB*, and *DE*. Join *HC*, *CB*, *MF*, and *FE*. I.12

Since the square on *HA* equals the sum of the squares on *HK* and *KA*, and the sum of the squares on *KC* and *CA* equals the square on *KA*, therefore the square on *HA* equals the sum of the squares on *HK*, *KC*, and *CA*. I.47

But the square on *HC* equals the sum of the squares on *HK* and *KC*, therefore the square on *HA* equals the sum of the squares on *HC* and *CA*. Therefore the angle *HCA* is right. For the same reason the angle *DFM* is also right. I.47 I.48

Therefore the angle *ACH* equals the angle *DFM*. But the angle *HAC* equals the angle *MDF*. Therefore *MDF* and *HAC* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that opposite one of the equal angles, that is, *HA* equals *MD*, therefore they also have the remaining sides equal to the remaining sides respectively. Therefore *AC* equals *DF*. I.26

Similarly we can prove that *AB* also equals *DE.* Since then *AC* equals *DF*, and *AB* equals *DE,* the two sides *CA* and *AB* equal the two sides *FD* and *DE.* But the angle *CAB* also equals the angle *FDE,* therefore the base *BC* equals the base *EF*, the triangle equals the triangle, and the remaining angles to the remaining angles. Therefore the angle *ACB* equals the angle *DPE.* I.4

But the right angle *ACK* equals the right angle *DFN*, therefore the remaining angle *BCK* equals the remaining angle *EFN.* For the same reason the angle *CBK* also equals the angle *FEN.*

Therefore *BCK* and *EFN* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that adjacent to the equal angles, that is, *BC* equals *EF,* therefore the remaining sides equal the remaining sides. Therefore *CK* equals *FN.* I.26

But *AC* also equals *DF*, therefore the two sides *AC* and *CK* equal the two sides *DF* and *FN,* and they contain right angles. Therefore the base *AK* equals the base *DN.* I.4

And, since *AH* equals *DM,* therefore the square on *AH* equals the square on *DM.* But the sum of the squares on *AK* and *KH* equals the square on *AH*, for the angle *AKH* is right, and the sum of the squares on *DN* and *NM* equals the square on *DM,* for the angle *DNM* is right, therefore the sum of the squares on *AK* and *KH* equals the sum of the squares on *DN* and *NM.* And of these the square on *AK* equals the square on *DN,* therefore the remaining square on *KH* equals the square on *NM*. Therefore *HK* equals *MN.* I.47

And, since the two sides *HA* and *AK* equal the two sides *MD* and *DN* respectively, and the base *HK* equals the base *MN,* therefore the angle *HAK* equals the angle *MDN*. I.8

Therefore, *if there are two equal plane angles, and on their vertices there are set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points are taken at random and perpendiculars are drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines are joined to the vertices of the original angles, then they contain with the elevated straight lines equal angles.*

Corollary.

From this it is clear that, *if there are two equal plane angles, and if elevated straight lines set up on them which are equal and contain equal angles with the original straight lines respectively, then the perpendiculars drawn from their ends to the planes in which are the original angles equal one another.*

Q. E. D.

Guide

The situation described here occurs in the next proposition, and the corollary is used there to show two parallelepipeds have the same height.

Euclid's proof is quite long. Various authors have substituted shorter ones.

Next proposition: XI.36

Previous: XI.34

Book XI introduction

Select from Book XI

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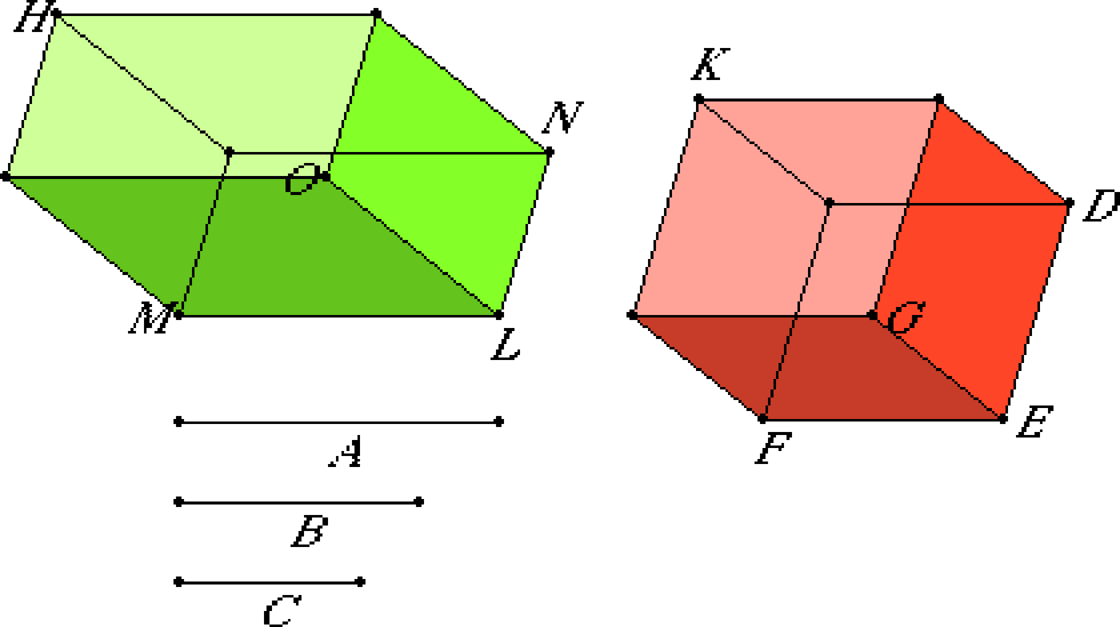
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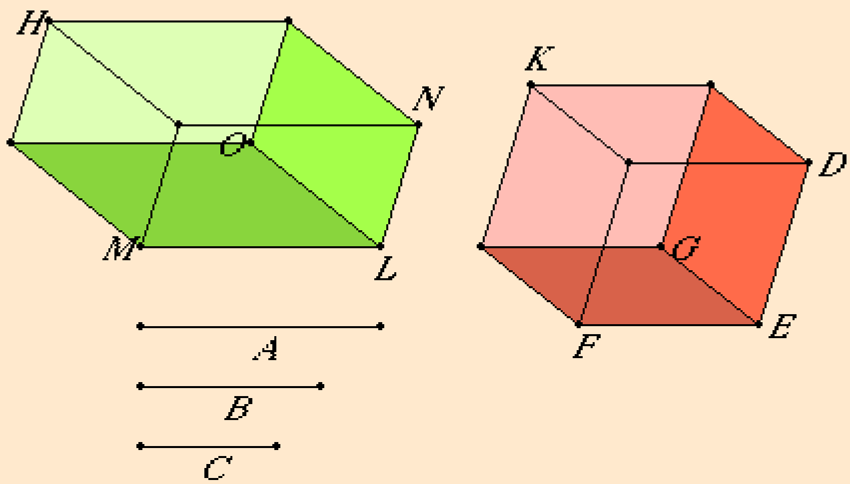
Euclid's Elements
Book XI
Proposition 36

If three straight lines are proportional, then the parallelepipedal solid formed out of the three equals the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.

Let *A, B,* and *C* be three straight lines in proportion, so that *A* is to *B* as *B* is to *C.*

I say that the solid formed out of *A, B,* and *C* equals the solid on *B* which is equilateral, but equiangular with the aforesaid solid.



Set out the solid angle at *E* contained by the angles *DEG, GEF,* and *FED,* and make each of the straight lines *DE, GE,* and *EF* equal to *B.* Complete the parallelepipedal solid *EK.* Make *LM* equal to *A.* Construct a solid angle at the point *L* on the straight line *LM* equal to the solid angle at *E,* namely that contained by *NLO, OLM,* and *MLN.* Make *LO* equal to *B,* and *LN* equal to *C.* I.3

Now, since *A* is to *B* as *B* is to *C,* while *A* equals *LM,* and *B* equals each of the straight lines *LO, ED,* and *C* to *LN,* therefore *LM* is to *EF* as *DE* is to *LN.* Thus the sides about the equal angles *NLM, DEF* are reciprocally proportional, therefore the parallelogram *MN* equals the parallelogram *DF.* VI.4

And, since the angles *DEF* and *NLM* are two plane rectilinear angles, and on them the elevated straight lines *LO* and *EG* are set up which equal one an other and contain equal angles with the original straight lines respectively, therefore the perpendiculars drawn from the points *G* and *O* to the planes through *NL* and *LM* and through *DE* and *EF* equal one another, therefore the solids *LH* and *EK* are of the same height. XI.35,Cor

But parallelepipedal solids on equal bases and of the same height equal one another, therefore the solid *HL* equals the solid *EK.* XI.31

And *LH* is the solid formed out of *A, B,* and *C,* and *EK* is the solid on *B,* therefore the parallelepipedal solid formed out of *A, B,* and *C* equals the solid on *B* which is equilateral, but equiangular with the aforesaid solid.

Therefore, *if three straight lines are proportional, then the parallelepipedal solid formed out of the three equals the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.*

Q.E.D.

Guide

This straightforward proof depends on viewing the two parallelepipeds as having the same height, which they do if the base of the first is taken to be the parallelogram *MN* and the base of the second the parallelogram *DF*.

Next proposition: XI.37

Previous: XI.35

Book XI introduction

Select from Book XI

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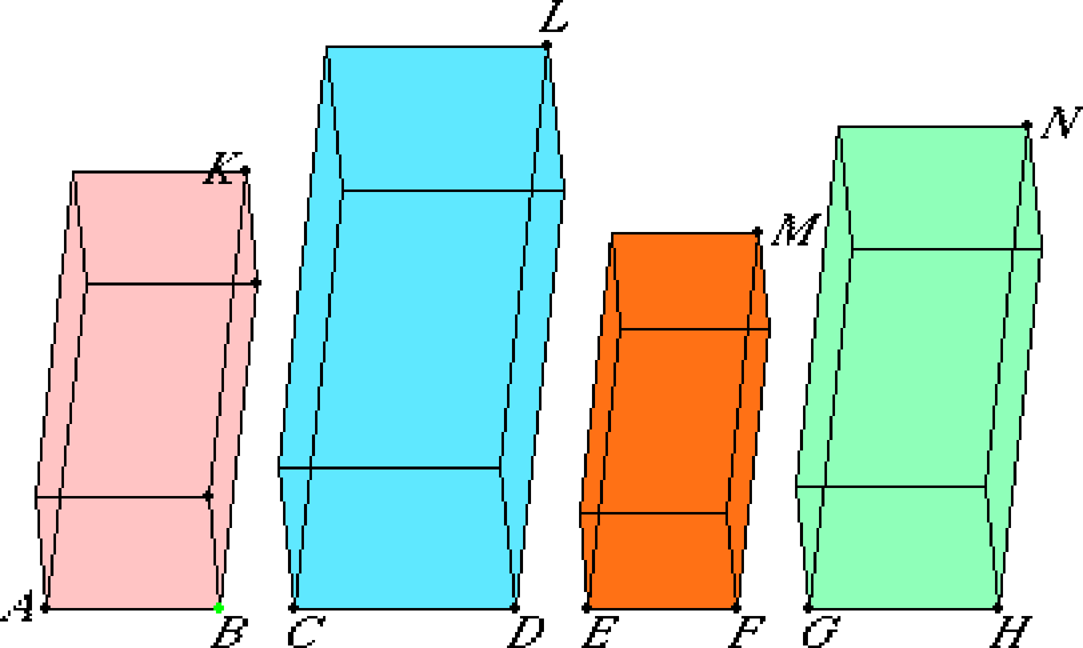

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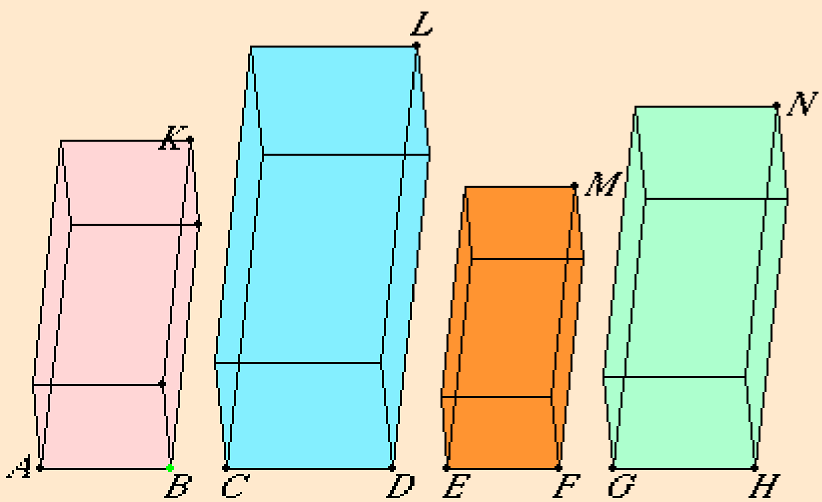
Euclid's Elements
Book XI
Proposition 37

If four straight lines are proportional, then parallelepipedal solids on them which are similar and similarly described are also proportional; and, if the parallelepipedal solids on them which are similar and similarly described are proportional, then the straight lines themselves are also proportional.

Let *AB*, *CD*, *EF*, and *GH* be four straight lines in proportion, so that *AB* is to *CD* as *EF* is to *GH*, and let there be described on *AB*, *CD*, *EF*, and *GH* the similar and similarly situated parallelepipedal solids *KA*, *LC*, *ME*, *NG*.

I say that *KA* is to *LC* as *ME* is to *NG*.



Since the parallelepipedal solid *KA* is similar to *LC*, therefore *KA* has to *LC* the ratio triplicate of that which *AB* has to *CD*. For the same reason *ME* has to *NG* the ratio triplicate of that which *EF* has to *GH*. XI.33

And *AB* is to *CD* as *EF* is to *GH*. Therefore *AK* is to *LC* as *ME* is to *NG*.

Next as the solid *AK* is to the solid *LC*, so let the solid *ME* be to the solid *NG*.

I say that the straight line *AB* is to *CD* as *EF* is to *GH*. Since, again, *KA* has to *LC* the ratio triplicate of that which *AB* has to *CD*, and *ME* also has to *NG* the ratio triplicate of that which *EF* has to *GH*, and *KA* is to *LC* as *ME* is to *NG*, therefore *AB* is to *CD* as *EF* is to *GH*. XI.33

Therefore, *if four straight lines are proportional, then parallelepipedal solids on them which are similar and similarly described are also proportional; and, if the parallelepipedal solids on them which are similar and similarly described are proportional, then the straight lines themselves are also proportional.*

Q. E. D.

Guide

This proposition completes the theory of volumes of parallelepipeds.

In the proof of this proposition it is assumed that two ratios are equal if and only if their triplicate ratios are equal. The required proof is long and detailed, but not difficult.

Next proposition: XI.38

Previous: XI.36

Book XI introduction

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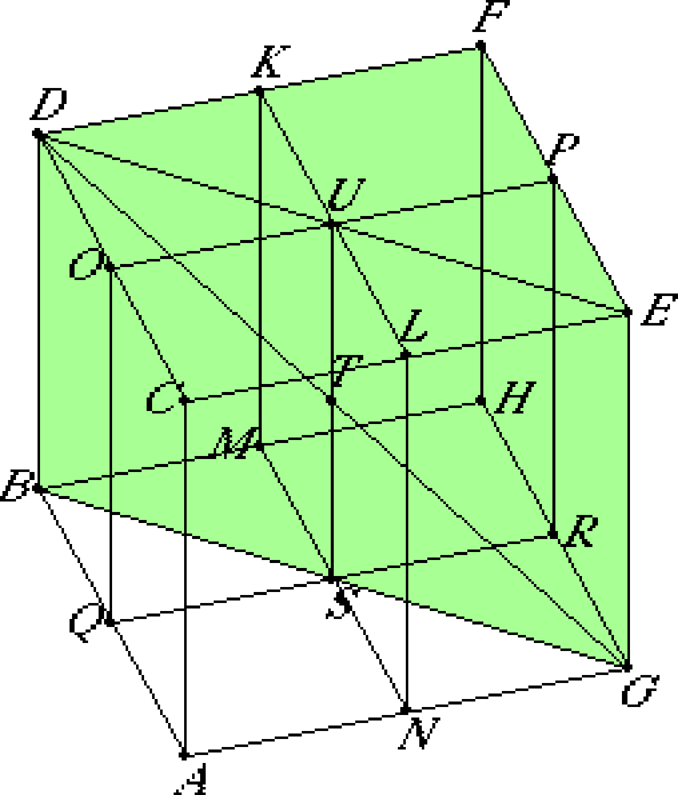

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Euclid's Elements
Book XI
Proposition 38

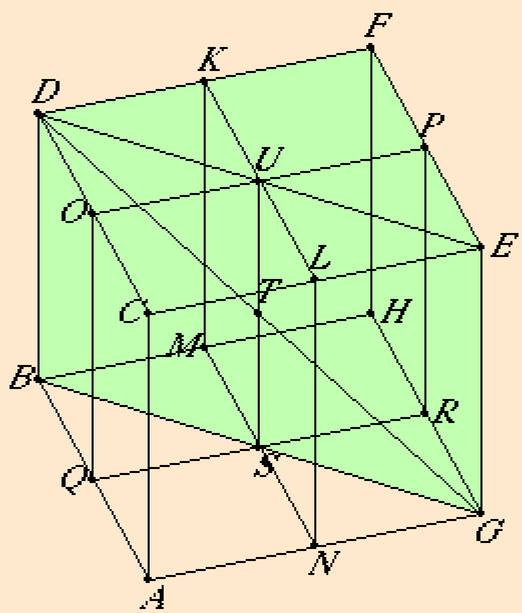
If the sides of the opposite planes of a cube are bisected, and the planes are carried through the points of section, then the intersection of the planes and the diameter of the cube bisect one another.

Let the sides of the opposite planes *CF* and *AH* of the cube *AF* be bisected at the points *K, L, M, N, O, Q, P,* and *R,* and through the points of section let the planes *KN* and *OR* be carried. Let *US* be the common section of the planes, and *DG* the diameter of the cube *AF.*

I say that *UT* equals *TS,* and *DT* equals *TG.*

Join *DU, UE, BS,* and *SG.*

Then, since *DO* is parallel to *PE,* therefore the alternate angles *DOU* and *UPE* equal one another. I.29



Since *DO* equals *PE,* and *OU* equals *UP,* and they contain equal angles, therefore the base *DU* equals the base *UE,* the triangle *DOU* equals the triangle *PUE,* and the remaining angles equal the remaining angles. Therefore the angle *OUD* equals the angle *PUE.* I.4

Therefore *DUE* is a straight line. For the same reason *BSG* is also a straight line, and *BS* equals *SG.* I.14

Now, since *CA* equals and is parallel to *DB,* while *CA* also equals and is parallel to *EG,* therefore *DB* equals and is parallel to *EG.* XI.9

And the straight lines *DE* and *BG* join their ends, therefore *DE* is parallel to *BG.* I.33

Therefore the angle *EDT* equals the angle *BGT,* for they are alternate, and the angle *DTU* equals the angle *GTS.* I.29 I.15

Therefore *DTU* and *GTS* are two triangles which have two angles equal to two angles and one side equal to one side, namely that opposite one of the equal angles, that is, *DU* equals *GS,* for they are the halves of *DE* and *BG,* therefore the remaining sides equal the remaining sides. Therefore *DT* equals *TG,* and *UT* equals *TS.* I.26

Therefore, *if the sides of the opposite planes of a cube are bisected, and the planes are carried through the points of section, then the intersection of the planes and the diameter of the cube bisect one another.*

Q. E. D.

Guide

This proposition takes care of a specific situation that occurs in proposition XIII.17. In XIII.17 a dodecahedron is constructed based on a cube, and the fact proven here in XI.38 is needed to show that the point *T* where *SU* intersects *DG* is the center of a sphere circumscribing the cube.

There are a couple of details missing from this proof. For instance, it is not shown that *KL* and *MN* actually lie in one plane, and that the line *SU* actually intersects the line *DG*.

Next proposition: XI.39

Previous: XI.37

Book XI introduction

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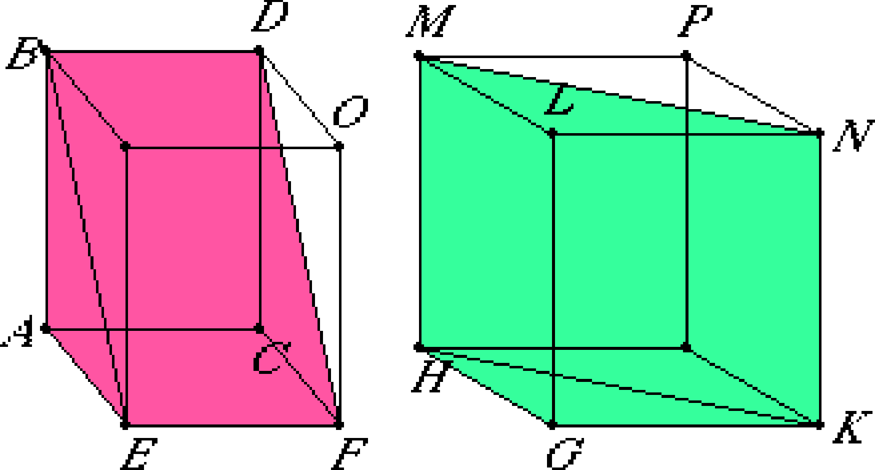

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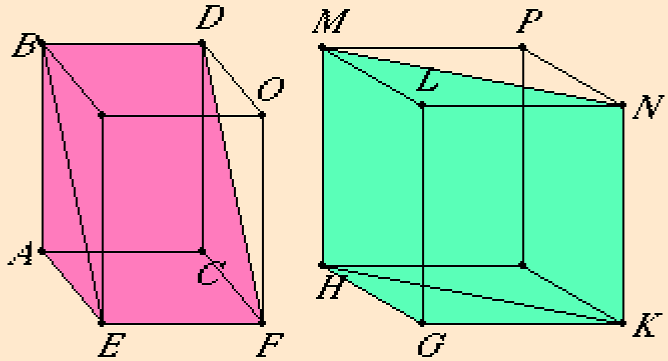
Euclid's Elements
Book XI
Proposition 39

If there are two prisms of equal height, and one has a parallelogram as base and the other a triangle, and if the parallelogram is double the triangle, then the prisms are equal.

Let *ABCDEF* and *GHKLMN* be two prisms of equal height, let one have the parallelogram *AF* as base, and the other the triangle *GHK,* and let the parallelogram *AF* be double the triangle *GHK.*

I say that the prism *ABCDEF* equals the prism *GHKLMN.*

Complete the solids *AO* and *GP*.



Since the parallelogram *AF* is double the triangle *GHK,* and the parallelogram *HK* is also double the triangle *GHK,* therefore the parallelogram *AF* equals the parallelogram *HK.* I.34

But parallelepipedal solids on equal bases of the same height equal one another, therefore the solid *AO* equals the solid *GP*. XI.31

And the prism *ABCDEF* is half of the solid *AO,* and the prism *GHKLMN* is half of the solid *GP,* therefore the prism *ABCDEF* equals the prism *GMKLMN.* XI.28

Therefore, *if there are two prisms of equal height, and one has a parallelogram as base and the other a triangle, and if the parallelogram is double the triangle, then the prisms are equal.*

Q. E. D.

Guide

This proposition is designed specifically to take care of a situation that occurs in propositions XII.3 and XII.4 on the way to proving XII.5 concerning the volume of a pyramid.

Both of the prisms in this proposition are triangular, but the base of the first is taken to be one of the parallelograms *ACFE* on its side while the base of the second is a triangular end *GHK.* To say that they have the height means the distance from the vertex *B* to the plane of the parallelogram *ACEF* is the same as the distance from the vertex *M* to the plane of the triangle *GHK.*

When the solids are completed, they are doubled to create two parallelepipeds of the same height and equal bases, which therefore are equal, and so are their halves, the original prisms.

Next book: Book XII introduction

Previous proposition: XI.38

Book XI introduction

Select from Book XI

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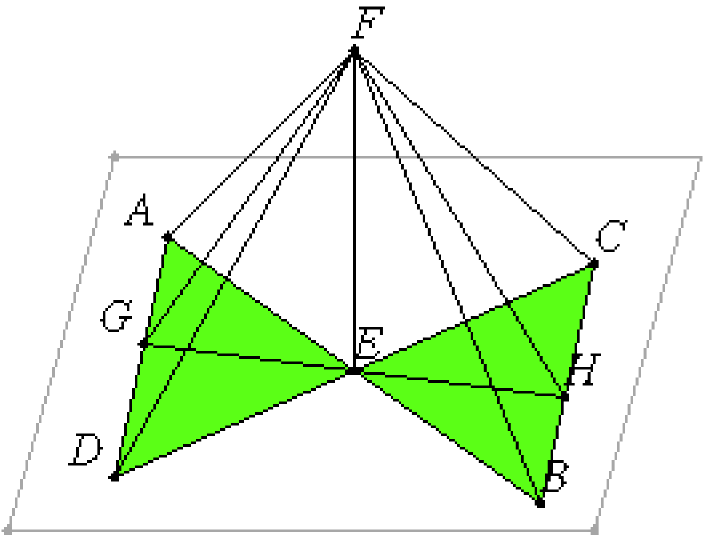
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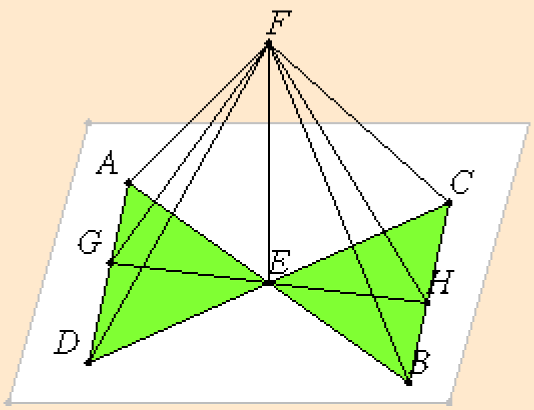
Euclid's Elements
Book XI
Proposition 4

If a straight line is set up at right angles to two straight lines which cut one another at their common point of section, then it is also at right angles to the plane passing through them.

For let a straight line *EF* be set up at right angles to the two straight lines *AB* and *CD* at *E*, the point at which the lines cut one another.

I say that *EF* is also at right angles to the plane passing through *AB* and *CD*.



Cut off *AE*, *EB*, *CE*, and *ED* equal to one another. Draw any straight line *GEH* across through *E* at random. Join *AD* and *CB*, and join *FA*, *FG*, *FD*, *FC*, *FH*, and *FB* from a point *F* taken at random on *EF*. XI.2 I.3

Now, since the two straight lines *AE* and *ED* equal the two straight lines *CE* and *EB* and contain equal angles, therefore the base *AD* equals the base *CB*, and the triangle *AED* equals the triangle *CEB*, so that the angle *DAE* equals the angle *EBC*. I.15 I.4

But the angle *AEG* also equals the angle *BEH*, therefore *AGE* and *BEH* are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely that adjacent to the equal angles, that is to say, *AE* equals *EB*. Therefore they also have the remaining sides equals to the remaining sides, that is, *GE* equals *EH*, and *AG* equals *BH*. I.15 I.26

And, since *AE* equals *EB*, while *FE* is common and at right angles, therefore the base *FA* equals the base *FB*. I.4

For the same reason, *FC* equals *FD*.

And, since *AD* equals *CB*, and *FA* also equals *FB*, the two sides *FA* and *AD* equal the two sides *FB* and *BC* respectively, and the base *FD* was proved equal to the the base *FC*, therefore the angle *FAD* also equals the angle *FBC*. I.8

And since, again, *AG* was proved equal to *BH*, and further, *FA* also equal to *FB*, the two sides *FA* and *AG* equal the two sides *FB* and *BH*, and the angle *FAG* was proved equal to the angle *FBH*, therefore the base *FG* equals the base *FH*. I.4

Again, since *GE* was proved equal to *EH*, and *EF* is common, the two sides *GE* and *EF* equal the two sides *HE* and *EF*, and the base *FG* equals the base *FH*, therefore the angle *GEF* equals the angle *HEF*. I.8

Therefore each of the angles *GEF* and *HEF* is right.

Therefore *FE* is at right angles to *GH* drawn at random through *E*.

Similarly we can prove that *FE* also makes right angles with all the straight lines which meet it and are in the plane of reference.

But a straight line is at right angles to a plane when it makes right angles with all the straight lines which meet it and are in that same plane, therefore *FE* is at right angles to the plane of reference. XI.Def.3

But the plane of reference is the plane through the straight lines *AB* and *CD*.

Therefore *FE* is at right angles to the plane through *AB* and *CD*.

Therefore *if a straight line is set up at right angles to two straight lines which cut one another at their common point of section, then it is also at right angles to the plane passing through them.*

Q. E. D.

Guide

This proposition says that if a line passing through a point is perpendicular to two other lines passing through that point, then it is perpendicular to all the lines which pass through that point and which lie in the plane of those two other lines.

After the preceding three dubious proofs, this one is a relief. It is a little long, but it is clear.

Near the beginning of the proof, proposition XI.2 is needed to conclude that the two lines *AB* and *CD* determine a plane. The line *GH* is to be any line that passes through *E* and lies in that plane. Then, by XI.2 again, the lines *AD* and *BC* lie in this plane.

Use of this proposition

This proposition is used frequently starting with the proof of the next proposition.

Next proposition: XI.5

Previous: XI.3

Book XI introduction

Select from Book XI

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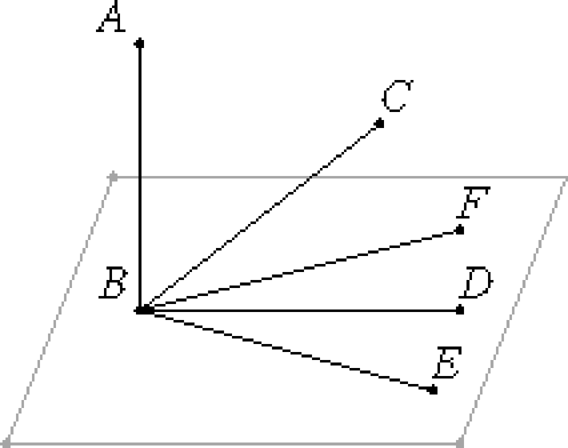
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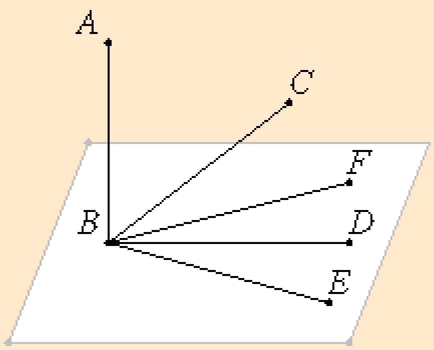
Euclid's Elements
Book XI
Proposition 5

If a straight line is set up at right angles to three straight lines which meet one another at their common point of section, then the three straight lines lie in one plane.

Let a straight line *AB* be set up at right angles to the three straight lines *BC*, *BD,* and *BE* at their intersection *B.*

I say that *BC*, *BD,* and *BE* lie in one plane.



For suppose that they do not, but, if possible, let *BD* and *BE* lie in the plane of reference and *BC* in one more elevated. Produce the plane through *AB* and *BC.* XI.3

It intersects the plane of reference in a straight line. Let the intersection be *BF.* Therefore the three straight lines *AB, BC,* and *BF* lie in one plane, namely that drawn through *AB* and *BC.*

Now, since *AB* is at right angles to each of the straight lines *BD* and *BE*, therefore *AB* is also at right angles to the plane through *BD* and *BE.* XI.4

But the plane through *BD* and *BE* is the plane of reference, therefore *AB* is at right angles to the plane of reference.

Thus *AB* also makes right angles with all the straight lines which meet it and lie in the plane of reference. XI.Def.3

But *BF,* which is the plane of reference, meets it, therefore the angle *ABF* is right. And, by hypothesis, the angle *ABC* is also right, therefore the angle *ABF* equals the angle *ABC,* and they lie in one plane, which is impossible.

Therefore the straight line *BC* is not in a more elevated plane. Therefore the three straight lines *BC, BC,* and *BE* lie in one plane.

Therefore, *if a straight line is set up at right angles to three straight lines which meet one another at their common point of section, then the three straight lines lie in one plane.*

Q. E. D.

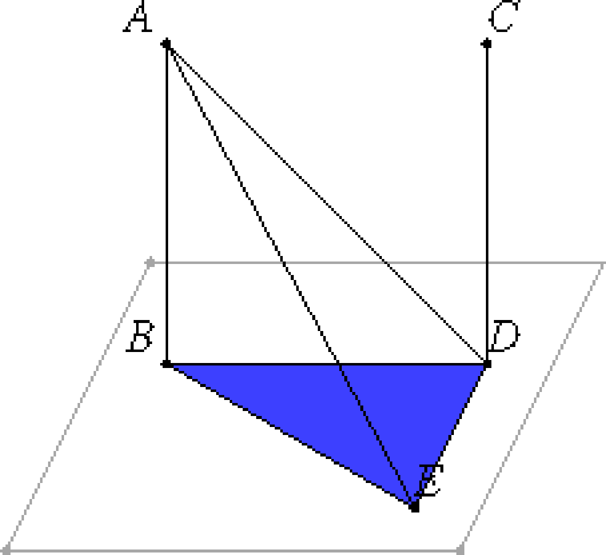
Guide

This proposition is used in the proof of the next.

Next proposition: XI.6

Select from Book XI

Previous: XI.4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Euclid's Elements

Book XI

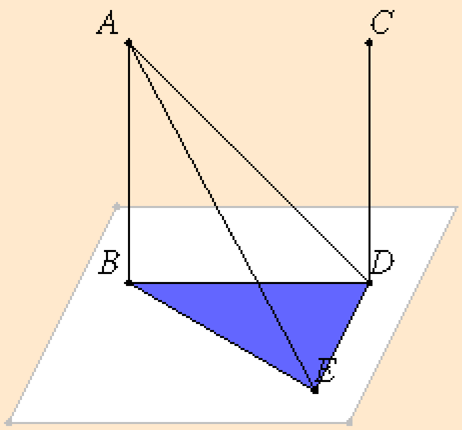
Proposition 6

If two straight lines are at right angles to the same plane, then the straight lines are parallel.

Let the two straight lines *AB* and *CD* be at right angles to the plane of reference.

I say that *AB* is parallel to *CD.*

Let them meet the plane of reference at the points *B* and *D.*



Join the straight line *BD.* Draw *DE* in the plane of reference at right angles to *BD,* and make *DE* equal to *AB.* I.11 I.3

Now, since *AB* is at right angles to the plane of reference, it also makes right angles with all the straight lines which meet it and lie in the plane of reference. XI.Def.3

But each of the straight lines *BD* and *BE* lies in the plane of reference and meets *AB,* therefore each of the angles *ABD* and *ABE* is right. For the same reason each of the angles *CDB* and *CDE* is also right.

And since *AB* equals *DE,* and *BD* is common, therefore the two sides *AB* and *BD* equal the two sides *ED* and *DB.* And they include right angles, therefore the base *AD* equals the base *BE* I.4

And, since *AB* equals *DE* while *AD* equals *BE,* the two sides *AB* and *BE* equal the two sides *ED* and *DA,* and *AE* is their common base, therefore the angle *ABE* equals the angle *EDA.* I.8

But the angle *ABE* is right, therefore the angle *EDA* is also right. Therefore *ED* is at right angles to *DA.*

But it is also at right angles to each of the straight lines *BD* and *DC,* therefore *ED* is set up at right angles to the three straight lines *BD, DA,* and *DC* at their intersection. Therefore the three straight lines *BD, DA,* and *DC* lie in one plane. XI.5

But in whatever plane *DB* and *DA* lie, *AB* also lies, for every triangle lies in one plane. XI.2

Therefore the straight lines *AB, BD,* and *DC* lie in one plane. And each of the angles *ABD* and *BDC* is right, therefore *AB* is parallel to *CD.* I.28

Therefore, *if two straight lines are at right angles to the same plane, then the straight lines are parallel.*

Q.E.D.

Guide

Euclid does not consider the possibility that the two lines meet the plane at one point, but that possibility can easily be eliminated. Indeed, that is the statement of proposition XI.13 which, therefore, should preceed this one.

A converse of this proposition is XI.8.

Use of this proposition

This proposition is used in the proofs of propositions XII.17, XIII.16, and XIII.17.

Next proposition: XI.7

Previous: XI.5

Book XI introduction

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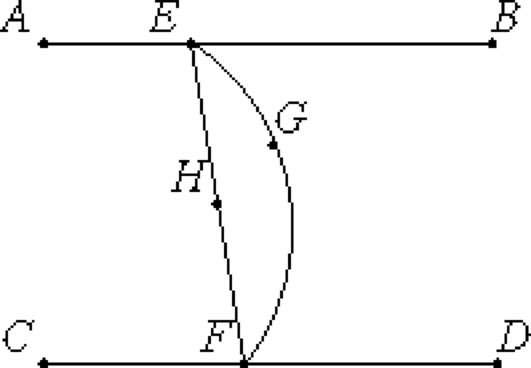
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Euclid's Elements

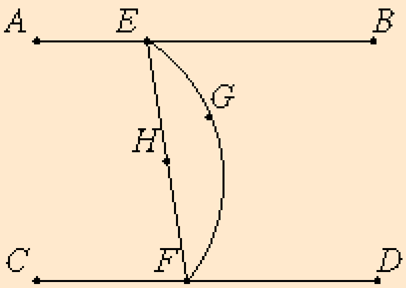
Book XI

Proposition 7

If two straight lines are parallel and points are taken at random on each of them, then the straight line joining the points is in the same plane with the parallel straight lines.

Let *AB* and *CD* be two parallel straight lines, and let points *E* and *F* be taken at random on them respectively.

I say that the straight line joining the points *E* and *F* lies in the same plane with the parallel straight lines.



For suppose it is not, but, if possible, let it be in a more elevated plane as *EGF*. Draw a plane through *EGF*. Its intersection with the plane of reference is a straight line. Let it be *EF*. XI.3

Therefore the two straight lines *EGF* and *EF* enclose an area, which is impossible. Therefore the straight line joined from *E* to *F* is not in a plane more elevated. Therefore the straight line joined from *E* to *F* lies in the plane through the parallel straight lines *AB* and *CD*.

Therefore, *if two straight lines are parallel and points are taken at random on each of them, then the straight line joining the points is in the same plane with the parallel straight lines.*

Q. E. D.

Guide

The existence of this proposition is a good argument that Euclid's definition I.Def.7 of a plane (it lies evenly with the straight lines on itself) does not mean that if two points lie in a plane, then the line joining them also lies in the plane. If it did, then this proposition would be true by definition, and no proof would be required at all.

Note that this proof assumes that every line lies in a plane, a conclusion that has not been justified.

Use of this proposition

This proposition is used in the proof of the next as well as proposition XII.17.

Next proposition: XI.8

Previous: XI.6

Book XI introduction

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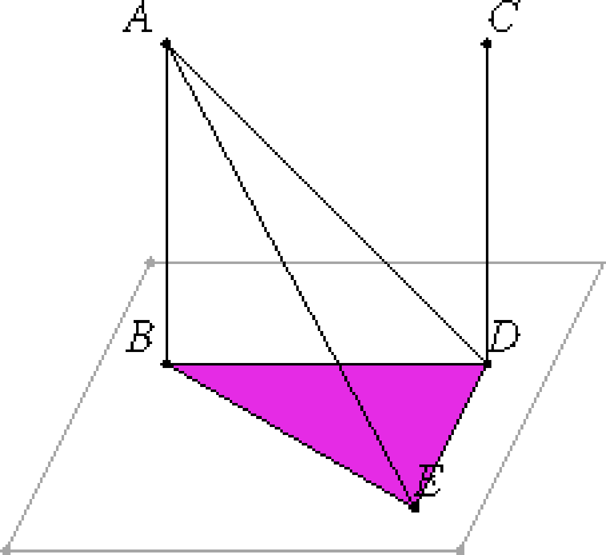

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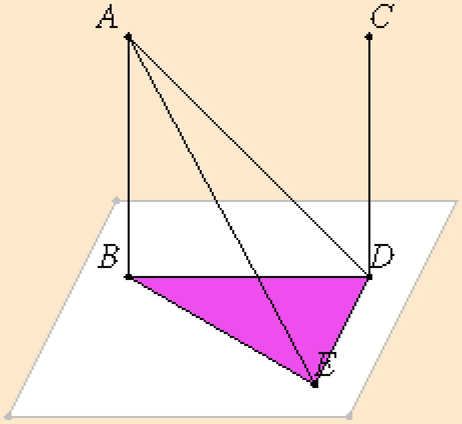
Euclid's Elements
Book XI
Proposition 8

If two straight lines are parallel, and one of them is at right angles to any plane, then the remaining one is also at right angles to the same plane.

Let *AB* and *CD* be two parallel straight lines, and let one of them, *AB*, be at right angles to the plane of reference.

I say that the remaining one, *CD*, is also at right angles to the same plane.



Let *AB* and *CD* meet the plane of reference at the points *B* and *D.* Join *BD.* Then *AB*, *CD*, and *BD* lie in one plane. XI.7

Draw *DE* in the plane of reference at right angles to *BD*, make *DE* equal to *AB*, and join *BE*, *AE*, and *AD* I.11 I.3

Now, since *AB* is at right angles to the plane of reference, therefore *AB* is also at right angles to all the straight lines which meet it and lie in the plane of reference. Therefore each of the angles *ABD* and *ABE* is right. XI.Def.3

And, since the straight line *BD* falls on the parallels *AB* and *CD*, therefore the sum of the angles *ABD* and *CDB* equals two right angles. I.29

But the angle *ABD* is right, therefore the angle *CDB* is also right. Therefore *CD* is at right angles to *BC*.

And since *AB* equals *DE*, and *BD* is common, the two sides *AB* and *BD* equal the two sides *ED* and *DB*, and the angle *ABD* equals the angle *EDB*, for each is right, therefore the base *AD* equals the base *BE*. I.4

And since *AB* equals *DE*, and *BE* equals *AD*, the two sides *AB* and *BE* equal the two sides *ED* and *DA* respectively, and *AE* is their common base, therefore the angle *ABE* equals the angle *EDA*. I.8

But the angle *ABE* is right, therefore the angle *EDA* is also right. Therefore *ED* is at right angles to *AD*. But it is also at right angles to *DB*. Therefore *ED* is also at right angles to the plane through *BD* and *DA*. XI.4

Therefore *ED* also makes right angles with all the straight lines which meet it and lie in the plane through *BD* and *DA*. But *DC* lies in the plane through *BD* and *DA* inasmuch as *AB* and *BD* lie in the plane through *BD* and *DA*, and *DC* also lies in the plane in which *AB* and *BD* lie.

Therefore *ED* is at right angles to *DC*, so that *CD* is also at right angles to *DE*. But *CD* is also at right angles to *BD*. Therefore *CD* is set up at right angles to the two straight lines *DE* and *DB* so that *CD* is also at right angles to the plane through *DE* and *DB*. XI.4

But the plane through *DE* and *DB* is the plane of reference, therefore *CD* is at right angles to the plane of reference.

Therefore, *if two straight lines are parallel, and one of them is at right angles to any plane, then the remaining one is also at right angles to the same plane.*

Q.E.D.

Guide

This is a converse of proposition XI.6.

This proposition is used in the proof of the next one as well as several others in this book.

