CALCULUS

OF

FINITE DIFFERENCES

BY

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INTRODUCTION

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INTRODUCTION

There is more than mere coincidence in the fact that the recent rapid growth in the theory and application of mathematical statistics has been accompanied by a revival in interest in the Calculus of Finite Differences. The reason for this phenomena is clear: the student of mathematical statistics must now regard the finite calculus as just as important a tool and prerequisite as the infinitesimal calculus.

To my mind, the progress that has been made to date in the development of the finite calculus has been marked and stimulated by the appearence of four outstanding texts.

The first of these was the treatise by George Boole that appeared in 1860. I do not mean by this to underestimate the valuable contributions of earlier writers on this subject or to overlook the elaborate work of Lacroix.[1] I merely wish to state that Boole was the first to present' this subject in a form best suited to the needs of student and teacher.

The second milestone was the remarkable work of Nörlund that appeared in 1924. This book presented the first rigorous treatment of the subject, and was written from the point of view of the mathematician rather than the statistician. It was most oportune.

Steffensen's *Interpolation,* the third of the four texts to which I have referred, presents an excellent treatment of one section of the Calculus of Finite Differences, namely interpolation and summation formulae, and merits the commendation of both mathematicians and statisticians.

I do not hesitate to predict that the fourth of the texts that

[1] Volume 3 of *Traité du Calcul Diffdrentiel et du Calcul Intégral,* entitled *Traité des différences et des séries.* S. F, Lacroix, 1819.

I have in mind, Professor Jordan's Calculus of Finite Differences, is destined to remain the classic treatment of this subject — especially for statisticians — for many years to come. Although an inspection of the table of contents reveals a coverage so extensive that the work of more than 600 pages might lead one at first to regard this book as an encyclopedia on the subject, yet a reading of any chapter of the text will impress the reader as a friendly lecture revealing an ununsual appreciation of both rigor and the computing technique so important to the statistician.

The author has made a most thorough study of the literature that has appeared during the last two centuries on the calculus of finite differences and has not hesitated in resurrecting forgotten journal contributions and giving them the emphasis that his long experience indicates they deserve in this day of mathematical statistics.

In a word, Professor Jordan's work is a most readable and detailed record of lectures on the Calculus of Finite Differences which will certainly appeal tremendously to the statistician and which could have been written only by one possessing a deep appreciation of mathematical statistics.

Harry C. Carver.

THE AUTHOR'S PREFACE

This book, a result of nineteen years' lectures on the Calculus of Finite Differences, Probability, and Mathematical Statistics in the Budapest University of Technical and Economical Sciences, and based on the venerable works of ***Stirling, Euler*** and ***Boole,*** has been written especially for practical use, with the object of shortening and facilitating the **labours** of the Computer. With this aim in view, some of the old and neglected, though useful, methods have been utilized and further developed: as for instance ***Stirling's*** methods of summation, ***Boole's*** symbolical methods, and ***Laplace's*** method of Generating Functions, which last is especially helpful for the resolution of equations of partial differences,

The great practical value of ***Newton's*** formula is shown; this is in general little appreciated by the Computer and the Statistician, who as a rule develop their functions in power series, although they are primarily concerned with the differences and sums of their functions, which in this case are hard to compute, but easy with the use of ***Newton's*** formula. Even for interpolation **it is** more advisable to employ ***Newton's*** expansion than to expand the function into a power series.

The importance of ***Stirling's*** numbers in Mathematical Calculus has not yet been fully **recognised,** and they are seldom used. This is especially due to the fact that different authors have reintroduced them under different definitions and notations, often not knowing, or not mentioning, that they deal with the same numbers. Since ***Stirling's*** numbers are as important as ***Bernoulli's,*** or even more so, they should occupy a central position in the Calculus of Finite Differences, The demonstration of a great number of formulae is considerably shortened by using these numbers, and many new formulae are obtained by their aid; for instance, those which express differences by derivatives or vice versâ; formulae for the operations $x\frac{d}{dx}$ and $x\Delta$, and many others; formulae for the inversion of certain sums, for changing the length of the interval of the differences, for summation of reciprocal powers, etc.

In this book the functions especially useful in the Calculus of Finite Differences, such as the Factorial, the Binomial Coefficient, the Digamma and Trigamma Functions, and the ***Bernoulli*** and ***Euler*** Polynomials are fully treated. Moreover two species of polynomials, even more useful, analogous to those of ***Bernoulli*** and ***Euler***, have been introduced; these are the ***Bernoulli*** polynomiali of the second kind (§ 89), and the ***Boole*** polynomials (§ 113).

Some new methods which permit great simplifications, will also be found, such as the method of interpolation without printed differences (§ 133), which reduces the cost and size of tables to a minimum. Though this formula has been especially deduced for Computers working with a calculating machine, it demands no more work of computation, even without this aid, than ***Everett's*** formula, which involves the use of the even differences. Of course, if a table contains both the odd and the even differences, then interpolation by ***Newton's*** formula is the shortest way, But there are very few tables which contain the first three differences, and hardly any with more than three, which would make the table too large and too expensive; moreover, the advantage of having the differences is not very great, provided one works with a machine, as has been shown, even in the case of linear interpolation (§ 133). So the printing of the differences may be considered as superfluous.

The construction of Tables has been thoroughly treated (§§ 126 and 133). This was by no means superfluous, since nearly ***all the existing tables*** are much too large in comparison with the precision they afford. A table ought to be constructed from the point of view of the interpolation formula which is to be employed. Indeed, if the degree of the interpolation, and the number of the decimals in the table, are given, then this determines the range or the interval of the table. But generally, as is shown, the range chosen is ten or twenty times too large, or the interval as much too small; and the table is therefore unnecessarily bulky. If the table were reduced to the proper dimensions, it would be easy and very useful to add another table for the inverse function.

A method of approximation by aid of orthogonal polynomials, which greatly simplifies the operations, is given. Indeed, the orthogonal polynomials are used only temporarily, and the result obtained is expressed by Newton's formula (§ 142), so that no tables are necessary for giving the numerical values of the orthogonal polynomials.

In § 143 an exceedingly simple method of graduation according to the principle of least squares is given, in which it is only necessary to compute certain *"orthogonal"* moments corresponding to the data.

