Complex Analysis

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Contents

1 The complex numbers

Proposition 1.1 The complex conjugation has the following properties:

(a) $\overline{z+w} = \overline{z} + \overline{w}$,

(b) $\overline{zw} = \overline{z}\,\overline{w}$,

(c) $\overline{z^{-1}} = \overline{z}^{-1}$, or $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$,

(d) $\overline{\overline{z}} = z$,

(e) $z + \overline{z} = 2\mathrm{Re}(z)$, and $z - \overline{z} = 2i\mathrm{Im}(z)$.

Proposition 1.2 The absolute value satisfies:

(a) $|z| = 0 \ \Leftrightarrow \ z = 0$,

(b) $|zw| = |z||w|$,

(c) $|\overline{z}| = |z|$,

(d) $|z^{-1}| = |z|^{-1}$,

(e) $|z + w| \leq |z| + |w|$, (triangle inequality).

Proposition 1.3 A subset $A \subset \mathbb{C}$ is closed iff for every sequence (a_n) in A that converges in $\mathbb{C}$ the limit $a = \lim_{n\to\infty} a_n$ also belongs to A.
We say that A contains all its limit points.

Proposition 1.4 Let $\mathcal{O}$ denote the system of all open sets in $\mathbb{C}$. Then

(a) $\emptyset \in \mathcal{O}$, $\mathbb{C} \in \mathcal{O}$,

(b) $A, B \in \mathcal{O} \;\Rightarrow\; A \cap B \in \mathcal{O}$,

(c) $A_i \in \mathcal{O}$ for every $i \in I$ implies $\bigcup_{i \in I} A_i \in \mathcal{O}$.

Proposition 1.5 For a subset $K \subset \mathbb{C}$ the following are equivalent:

(a) K is compact.

(b) Every sequence (z_n) in K has a convergent subsequence with limit in K.

Theorem 1.6 Let $S \subset \mathbb{C}$ be compact and $f\colon S \to \mathbb{C}$ be continuous. Then

(a) $f(S)$ is compact, and

(b) there are $z_1, z_2 \in S$ such that for every $z \in S$,

$$|f(z_1)| \ \leq \ |f(z)| \ \leq \ |f(z_2)|.$$

2 Holomorphy

Proposition 2.1 Let $D \subset \mathbb{C}$ be open. If f, g are holomorphic in D, then so are λf for $\lambda \in \mathbb{C}$, $f+g$, and fg. We have

$$(\lambda f)' = \lambda f', \quad (f+g)' = f' + g',$$

$$(fg)' = f'g + fg'.$$

Let f be holomorphic on D and g be holomorphic on E, where $f(D) \subset E$. Then $g \circ f$ is holomorphic on D and

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Finally, if f is holomorphic on D and $f(z) \neq 0$ for every $z \in D$, then $\frac{1}{f}$ is holomorphic on D with

$$(\frac{1}{f})'(z) = -\frac{f'(z)}{f(z)^2}.$$

Theorem 2.2 (Cauchy-Riemann Equations)
Let $f = u + iv$ be complex differentiable at $z = x + iy$. Then the partial derivatives u_x, u_y, v_x, v_y all exist and satisfy

$$u_x = v_y, \quad u_y = -v_x.$$

Proposition 2.3 Suppose f is holomorphic on a disk D.

(a) If $f' = 0$ in D, then f is constant.

(b) If $|f|$ is constant, then f is constant.

3 Power Series

Proposition 3.1 Let (a_n) be a sequence of complex numbers.

(a) Suppose that $\sum a_n$ converges. Then the sequence (a_n) tends to zero. In particular, the sequence (a_n) is bounded.

(b) If $\sum |a_n|$ converges, then $\sum a_n$ converges. In this case we say that $\sum a_n$ converges absolutely.

(c) If the series $\sum b_n$ converges with $b_n \geq 0$ and if there is an $\alpha > 0$ such that $b_n \geq \alpha |a_n|$, then the series $\sum a_n$ converges absolutely.

Proposition 3.2 If a powers series $\sum c_n z^n$ converges for some $z = z_0$, then it converges absolutely for every $z \in \mathbb{C}$ with $|z| < |z_0|$. Consequently, there is an element R of the interval $[0, \infty]$ such that

(a) for every $|z| < R$ the series $\sum c_n z^n$ converges absolutely, and

(b) for every $|z| > R$ the series $\sum c_n z^n$ is divergent.