Next proposition: XI.9

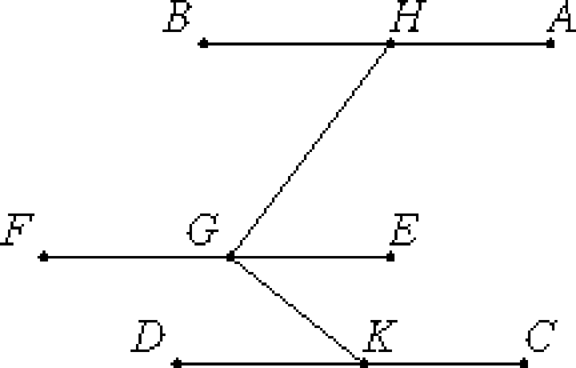
Previous: XI.7

Book XI introduction

Select from Book XI

Select book

Select topic


B
H
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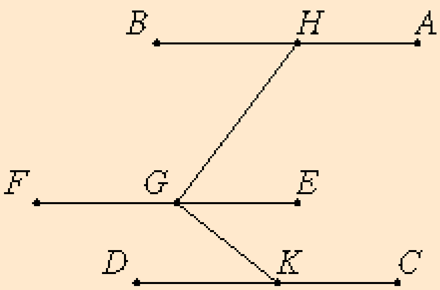
C

Euclid's Elements
Book XI
Proposition 9

Straight lines which are parallel to the same straight line but do not lie in the same plane with it are also parallel to each other.

Let each of the straight lines *AB* and *CD* be parallel to *EF*, but not in the same plane with it.

I say that *AB* is parallel to *CD.*



Let a point *G* be taken at random on *EF*, and from it draw *GH* in the plane through *EF* and *AB* at right angles to *EF*, and *GK* in the plane through *EF* and *CD* again at right angles to *EF*. I.11

Now, since *EF* is at right angles to each of the straight lines *GH* and *GK*, therefore *EF* is also at right angles to the plane through *GH* and *GK.* XI.4

And *EF* is parallel to *AB*, therefore *AB* is also at right angles to the plane through *HG* and *GK.* XI.8

For the same reason *CD* is also at right angles to the plane through *HG* and *GK.* Therefore each of the straight lines *AB* and *CD* is at right angles to the plane through *HG* and *GK.*

But if two straight lines are at right angles to the same plane, then the straight lines are parallel. Therefore *AB* is parallel to *CD.* XI.6

Therefore, *straight lines which are parallel to the same straight line but do not lie in the same plane with it are also parallel to each other*.

Q. E. D.

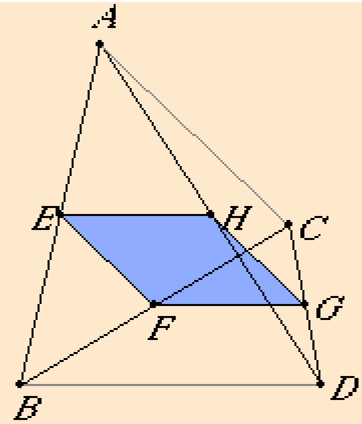
Guide

Note that this proposition is the three-dimensional analogue to proposition I.30.

The Varignon parallelogram of space quadrilaterals

Consider a quadrilateral *ABCD* whose four vertices may or may not lie in a plane. Let *E*, *F*, *G*, and *H* be the midpoints of the sides *AB*, *BC*, *CD*, and *DA*, respectively. Then the quadrilateral *EFGH* lies in a plane and is a parallelogram, called the Varignon parallelogram. Varignon (1654-1722) showed the area of a planar quadrilateral is twice the area of this parallelogram.

The proof that *EFGH* is a parallelogram relies on this proposition XI.9 to show the sides are parallel, since it is readily shown that both *EF* and *HG* are parallel to the line *AC*, and both *FG* and *EH* are parallel to the line *BD.*



As a corollary, it follows that the lines joining the midpoints of an arbitrary quadrilateral are concurrent and bisect each other, even if the four sides of the quadrilateral do not lie in a plane. (These are the lines *EG* and *FH* which are not drawn in the diagram.)

Use of this proposition

This proposition is used in the proof of the next proposition as well as others in this and the next book.

Next proposition: XI.10

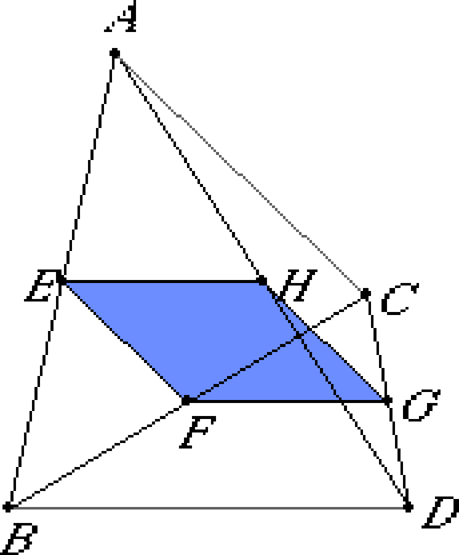
Previous: XI.8

Book XI introduction

Select from Book XI

Select book

Select topic



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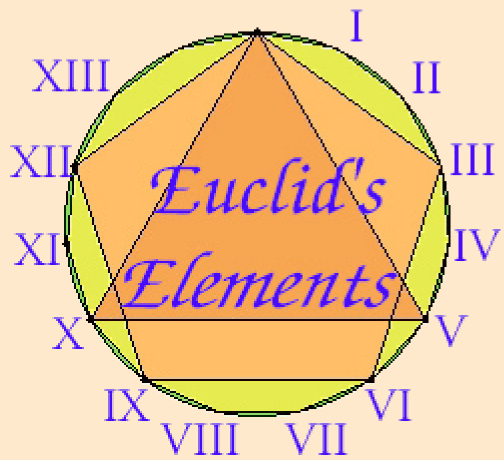
Book XII



Book XII

Table of contents

Propositions

Proposition 1.
Similar polygons inscribed in circles are to one another as the squares on their diameters.

Proposition 2.
Circles are to one another as the squares on their diameters.

Lemma for XII.2.

Proposition 3.
Any pyramid with a triangular base is divided into two pyramids equal and similar to one another, similar to the whole, and having triangular bases, and into two equal prisms, and the two prisms are greater than half of the whole pyramid.

Proposition 4.
If there are two pyramids of the same height with triangular bases, and each of them is divided into two pyramids equal and similar to one another and similar to the whole, and into two equal prisms, then the base of the one pyramid is to the base of the other pyramid as all the prisms in the one pyramid are to all the prisms, being equal in multitude, in the other pyramid.

Lemma for XII.4.

Proposition 5.
Pyramids of the same height with triangular bases are to one another as their bases.

Proposition 6.
Pyramids of the same height with polygonal bases are to one another as their bases.

Proposition 7.
Any prism with a triangular base is divided into three pyramids equal to one another with triangular bases.

Corollary. Any pyramid is a third part of the prism with the same base and equal height.

Proposition 8.
Similar pyramids with triangular bases are in triplicate ratio of their corresponding sides.

Corollary. Similar pyramids with polygonal bases are also to one another in triplicate ratio of their corresponding sides.

Proposition 9.
In equal pyramids with triangular bases the bases are reciprocally proportional to the heights; and those pyramids are equal in which the bases are reciprocally proportional to the heights.

Proposition 10.
Any cone is a third part of the cylinder with the same base and equal height.

Proposition 11.
Cones and cylinders of the same height are to one another as their bases.

Proposition 12.
Similar cones and cylinders are to one another in triplicate ratio of the diameters of their bases.

Proposition 13.
If a cylinder is cut by a plane parallel to its opposite planes, then the cylinder is to the cylinder as the axis is to the axis.

Proposition 14.
Cones and cylinders on equal bases are to one another as their heights.

Proposition 15.
In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.

Proposition 16.
Given two circles about the same center, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle.

Proposition 17.
Given two spheres about the same center, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Corollary to XII.17.

Proposition 18.
Spheres are to one another in triplicate ratio of their respective diameters.

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Index of /~djoyce/java/elements/bookXII

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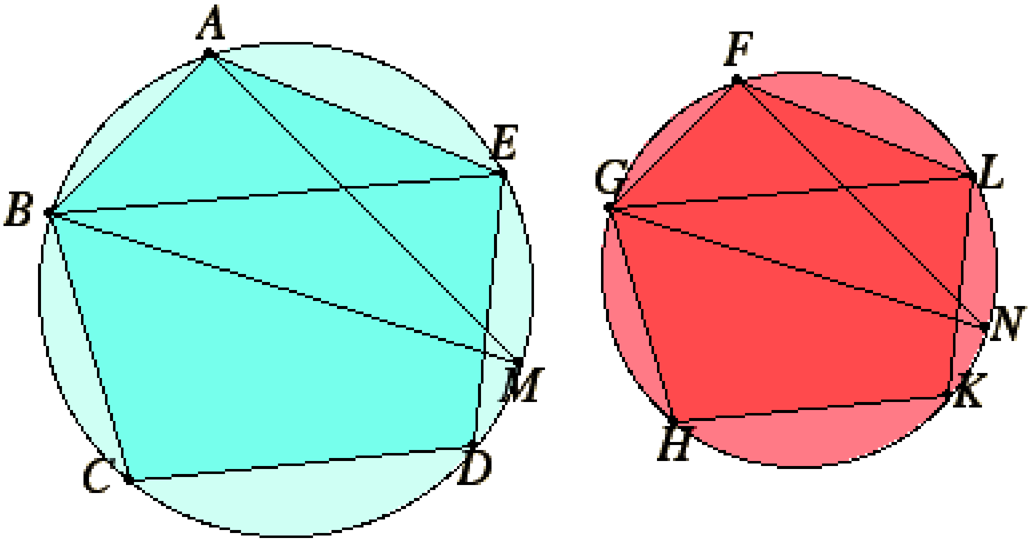
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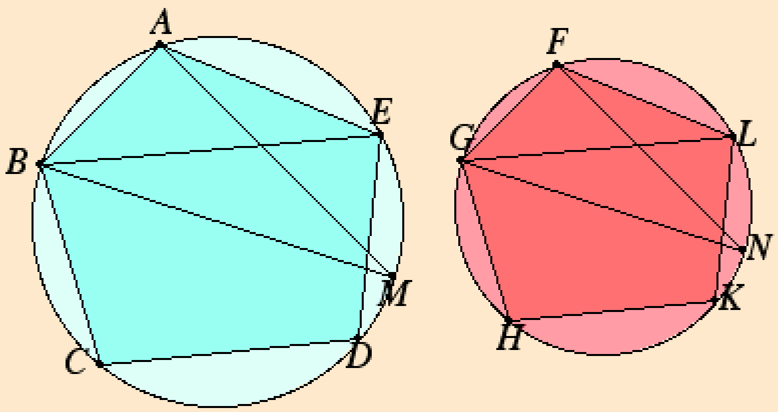
Euclid's Elements
Book XII
Proposition 1

Similar polygons inscribed in circles are to one another as the squares on their diameters.

Let *ABC* and *FGH* be circles, let *ABCDE* and *FGHKL* be similar polygons inscribed in them, and let *BM* and *GN* be diameters of the circles.

I say that the square on *BM* is to the square on *GN* as the polygon *ABCDE* is to the polygon *FGHKL.*

Join *BE*, *AM*, *GL*, and *FN.*



Now, since the polygon *ABCDE* is similar to the polygon *FGHKL,* therefore the angle *BAE* equals the angle *GFL,* and *BA* is to *AE* as *GF* is to *FL.* VI.Def.1

Thus *BAE* and *GFL* are two triangles which have one angle equal to one angle, namely the angle *BAE* equal to the angle *GFL,* and the sides about the equal angles proportional, therefore the triangle *ABE* is equiangular with the triangle *FGL.* Therefore the angle *AEB* equals the angle *FLG.* VI.6

But the angle *AEB* equals the angle *AMB,* for they stand on the same circumference, and the angle *FLG* equals the angle *FNG,* therefore the angle *AMB* also equals the angle *FNG.* III.27

But the right angle *BAM* also equals the right angle *GFN,* therefore the remaining angle equals the remaining angle. Therefore the triangle *ABM* is equiangular with the triangle *FGN.* III.31 I.32

Therefore, proportionally *BM* is to *GN* as *BA* is to *GF.* VI.4

But the ratio of the square on *BM* to the square on *GN* is duplicate of the ratio of *BM* to *GN,* and the ratio of the polygon *ABCDE* to the polygon *FGHKL* is duplicate of the ratio of *BA* to *GF.* VI.20

Therefore the square on *BM* is to the square on *GN* as the polygon *ABCDE* is to the polygon *FGHKL.*

Therefore, *similar polygons inscribed in circles are to one another as the squares on their diameters.*

Q.E.D.

Guide

Proposition VI.20 states that the ratio of similar polygons is duplicate the ratio of their corresponding sides, so all that is needed is that the corresponding sides are proportional to the diameters of the circumscribed circles, a result that constitutes the bulk of the straightforward proof.

This proposition is in preparation for the next in which it is shown that circles are proportional to the squares on their diameters. The connection is that the circles can be arbitrarily closely approximated by polygons, so that if the polygons are proportional to the squares, then so will the circles be proportional to the squares. The difficulty in that proof coming up is to make that argument rigorous.

Next proposition: XII.2

Book XII introduction

Select from Book XII

Select book

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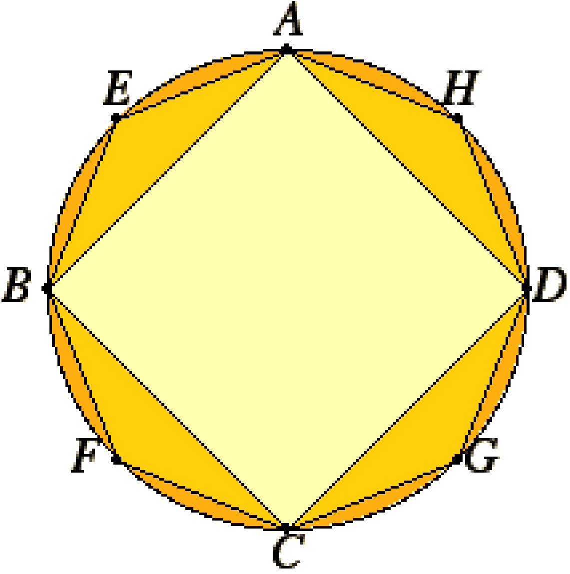
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Euclid's Elements
Book XII
Proposition 10

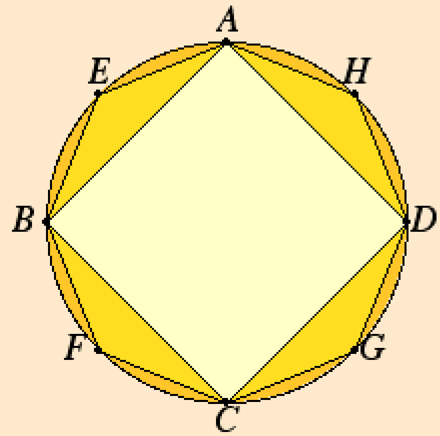
Any cone is a third part of the cylinder with the same base and equal height.

Let a cone have the same base, namely the circle *ABCD*, with a cylinder and equal height.

I say that the cone is a third part of the cylinder, that is, that the cylinder is triple the cone.

For if the cylinder is not triple the cone, then the cylinder will be either greater than triple or less than triple the cone.

First let it be greater than triple.



Inscribe the square *ABCD* in the circle *ABCD.* Then the square *ABCD* is greater than half of the circle *ABCD.* From the square *ABCD* set up a prism of equal height with the cylinder. IV.6

Then the prism so set up is greater than the half of the cylinder, for if we also circumscribe a square about the circle *ABCD,* the square inscribed in the circle *ABCD* is half of that circumscribed about it, and the solids set up from them are parallelepipedal prisms of equal height, while parallelepipedal solids of the same height are to one another as their bases, therefore also the prism set up on the square *ABCD* is half of the prism set up from the square circumscribed about the circle *ABCD,* and the cylinder is less than the prism set up from the square circumscribed about the circle *ABCD,* therefore the prism set up from the square *ABCD* and of equal height with the cylinder is greater than the half of the cylinder. IV.7 XI.32 XI.28 or XII.6 and XII.7.Cor

Bisect the circumferences *AB, BC, CD,* and *DA* at the points *E, F, G,* and *H,* and join *AE, EB, BF, FC, CG, GD, DH,* and *HA.* Then each of the triangles *AEB, BFC, CGD,* and *DHA* is greater than the half of that segment of the circle *ABCD* about it, as we proved before. XII.2

On each of the triangles *AEB, BFC, CGD,* and *DHA* set prisms up of equal height with the cylinder. Then each of the prisms so set up is greater than the half part of that segment of the cylinder about it, for if we draw through the points *E, F, G,* and *H* parallels to *AB, BC, CD,* and *DA,* complete the parallelograms on *AB, BC, CD,* and *DA,* and set up from them parallelepipedal solids of equal height with the cylinder, then the prisms on the triangles *AEB, BFC, CGD,* and *DHA* are halves of the several solids set up, and the segments of the cylinder are less than the parallelepipedal solids set up, hence also the prisms on the triangles *AEB, BFC, CGD,* and *DHA* are greater than half of the segments of the cylinder about them. I.31

Thus, bisecting the circumferences that are left, joining straight lines, setting up on each of the triangles prisms of equal height with the cylinder, and doing this repeatedly, we shall leave some segments of the cylinder which are less than the excess by which the cylinder exceeds the triple the cone. X.1

Let such segments be left, and let them be *AE, EB, BF, FC, CG, GD, DH,* and *HA.* Therefore the remainder, the prism with polygonal base *AEBFCGDH* and the same height as that of the cylinder, is greater than triple the cone.

But the prism with polygonal base *AEBFCGDH* and the same height as that of the cylinder is triple the pyramid with polygonal base *AEBFCGDH* and the same vertex as that of the cone. Therefore the pyramid with the polygonal base *AEBFCGDH* and the same vertex as that of the cone is greater than the cone with circular base *ABCD.* XII.7,Cor

But it is also less, for it is enclosed by it, which is impossible.

Therefore the cylinder is not greater than triple the cone.

I say next that neither is the cylinder less than triple the cone,

For, if possible, let the cylinder be less than triple the cone. Therefore, inversely, the cone is greater than a third part of the cylinder.

Inscribe the square *ABCD* in the circle *ABCD.* Therefore the square *ABCD* is greater than the half of the circle *ABCD.* IV.6

Now set up from the square *ABCD* a pyramid with the same vertex as the cone. Therefore the pyramid so set up is greater than half of the cone, for, as we proved before, if we circumscribe a square about the circle, then the square *ABCD* is half of the square circumscribed about the circle, and if we set up from the squares parallelepipedal solids of equal height with the cone, which are also called prisms, then the solid set up from the square *ABCD* is half of that set up from the square circumscribed about the circle, for they are to one another as their bases. XI.32

Hence the thirds of them are also in that ratio. Therefore the pyramid with the square base *ABCD* is half of the pyramid set up from the square circumscribed about the circle.

And the pyramid set up from the square about the circle is greater than the cone, for it encloses it.

Therefore the pyramid with the square base *ABCD* and the same vertex as that of the cone is greater than the half of the cone.

Bisect the circumferences *AB*, *BC, CD*, and *DA* at the points *E*, *F*, *G*, and *H*, and join *AE, EB, BF, FC, CG, GD, DH,* and *HA* be joined. Then each of the triangles *AEB, BFC, CGD,* and *DHA* is greater than the half part of that segment of the circle *ABCD* about it.

Now, on each of the triangles *AEB, BFC, CGD,* and *DHA* set pyramids up with the same vertex as the cone. Therefore each of the pyramids so set up is, in the same manner, greater than the half part of that segment of the cone about it.

Thus, by bisecting the circumferences that are left, joining straight lines, setting up pyramids on each of the triangles with the same vertex as the cone, and doing this repeatedly, we shall leave some segments of the cone which will be less than the excess by which the cone exceeds the third part of the cylinder. X.1

Let such be left, and let them be the segments on *AE, EB, BF, FC, CG, GD, DH,* and *HA.* Therefore the remainder, the pyramid with the polygonal base *AEBFCGDH* and the same vertex as that of the cone, is greater than a third part of the cylinder.

But the pyramid with the polygonal *AEBFCGDH* base and the same vertex as that of the cone is a third part of the prism with the polygonal base *AEBFCGDH* and the same height as that of the cylinder, therefore the prism with the polygonal base *AEBFCGDH* and the same height as that of the cylinder is greater than the cylinder with the circular base *ABCD.*

But it is also less, for it is enclosed by it, which is impossible.

Therefore the cylinder is not less than triple the cone.

But it was proved that neither is it greater than triple. Therefore the cylinder is triple the cone, hence the cone is a third part of the cylinder.

Therefore, *any cone is a third part of the cylinder with the same base and equal height.*

Q.E.D.

Guide

This and the next five propositions deal with the volumes of cones and cylinders. This proposition is fundamental in that it relates the volume of a cone to that of the circumscribed cylinder so that whatever is said about the volumes cylinder can be converted into a statement about volumes of cones and vice versa.

In XII.11, the next proposition, cones of the same height are shown to be proportional to their bases, and therefore cylinders of the same height are proportional to their bases. In XII.12 similar cones are shown to be in the triplicate ratio of the diameters of their bases, therefore the analogous statement holds for cylinders. In XII.14 cylinders on equal bases are shown to be proportional to their heights, therefore the analogous statement holds for cones. And in XII.15 it is shown that equal cylinders are those whose bases are reciprocally proportional to their heights, and as Euclid says, "the same it true for the cones also."

Next proposition: XII.11

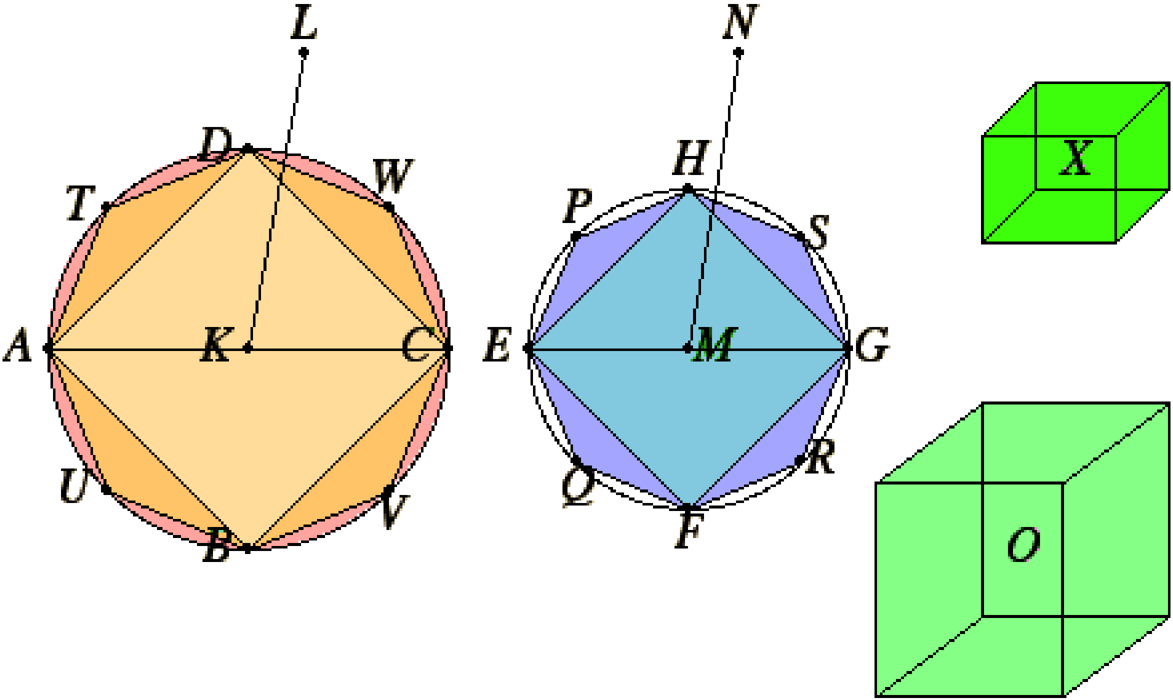
Previous: XII.9

Book XII introduction

Select from Book XII

Select book

Select topic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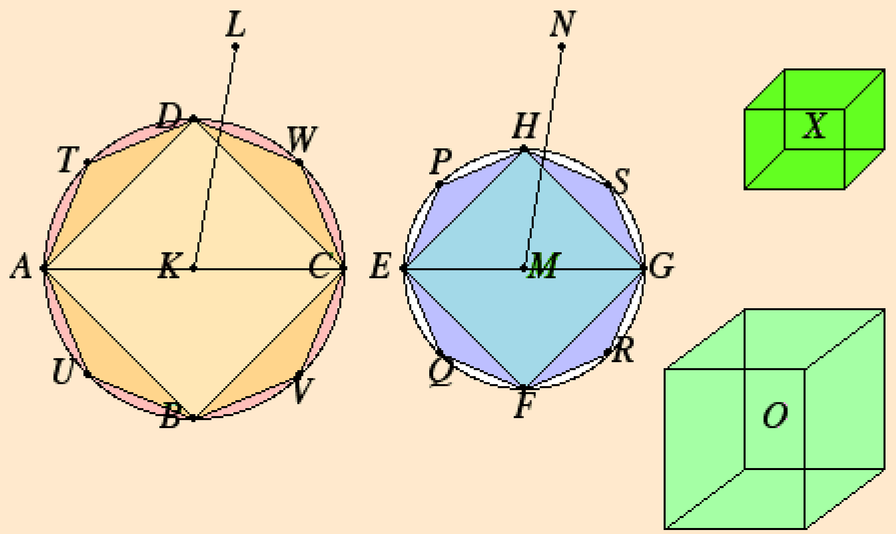
Euclid's Elements
Book XII
Proposition 11

Cones and cylinders of the same height are to one another as their bases.

Let there be cones and cylinders of the same height, let the circles *ABCD* and *EFGH* be their bases, *KL* and *MN* their axes, and *AC* and *EG* the diameters of their bases.

I say that the circle *ABCD* is to the circle *EFGH* as the cone *AL* is to the cone *EN*.



For, if not, then the circle *ABCD* is to the circle *EFGH* as the cone *AL* is either to some solid less than the cone *EN* or to a greater.

First, let it be in that ratio to a less solid *O,* and let the solid *X* be equal to that by which the solid *O* is less than the cone *EN.* Therefore the cone *EN* equals the sum of the solids *O* and *X.*

Inscribe the square *EFGH* in the circle *EFGH.* Therefore the square is greater than the half of the circle. IV.6

Set up from the square *EFGH* a pyramid of equal height with the cone. Therefore the pyramid so set up is greater than the half of the cone, for if we circumscribe a square about the circle, and set up from it a pyramid of equal height with the cone, then the inscribed pyramid is half of the circumscribed pyramid, for they are to one another as their bases, while the cone is less than the circumscribed pyramid. XII.6

Bisect the circumferences *EF, FG, GH,* and *HE* at the points *P, Q, R,* and *S,* and join *HP, PE, EQ, QF, FR, RG, GS,* and *SH.*

Therefore each of the triangles *HPE, EQF, FRG,* and *GSH* is greater than the half of that segment of the circle about it.

Set up on each of the triangles *HPE, EQF, FRG,* and *GSH* a pyramid of equal height with the cone. Therefore each of the pyramids so set up is also greater than the half of that segment of the cone about it.

Thus, bisecting the circumferences which are left, joining straight lines, setting up on each of the triangles pyramids of equal height with the cone, and doing this repeatedly, we shall leave some segments of the cone which are less than the solid *X.* X.1

Let such be left, and let them be the segments on *HP*, *PE*, *EQ*, *QF*, *FR*, *RG*, *GS*, and *SH*. Therefore the remainder, the pyramid with the polygonal base *HPEQFRGS* and the same height as that of the cone, is greater than the solid *O*.

Now inscribe in the circle *ABCD* the polygon *DTAUBVCW* similar and similarly situated to the polygon *HPEQFRGS*, and on it set up a pyramid of equal height with the cone *AL*.

Since then the square on *AC* is to the square on *EG* as the polygon *DTAUBVCW* is to the polygon *HPEQFRGS*, while the square on *AC* is to the square on *EG* as the circle *ABCD* is to the circle *EFGH*, therefore the circle *ABCD* is to the circle *EFGH* as the polygon *DTAUBVCW* is to the polygon *HPEQFRGS*. XII.1 XII.2

But the circle *ABCD* is to the circle *EFGH* as the cone *AL* is to the solid 0, and the polygon *DTAUBVCW* is to the polygon *HPEQFRGS* as the pyramid with the polygonal base *DTAUBVCW* and the vertex *L* is to the pyramid with the polygonal base *HPEQFRGS* and the vertex *N*. XII.6

Therefore the cone *AL* is to the solid *O* as the pyramid with the polygonal base *DTAUBVCW* and vertex *L* is to the pyramid with the polygonal base *HPEQFRGS* and vertex *N*. Therefore, alternately the cone *AL* is to the pyramid in it as the solid *O* is to the pyramid in the cone *EN*. V.11 V.16

But the cone *AL* is greater than the pyramid in it, therefore the solid *O* is also greater than the pyramid in the cone *EN*.

But it is also less, which is absurd.

Therefore the cone *AL* is not to any solid less than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

Similarly we can prove that neither is the cone *EN* to any solid less than the cone *AL* as the circle *EFGH* is to the circle *ABCD*.

I say next that neither is the cone *AL* to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

For, if possible, let it be in that ratio to a greater solid *O*. Therefore, inversely the circle *EFGH* is to the circle *ABCD* as the solid *O* is to the cone *AL*.

But the solid *O* is to the cone *AL* as the cone *EN* to some solid less than the cone *AL*, therefore the circle *EFGH* is to the circle *ABCD* as the cone *EN* is to some solid less than the cone *AL*, which was proved impossible.

Therefore the cone *AL* is not to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*.

But it was proved that neither is it in this ratio to a less solid, therefore the circle *ABCD* is to the circle *EFGH* as the cone *AL* is to the cone *EN*.

But the cone is to the cone as the cylinder is to the cylinder, for each is triple each. Therefore the circle *ABCD* is to the circle *EFGH* as are the cylinders on them of equal height. XII.10

Therefore, *cones and cylinders of the same height are to one another as their bases.*

Q.E.D.

Guide

Use of this proposition

This proposition is used in the proofs of XII.13 and XII.14 when the cylinders under question have the same height and equal bases, and in the proof of XII.15 for cylinders of different heights.

Next proposition: XII.12

Previous: XII.10

Book XII introduction

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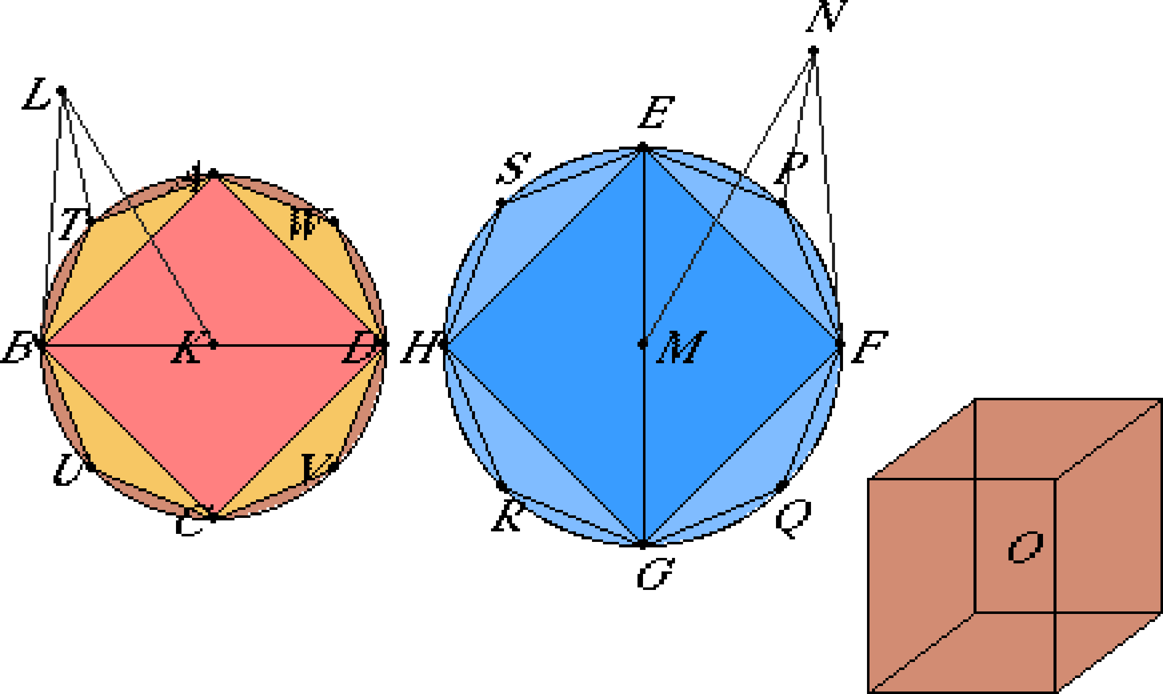
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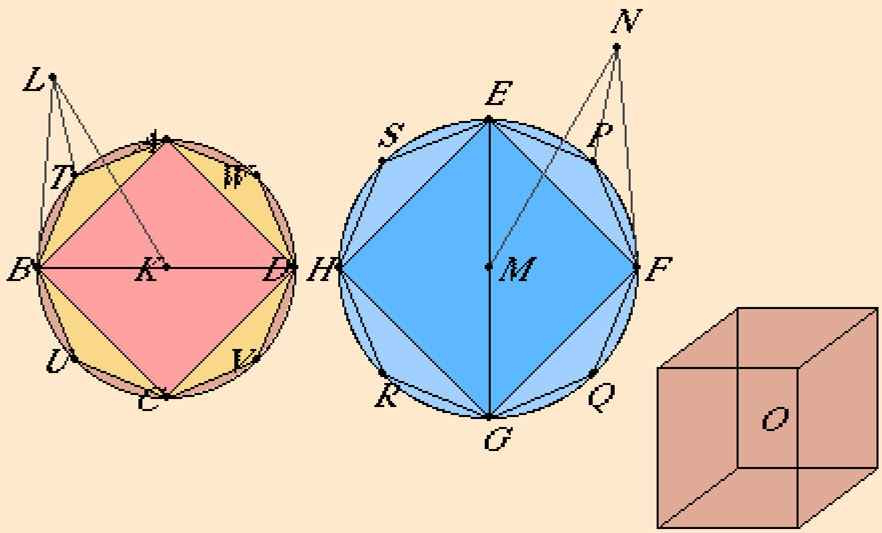
Euclid's Elements
Book XII
Proposition 12

Similar cones and cylinders are to one another in triplicate ratio of the diameters of their bases.

Let there be similar cones and cylinders, let the circles *ABCD* and *EFGH* be their bases, *BD* and *FH* the diameters of the bases, and *KL* and *MN* the axes of the cones and cylinders.

I say that the cone with circular base *ABCD* and vertex *L* has to the cone with circular base *EFGH* and vertex *N* the ratio triplicate of that which *BD* has to *FH*.



For, if the cone *ABCDL* does not have to the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*, then the cone *ABCDL* has that triplicate ratio either to some solid less than the cone *EFGHN* or to a greater.

First, let it have that triplicate ratio to a less solid *O*. Inscribe the square *EFGH* in the circle *EFGH*. Therefore the square *EFGH* is greater than the half of the circle *EFGH*. IV.6

Now set up on the square *EFGH* a pyramid with the same vertex as the cone. Therefore the pyramid so set up is greater than the half part of the cone.Bisect the circumferences *EF, FG, GH,* and *HE* at the points *P, Q, R,* and *S,* and join *EP, PF, FQ, QG, GR, RH, HS,* and *SE.*

Therefore each of the triangles *EPF, FQG, GRH,* and *HSE* is also greater than the half part of that segment of the circle *EFGH* about it.

Now set up on each of the triangles *EPF, FQG, GRH,* and *HSE* a pyramid with the same vertex as the cone.

Therefore each of the pyramids so set up is also greater than the half part of that segment of the cone about it.

Thus, bisecting the circumferences so left, joining straight lines, setting up on each of the triangles pyramids with the same vertex as the cone, and doing this repeatedly, we shall leave some segments of the cone which are less than the excess by which the cone *EFGHN* exceeds the solid *O*. X.1

Let such be left, and let them be the segments on *EP, PF, FQ, QG, GR, RH, HS,* and *SE.* Therefore the remainder, the pyramid with the polygonal base *EPFQGRHS* and vertex *N,* is greater than the solid *O.*

Now inscribe in the circle *ABCD* the polygon *ATBUCVDW* similar and similarly situated to the polygon *EPFQGRHS*, and set up on the polygon *ATBUCVDW* a pyramid with the same vertex as the cone.

Let *LBT* be one off the triangles containing the pyramid with polygonal base *ATBUCVDW* and vertex *L*, and let *NFP* be one of the triangles containing the pyramid with polygonal base *EPFQGRHS* and vertex *N*. Join *KT* and *MP*.

Now, since the cone *ABCDL* is similar to the cone *EFGHN*, therefore *BD* is to *FH* as the axis *KL* is to the axis *MN*. XI.Def.24

But *BD* is to *FH* as *BK* is to *FM*, therefore *BK* is to *FM* as *KL* to *MN*. And, alternately *BK* is to *KL* as *FM* is to *MN*. V.16

And the sides are proportional about equal angles, namely the angles *BKL* and *FMN*, therefore the triangle *BKL* is similar to the triangle *FMN*. VI.6

Again, since *BK* is to *KT* as *FM* is to *MP*, and they are about equal angles, namely the angles *BKT* and *FMP*, for whatever part the angle *BKT* is of the four right angles at the center *K*, it is the same part as the angle *FMP* of the four right angles at the center *M*. Then, since the sides are proportional about equal angles, therefore the triangle *BKT* is similar to the triangle *FMP*. VI.6

Again, since it was proved that *BK* is to *KL* as *FM* is to *MN*, while *BK* equals *KT*, and *FM* equals *PM*, therefore *TK* is to *KL* as *PM* is to *MN*. And the sides are proportional about equal angles, namely the angles *TKL* and *PMN*, for they are right, therefore the triangle *LKT* is similar to the triangle *NMP*. VI.6

And since the triangles *LKB* and *NMF* are similar, therefore *LB* is to *BK* as *NF* is to *FM*. And since the triangles *BKT* and *FMP* are similar, therefore *KB* is to *BT* as *MF* is to *FP*. Therefore, *ex aequali*, *LB* is to *BT* as *NF* is to *FP*. VI.6

Again, since the triangles *LTK* and *NPM* are similar, therefore *LT* is to *TK* as *NP* is to *PM*, and since the triangles *TKB* and *PMF* are similar, therefore *KT* is to *TB* as *MP* is to *PF*. Therefore, *ex aequali*, *LT* is to *TB* as *NP* is to *PF*. VI.6

But it was also proved that *TB* is to *BL* as *PF* is to *FN*. Therefore, *ex aequali*, *TL* is to *LB* as *PN* is to *NF*. V.22

Therefore in the triangles *LTB* and *NPF* the sides are proportional. Therefore the triangles *LTB* and *NPF* are equiangular, hence they are also similar. VI.5 VI.Def.1

Therefore the pyramid with triangular base *BKT* and vertex *L* is similar to the pyramid with triangular base *FMP* and vertex *N*, for they are contained by similar planes equal in multitude. XI.Def.9

But similar pyramids with triangular bases are to one another in the triplicate ratio of their corresponding sides. XII.8

Therefore the pyramid *BKTL* has to the pyramid *FMPN* the ratio triplicate of that which *BK* has to *FM*.

Similarly, by joining straight lines from *A*, *W*, *D*, *V*, *C*, and *U* to *K*, and from *E*, *S*, *H*, *R*, *G*, and *Q* to *M*, and setting up on each of the triangles pyramids with the same vertex as the cones, we can prove that each of the similarly arranged pyramids also has to each similarly arranged pyramid the ratio triplicate of that which the corresponding side *BK* has to the corresponding side *FM*, that is, which *BD* has to *FH*.

And one of the antecedents is to one of the consequents as all the antecedents are to all the consequents, therefore the pyramid *BKTL* is to the pyramid *FMPN* as the whole pyramid with polygonal base *ATBUCVDW* and vertex *L* is to the whole pyramid with polygonal base *EPFQGRHS* and vertex *N*, hence the pyramid with base *ATBUCVDW* and vertex *L* has to the pyramid with polygonal base *EPFQGRHS* and vertex *N* the ratio triplicate of that which *BD* has to *FH*. V.12

But, by hypothesis, the cone with circular base *ABCD* and vertex *L* also has to the solid *O* the ratio triplicate of that which *BD* has to *FH*, therefore the cone with circular base *ABCD* and vertex *L* is to the solid 0 as the pyramid with polygonal base *ATBUCVDW* and vertex *L* is to the pyramid with polygonal base *EPFQGRHS* and vertex *N*. Therefore, alternately the cone with circular base *ABCD* and vertex *L* is to the pyramid contained in it with polygonal base *ATBUCVDW* and vertex *L* as the solid *O* is to the pyramid with the polygonal base *EPFQGRHS* and vertex *N*. V.16

But the said cone is greater than the pyramid in it, for it encloses it. Therefore the solid *O* is also greater than the pyramid with polygonal base *EPFQGRHS* and vertex *N*. But it is also less, which is impossible.

Therefore the cone with circular base *ABCD* and vertex *L* does not have to any solid less than the cone of with circular base *EFGH* and vertex *N* the ratio triplicate of that which *BD* has to *FH*.

Similarly we can prove that neither has the cone *EFGHN* to any solid less than the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*.

I say next that neither has the cone *ABCDL* to any solid greater than the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

For, if possible, let it have that ratio to a greater solid*O*. Therefore, inversely, the solid *O* has to the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*. But the solid *O* is to the cone *ABCDL* as the cone *EFGHN* is to some solid less than the cone *ABCDL*.

Therefore the cone *EFGHN* also has to some solid less than the cone *ABCDL* the ratio triplicate of that which *FH* has to *BD*, which was proved impossible.

Therefore the cone *ABCDL* does not have to any solid greater than the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

But it was proved that neither has it this ratio to a less solid than the cone *EFGHN*. Therefore the cone *ABCDL* has to the cone *EFGHN* the ratio triplicate of that which *BD* has to *FH*.

But the cone is to the cone as the cylinder is to the cylinder, for the cylinder with the same base as the cone and of equal height with it is triple the cone. Therefore the cylinder also has to the cylinder the ratio triplicate of that which *BD* has to *FH*. XII.10

Therefore, *similar cones and cylinders are to one another in triplicate ratio of the diameters of their bases.*

Q.E.D.

Guide

An alternate proof would use the previous proposition (cylinders of the same height are proportional to their bases) and XII.14 (cylinders on equal bases are proportional to their heights), which doesn't depend on this one. Instead Euclid proves this proposition afresh in a manner like that of the previous proposition but necessarily more complicated.

This proposition is not used in later ones.