In the Chapter dealing with the numerical resolution of equations, stress has been laid on the rule of *False Posifion*, which, with the slight modification given (§§ 127 and 149, and Example 1, in § 134), enables us to attain the required precision in a very few steps, so that it is preferable, for the Computer, to every other method,

The Chapters on the Equations of Differences give only those methods which really lead to practical results. The Equations of Partial Differences have been especially considered. The method shown for the determination of the necessary initial conditions will be found very useful (§ 181). The very seldom used, but advantageous, way of solving Equations of Partial Differences by *Laplace's* method of Generating -Functions has been dealt with and somewhat further developed (§ 183), and examples given. The neglected method of *Fourier, Lagrange* and *Ellis* (§ *184)* has been treated in the same way.

Some formulae of Mathematical Analysis are briefly mentioned, with the object of giving as far as possible everything necessary for the Computer.

Unfamiliar notations, which make the reading of mathematical texts difficult and disagreeable, have been as far as possible avoided. The principal notations used are given on pp. xix-xxii. To obviate another difficulty of reading the works on Finite Differences, in which nearly every author uses other definitions and notations, these are given, for all the principal authors, in the respective paragraphs in the Bibliographical Notes.

Though this book has been written as has been said above, especially for the use of the computer, nevertheless it may be considered as an introductory volume to Mathematical Statistics and to the Calculus of Probability.

I owe a debt of gratitude to my friend and colleague Mr. ***A. Szücs,*** Professor of Mathematics in the University of Budapest, who read the proofs and made many valuable suggestions; moreover to Mr. ***Philip Redmond,*** who kindly revised the text from the point of view of English.

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NOTATIONS AND DEFINITIONS

[illegible]

CHAPTER I.

ON OPERATIONS.

§ 1. Historical and Bibliographical Notes. The most important conception of Mathematical Analysis is that of the function. If to a given value of x a certain value of y correspond, we say that y is a function of the independent variable x.

Two sorts of functions are to be distinguished. First, functions in which the variable x may take every possible value in a given interval; that is, the variable is continuous. These functions belong to the domain of ***Infinitesimal*** Calculus. Secondly, functions in which the variable x takes only the given values $x_0, x_1, x_2, \ldots x_n$; then the variable is discontinuous. To such functions the methods of Infinitesimal Calculus are not applicable. The ***Calculus of Finite Differences*** deals especially with such functions, but it may be applied to both categories.

The origin of this Calculus may be ascribed to ***Brook Taylor's*** Methodus Incrementorum (London , 1717), but the real founder of the theory was ***Jacob Stirling,*** who in his Methodus Differentialis (London , 1730) solved very advanced questions, and gave useful methods, introducing the famous ***Stirling*** numbers; these, though hitherto neglected, will form the backbone of the Calculus of Finite Differences.

The first treatise on this Calculus is contained in ***Leonhardo Eulero,*** Institutiones Calculi Differentialis (Academiae Imperialis Scientiarum Petropolitanae, 1755. See also Opera Omnia, Series I. Vol. X. 1913) in which he was the first to introduce the symbol Δ for the differences, which is universally used now. From the early works on this subject the interesting article "Difference" in the Encyclopédie Mtthodique (Paris, 1784), written by l'Abbé *Charles Bossut,* should be mentioned, also, *F. S. Lacroix's* "Traité des differences et series" Paris, 1800.

§ 2. Definition of **the differences,** A function *f(x)* is given for $x = x_0, x_1, x_2, \ldots, x_n$. In the general case these values are not equidistant. To deal with such functions, the *"Divided Differences"* have been introduced, We shall see them later (§ 9). *Newton's* general interpolation formula is based on these differences. The Calculus, when working with divided differences, is always complicated, The real advantages of the theory of Finite Differences are shown only if the values of the variable x are equidistant; that is if

$$x_{i+1} - x_i = h$$

where h is independent of i.

In this case, the first difference of *f(x)* will be defined by the increment of *f(x)* corresponding to a given increment h of the variable x. Therefore, denoting the first difference by Δ we shall have

$$\Delta f(x) = f(x+h) - f(x).$$

The symbol Δ is not complete; in fact the independent variable and its increment should also be indicated. For instance thus:

$$\underset{x,h}{\Delta}$$

This must be done every time if there is any danger of a misunderstanding, and therefore Δ must be considered as an abbreviation of the symbol above.

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[1] **The most important treatises on the Calculus of Finite Differences are the following:**

George *Boole,* A treatise on the Calculus of Finite Differences, Cambridge, 1860.
***A. A. Markoff,* Differenzenrechnung, Leipzig, 18%.**
***D. Selivanoff,* Lehrbuch der Differenzenrechnung, Leipzig, 1904.**
E. T. *Whittaker* and G. Robinson, Calculus of Observations, London, 1924.
***N. E. Nörlund,* Differenzenrechnung, Berlin, 1924.**
***J. F. Sfeffensen,* Interpolation, London, 1927.**
***J. B. Scarborough,* Numerical Mathematical Analysis, Baltimore, 1930.**
G. *Kowalewski,* Interpolation und genäherte Quadratur, Leipzig. 1932.
***L. M. Milne-Thomson.* The Calculus of Finite Differences, London, 1933.**

Often the independent variable is obvious, but not the increment; then we shall write $\underset{h}{\Delta}$, omitting h only in the case of $h = 1$.

If the increment of x is equal to one, then the formulae of the Calculus are much simplified. Since it is always possible to introduce into the function $f(x)$ a new variable whose increment is equal to one, we shall generally do so. For instance if $y=f(x)$ and the increment of x is h, then we put $x=a+h\xi$; from this it follows that $\Delta\xi=1$; that is, ξ will increase by one if x increases by h. Therefore, starting from $f(x)$ we find

$$f(x) = f(a+\xi h) = F(\xi)$$

and operate on $F(\xi)$; putting finally into the results obtained $(x-a)/h$ instead of ξ.