The number R is called the **radius of convergence** of the power series $\sum c_n z^n$.

For every $0 \leq r < R$ the series converges *uniformly* on the closed disk $\overline{D_r}(0)$.

Lemma 3.3 The power series $\sum_n c_n z^n$ and $\sum_n c_n n z^{n-1}$ have the same radius of convergence.

Theorem 3.4 Let $\sum_n c_n z^n$ have radius of convergence $R > 0$. Define f by

$$f(z) \;=\; \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < R.$$

Then f is holomorphic on the disk $D_R(0)$ and

$$f'(z) \;=\; \sum_{n=0}^{\infty} c_n n z^{n-1}, \qquad |z| < R.$$

Proposition 3.5 Every rational function $\frac{p(z)}{q(z)}$, $p, q \in \mathbb{C}[z]$, can be written as a convergent power series around $z_0 \in \mathbb{C}$ if $q(z_0) \neq 0$.

Lemma 3.6 There are polynomials $g_1, \ldots g_n$ with

$$\frac{1}{\prod_{j=1}^{n}(z-\lambda_j)^{n_j}} \;=\; \sum_{j=1}^{n} \frac{g_j(z)}{(z-\lambda_j)^{n_j}}.$$

Theorem 3.7

(a) e^z is holomorphic in $\mathbb{C}$ and

$$\frac{\partial}{\partial z} e^z \;=\; e^z.$$

(b) For all $z, w \in \mathbb{C}$ we have

$$e^{z+w} \;=\; e^z e^w.$$

(c) $e^z \neq 0$ for every $z \in \mathbb{C}$ and $e^z > 0$ if z is real.

(d) $|e^z| = e^{\mathrm{Re}(z)}$, so in particular $|e^{iy}| = 1$.

Proposition 3.8 The power series

$$\cos z \;=\; \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}, \quad \sin z \;=\; \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

converge for every $z \in \mathbb{C}$. We have

$$\frac{\partial}{\partial z}\cos z \;=\; -\sin z, \quad \frac{\partial}{\partial z}\sin z \;=\; \cos z,$$

as well as

$$e^{iz} \;=\; \cos z + i\sin z,$$
$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}), \quad \sin z \;=\; \frac{1}{2i}(e^{iz}-e^{-iz}).$$

Proposition 3.9 We have

$$e^{z+2\pi i} \;=\; e^z$$

and consequently,

$$\cos(z+2\pi) \;=\; \cos z, \quad \sin(z+2\pi) \;=\; \sin z$$

for every $z \in \mathbb{C}$. Further, $e^{z+\alpha} = e^z$ holds for every $z \in \mathbb{C}$ iff it holds for one $z \in \mathbb{C}$ iff $\alpha \in 2\pi i\mathbb{Z}$.

4 Path Integrals

Theorem 4.1 Let γ be a path and let $\tilde{\gamma}$ be a reparametrization of γ. Then

$$\int_\gamma f(z)dz \;=\; \int_{\tilde{\gamma}} f(z)dz.$$

Theorem 4.2 (Fundamental Theorem of Calculus) Suppose that $\gamma : [a,b] \to D$ is a path and F is holomorphic on D, and that F' is continuous. Then

$$\int_\gamma F'(z)dz \;=\; F(\gamma(b)) - F(\gamma(a)).$$

Proposition 4.3 Let $\gamma : [a, b] \to \mathbb{C}$ be a path and $f : \mathrm{Im}(\gamma) \to \mathbb{C}$ continuous. Then

$$\left| \int_\gamma f(z)dz \right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\, dt.$$

In particular, if $|f(z)| \le M$ for some $M > 0$, then $\left|\int_\gamma f(z)dz\right| \le M\mathrm{length}(\gamma)$.

Theorem 4.4 Let γ be a path and let $f_1, f_2, \ldots$ be continuous on γ^*. Assume that the sequence f_n converges uniformly to f. Then

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz.$$

Proposition 4.5 Let $D \subset \mathbb{C}$ be open. Then D is connected iff it is path connected.

Proposition 4.6 Let $f : D \to \mathbb{C}$ be holomorphic where D is a region. If $f' = 0$, then f is constant.