Next proposition: XII.13

Previous: XII.11

Book XII introduction

Select from Book XII

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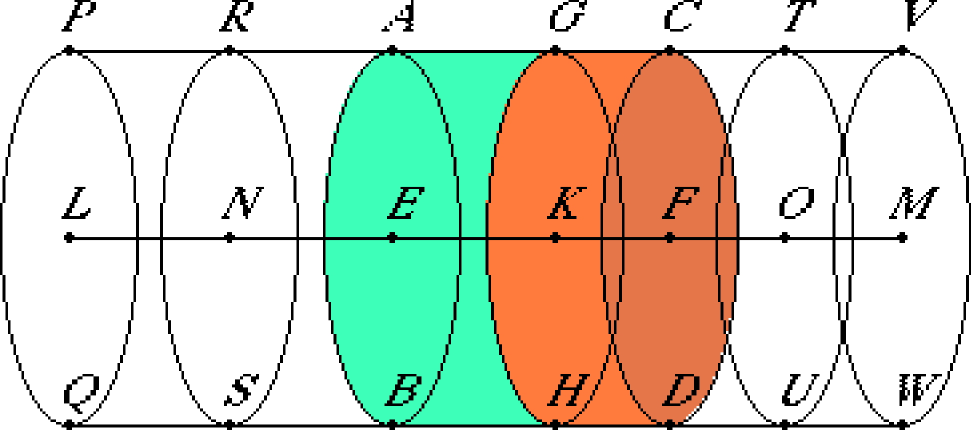
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Euclid's Elements

Book XII

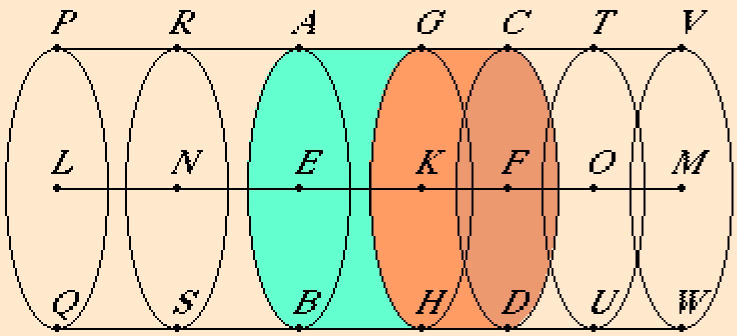
Proposition 13

If a cylinder is cut by a plane parallel to its opposite planes, then the cylinder is to the cylinder as the axis is to the axis.

Let the cylinder *AD* be cut by the plane *GH* parallel to the opposite planes *AB* and *CD.* Let the plane *GH* meet the axis at the point *K.*

I say that the cylinder *BG* is to the cylinder *GD* as the axis *EK* is to the axis *KF.*

Produce the axis *EF* in both directions to the points *L* and *M.* Set out any number whatever of axes *EN* and *NL* equal to the axis *EK,* and any number whatever *FO* and *OM* equal to *FK.* Construct the cylinder *PW* on the axis *LM* with the circles *PQ* and *VW* as bases.



Carry the planes through the points *N* and *O* parallel to *AB* and *CD* and to the bases of the cylinder *PW,* and let them produce the circles *RS* and *TU* about the centers *N, O.*

Then, since the axes *LN, NE,* and *EK* equal one another, therefore the cylinders *QR, RB,* and *BG* are to one another as their bases. XII.11

But the bases are equal, therefore the cylinders *QR, RB,* and *BG* also equal one another.

Since then the axes *LN, NE,* and *EK* equal one another, and the cylinders *QR, RB,* and *BG* also equal one another, and the multitude of the former equals the multitude of the latter, therefore, the multiple the axis *KL* is of the axis *EK* is the same multiple the cylinder *QG* is of the cylinder *GB.*

For the same reason, the multiple the axis *MK* is of the axis *KF* is the same multiple the cylinder *WG* is of the cylinder *GD.*

And, if the axis *KL* equals the axis *KM,* then the cylinder *QG* also equals the cylinder *GW*; if the axis is greater than the axis, then the cylinder is also greater than the cylinder; and if less, less. Thus, there being four magnitudes, the axes *EK* and *KF* and the cylinders *BG* and *GD,* there have been taken equimultiples of the axis *EK* and of the cylinder *BG,* namely the axis *LK* and the cylinder *QG,* and equimultiples of the axis V.Def.5 *KF* and of the cylinder *GD,* namely the axis *KM* and the cylinder *GW,* and it has been proved that, if the axis *KL* is in excess of the axis *KM,* the cylinder *QG* is also in excess of the cylinder *GW*; if equal, equal; and if less, less. Therefore the axis *EK* is to the axis *KF* as the cylinder *BG* is to the cylinder *GD.*

Therefore, *if a cylinder is cut by a plane parallel to its opposite planes, then the cylinder is to the cylinder as the axis is to the axis.*

Q.E.D.

Guide

Use of this proposition

This proposition is preliminary to the next in which it is shown that cylinders on equal bases are proportional to their heights. It is also used in the proposition following that.

Next proposition: XII.14

Previous: XII.12

Book XII introduction

Select from Book XII

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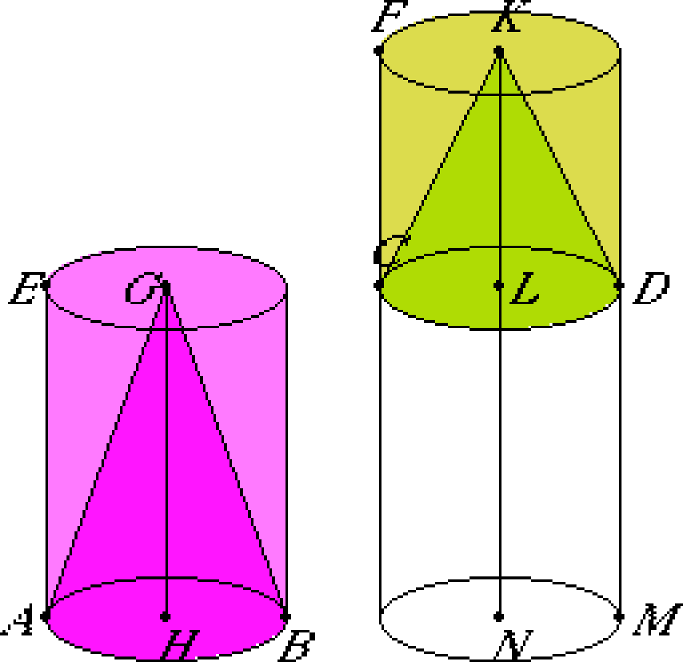

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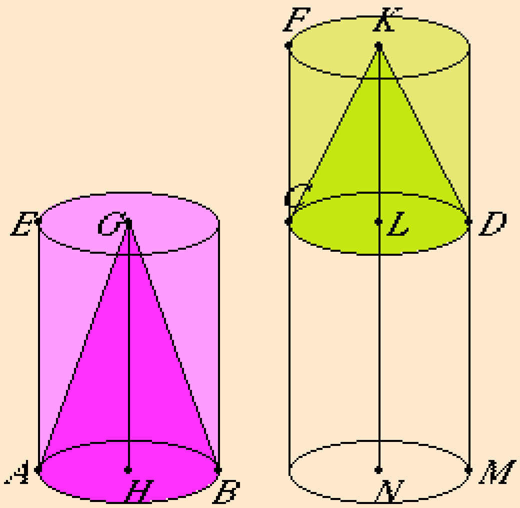
Euclid's Elements
Book XII
Proposition 14

Cones and cylinders on equal bases are to one another as their heights.

Let *EB* and *FD* be cylinders on equal bases, the circles *AB* and *CD.*

I say that the cylinder *EB* is to the cylinder *FD* as the axis *GH* is to the axis *KL.*



Produce the axis *KL* to the point *N*, make *LN* equal to the axis *GH*, and construct the cylinder *CM* about *LN* as the axis. I.3

Then, since the cylinders *EB* and *CM* are of the same height, therefore they are to one another as their bases. XII.11

But the bases equal one another, therefore the cylinders *EB* and *CM* are also equal.

And, since the cylinder *FM* has been cut by the plane *CD* parallel to its opposite planes, therefore the cylinder *CM* is to the cylinder *FD* as the axis *LN* is to the axis *KL.* XII.13

But the cylinder *CM* equals the cylinder *EB*, and the axis *LN* equals the axis *GH*, therefore the cylinder *EB* is to the cylinder *FD* as the axis *GH* is to the axis *KL.*

But the cylinder *EB* is to the cylinder *FD* as the cone *ABG* is to the cone *CDK.* Therefore the axis *GH* is to the axis *KL* as the cone *ABG* is to the cone *CDK* and as the cylinder *EB* is to the cylinder *FD.* XII.10

Therefore, *cones and cylinders on equal bases are to one another as their heights.*

Q.E.D.

Guide

Back in proposition XII.11 cones and cylinders were shown to be proportional to their bases, and this proposition shows that they are proportional to their heights. The next proposition relates the volume to the base and height in a different way by fixing the volume so that the base and height are reciprocally proportional.

Next proposition: XII.15

Select from Book XII

Previous: XII.13

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[Book XII introduction](#)

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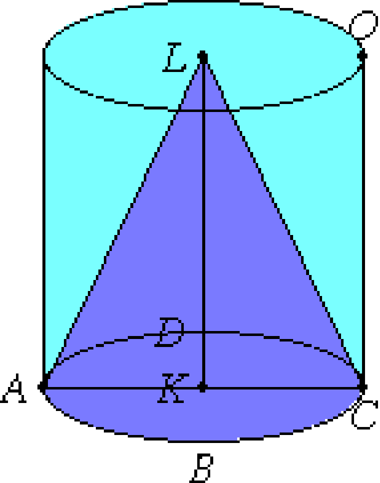
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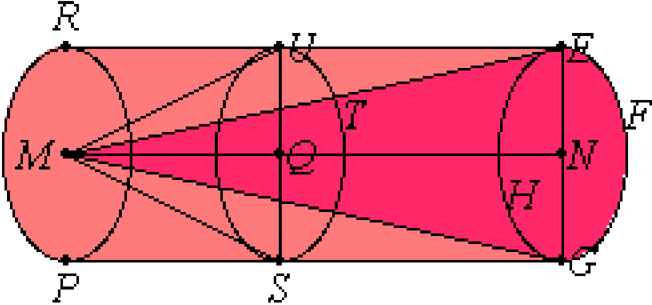
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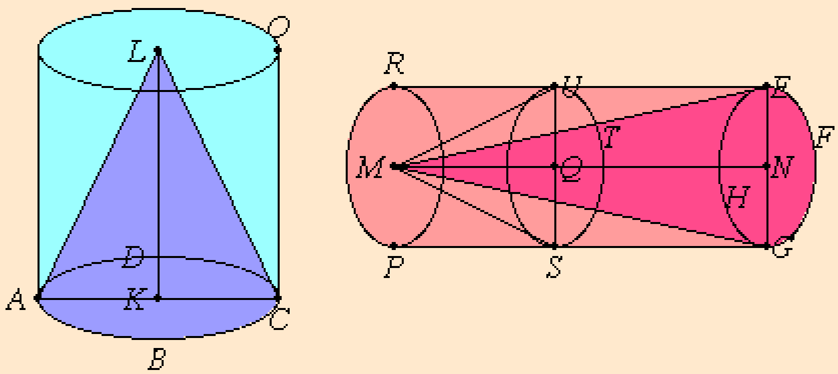
Euclid's Elements
Book XII
Proposition 15

In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.

Let there be equal cones and cylinders with the circular bases *ABCD* and *EFGH*. Let *AC* and *EG* be the diameters of the bases, and *KL* and *MN* the axes, which are also the heights of the cones or cylinders

Complete the cylinders *AO* and *EP*.

I say that in the cylinders *AO* and *EP* the bases are reciprocally proportional to the heights, that is, the base *ABCD* is to the base *EFGH* as the height *MN* is to the height *KL*.



For the height *LK* is either equal to the height *MN* or unequal.

First, let it be equal.

Now the cylinder *AO* also equals the cylinder *EP*. But cones and cylinders of the same height are to one another as their bases, therefore the base *ABCD* equals the base *EFGH*. XII.11

Hence, reciprocally, the base *ABCD* is to the base *EFGH* as the height *MN* is to the height *KL*.

Next, let the height *LK* be unequal to *MN*, and let *MN* be greater.

Cut *QN* off the height *MN* equal to *KL*. Through the point *Q* let the cylinder *EP* be cut by the plane *TUS* parallel to the planes of the circles *EFGH* and *RP*. Erect the cylinder *ES* from the circle *EFGH* as base and with height *NQ*.

Now, since the cylinder *AO* equals the cylinder *EP*, therefore the cylinder *AO* is to the cylinder *ES* as the cylinder *EP* is to the cylinder *ES*. V.7

But the cylinder *AO* is to the cylinder *ES* as the base *ABCD* is to the base *EFGH*, for the cylinders *AO* and *ES* are of the same height. And the cylinder *EP* is to the cylinder *ES* as the height *MN* is to the height *QN*, for the cylinder *EP* is cut by a plane parallel to its opposite planes. Therefore the base *ABCD* is to the base *EFGH* as the height *MN* is to the height *QN*. XII.11 XII.13 V.11

But the height *QN* equals the height *KL*, therefore the base *ABCD* is to the base *EFGH* as the height *MN* is to the height *KL*.

Therefore in the cylinders *AO* and *EP* the bases are reciprocally proportional to the heights.

Next, in the cylinders *AO* and *EP* let the bases be reciprocally proportional to the heights, that is, as the base *ABCD* is to the base *EFGH,* so let the height *MN* be to the height *KL*.

I say that the cylinder *AO* equals the cylinder *EP*.

With the same construction, since the base *ABCD* is to the base *EFGH* as the height *MN* is to the height *KL*, and the height *KL* equals the height *QN*, therefore the base *ABCD* is to the base *EFGH* as the height *MN* is to the height *QN*.

But the base *ABCD* is to the base *EFGH* as the cylinder *AO* is to the cylinder *ES,* for they have the same height. And the height *MN* is to *QN* as the cylinder *EP* is to the cylinder *ES,* therefore the cylinder *AO* is to the cylinder *ES* as the cylinder *EP* is to the cylinder *ES.* XII.11 XII.13 V.11

Therefore the cylinder *AO* equals the cylinder *EP*. V.9

And the same is true for the cones also. XII.10

Therefore, *in equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.*

Q.E.D.

Guide

This proof of this proposition applies to a more general situation than cones and cylinders. Whenever a magnitude x is proportional to two other magnitudes y and z, that is to say when y is fixed then x is proportional to z and when x is fixed then y is proportional to z, it follows that when x is fixed then y and z are reciprocally proportional.

This proposition completes the theory of the volumes of cones and cylinders. The remaining three propositions in this book concern the volume of spheres.

Next proposition: XII.16

Previous: XII.14

Book XII introduction

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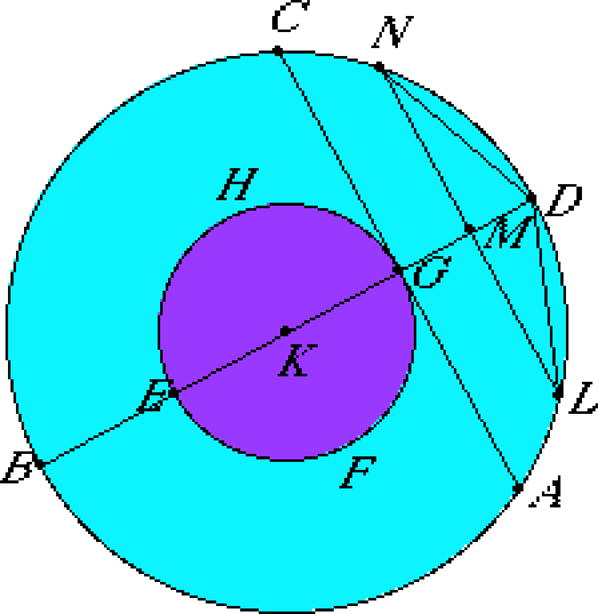
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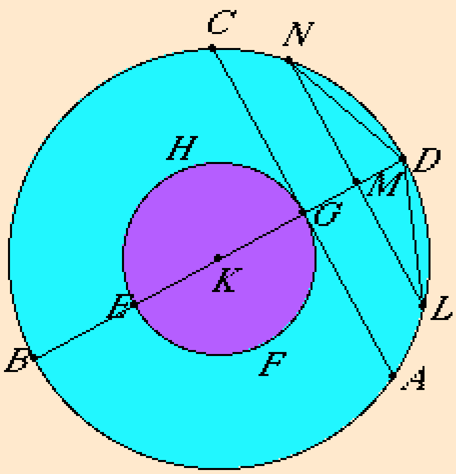
Euclid's Elements
Book XII
Proposition 16

Given two circles about the same center, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle.

Let *ABCD* and *EFGH* be the two given circles about the same center *K*.

It is required to inscribe in the greater circle *ABCD* an equilateral polygon with an even number of sides which does not touch the circle *EFGH*.



Draw the straight line *BKD* through the center *K*, and draw *GA* from the point *G* at right angles to the straight line *BD*, and carry it through to *C*. I.11

Therefore *AC* touches the circle *EFGH*. III.16,Cor

Then, bisecting the circumference *BAD*, bisecting the half of it, and doing this repeatedly, we shall leave a circumference less than *AD*. X.1

Let such be left, and let it be *LD*.

Draw *LM* from *L* perpendicular to *BD*, and carry it through to *N*. Join *LD* and *DN*. I.12

Therefore *LD* equals *DN*. III.3 I.4

Now, since *LN* is parallel to *AC*, and *AC* touches the circle *EFGH*, therefore *LN* does not touch the circle *EFGH*. Therefore *LD* and *DN* are far from touching the circle *EFGH*.

If, then, we fit into the circle *ABCD* straight lines equal to the straight line *LD* and placed repeatedly, then there is inscribed in the circle *ABCD* an equilateral polygon with an even number of sides which does not touch the lesser circle *EFGH*.

Q.E.F.

Guide

The purpose of this construction is to separate the two concentric circles by a polygon so that a three-dimensional construction can be made in the next proposition to separate two concentric spheres.

This construction will actually generate a polygon whose number of sides is a power of 2 such as 8, 16, 32, etc. The next proposition requires a polygon where the number of sides is not just even, but a multiple of 4, which conveniently this construction generates.

Furthermore, the next proposition requires not just that the polygon not touch the inner circle, but the chords joining alternate vertices also not touch the inner circle, which again this construction satisfies.

Next proposition: XII.17

Previous: XII.15

Book XII introduction

Select from Book XII

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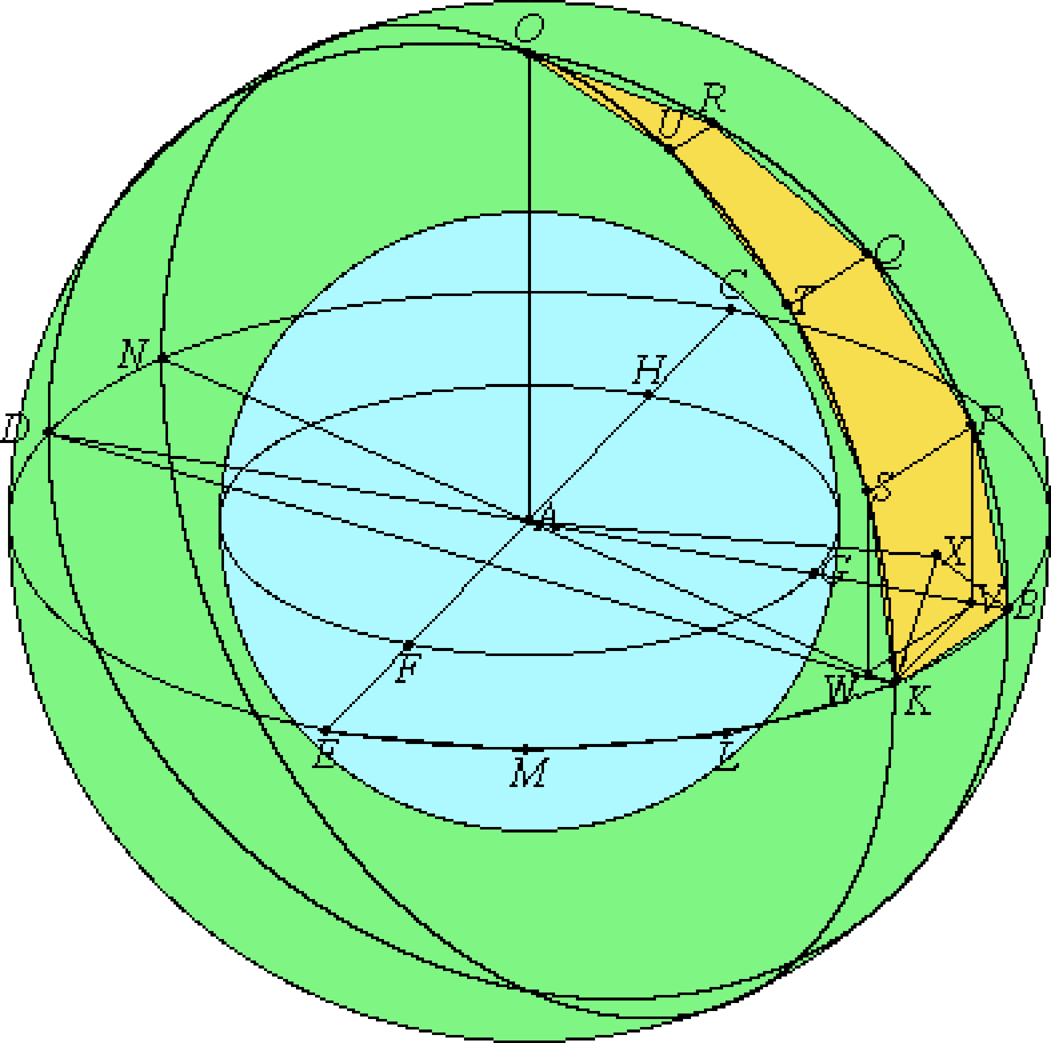
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Euclid's Elements
Book XII
Proposition 17

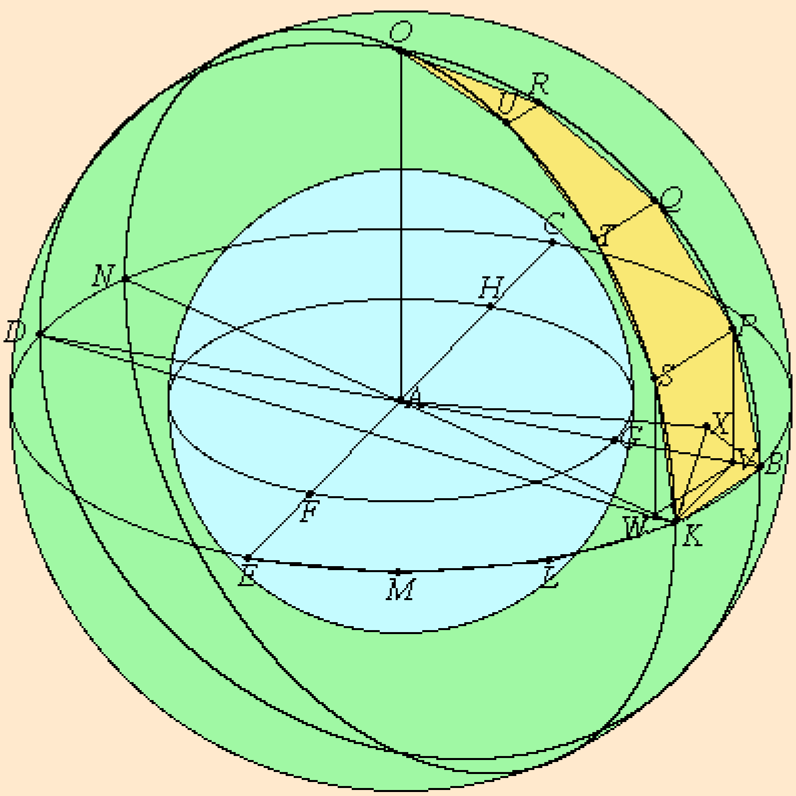
Given two spheres about the same center, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Let there be two spheres about the same center *A.*

It is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Cut the spheres by any plane through the center. Then the sections are circles, for as a sphere is produced by the diameter remaining fixed and the semicircle being carried round it, hence, in whatever position we conceive the semicircle to be, the plane carried through it produces a circle on the circumference of the sphere. XI.Def.14

And it is clear that this circle is the greatest possible, for the diameter of the sphere, which is of course the diameter both of the semicircle and of the circle, is greater than all the straight lines drawn across in the circle or the sphere.



Let then *BCDE* be the circle in the greater sphere, and *FGH* the circle in the lesser sphere. Draw two diameters in them, *BD* and *CE,* at right angles to one another. I.11

Then, given the two circles *BCDE* and *FGH* about the same center, inscribe in the greater circle *BCDE* an equilateral polygon with an even number of sides which does not touch the lesser circle *FGH.* XII.16

Let *BK*, *KL*, *LM,* and *ME* be its sides in the quadrant *BE.* Join *KA* and carry it through to *N.* Set *AO* up from the point *A* at right angles to the plane of the circle *BCDE,* and let it meet the surface of the sphere XI.12

at *O.*

Carry planes through *AO* and each of the straight lines *BD* and *KN.* They make the greatest circles on the surface of the sphere for the reason stated.

Let them make such, and in them let *BOD* and *KON* be the semicircles on *BD* and *KN.*

Now, since *OA* is at right angles to the plane of the circle *BCDE,* therefore all the planes through *OA* are also at right angles to the plane of the circle *BCDE.* Hence the semicircles *BOD* and *KON* are also at right angles to the plane of the circle *BCDE.* XI.18

And, since the semicircles *BED, BOD,* and *KON* are equal, for they are on equal diameters *BD* and *KN,* therefore the quadrants *BE, BO,* and *KO* equal one another.

Therefore there are as many straight lines in the quadrants *BO* and *KO* equal to the straight lines *BK, KL, LM,* and *ME* as there are sides of the polygon in the quadrant *BE.*

Inscribe them as *BP, PQ, QR,* and *RO* and as *KS, ST, TU,* and *UO.* Join *SP, TQ, UR,* and draw perpendiculars from *P* and *S* to the plane of the circle *BCDE.* IV.1 XI.11

These will fall on *BD* and *KN,* the common sections of the planes, for the planes of *BOD* and *KON* are also at right angles to the plane of the circle *BCDE.* cf.XI.Def.4

Let them so fall as *PV* and *SW,* and join *WV.*

Now since, in the equal semicircles *BOD* and *KON,* equal straight lines *BP* and *KS* have been cut off, and the perpendiculars *PV* and *SW* have been drawn, therefore *PV* equals *SW,* and *BV* equals *KW.* III.27 I.26

But the whole *BA* also equals the whole *KA,* therefore the remainder *VA* also equals the remainder *WA.* Therefore *BV* is to *VA* as *KW* is to *WA.* Therefore *WV* is parallel to *KB.* VI.2

And, since each of the straight lines *PV* and *SW* is at right angles to the plane of the circle *BCDE,* therefore *PV* is parallel to *SW.* XI.6

But it was also proved equal to it, therefore *WV* and *SP* are equal and parallel. I.33

And, since *WV* is parallel to *SP,* and *WV* is parallel to *KB,* therefore *SP* is also parallel to *KB.* XI.9

And *BP* and *KS* join their ends, therefore the quadrilateral *KBPS* is in one plane, for if two straight lines are parallel, and points are taken at random on each of them, then the straight line joining the points is in the same plane with the parallels. For the same reason each of the quadrilaterals *SPQT* and *TQRU* is also in one plane. XI.7

But the triangle *URO* is also in one plane. If then we join straight lines from the points *P, S, Q, T, R,* and *U* to *A,* then there will be constructed a certain polyhedral solid figure between the circumferences *BO* and *KO* consisting of pyramids of which the quadrilaterals *KBPS, SPQT,* and *TQRU* and the triangle *URO* are the bases and the point *A* is the vertex. XI.2

And, if we make the same construction in the case of each of the sides *KL, LM,* and *ME* as in the case of *BK,* and further, in the case of the remaining three quadrants, then there will be constructed a certain polyhedral figure inscribed in the sphere and contained by pyramids, of which the said quadrilaterals and the triangle *URO,* and the others corresponding to them, are the bases and the point *A* is the vertex.

I say that the said polyhedron does not touch the lesser sphere at the surface on which the circle *FGH* is.

Draw *AX* from the point *A* perpendicular to the plane of the quadrilateral *KBPS,* and let it meet the plane at the point *X.* Join *XB* and *XK.* XI.11

Then, since *AX* is at right angles to the plane of the quadrilateral *KBPS,* therefore it is also at right angles to all the straight lines which meet it and are in the plane of the quadrilateral. Therefore *AX* is at right XI.Def.3

angles to each of the straight lines *BX* and *XK*.

And, since *AB* equals *AK*, therefore the square on *AB* equals the square on *AK*. And the sum of the squares on *AX* and *XB* equals the square on *AB*, for the angle at *X* is right, and the sum of the squares on *AX* and *XK* equals the square on *AK*. I.47

Therefore the sum of the squares on *AX* and *XB* equals the sum of the squares on *AX* and *XK*.

Subtract the square on *AX* from each, therefore the remainder, the square on *BX*, equals the remainder, the square on *XK*. Therefore *BX* equals *XK*.

Similarly we can prove that the straight lines joined from *X* to *P* and *S* are equal to each of the straight lines *BX* and *XK*.

Therefore the circle with center *X* and radius on of the straight lines *XB* or *XK* passes through *P* and *S* also, and *KBPS* is a quadrilateral in a circle.

Now, since *KB* is greater than *WV*, and *WV* equals *SP*, therefore *KB* is greater than *SP*. But *KB* equals each of the straight lines *KS* and *BP*, therefore each of the straight lines *KS* and *BP* is greater than *SP*. And, since *KBPS* is a quadrilateral in a circle, and *KB*, *BP*, and *KS* are equal, and *PS* less, and *BX* is the radius of the circle, therefore the square on *KB* is greater than double the square on *BX*.

Draw *KZ* from *K* perpendicular to *BV*. I.12

Then, since *BD* is less than double *DZ*, and *BD* is to *DZ* as the rectangle *DB* by *BZ* is to the rectangle *DZ* by *ZB*, therefore if a square is described on *BZ* and the parallelogram on *ZD* is completed, then the rectangle *DB* by *BZ* is also less than double the rectangle *DZ* by *ZB*. And, if *KD* is joined, then the rectangle *DB* by *BZ* equals the square on *BK*, and the rectangle *DZ* by *ZB* equals the square on *KZ*. Therefore the square on *KB* is less than double the square on *KZ*. I.46 III.31 VI.18,Cor

But the square on *KB* is greater than double the square on *BX*, therefore the square on *KZ* is greater than the square on *BX*. And, since *BA* equals *KA*, therefore the square on *BA* equals the square on *AK*.

And the sum of the squares on *BX* and *XA* equals the square on *BA*, and the sum of the squares on *KZ* and *ZA* equals the square on *KA*, therefore the sum of the squares on *BX* and *XA* equals the sum of the squares on *KZ* and *ZA*, and of these the square on *KZ* is greater than the square on *BX*, therefore the remainder, the square on *ZA*, is less than the square on *XA*. I.47

Therefore *AX* is greater than *AZ*. Therefore *AX* is much greater than *AG*.

And *AX* is the perpendicular on one base of the polyhedron, and *AG* on the surface of the lesser sphere, hence the polyhedron does not touch the lesser sphere on its surface.

Therefore, given two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface.

Corollary.

But *if in another sphere a polyhedral solid is inscribed similar to the solid in the sphere BCDE, then the polyhedral solid in the sphere BCDE has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere BCDE has to the diameter of the other sphere.*

For, the solids being divided into their pyramids similar in multitude and arrangement, the pyramids will be similar.

But similar pyramids are to one another in the triplicate ratio of their corresponding sides, therefore the pyramid with the quadrilateral base *KBPS* and the vertex *A* has to the similarly arranged pyramid in the other sphere the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of that which the radius *AB* of the sphere about *A* as center has to the radius of the other sphere. XII.18,Cor.

Similarly each pyramid of those in the sphere about *A* as center has to each similarly arranged pyramid of those in the other sphere the ratio triplicate of that which *AB* has to the radius of the other sphere.

And one of the antecedents is to one of the consequents as all the antecedents are to all the consequents, hence the whole polyhedral solid in the sphere about *A* as center has to the whole polyhedral solid in the other sphere the ratio triplicate of that which *AB* has to the radius of the other sphere, that is, of that which the diameter *BD* has to the diameter of the other sphere. V.12

Q.E.F.

Guide

The purpose of this proposition and its corollary is to separate concentric spheres so that it can be proved in the next proposition XII.18 that spheres are to each other in triplicate ratios of their diameters.

The argument that the intersection of a sphere and a plane through its center is a circle is weak. It has not been shown that the sphere is generated by taking any of its diameters and rotating a semicircle on that diameter about the diameter. Even the very concept of rotation about an axis has not been formalized.

Next proposition: XII.18

Previous: XII.16

Book XII introduction

Select from Book XII

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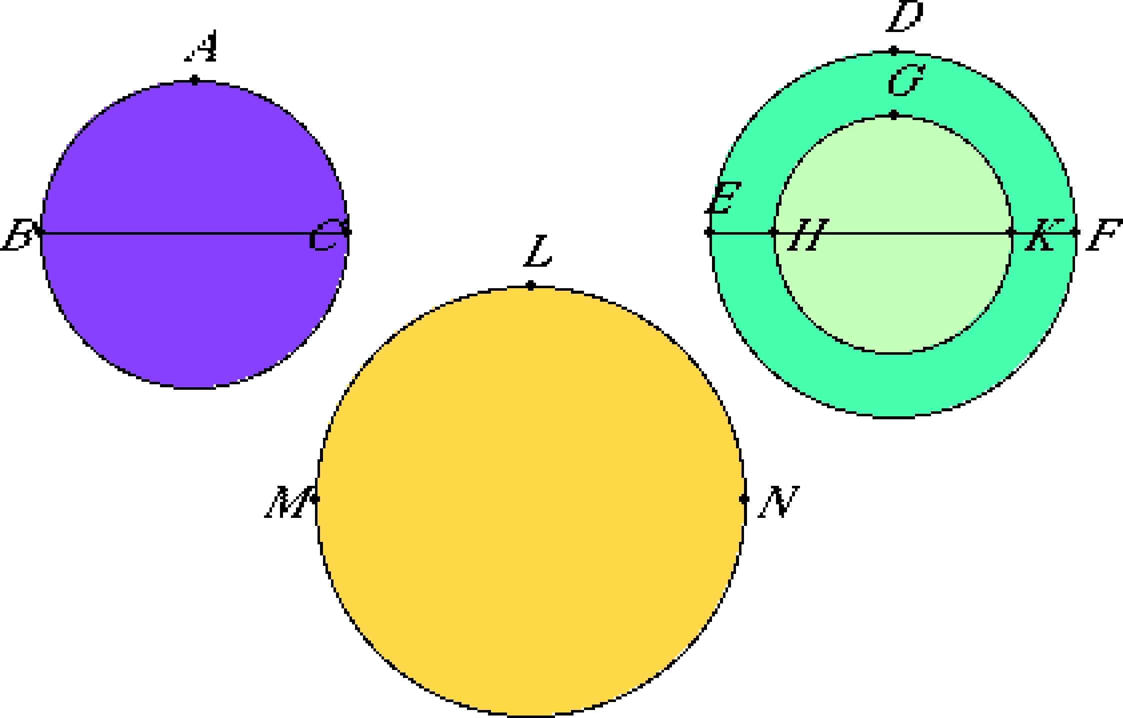
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Euclid's Elements
Book XII
Proposition 18

Spheres are to one another in triplicate ratio of their respective diameters.

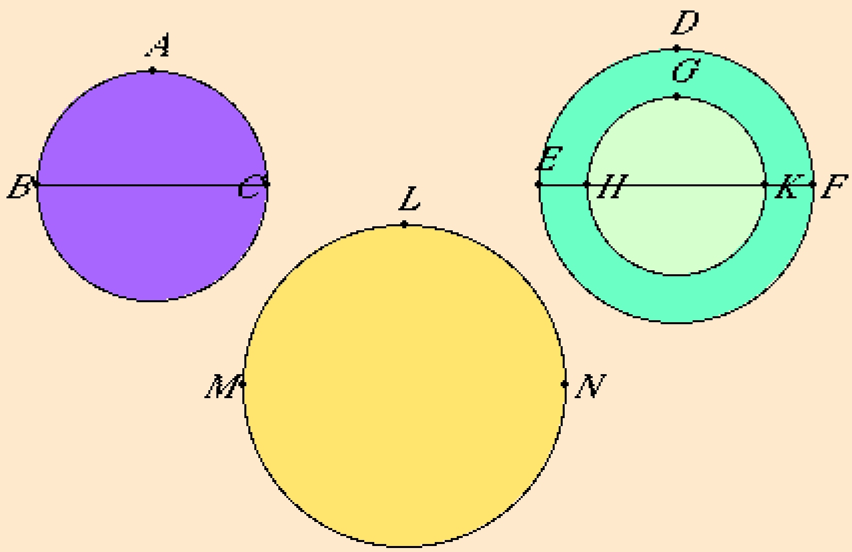
Let the *ABC* and *DEF* be spheres, and let *BC* and *EF* be their diameters.

I say that the sphere *ABC* has to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

For, if the sphere *ABC* has not to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*, then the sphere *ABC* has either to some less sphere than the sphere *DEF*, or to a greater, the ratio triplicate of that which *BC* has to *EF*.

First, let it have that ratio to a less sphere *GHK*.

Let *DEF* be about the same center with *GHK*. Inscribe in the greater sphere *DEF* a polyhedral solid which does not touch the lesser sphere *GHK* at its surface. XII.17



Also inscribe in the sphere *ABC* a polyhedral solid similar to the polyhedral solid in the sphere *DEF*. Therefore the polyhedral solid in *ABC* has to the polyhedral solid in *DEF* the ratio triplicate of that which *BC* has to *EF*. XII.17,Cor.

But the sphere *ABC* also has to the sphere *GHK* the ratio triplicate of that which *BC* has to *EF*, therefore the sphere *ABC* is to the sphere *GHK* as the polyhedral solid in the sphere *ABC* is to the polyhedral solid in the sphere *DEF*, and, alternately the sphere *ABC* is to the polyhedron in it as the sphere *GHK* is to the polyhedral solid in the sphere *DEF*. V.16

But the sphere *ABC* is greater than the polyhedron in it, therefore the sphere *GHK* is also greater than the polyhedron in the sphere *DEF*. V.14

But it is also less, for it is enclosed by it. Therefore the sphere *ABC* has not to a less sphere than the sphere *DEF* the ratio triplicate of that which the diameter *BC* has to *EF*.

Similarly we can prove that neither has the sphere *DEF* to a less sphere than the sphere *ABC* the ratio triplicate of that which *EF* has to *BC*.

I say next that neither has the sphere *ABC* to any greater sphere than the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

For, if possible, let it have that ratio to a greater, *LMN*. Therefore, inversely, the sphere *LMN* has to the sphere *ABC* the ratio triplicate of that which the diameter *EF* has to the diameter *BC*.

But, since *LMN* is greater than *DEF*, therefore the sphere *LMN* is to the sphere *ABC* as the sphere *DEF* is to some less sphere than the sphere *ABC*, as was before proved. XII.2,Lemma

Therefore the sphere *DEF* also has to some less sphere than the sphere *ABC* the ratio triplicate of that which *EF* has to *BC*, which was proved impossible.

Therefore the sphere *ABC* has not to any sphere greater than the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

But it was proved that neither has it that ratio to a less sphere.

Therefore the sphere *ABC* has to the sphere *DEF* the ratio triplicate of that which *BC* has to *EF*.

Therefore, *spheres are to one another in triplicate ratio of their respective diameters.*

Q.E.D.

Guide

This proposition completes Book XII.

Although this is an important proposition, it is just the beginning of the study of volumes of spheres. The arguments given in this proof are fairly convincing that any two similar solids are to each other in triplicate ratio of their linear parts. One difficulty is defining just what similar solids are.

The volume of a sphere

Euclid proved in proposition XII.10 that the cone with the same base and height as a cylinder was one third of the cylinder, but he could not find the ratio of a sphere to the circumscribed cylinder.

In the century after Euclid, Archimedes solved this problem as well as the much more difficult problem of the surface area of a sphere. He showed that the ratio of the sphere to the cylinder is 4:3. Since the volume of the cylinder is proportional to its base and its height, it follows that the volumes of spheres, cylinders, and cones can be found in terms of areas of circles. In algebraic terms, if we let pi stand for the ratio of a circle to the square on its radius, then the volume of a cylinder of radius *r* and height *h* is pi r^2h; the volume of an inscribed cone is pi $r^2h/3$; and the volume of a sphere of radius *r* is 4 pi $r^3/3$.

Next book: Book XIII

Previous: XII.17

Book XII introduction

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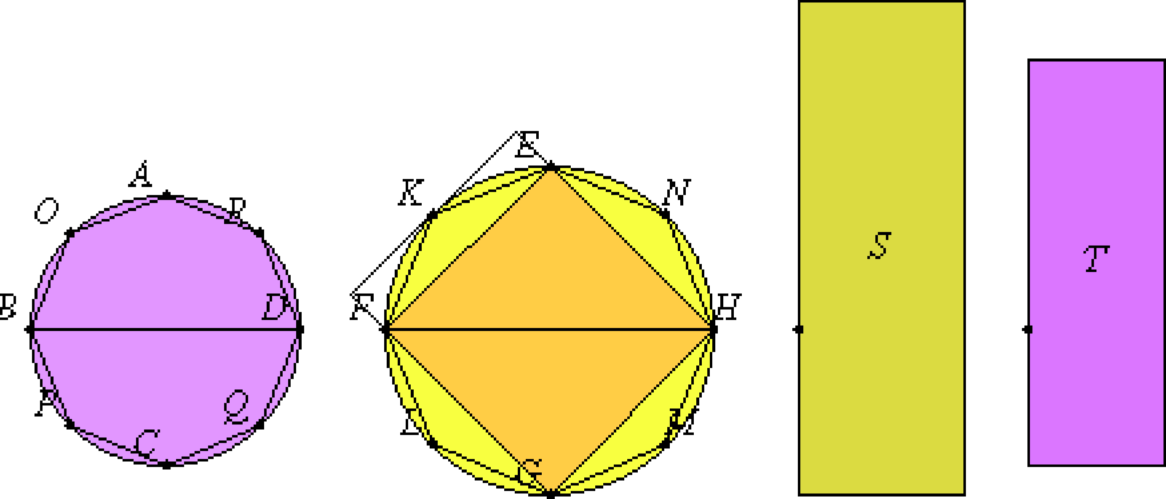
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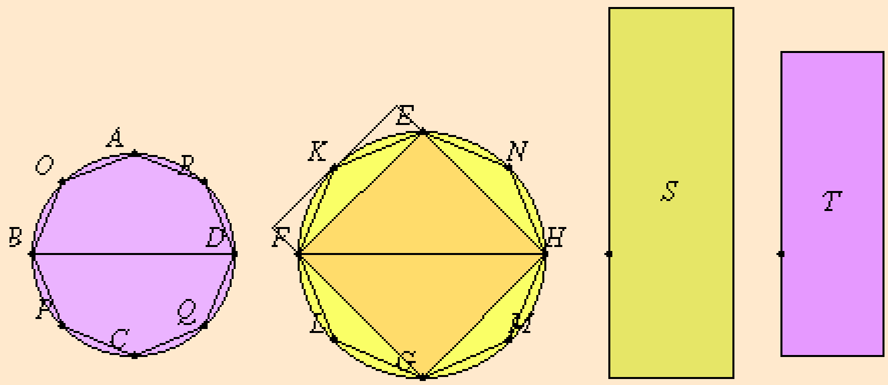
Euclid's Elements
Book XII
Proposition 2

Circles are to one another as the squares on their diameters.

Let *ABCD* and *EFGH* be circles, and let *BD* and *FH* be their diameters.

I say that the circle *ABCD* is to the circle *EFGH* as the square on *BD* is to the square on *FH*.



For, if the square on *BD* is not to the square on *FH* as the circle *ABCD* is to the circle *EFGH*, then as the square on *BD* is to the square on *FH*, the circle *ABCD* is either to some less area than the circle *EFGH*, or to a greater area.