We shall call second difference of $f(x)$ the difference of its first difference. Denoting it by $\underset{h}{\Delta}^2 f(x)$ we have

$$\underset{h}{\Delta}^2 f(x) = \underset{h}{\Delta}[\underset{h}{\Delta} f(x)] = \underset{h}{\Delta} f(x+h) - \underset{h}{\Delta} f(x) =$$
$$= f(x+2h) - 2f(x+h) + f(x).$$

In the same manner the n-th difference of $f(x)$ will be defined by

$$\underset{h}{\Delta}^n f(x) = \underset{h}{\Delta}[\underset{h}{\Delta}^{n-1} f(x)] = \underset{h}{\Delta}^{n-1} f(x+h) - \underset{h}{\Delta}^{n-1} f(x).$$

Remark. In Infinitesimal Calculus the first derivative of a function $f(x)$ generally denoted by $\frac{df(x)}{dx}$, or more briefly by $\mathrm{D}f(x)$ (if there can be no misunderstanding), is given by

$$\mathrm{D}f(x) = \lim_{h=0} \frac{f(x+h) - f(x)}{h} = \lim_{h=0} \frac{\underset{h}{\Delta} f(x)}{h}.$$

Moreover it is shown that the n-th derivative of $f(x)$ is

$$\mathrm{D}^n f(x) = \lim_{h=0} \frac{\underset{h}{\Delta}^n f(x)}{h^n}.$$

If a function of continuous variable is given, we may determine the derivatives and the integral of the function by using the methods of Infinitesimal Calculus. From the point of view of the Calculus of Finite Differences these functions are treated exactly in the same manner as those of a discontinuous variable; wc may determine the differences, and the sum of the function; but the increment h and the beginning of the intervals must be given, For instance, log x may be given by a table from $x=1000$ to $x=10000$, where $h=1$.

Generally we write the values of the function in the first column of a table, the first differences in the second column, the second differences in the third, and so on.

If we begin the first column with $f(a)$, then we shall write the first difference $\underset{h}{\Delta} f$ (a) in the line between $f(a)$ and $f\,(a+h)$; the second difference $\Delta^2 f(a)$ will be put into the row between $\underset{h}{\Delta} f(a)$ and $\underset{h}{\Delta} f(a+h)$ and so on. We have.

$f(a)$					
	$\underset{h}{\Delta} f(a)$				
$f(a+h)$		$\underset{h}{\Delta}^2 f(a)$			
	$\underset{h}{\Delta} f(a+h)$		$\underset{h}{\Delta}^3 f(a)$		
$f(a+2h)$		$\underset{h}{\Delta}^2 f(a+h)$		$\underset{h}{\Delta}^4 f(a)$	
	$\underset{h}{\Delta} f(a+2h)$		$\underset{h}{\Delta}^3 f(a+h)$		$\underset{h}{\Delta}^5 f(a)$
$f(a+3h)$		$\underset{h}{\Delta}^2 f(a+2h)$		$\underset{h}{\Delta}^4 f(a+h)$	
	$\underset{h}{\Delta} f(a+3h)$		$\underset{h}{\Delta}^3 f(a+2h)$		
$f(a+4h)$		$\underset{h}{\Delta}^2 f(a+3h)$			
	$\underset{h}{\Delta} f(a+4h)$				
$f(a+5h)$					

It should be noted that proceeding in this way, the expressions with the same argument are put in a descending line; and that the arguments in each horizontal line are decreasing. The reason is that the notation used above for the differences is nof symmetric with respect to the argument.

Differences of functions with negative arguments. If we have

$$\underset{h}{\Delta} f(x) = f(x+h) - f(x) = \varphi(x)$$

then according to our definition

$$Qf(-x) = f(-x-h) - f(-x).$$

From this it follows that

$$\underset{h}{\Delta} f(-x) = -\varphi(-x-h)$$

that is, the argument $-x$ of the function is diminished by h.

In the same manner we should obtain

$$\underset{h}{\Delta}^m f(-x) = (-1)^m \varphi'(-x-mh); \quad \varphi'(x) = \underset{h}{\Delta}^m f(x),$$

This formula will be very useful in the following.

Difference of a sum. It is easy to show that

$$\Delta[f_1(x) + f_2(x) + \ldots + f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \ldots + \Delta f_n(x)$$

moreover if C is a constant that

$$\Delta Cf(x) = C\Delta f(x).$$

According to these rules the difference of a polynomial is

$$\Delta[a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n] = a_1 \Delta x + a_2 \Delta x^2 + \ldots + a_n \Delta x^n.$$

§ 3, Operation of displacement.[2] An important operation

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[2] *Boole* denoted this operation by *D;* but since *D* is now universally used as a symbol for derivation, it had to be changed. *De la Vallée Poussin* in his Cours d'Analyse Infinitésimale [1922, Tome II, p. 329] denoted this operation by Pseudodelta ∇. We have adopted here **E** since this is generally used in England. So for instance in

W. F. Sheppard, Encyclopaedia Britannica, the 11-th edition, 1910, in voc. Differences (Calculus of., .) Vol. VIII p. 223.

E. T. Whittaker and G. *Robinson,* loc. cit. 1, p. 4.

J. E. Steffensen, loc. cit. 1. p. 4.

L. M. Milne-Thomson. loc. cit. 1, p. 31.

This operation has already been considered by *L. F. A. Arbogast* [Du calcul des derivations, Strasbourg, 1800]; he called it an operation of "état varié". *F. Casorati* proposed for this operation the symbol Θ which was also used by *Pincherle* [loc. cit. 4].

C. *Jordan,* in his Cours d'Analyse [Second edition tom. I, p. 115] has also introduced this operation and deduced several formulae by aid of symbolical methods. His notation was

$$f_n = f(x + nh).$$

was introduced into the Calculus of Finite Differences by *Boole* [loc. cit. 1. p. 16], the operation of displacement. This consists, f(x) being given, in increasing the variable x by h. Denoting the operation by E we have

$$\mathsf{E}f(x) = f(x+h).$$

This symbol must also be considered as an abbreviation of $\underset{x,h}{\mathsf{E}}$.

The operation E^2 will be defined by

$$\mathsf{E}^2 f(x) = \mathsf{E}\,[\,\mathsf{E}f(x)] = Ef(x+h) = f\,(x+2h)$$

and in the same way

$$\mathsf{E}^n f(x) = E[\,\mathsf{E}^{n-1} f(x)\,] = f(x+nh).$$

It is easy to extend this operation to negative indices of E so that we have

$$\frac{1}{\mathsf{E}} f(x) = \mathsf{E}^{-1} f(x) = f(x-h)$$

$$\frac{1}{\mathsf{E}^n} f(x) = \mathsf{E}^{-n} f(x) = f(x-nh).$$

§ 4. **Operation of the Mean.** The operation of the mean introduced by *Sheppard* (loc. cit. 2) corresponds to the system of Central Differences which we shall see later. We shall denote the operation corresponding to the system of differences considered in § 2, by M.[3] Its definition is

7 $$\mathsf{M}f(x) = \tfrac{1}{2}[f(x) + f(x+h)\,].$$

The operation M will be defined in the same manner by

$$\mathsf{M}^n f(x) = \mathsf{M}[\mathsf{M}^{n-1} f(x)\,] = \tfrac{1}{2}\,[\mathsf{M}^{n-1} f(x) + \mathsf{M}^{n-1} f(x+h)\,].$$

Of course the notation M is an abbreviation of $\underset{x,h}{\mathsf{M}}$.