5 Cauchy's Theorem

Proposition 5.1 Let γ be a path. Let σ be a path with the same image but with reversed orientation. Let f be continuous on γ^*. Then

$$\int_\sigma f(z)dz \;=\; -\int_\gamma f(z)dz.$$

Theorem 5.2 (Cauchy's Theorem for triangles)
Let γ be a triangle and let f be holomorphic on an open set that contains γ and the interior of γ. Then

$$\int_\gamma f(z)dz \;=\; 0.$$

Theorem 5.3 (Fundamental theorem of Calculus II)
Let f be holomorphic on the star shaped region D. Let z_0 be a central point of D. Define

$$F(z) \;=\; \int_{z_0}^{z} f(\zeta)d\zeta,$$

where the integral is the path integral along the line segment $[z_0, z]$. Then F is holomorphic on D and

$$F' \;=\; f.$$

Theorem 5.4 (Cauchy's Theorem for $\star$-shaped D)
Let D be star shaped and let f be holomorphic on D. Then for every closed path γ in D we have

$$\int_\gamma f(z)dz \;=\; 0.$$

6 Homotopy

Theorem 6.1 Let D be a region and f holomorphic on D. If γ and $\tilde{\gamma}$ are homotopic closed paths in D, then

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

Theorem 6.2 (Cauchy's Theorem)
Let D be a simply connected region and f holomorphic on D. Then for every closed path γ in D we have

$$\int_\gamma f(z)dz = 0.$$

Theorem 6.3 Let D be a simply connected region and let f be holomorphic on D. Then f has a primitive, i.e., there is $F \in \mathrm{Hol}(D)$ such that

$$F' = f.$$

Theorem 6.4 Let D be a simply connected region that does not contain zero. Then there is a function $f \in \mathrm{Hol}(D)$ such that $e^f(z) = z$ for each $z \in D$ and

$$\int_{z_0}^{z} \frac{1}{w}\, dw = f(z) - f(z_0), \quad z, z_0 \in D.$$

The function f is uniquely determined up to adding $2\pi i k$ for some $k \in \mathbb{Z}$. Every such function is called a **holomorphic logarithm** for D.

Theorem 6.5 Let D be simply connected and let g be holomorphic on D. [Assume that also the derivative g' is holomorphic on D.] Suppose that g has no zeros in D. Then there exists $f \in \mathrm{Hol}(D)$ such that

$$g = e^f.$$

The function f is uniquely determined up to adding a constant of the form $2\pi ik$ for some $k \in \mathbb{Z}$. Every such function f is called a **holomorphic logarithm** of g.

Proposition 6.6 Let D be a region and $g \in \mathrm{Hol}(D)$. Let $f : D \to \mathbb{C}$ be continuous with $e^f = g$. then f is holomorphic, indeed it is a holomorphic logarithm for g.

Proposition 6.7 (standard branch of the logarithm)
The function

$$\log(z) \;=\; \log(re^{i\theta}) \;=\; \log_{\mathbb{R}}(r) + i\theta,$$

where $r > 0$, $\log_{\mathbb{R}}$ is the real logarithm and $-\pi < \theta < \pi$, is a holomorphic logarithm for $\mathbb{C} \setminus (-\infty, 0]$. The same formula for, say, $0 < \theta < 2\pi$ gives a holomorphic logarithm for $\mathbb{C} \setminus [0, \infty)$.

More generally, for any simply connected D that does not contain zero any holomorphic logarithm is of the form

$$\log_D(z) \;=\; \log_{\mathbb{R}}(|z|) + i\theta(z),$$

where θ is a continuous function on D with $\theta(z) \in \arg(z)$.

Proposition 6.8 For $|z| < 1$ we have

$$\log(1 - z) \;=\; -\sum_{n=1}^{\infty} \frac{z^n}{n},$$

or, for $|w - 1| < 1$ we have

$$\log(w) \;=\; -\sum_{n=1}^{\infty} \frac{(1 - w)^n}{n}.$$

Theorem 6.9 Let $\gamma : [a, b] \to \mathbb{C}$ be a closed path with $0 \notin \gamma^*$. Then $n(\gamma, 0)$ is an integer.