First, let it be in that ratio to a less area *S*.

Inscribe the square *EFGH* in the circle *EFGH*. Then the inscribed square is greater than the half of the circle *EFGH*, for if through the points *E*, *F*, *G*, and *H* we draw tangents to the circle, then the square *EFGH* is half the square circumscribed about the circle, and the circle is less than the circumscribed square, hence the inscribed square *EFGH* is greater than the half of the circle *EFGH*. IV.6 III.17

Bisect the circumferences *EF*, *FG*, *GH*, and *HE* at the points *K*, *L*, *M*, and *N*. Join *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, and *NE*.

Therefore each of the triangles *EKF*, *FLG*, *GMH*, and *HNE* is also greater than the half of the segment of the circle about it, for if through the points *K*, *L*, *M*, and *N* we draw tangents to the circle and complete the parallelograms on the straight lines *EF*, *FG*, *GH*, and *HE*, then each of the triangles *EKF*, *FLG*, *GMH*, and *HNE* is half of the parallelogram about it, while the segment about it is less than the parallelogram, hence each of the triangles *EKF*, *FLG*, *GMH*, and *HNE* is greater than the half of the segment of the circle about it. III.17

Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this repeatedly, we shall leave some segments of the circle which will be less than the excess by which the circle *EFGH* exceeds the area *S*.

For it was proved in the first theorem of the tenth book that if two unequal magnitudes are set out, and if from the greater there is subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this is done repeatedly, then there will be left some magnitude which is less than the lesser magnitude set out. X.1

Let segments be left such as described, and let the segments of the circle *EFGH* on *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, and *NE* be less than the excess by which the circle *EFGH* exceeds the area *S*.

Therefore the remainder, the polygon *EKFLGMHN,* is greater than the area *S.*

Now inscribe in the circle *ABCD* the polygon *AOBPCQDR* similar to the polygon *EKFLGMHN.*

Therefore the square on *BD* is to the square on *FH* as the polygon *AOBPCQDR* is to the polygon *EKFLGMHN.* XII.1

But the square on *BD* is to the square on *FH* as the circle *ABCD* to the area *S,* therefore the circle *ABCD* is to the area *S* as the polygon *AOBPCQDR* is to the polygon *EKFLGMHN.* Therefore, alternately the circle *ABCD* is to the polygon inscribed in it as the area *S* is to the polygon *EKFLGMHN.* V.11 V.16

But the circle *ABCD* is greater than the polygon inscribed in it, therefore the area *S* is also greater than the polygon *EKFLGMHN.* But it is also less, which is impossible.

Therefore the square on *BD* is to the square on *FH* not as the circle *ABCD* is to any area less than the circle *EFGH.*

Similarly we can prove that the circle *EFGH* is to any area less than the circle *ABCD* not as the square on *FH* is to the square on *BD.*

I say next that neither is the circle *ABCD* to any area greater than the circle *EFGH* as the square on *BD* is to the square on *FH.*

For, if possible, let it be in that ratio to a greater area *S.* Therefore, inversely the square on *FH* is to the square on *DB* as the area *S* is to the circle *ABCD.*

But the area *S* is to the circle *ABCD* as the circle *EFGH* is to some area less than the circle *ABCD,* therefore the square on *FH* is to the square on *BD* as the circle *EFGH* is to some area less than the circle *ABCD,* which was proved impossible. Therefore the square on *BD* is to the square on *FH* not as the circle *ABCD* to any area greater than the circle *EFGH.* Lemma V.11

And it was proved that neither is it in that ratio to any area less than the circle *EFGH,* therefore the square on *BD* is to the square on *FH* as the circle *ABCD* is to the circle *EFGH.*

Q.E.D.

Lemma

I say that, the area *S* being greater than the circle *EFGH* the area *S* is to the circle *ABCD* as the circle *EFGH* is to some area less than the circle *ABCD.*

For let it be contrived that the area *S* is to the circle *ABCD* as the circle *EFGH* to the area *T.*

I say that the area *T* is less than the circle *ABCD.*

Since the area *S* is to the circle *ABCD* as the circle *EFGH* is to the area *T,* therefore, alternately the area *S* is to the circle *EFGH* as the circle *ABCD* is to the area *T.* V.16

But the area *S* is greater than the circle *EFGH,* therefore the circle *ABCD* is also greater than the area *T.*

Hence the area *S* is to the circle *ABCD* as the circle *EFGH* is to some area less than the circle *ABCD.*

Therefore, *circles are to one another as the squares on their diameters.*

Q.E.D.

Guide

In the last proposition it was shown that similar polygons inscribed in circles are proportional to the squares on the diameters of the circles. By approximating circles closely by similar polygons, the proportion is carried over to the circles.

The form of the proof is a double proof by contradiction. There are three cases when comparing the ratio of the squares *BD*:*FH* to the ratio of the circles *ABCD*:*EFGH.* One case is that the ratio of the squares *BD*:*FH* equals *ABCD*:*S* where *S* is some area less than circle *EFGH.* Most of the proof is spent refuting this case. The second case is that the ratio of the squares *BD*:*FH* equals *ABCD*:*S* where this time *S* is some area greater than circle *EFGH.* This is inverted to a statement that the ratio of the squares *FH*:*BD* equals *EFGH* to some area less than circle *ABCD,* which is the first case already already shown not to occur. That leaves only the third case that the ratio of the squares *BD*:*FH* equals the ratio of the circles *ABCD*:*EFGH.* (Actually, there is a gap in the proof at this last step; it was never shown that these are the only three cases. It may be true that there are three cases, that the ratio of the squares is greater, less, or equal to the ratio of the circles, but the three cases of in the proof are one step removed from these three cases.)

Approximation by polygons

The first case is disposed of by approximating the circles by very close polygons. To begin with a square *EFGH* is inscribed in the circle *EFGH,* and it is shown that the remainder is less than half the circle. Next the circumferences are bisected to construct an octagon *EKFLGMHN,* and the remainder of the circle is shown to be less than half the old remainder. Continuing, polygons of 16, 32, 64, etc., sides are constructed and each one leaves a remainder less than half the previous remainder.

Now the circle *EFGH* exceeds the area *S* by some finite amount, and by the principle of proposition X.1, at some stage mentioned above, the remainder will be less than the excess of circle *EFGH* over *S.* For the rest of the proof, that stage is taken as that of the polygon *EKFLGMHN,* that is, circle

$$EFGH - \text{polygon } EKFLGMHN < \text{circle } EFGH - \text{area } S,$$

and so area *S* < polygon *EKFLGMHN.*

The similar polygon *AOBPCQDR* is inscribed in the other circle *ABCD.* Then

$$\begin{aligned}\text{circle } ABCD : \text{area } S &= BC^2 : FH^2 \\ &= \text{polygon } AOBPCQDR : \text{polygon } EKFLGMHN.\end{aligned}$$

Alternately, circle *ABCD* : polygon *AOBPCQDR* = area *S* : polygon *EKFLGMHN.* But circle *ABCD* > polygon *AOBPCQDR,* so area *S* > polygon *EKFLGMHN,* contradicting the statement above that the area *S* is less than the polygon.

Principle of exhaustion

The approximation of a figure by a sequence of figures inside it is sometimes called the "principle of exhaustion." The important point of this principle is that the sequence of approximations can be made so that the difference between the original figure and the inscribed figure decreases by at least half at each step of the sequence.

This principle is used in several later propositions in this book. Proposition XII.5 uses it to show that pyramids of the same height with triangular bases are proportional to their bases. The pyramids are approximated by a union of similar triangular prisms. Proposition XII.10 uses the principle of exhaustion to show that a cone inscribed in a cylinder is one-third of the cylinder. The cone is approximated by inscribed pyramids while the cylinder is approximated by inscribed prisms. Pyramids inscribed in cones are similarly used in XII.11 and XII.12. Finally, the principle of exhaustion is used in proposition XII.18 to show spheres are to one another in triplicate ratio of their diameters. There the spheres are exhausted by inscribed polyhedra.

Next proposition: XII.3

Previous: XII.1

Book XII introduction

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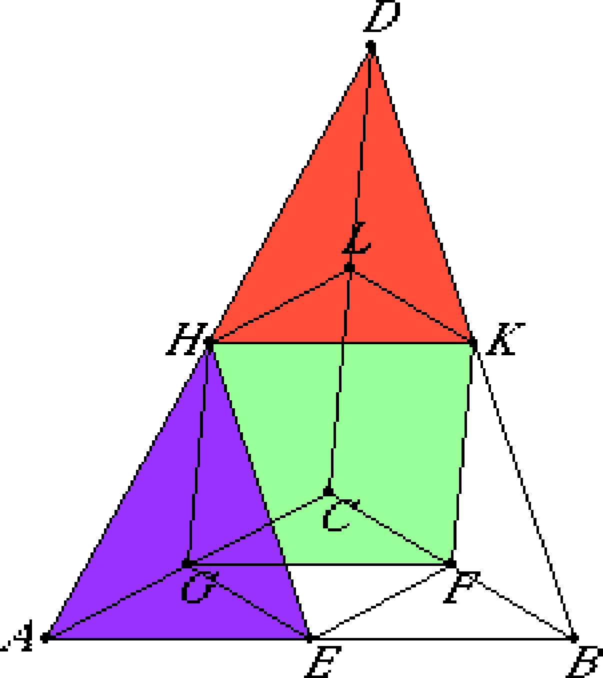
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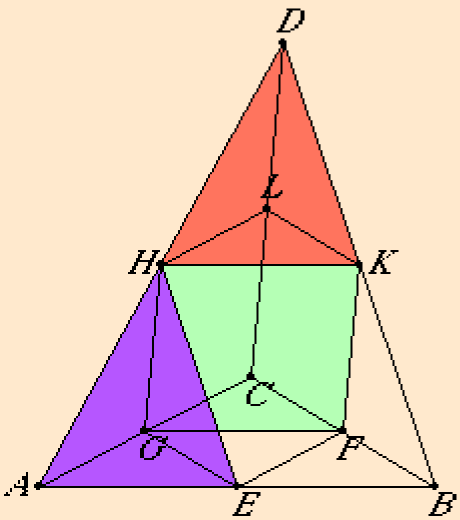
Euclid's Elements
Book XII
Proposition 3

Any pyramid with a triangular base is divided into two pyramids equal and similar to one another, similar to the whole, and having triangular bases, and into two equal prisms, and the two prisms are greater than half of the whole pyramid.

Let there be a pyramid of with the triangular base *ABC* and vertex *D*.

I say that the pyramid *ABCD* is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms, and the two prisms are greater than the half of the whole pyramid.



Bisect *AB*, *BC*, *CA*, *AD*, *DB*, and *DC* at the points *E*, *F*, *G*, *H*, *K*, and *L*. Join *HE*, *EG*, *GH*, *HK*, *KL*, *LH*, *KF*, and *FG* be joined.

Since *AE* equals *EB*, and *AH* equals *DH*, therefore *EH* is parallel to *DB*. For the same reason *HK* is also parallel to *AB*. Therefore *HEBK* is a parallelogram. Therefore *HK* equals *EB*. VI.2 I.34

But *EB* equals *EA*, therefore *AE* also equals *HK*.

But *AH* also equals *HD*, therefore the two sides *EA* and *AH* equal the two sides *KH*, *HD* respectively, and the angle *EAH* equals the angle *KHD*, therefore the base *EH* equals the base *KD*. I.4

Therefore the triangle *AEH* equals and is similar to the triangle *HKD*. For the same reason the triangle *AHG* also equals and is similar to the triangle *HLD*.

Now, since two straight lines *EH* and *HG* meeting one another are parallel to two straight lines *KD* and *DL* meeting one another and are not in the same plane, therefore they contain equal angles. Therefore the angle *EHG* equals the angle *KDL*. XI.10

And, since the two straight lines *EH* and *HG* equal the two *KD* and *DL* respectively, and the angle *EHG* equals the angle *KDL*, therefore the base *EG* equals the base *KL*. Therefore the triangle *EHG* equals and is similar to the triangle *KDL*. For the same reason the triangle *AEG* also equals and is similar to the triangle *HKL*. I.4

Therefore the pyramid with triangular base *AEG* and vertex *H* equals and is similar to the pyramid with triangular base *HKL* and the vertex *D*. XI.Def.10

And, since *HK* is parallel to *AB*, one of the sides of the triangle *ADB*, the triangle *ADB* is equiangular to the triangle *DHK*, and they have their sides proportional, therefore the triangle *ADB* is similar to the I.29 VI.Def.1

triangle *DHK.* For the same reason the triangle *DBC* is also similar to the triangle *DKL,* and the triangle *ADC* is similar to the triangle *DLH.*

Now, since the two straight lines *BA* and *AC* meeting one another are parallel to the two straight lines *KH* and *HL* meeting one another not in the same plane, therefore they contain equal angles. Therefore the angle *BAC* equals the angle *KHL.* XI.10

And *BA* is to *AC* as *KH* is to *HL,* therefore the triangle *ABC* is similar to the triangle *HKL.*

Therefore the pyramid with the triangular base *ABC* and vertex *D* is similar to the pyramid with the triangular base *HKL* and vertex *D.*

But the pyramid with the triangular base *HKL* and vertex *D* was proved similar to the pyramid with the triangular base *AEG* and the vertex *H.* Therefore each of the pyramids *AEGH* and *HKLD* is similar to the whole pyramid *ABCD.*

Next, since *BF* equals *FC,* therefore the parallelogram *EBFG* is double the triangle *GFC.* And since, if there are two prisms of equal height, and one has a parallelogram as base and the other a triangle, and if the parallelogram is double the triangle, then the prisms are equal. Therefore the prism contained by the two triangles *BKF* and *EHG,* and the three parallelograms *EBFG, EBKH,* and *HKFG* equals the prism contained by the two triangles *GFC* and *HKL* and the three parallelograms *KFCL, LCGH,* and *HKFG.* XI.39

And it is clear that each of the prisms, namely that with the parallelogram *EBFG* the base and the straight line *HK* its opposite, and that with the triangle *GFC* the base and the triangle *HKL* its opposite, is greater than each of the pyramids with the triangular bases *AEG* and *HKL* and vertices *H* and *D,* for, if we join the straight lines *EF* and *EK,* the prism with the parallelogram *EBFG* the base and the straight line *HK* opposite is greater than the pyramid with the triangular base *EBF* and vertex *K.*

But the pyramid with the triangular base *EBF* and vertex *A* equals the pyramid with the triangular base *AE* and the vertex *H,* for they are contained by equal and similar planes.

Hence the prism with the parallelogram *EBF* the base and the straight line *HK* opposite is greater than the pyramid with the triangular base *AE* and vertex *H.* But the prism with the parallelogram *EBF* the base and the straight line *HK* opposite equals the prism with the triangle *GFC* the base and the triangle *HKL* opposite, and the pyramid with the triangular base *AEG* and vertex *H* equals the pyramid with the triangular base *HKL* and vertex *D.*

Therefore the said two prisms are greater than the said two pyramids with the triangular bases *AEG* and *HKL* and vertices *H* and *D.* Therefore the whole pyramid with the triangular base *ABC* and vertex *D* has been divided into two pyramids equal to one another and into two equal prisms, and the two prisms are greater than the half of the whole pyramid.

Therefore, *any pyramid with a triangular base is divided into two pyramids equal and similar to one another, similar to the whole, and having triangular bases, and into two equal prisms, and the two prisms are greater than half of the whole pyramid.*

Q.E.D.

Guide

This and the next six propositions deal with volumes of pyramids. The first two of these lay the foundations for XII.5 (pyramids are proportional to their bases). In the last book it was shown in XI.32 that parallelepipeds of the same height are proportional to their bases, and XI.28 (a triangular prism is half a parallelepiped) implies that this proportionality can be carried over to prisms with triangular bases. It is not so easy to carry the proportionality over to pyramids with triangular bases. But that is what is done in XII.3 through XII.5.

The basic observation is in this proposition: most of a triangular based pyramid can be filled up by two congruent prisms leaving less than half to two smaller similar pyramids. Next, if each of these two smaller pyramids are filled up by two smaller prisms leaving two even smaller pyramids in each, then the four even smaller pyramids that remain are less then 1/4 of the original pyramid. Partitioning those four again yields eight with a total volume less then 1/8 of the original pyramid. And so on. Since the desired proportionality holds for prisms, and pyramids can be partitioned nearly all into prisms, therefore the desired proportionality will hold for pyramids.

This process is used and clarified in XII.5. The intermediate proposition XII.4 supplies a important technical result needed in XII.5.

Next proposition: XII.4

Previous: XII.2

Book XII introduction

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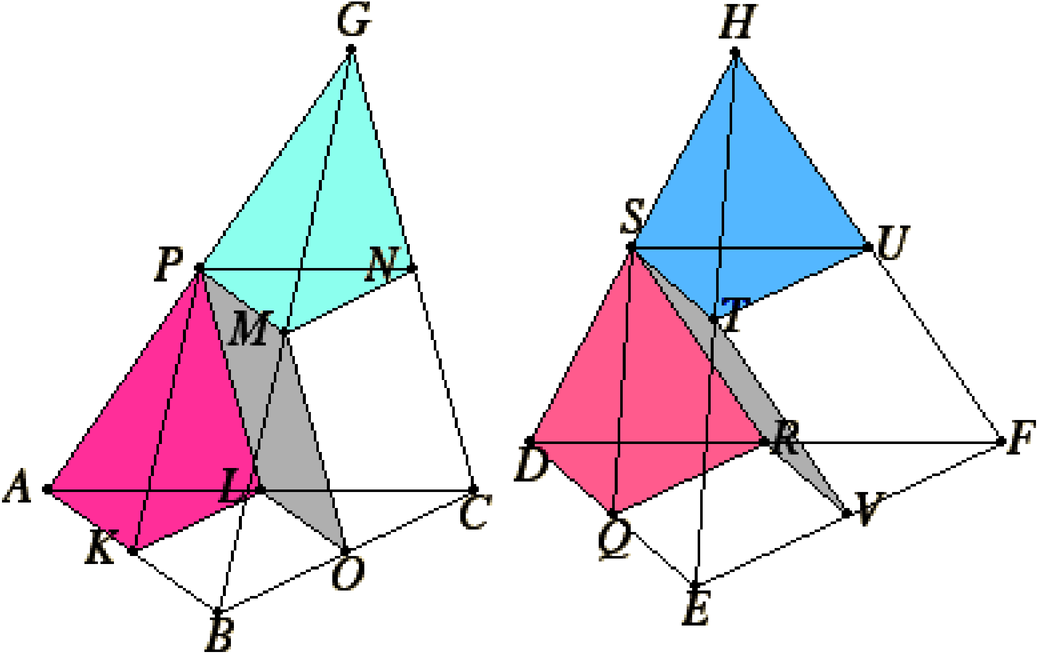
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Euclid's Elements

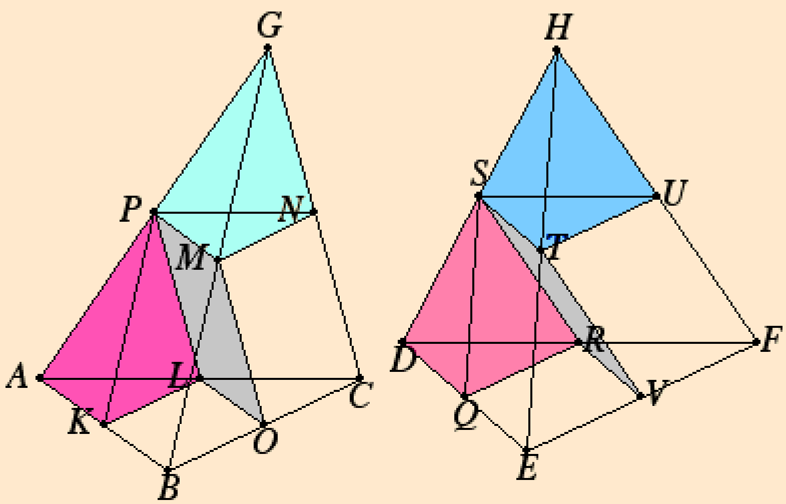
Book XII

Proposition 4

If there are two pyramids of the same height with triangular bases, and each of them is divided into two pyramids equal and similar to one another and similar to the whole, and into two equal prisms, then the base of the one pyramid is to the base of the other pyramid as all the prisms in the one pyramid are to all the prisms, being equal in multitude, in the other pyramid.

Let there be two pyramids of the same height with triangular bases *ABC* and *DEF* the points *G* and *H* the vertices, and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms. XII.3

I say that the base *ABC* is to the base *DEF* as all the prisms in the pyramid *ABCG* to all the prisms, being equal in multitude, in the pyramid *DEFH*.



Since *BO* equals *OC*, and *AL* equals *LC*, therefore *LO* is parallel to *AB*, and the triangle *ABC* is similar to the triangle *LOC*. For the same reason the triangle *DEF* is also similar to the triangle *RVF*.

And, since *BC* is double *CO*, and *EF* double *FV*, therefore *BC* is to *CO* as *EF* is to *FV*.

And on *BC* and *CO* are described the similar and similarly situated rectilinear figures *ABC* and *LOC*, and on *EF* and *FV* the similar and similarly situated figures *DEF* and *RVF*, therefore the triangle *ABC* is to the triangle *LOC* as the triangle *DEF* is to the triangle *RVF*. VI.22

Therefore, alternately the triangle *ABC* is to the triangle *DEF* as the triangle *LOC* is to the triangle *RVF*. But the triangle *LOC* is to the triangle *RVF* as the prism with the triangle *LOC* the base and *PMN* opposite is to the prism with the triangle *RVF* the base and *STU* opposite. V.16 Lemma below

Therefore the triangle *ABC* is to the triangle *DEF* as the prism with the triangle *LOC* the base and *PMN* opposite is to the prism with the triangle *RVF* the base and *STU* opposite.

But the said prisms are to one another as the prism with the parallelogram *KBOL* the base and the straight line *PM* opposite is to the prism with the parallelogram *QEVR* the base and the straight line *ST* opposite. XI.39

Therefore the two prisms, that with the parallelogram *KBOL* the base and *PM* opposite, and that with the triangle *LOC* the base and *PMN* opposite, are to the prisms with *QEVR* the base and the straight line *ST* V.12

opposite and with the triangle *RVF* the base and *STU* opposite in the same ratio.

Therefore the base *ABC* is to the base *DEF* as the said two prisms are to the said two prisms.

And similarly, if the pyramids *PMNG* and *STUH* are divided into two prisms and two pyramids, then the base *PMN* is to the base *STU* as the two prisms in the pyramid *PMNG* are to the two prisms in the pyramid *STUH*.

But the base *PMN* is to the base *STU* as the base *ABC* is to the base *DEF*, for the triangles *PMN* and *STU* equal the triangles *LOC* and *RVF* respectively.

Therefore the base *ABC* is to the base *DEF* as the four prisms are to the four prisms. And similarly, if we divide the remaining pyramids into two pyramids and into two prisms, then the base *ABC* is to base the *DEF* as all the prisms in the pyramid *ABCG* are to all the prisms, being equal in multitude, in the pyramid *DEFH*.

Lemma

But that the triangle *LOC* is to the triangle *RVF* as the prism with the triangle *LOC* the base and *PMN* opposite is to the prism with the triangle *RVF* the base and *STU* opposite, we must prove as follows.

In the same figure draw perpendiculars from *G* and *H* to the planes *ABC* and *DEF*. These are, of course, equal since the pyramids are of equal height by hypothesis. XI.11

Now, since the two straight lines *GC* and the perpendicular from *G* are cut by the parallel planes *ABC* and *PMN*, therefore they are cut in the same ratios. XI.17

And *GC* is bisected by the plane *PMN* at *N*, therefore the perpendicular from *G* to the plane *ABC* is also bisected by the plane *PMN*. For the same reason the perpendicular from *H* to the plane *DEF* is also bisected by the plane *STU*.

And the perpendiculars from *G* and *H* to the planes *ABC* and *DEF* are equal, therefore the perpendiculars from the triangles *PMN* and *STU* to the planes *ABC* and *DEF* are also equal.

Therefore the prisms with the triangles *LOC* and *RVF* the bases, and *PMN* and *STU* opposite, are of equal height.

Hence also the parallelepipedal solids described from the said prisms are of equal height and are to one another as their bases. Therefore their halves, namely the said prisms, are to one another as the base *LOC* is to the base *RVF*. XI.32 XI.28

Therefore, *if there are two pyramids of the same height with triangular bases, and each of them is divided into two pyramids equal and similar to one another and similar to the whole, and into two equal prisms, then the base of the one pyramid is to the base of the other pyramid as all the prisms in the one pyramid are to all the prisms, being equal in multitude, in the other pyramid.*

Q.E.D.

Guide

This proposition is subordinate to the next, XII.5, in which two pyramids with triangular bases and the same height are shown to be proportional to their bases. Its proof proceeds by partitioning each of the two original pyramids into the two- pyramid-two-prism division of the previous proposition, then doing the same partition to the two smaller pyramids, then to the four even smaller pyramids, until a sufficiently small part of each original pyramid remains in whatever tiny pyramids there are while a sufficiently large part of each is composed of various sized prisms.

This proposition, at least in the last paragraph, considers that situation and concludes that the base of the first pyramid is to the second as the union of the various sized prisms in the first pyramid is to the union of the various sized prisms in the second pyramid. This is the crucial step in the proof of XII.5.

Next proposition: XII.5

Previous: XII.3

Book XII introduction

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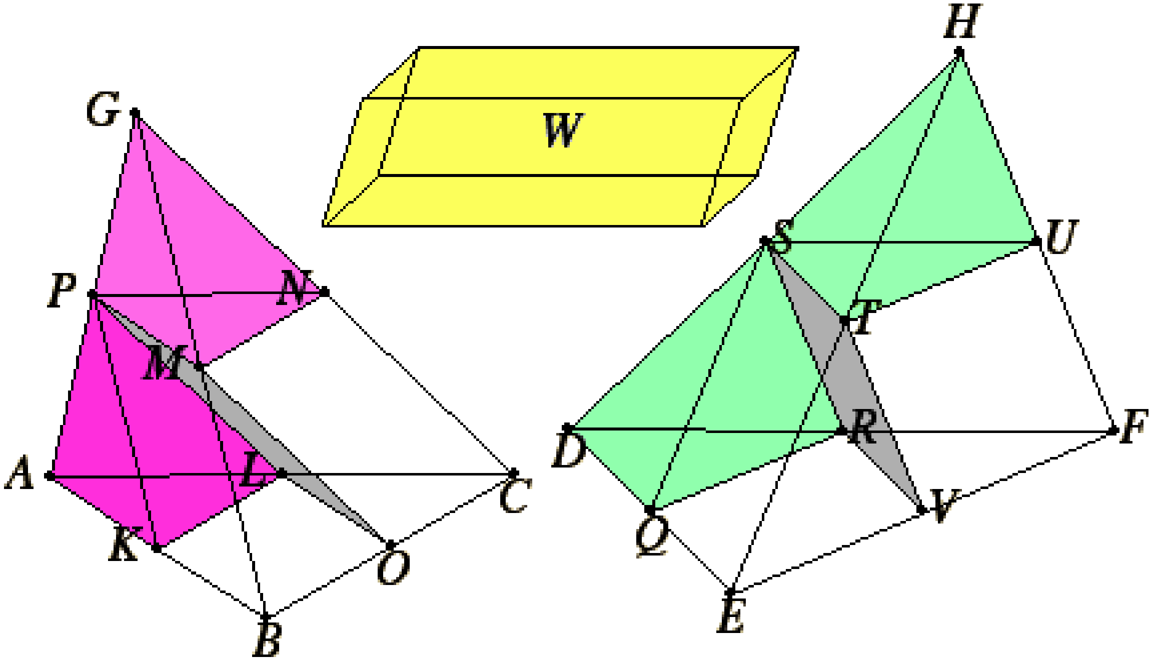
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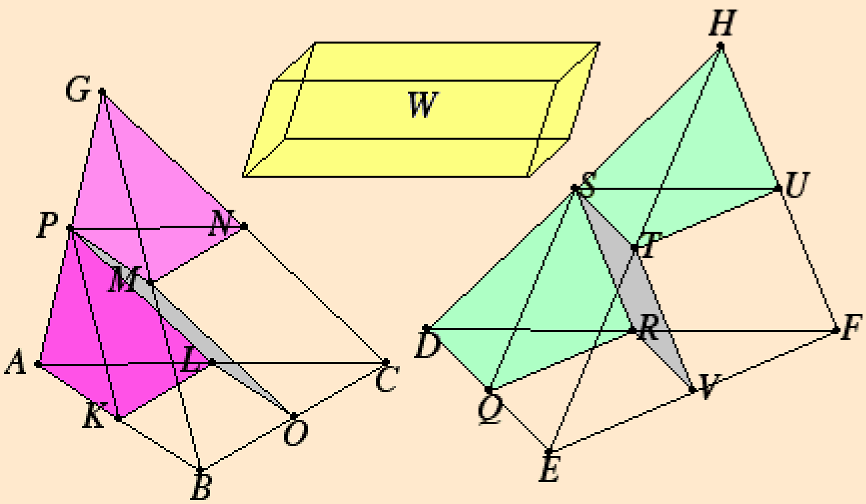
Euclid's Elements
Book XII
Proposition 5

Pyramids of the same height with triangular bases are to one another as their bases.

Let there be pyramids of the same height with triangular bases *ABC* and *DEF* and vertices *G* and *H*.

I say that the base *ABC* is to the base *DEF* as the pyramid *ABCG* is to the pyramid *DEFH*.



For, if the pyramid *ABCG* is not to the pyramid *DEFH* as the base *ABC* is to the base *DEF*, then the base *ABC* is to the base *DEF* as the pyramid *ABCG* is either to some solid less than the pyramid *DEFH* or to a greater solid.

Let it, first, be in that ratio to a less solid *W*.

Divide the pyramid *DEFH* into two pyramids equal to one another and similar to the whole and into two equal prisms.

Then the two prisms are greater than the half of the whole pyramid. XII.3

Again, divide the pyramids arising from the division similarly, and let this be done repeatedly until there are left over from the pyramid *DEFH* some pyramids which are less than the excess by which the pyramid *DEFH* exceeds the solid *W*. X.1

Let such be left, and let them be, for the sake of argument, *DQRS* and *STUH*. Therefore the remainders, the prisms in the pyramid *DEFH*, are greater than the solid *W*.

Divide the pyramid *ABCG* similarly, and a same number of times, with the pyramid *DEFH*. Therefore the base *ABC* is to the base *DEF* as the prisms in the pyramid *ABCG* are to the prisms in the pyramid *DEFH*. XII.4

But the base *ABC* is to the base *DEF* as the pyramid *ABCG* is to the solid *W*, therefore the pyramid *ABCG* is to the solid *W* as the prisms in the pyramid *ABCG* are to the prisms in the pyramid *DEFH*. V.11
Therefore, alternately the pyramid *ABCG* is to the prisms in it as the solid *W* is to the prisms in the pyramid *DEFH*. V.16

But the pyramid *ABCG* is greater than the prisms in it, therefore the solid *W* is also greater than the prisms in the pyramid *DEFH*.

But it is also less, which is impossible.

Therefore the prism *ABCG* is not to any solid less than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

Similarly it can be proved that neither is the pyramid *DEFH* to any solid less than the pyramid *ABCG* as the base *DEF* is to the base *ABC*.

I say next that neither is the pyramid *ABCG* to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

For, if possible, let it be in that ratio to a greater solid *W*.

Therefore, inversely the base *DEF* is to the base *ABC* as the solid *W* is to the pyramid *ABCG*.

But it was proved before that the solid *W* is to the solid *ABCG* as the pyramid *DEFH* is to some solid less than the pyramid *ABCG*. Therefore the base *DEF* is to the base *ABC* as the pyramid *DEFH* is to some solid less than the pyramid *ABCG*, which was proved absurd. XII.2,Lemma V.11

Therefore the pyramid *ABCG* is not to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*.

But it was proved that neither is it in that ratio to a less solid.

Therefore the base *ABC* is to the base *DEF* as the pyramid *ABCG* is to the pyramid *DEFH*.

Therefore, *pyramids of the same height with triangular bases are to one another as their bases.*

Q.E.D.

Guide

Use of this theorem

The next proposition generalizes this one so that the bases of the pyramids may be any polygons, not just triangles. In the following proposition, this proposition is used to show that a prism can be dissected into three equal (but not congruent) prisms.

Next proposition: XII.6

Previous: XII.4

Book XII introduction

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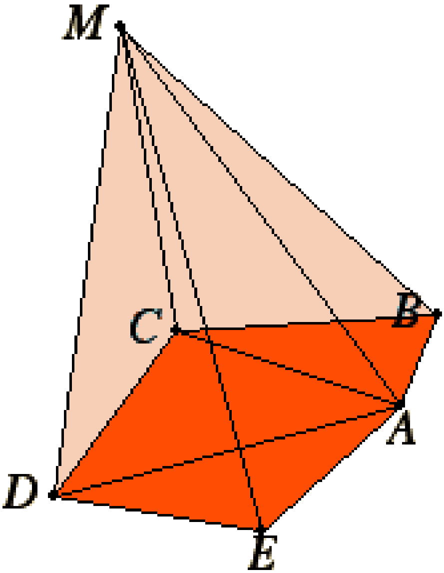
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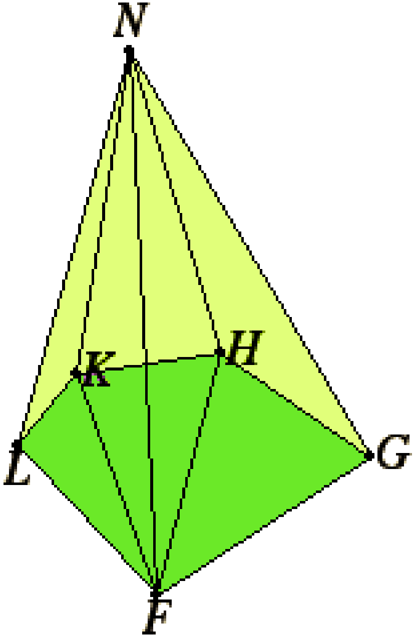
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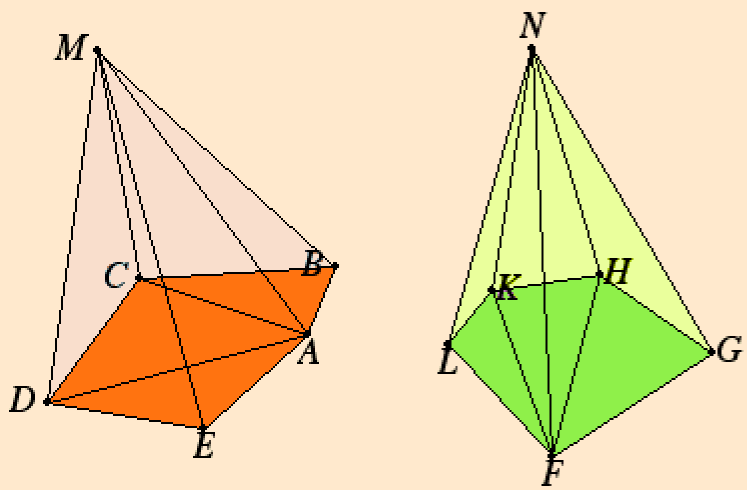
Euclid's Elements
Book XII
Proposition 6

Pyramids of the same height with polygonal bases are to one another as their bases.

Let there be pyramids of the same height with the polygonal bases *ABCDE* and *FGHKL* and vertices *M* and *N.*

I say that the base *ABCDE* is to the base *FGHKL* as the pyramid *ABCDEM* is to the pyramid *FGHKLN.*

Join *AC*, *AD*, *FH,* and *FK.*



Since then *ABCM* and *ACDM* are two pyramids with triangular bases and equal height, therefore they are to one another as their bases. Therefore the base *ABC* is to the base *ACD* as the pyramid *ABCM* is to the pyramid *ACDM.* And, taken together, the base *ABCD* is to the base *ACD* as the pyramid *ABCDM* is to the pyramid *ACDM.* XII.5 V.18

But the base *ACD* is to the base *ADE* as the pyramid *ACDM* is to the pyramid *ADEM.* XII.5

Therefore, *ex aequali*, the base *ABCD* is to the base *ADE* as the pyramid *ABCDM* is to the pyramid *ADEM.* V.22

And again, taken together, the base *ABCDE* is to the base *ADE* as the pyramid *ABCDEM* is to the pyramid *ADEM.* Similarly also it can be proved that the base *FGHKL* is to the base *FGH* as the pyramid *FGHKLN* is to the pyramid *FGHN.* V.18

And, since *ADEM* and *FGHN* are two pyramids with triangular bases and equal heights, therefore the base *ADE* is to the base *FGH* as the pyramid *ADEM* is to the pyramid *FGHN.* XII.5

But the base *ADE* is to the base *ABCDE* as the pyramid *ADEM* is to the pyramid *ABCDEM.* Therefore, *ex aequali*, the base *ABCDE* is to the base *FGH* as the pyramid *ABCDEM* is to the pyramid *FGHN.* V.22

But further the base *FGH* is to the base *FGHKL* as the pyramid *FGHN* is to the pyramid *FGHKLN.* Therefore also, *ex aequali*, the base *ABCDE* is to the base *FGHKL* as the pyramid *ABCDEM* is to the pyramid *FGHKLN* V.22

Therefore, *pyramids of the same height with polygonal bases are to one another as their bases.*

Q.E.D.

Guide

It is important to notice that the bases of the pyramids under consideration need not be similar, indeed they may have different numbers of sides.

Use of this proposition

This proposition will be used in XII.10 and XII.11 which concern the volumes of cones and cylinders.

Next proposition: XII.7

Previous: XII.5

Book XII introduction

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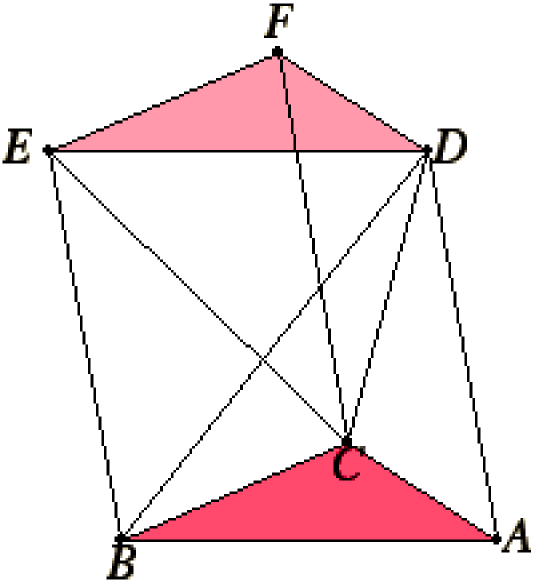
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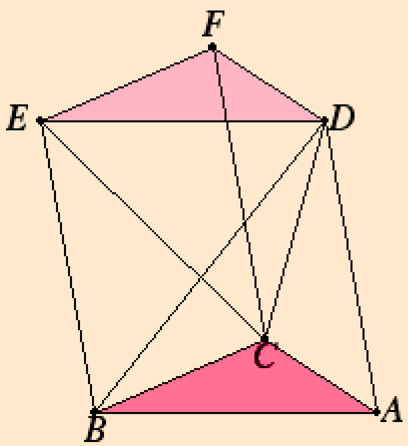
Euclid's Elements
Book XII
Proposition 7

Any prism with a triangular base is divided into three pyramids equal to one another with triangular bases.

Let there be a prism with the triangular base *ABC* and *DEF* opposite.

I say that the prism *ABCDEF* is divided into three pyramids equal to one another, which have triangular bases.



Join *BD, EC,* and *CD.*

Since *ABED* is a parallelogram, and *BD* is its diameter, therefore the triangle *ABD* equals the triangle *EBD.* Therefore the pyramid with triangular base *ABD* and vertex *C* equals the pyramid with triangular base *DEB* and vertex *C.* I.34 XII.5

But the pyramid with triangular base *DEB* and vertex *C* is identical with the pyramid with triangular base *EBD* and vertex *D,* for they are contained by the same planes.

Therefore the pyramid with triangular base *ABD* vertex *C* is also equal to the pyramid with triangular base *EBC* and vertex *D.*

Again, since *FCBE* is a parallelogram, and *CE* is its diameter, therefore the triangle *CEF* equals the triangle *CBE.* I.34

Therefore the pyramid with triangular base *BCE* and vertex *D* equals the pyramid with triangular base *ECF* and vertex *D.* XII.5

But the pyramid with triangular base *BCE* and vertex *D* was proved equal to the pyramid with triangular base *ABD* and vertex *C,* therefore the pyramid with triangular base *CEF* and vertex *D* equals the pyramid with triangular base *ABD* and vertex *C.* Therefore the prism *ABCDEF* is divided into three pyramids equal to one another which have triangular bases.

And, since the pyramid with triangular base *ABD* and vertex *C* is identical with the pyramid with triangular base *CAB* and vertex *D,* for they are contained by the same planes, while the pyramid with triangular base *ABD* vertex *C* was proved to be a third of the prism with triangular base *ABC* and *DEF* opposite, therefore the pyramid with triangular base *ABC* and vertex *D* is a third of the prism with the same base *ABC,* and *DEF* opposite.

Therefore, *any prism with a triangular base is divided into three pyramids equal to one another with triangular bases.*

Corollary.

From this it is clear that *any pyramid is a third part of the prism with the same base and equal height.*

Q.E.D.

Guide

The proof of this proposition is easier than it looks. The triangles *ABD* and *EBD* are equal. Now since the pyramids *ABDC* and *DEBC* have equal bases and the same altitude, by XII.5, they are equal pyramids. A similar argument shows pyramids *BCED*and *ECFD* are equal. But *DEBC* and *BCED* are the same pyramid named differently. So the prism is divided into three equal pyramids.

Use of this proposition

This proposition is used in the next two propositions about volumes of pyramids and in XII.10 following them about volumes of cones and cylinders.