Returning to our table of § 2, we may write into the first column between $f(a)$ and $f(a+h)$ the number $\mathsf{M}f\,(a)$; between

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[3] *Sheppard* denoted the central difference by δ and the corresponding central mean by μ. Since Δ corresponds to δ it is logical that M should correspond to μ. *Thiele* introduced for the central mean the symbol $\square$, which has also been adopted by *Steffensen* [loc. cit. 1, p, 10]. *Nörlund* [loc. cit. 1. p. 31] denoted our mean by the symbol Pseudodelta ∇. This also has been adopted by *Milne-Thomson* [loc. cit. 1, p. 31]. We have seen that other authors have already used the symbol ∇ for the operation of displacement, therefore it is not practical to use it for another operation.

$f(a+h$ and $f(a+2h)$ the number $\mathsf{M}f(a+h)$ and so on. In the second column we put $\mathsf{M}\Delta f$ (a) between $\Delta f(a)$ and $\Delta f\,(a+h)$; continuing in this manner we shall obtain for instance the following lines of our table

. . . . . .

$f(a+2h)$	$\mathsf{M}\Delta f(a+h)$	$\Delta^2 f(a+h)$	$\mathsf{M}\Delta^3 f(a)$	$\Delta^4 f(a)$
$\mathsf{M}f\,(a+2h)$	$\Delta f(a+2h)$	$\mathsf{M}\Delta^2 f(a+h)$	$\Delta^3 f(a+h)$	$\mathsf{M}\Delta^4 f(a)$
$f(a+3h)$	$\mathsf{M}\Delta f\,(a+2h)$	$\Delta^2 f\,(a+2h)$	$\mathsf{M}\Delta^3 f(a+h)$	$\Delta^4 f(a+h)$
$\mathsf{M}f\,(a+3h)$	$\Delta f(a+3h)$	$\mathsf{M}\Delta^2 f\,(a+2h)$	$\Delta^3 f\,(a+2h)$	$\mathsf{M}\Delta^4 f(a+h)$

§ 5. **Symbolical Calculus.**[4] It is easy to show that the considered operations represented by the symbols Δ, D, M, and E are distributive; for instance that we have

$$\Delta^n[f(x) + \varphi(x) + \psi(x)] = \Delta^n f(x) + \Delta^n\varphi(x) + \Delta^n\psi(x).$$

Moreover they are commutative, for instance

$$\mathsf{E}^n\mathsf{E}^m\,f(x) = \mathsf{E}^m\mathsf{E}^n f(x) = \mathsf{E}^{n+m} f(x)$$

and

$$\Delta^m\ \mathsf{E}^n f(x) = \mathsf{E}^n\,\Delta^m f(x).$$

The constant k may be considered as the symbol of multiplication by k; this symbol will obviously share the properties above mentioned. For instance we have:

$$\Delta k\ f(x) = k\ \Delta\ f(x)\,.$$

Therefore we conclude that with respect to addition, subtraction and multiplication these symbols behave as if they were algebraic quantities. A polynomial formed of them represents an operation, Several such polynomials may be united by addition, subtraction, or multiplication, For instance

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[4] Symbolical methods were first applied by *Boole* in his Treatise on Differential Equations, London, 1859 (third edition 1872, pp. 381-461 and in loc. cit. 1. p. 16).

On the Calculus of Symbols there is a remarkable chapter in *Steffensen*'s loc. cit. 1. p. 178—202., and in S. *Pincherle,* Equations et Operations Fonctionnelles, Encyclopédie des Sciences Mathematiques (French edition) 1912, Tome II, Vol. 5, pp. 1-81.

$$(a_0+a_1\Delta+a_2\Delta^2+\dots+a_n\Delta^n)(b_0+b_1\Delta+b_2\Delta^2+\dots+b_m\Delta^m) =$$
$$= \sum_{\nu=0}^{n+1}\sum_{\mu=0}^{m+1} a_\nu b_\mu \Delta^{\nu+\mu}$$

Remark. The operation E^{-n} behaves exactly like E^m; we have

$$E^{-n}E^m = E^m E^{-n} = E^{m-n}$$

Therefore in the case of the displacement operation E, division, or multiplication with negative powers of E will be permitted, exactly as with positive powers. Moreover

$$E[f(x)\varphi(x)\psi(x)] = f(x+h)\varphi(x+h)\psi(x+h) = Ef(x)\,E\varphi(x)\,E\psi(x)$$

These are not true for the other symbols introduced.

§ 6. Symbolical Methods, Starting from the definitions of the operations it is easy to see that their Symbols are connected, for instance by the following relations

$$E = 1+\Delta\,;\quad M = \tfrac12(1+E)\,;\quad M = 1+\tfrac12\Delta\,;$$
$$M = E-\tfrac12\Delta\,.$$

To prove the first let us write

$$(1+\underset{h}{\Delta})f(x) = f(x)+\underset{h}{\Delta}f(x) = f(x)+f(x+h)-f(x) =$$
$$= f(x+h) = \underset{h}{E}\,f(x)$$

the others are shown in the same way.