Theorem 6.10 Let D be a region. The following are equivalent:

(a) D is simply connected,

(b) $n(\gamma, z) = 0$ for every $z \notin D$, γ closed path in D,

(c) $\int_\gamma f(z)dz = 0$ for every closed path γ in D and every $f \in \mathrm{Hol}(D)$,

(d) every $f \in \mathrm{Hol}(D)$ has a primitive,

(e) every $f \in \mathrm{Hol}(D)$ without zeros has a holomorphic logarithm.

7 Cauchy's Integral Formula

Theorem 7.1 (Cauchy's integral formula)
Let D be an open disk an let f be holomorphic in a neighbourhood of the closure $\bar{D}$. Then for every $z \in D$ we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw.$$

Theorem 7.2 (Liouville's theorem)
Let f be holomorphic and bounded on $\mathbb{C}$. Then f is constant.

Theorem 7.3 (Fundamental theorem of algebra)
Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Theorem 7.4 Let D be a disk and f holomorphic in a neighbourhood of $\bar{D}$. Let $z \in D$. Then all higher derivatives $f^{(n)}(z)$ exist and satisfy

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^{n+1}}dw.$$

Corollary 7.5 Suppose f is holomorphic in an open set D. Then f has holomorphic derivatives of all orders.

Theorem 7.6 (Morera's Theorem)
Suppose f is continuous on the open set $D \subset \mathbb{C}$ and that $\int_{\triangle} f(w)dw = 0$ for every triangle $\triangle$ which together with its interior lies in D. Then $f \in \mathrm{Hol}(D)$.

Theorem 7.7 Let $a \in \mathbb{C}$. Let f be holomorphic in the disk $D = D_R(a)$ for some $R > 0$. Then there exist $c_n \in \mathbb{C}$ such that for $z \in D$ the function f can be represented by the following convergent power series,

$$f(z) \;=\; \sum_{n=0}^{\infty} c_n (z-a)^n.$$

The constants c_n are given by

$$c_n \;=\; \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(w)}{(w-a)^{n+1}} dw \;=\; \frac{f^{(n)}(a)}{n!},$$

for every $0 < r < R$.

Proposition 7.8 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be complex power series with radii of convergence R_1, R_2. Then the power series

$$h(z) \;=\; \sum_{n=0}^{\infty} c_n z^n, \quad where \quad c_n \;=\; \sum_{k=0}^{n} a_k b_{n-k}$$

has radius of convergence at least $R = \min(R_1, R_2)$ and $h(z) = f(z)g(z)$ for $|z| < R$.

Theorem 7.9 (Identity theorem for power series)
Let $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ be a power series with radius of convergence $R > 0$. Suppose that there is a sequence $z_j \in \mathbb{C}$ with $0 < |z_j| < R$ and $z_j \to z_0$ as $j \to \infty$, as well as $f(z_j) = 0$. Then $c_n = 0$ for every $n \geq 0$.

Corollary 7.10 (Identity theorem for holomorphic functions)
Let D be a region. If two holomorphic functions f, g on D coincide on a set $A \subset D$ that has a limit point in D, then $f = g$.

Theorem 7.11 (Local maximum principle)
Let f be holomorphic on the disk $D = D_R(a)$, $a \in \mathbb{C}$, $R > 0$. If $|f(z)| \leq |f(a)|$ for every $z \in D$, then f is constant.

"A holomorphic function has no proper local maximum."

Theorem 7.12 (Global maximum principle)
Let f be holomorphic on the bounded region D and continuous on $\bar{D}$. Then $|f|$ attains its maximum on the boundary $\partial D = \bar{D} \setminus D$.

8 Singularities

Theorem 8.1 (Laurent expansion)
Let $a \in \mathbb{C}$, $0 < R < S$ and let

$$A = \{z \in \mathbb{C} : R < |z - a| < S\}.$$

Let $f \in \mathrm{Hol}(A)$. For $z \in A$ we have the absolutely convergent expansion (Laurent series):

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(w)}{(w-a)^{n+1}} dw$$

for every $R < r < S$.

Proposition 8.2 Let $a \in \mathbb{C}$, $0 < R < S$ and let

$$A = \{z \in \mathbb{C} : R < |z - a| < S\}.$$

Let $f \in \mathrm{Hol}(A)$ and assume that

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - a)^n.$$

Then $b_n = c_n$ for all n, where c_n is as in Theorem 8.1.