Next proposition: XII.8

Previous: XII.6

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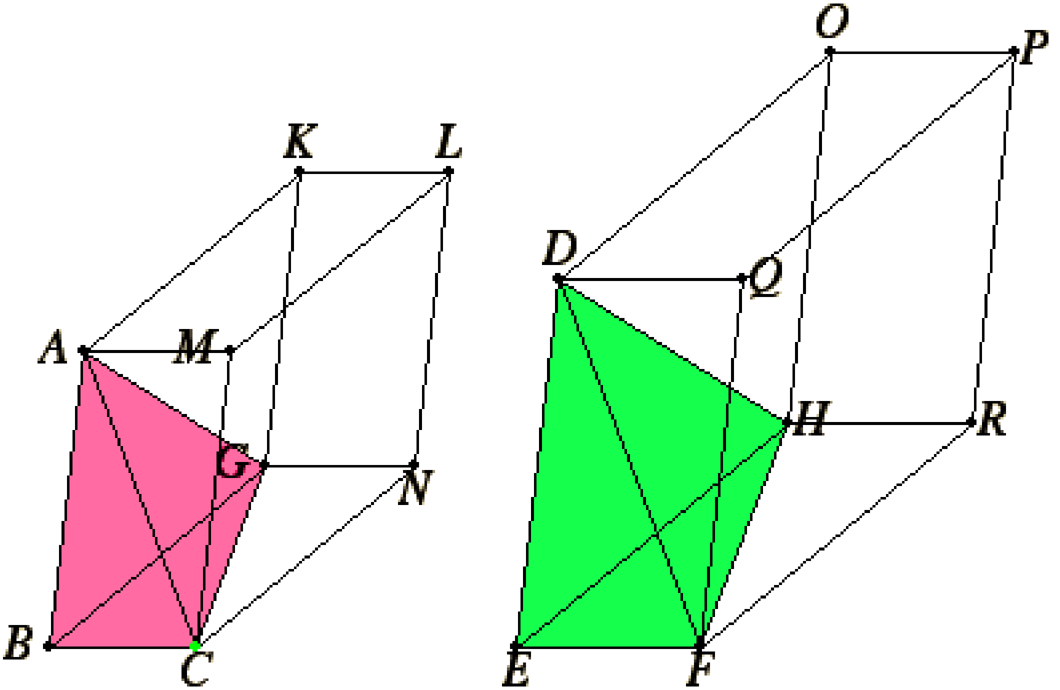

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Euclid's Elements

Book XII

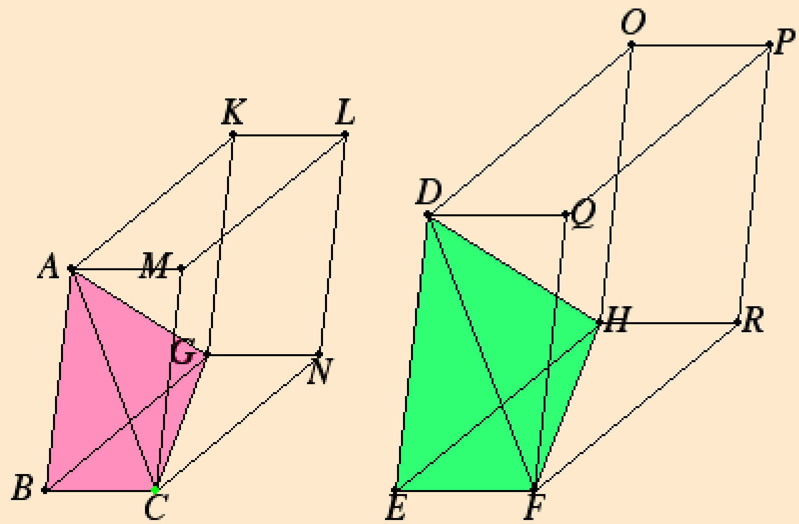
Proposition 8

Similar pyramids with triangular bases are in triplicate ratio of their corresponding sides.

Let there be similar and similarly situated pyramids with triangular bases *AB* and *DEF* vertices *G* and *H*.

I say that the pyramid *ABCG* has to the pyramid *DEFH* the ratio triplicate of that which *BC* has to *EF*.

Complete the parallelepipedal solids *BGML* and *EHQP*.



Now, since the pyramid *ABCG* is similar to the pyramid *DEFH*, therefore the angle *ABC* equals the angle *DEF*, the angle *GBC* equals the angle *HEF*, the angle *ABG* equals the angle *DEH*, and *AB* is to *DE* as *BC* is to *EF*, and as *BG* is to *EH*.

And since *AB* is to *DE* as *BC* is to *EF*, and the sides are proportional about equal angles, therefore the parallelogram *BM* is similar to the parallelogram *EQ*. For the same reason *BN* is also similar to *ER*, and *BR* similar to *EO*.

Therefore the three parallelograms *MB*, *BK*, and *BN* are similar to the three *EQ*, *EO*, and *ER*. But the three parallelograms *MB*, *BK*, and *BN* are equal and similar to their three opposites, and the three *EQ*, *EO*, and *ER* are equal and similar to their three opposites. XI.24

Therefore the solids *BGML* and *EHQP* are contained by similar planes equal in multitude. Therefore the solid *BGML* is similar to the solid *EHQP*.

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides. Therefore the solid *BGML* has to the solid *EHQP* the ratio triplicate of that which the corresponding side *BC* has to the corresponding side *EF*. XI.33

But the solid *BGML* is to the solid *EHQP* as the pyramid *ABCG* is to the pyramid *DEFH*, for the pyramid is a sixth part of the solid, because the prism which is half of the parallelepipedal solid is also triple the pyramid. XI.28 XII.7

Therefore the pyramid *ABCG* has to the pyramid *DEFH* the ratio triplicate of that which *BC* has to *EF*.

Q.E.D.

Corollary.

From this it is clear that *similar pyramids with polygonal bases are also to one another in the triplicate ratio of their corresponding sides.*

For, if they are divided into the pyramids contained in them which have triangular bases, by virtue of the fact that the similar polygons forming their bases are also divided into similar triangles equal in multitude and corresponding to the wholes, then the one pyramid with a triangular base in the one complete pyramid is to the one pyramid with a triangular base in the other complete pyramid as all the pyramids with triangular bases contained in the one pyramid is to all the pyramids with triangular bases contained in the other pyramid, that is, the pyramid itself with a polygonal base, to the pyramid with a polygonal base.

VI.20

V.12

But the pyramid with a triangular base is to the pyramid with a triangular base in the triplicate ratio of the corresponding sides, therefore also the pyramid with a polygonal base has to the pyramid with a similar base the ratio triplicate of that which the side has to the side.

Therefore, *similar pyramids with triangular bases are in triplicate ratio of their corresponding sides.*

Guide

Heath gives a good argument that the corollary was added later.

Use of this propostion

This proposition is used to show similar cones are in triplicate ratio of the diameters of their bases in proposition XII.12. Also, the corollary justifies a statement in the corollary of XII.17 concerning similar solids.

Next proposition: XII.9

Previous: XII.7

Book XII introduction

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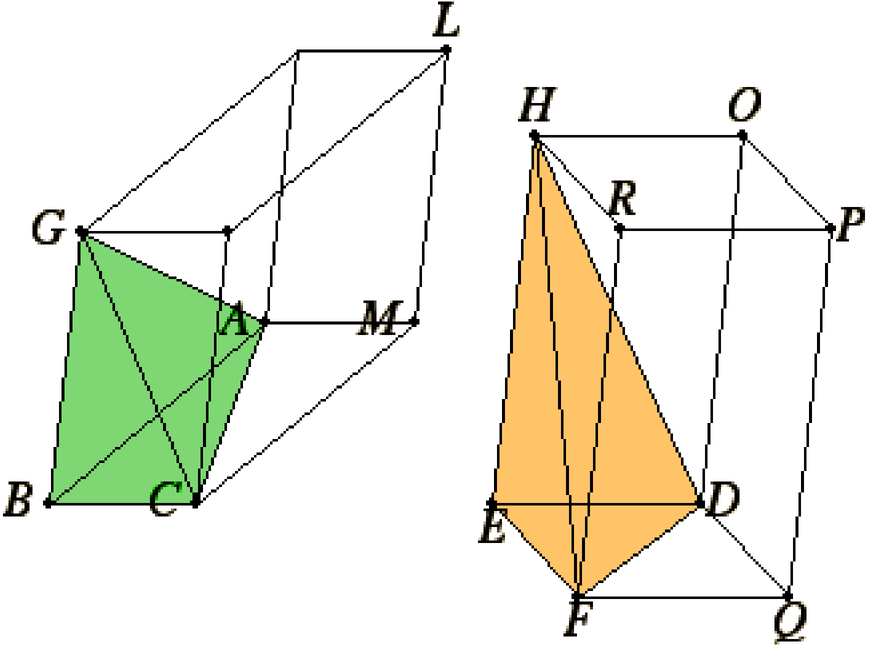

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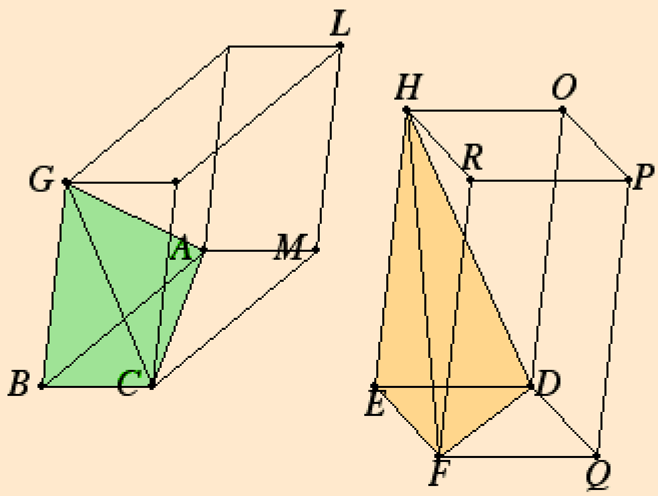
Euclid's Elements
Book XII
Proposition 9

In equal pyramids with triangular bases the bases are reciprocally proportional to the heights; and those pyramids are equal in which the bases are reciprocally proportional to the heights.

Let there be equal pyramids with triangular bases *ABC* and *DEF* and vertices *G* and *H*.

I say that in the pyramids *ABCG* and *DEFH* the bases are reciprocally proportional to the heights, that is the base *ABC* is to the base *DEF* as the height of the pyramid *DEFH* is to the height of the pyramid *ABCG*.

Complete the parallelepipedal solids *BGML* and *EHQP*.



Now, since the pyramid *ABCG* equals the pyramid *DEFH*, and the solid *BGML* is six times the pyramid *ABCG*, and the solid *EHQP* six times the pyramid *DEFH*, therefore the solid *BGML* equals the solid *EHQP*. XII.7,Cor

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights, therefore the base *BM* is to the base *EQ* as the height of the solid *EHQP* is to the height of the solid *BGML*. XI.34

But the base *BM* is to *EQ* as the triangle *ABC* is to the triangle *DEF*. Therefore the triangle *ABC* is to the triangle *DEF* as the height of the solid *EHQP* is to the height of the solid *BGML*. I.34 V.11

But the height of the solid *EHQP* is identical with the height of the pyramid *DEFH*, and the height of the solid *BGML* is identical with the height of the pyramid *ABCG*, therefore the base *ABC* is to the base *DEF* as the height of the pyramid *DEFH* is to the height of the pyramid *ABCG*.

Therefore in the pyramids *ABCG* and *DEFH* the bases are reciprocally proportional to the heights.

Next, in the pyramids *ABCG* and *DEFH* let the bases be reciprocally proportional to the heights, that is, as the base *ABC* is to the base *DEF*, so let the height of the pyramid *DEFH* be to the height of the pyramid *ABCG*.

I say that the pyramid *ABCG* equals the pyramid *DEFH*.

With the same construction, since the base *ABC* is to the base *DEF* as the height of the pyramid *DEFH* is to the height of the pyramid *ABCG*, while the base *ABC* is to the base *DEF* as the parallelogram *BM* is to the parallelogram *EQ*, therefore the parallelogram *BM* is to the parallelogram *EQ* as the height of the pyramid *DEFH* is to the height of the pyramid *ABCG*. V.11

But the height of the pyramid *DEFH* is identical with the height of the parallelepiped *EHQP*, and the height of the pyramid *ABCG* is identical with the height of the parallelepiped *BGML*, therefore the base *BM* is to the base *EQ* as the height of the parallelepiped *EHQP* is to the height of the parallelepiped *BGML*.

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal, therefore the parallelepipedal solid *BGML* equals the parallelepipedal solid *EHQP*. XI.34

And the pyramid *ABCG* is a sixth part of *BGML*, and the pyramid *DEFH* a sixth part of the parallelepiped *EHQP*, therefore the pyramid *ABCG* equals the pyramid *DEFH*.

Therefore, *in equal pyramids with triangular bases the bases are reciprocally proportional to the heights; and those pyramids are equal in which the bases are reciprocally proportional to the heights.*

Q.E.D.

Guide

The pyramids with triangular bases are one-third of the prisms with triangular bases, which aren't drawn, and the prisms are half of the parallelepipeds. Since the analogous proposition XI.34 holds for parallelepipeds, this proposition holds for pyramids.

When a similar situation appears later where cones are one-third of cylinders, Euclid simply says the same holds for cones, too, with no details whatsoever.

This proposition completes the theory of volumes for pyramids. The next few propositions treat the theory of volumes of cones and cylinders.

Next proposition: XII.10

Previous: XII.8

Book XII introduction

Select from Book XII

Select book

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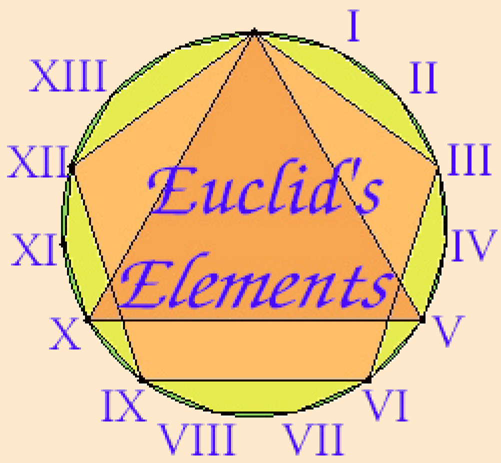
Book XIII



Book XIII

Table of contents

Propositions

Proposition 1.
If a straight line is cut in extreme and mean ratio, then the square on the greater segment added to the half of the whole is five times the square on the half.

Proposition 2.
If the square on a straight line is five times the square on a segment on it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

Lemma for XIII.2.

Proposition 3.
If a straight line is cut in extreme and mean ratio, then the square on the sum of the lesser segment and the half of the greater segment is five times the square on the half of the greater segment.

Proposition 4.
If a straight line is cut in extreme and mean ratio, then the sum of the squares on the whole and on the lesser segment is triple the square on the greater segment.

Proposition 5.
If a straight line is cut in extreme and mean ratio, and a straight line equal to the greater segment is added to it, then the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment.

Proposition 6.
If a rational straight line is cut in extreme and mean ratio, then each of the segments is the irrational straight line called apotome.

Proposition 7.
If three angles of an equilateral pentagon, taken either in order or not in order, are equal, then the pentagon is

equiangular.

Proposition 8.
If in an equilateral and equilateral pentagon straight lines subtend two angles are taken in order, then they cut one another in extreme and mean ratio, and their greater segments equal the side of the pentagon.

Proposition 9.
If the side of the hexagon and that of the decagon inscribed in the same circle are added together, then the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

Proposition 10.
If an equilateral pentagon is inscribed ina circle, then the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon inscribed in the same circle.

Proposition 11.
If an equilateral pentagon is inscribed in a circle which has its diameter rational, then the side of the pentagon is the irrational straight line called minor.

Proposition 12.
If an equilateral triangle is inscribed in a circle, then the square on the side of the triangle is triple the square on the radius of the circle.

Proposition 13.
To construct a pyramid, to comprehend it in a given sphere; and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Lemma for XIII.13.

Proposition 14.
To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double the square on the side of the octahedron.

Proposition 15.
To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple the square on the side of the cube.

Proposition 16.
To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the icosahedron is the irrational straight line called minor.

Corollary. The square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle.

Proposition 17.
To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the dodecahedron is the irrational straight line called apotome.

Corollary. When the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron.

Proposition 18.
To set out the sides of the five figures and compare them with one another.

Remark.

No other figure, besides the said five figures, can be constructed by equilateral and equiangular figures equal to one another.

Lemma. The angle of the equilateral and equiangular pentagon is a right angle and a fifth.

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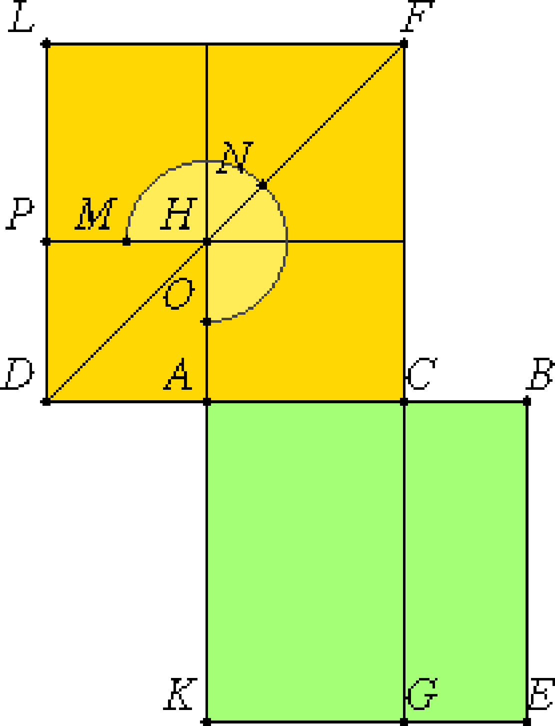
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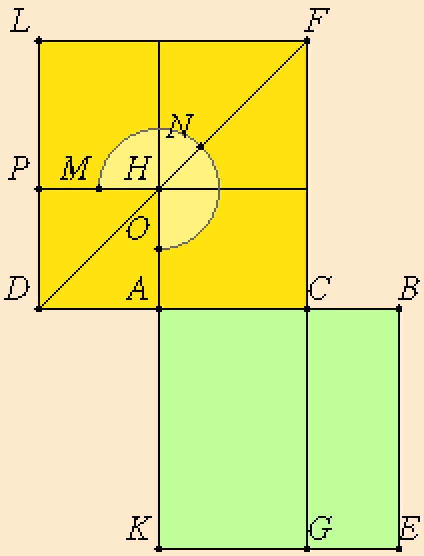
Euclid's Elements
Book XIII
Proposition 1

If a straight line is cut in extreme and mean ratio, then the square on the greater segment added to the half of the whole is five times the square on the half.

Let the straight line *AB* be cut in extreme and mean ratio at the point *C*, and let *AC* be the greater segment. Produce the straight line *AD* in a straight line with *CA*, and make *AD* half of *AB*.

I say that the square on *CD* is five times the square on *AD*.



Describe the squares *AE* and *DF* on *AB* and *DC*, draw the figure in *DF*, and carry *FC* through to *G*. I.46

Now, since *AB* is cut in extreme and mean ratio at *C*, therefore the rectangle *AB* by *BC* equals the square on *AC*. And *CE* is the rectangle *AB* by *BC*, and *FH* is the square on *AC*, therefore *CE* equals *FH*. VI.Def.3 VI.17

And, since *BA* is double *AD*, while *BA* equals *KA*, and *AD* equals *AH*, therefore *KA* is also double *AH*.

But *KA* is to *AH* as *CK* is to *CH*, therefore *CK* is double *CH*. But the sum of *LH* and *HC* is also double *CH*. Therefore *KC* equals the sum of *LH* and *HC*. VI.1

But *CE* was also proved equal to *HF*, therefore the whole square *AE* equals the gnomon *MNO*.

And, since *BA* is double *AD*, therefore the square on *BA* is quadruple the square on *AD*, that is, *AE* is quadruple *DH*.

But *AE* equals the gnomon *MNO*, therefore the gnomon *MNO* is also quadruple *AP*. Therefore the whole *DF* is five times *AP*.

And *DF* is the square on *DC*, and *AP* the square on *DP*, therefore the square on *CD* is five times the square on *DA*.

Therefore, *if a straight line is cut in extreme and mean ratio, then the square on the greater segment added to the half of the whole is five times the square on the half.*

Q.E.D.

Guide

Use of this theorem

This proposition is used in the proofs of XIII.6 and XIII.11. Those propositions are in turn used to make conclusions about the sides of the icosahedron and dodecahedron constructed in propositions XIII.16 and XIII.17.

Next proposition: XIII.2

Book XIII introduction

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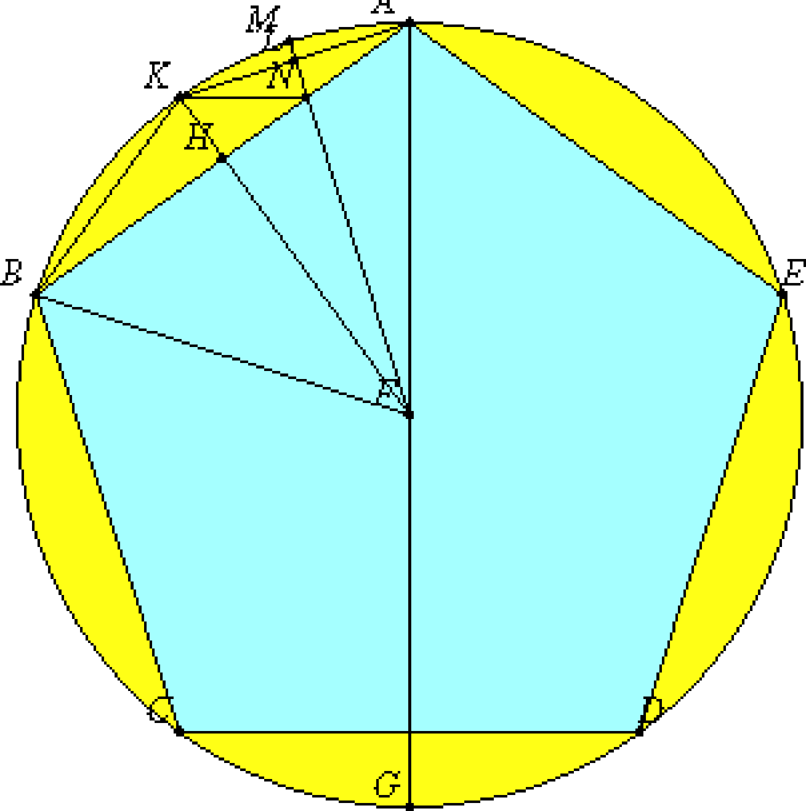

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Euclid's Elements

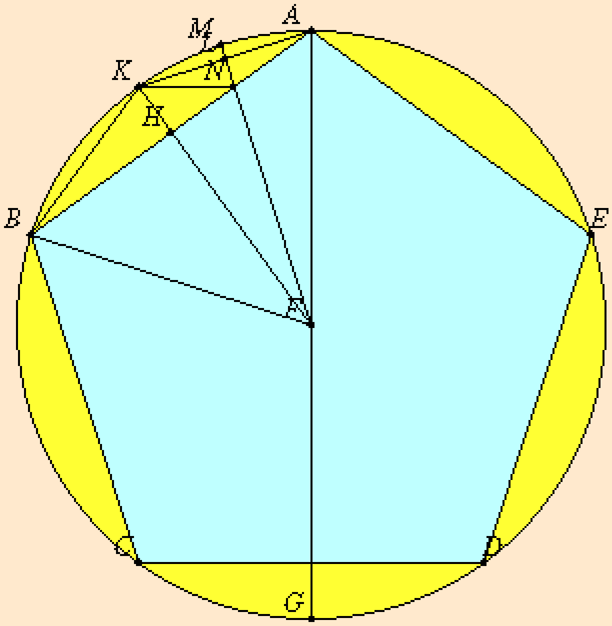
Book XIII

Proposition 10

If an equilateral pentagon is inscribed in a circle, then the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon inscribed in the same circle.

Let *ABCDE* be a circle, and let the equilateral pentagon *ABCDE* be inscribed in the circle *ABCDE.*

I say that the square on the side of the pentagon *ABCDE* equals the sum of the squares on the side of the hexagon and on that of the decagon inscribed in the circle *ABCDE.*



Take the center *F* of the circle, join *AF* and carry it through to the point *G,* and join *FB.* Draw *FH* from *F* perpendicular to *AB* and carry it through to *K,* join *AK* and *KB,* draw *FL* from *F* perpendicular to *AK,* carry it through to *M,* and join *KN.* III.1 I.12

Since the circumference *ABCG* equals the circumference *AEDG,* and in them *ABC* equals *AED,* therefore the remainder, the circumference *CG,* equals the remainder *GD.*

But *CD* belongs to a pentagon, therefore *CG* belongs to a decagon.

And, since *FA* equals *FB,* and *FH* is perpendicular, therefore the angle *AFK* equals the angle *KFB.* I.5 I.26

Hence the circumference *AK* equals *KB.* Therefore the circumference *AB* is double the circumference *BK.* Therefore the straight line *AK* is a side of a decagon. For the same reason *AK* is double *KM.* III.26

Now, since the circumference *AB* is double the circumference *BK,* while the circumference *CD* equals the circumference *AB,* therefore the circumference *CD* is also double the circumference *BK.*

But the circumference *CD* is also double *CG,* therefore the circumference *CG* equals the circumference *BK.* But *BK* is double *KM,* since *KA* is so also, therefore *CG* is also double *KM.*

But, further, the circumference *CB* is also double the circumference *BK,* for the circumference *CB* equals *BA.* Therefore the whole circumference *GB* is also double *BM.* Hence the angle *GFB* is double the angle *BFM.* VI.33

But the angle *GFB* is double the angle *FAB*, for the angle *FAB* equals the angle *ABF*. Therefore the angle *BFN* equals the angle *FAB*.

But the angle *ABF* is common to the two triangles *ABF* and *BFN*, therefore the remaining angle *AFB* equals the remaining angle *BNF*. Therefore the triangle *ABF* is equiangular with the triangle *BFN*. I.32

Therefore, proportionally the straight line *AB* is to *BF* as *FB* is to *BN*. Therefore the rectangle *AB* by *BN* equals the square on *BF*. VI.4 VI.17

Again, since *AL* equals *LK*, while *LN* is common and at right angles, therefore the base *KN* equals the base *AN*. Therefore the angle *LKN* also equals the angle *LAN*. I.4

But the angle *LAN* equals the angle *KBN*, therefore the angle *LKN* also equals the angle *KBN*.

And the angle at *A* is common to the two triangles *AKB* and *AKN*. Therefore the remaining angle *AKB* equals the remaining angle *KNA*. I.32

Therefore the triangle *KBA* is equiangular with the triangle *KNA*. Therefore, proportionally the straight line *BA* is to *AK* as *KA* is to *AN*. VI.1

Therefore the rectangle *BA* by *AN* equals the square on *AK*. VI.17

But the rectangle *AB* by *BN* was also proved equal to the square on *BF*, therefore the sum of the rectangle *AB* by *BN* and the rectangle *BA* by *AN*, that is, the square on *BA*, equals the sum of the squares on *BF* and *AK*. II.2

And *BA* is a side of the pentagon, *BF* of the hexagon, and *AK* of the decagon. IV.15,Cor.

Therefore, *if an equilateral pentagon is inscribed in a circle, then the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon inscribed in the same circle.*

Q.E.D.

Guide

Use of this proposition

This result is used in XIII.16 for the construction of an icosahedron and later in XIII.18 when an icosahedron is compared to the other four regular polyhedra.

Next proposition: XIII.11

Previous: XIII.9

Book XIII introduction

Select from Book XIII

Select book

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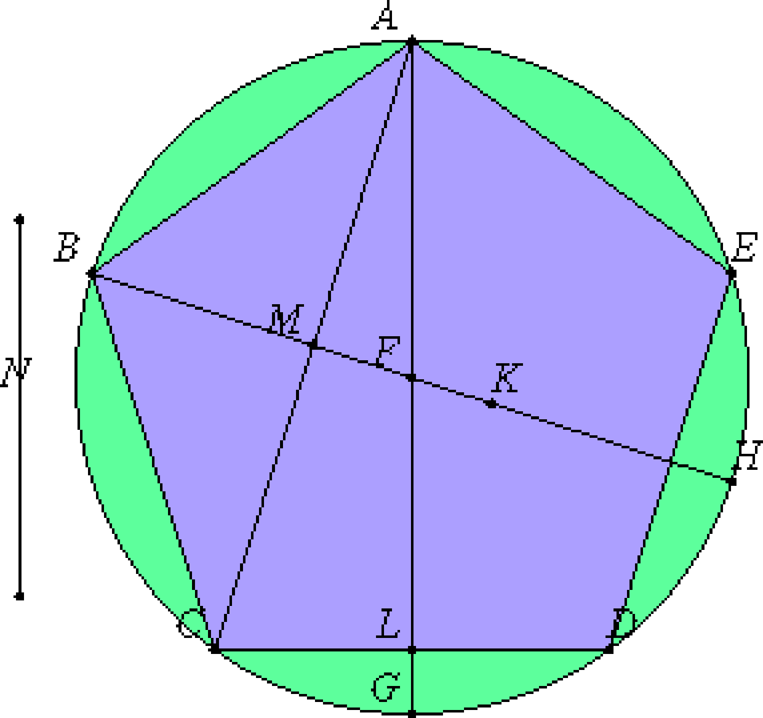
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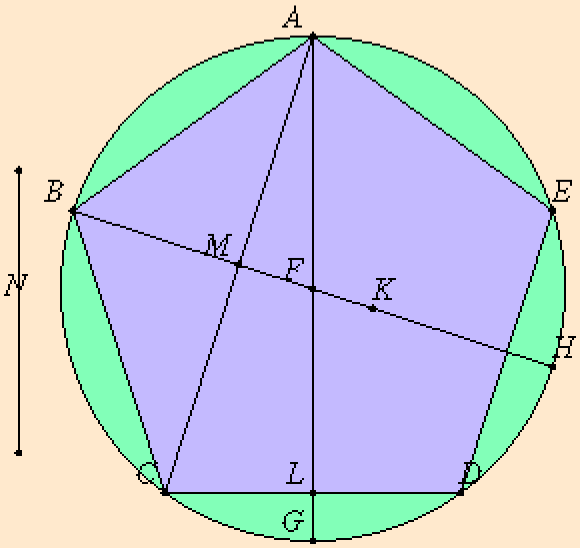
Euclid's Elements
Book XIII
Proposition 11

If an equilateral pentagon is inscribed in a circle which has its diameter rational, then the side of the pentagon is the irrational straight line called minor.

In the circle *ABCDE* which has its diameter rational let the equilateral pentagon *ABCDE* be inscribed.

I say that the side of the pentagon is the irrational straight line called minor.



Take the center *F* of the circle, join *AF* and *FB* and carry them through to the points *G* and *H,* join *AC,* and make *FK* a fourth part of *AF.* III.1 VI.9

Now *AF* is rational, therefore *FK* is also rational. But *BF* is also rational, therefore the whole *BK* is rational.

And, since the circumference *ACG* equals the circumference *ADG,* and in them *ABC* equals *AED,* therefore the remainder *CG* equals the remainder *GD.*

And, if we join *AD,* then we conclude that the angles at *L* are right, and *CD* is double *CL.*

For the same reason the angles at *M* are also right, and *AC* is double *CM.*

Since then the angle *ALC* equals the angle *AMF,* and the angle *LAC* is common to the two triangles *ACL* and *AMF,* therefore the remaining angle *ACL* equals the remaining angle *MFA.* I.32

Therefore the triangle *ACL* is equiangular with the triangle *AMF.* Therefore, proportionally *LC* is to *CA* as *MF* is to *FA.* Taking the doubles of the antecedents, therefore double *LC* is to *CA* as double *MF* to *FA.*

But double *MF* is to *FA* as *MF* is to the half of *FA,* therefore also double *LC* is to *CA* as *MF* is to the half of *FA.*

Taking the halves of the consequents, therefore double *LC* is to the half of *CA* as *MF* to the fourth of *FA.*

And *DC* is double *LC, CM* is half of *CA,* and *FK* is a fourth part of *FA,* therefore *DC* is to *CM* as *MF* to *FK.*

Taken together, the sum of *DC* and *CM* is to *CM* as *MK* to *KF.* Therefore the square on the sum of *DC* and *CM* is to the square on *CM* as the square on *MK* is to the square on *KF.* V.18

And since, when the straight line opposite two sides of the pentagon *AC* is cut in extreme and mean ratio, the greater segment equals the side of the pentagon, that is, *DC,* while the square on the greater segment added to the half of the whole is five times the square on the half of the whole, and *CM* is half of the whole *AC,* therefore the square on *DC* and *CM* taken as one straight line is five times the square on *CM.* [XIII.8](#) [XIII.1](#)

But it was proved that the square on *DC* and *CM* taken as one straight line is to the square on *CM* as the square on *MK* to the square on *KF,* therefore the square on *MK* is five times the square on *KF.*

But the square on *KF* is rational, for the diameter is rational, therefore the square on *MK* is also rational. Therefore *MK* is rational.

And, since *BF* is quadruple *FK,* therefore *BK* is five times *KF.* Therefore the square on *BK* is twenty-five times the square on *KF.*

But the square on *MK* is five times the square on *KF,* therefore the square on *BK* is five times the square on *KM.* Therefore the square on *BK* has not to the square on *KM* the ratio which a square number has to a square number. Therefore *BK* is incommensurable in length with *KM.* X.9

And each of them is rational. Therefore *BK* and *KM* are rational straight lines commensurable in square only.

But, if from a rational straight line there is subtracted a rational straight line which is commensurable with the whole in square only, then the remainder is irrational, namely an apotome, therefore *MB* is an apotome and *MK* the annex to it. X.73

I say next that *MB* is also a fourth apotome.

Let the square on *N* be equal to that by which the square on *BK* is greater than the square on *KM.* Therefore the square on *BK* is greater than the square on *KM* by the square on *N.*

And, since *KF* is commensurable with *FB,* taken together, *KB* is commensurable with *FB.* But *BF* is commensurable with *BH,* therefore *BK* is also commensurable with *BH.* X.15 X.12

And, since the square on *BK* is five times the square on *KM,* therefore the square on *BK* has to the square on *KM* the ratio which 5 has to 1. Therefore, in conversion, the square on *BK* has to the square on *N* the ratio which 5 has to 4, and this is not the ratio which a square number has to a square number. Therefore *BK* is incommensurable with *N.* Therefore the square on *BK* is greater than the square on *KM* by the square on a straight line incommensurable with *BK.* V.19,Cor. X.9

Since then the square on the whole *BK* is greater than the square on the annex *KM* by the square on a straight line incommensurable with *BK,* and the whole *BK* is commensurable with the rational straight line, *BH,* set out, therefore *MB* is a fourth apotome. X.Def.III.4

But the rectangle contained by a rational straight line and a fourth apotome is irrational, and its square root is irrational, and is called minor. X.94

But the square on *AB* equals the rectangle *HB* by *BM,* because, when *AH* is joined, the triangle *ABH* is equiangular with the triangle *ABM,* and *HB* is to *BA* as *AB* is to *BM.*

Therefore the side *AB* of the pentagon is the irrational straight line called minor.

Therefore, *if an equilateral pentagon is inscribed in a circle which has its diameter rational, then the side of the pentagon is the irrational straight line called minor.*

Q.E.D.

Guide

Use of this proposition

This proposition is needed in XIII.16 after the construction of a dodecahedron to show the side of a pentagonal face is the irrational straight line called minor.

Next proposition: XIII.12

Previous: XIII.10

Book XIII introduction

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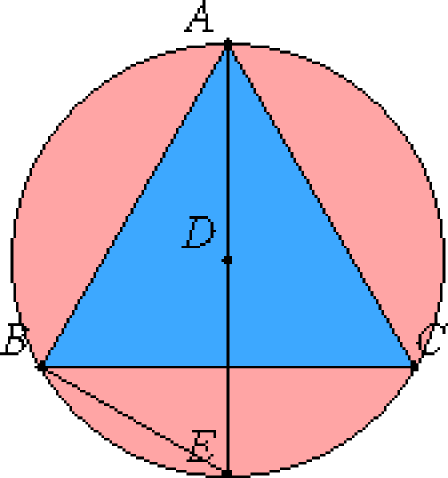
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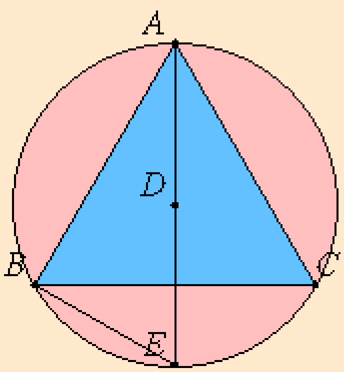
Euclid's Elements
Book XIII
Proposition 12

If an equilateral triangle is inscribed in a circle, then the square on the side of the triangle is triple the square on the radius of the circle.

Let *ABC* be a circle, and let the equilateral triangle *ABC* be inscribed in it.

I say that the square on one side of the triangle *ABC* is triple the square on the radius of the circle.



Take the center *D* of the circle *ABC*, join *AD* and carry it through to *E*, and join *BE*. III.1

Then, since the triangle *ABC* is equilateral, therefore the circumference *BEC* is a third part of the circumference of the circle *ABC*. Therefore the circumference *BE* is a sixth part of the circumference of the circle. Therefore the straight line *BE* belongs to a hexagon. Therefore it equals the radius *DE*. IV.15,Cor.

And, since *AE* is double *DE*, therefore the square on *AE* is quadruple the square on *ED*, that is, of the square on *BE*.

But the square on *AE* equals the sum of the squares on *AB* and *BE*. Therefore the sum of the squares on *AB* and *BE* is quadruple the square on *BE*. III.31 I.47

Therefore, taken separately, the square on *AB* is triple the square on *BE*. But *BE* equals *DE*, therefore the square on *AB* is triple the square on *DE*.

Therefore the square on the side of the triangle is triple the square on the radius.

Therefore, *if an equilateral triangle is inscribed in a circle, then the square on the side of the triangle is triple the square on the radius of the circle.*

Q.E.D.

Guide

Use of this proposition

This result is needed in proposition XIII.13 coming next to show that the construction given there produces a regular tetrahedron.

Next proposition: XIII.13 Select from Book XIII

Previous: XIII.11

Select book

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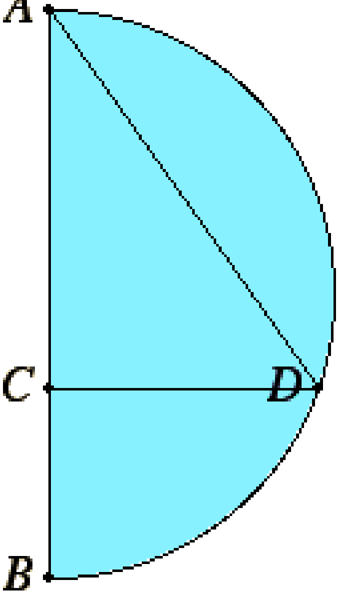
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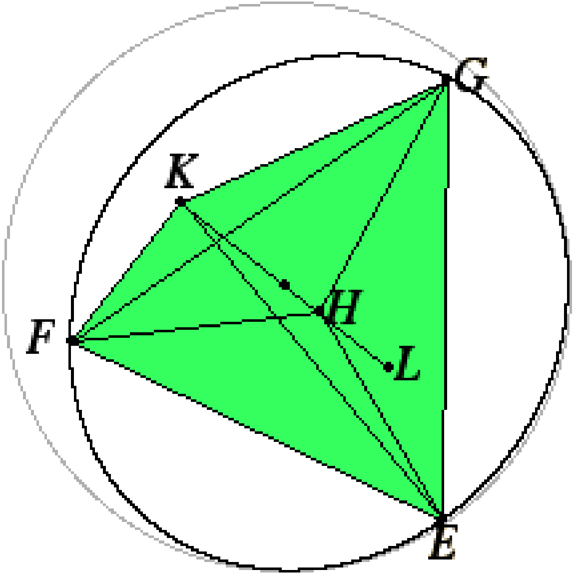
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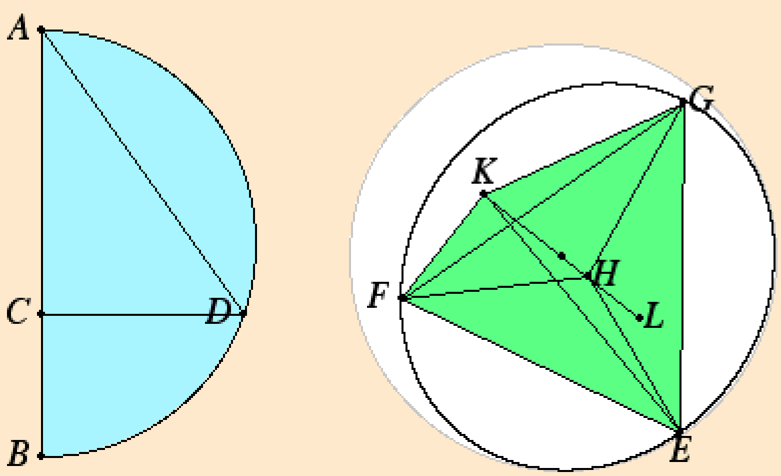
Euclid's Elements
Book XIII
Proposition 13

To construct a pyramid, to comprehend it in a given sphere; and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Set out the diameter *AB* of the given sphere, cut it at the point *C* so that *AC* is double *CB*, describe the semicircle *ADB* on *AB*, draw *CD* from the point *C* at right angles to *AB*, and join *DA*. VI.9 I.11

Set out the circle *EFG* with radius equal to *DC*, inscribe the equilateral triangle *EFG* in the circle *EFG*, take the center *H* of the circle, and join *EH*, *HF*, and *HG*. I.1 IV.2

Set *HK* up from the point *H* at right angles to the plane of the circle *EFG*, cut off *HK* equal to the straight line *AC* from *HK*, and join *KE*, *KF*, and *KG*. XI.12 I.3



Now, since *KH* is at right angles to the plane of the circle *EFG*, therefore it makes right angles with all the straight lines which meet it and are in the plane of the circle *EFG*. But each of the straight lines *HE*, *HF*, and *HG* meets it, therefore *HK* is at right angles to each of the straight lines *HE*, *HF*, and *HG*. XI.Def.3

And, since *AC* equals *HK*, and *CD* equals *HE*, and they contain right angles, therefore the base *DA* equals the base *KE*. For the same reason each of the straight lines *KF* and *KG* also equals *DA*. Therefore the three straight lines *KE*, *KF*, and *KG* equal one another. I.4

And, since *AC* is double *CB*, therefore *AB* is triple *BC*.

But that *AB* is to *BC* as the square on *AD* is to the square on *DC* will be proved afterwards.

Therefore the square on *AD* is triple the square on *DC*. But the square on *FE* is also triple the square on *EH*, and *DC* equals *EH*, therefore *DA* also equals *EF*. XIII.12

But *DA* was proved equal to each of the straight lines *KE*, *KF*, and *KG*, therefore each of the straight lines *EF*, *FG*, and *GE* also equals each of the straight lines *KE*, *KF*, and *KG*. Therefore the four triangles *EFG*, *KEF*, *KFG*, and *KEG* are equilateral.

Therefore a pyramid has been constructed out of four equilateral triangles, the triangle *EFG* being its base and the point *K* its vertex.

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Produce the straight line *HL* in a straight line with *KH*, and make *HL* equal to *CB*. I.3

Now, since *AC* is to *CD* as *CD* is to *CB*, while *AC* equals *KH*, *CD* equals *HE*, and *CB* equals *HL*, therefore *KH* is to *HE* as *EH* is to *HL*. Therefore the rectangle *KH* by *HL* equals the square on *EH*. VI.8,Cor. VI.17

And each of the angles *KHE*, *EHL* is right, therefore the semicircle described on *KL* passes through *E* also. cf. VI.8 III.31

If then, *KL* remaining fixed, the semicircle is carried round and restored to the same position from which it began to be moved, then it also passes through the points *F* and *G*, since, if *FL* and *LG* are joined, then the angles at *F* and *G* similarly become right angles, and the pyramid is comprehended in the given sphere. For *KL*, the diameter of the sphere, equals the diameter *AB* of the given sphere, since *KH* was made equal to *AC*, and *HL* to *CB*.

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Since *AC* is double *CB*, therefore *AB* is triple *BC*, and, in conversion, *BA* is one and a half times *AC*.

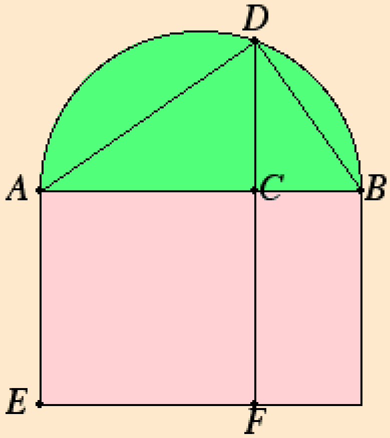
But *BA* is to *AC* as the square on *BA* is to the square on *AD*. Therefore the square on *BA* is also one and a half times the square on *AD*. And *BA* is the diameter of the given sphere, and *AD* equals the side of the pyramid. Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Q.E.F.