Differences expressed by successive values of the function. From the first relation we deduce

$$\Delta^m = (E-1)^m = \sum_{\nu=0}^{m+1}(-1)^\nu\binom{m}{\nu}E^{m-\nu}$$

Knowing the successive values of the function, this formula gives its m-th difference. We may write it as follows

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[5] Our notation of the sums is somewhat different from the ordinary notation. We denote

$$f(0)+f(1)+f(2)+\dots+f(n-1) = \sum_{x=0}^{n} f(x)$$

that is, the value $f(0)$ corresponding to the lower limit is included in the sum. but not the value $f(n)$ corresponding to the upper limit. The reason for tbis notation will be given in the paragraph dealing with sums.

$$\Delta^m f(x) = f(x+mh) - \binom{m}{1} f(x+mh-h) + \binom{m}{2} f(x+mh-2h) + \ldots + (-1)^m \binom{m}{m} f(x)$$

Differences expressed by means. Starting from the third formula we have

$$\Delta^m = [2(\mathbf{M}-1)]^m = 2^m \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m}{\nu} \mathbf{M}^{m-\nu}$$

Performed on *f(x),* this operation gives

$$\Delta^m f(x) = 2^m \left[\mathbf{M}^m f(x) - \binom{m}{1} \mathbf{M}^{m-1} f(x) + \binom{m}{2} \mathbf{M}^{m-2} f(x) - \ldots + (-1)^m \binom{m}{m} f(x)\right].$$

The differences may be expressed by means; moreover, 'using the fourth formula

$$\Delta^m = [2(\mathbf{E}-\mathbf{M})]^m = 2^m \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m}{\nu} \mathbf{E}^{m-\nu} \mathbf{M}^\nu$$

this will give

$$\Delta^m f(x) = 2^m \left[f(x+mh) - \binom{m}{1} \mathbf{M} f(x+mh-h) + \binom{m}{2} \mathbf{M}^2 f(x+mh-2h) + \ldots + (-1)^m \binom{m}{m} \mathbf{M}^m f(x)\right].$$

This formula becomes especially useful if *f(x)* is such a function that we have $\mathbf{M}^\nu f(x) = F(x+\nu)$; since then we have

$$\mathbf{E}^{m-\nu} \mathbf{M}^\nu f(x) = F(x+m)$$

that is, the *argument* of $F(x)$ is independent of ν. This is often important.

Example. Let $f(x) = \binom{n}{x}$. As we shall see later, we have

$$\mathbf{M}^\nu \binom{n}{x} = \frac{1}{2^\nu} \binom{n+\nu}{x+\nu}$$

and therefore

$$\Delta^m \binom{n}{x} = \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m}{\nu} 2^{m-\nu} \binom{n+\nu}{x+m}$$

The function expressed by differences. From the first relation it follows that

$$\mathbf{E}^m = (1+\Delta)^m = \sum_{\nu=0}^{m+1} \binom{m}{\nu} \Delta^\nu$$

Executed on f(x) this gives

$$f(x+mh) = f(x) + \binom{m}{1} \Delta f(x) + \binom{m}{2} \Delta^2 f(x) + \ldots +$$
$$+ \binom{m}{m} \Delta^m f(x) \, .$$

The function expressed by means. From the second relation we deduce that

$$\mathbf{E}^m = (2\mathbf{M}-1)^m = \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m}{\nu} (2\mathbf{M})^{m-\nu}$$

The operation executed on *f(x)* gives

$$f(x+mh) = 2^m \mathbf{M}^m f(x) - \binom{m}{1} 2^{m-1} \mathbf{M}^{m-1} f(x) + \binom{m}{2} 2^{m-2} \mathbf{M}^{m-2} f(x) -$$
$$+ \ldots + (-1)^m \binom{m}{m} f(x) \, .$$

Means expressed by successive values of the function. The second relation gives

$$\mathbf{M}^m = [\tfrac{1}{2}(1+\mathbf{E})]^m = \frac{1}{2^m} \sum_{\nu=0}^{m+1} \binom{m}{\nu} \mathbf{E}^\nu$$

and therefore we have

$$\mathbf{M}^m f(x) = \frac{1}{2^m} \Big[f(x) + \binom{m}{1} f(x+h) + \binom{m}{2} f(x+2h) + \ldots +$$
$$+ \binom{m}{m} f(x+mh) \Big] \, .$$

Means expressed by differences. From the third relation we get

$$\mathbf{M}^m = (1+\tfrac{1}{2}\Delta)^m = \sum_{\nu=0}^{m+1} \binom{m}{\nu} \frac{1}{2^\nu} \Delta^\nu$$

Hence

$$\mathsf{M}^m f(x) = f(x) + \binom{m}{1} \frac{\underset{h}{\Delta} f(x)}{2} + \binom{m}{2} \frac{\underset{h}{\Delta^2} f(x)}{2^2} + \dots + \binom{m}{m} \frac{\underset{h}{\Delta^m} f(x)}{2^m}$$

Expansion of a function by **symbolical** *methods.* We have

$$\mathsf{E}^x = (1+\Delta)^x = \sum_{\nu=0}^{x+1} \binom{x}{\nu} \Delta^\nu$$

since the operation E^x performed on $f(z)$ gives $f(x)$ for $z=0$ if $h=1$ Tberefore this is the symbolical expression of *Newton's* formula

$$(1) \qquad f(x) = f(0) + \binom{x}{1} \Delta f(0) + \binom{x}{2} \Delta^2 f(0) + \dots$$

Hitherto we have only defined operations combined by polynomial relations (except in the case of E^{-n}); therefore the above demonstration assumes that x is a positive integer.

But the significance of the operation E^x is obvious for any value of x; indeed if $h=1$ we always have

$$\mathsf{E}^x f(0) = f(x).$$

To prove that formula (1) holds too for any values of x it is necessary to show that the operation E^x is identical with that corresponding to the series

$$\mathsf{E}'^x = 1 + \binom{x}{1} \Delta + \binom{x}{2} \Delta^2 + \dots + \binom{x}{n} \Delta^n + \dots\dots$$

This is obviously impossible if the series

$$f(0) + \binom{x}{1} \Delta f(0) + \dots + \binom{x}{n} \Delta^n f(0) + \dots$$

is divergent. On the other hand if $f(x)$ is a polynomial of degree n then $\Delta^{n+1} f(x) = 0$ and the corresponding series is finite. *Steffensen* [loc. cit. 1. p. 184] has. shown that in such cases the expansions are justified. If we limit the use of the symbolical expansions to these cases, their application becomes somewhat restricted; but as *Steffensen* remarked, these expansions are nevertheless of considerable use, since the form of an interpolation or summation formula does not depend on whether the function is a polynomial or not (except the remainder term)

If certain conditions are satisfied, the expansion of operation symbols into infinite series may be permitted even if the function to which the operations are applied, is not a polynomial when the corresponding series is convergent. [*Pincherle* **loc.** cit. **4.**]

To obtain *Newton's backward formula* let us remark that $\mathbf{E}-\Delta = 1$ and start from

$$\mathbf{E}^x = \left(\frac{\mathbf{E}}{\mathbf{E}-\Delta}\right)^x = \frac{1}{\left(1-\frac{\Delta}{\mathbf{E}}\right)^x}$$

Expanding the denominator into an infinite series we get

$$\mathbf{E}^x = \sum_{\nu=0}^{\infty} \binom{x+\nu-1}{\nu} \frac{\Delta^\nu}{\mathbf{E}^\nu} \tag{2}$$

According to what has been said, this formula is applicable, first if x is a positive integer, if f(x) is a polynomial, and in the general case if the series

$$f(x) = f(0) + \binom{x}{1} \Delta f(-1) + \binom{x+1}{2} \Delta^2 f(-2) +$$
$$+ \binom{x+2}{3} \Delta^3 f(-3) + , \ldots$$

is convergent, and *f(x)* satisfies certain conditions.