Theorem 8.3

(a) Let $f \in \mathrm{Hol}(D_r(a))$. Then f has a zero of order k at a iff

$$\lim_{z \to a} (z-a)^{-k} f(z) \;=\; c,$$

where $c \neq 0$.

(b) Let $f \in \mathrm{Hol}(D_r'(a))$. Then f has a pole of order k at a iff

$$\lim_{z \to a} (z-a)^{k} f(z) \;=\; d,$$

where $d \neq 0$.

Corollary 8.4 Suppose f is holomorphic in a disk $D_r(a)$. Then f has a zero of order k at a if and only if $\frac{1}{f}$ has a pole of order k at a.

9 The Residue Theorem

Lemma 9.1 Let D be simply connected and bounded. Let $a \in D$ and let f be holomorphic in $D \setminus \{a\}$. Assume that f extends continuously to ∂D. Let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

be the Laurent expansion of f around a. Then

$$\int_{\partial D} f(z)dz = 2\pi i \; c_{-1}.$$

Theorem 9.2 (Residue Theorem)
Let D be simply connected and bounded. Let f be holomorphic on D except for finitely many points $a_1, \ldots, a_n \in D$. Assume that f extends continuously to ∂D. Then

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{z=a_k} f(z) = 2\pi i \sum_{z \in D} \mathrm{res}_z f(z).$$

Proposition 9.3 Let $f(z) = \frac{p(z)}{q(z)}$, where p, q are polynomials. Assume that q has no zero on $\mathbb{R}$ and that $1 + \deg p < \deg q$. Then

$$\int_{-\infty}^{\infty} f(x)dx \;=\; 2\pi i \sum_{z:\mathrm{Im}(z)>0} \mathrm{res}_z f(z).$$

Theorem 9.4 (Counting zeros and poles)
Let D be simply connected and bounded. Let f be holomorphic in a neighbourhood of $\bar{D}$, except for finitely many poles in D. Suppose that f is non-zero on ∂D. Then

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz \;=\; \sum_{z\in D} \mathrm{ord}_z f(z) \;=\; N - P,$$

where N is the number of zeros of f, counted with multiplicity, and P is the number of poles of f, counted with multiplicity.

Theorem 9.5 (Rouché)
Let D be simply connected and bounded. Let f, g be holomorphic in $\bar{D}$ and suppose that $|f(z)| > |g(z)|$ on ∂D. Then f and $f + g$ have the same number of zeros in D, counted with multiplicities.

Lemma 9.6 If f has a simple pole at z_0, then

$$\text{res}_{z_0} f(z) \;=\; \lim_{z \to z_0} (z - z_0) f(z).$$

If f has a pole at z_0 of order $k > 1$. then

$$\text{res}_{z_0} f(z) \;=\; \frac{1}{(k-1)!} g^{(k-1)}(z_0),$$

where $g(z) = (z - z_0)^k f(z)$.

Lemma 9.7 Let f have a simple pole at z_0 of residue c. For $\varepsilon > 0$ let

$$\gamma_\varepsilon(t) \ = \ z_0 + \varepsilon e^{it}, \qquad t \in [t_1, t_2],$$

where $0 \le t_1 < t_2 \le 2\pi$. Then

$$\lim_{\varepsilon \to o} \int_{\gamma_\varepsilon} f(z)dz \ = \ ic(t_2 - t_1).$$

Proposition 9.8

$$\int_0^\infty \frac{\sin x}{x} dx \ = \ \frac{\pi}{2}.$$

10 Construction of functions

Lemma 10.1 If $\prod_j z_j$ exists and is not zero, then $z_n \to 1$.

Proposition 10.2 The product $\prod_j z_j$ converges to a non-zero number $z \in \mathbb{C}$ if and only if the sum $\sum_{j=1}^{\infty} \log z_j$ converges. In that case we have

$$\exp(\left(\sum_{j=1}^{\infty} \log z_j\right) = \prod_j z_j = z.$$

Proposition 10.3 The sum $\sum_n \log z_n$ converges absolutely if and only if the sum $\sum_n (z_n - 1)$ converges absolutely.