Lemma

It is to be proved that *AB* is to *BC* as the square on *AD* is to the square on *DC*.

Set out the figure of the semicircle, join *DB*, describe the square *EC* on *AC*, and complete the parallelogram *FB*. I.46



Since the triangle *DAB* is equiangular with the triangle *DAC*, therefore *BA* is to *AD* as *DA* is to *AC*. Therefore the rectangle *BA* by *AC* equals the square on *AD*. VI.8 VI.4 VI.17

And since *AB* is to *BC* as *EB* is to *BF*, and *EB* is the rectangle *BA* by *AC*, for *EA* equals *AC*, and *BF* is the rectangle *AC* by *CB*, therefore *AB* is to *BC* as the rectangle *BA* by *AC* is to the rectangle *AC* by *CB*. VI.1

And the rectangle *BA* by *AC* equals the square on *AD*, and the rectangle *AC* by *CB* equals the square on *DC*, for the perpendicular *DC* is a mean proportional between the segments *AC* and *CB* of the base, because the angle *ADB* is right. Therefore *AB* is to *BC* as the square on *AD* is to the square on *DC*. VI.8,Cor.

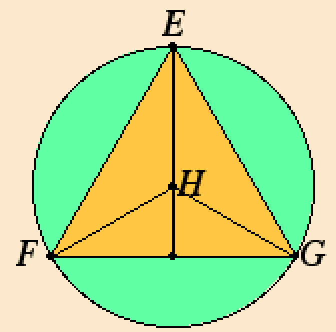
Q.E.D.

Guide

This figure is usually called a regular tetrahedron, that is, a solid figure contained by four equal and equilateral triangles. Euclid simply calls it "a pyramid" with the understanding that by that he means not just any pyramid, but a regular tetrahedron. A similar ambiguity occured in ancient Greek when the word "tetragon" was used. It meant either any four-angled figure or specifically a square, depending on the context.

Summary of the construction

Standardize the radius of the sphere at 1 unit, so that $AB = 2$. Then cut AB at C so that $AC = 4/3$ and $BC = 2/3$. Let DC be their mean proportional $(2/3)\sqrt{2}$. Then $AD = (2/3)\sqrt{6}$. This line AD will end up being the length of the side of the tetrahedron. Note that it has the correct value so that "the square on the diameter of the sphere is one and a half times the square on the side of the pyramid."



Set out the circle EFG of radius $EH = (2/3)\sqrt{2}$, and inscribe in that circle an equilateral triangle. Then each side of the triangle will be $(2/3)\sqrt{6}$, the same as AD (XIII.12).

Make HK of length 4/3 and perpendicular to the plane of the triangle, and connect KE, KF, and KG. Then K lies on the surface of the sphere. And since the triangle HKE is a right triangle, therefore its hypotenuse $KE = (2/3)\sqrt{6}$, the same as AD. Likewise KF and KG have the same length. That constructs the tetrahedron in the sphere.

Coordinates for the vertices of the tetrahedron

A cube can be easily constructed from a tetrahedron since the four vertices of a tetrahedron are four of the eight vertices of a cube. See proposition XIII.15. That being the case, an obvious coordinate system will make eight vertices of a cube have the coordinates

$$(1,1,1)\quad(1,1,-1)\quad(1,-1,1)\quad(1,-1,-1)\quad(-1,1,1)\quad(-1,1,-1)\quad(-1,-1,1)\quad(-1,-1,-1)$$

that is, all eight combinations of –1 and 1 in all three coordinates. The radius for such a cube is $\sqrt{3}$, so if a unit sphere is desired, then all the coordinates would have to be divided by $\sqrt{3}$.

The tetrahedron has only half of these eight vertices, and they can be chosen to be

$$(1,1,1)\quad(1,-1,-1)\quad(-1,1,-1)\quad(-1,-1,1)$$

that is, the points which have an odd number of positive coordinates. There is another tetrahedron which has as its vertices the remaining four points which have an even number of positive coordinates.

Use of this construction

Constructing this regular tetrahedron is an end in itself. In the last proposition of the *Elements* XIII.18, the five regular polyhedra are compared, and this construction is needed there as well as constructions of the other four regular polyhedra.

Next proposition: XIII.14   Select from Book XIII

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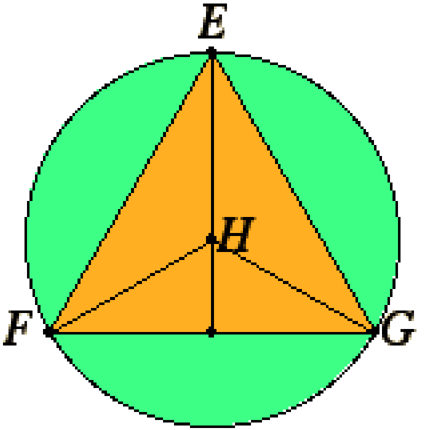
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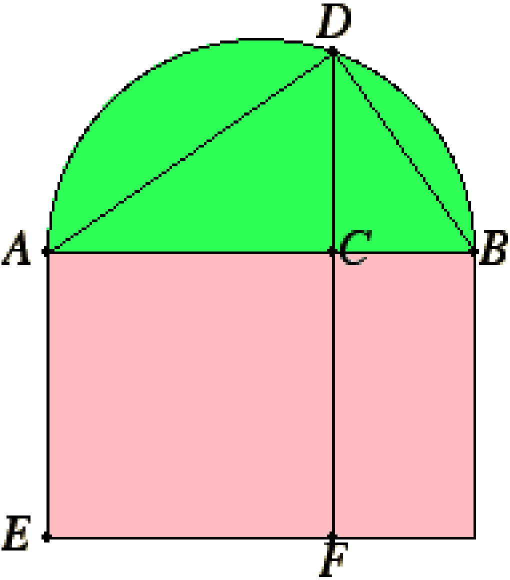
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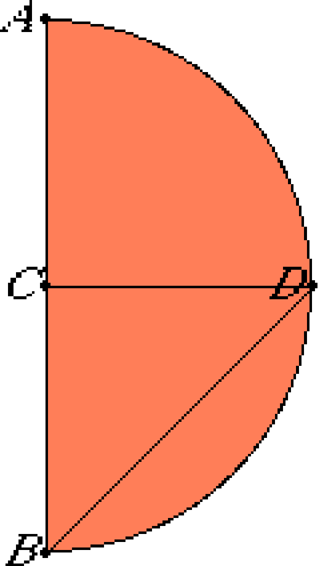
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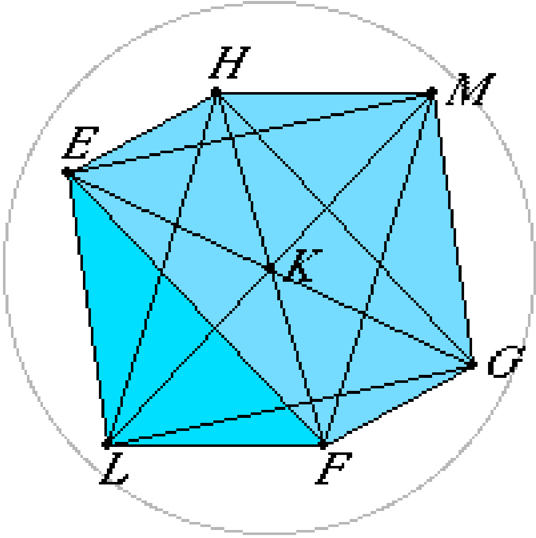
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Euclid's Elements

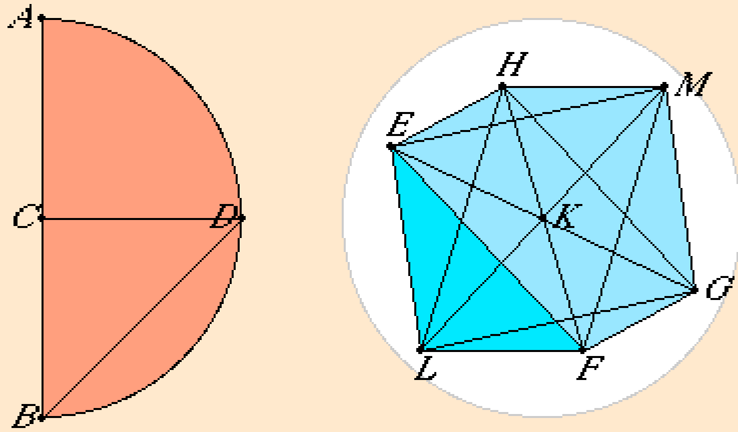
Book XIII

Proposition 14

To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double the square on the side of the octahedron.

Set out the diameter *AB* of the given sphere, bisect it at *C,* describe the semicircle *ADB* on *AB,* draw *CD* from *C* at right angles to *AB,* and join *DB.* I.11

Set out the square *EFGH,* having each of its sides equal to *DB,* join *HF* and *EG,* set up the straight line *KL* from the point *K* at right angles to the plane of the square *EFGH,* and carry it through to the other side of the plane *KM.* I.46 XI.12



Cut off *KL* and *KM* from the straight lines *KL* and *KM* respectively equal to one of the straight lines *EK, FK, GK,* or *HK,* and join *LE, LF, LG, LH, ME, MF, MG,* and *MH.* I.3

Then, since *KE* equals *KH,* and the angle *EKH* is right, therefore the square on *HE* is double the square on *EK.* Again, since *LK* equals *KE,* and the angle *LKE* is right, therefore the square on *EL* is double the square on *EK.* I.47

But the square on *HE* was also proved double the square on *EK,* therefore the square on *LE* equals the square on *EH.* Therefore *LE* equals *EH.* For the same reason *LH* also equals *HE.*

Therefore the triangle *LEH* is equilateral. XI.Def.26

Similarly we can prove that each of the remaining triangles of which the sides of the square *EFGH* are the bases and the points *L* and *M* are the vertices, is equilateral, therefore an octahedron has been constructed which is contained by eight equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double the square on the side of the octahedron.

Since the three straight lines *LK, KM,* and *KE* equal one another, therefore the semicircle described on *LM* passes through *E.* And for the same reason, if, *LM* remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, then it also passes through the points *F, G,* and *H,* and the octahedron will be comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since *LK* equals *KM*, while *KE* is common, and they contain right angles, therefore the base *LE* equals the base *EM*. I.4

And, since the angle *LEM* is right, for it is in a semicircle, therefore the square on *LM* is double the square on *LE*. III.31 I.47

Again, since *AC* equals *CB*, therefore *AB* is double *BC*. But *AB* is to *BC* as the square on *AB* is to the square on *BD*, therefore the square on *AB* is double the square on *BD*.

But the square on *LM* was also proved double the square on *LE*. And the square on *DB* equals the square on *LE*, for *EH* was made equal to *DB*. Therefore the square on *AB* equals the square on *LM*. Therefore *AB* equals *LM*.

And *AB* is the diameter of the given sphere, therefore *LM* equals the diameter of the given sphere.

Therefore the octahedron has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is double the square on the side of the octahedron.

Q.E.F.

Guide

Of the five regular polyhedra to be constructed in a sphere, the octahedron has the easiest construction. Relative to the center of the sphere *K*, the lines to the six verices *KE*, *KF*, *KG*, *KH*, *KL*, and *KM* form three mutually perpendicular diameters. Also, the 12 sides group into three groups of four lines, each group forming the vertices of a square —*EFGH*, *EMGL*, and *FMHL*.

Since the center of each square is the center of the sphere, therefore two sides, *EF* and *FG*, along with the one diameter *EG* of the octahedron form a 45°-45°-90° triangle. Thus, the square on the diameter of the sphere is twice the square on the side of the octahedron.

Coordinates for the vertices of the octahedron

If the sphere circumscribing the octahedron is the unit sphere, then a natural coordinate system to impose would have the three coordinate axes be the three perpendicular diameters. Then the points a unit distance from the origin are the six vertices of the octahedron, namely,

(1,0,0)   (–1,0,0)   (0,1,0)   (0,–1,0)   (0,0,1)   (0,0,–1)

Duals of the regular polyhedra

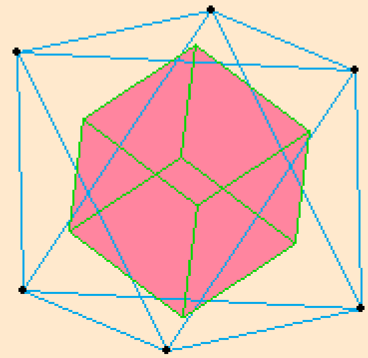
As will be shown in proposition XIII.18, there are exactly five regular polyhedra. The accompanying table lists these five polyhedra along with the numbers of the their faces, edges, and vertices. Their names are taken from the number of their faces, except, of course, the cube, which otherwise would be called a hexahedron.

Polyhedron	Faces	Edges	Vertices
tetrahedron	4	6	4
octahedron	8	12	6
cube	6	12	8
icosahedron	20	30	12

dodecahedron	12	30	20

Note that there are two pairs of polyhedra in this table where the numbers are related. One pair is the octahedron and cube, the other is the icosahedron and dodecahedron. For these pairs the number of faces of one of the pair equals the number of vertices of the other, and both of the pair have the same number of edges. These are the pairs of "duals." The numbers for the tetrahedron indicate that it dual to itself.

We can see the correspondence between the parts of one of these polyhedra and the parts of its dual. Consider the octahedron. Place a point in the circumcenter of each of the eight faces. Connect two of these points if the faces that contain them share an edge. For each of the six vertices of the octahedron, connect the four circumcenters of the adjacent faces to make a square. What results is a cube with six vertices, 12 edges, and eight faces.



An analogous construction for the cube yields an octahedron. Likewise the constructions for the icosahedron and dodecahedron yield each other, and the construction for a tetrahedron yields another tetrahedron.

Use of this construction

Constructing this octahedron is an end in itself. The construction is also used in proposition XIII.18 where the five regular polyhedra are compared.

Next proposition: XIII.15

Previous: XIII.13

Book XIII introduction

Select from Book XIII

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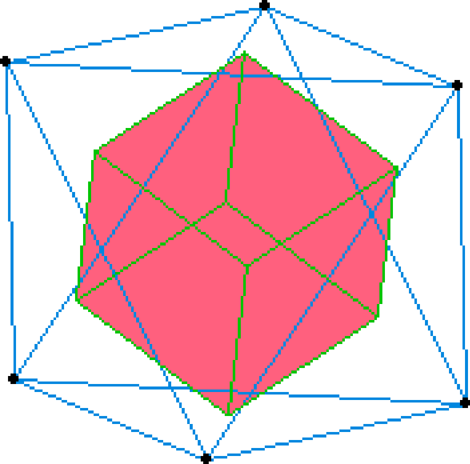

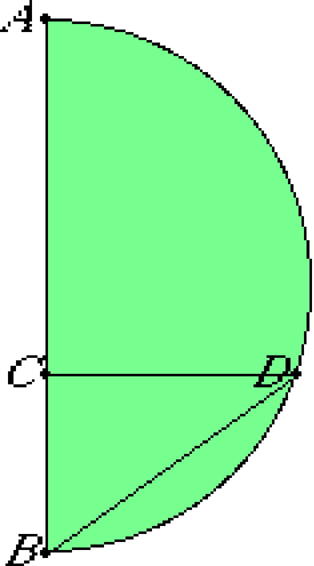
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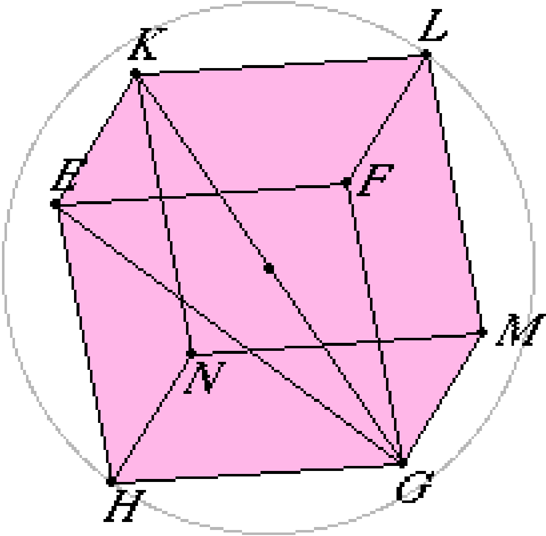
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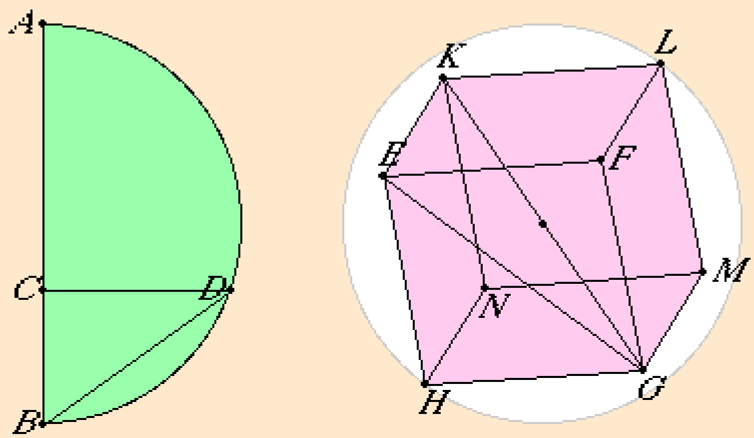
Euclid's Elements
Book XIII
Proposition 15

To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple the square on the side of the cube.

Set out the diameter *AB* of the given sphere, and cut it at *C* so that *AC* is double *CB.* Describe the semicircle *ADB* on *AB,* draw *CD* from *C* at right angles to *AB,* and join *DB.* Set out the square *EFGH* having its side equal to *DB,* draw *EK, FL, GM,* and *HN* from *E, F, G,* and *H* at right angles to the plane of the square *EFGH,* and cut *EK, FL, GM,* and *HN* off from *EK, FL, GM,* and *HN* respectively equal to one of the straight lines *EF, FG, GH,* or *HE.* Join *KL, LM, MN,* and *NK.* VI.9 I.11 I.46 XI.12 I.3



Therefore the cube *FN* has been constructed which is contained by six equal squares. XI.Def.25

It is then required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is triple the square on the side of the cube.

Join *KG* and *EG.*

Then, since the angle *KEG* is right, for *KE* is also at right angles to the plane *EG* and of course to the straight line *EG* also, therefore the semicircle described on *KG* passes through the point *E.* XI.Def.3

Again, since *GF* is at right angles to each of the straight lines *FL* and *FE,* therefore *GF* is also at right angles to the plane *FK.* Hence also, if we join *FK,* then *GF* will be at right angles to *FK.* For this reason the semicircle described on *GK* also passes through *F.*

Similarly it also passes through the remaining angular points of the cube.

If then, *KG* remaining fixed, the semicircle is carried round and restored to the same position from which it began to be moved, then the cube is comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since *GF* equals *FE,* and the angle at *F* is right, therefore the square on *EG* is double the square on *EF.* But *EF* equals *EK,* therefore the square on *EG* is double the square on *EK.* Hence the sum of the squares on *GE* and *EK,* that is the square on *GK,* is triple the square on *EK.* I.47

And, since *AB* is triple *BC,* while *AB* is to *BC* as the square on *AB* is to the square on *BD,* therefore the square on *AB* is triple the square on *BD.*

But the square on *GK* was also proved triple the square on *KE.* And *KE* was made equal to *DB,* therefore *KG* also equals *AB.* And *AB* is the diameter of the given sphere, therefore *KG* also equals the diameter of the given sphere.

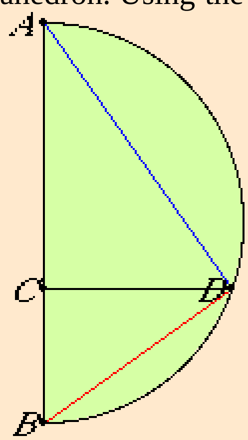
Therefore the cube has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is triple the square on the side of the cube.

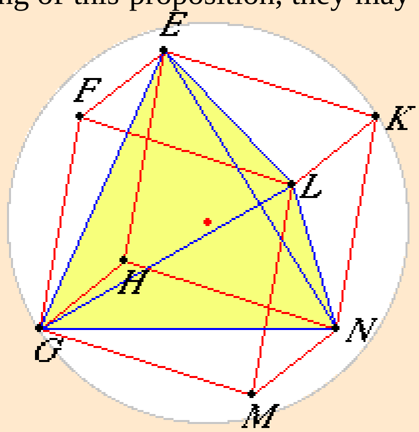
Q.E.F.

Guide

Special relationships between regular tetrahedra and cubes

Note that the beginning of this construction of a cube is the same as that for the tetrahedron in proposition XIII.13, namely, the points *C* and *D* are the same. The difference is that the line *AD* is the edge of a tetrahedron while the line *BD* is the edge of a cube. Following through the construction, you will see that four of the eight vertices of the cube are the four vertices of the tetrahedron. Using the labelling of this proposition, they may be taken as *E, G, L,* and *N.*





Alternatively, the other four vertices of the cube, *F, H, K,* and *M,* form the vertices of a regular tetrahedron. See the Guide to XIII.13 for more on this connection which involves placing coordinates on the vertices of the cube and tetrahedron.

The volumes of the tetrahedron and cube are easily compared. When the tetrahedron is removed from the cube, there are four remaining pyramids, *EGHN* is one of them. By proposition XII.9 the volume of each pyramid is one-third of the volume of a prism, for instance, *EGHN* is one-third of the triangular prism *EFKHGN,* which in turn is half of the cube. Therefore each pyramid is one-sixth of the cube. Since the four pyramids together make four-sixths of the cube, that leaves one-third of the cube for the regular tetrahedron *EGLN.*

Use of this construction

Constructing a cube is an end in itself, but Euclid also starts with a cube to construct a dodecahedron in proposition XIII.17. Finally, this construction is used in XIII.18 where the five regular polyhedra are compared.

Next proposition: XIII.16

Previous: XIII.14

Book XIII introduction

Select from Book XIII

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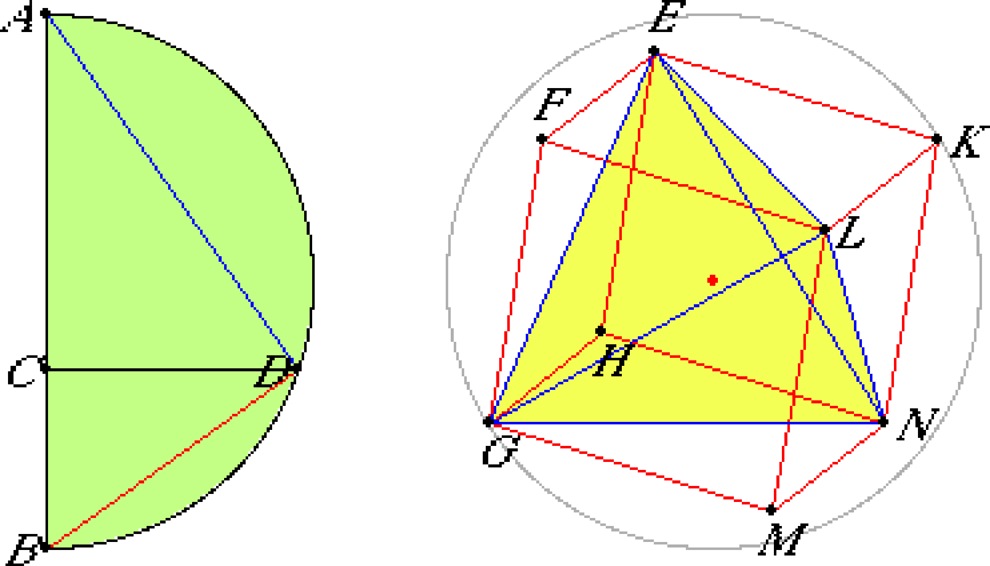
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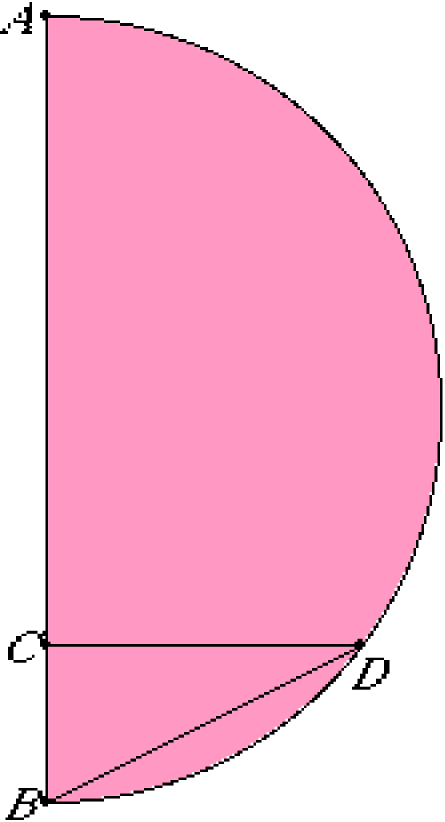
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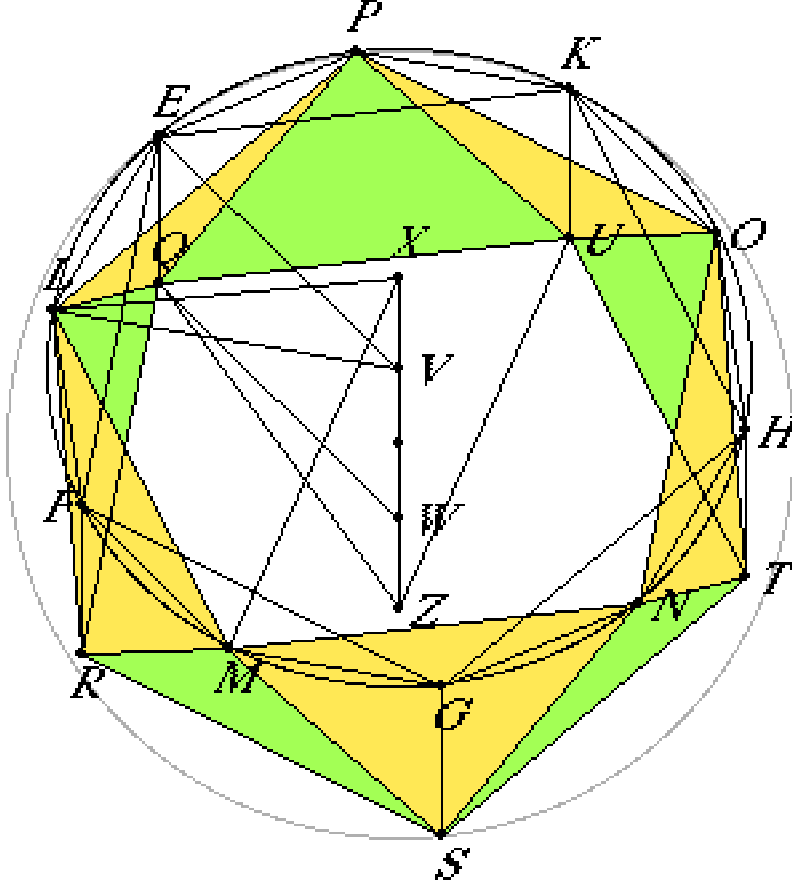
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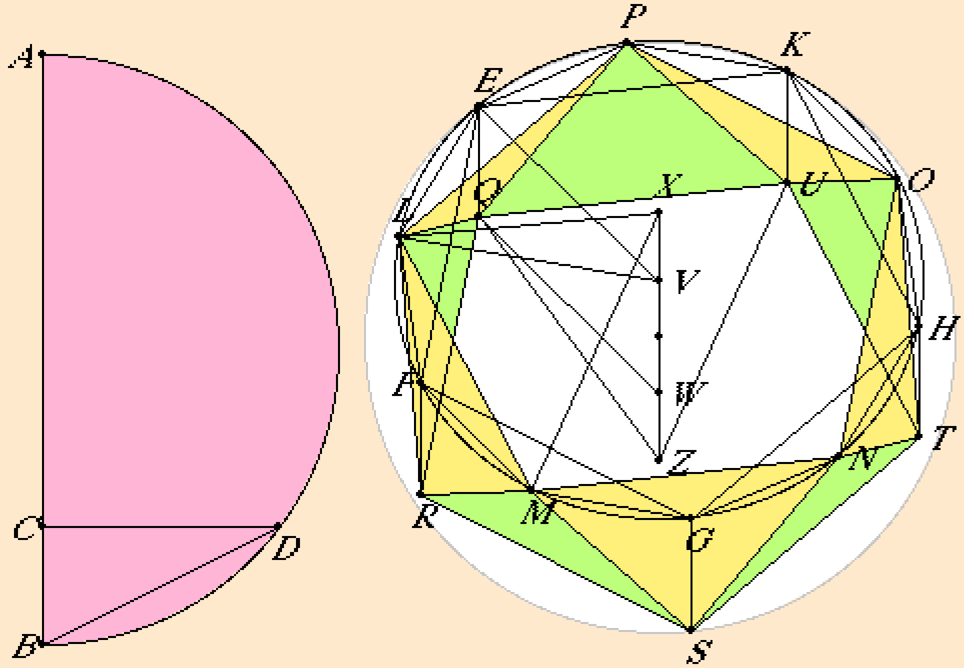
Euclid's Elements
Book XIII
Proposition 16

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the icosahedron is the irrational straight line called minor.

Set out the diameter *AB* of the given sphere, and cut it at *C* so that *AC* is quadruple *CB,* describe the semicircle *ADB* on *AB,* draw the straight line *CD* from *C* at right angles to *AB,* and join *DB.* VI.9 I.11

Set out the circle *EFGHK,* and let its radius be equal to *DB.* Inscribe the equilateral and equiangular pentagon *EFGHK* in the circle *EFGHK,* bisect the circumferences *EF, FG, GH, HK,* and *KE* at the points *L, M, N, O,* and *P,* and join *LM, MN, NO, OP, PL,* and *EP*. IV.11 I.9



Therefore the pentagon *LMNOP* is also equilateral, and the straight line *EP* belongs to a decagon.

Now from the points *E, F, G, H,* and *K* set up the straight lines *EQ, FR, GS, HT,* and *KU* at right angles to the plane of the circle, and make them equal to the radius of the circle *EFGHK*. Join *QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP,* and *PQ.* XI.12 I.3

Now, since each of the straight lines *EQ* and *KU* is at right angles to the same plane, therefore *EQ* is parallel to *KU.*

But it is also equal to it, and the straight lines joining those ends of equal and parallel straight lines which are in the same direction are equal and parallel. Therefore *QU* is equal and parallel to *EK.* I.33

But *EK* belongs to an equilateral pentagon, therefore *QU* also belongs to the equilateral pentagon inscribed in the circle *EFGHK.*

For the same reason each of the straight lines *QR, RS, ST,* and *TU* also belongs to the equilateral pentagon inscribed in the circle *EFGHK.* Therefore the pentagon *QRSTU* is equilateral.

And, since *QE* belongs to a hexagon, and *EP* to a decagon, and the angle *QEP* is right, therefore *QP* belongs to a pentagon, for the square on the side of the pentagon equals the sum of the square on the side of the hexagon and the square on the side of the decagon inscribed in the same circle. XIII.10

For the same reason *PU* is also a side of a pentagon. But *QU* also belongs to a pentagon, therefore the triangle *QPU* is equilateral. For the same reason each of the triangles *QLR, RMS, SNT,* and *TOU* is also equilateral.

And, since each of the straight lines *QL* and *QP* was proved to belong to a pentagon, and *LP* also belongs to a pentagon, therefore the triangle *QLP* is equilateral.

For the same reason each of the triangles *LRM, MSN, NTO,* and *OUP* is also equilateral.

Take the center *V* of the circle *EFGHK,* set *VZ* up from *V* at right angles to the plane of the circle, and produce it in the other direction *VX.* Cut off *VW,* the side of a hexagon, and each of the straight lines *VX* and *WZ,* sides of a decagon. Join *QZ, QW, UZ, EV, LV, LX,* and *XM.* III.1 XI.12

Now, since each of the straight lines *VW* and *QE* is at right angles to the plane of the circle, therefore *VW* is parallel to *QE.* But they are also equal, therefore *EV* and *QW* are also equal and parallel. XI.6 I.33

But *EV* belongs to a hexagon, therefore *QW* also belongs to a hexagon. And, since *QW* belongs to a hexagon, and *WZ* to a decagon, and the angle *QWZ* is right, therefore *QZ* belongs to a pentagon. XIII.10

For the same reason *UZ* also belongs to a pentagon, for if we join *VK* and *WU,* then they will be equal and opposite, and *VK,* being a radius, belongs to a hexagon, therefore *WU* also belongs to a hexagon. But *WZ* belongs to a decagon, and the angle *UWZ* is right, therefore *UZ* belongs to a pentagon. IV.15,Cor. XIII.10

But *QU* also belongs to a pentagon, therefore the triangle *QUZ* is equilateral. For the same reason each of the remaining triangles of which the straight lines *QR, RS, ST,* and *TU* are the bases, and the point *Z* the vertex, is also equilateral.

Again, since *VL* belongs to a hexagon, and *VX* to a decagon, and the angle *LVX* is right, therefore *LX* belongs to a pentagon. XIII.10

For the same reason, if we join *MV,* which belongs to a hexagon, *MX* is also inferred to belong to a pentagon.

But *LM* also belongs to a pentagon, therefore the triangle *LMX* is equilateral.

XI.Def.27

Similarly it can be proved that each of the remaining triangles of which *MN, NO, OP,* and *PL* are the bases and the point *X* the vertex, is also equilateral.

Therefore an icosahedron has been constructed which is contained by twenty equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the side of the icosahedron is the irrational straight line called minor.

Since *VW* belongs to a hexagon, and *WZ* to a decagon, therefore *VZ* is cut in extreme and mean ratio at *W,* and *VW* is its greater segment. Therefore as *ZV* is to *VW* as *VW* is to *WZ.* XIII.9

But *VW* equals *VE,* and *WZ* equals *VX,* therefore *ZV* is to *VE* as *EV* is to *VX.*

And the angles *ZVE* and *EVX* are right, therefore, if we join the straight line *EZ,* then the angle *XEZ* will be right since the triangles *XEZ* and *VEZ* are similar.

For the same reason, since *ZV* is to *VW* as *VW* is to *WZ,* and *ZV* equals *XW,* and *VW* equals *WQ,* therefore *XW* is to *WQ* as *QW* is to *WZ.*

And for this reason again, if we join *QX,* then the angle at *Q* will be right, therefore the semicircle described on *XZ* will also pass through *Q.* VI.8 III.31

And if, *XZ* remaining fixed, the semicircle is carried round and restored to the same position from which it began to be moved, then it will pass through *Q* and the remaining angular points of the icosahedron, and the icosahedron will have been comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

Bisect *VW* at *A'.* I.9

Then, since the straight line *VZ* is cut in extreme and mean ratio at *W,* and *ZW* is its lesser segment, therefore the square on *ZW* added to the half of the greater segment, that is *WA',* is five times the square on the half of the greater segment. Therefore the square on *ZA'* is five times the square on *A'W.* XIII.3

And *ZX* is double *ZA',* and *VW* is double *A'W,* therefore the square on *ZX* is five times the square on *WV.* And, since *AC* is quadruple *CB,* therefore *AB* is five times *BC.*

But *AB* is to *BC* as the square on *AB* is to the square on *BD,* therefore the square on *AB* is five times the square on *BD.* VI.8 V.Def.9

But the square on *ZX* was also proved to be five times the square on *VW.* And *DB* equals *VW,* for each of them equals the radius of the circle *EFGHK,* therefore *AB* also equals *XZ.* And *AB* is the diameter of the given sphere, therefore *XZ* also equals the diameter of the given sphere.

Therefore the icosahedron has been comprehended in the given sphere.

I say next that the side of the icosahedron is the irrational straight line called minor.

Since the diameter of the sphere is rational, and the square on it is five times the square on the radius of the circle *EFGHK,* therefore the radius of the circle *EFGHK* is also rational, hence its diameter is also rational. But, if an equilateral pentagon is inscribed in a circle which has its diameter rational, then the side of the pentagon is the irrational straight line called minor. XIII.11

And the side of the pentagon *EFGHK* is the side of the icosahedron.

Therefore the side of the icosahedron is the irrational straight line called minor.

Corollary.

From this it is clear that *the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle.*

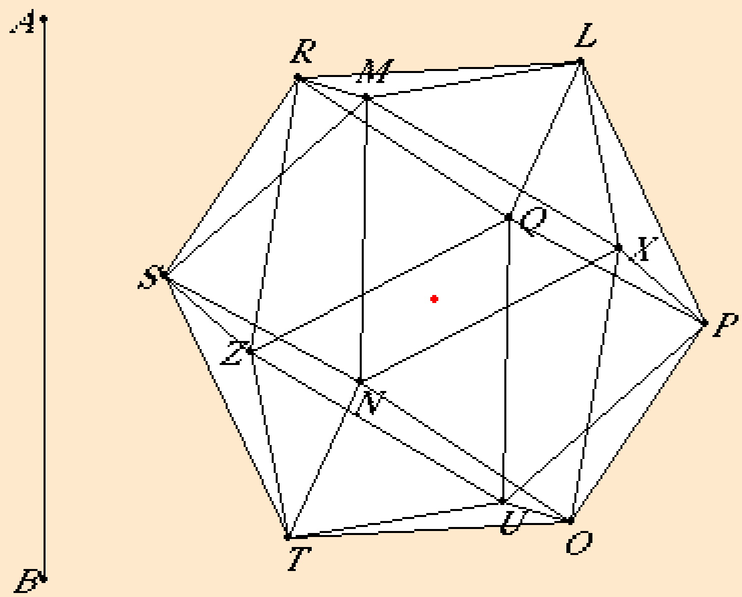
Q.E.F.

Guide

The icosahedron

The regular icosahedron is composed of 20 faces, each face an equilateral triangle, with five triangles meeting at each vertex. There are 12 vertices, and there are 30 edges.

Unlike most of the Euclid's illustrations, the diagram he used for this proposition is highly schematic; it is not intended to be an accurate projection of the icosahedron. Of course, it could be that the diagram changed over the centuries of copying, but his diagram has the advantage of spreading out the vertices to be readable. The figure shown in the proof above is a standard orthogonal projection of the icosahedron. Directly below the same icosahedron is shown without all the auxiliary lines.



Use of this construction

Constructing a icosahedron is an end in itself. This construction and the corollary are also used in XIII.18 where the five regular polyhedra are compared.

Next proposition: XIII.17

Previous: XIII.15

Book XIII introduction

Select from Book XIII

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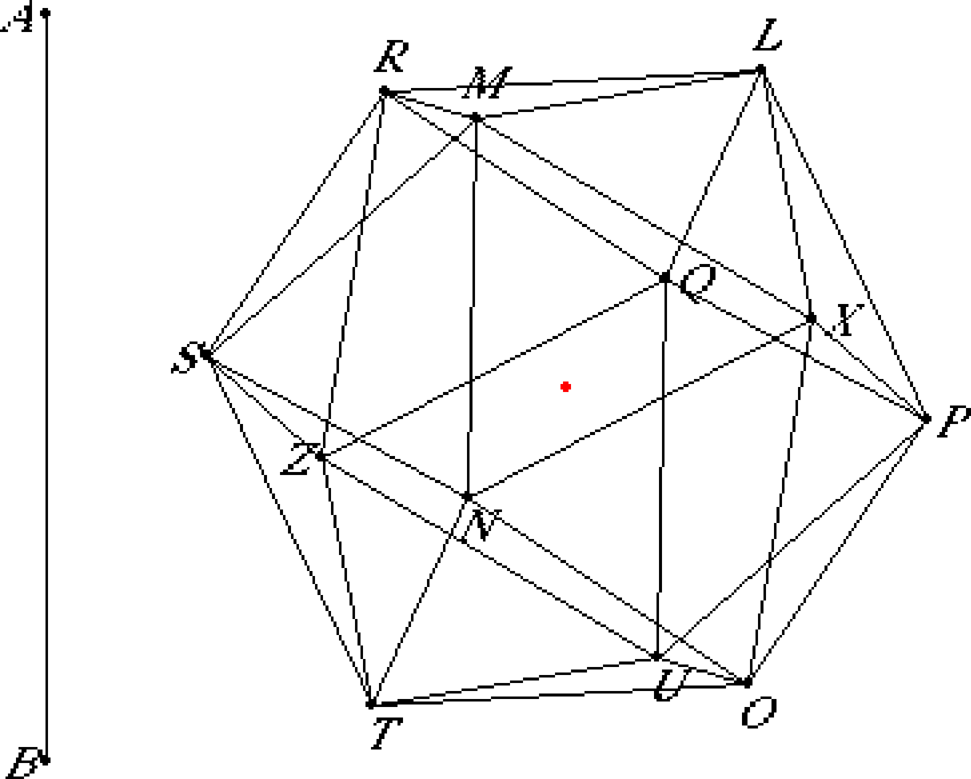

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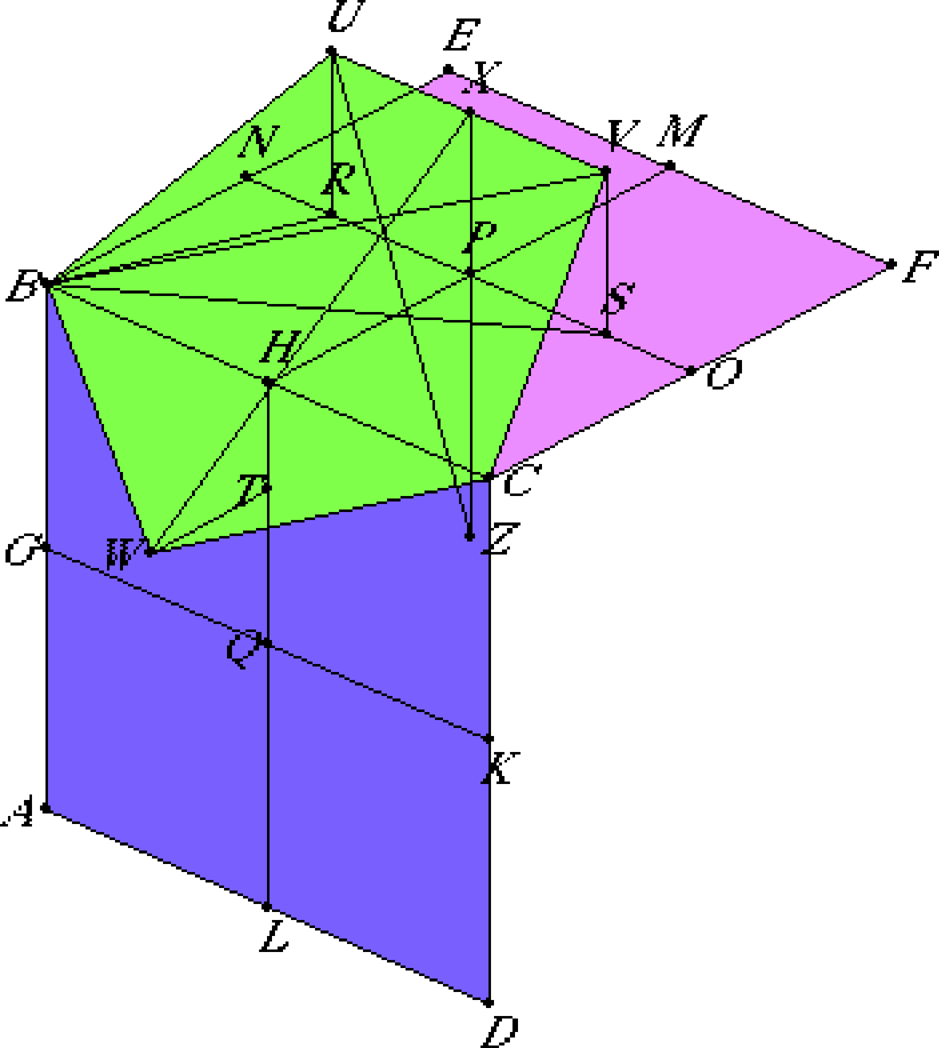

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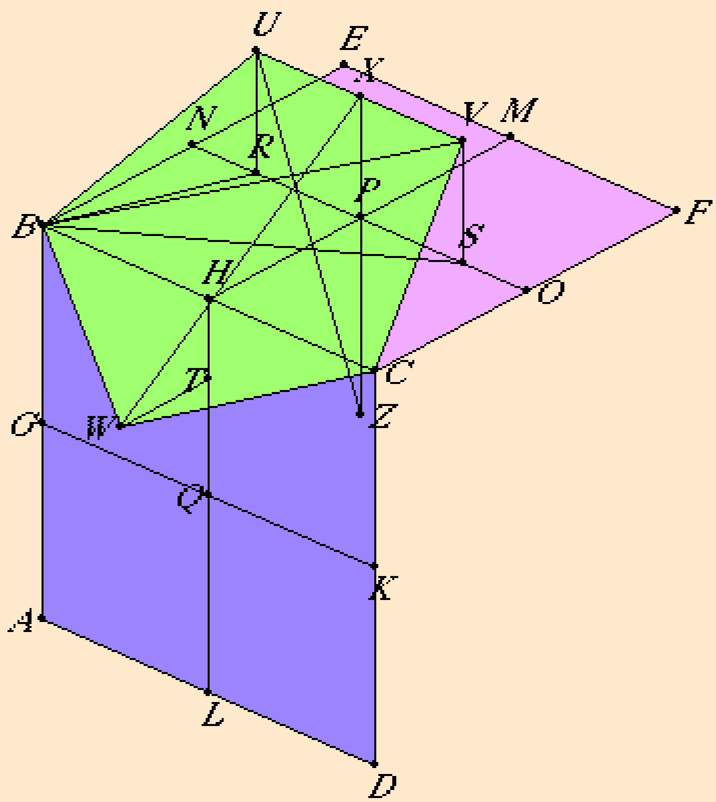
Euclid's Elements
Book XIII
Proposition 17

To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the square on the side of the dodecahedron is the irrational straight line called apotome.