An interesting particular case of (2) is obtained for $x=1$

$$\mathbf{E} = \sum_{\nu=0}^{\infty} \left(\frac{\Delta}{\mathbf{E}}\right)^\nu$$

and if the operation is performed on f(x), we have

$$f(x+h) = \sum_{\nu=0}^{\infty} \underset{h}{\Delta^\nu} f(x-\nu h)$$

We may obtain a somewhat modified formula when starting from

$$\mathbf{E}^{x+1} = \frac{1}{\left(1-\frac{\Delta}{\mathbf{E}}\right)^{x+1}} .$$

Expanding and dividing by $\mathbf{E}$, we get

$$\mathbf{E}^x = \sum_{\nu=0}^{\infty} \binom{x+\nu}{\nu} \frac{\Delta^\nu}{\mathbf{E}^{\nu+1}}$$

In the same manner we have, if f(x) is a polynomial or if it satisfies certain conditions:

$$\frac{1}{\mathbf{E}^n} = \frac{1}{(1+\Delta)^n} = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{n+\nu-1}{\nu} \Delta^\nu$$

that is

$$f(x-nh) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{n+\nu-1}{\nu} \Delta_h^\nu f(x)$$

Expansion of an alternate function. A function $(-1)^x f(x)$ defined for x = 0, 1, 2, 3, . . . is called an alternate function. Starting from the second formula we deduce

$$(-1)^x \mathbf{E}^x = (1-2\mathbf{M})^x = \sum_{\nu=0}^{x+1} (-1)^\nu \binom{x}{\nu} 2^\nu \mathbf{M}^\nu$$

This formula may only he applied if x is a positive integer, since otherwise even in the case of polynomials it is divergent.

It has been mentioned that the Calculus of Finite Differences deals also with functions of a continuous variable. In this case it is possible to determine both the derivatives and the differences and to establish relations between these quantities.

Relation between differences and derivatives. If we write *Taylor's* series in the following manner:

$$f(x+h) = f(x) + h\mathbf{D}f(x) + \frac{h^2}{2!}\mathbf{D}^2 f(x) + \frac{h^3}{3!}\mathbf{D}^3 f(x) + \dots$$

Written symbolically it will be

$$\underset{h}{\mathbf{E}} = \sum_{\nu=0}^{\infty} \frac{h^\nu \mathbf{D}^\nu}{\nu!} = e^{h\mathrm{D}}$$

and therefore

(3) $$\underset{h}{\Delta} = e^{h\mathrm{D}} - 1.$$

This formula was found by *Lagrange;* it gives the first difference in terms of the derivatives If we expand the second member into a series of powers of hD, and multiply this series by itself, and apply *Cauchy's* rule of multiplication we obtain Δ^2; multiplying again we get Δ^3, and so on. **We** could express in this way the m -th difference by derivatives, but we will obtain this later in a shorter way by aid of ***Stirling*** numbers (§ 67).

Starting from $e^{hD} = 1 + \underset{h}{\Delta}$ we could write formally

$$hD = \log(1+\underset{h}{\Delta}) \tag{4}$$

In this form the second member has no meaning, but expanding it into a power series it will acquire one:

$$hD = \underset{h}{\Delta} - \tfrac{1}{2}\underset{h}{\Delta}^2 + \tfrac{1}{3}\underset{h}{\Delta}^3 - \tfrac{1}{4}\underset{h}{\Delta}^4 + \cdots$$

This formula gives the first derivative expressed by differences. It holds if the function is a polynomial, or if the series is convergent and satisfies certain conditions.

Again applying *Cauchy's* rule we obtain D^2, then D^3, and so on. **We** could thus get the expression of the m -th derivative, but we shall determine it in another way (§ 56).

§ 7. Receding Differences. Some authors have introduced besides the differences considered in the preceding paragraph, called also ***advancing differences,*** others, the ***receding differences.*** defined by

$$\Delta' f(x) = f(x) - f(x-h).$$

The symbol Δ' *is* that used by *Sheppard*[6]; in ***our*** notation this will be

$$\Delta' = \frac{\Delta}{E} \quad \text{and} \quad (\Delta')^n = \frac{\Delta^n}{E^n}$$

Since the formulae containing symbols of receding differences are very easily expressed by formulae of advancing differences, and since there is no advantage whatever in introducing the receding differences, they are only mentioned here.

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[6] *Sheppard* [loc. cit. 3]; different notations have been proposed for the receding differences. Steffensen [loc. cit. 1, pp. 224] puts

$$\nabla f(x) = f(x) - f(x-1).$$

§ 8. Central Differences. If we accept the notation of the advancing or that of the receding differences, there will always be a want of symmetry in the formulae obtained. Indeed, to have symmetrical formulae the argument of the difference

$$f(x+h) - f(x)$$

should be $x+\frac{1}{2}h$; and therefore the difference corresponding to the argument x would be

$$f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h)$$

This has been adopted in the system called *Central Differences*. Different notations have been proposed; we will adopt *Sheppard's* notation, which is the following for the first central difference[7]

$$\delta f(x) = f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h)$$

and for the first *Central Mean*

$$\mu f(x) = \tfrac{1}{2}[f(x+\tfrac{1}{2}h) + f(x-\tfrac{1}{2}h)].$$

From the above formulae it follows immediately that

$$\delta f(x) = \Delta f(x-\tfrac{1}{2}h) \qquad \text{and} \qquad \mu f(x) = \mathbf{M} f(x-\tfrac{1}{2}h).$$

Moreover from these relations we easily deduce the following:

$$\delta^2 f(x) = \Delta^2 f(x-h); \qquad \delta^{2n} f(x) = \Delta^{2n} f(x-nh)$$

$$\mu^2 f(x) = \mathbf{M}^2 f(x-h); \qquad \mu^{2n} f(x) = \mathbf{M}^{2n} f(x-nh)$$

$$\mu\delta f(x) = \mathbf{M}\Delta f(x-h); \qquad \mu\delta^{2n+1} f(x) = \mathbf{M}\Delta^{2n+1} f(x-nh-h).$$