Lemma 10.4 If $|z| \leq 1$ and $p \geq 0$ then

$$|E_p(z) - 1| \;\leq\; |z|^{p+1}.$$

Theorem 10.5 Let (a_n) be a sequence of complex numbers such that $|a_n| \to \infty$ as $n \to \infty$ and $a_n \neq 0$ for all n. If p_n is a sequence of integers ≥ 0 such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} \;<\; \infty$$

for every $r > 0$, then

$$f(z) \;=\; \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n} \right)$$

converges and is an entire function (=holomorphic on entire $\mathbb{C}$) with zeros exactly at the points a_n. The order of a zero at a equals the number of times a occurs as one of the a_n.

Corollary 10.6 Let (a_n) be a sequence in $\mathbb{C}$ that tends to infinity. Then there exists an entire function that has zeros exactly at the a_n.

Theorem 10.7 (Weierstraß Factorization Theorem)
Let f be an entire function. Let a_n be the sequence of zeros repeated with multiplicity. Then there is an entire function g and a sequence $p_n \geq 0$ such that

$$f(z) \; = \; z^m e^{g(z)} \prod_n E_{p_n}\left(\frac{z}{a_n}\right).$$

Theorem 10.8 Let D be a region and let (a_j) be a sequence in D with no limit point in D. then there is a holomorphic function f on D whose zeros are precisely the a_j with the multiplicities of the occurrence.

Theorem 10.9 For every principal parts distribution (h_n) on $\mathbb{C}$ there is a meromorphic function f on $\mathbb{C}$ with the given principal parts.

Theorem 10.10 Let $f \in \mathrm{Mer}(\mathbb{C})$ with principal parts (h_n). then there are polynomials p_n such that

$$f \;=\; g + \sum_n (h_n - p_n)$$

for some entire function g.

Theorem 10.11 For every $z \in \mathbb{C}$ we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{2z}{z^2 - n^2} \right)$$

and the sum converges locally uniformly in $\mathbb{C} \setminus \mathbb{Z}$.

Lemma 10.12 If $f \in \mathrm{Hol}(D)$ for a region D and if

$$f(z) = \prod_{n=1}^{\infty} f_n(z),$$

where the product converges locally uniformly, then

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n(z)},$$

and the sum converges locally uniformly in $D \setminus \{\text{zeros of } f\}$.

Theorem 10.13

$$\sin \pi z \ = \ \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

11 Gamma & Zeta

Proposition 11.1 The Gamma function extends to a holomorphic function on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$. At $z = -k$ it has a simple pole of residue $(-1)^k/k!$.

Theorem 11.2 The Γ-function satisfies

$$\Gamma(z) \;=\; \frac{e^{-\gamma z}}{z} \prod_{j=1}^{\infty} (1 + \frac{z}{j})^{-1} e^{z/j}.$$

Theorem 11.3

$$\frac{\Gamma'}{\Gamma}(z) \;=\; -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}.$$

Theorem 11.4 The function $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole of residue 1 at $s = 1$ and is holomorphic elsewhere.

Theorem 11.5 The Riemann zeta function satisfies

$$\zeta(s) \;=\; \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

We have the functional equation

$$\zeta(1-s) \;=\; (2\pi)^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s).$$

$\zeta(s)$ has no zeros in $\mathrm{Re}(s) > 1$. It has zeros at $s = -2, -4, -6, \ldots$ called the trivial zeros. All other zeros lie in $0 \le \mathrm{Re}(s) \le 1$.

12 The upper half plane

Theorem 12.1 Every biholomorphic automorphism of $\mathbb{H}$ is of the form $z \mapsto g.z$ for some $g \in \mathrm{SL}_2(\mathbb{R})$.

Lemma 12.2 (Schwarz's Lemma)
Let $\mathbb{D} = D_1(0)$ and let $f \in \mathrm{Hol}(\mathbb{D})$. Suppose that

(a) $|f(z)| \leq 1$ for $z \in \mathbb{D}$,

(b) $f(0) = 0$.

Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$. Moreover, if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \in \mathbb{D}$, $z \neq 0$, then there is a constant c, $|c| = 1$ such that $f(z) = cz$ for every $z \in \mathbb{D}$.

Proposition 12.3 If $|a| < 1$, then ϕ_a is a biholomorphic map of $\mathbb{D}$ onto itself. It is self-inverse, i.e., $\phi_a \phi_a = \text{Id}$.