Let *ABCD* and *CBEF*, two planes of the aforesaid cube at right angles to one another, be set out. Bisect the sides *AB*, *BC*, *CD*, *DA*, *EF*, *EB*, and *FC* at *G*, *H*, *K*, *L*, *M*, *N*, and *O* respectively, and join *GK*, *HL*, *MH*, and *NO*. Cut the straight lines *NP*, *PO*, and *HQ* in extreme and mean ratio at the points *R*, *S*, and *T* respectively, and let *RP*, *PS*, and *TQ* be their greater segments. Set up *RU*, *SV*, and *TW* from the points *R*, *S*, and *T* at right angles to the planes of the cube towards the outside of the cube, and make them equal to *RP*, *PS*, and *TQ*. Join *UB*, *BW*, *WC*, *CV*, and *VU*. XIII.15 I.10 II.11/VI.30 XI.11 I.3

I say that the pentagon *UBWCV* is equilateral, in one plane, and equiangular.



Join *RB*, *SB*, and *VB*.

Then, since the straight line *NP* is cut in extreme and mean ratio at *R*, and *RP* is the greater segment, therefore the sum of the squares on *PN* and *NR* is triple the square on *RP*. XIII.4

But *PN* equals *NB*, and *PR* equals *RU*, therefore the sum of the squares on *BN* and *NR* is triple the square on *RU*.

But the square on *BR* equals the sum of the squares on *BN* and *NR*, therefore the square on *BR* is triple the square on *RU*. Hence the sum of the squares on *BR* and *RU* is quadruple the square on *RU*. I.47

But the square on *BU* equals the sum of the squares on *BR* and *RU*, therefore the square on *BU* is quadruple the square on *RU.* Therefore *BU* is double *RU.*

But *VU* is also double *UR,* for *SR* is also double *PR,* that is, of *RU,* therefore *BU* equals *UV.*

Similarly it can be proved that each of the straight lines *BW, WC,* and *CV* also equals each of the straight lines *BU* and *UV.* Therefore the pentagon *BUVCW* is equilateral.

I say next that it is also in one plane.

Draw *PX* from *P* parallel to each of the straight lines *RU* and *SV* and toward the outside of the cube, and join *XH* and *HW.* I.31

I say that *XHW* is a straight line.

Since *HQ* is cut in extreme and mean ratio at *T,* and *QT* is its greater segment, therefore *HQ* is to *QT* as *QT* is to *TH.* But *HQ* equals *HP,* and *QT* equals each of the straight lines *TW* and *PX,* therefore *HP* is to *PX* as *WT* is to *TH.* XI.6

And *HP* is parallel to *TW,* for each of them is at right angles to the plane *BD,* and *TH* is parallel to *PX,* for each of them is at right angles to the plane *BF.*

But if two triangles *XPH* and *HTW,* which have two sides proportional to two sides are placed together at one angle so that their corresponding sides are also parallel, then the remaining straight lines are in a straight line, therefore *XH* is in a straight line with *HW.* VI.32

But every straight line is in one plane, therefore the pentagon *UBWCV* is in one plane. XI.1

I say next that it is also equiangular.

Since the straight line *NP* is cut in extreme and mean ratio at *R,* and *PR* is the greater segment, while *PR* equals *PS,* therefore *NS* is also cut in extreme and mean ratio at *P,* and *NP* is the greater segment. Therefore the sum of the squares on *NS* and *SP* is triple the square on *NP.* XIII.5 XIII.4

But *NP* equals *NB,* and *PS* equals *SV,* therefore the squares on *NS* and *SV* is triple the square on *NB.* Hence the sum of the squares on *VS, SN,* and *NB* is quadruple the square on *NB.*

But the square on *SB* equals the sum of the squares on *SN* and *NB,* therefore the sum of the squares on *BS* and *SV,* that is, the square on *BV,* for the angle *VSB* is right, is quadruple the square on *NB.* Therefore *VB* is double *BN.*

But *BC* is also double *BN,* therefore *BV* equals *BC.*

And, since the two sides *BU* and *UV* equal the two sides *BW* and *WC,* and the base *BV* equals the base *BC,* therefore the angle *BUV* equals the angle *BWC.* I.8

Similarly we can prove that the angle *UVC* also equals the angle *BWC.* Therefore the three angles *BWC, BUV,* and *UVC* equal one another.

But if in an equilateral pentagon three angles equal one another, then the pentagon is equiangular, therefore the pentagon *BUVCW* is equiangular. XIII.7

And it was also proved equilateral, therefore the pentagon *BUVCW* is equilateral and equiangular, and it is on one side *BC* of the cube.

XI.Def.28

Therefore, if we make the same construction in the case of each of the twelve sides of the cube, a solid figure will be constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron.

It is now required to comprehend it in the given sphere, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

Produce *XP*, and let the produced straight line be *XZ*.

Therefore *PZ* meets the diameter of the cube, and they bisect one another, for this has been proved in the last theorem but one of the eleventh book. XI.38

Let them cut at *Z*. Therefore *Z* is the center of the sphere which comprehends the cube, and *ZP* is half of the side of the cube.

Join *UZ*.

Now, since the straight line *NS* is cut in extreme and mean ratio at *P*, and *NP* is its greater segment, therefore the sum of the squares on *NS* and *SP* is triple the square on *NP*. XIII.4

But *NS* equals *XZ*, for *NP* also equals *PZ*, and *XP* equals *PS*.

But *PS* also equals *XU*, since it also equals *RP*. Therefore the sum of the squares on *ZX* and *XU* is triple the square on *NP*.

But the square on *UZ* equals the sum of the squares on *ZX* and *XU*, therefore the square on *UZ* is triple the square on *NP*.

But the square on the radius of the sphere which comprehends the cube is also triple the square on the half of the side of the cube, for it has previously been shown how to construct a cube and comprehend it in a sphere, and to prove that the square on the diameter of the sphere is triple the square on the side of the cube. XIII.15

But, if the whole is so related to the whole as the half to the half also, and *NP* is half of the side of the cube, therefore *UZ* equals the radius of the sphere which comprehends the cube.

And *Z* is the center of the sphere which comprehends the cube, therefore the point *U* is on the surface of the sphere.

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere, therefore the dodecahedron has been comprehended in the given sphere.

I say next that the side of the dodecahedron is the irrational straight line called apotome.

Since, when *NP* is cut in extreme and mean ratio, *RP* is the greater segment, and, when *PO* is cut in extreme and mean ratio, *PS* is the greater segment, therefore, when the whole *NO* is cut in extreme and mean ratio, *RS* is the great er segment.

Thus, since *NP* is to *PR* as *PR* is to *RN*, the same is true of the doubles also, for parts have the same ratio as their equimultiples, therefore *NO* is to *RS* as *RS* is to the sum of *NR* and *SO*. But *NO* is greater than *RS*, therefore *RS* is also greater than the sum of *NR* and *SO*, therefore *NO* is cut in extreme and mean ratio, and *RS* is its greater segment. V.15

But *RS* equals *UV*, therefore, when *NO* is cut in extreme and mean ratio, *UV* is the greater segment. And, since the diameter of the sphere is rational, and the square on it is triple the square on the side of the cube, therefore *NO*, being a side of the cube, is rational.

But if a rational line is cut in extreme and mean ratio, each of the segments is an irrational apotome.

Therefore *UV*, being a side of the dodecahedron, is an irrational apotome. XIII.6

Q.E.F.

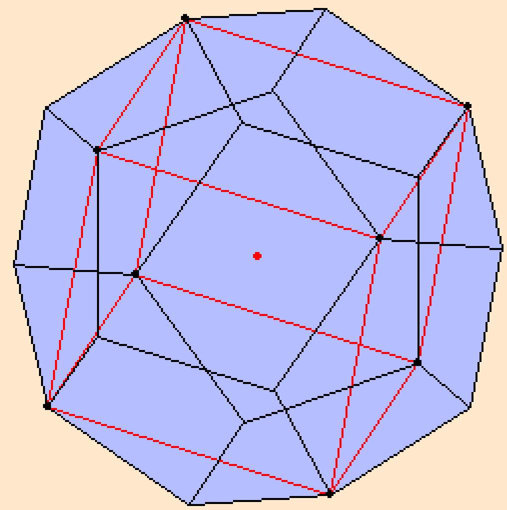
Corollary.

From this it is clear that *when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron.*

Q.E.D.

Guide

Cubes and regular dodecahedra



Euclid's construction of a dodecahedron is particularly easy because he circumscribed his dodecahedron about a cube. Just as a regular tetrahedron can be circumscribed by a cube, a cube can be circumscribed by a regular dodecahedron, indeed, two regular dodecahedra. Also, each cube circumscribes two regular tetrahedra, and a regular dodecahedron circumscribes five cubes, and also ten tetrahedra.

Coordinates for the vertices of the dodecahedron

We can specify a coordinate system so that the center of the sphere is located at the origin and the eight vertices of the cube are located at

(1,1,1) (1,1,–1) (1,–1,1) (1,–1,–1) (–1,1,1) (–1,1,–1) (–1,–1,1) (–1,–1,–1)

The points A a through F in Euclid's construction may be assigned six of these coordinates.

$$A = (–1,–1,–1),\ B = (–1,1,–1),\ C = (1,1,–1),\ D = (1,–1,–1),\ E = (–1,1,1),\ F = (1,1,1).$$

After bisecting the sides, the points G through Q receive the following coordinates.

$$G = (–1,0,–1),\ H = (0,1,–1),\ K = (1,0,–1),\ L = (0,–1,–1),\ M = (0,1,1),$$
$$N = (–1,1,0),\ O = (1,1,0),\ P = (0,1,0),\ Q = (0,0,–1).$$

The points R, S, and T cut the lines they're on into extreme and mean ratios, so they have these coordinates:

$$R = (–x,1,0),\ S = (x,1,0),\ T = (0,x,–1),$$

where x equals $(\sqrt{5} – 1)/2$. Finally, points U, V, and W are outside the original cube with the coordinates

$$U = (–x,1+x,0),\ V = (x,1+x,0),\ W = (0,x,–1–x),$$

Use of this construction

Constructing a dodecahedron is an end in itself. This construction and the corollary are also used in XIII.18 where the five regular polyhedra are compared.

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Previous: XIII.16

Book XIII introduction

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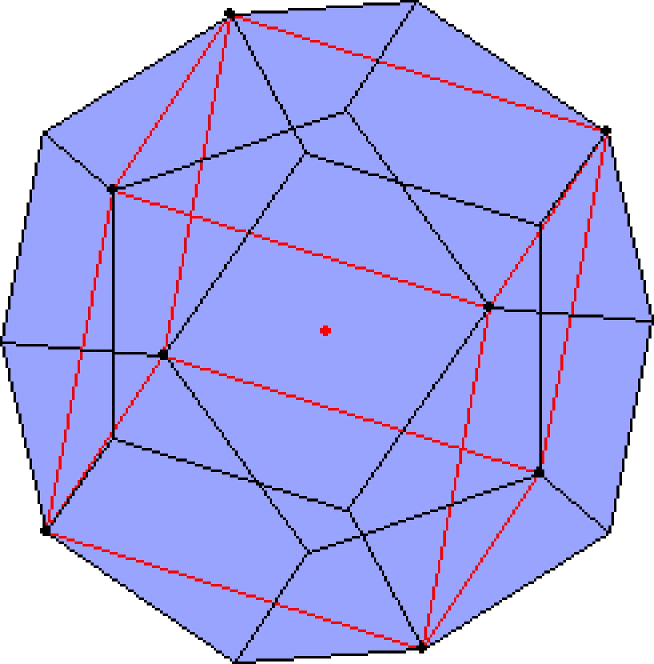

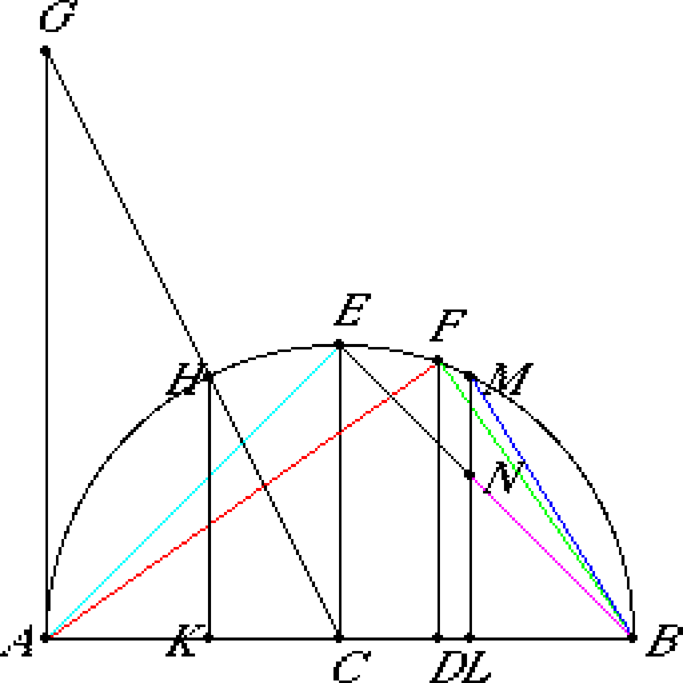
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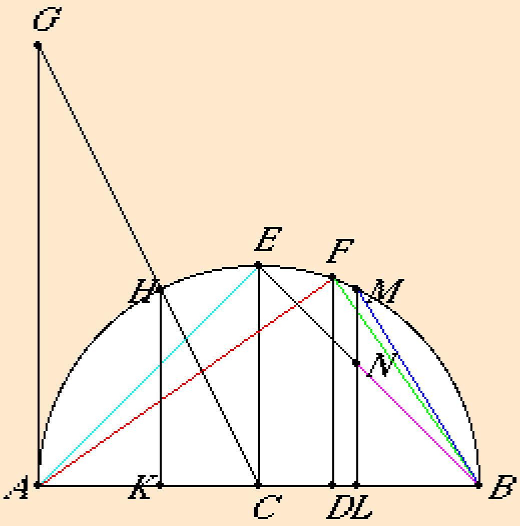
Euclid's Elements
Book XIII
Proposition 18

To set out the sides of the five figures and compare them with one another.

Set out *AB* the diameter of the given sphere, and cut it at *C* so that *AC* equals *CB,* and at *D* so that *AD* is double *DB.* Describe the semicircle *AEB* on *AB,* draw *CE* and *DF* from *C* and *D* at right angles to *AB,* and join *AF, FB,* and *EB.* I.11

Then, since *AD* is double *DB,* therefore *AB* is triple *BD.* In conversion, therefore, *BA* is one and a half times *AD.*



But *BA* is to *AD* as the square on *BA* is to the square on *AF*, for the triangle *AFB* is equiangular with the triangle *AFD.* Therefore the square on *BA* is one and a half times the square on *AF*. V.Def.9 VI.8

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid. And *AB* is the diameter of the sphere, therefore *AF* equals the side of the pyramid. XIII.13

Again, since *AD* is double *DB,* therefore *AB* is triple *BD.* But *AB* is to *BD* as the square on *AB* to the square on *BF*, therefore the square on *AB* is triple the square on *BF.* V.Def.9 VI.8

But the square on the diameter of the sphere is also triple the square on the side of the cube. And *AB* is the diameter of the sphere, therefore *BF* is the side of the cube. XIII.15

And, since *AC* equals *CB,* therefore *AB* is double *BC.* But *AB* is to *BC* as the square on *AB* to the square on *BE,* therefore the square on *AB* is double the square on *BE.*

But the square on the diameter of the sphere is also double the square on the side of the octahedron. And *AB* is the diameter of the given sphere, therefore *BE* is the side of the octahedron. XIII.14

Next, draw *AG* from the point *A* at right angles to the straight line *AB,* make *AG* equal to *AB,* join *GC,* and draw *HK* from *H* perpendicular to *AB.* I.11 I.3 I.12

Then, since *GA* is double *AC,* for *GA* equals *AB* and *GA* is to *AC* as *HK* is to *KC,* therefore *HK* is also double *KC.*

Therefore the square on *HK* is quadruple the square on *KC,* therefore the sum of the squares on *HK* and *KC,* that is, the square on *HC,* is five times the square on *KC.*

But *HC* equals *CB,* therefore the square on *BC* is five times the square on *CK.* And, since *AB* is double *CB,* and, in them, *AD* is double *DB,* therefore the remainder *BD* is double the remainder *DC.*

Therefore *BC* is triple *CD,* therefore the square on *BC* is nine times the square on *CD.* But the square on *BC* is five times the square on *CK,* therefore the square on *CK* is greater than the square on *CD.* Therefore *CK* is greater than *CD.*

Make *CL* equal to *CK,* draw *LM* from *L* at right angles to *AB,* and join *MB.* I.3 I.11

Now, since the square on *BC* is five times the square on *CK,* and *AB* is double *BC,* and *KL* is double *CK,* therefore the square on *AB* is five times the square on *KL.* But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described. And *AB* is the diameter of the sphere, therefore *KL* is the radius of the circle from which the icosahedron has been described. Therefore *KL* is a side of the hexagon in the said circle. XIII.16,Cor. IV.15,Cor.

And, since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle, and *AB* is the diameter of the sphere, while *KL* is a side of the hexagon, and *AK* equals *LB,* therefore each of the straight lines *AK* and *LB* is a side of the decagon inscribed in the circle from which the icosahedron has been described. XIII.16,Cor.

And, since *LB* belongs to a decagon, and *ML* to a hexagon, for *ML* equals *KL,* since it also equals *HK* being the same distance from the center and each of the straight lines *HK* and *KL* is double *KC,* therefore *MB* belongs to a pentagon. XIII.10

But the side of the pentagon is the side of the icosahedron, therefore *MB* belongs to the icosahedron. XIII.16

Now, since *FB* is a side of the cube, cut it in extreme and mean ratio at *N,* and let *NB* be the greater segment. Therefore *NB* is a side of the dodecahedron. XIII.17,Cor.

And, since the square on the diameter of the sphere was proved to be one and a half times the square on the side *AF* of the pyramid, double the square on the side *BE* of the octahedron and triple the side *FB* of the cube, therefore, of parts of which the square on the diameter of the sphere contains six, the square on the side of the pyramid contains four, the square on the side of the octahedron three, and the square on the side of the cube two.

Therefore the square on the side of the pyramid is four-thirds of the square on the side of the octahedron, and double the square on the side of the cube, and the square on the side of the octahedron is one and a half times the square on the side of the cube.

The said sides, therefore, of the three figures, I mean the pyramid, the octahedron and the cube, are to one another in rational ratios.

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in rational ratios either to one another or to the aforesaid sides, for they are irrational, the one being minor and the other an apotome. XIII.16 XIII.17

That the side *MB* of the icosahedron is greater than the side *NB* of the dodecahedron we can prove thus.

Since the triangle *FDB* is equiangular with the triangle *FAB,* proportionally *DB* is to *BF* as *BF* is to *BA.* VI.8 VI.4

And, since the three straight lines are proportional, the first is to the third as the square on the first is to the square on the second, therefore *DB* is to *BA* as the square on *DB* is to the square on *BF.* Therefore, inversely *AB* is to *BD* as the square on *FB* is to the square on *BD.* V.Def.9 VI.20,Cor.

But *AB* is triple *BD,* therefore the square on *FB* is triple the square on *BD.*

But the square on *AD* is also quadruple the square on *DB,* for *AD* is double *DB,* therefore the square on *AD* is greater than the square on *FB.* Therefore *AD* is greater than *FB.* Therefore *AL* is by far greater than *FB.*

And, when *AL* is cut in extreme and mean ratio, *KL* is the greater segment, for *LK* belongs to a hexagon, and *KA* to a decagon, and, when *FB* is cut in extreme and mean ratio, *NB* is the greater segment, therefore *KL* is greater than *NB*. XIII.9

But *KL* equals *LM*, therefore *LM* is greater than *NB*.

Therefore *MB*, which is a side of the icosahedron, is by far greater than *NB* which is a side of the dodecahedron.

Q.E.F.

Remark

I say next that *no other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another*.

For a solid angle cannot be constructed with two triangles, or indeed planes.

With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron, and with five the angle of the icosahedron, but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point, for, the angle of the equilateral triangle being two-thirds of a right angle, the six would be equal to four right angles, which is impossible, for any solid angle is contained by angles less than four right angles. XI.21

For the same reason, neither can a solid angle be constructed by more than six plane angles.

By three squares the angle of the cube is contained, but by four it is impossible for a solid angle to be contained, for they would again be four right angles.

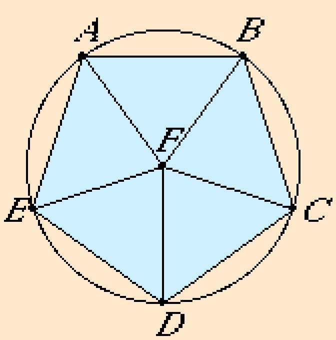
By three equilateral and equiangular pentagons the angle of the dodecahedron is contained, but by four such it is impossible for any solid angle to be contained, for, the angle of the equilateral pentagon being a right angle and a fifth, the four angles would be greater than four right angles, which is impossible.

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity.

Q.E.D.

Lemma

But that the angle of the equilateral and equiangular pentagon is a right angle and a fifth we must prove thus.



Let *ABCDE* be an equilateral and equiangular pentagon. Circumscribe the circle *ABCDE* about it, take its center *F*, and join *FA*, *FB*, *FC*, *FD*, and *FE*. IV.14

Therefore they bisect the angles of the pentagon at *A*, *B*, *C*, *D*, and *E*. And, since the angles at *F* equal four right angles and are equal, therefore one of them, as the angle *AFB*, is one right angle less a fifth. Therefore the remaining angles *FAB* and *ABF* consist of one right angle and a fifth.

But the angle *FAB* equals the angle *FBC*, therefore the whole angle *ABC* of the pentagon consists of one right angle and a fifth.

Q.E.D.

Guide

Summary of the regular polyhedra

In the following table, *d* is the diameter of the sphere in which each regular polygon is inscribed while *s* is the side of the polygon. Then d^2/s^2 is the ratio of the square of the diameter to the square of the side.

Polyhedron	construction	d^2/s^2
tetrahedron	XIII.13	3/2
octahedron	XIII.14	2
cube	XIII.15	3
icosahedron	XIII.16	$(2\sqrt{5})/(\sqrt{5}-1)$
dodecahedron	XIII.17	$(3-\sqrt{5})/6$

Previous: XIII.17

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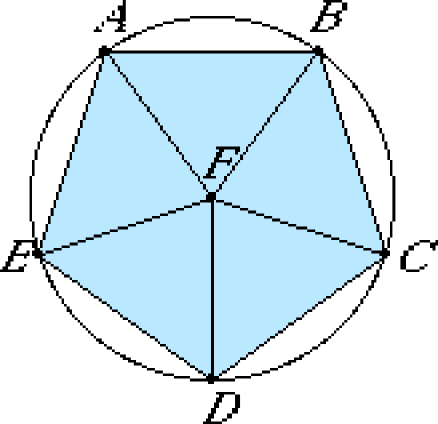

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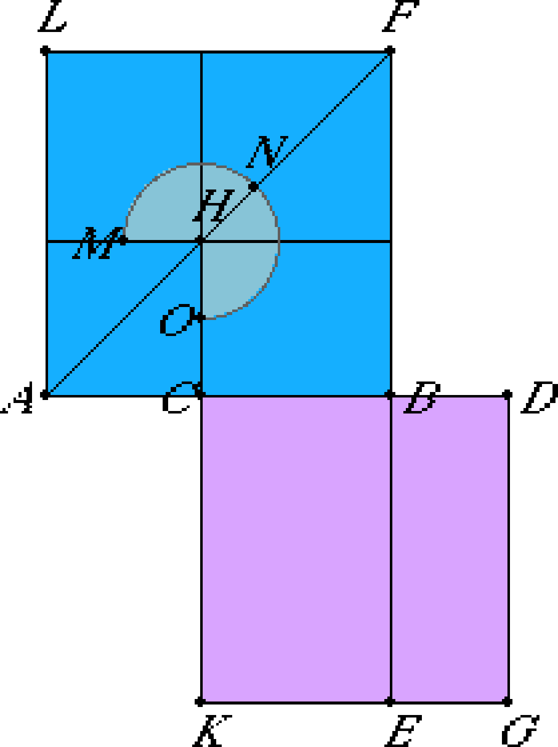
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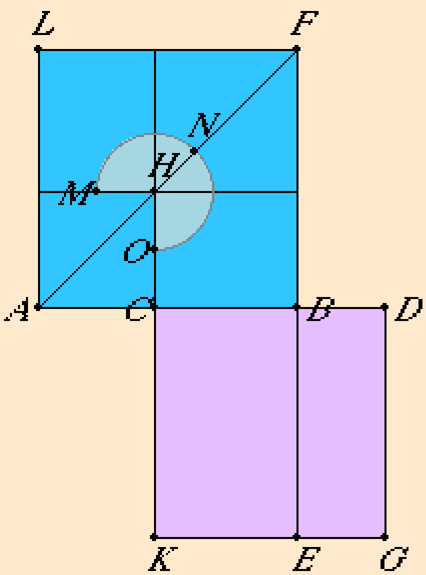
Euclid's Elements
Book XIII
Proposition 2

If the square on a straight line is five times the square on a segment on it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

Let the square on the straight line *AB* be five times the square on the segment *AC* of it, and let *CD* be double *AC*.

I say that, when *CD* is cut in extreme and mean ratio, then the greater segment is *CB*.



Describe the squares *AF* and *CG* on *AB* and *CD* respectively, draw the figure in *AF*, and draw *BE* through. I.46

Now, since the square on *BA* is five times the square on *AC*, therefore *AF* is five times *AH*. Therefore the gnomon *MNO* is quadruple *AH*.

And, since *DC* is double *CA*, therefore the square on *DC* is quadruple the square on *CA*, that is, *CG* is quadruple *AH*. But the gnomon *MNO* is also quadruple *AH*, therefore the gnomon *MNO* equals *CG*.

And, since *DC* is double *CA*, while *DC* equals *CK*, and *AC* equals *CH*, therefore *KB* is also double *BH*. VI.1

But the sum of *LH* and *HB* is also double *HB*, therefore *KB* equals the sum of *LH* and *HB*.

But the whole gnomon *MNO* was also proved equal to the whole *CG*, therefore the remainder *HF* equals *BG*.

And *BG* is the rectangle *CD* by *DB*, for *CD* equals *DG*, and *HF* is the square on *CB*, therefore the rectangle *CD* by *DB* equals the square on *CB*.

Therefore *DC* is to *CB* as *CB* is to *BD*. But *DC* is greater than *CB*, therefore *CB* is also greater than *BD*.

Therefore, when the straight line *CD* is cut in extreme and mean ratio, *CB* is the greater segment.

Q.E.D.

Lemma

That the double *AC* is greater than *BC* is to be proved thus.

If not, let *BC* be, if possible, double *CA*.

Therefore the square on *BC* is quadruple the square on *CA*. Therefore the sum of the squares on *BC* and *CA* is five times the square on *CA.* But, by hypothesis, the square on *BA* is also five times the square on *CA*

Therefore the square on *BA* equals the sum of the squares on *BC* and *CA,* which is impossible. II.4

Therefore *CB* is not double *AC.*

Similarly we can prove that neither is a straight line less than *CB* double *CA,* for the absurdity is much greater.

Therefore the double *AC* is greater than *CB.*

Therefore, *if the square on a straight line is five times the square on a segment on it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.*

Q.E.D.

Guide

This proposition is not used in the rest of the *Elements.* Apparently, it is only included because it is the converse of the previous proposition XIII.1.

Next proposition: XIII.3

Previous: XIII.1

Book XIII introduction

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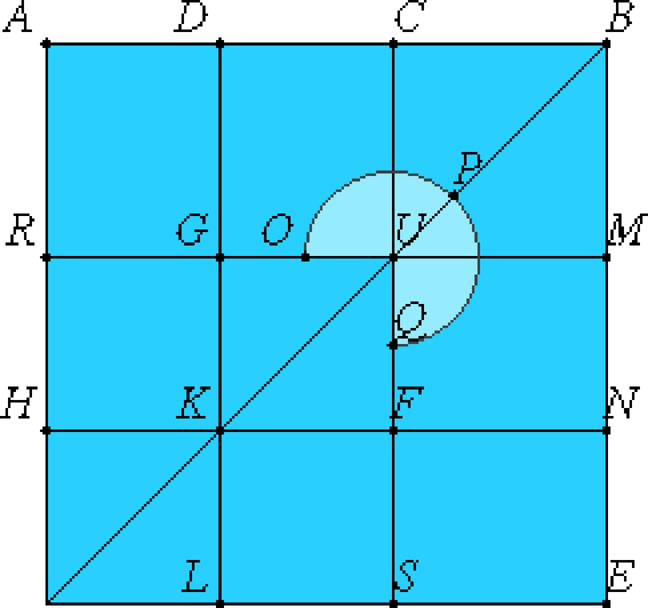
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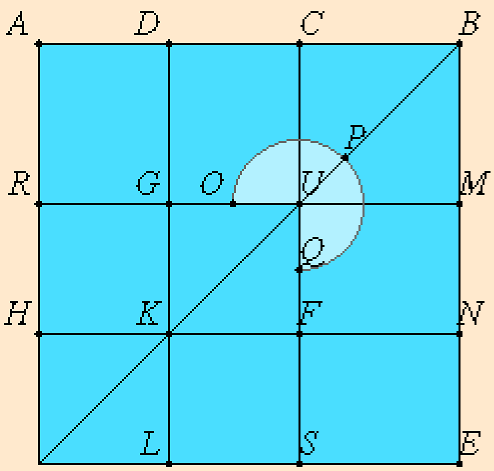
Euclid's Elements
Book XIII
Proposition 3

If a straight line is cut in extreme and mean ratio, then the square on the sum of the lesser segment and the half of the greater segment is five times the square on the half of the greater segment.

Cut any straight line *AB* in extreme and mean ratio at the point *C,* and let *AC* be the greater segment. Bisect *AC* at *D.*

I say that the square on *BD* is five times the square on *DC.*



Describe the square *AE* on *AB,* and draw the figure. I.46

Since *AC* is double *DC,* therefore the square on *AC* is quadruple the square on *DC,* that is, *RS* is quadruple *FG.*

And, since the rectangle *AB* by *BC* equals the square on *AC,* and *CE* is the rectangle *AB* by *BC,* therefore *CE* equals *RS.*

But *RS* is quadruple *FG,* therefore *CE* is also quadruple *FG.*

Again, since *AD* equals *DC,* therefore *HK* also equals *KF.*

Hence the square *GF* equals the square *HL.*

Therefore *GK* equals *KL,* that is *MN* equals *NE,* hence *MF* equals *FE.*

But *MF* equals *CG,* therefore *CG* equals *FE.*

Add *CN* to each, therefore the gnomon *OPQ* equals *CE.*

But *CE* was proved quadruple *GF,* therefore the gnomon *OPQ* is also quadruple the square *FG.* Therefore the sum of the gnomon *OPQ* and the square *FG* is five times *FG.*

But the sum of the gnomon *OPQ* and the square *FG* is the square *DN.* And *DN* is the square on *DB,* and *GF* is the square on *DC.* Therefore the square on *DB* is five times the square on *DC.*

Therefore, *if a straight line is cut in extreme and mean ratio, then the square on the sum of the lesser segment and the half of the greater segment is five times the square on the half of the greater segment.*

Q.E.D.

Guide

Use of this proposition

This result is needed in proposition XIII.16 to show the icosahedron is inscribed in the given sphere.

Next proposition: XIII.4

Previous: XIII.2

Book XIII introduction

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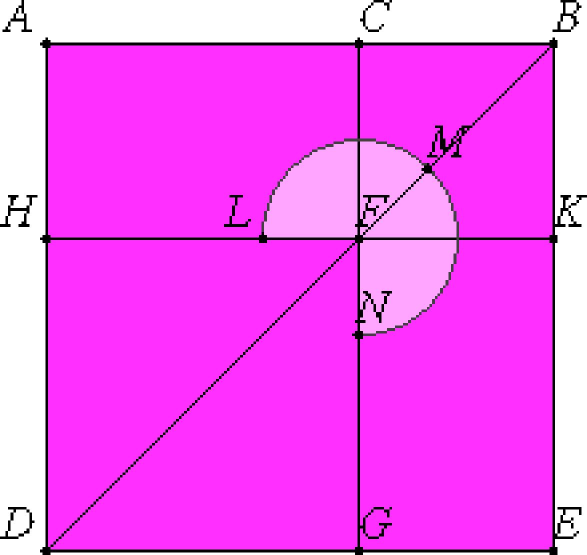
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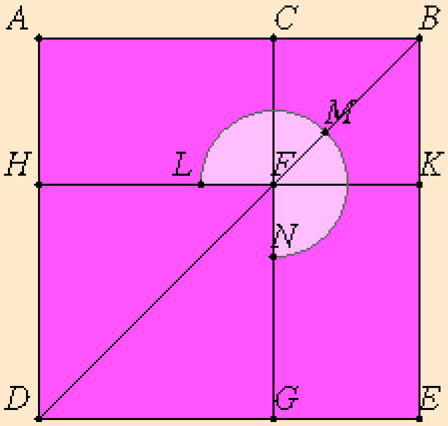
Euclid's Elements
Book XIII
Proposition 4

If a straight line is cut in extreme and mean ratio, then the sum of the squares on the whole and on the lesser segment is triple the square on the greater segment.

Let *AB* be a straight line cut in extreme and mean ratio at *C,* and let *AC* be the greater segment.

I say that the sum of the squares on *AB* and *BC* is triple the square on *CA.*



Describe the square *ADEB* on *AB,* and draw the figure. I.46

Since, then, *AB* is cut in extreme and mean ratio at *C,* and *AC* is the greater segment, therefore the rectangle *AB* by *BC* equals the square on *AC.* VI.Def.3 VI.17

And *AK* is the rectangle *AB* by *BC,* and *HG* is the square on *AC,* therefore *AK* equals *HG.*

And, since *AF* equals *FE,* add *CK* to each, therefore the whole *AK* equals the whole *CE.* Therefore the sum of *AK* and *CE* is double *AK.*

But the sum of *AK* and *CE* is the sum of the gnomon *LMN* and the square *CK,* therefore the sum of the gnomon *LMN* and the square *CK* is double *AK.*

But, further, *AK* was also proved equal to *HG,* therefore the sum of the gnomon *LMN* and the squares *CK* and *HG* is triple the square *HG.*

And the sum of the gnomon *LMN* and the squares *CK* and *HG* is the sum of the whole square *AE* and *CK,* which are the squares on *AB* and *BC,* while *HG* is the square on *AC.*

Therefore the sum of the squares on *AB* and *BC* is triple the square on *AC.*

Therefore, *if a straight line is cut in extreme and mean ratio, then the sum of the squares on the whole and on the lesser segment is triple the square on the greater segment.*

Q.E.D.

Guide

Use of this proposition

This and the next three propositions are all preparatory to the construction of a dodecahedron in proposition XIII.17.

Next proposition: XIII.5

Previous: XIII.3

Book XIII introduction

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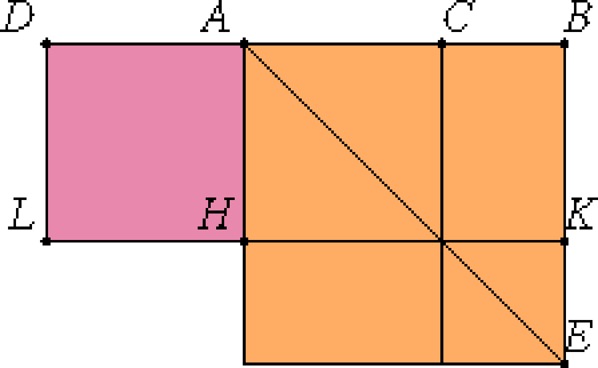
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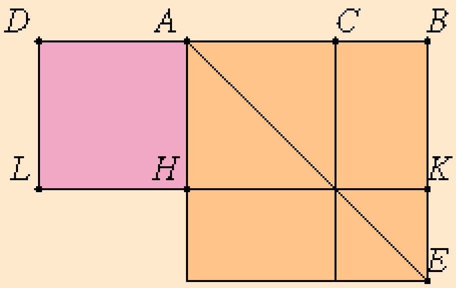
Euclid's Elements
Book XIII
Proposition 5

If a straight line is cut in extreme and mean ratio, and a straight line equal to the greater segment is added to it, then the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment.

Let the straight line *AB* be cut in extreme and mean ratio at the point *C*, let *AC* be the greater segment, and let *AD* be equal to *AC*.

I say that the straight line *DB* is cut in extreme and mean ratio at *A*, and the original straight line *AB* is the greater segment.



Describe the square *AE* on *AB*, and draw the figure. I.46

Since *AB* is cut in extreme and mean ratio at *C*, therefore the rectangle *AB* by *BC* equals the square on *AC*. VI.Def.3 VI.17

And *CE* is the rectangle *AB* by *BC*, and *CH* is the square on *AC*, therefore *CE* equals *HC*.

But *HE* equals *CE*, and *DH* equals *HC*, therefore *DH* also equals *HE*. Therefore the whole *DK* is equal to the whole *AE*.

And *DK* is the rectangle *BD* by *DA*, for *AD* equals *DL*, and *AE* is the square on *AB*, therefore the rectangle *BD* by *DA* equals the square on *AB*.

Therefore *DB* is to *BA* as *BA* is to *AD*. And *DB* is greater than *BA*, therefore *BA* is also greater than *AD*. VI.17 V.14

Therefore *DB* has been cut in extreme and mean ratio at *A*, and *AB* is the greater segment.

Therefore, *if a straight line is cut in extreme and mean ratio, and a straight line equal to the greater segment is added to it, then the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment.*

Q.E.D.

Guide

Use of this proposition

This proposition is used XIII.17 where a dodecahedron is constructed.

Next proposition: XIII.6

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Euclid's Elements

Book XIII

Proposition 6

If a rational straight line is cut in extreme and mean ratio, then each of the segments is the irrational straight line called apotome.

Let *AB* be a rational straight line cut in extreme and mean ratio at *C*, and let *AC* be the greater segment.

I say that each of the straight lines *AC* and *CB* is the irrational straight line called apotome.

Produce *BA*, and make *AD* half of *BA*.

Since, then, the straight line *AB* is cut in extreme and mean ratio, and to the greater segment *AC* is added *AD* which is half of *AB*, therefore the square on *CD* is five times the square on *DA*. XIII.1

Therefore the square on *CD* has to the square on *DA* the ratio which a number has to a number, therefore the square on *CD* is commensurable with the square on *DA* X.6

But the square on *DA* is rational, for *DA* is rational being half of *AB* which is rational, therefore the square on *CD* is also rational. Therefore *CD* is also rational. X.Def.4

And, since the square on *CD* has not to the square on *DA* the ratio which a square number has to a square number, therefore *CD* is incommensurable in length with *DA*. Therefore *CD* and *DA* are rational straight lines commensurable in square only. Therefore *AC* is an apotome. X.9 X.73

Again, since *AB* is cut in extreme and mean ratio, and *AC* is the greater segment, therefore the rectangle *AB* by *BC* equals the square on *AC*. VI.Def.3 VI.17

Therefore the square on the apotome *AC*, if applied to the rational straight line *AB*, produces *BC* as breadth. But the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome, therefore *CB* is a first apotome. And *CA* was also proved to be an apotome. X.97

Therefore, *if a rational straight line is cut in extreme and mean ratio, then each of the segments is the irrational straight line called apotome.*

Q.E.D.

Guide

Heath argues that this proposition was interpolated.

Use of this proposition

This proposition is used after the construction of a dodecahedron in XIII.17 to show that the side of a pentagonal face is the irrational straight line called apotome.

Next proposition: XIII.7

Previous: XIII.5

Book XIII introduction

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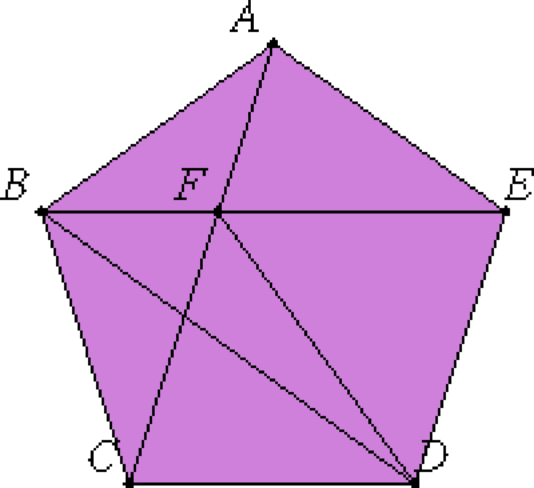
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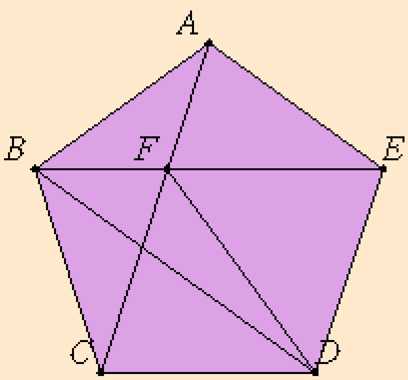
Euclid's Elements
Book XIII
Proposition 7

If three angles of an equilateral pentagon, taken either in order or not in order, are equal, then the pentagon is equiangular.