The connection between the formulae of the advancing and those of the central differences may be established easily by the calculus of symbols. Starting from

$$\delta = \frac{\Delta}{\mathbf{E}^{1/2}} \qquad \text{and} \qquad \mu = \frac{\mathbf{M}}{\mathbf{E}^{1/2}}$$

---

[7] An excellent monograph on Central Differences is found in Sheppard [loc. cit. 3. p. 224]. Of the other notations let us mention *Joffe*'s [*see Whittaker and* Robinson loc. cit. 1. p. 36]

$$\underset{\circ}{\triangle} f(x) = f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2}).$$

The notation of the Central Mean introduced by *Thiele* and adopted by *Steffensen* [loc. cit. 1. p. 10] is the following

$$\square f(x) = \tfrac{1}{2}[f(x+\tfrac{1}{2}) + f(x-\tfrac{1}{2})].$$

we obtain

$$\delta^{2n} = \frac{\Delta^{2n}}{E^n} ; \ \delta^{2n+1} = \frac{\Delta^{2n+1}}{E^{n+1/2}} ; \ \mu^{2n} = \frac{M^{2n}}{E^n}$$

and so on.

Working with central differences there is but one difficulty, since generally the function $f(x)$ is given only for values such as $f(x+nh)$ where n is a positive or negative integer. In such cases $f(x+\frac{1}{2}h)$ or $f(x-\frac{1}{2}h)$ has no meaning, nor have δf (x) and μf (x). But there is no difficulty in determining $\delta^{2n} f$ (x), $\mu^{2n} f$ (x) and $\mu\delta^{2n+1} f(x)$.

Returning to our table of differences (§ 4) written in the advancing system, we find for instance the following line

$$f(a+2h) \quad M\Delta f(a+h) \quad \Delta^2 f(a+h) \quad M\Delta^3 f(a) \quad \Delta^4 f(a) \ldots ,$$

If in this table we had used the central difference notation the numbers of the above line would have been denoted by

$$f(a+2h) \quad \mu\delta f(a+2h) \quad \delta^2 f(a+2h) \quad \mu\delta^3 f(a+2h) \quad \delta^4 f(a+2h) \ldots$$

That is, the argument would have remained unchanged along the line. This is a simplification.

Since in the notation of central differences the odd ones are meaningless, the difference

$$\Delta^{2n+1} = E^{n+1/2}\delta^{2n+1}$$

must be expressed by aid of the even differences δ^{2m} and by $\mu\delta^{2m+1}$

To obtain an expression for Δ we will start from the above symbolical expressions, writing

$$E\delta^2 + 2E\mu\delta = \Delta^2 + 2MA = \Delta(\Delta+E+1) = 2AE$$

From this we get the important relations

$$(1) \quad \Delta = \mu\delta + \tfrac{1}{2}\delta^2 \quad \text{and} \quad A^{2n+1} = E^n[\mu\delta^{2n+1} + \tfrac{1}{2}\delta^{n+2}].$$

Symbolical methods. We have seen in § 6 that $E = e^{hD}$ and A $= e^{hD} - 1$. Putting $h=\omega$, to avoid mistakes in the hyperbolic formulae, we deduce from the definitions

$$\delta = \frac{\Delta}{E^{1/2}} = \frac{e^{\omega D} - 1}{e^{1/2\,\omega D}} = e^{1/2\,\omega D} - e^{-1/2\,\omega D} = 2 \sinh \tfrac{1}{2}\omega D$$

and

$$\mu = \frac{M}{E^{1/2}} = \frac{1+E}{2E^{1/2}} = \frac{1+e^{\omega D}}{2e^{1/2\,\omega D}} = \cosh \tfrac{1}{2}\omega D.$$

Starting from these formulae we may determine any symbolical central difference expression in terms of the derivatives, for instance δ^{2n}, $\mu\delta^{2n+1}$ and others.

To obtain δ^{2n} we use the expansion of $(\sinh x)^{2n}$ into a power series. The simplest way of obtaining this series is to express first $(\sinh x)^{2n}$ by a sum of $\cosh \nu x$. Since

$$2 \sinh x = e^{x} - e^{-x}$$

we have

$$(2 \sinh x)^{2n} = \sum_{\nu=0}^{2n} (-1)^{\nu} \binom{2n}{\nu} e^{2(n-\nu)x}.$$

In the second member the terms corresponding to ν and $2n-\nu$ combined give

$$e^{2(n-\nu)x} + e^{-2(n-\nu)x} = 2 \cosh 2(n-\nu)x$$

so that we have

$$(2 \sinh x)^{2n} = 2 \sum_{\nu=0}^{n} (-1)^{\nu} \binom{2n}{\nu} \cosh 2(n-\nu)x + (-1)^{n} \binom{2n}{n}.$$

From this it follows that

$$\left[D^{2m+1}(2 \sinh x)^{2n}\right]_{x=0} = 0.$$

For $m>0$ we find

(2) $$\left[D^{2m}(2 \sinh x)^{2n}\right]_{x=0} = 2 \sum_{\nu=0}^{n} (-1)^{\nu} \binom{2n}{\nu} (2n-2\nu)^{2m};$$

therefore the required expansion will be

$$(2 \sinh x)^{2n} = 2 \sum_{m=1}^{\infty} \frac{x^{2m}}{(2m)!} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{2n}{\nu} (2n-2\nu)^{2m}.$$

Finally putting $x = \frac{1}{2}\omega\mathbf{D}$ we have

$$\delta^{2n} = 2 \sum_{m=1}^{\infty} \frac{(\omega\mathbf{D})^{2m}}{(2m)!} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{2n}{\nu} (n-\nu)^{2m}.$$

Expansion of $\mu\delta^{2n+1}$ *in powers of the derivative-symbol* $\mathbf{D}$
Futting again $\frac{1}{2}\omega\mathbf{D} = x$, we have

$$\mu\delta^{2n+1} = 2^{2n+1} (\sinh x)^{2n+1} \cosh x$$

The first derivative will be

$$\mathbf{D}\mu\delta^{2n+1} = (n+1)\ (2 \sinh x)^{2n+2} + 2(2n+1)\ (2 \sinh x)^{2n}$$