Theorem 12.4 Let $f \colon \mathbb{D} \to \mathbb{D}$ be holomorphic and bijective with $f(a) = 0$. Then there is a $c \in \mathbb{C}$ with $|c| = 1$ such that $f = c\phi_a$.

Lemma 12.5 The map $\tau(z) = \frac{z-i}{z+i}$ maps $\mathbb{H}$ biholomorphically to $\mathbb{D}$. Its inverse is $\tau^{-1}(w) = i\frac{w+1}{w-1}$.

Proposition 12.6 F is a *fundamental domain* for the action of Γ on $\mathbb{H}$. This means

(a) For every $z \in \mathbb{H}$ there is $\gamma \in \Gamma$ such that $\gamma z \in F$.

(b) If $z, w \in F$, $z \neq w$ and there is $\gamma \in \Gamma$ with $\gamma z = w$, then $z, w \in \partial F$.

Proposition 12.7 Let $k > 1$. The Eisenstein series $G_k(z)$ is a modular form of weight $2k$. We have $G_k(\infty) = 2\zeta(2k)$, where ζ is the Riemann zeta function.

Theorem 12.8 Let $f \neq 0$ be a modular form of weight $2k$. Then

$$v_\infty(f) + \sum_{z \in \Gamma \backslash \mathbb{H}} \frac{1}{e_z} v_z(f) \;=\; \frac{k}{6}.$$

13 Conformal mappings

Theorem 13.1 Let D be a region and $f\colon D \to \mathbb{C}$ a map. Let $z_0 \in D$. If $f'(z_0)$ exists and $f'(z_0) \neq 0$, then f preserves angles at z_0.

Lemma 13.2 If $f \in \mathrm{Hol}(D)$ and η is defined on $D \times D$ by

$$\eta(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & w \neq z, \\ f'(z) & w = z, \end{cases}$$

then η is continuous.

Theorem 13.3 Let $f \in \mathrm{Hol}(D)$, $z_0 \in D$ and $f'(z_0) \neq 0$. then D contains a neighbourhood V of z_0 such that

(a) f is injective on V,

(b) $W = f(V)$ is open,

(c) if $g \colon W \to V$ is defined by $g(f(z)) = z$, then $g \in \mathrm{Hol}(W)$.

Theorem 13.4 Let D be a region, $f \in \mathrm{Hol}(D)$. non-constant, $z_0 \in D$ and $w_0 = f(z_0)$. Let m be the order of the zero of $f(z) - w_0$ at z_0.
then there exists a neighbourhood V of z_0, $V \subset D$, and $\varphi \in \mathrm{Hol}(D)$, such that

(a) $f(z) = z_0 + \varphi(z)^m$,

(b) φ' has no zero in V and is an invertible mapping of V onto a disk $D_r(0)$.

Theorem 13.5 Let D be a region, $f \in \mathrm{Hol}(D)$, f injective. Then for every $z \in D$ we have $f'(z) \neq 0$ and the inverse of f is holomorphic.

Theorem 13.6 Let $\mathcal{F} \subset \mathrm{Hol}(D)$ and assume that $\mathcal{F}$ is uniformly bounded on every compact subset of D. Then $\mathcal{F}$ is normal.

Theorem 13.7 (Riemann mapping theorem)
Every simply connected region $D \neq \mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D}$.

14 Simple connectedness

Theorem 14.1 Let D be a region. The following are equivalent:

(a) D is simply connected,

(b) $n(\gamma, z) = 0$ for every $z \notin D$, γ closed path in D,

(c) $\hat{\mathbb{C}} \setminus D$ is connected,

(d) For every $f \in \mathrm{Hol}(D)$ there exists a sequence of polynomials p_n that converges to f locally uniformly,

(e) $\int_\gamma f(z)dz = 0$ for every closed path γ in D and every $f \in \mathrm{Hol}(D)$,

(f) every $f \in \mathrm{Hol}(D)$ has a primitive,

(g) every $f \in \mathrm{Hol}(D)$ without zeros has a holomorphic logarithm,

(h) every $f \in \mathrm{Hol}(D)$ without zeros has a holomorphic square root,

(i) either $D = \mathbb{C}$ or there is a biholomorphic map $f\colon \mathbb{D} \to D$,

(j) D is homeomorphic to the unit disk $\mathbb{D}$.