First, let three angles *A, B,* and *C* taken in order in the equilateral pentagon *ABCDE* be equal to one another.

I say that the pentagon *ABCDE* is equiangular.



Join *AC, BE,* and *FD.*

Now, since the two sides *CB* and *BA* equal the two sides *BA* and *AE* respectively, and the angle *CBA* equals the angle *BAE,* therefore the base *AC* equals the base *BE,* the triangle *ABC* equals the triangle *ABE,*and the remaining angles equal the remaining angles, namely those opposite the equal sides, that is, the angle *BCA* equals the angle *BEA,* and the angle *ABE* equals the angle *CAB.* I.4

Hence the side *AF* also equals the side *BF.* I.6

But the whole *AC* equals the whole *BE,* therefore the remainder *FC* equals the remainder *FE.* But *CD* also equals *DE.* Therefore the two sides *FC* and *CD* equal the two sides *FE* and *ED,* and the base *FD* is common to them, therefore the angle *FCD* equals the angle *FED.* I.8

But the angle *BCA* was also proved equal to the angle *AEB,* therefore the whole angle *BCD* equals the whole angle *AED.* And, by hypothesis, the angle *BCD* equals the angles at *A* and *B,* therefore the angle *AED* also equals the angles at *A* and *B.* Similarly we can prove that the angle *CDE* also equals the angles at *A, B,* and *C.* Therefore the pentagon *ABCDE* is equiangular.

Next, let the given equal angles not be angles taken in order, but let the angles at the points *A, C,* and *D* be equal.

I say that in this case too the pentagon *ABCDE* is equiangular.

Join *BD.*

Then, since the two sides *BA* and *AE* equal the two sides *BC* and *CD,* and they contain equal angles, therefore the base *BE* equals the base *BD,* the triangle *ABE* equals the triangle *BCD,* and the remaining angles equal the remaining angles, namely those opposite the equal sides. Therefore the angle *AEB* equals the angle *CDB.* I.4

But the angle *BED* also equals the angle *BDE,* since the side *BE* equals the side *BD.* I.5

Therefore the whole angle *AED* equals the whole angle *CDE.*

But the angle *CDE* is, by hypothesis, equal to the angles at *A* and *C,* therefore the angle *AED* also equals the angles at *A* and *C.*

For the same reason the angle *ABC* also equals the angles at *A, C,* and *D.* Therefore the pentagon *ABCDE* is equiangular.

Therefore, *if three angles of an equilateral pentagon, taken either in order or not in order, are equal, then the pentagon is equiangular.*

Q.E.D.

Guide

Use of this proposition

This proposition is needed in XIII.17 to show that the dodecahedron constructed there has equiangular pentagons as faces.

Next proposition: XIII.8

Previous: XIII.6

Book XIII introduction

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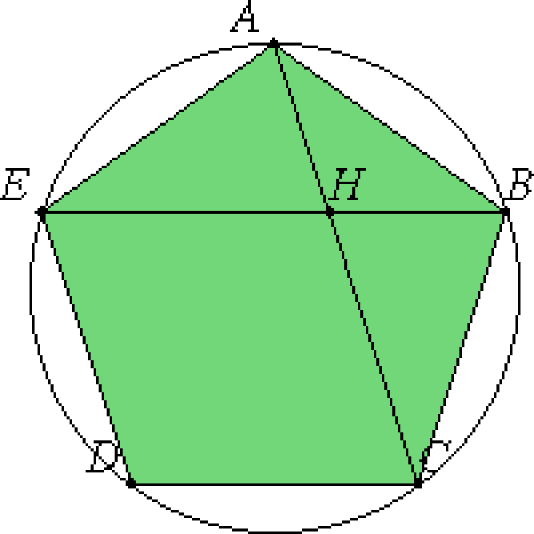

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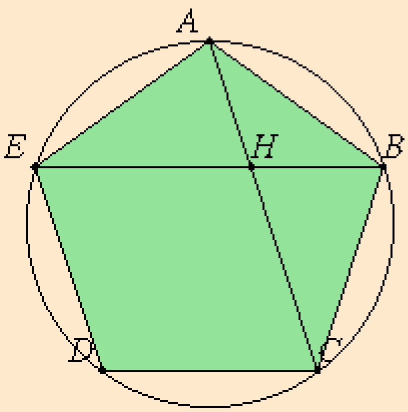
Euclid's Elements
Book XIII
Proposition 8

If in an equilateral and equilateral pentagon straight lines subtend two angles are taken in order, then they cut one another in extreme and mean ratio, and their greater segments equal the side of the pentagon.

In the equilateral and equiangular pentagon *ABCDE* let the straight lines *AC* and *BE,* cutting one another at the point *H,* subtend two angles taken in order, the angles at *A* and *B.*

I say that each of them has been cut in extreme and mean ratio at the point *H,* and their greater segments equal the side of the pentagon.



Circumscribe the circle *ABCDE* about the pentagon *ABCDE.* IV.14

Then, since the two straight lines *EA* and *AB* equal the two lines *AB* and *BC,* and they contain equal angles, therefore the base *BE* equals the base *AC,* the triangle *ABE* equals the triangle *ABC,* and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. I.4

Therefore the angle *BAC* equals the angle *ABE.* Therefore the angle *AHE* is double the angle *BAH.* I.32

But the angle *EAC* is also double the angle *BAC,* for the circumference *EDC* is also double the circumference *CB.* III.28 VI.33

Therefore the angle *HAE* equals the angle *AHE.* Hence the straight line *HE* also equals *EA,* that is, *AB.* I.6

And, since the straight line *BA* equals *AE,* therefore the angle *ABE* also equals the angle *AEB.* I.5

But the angle *ABE* was proved equal to the angle *BAH,* therefore the angle *BEA* also equals the angle *BAH.*

And the angle *ABE* is common to the two triangles *ABE* and *ABH,* therefore the remaining angle *BAE* equals the remaining angle *AHB.* Therefore the triangle *ABE* is equiangular with the triangle *ABH.* I.32

Therefore, proportionally *EB* is to *BA* as *AB* is to *BH.* VI.4

But *BA* equals *EH,* therefore *BE* is to *EH* as *EH* is to *HB.*

And *BE* is greater than *EH,* therefore *EH* is also greater than *HB.* VI.14

Therefore *BE* has been cut in extreme and mean ratio at *H,* and the greater segment *HE* equals the side of the pentagon.

Similarly we can prove that *AC* has also been cut in extreme and mean ratio at *H,* and its greater segment *CH* equals the side of the pentagon.

Therefore, *if in an equilateral and equilateral pentagon straight lines subtend two angles are taken in order, then they cut one another in extreme and mean ratio, and their greater segments equal the side of the pentagon.*

Q.E.D.

Guide

Use of this proposition

This proposition is used in the proof of XIII.11 to establish that the side of a regular pentagon inscribed in a circle with rational diameter is the irrational straight line called minor.

Next proposition: XIII.9

Previous: XIII.7

Book XIII introduction

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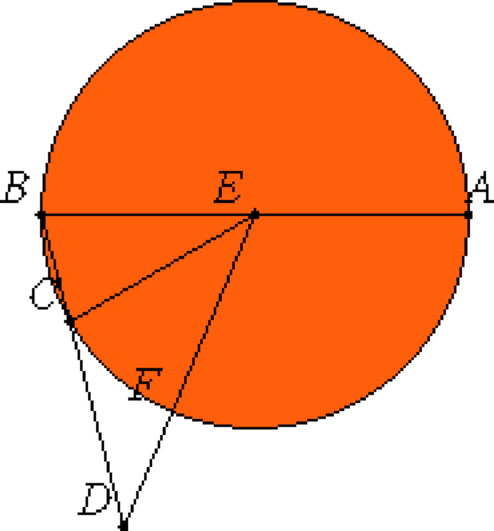
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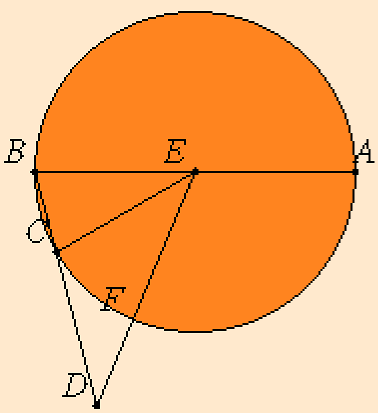
Euclid's Elements
Book XIII
Proposition 9

If the side of the hexagon and that of the decagon inscribed in the same circle are added together, then the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

Let *ABC* be a circle, and of the figures inscribed in the circle *ABC* let *BC* be the side of a decagon, and *CD* that of a hexagon, and let them be in a straight line.

I say that the whole straight line *BD* is cut in extreme and mean ratio, and *CD* is its greater segment.



Take the center *E* of the circle, join *EB*, *EC*, and *ED*, and carry *BE* through to *A*. III.1

Since *BC* is the side of an equilateral decagon, therefore the circumference *ACB* is five times the circumference *BC*. Therefore the circumference *AC* is quadruple *CB*.

But the circumference *AC* is to *CB* as the angle *AEC* is to the angle *CEB*. Therefore the angle *AEC* is quadruple the angle *CEB*. VI.33

And, since the angle *EBC* equals the angle *ECB*, therefore the angle *AEC* is double the angle *ECB*. I.5 I.32

And, since the straight line *EC* equals *CD*, for each of them equals the side of the hexagon inscribed in the circle *ABC*. Therefore the angle *CED* also equals the angle *CDE*. Therefore the angle *ECB* is double the angle *EDC*. IV.15,Cor. I.5 I.32

But the angle *AEC* was proved double the angle *ECB*, therefore the angle *AEC* is quadruple the angle *EDC*. And the angle *AEC* was also proved quadruple the angle *BEC*, therefore the angle *EDC* equals the angle *BEC*.

But the angle *EBD* is common to the two triangles *BEC* and *BED*, therefore the remaining angle *BED* equals the remaining angle *ECB*. Therefore the triangle *EBD* is equiangular with the triangle *EBC*. I.32

Therefore, proportionally *DB* is to *BE* as *EB* is to *BC*. VI.4

But *EB* equals *CD*. Therefore *BD* is to *DC* as *DC* is to *CB*. And *BD* is greater than *DC*, therefore *DC* is also greater than *CB*.

Therefore the straight line *BD* is cut in extreme and mean ratio, and *DC* is its greater segment.

Therefore, *if the side of the hexagon and that of the decagon inscribed in the same circle are added together, then the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.*

Q.E.D.

Guide

Use of this proposition

This result is used in the construction of an icosahedron in propositions XIII.16 and XIII.18.

Next proposition: XIII.10

Previous: XIII.8

Book XIII introduction

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About the Text

This text of this version of Euclid's *Elements* is similar to Heath's edition which he translated from Heiberg's definitive edition in Greek, but it is slightly less literal to make it more readable.

- Heath, Sir Thomas Little (1861-1940)
  The thirteen books of Euclid's Elements translated from the text of Heiberg with introduction and commentary. Three volumes. University Press, Cambridge, 1908. Second edition: University Press, Cambridge, 1925. Reprint: Dover Publ., New York, 1956. Reviewed: *Isis* 10 (1928), 60-62.

- Heiberg, J. L. (Johan Ludwig) (1854-1928), and H. Menge.
  Euclidis opera omnia. 8 vol. & supplement, in Greek. Teubner, Leipzig, 1883-1916. Edited by J. L. Heiberg and H. Menge.

Heath's excellent critical commentary is as important as the text itself, and since Heath's edition is in publication (Dover), a purchase of that edition is recommended. The text of Heath's translation of Euclid's *Elements* is also available on-line at the Perseus Project at Tuft's University starting with the first definition of book I. Not just Heath's translation, but his commentary as well as the Greek text is available at the Perseus Project. (The figures are not included.)

Other versions of Euclid's *Elements*

I have also used other versions of the *Elements* for this translation including

- Peyrard, F.
  Les Oeuvres d'Euclide, en Grec, en Latin et en Français. Three volumes. M. Patris, Paris, 1814.

- Todhunter's edition (1862) of Simson's translation (various editions from 1756-1830) which was published in 1933 by Dent, London, with an introduction by Heath.

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D.E.Joyce
Clark University

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Mirror sites

I am formulating a policy on mirroring Euclid's *Elements*.

Euclid's *Elements* Introduction

David E. Joyce
Department of Mathematics and Computer Science
Clark University
Worcester, MA 01610

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Definition 1

Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.

The words in this definition do not quite express its entire intent. It is apparent from its use that the notion of similarity assumes a specific correspondence of consecutive vertices and sides. Consider, for instance, pentagons.

In order for the pentagons *ABCDE* and *FGHKL* to be similar, it is required that

1. corresponding angles taken in order are equal, that is, $A = F$, $B = G$, $C = H$, $D = K$, and $E = L$, and
2. the sides about their equal angles are proportional in the same order:

$$\begin{aligned} EA:AB &= LF:FG, \\ AB:BC &= FG:GH, \\ BC:CD &= GH:HK, \\ CD:EF &= HK:KL, \text{ and} \\ EA:AB &= KL:LF. \end{aligned}$$

It wouldn't be allowed, for instance, if the angles of one figure equalled the angles of the other, but in some haphazard order. And it wouldn't be allowed for the orders of the terms in the proportions to be permuted, or inverted, for instance, the second proportion could not be $AB:BC = GH:FG$.

Use of this definition

Propositions VI.4 and VI.5 give two criteria for two triangles to be similar. Proposition VI.4 says that condition 1 implies similarity, while VI.5 says condition 2 implies similarity. Proposition VI.6 is a side-angle-side similarity theorem, and VI.7 is a side-side-angle similarity theorem. Many of the other propositions in this and later books involve similarity in one way or another.

Next definition: VI.Def.2

Book VI introduction

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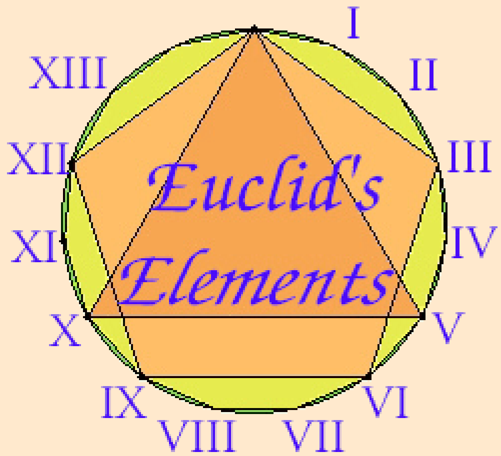
Euclid's Elements



Introduction

Euclid's *Elements* form one of the most beautiful and influential works of science in the history of humankind. Its beauty lies in its logical development of geometry and other branches of mathematics. It has influenced all branches of science but none so much as mathematics and the exact sciences. The *Elements* have been studied 24 centuries in many languages starting, of course, in the original Greek, then in Arabic, Latin, and many modern languages.

I'm creating this version of Euclid's *Elements* for a couple of reasons. The main one is to rekindle an interest in the *Elements*, and the web is a great way to do that. Another reason is to show how Java applets can be used to illustrate geometry. That also helps to bring the *Elements* alive.

The text of all 13 Books is complete, and all of the figures are illustrated using the Geometry Applet, even those in the last three books on solid geometry that are three-dimensional. I still have a lot to write in the guide sections and that will keep me busy for quite a while.

This edition of Euclid's *Elements* uses a Java applet called the Geometry Applet to illustrate the diagrams. If you enable Java on your browser, then you'll be able to dynamically change the diagrams. In order to see how, please read Using the Geometry Applet before moving on to the Table of Contents.

I often hear that geometry is no longer taught well here in the United States high schools. (I also understand that it is not taught at all in some high schools.) This is a major problem because deductive logic is learned almost exclusively in geometry. Without understanding logic, students will have difficulty in their daily lives, and difficulty in college, if they go on to college.

Modern mathematics and science use deductive logic as a primary tool of understanding. In mathematics, especially, nothing is considered to be known until it is proved.

One contributing factor, perhaps the major contributing factor, to the downfall of geometry education in the United States is the way it is presented in text books. If the logic is not presented in text books, it is very hard for the teacher to insert it in class.

A recent text book, Prentice-Hall's *Geometry: tools for a changing world* shows how poor geometry education is today. For details, see my review of the book.

For a broader criticism of mathematics education in the United States, see the site Mathematically Correct.

D.E.Joyce
Clark University

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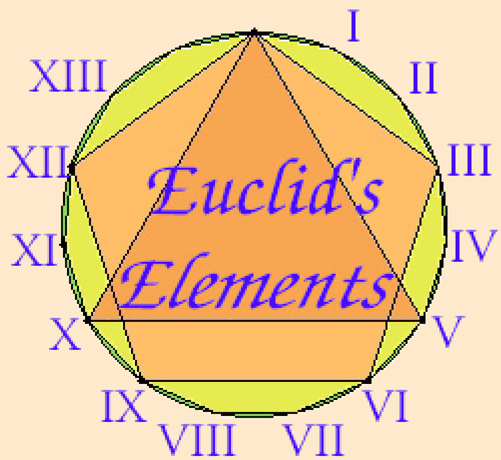

Euclid

Little is known about Euclid's actual life. He was living in Alexandria about 300 B.C.E. based on a passage in Proclus' *Commentary on the First Book of Euclid's Elements*. Indeed, much of what is known or conjectured is based on what Proclus says. After mentioning two students of Plato, Proclus writes

> All those who have written histories bring to this point their account of the development of this science. Not long after these men came Euclid, who brought together the *Elements,* systematizing many of the theorems of Eudoxus, perfecting many of those of Theatetus, and putting in irrefutable demonstrable form propositions that had been rather loosely established by his predecessors. He lived in the time of Ptolemy the First, for Archimedes, who lived after the time of the first Ptolemy, mentions Euclid. It is also reported that Ptolemy once asked Euclid if there was not a shorter road to geometry that through the *Elements,* and Euclid replied that there was no royal road to geometry. He was therefore later than Plato's group but earlier than Eratosthenes and Archimedes, for these two men were contemporaries, as Eratosthenes somewhere says. Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures. (Proclus, ed. Friedlein, p. 68, tr. Morrow)

It is apparent that Proclus had no direct evidence for when Euclid lived, but managed to place him between Plato's students and Archimedes, putting him, very roughly, about 300 B.C.E. Proclus lived about 800 years later, in the fifth century C.E.

There are a few other historical comments about Euclid. The most important being Pappus' (fourth century C.E.) comment that Apollonius (third century B.C.E.) studied "with the students of Euclid at Alexandria."

Thus, we know almost nothing about Euclid's life. But we have more of his writings than any other ancient mathematician. Besides the *Elements*, there are the *Data*, *On Divisions of Figures*, the *Phaenomena*, and the *Optics*. All are included in the *Euclidis opera omnia* of Heiberg and Menge (see below) in Greek and translated into Latin. Other translations are listed below. Euclid also wrote other books which no longer exist but were mentioned by later writers. They include *Surface Loci*, *Porisms*, *Conics*, and the *Pseudaria* (that is, the *Book of Fallacies*).

- Archibald, Raymond Clare (1875-1957).
  Euclid's book on division of figures. Cambridge University Press, Cambridge, 1915.

- Berggren, J. L. *Euclid's Phaenomena: a translation and study of a Hellenistic treatise in spherical astronomy.* Garland, 1996?

- Bretschneider, Karl Anton. *Die Geometrie und die Geometer vor Eukleides; ein historischer Versuch.* Teubner, Leipzig, 1870.

- Busard, H.L.L. *First Latin translation of Euclid's "Elements" commonly ascribe to Adelard of Bath.* Pontifical

Institute.

- Chasles, M. (Michel) (1793-1880)
Les trois livres de porismes d'Euclide, rétablis ... d'aprés la notice ... de Pappus. Mallet-Bachelier, Paris, 1860.

- Frankland, William Barrett. *The first book of Euclid's Elements with a commentary based principally upon that of Proclus Diadochus.* Cambridge Univ Press, New York, 1905.

- Heath, Sir Thomas Little (1861-1940)
The thirteen books of Euclid's Elements translated from the text of Heiberg with introduction and commentary. Three volumes. University Press, Cambridge, 1908. Second edition: University Press, Cambridge, 1925. Reprint: Dover Publ., New York, 1956. Reviewed: *Isis* 10 (1928),60-62.

  The text of Heath's translation of Euclid's *Elements* is also available on-line at the Perseus Project at Tuft's University starting with the first definition of book I.

- Heiberg, J. L. (Johan Ludwig) (1854-1928)
Euclidis opera omnia. 8 vol. & supplement. 1883-1916. Edited by J. L. Heiberg and H. Menge.

- Kayas, G. J. *Les Eléments.* CNRS, 1978.

- Knorr, Wilbur Richard *The evolution of the Euclidean elements. A study of the theory of incommensurable magnitudes and its significance for Greek geometry.* Synthese Historical Library, vol. 15. Reidel, Dordrecht-Boston, 1975. Reviewed: *MR* 57#12003.

- Morrow, Glenn R. *Proclus: A commentary on the first book of Euclid's elements.* Translated by G. R. Morrow. Princeton Univ Press, Princeton, 1970.

- Mueller, Ian. *Philosophy of mathematics and deductive structure in Euclid's Elements.* MIT Press, Cambridge, Mass., 1981.

- Schmidt, Robert. *Euclid's Recipients, commonly called the Data*. Golden Hind Press, 1988.

- Taisbak, C. M. *Colored quadrangles. A guide to the tenth book of Euclid's Elements.* Opuscula Graecolatina, 24. Museum Tusculanum Press, Copenhagen, 1982. Reviewed: *MR* 84i:01022.

- Thomas-Stanford, Charles *Early editions of Euclid's Elements.* Bibliographical Society, London, 1926. Reviewed: *Isis* 10 (1928), 59-60.

- Thomson, William. *Pappus' commentary on Euclid's Elements.* Cambridge, 1930. Review: *Isis* 16 (1931), 132-136.

Table of Contents

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A review of *Geometry: tools for a changing world*

Geometry: tools for a changing world by Laurie E. Bass, Basia Rinesmith Hall, Art Johnson, and Dorothy F. Wood, with contributing author Simone W. Bess, published by Prentice-Hall, 1998.

This textbook is on the list of accepted books for the states of Texas and New Hampshire. It's a glitzy book filled with pictures to keep the attention of the students. That's fine. It's the content that bothers me, in particular, the lack of logical content.

Chapter 1 introduces postulates on page 14 as accepted statements of facts. The four postulates stated there involve points, lines, and planes. Unfortunately, the first two are redundant. Postulate 1-1 says 'through any two points there is exactly one line,' and postulate 1-2 says 'if two lines intersect, then they intersect in exactly one point.' The second one should not be a postulate, but a theorem, since it easily follows from the first. And what better time to introduce logic than at the beginning of the course. No statement should be taken as a postulate when it can be proved, especially when it can be easily proved.

A number of definitions are also given in the first chapter. Later postulates deal with distance on a line, lengths of line segments, and angles.

The book does not properly treat constructions. Constructions can be either postulates or theorems, depending on whether they're assumed or proved. For instance, postulate 1-1 above is actually a construction. On pages 40 through 42 four constructions are given: 1) to cut a line segment equal to a given line segment, 2) to construct an angle equal to a given angle, 3) to construct a perpendicular bisector of a line segment, and 4) to bisect an angle. Later in the book, these constructions are used to prove theorems, yet they are not proved here, nor are they proved later in the book. There is no indication whether they are to be taken as postulates (they should not, since they can be proved), or as theorems. At the very least, it should be stated that they are theorems which will be proved later.

Next, the concept of theorem is given: a statement with a proof, where a proof is a convincing argument that uses deductive reasoning. The theorem "vertical angles are congruent" is given with a proof. It is followed by a two more theorems either supplied with proofs or left as exercises.

Also in chapter 1 there is an introduction to plane coordinate geometry. Unfortunately, there is no connection made with plane synthetic geometry. Here in chapter 1, a distance formula is asserted with neither logical nor intuitive justification. Of course, the justification is the Pythagorean theorem, and that's not discussed until chapter 5. In that chapter there is an exercise to prove the distance formula from the Pythagorean theorem. The Pythagorean theorem itself gets proved in yet a later chapter.

In summary, the constructions should be postponed until they can be justified, and then they should be justified. The same for coordinate geometry.

Chapter 2 begins with theorem that the internal angles of a triangle sum to 180°. The proof is postponed until an exercise in chapter 7, and is based on two postulates on parallels. (Chapter 7 suffers from unnecessary postulates.) Other theorems that follow from the angle sum theorem are given as exercises to prove with outlines.

In a return to coordinate geometry it is implicitly assumed that a linear equation is the equation of a straight line. A proof would depend on the theory of similar triangles in chapter 10. At this point it is suggested that one can conclude that parallel lines have equal slope, and that the product the slopes of perpendicular lines is -1. The only justification given is by experiment. (A proof would require the theory of parallels.) What's worse is what comes next on the page 85:

> 10. If line $k \parallel$ line l and line $r \parallel$ line l, what is the relationship between lines k and r?
>
> 11. If line t is perpendicular to line k and line s is perpendicular to line k, what is the relationship between lines t

and *s*?

Questions 10 and 11 demonstrate the following theorems.

Theorem 2-5. Two lines parallel to a third line are parallel to each other.

Theorem 2-6. In a plane, two lines perpendicular to a third line are parallel to each other.

Questions 10 and 11 do not demonstrate the following theorems. The students are not asked to prove 10 and 11, and even if they were, they would depend on the unproven statements about coordinate geometry slopes of lines. Thus, we have two purported theorems totally without justification. A proper justification can only be given after chapter 10.

(And this occurs in the section in which 'conjecture' is discussed. "Test your conjecture by graphing several equations of lines where the values of *m* are the same." What's the proper conclusion? That theorems may be justified by looking at a few examples?)

In summary, the material in chapter 2 should be postponed until after elementary geometry is developed.

Chapter 3 is about isometries of the plane. The entire chapter is entirely devoid of logic. How are the theorems proved? "The Work Together illustrates the two properties summarized in the theorems below. Theorem 3-1: A composition of reflections in two parallel lines is a translation. ..." Moving a bunch of paper figures around in a "work together" does not constitute a justification of a theorem.

The theorems can be proven once a little actual geometry is presented, but that's not done until the last half of the book. A little honesty is needed here. Why not tell them that the proofs will be postponed until a later chapter? Or that we just don't have time to do the proofs for this chapter. Even better: don't label statements as theorems (like many other unproved statements in the chapter). Like the theorems in chapter 2, those in chapter 3 cannot be proved until after elementary geometry is developed.

In summary, either this chapter should be inserted in the proper place in the course, or else tossed out entirely. Since there's a lot to learn in geometry, it would be best to toss it out.

Chapter 4 begins the study of triangles. The first theorem states that base angles of an isosceles triangle are equal. But the proof doesn't occur until chapter 8. It's getting a bit much. Almost every proof is being omitted or postponed until several chapters later. And when the proofs do come, they depend on constructions that were never justified. For instance, this first theorem is proved in chapter 8 using an angle bisector, but the construction for an angle bisector (chap.1) is given without justification.

The proofs of the next two theorems are postponed until chapter 8. The next four theorems which only involve addition and subtraction of angles appear with their proofs (which depend on the angle sum of a triangle whose proof doesn't occur until chapter 7).

Another theorem in this chapter states that the line joining the midpoints of two sides of a triangle is parallel to the third and half its length. A coordinate proof is given, but as the properties of coordinates are never proved, the proof is unsatisfactory. An actual proof can be given, but not until the basic properties of triangles and parallels are proven.

The text again shows contempt for logic in the section on triangle inequalities. In a silly "work together" students try to form triangles out of various length straws. Rather than try to figure out the relations between the sides of a triangle for themselves, they're led by the nose to "conjecture about the sum of the lengths of two sides of a triangle compared to the length of the third side. Think and discuss. Triangle Inequality Theorem. Your observations from the Work Together suggest the following theorem," and the statement of the theorem follows. Honesty out the window. Can any student armed with this book prove this theorem? Can a teacher? Can the authors? Is it possible to prove it without using the postulates of chapter eight? No. What is this theorem doing here?

The next two theorems depend on that one, and their proofs are either given or left as exercises, but the following four are not proved in any way. Theorem 4-12 says a point on a perpendicular bisector is equidistant from the ends, and the next theorem is its converse. What's the justification? Draw the figure and measure the lines. That's no justification. But it's easy to prove. All you need is a little basic geometry. Here we are on page 219 of a 633-page book (excluding the appendices and index), and, still, the students haven't seen the basics of geometry!

For the 19 theorems in chapter 4, three proofs are postponed until a later chapter; two proofs are given but depend on results in later chapters; two proofs are validly given or left as exercises; one is given with a coordinate proof (and therefore, ultimately unproved); one cannot be proved until a later chapter, and the next two depend on it; four more proofs are not given but would depend on results in later chapters, and the next two given proofs depend on them; and the last two proofs are not given (but would depend on later results).

In summary, chapter 4 is a dismal chapter. It's a prime example of what should never be done in a mathematics book. It shows no respect for logic, and unproved and proved theorems are not distinguished. ("Unproved theorem" is an oxymoron. If it's not proved, then it's not a theorem.) But the material is important and should be treated with respect. It should all be postponed (except the theorems on right angles) until after the basic theory of triangles and parallels is developed.

Chapter 5 is about areas, including the Pythagorean theorem. It begins with postulates about area: the area of a square is the square of the length of its side, congruent figures have equal area, and the area of a region is the sum of the areas of its nonoverlapping parts. A theorem follows: the area of a rectangle is the product of its base and height. There is no proof given, not even a "work together" piecing together squares to make the rectangle. An actual proof is difficult. It would depend either on limiting processes (which are inappropriate at this level), or the construction of a square equal to a rectangle (which could be done much later in the text). It would be just as well to make this theorem a postulate and drop the first postulate about a square.

The next two theorems about areas of parallelograms and triangles come with proofs. Then come the Pythagorean theorem and its converse. In a "work together" students try to piece together triangles and a square to come up with the ancient Chinese proof of the theorem. "The Work Together presents a justification of the well-known right triangle relationship called the Pythagorean Theorem." At this time, however, it's not a complete proof since the theory of parallels has not been done; the existence of any square requires the parallel postulate. Garfield's proof of the Pythagorean theorem is offered in chapter 5, and other proofs later, but all depend on the theory of parallels discussed in chapter 7. The converse is apparently not proved anywhere in the book.

Next 45°-45°-90° and 30°-60°-90° triangles are solved, and areas of trapezoids and regular polygons are found.

The tenth theorem in the chapter claims the circumference of a circle is pi times the diameter. It is apparent (but not explicit) that pi is defined in this theorem as the ratio of circumference of a circle to its diameter. So the content of the theorem is that all circles have the same ratio of circumference to diameter. This theorem is not proven. It would require the basic geometry that won't come for a couple of chapters yet, and it would require a definition of length of a curve and limiting processes. It would be nice if a statement were included that the proof the the theorem is beyond the scope of the course.

Theorem 5-12 states that the area of a circle is pi times the square of the radius. A "work together" has students cutting pie-shaped pieces from a circle and arranging them alternately to form a rough rectangle. That idea is the best justification that can be given without using advanced techniques.

In summary, chapter 5 could be fairly good, but it should be postponed until after the Pythagorean theorem can be proved.

Chapter 6 is on surface areas and volumes of solids. It is strange that surface areas and volumes are treated while the basics of solid geometry are ignored.

There are 11 theorems, the only ones that can be proved without advanced mathematics are the ones on the surface area of a right prism (box) and a regular pyramid. Most of the theorems are given with little or no justification. The area of a cylinder is justified by unrolling it; the area of a cone is unjustified; Cavalieri's principle is stated as a theorem but not proved (it can't be proved without advanced mathematics, better to make it a postulate); the volumes of prisms and cylinders are found using Cavalieri's principle; and the volumes of pyramids and cones are stated without justification. The only argument for the surface area of a sphere involves wrapping yarn around a ball, and that's unlikely to get within 10% of the formula. Finally, a limiting argument is given for the volume of a sphere, which is the best that can be done at this level.

In summary, there is little mathematics in chapter 6. Most of the results require more than what's possible in a first course in geometry. Surface areas and volumes should only be treated after the basics of solid geometry are covered. Alternatively, surface areas and volumes may be left as an application of calculus.

Chapter 7 is on the theory of parallel lines. It begins by postulating that corresponding angles made by a transversal cutting two parallel lines are equal. One postulate is enough, but for some reason two others are also given: the converse to the first postulate, and Euclid's parallel postulate (actually Playfair's postulate). A proliferation of unnecessary postulates is not a good thing. One postulate should be selected, and the others made into theorems.

Four theorems follow, each being proved or left as exercises. Then there are three constructions for parallel and perpendicular lines. Proofs of the constructions are given or left as exercises. But the constructions depend on earlier constructions which still have not been proved, and cannot be proved until the basic theory of triangles is developed in the next chapter.

In summary, postpone the presentation of parallel lines until after chapter 8, and select only one postulate for parallel lines.

Chapter 8 finally begins the basic theory of triangles at page 406, almost two-thirds of the way through the book. This chapter suffers from one of the same problems as the last, namely, too many postulates. The three congruence theorems for triangles, SSS, SAS, and ASA, are *all* taken as postulates. One is enough. The other two should be theorems.

There are only two theorems in this very important chapter. There's a trivial proof of AAS (by now the internal angle sum of a triangle has been demonstrated). Then the Hypotenuse-Leg congruence theorem for right triangles is proved. Some of the theorems of earlier chapters are finally proved, but the original constructions of chapter 1 aren't.

In summary, this should be chapter 1, not chapter 8. Results in all the earlier chapters depend on it. The book is backwards. How did geometry ever become taught in such a backward way? Done right, the material in chapters 8 and 7 and the theorems in the earlier chapters that depend on it, should form the bulk of the course. Much more emphasis should be placed here.

Chapter 9 is on parallelograms and other quadrilaterals. Nearly every theorem is proved or left as an exercise. By this time the students should be doing their own proofs with bare hints or none at all, but several of the exercises have almost complete outlines for proofs. Only one theorem has no proof (base angles of isosceles trapezoids, and one is given by way of coordinates.

Too much is included in this chapter. The sections on rhombuses, trapezoids, and kites are not important and should be omitted.

Chapter 10 is on similarity and similar figures. (Appropriately for this level, the difficulties of proportions are buried in the implicit assumptions of real numbers.) One postulate is taken: triangles with equal angles are similar (meaning proportional sides). The first five theorems are are accompanied by proofs or left as exercises. The proofs are omitted for the theorems which say similar plane figures have areas in duplicate ratios, and similar solid figures have areas in duplicate ratios and volumes in triplicate rations. At least there should be a proof that similar triangles have areas in duplicate ratios; that's easy since the areas of triangles are already known.

Chapter 11 covers right-triangle trigonometry. It's hard to see how there's any time left for trigonometry in a course on geometry, but at least it should be possible to prove the basic facts of trigonometry once the theory of similar triangles is done. The section of angles of elevation and depression need not appear, and the section of vectors omitted. (By the way, who ever calls the sum of two vectors the "resultant" of the two vectors?) The one theorem of the chapter (area of triangle = 1/2 bc sin A) is given for acute triangles.

As the trig functions for obtuse angles aren't covered, and applications of trig to non-right triangles aren't mentioned, it would probably be better to remove this chapter entirely.

Chapter 12 discusses some geometry of the circle, in particular, properties of radii, chords, secants, and tangents.

There are 16 theorems, some with proofs, some left to the students, some proofs omitted. This is one of the better chapters in the book.

Final conclusion. Much more emphasis should be placed on the logical structure of geometry. Postulates should be carefully selected, and clearly distinguished from theorems. Every theorem should be proved, or left as an exercise, or noted as having a proof beyond the scope of the course. Very few theorems, or none at all, should be stated with proofs forthcoming in future chapters.

It should be emphasized that "work togethers" do not substitute for proofs. They can lead to an understanding of the statement of the theorem, but few of them lead to proofs of the theorem. It must be emphasized that examples do not justify a theorem.

David E. Joyce

June, 1998

Guide

Help

Notice

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To the Introduction (which is still where it always was).

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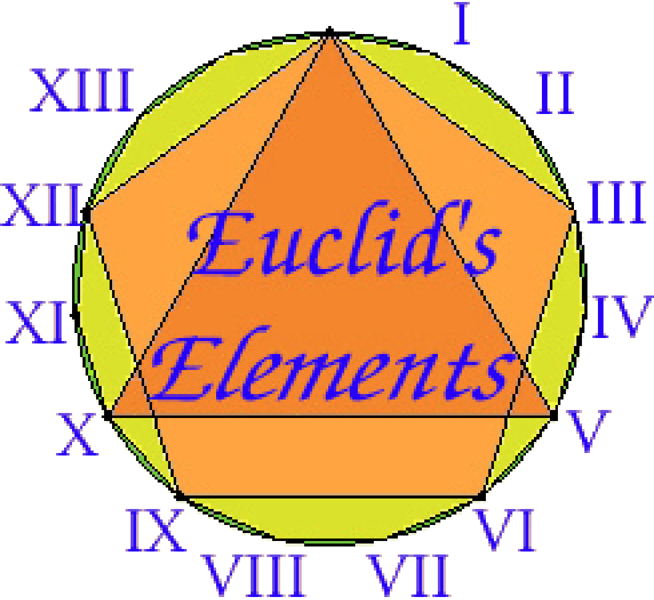
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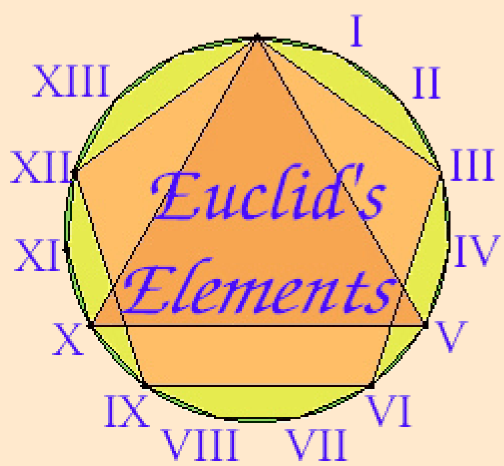

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D.E.Joyce
Clark University

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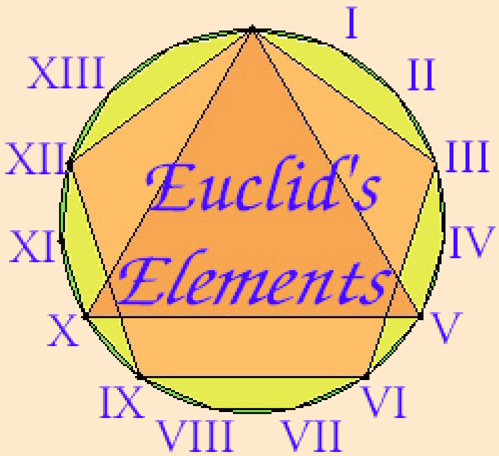

A Quick Trip through the *Elements*

To get an idea of what's in the *Elements*, here are a few highlights in the order that they appear:

Book I on basic plane geometry

- Def. I.23, definition of parallel lines, one of many definitions in Book I
- Post. I.5, the parallel postulate
- Common notions, the axioms for magnitudes
- Prop. I.1, the first proposition which shows how to construct an equilateral triangle
- The congruence theorems for triangles: Prop. I.4, side-angle-side, Prop. I.8, side-side-side, and Prop. I.26, angle-side-angle
- Propositions on isosceles triangles: Prop. I.5, equal angles imply equal sides, and the converse, Prop. I.6, equal sides imply equal angles
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- ~~Prop. XIII.11, when a pentagon, hexagon, and decagon are inscribed in a circle, the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon~~
- ~~Constructions of regular polyhedra XIII.13, XIII.15, XIII.14, XIII.16, and XIII.17~~
- ~~These five are shown to be the only regular solids in proposition XIII.18.~~

These pages are located at http://aleph0.clarku.edu/~djoyce/ java/elements/elements.html.

Using the Geometry Applet

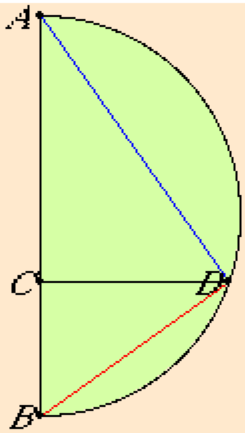
The Geometry Applet is used to illustrate the figures in the *Elements*. With the help of this applet, you can manipulate the figures by dragging points.

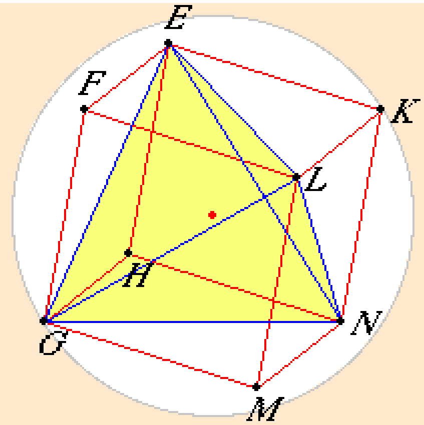
In order to take advantage of this applet, be sure that you have enabled Java on your browser. If you disable Java, or if your browser is not Java-capable, then the illustrations in the elements will still appear, but as plain images.

If you click on a point in the figure, you can usually move it in some way. The free points, usually colored red, can be freely dragged about, and as they move, the rest of the diagram (except the other free points) will adjust appropriately. Sliding points, usually colored orange, can be dragged about like the free points, except their motion is limited to either a straight line, a circle, a plane, or a sphere, depending on the point. Other points can be dragged to translate the entire diagram. But if a pivot point appears, usually colored green, then the diagram will be rotated and scaled around that pivot point.

Take, for example, the figure below showing the relation between a tetrahedron and a cube inscribed in a sphere. The diameter of the sphere has length *AB*, and you can drag the endpoints *A* and *B* to change the size of the sphere. The side of the cube has length *BD*, and the side of the tetrahedron has length *AD*. The cube is drawn with red edges while the tetrahedron is shaded light blue and drawn with blue edges. The center of the sphere is the red dot, and you can drag it to move the sphere around. The point *E* can be dragged anywhere on the surface of the sphere. The point *F* has to be at length *BD* from *E* on the surface of the sphere, and so it drags along a certain circle on the sphere. The rest of the cube and tetrahedron are then determined. (See proposition XIII.15 for background on the mathematics.)

If your browser doesn't deal with java applets, then the illustrations in the *Elements* will still appear but only as plain images and can't be manipulated. Those images were captured from the running Geometry Applet.





Note that you can't drag a point off the diagram, but frequently parts of the diagram will be moved off as you drag other points around. But if you type **r** or the **space key** while the cursor is over the diagram, then the diagram will be reset to its original configuration.

You can also lift the figure off the page into a separate window. When you type **u** or **return** the figure is moved to its own window. Typing **d** or **return** while the cursor is over the original window will return the diagram to the page. Note that you can resize the floating window to make the diagram larger.

Note that most web browsers do not allow printers to print images created by Java applets. If you want to print the images, turn off Java in your browser then the plain images that appear can be printed.

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These pages are located at http://aleph0.clarku.edu/~djoyce/ java/elements/elements.html.

References on the web

Besides this version of Euclid's *Elements* located at http://aleph0.clarku.edu/~djoyce/java/elements/elements.html, there is the text of Heath's translation of Euclid's *Elements* on-line at the Perseus Project at Tuft's University starting with the first definition of book I. Not just Heath's translation, but his commentary as well as the Greek text is available at the Perseus Project.

In 1847 Oliver Byrne designed a wonderful version of the Elements with an imaginative use of color to illustrate geometry. Bill Casselman has made Book I available at the University of British Columbia.

Another place you can get the statements of the propositions, but not their proofs is Richard Carr's pages on Euclid's *Elements*.

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February, 1996.

David E. Joyce