Using formula (2) we obtain the *2m* -th derivative of this expression

$$[\mathbf{D}^{2m-1} \mu\delta^{2n+1}]_{x=0} = 2(n+1) \sum_{\nu=0} (-1)^{\nu} \binom{2n+2}{\nu} (2n+2-2\nu)^{2m}$$

$$+ 4(2n+1) \sum_{\nu=0} (-1)^{\nu} \binom{2n}{\nu} (2n-2\nu)^{2m}$$

and finally

$$\mu\delta^{2n+1} = \sum_{m=n} \frac{(\omega\mathbf{D})^{2m+1}}{(2m+1)!} \Big[(n+1) \sum_{\nu=0} (-1)^{\nu} \binom{2n+2}{\nu} (n+1-\nu)^{2m} +$$

$$+ 2(2n+1) \sum_{\nu=0} (-1)^{\nu} \binom{2n}{\nu} (n-\nu)^{2m}.$$

§ 9. **Divided differences.** So far we have supposed that the function $f(x)$ is given for $x = x_0, x_1, x_2, \ldots, x_n$ and that the interval

$$x_{i+1} - x_i = h$$

is independent of *i. Now we* deal with the general problem where the system of $x_0, x_1, \ldots, x_n$ may be anything whatever.

By the first divided difference of $f(x_i)$, denoted by $\mathfrak{D}f\ (x_i)$ the following quantity is understood:

$$\mathfrak{D}f\ (x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

The second divided difference of $f(x_i)$ *is*

$$\mathfrak{D}^2 f\ (x_i) = \frac{\mathfrak{D}f(x_{i+1}) - \mathfrak{D}f(x_i)}{x_{i+2} - x_i}$$

and so on; the m -th divided difference is

$$\mathfrak{D}^m f(x_i) = \frac{\mathfrak{D}^{m-1} f(x_{i+1}) - \mathfrak{D}^{m-1} f(x_i)}{x_{i+m} - x_i}$$

From these equations we may deduce successively

$$\mathfrak{D} f(x_0) = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0},$$

$$\mathfrak{D}^2 f(x_0) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)},$$

and so on. Putting $\omega_m(x) = (x-x_0)(x-x_1)\ldots(x-x_m)$ we may give to the general term $\mathfrak{D}^m f(x_0)$ the following form

$$(1)\qquad \mathfrak{D}^m f(x_0) = \frac{f(x_0)}{D\omega_m(x_0)} + \frac{f(x_1)}{D\omega_m(x_1)} + \ldots\ldots + \frac{f(x_m)}{D\omega_m(x_m)}.$$

The divided differences may also be given by *Vandermonde's* determinant

$$\mathfrak{D}^m f(x_0) = \frac{\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^{m-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \ldots & x_1^{m-1} & f(x_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_m & x_m^2 & \ldots & x_m^{m-1} & f(x_m) \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^m \\ 1 & x_1 & x_1^2 & \ldots & x_1^m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_m & x_m^2 & \ldots & x_m^m \end{vmatrix}}$$

NOW we shall deduce an expression for $f(x_m)$ *by* aid of the divided differences.

First multiplying $\mathfrak{D} f(x_0)$ by $\omega_0(x_m)$, then $\mathfrak{D}^2 f(x_0)$ by $\omega_1(x_m)$ and so on: $\mathfrak{D}^\nu f(x_0)$ by $\omega_{\nu-1}(x_m)$ and finally $\mathfrak{D}^m f(x_0)$ by $\omega_{m-1}(x_m)$ we obtain

$$\sum_{\nu=1}^{m+1} \omega_{\nu-1}(x_m)\mathfrak{D}^\nu f(x_0) = f(x_0) \sum_{\nu=1}^{m+1} \frac{\omega_{\nu-1}(x_m)}{D\omega_\nu(x_0)} + \sum_{i=1}^{m+1} \sum_{\nu=i}^{m+1} \frac{\omega_{\nu-1}(x_m)}{D\omega_\nu(x_i)} f(x_i).$$

In consequence of the relation

$$\omega_{m-1}(x_m) = \mathbf{D}\omega_m(x_m)$$

it follows that the coefficient of $f(x_m)$ in the preceding equation is equal to one, since it can be shown that

$$\sum_{\nu=1}^{m+1} \frac{\omega_{\nu-1}\;(x_m)}{\mathbf{D}\;\omega_\nu\;(x_\nu)} = -1\;;$$

so that the coefficient of $f(x_0)$ is equal to -1. Moreover it can be demonstrated that

$$\sum_{\nu=1}^{m+1} \frac{\omega_{\nu-1}\;(x_m)}{\mathbf{D}\;\omega_\nu\;(x_i)} = 0 \qquad \text{if } m > i > o$$

therefore for these values the term $f(x_i)$ will vanish from the equation and we have

$$f\,(x_m) = f\,(x_0) + (x_m-x_0)\,\mathfrak{D}f\,(x_0) + (x_m-x_0)\,(x_m-x_1)\,\mathfrak{D}^2 f(x_0) +$$

$$(2) \qquad + (x_m-x_0)\,(x_m-x_1)\,(x_m-x_2)\,\mathfrak{D}^3 f(x_0) + \ldots .$$

$$+ (x_m-x_0)\,(x_m-x_1)\ldots(x_m-x_{m-1})\,\mathfrak{D}^m f(x_0).$$

This is the required expression of the function by aid of divided differences, Putting into this equation x instead of x_m and adding the remainder, we obtain *Newton's* expansion for functions given at unequal intervals

$$(3) \quad f(x) = f\,(x_0) + (x-x_0)\,\mathfrak{D}f\,(x_0) + (x-x_0)\,(x-x_1)\,\mathfrak{D}^2 f\,(x_0) + \ldots + (x-x_0)\;(x-x_1)\;\ldots\;(x-x_{m-1})\,\mathfrak{D}^m f(x_0) + R_{m+1}\,.$$

We shall see in § 123 that the remainder R_{m+1} of the series is equal to

$$R_{m+1} = \frac{(x-x_0)\,(x-x_1)\ldots(x-x_m)}{(m+1)\,!}\,\mathbf{D}^{m+1} f(\xi)$$

where ξ is included in the interval of x, x_0, $x_1, \ldots, x_m$.

Formula (3) may be written

$$f(x) = f(x_0) + \sum_{i=0}^{m} \omega_i(x)\,\mathfrak{D}^{i+1} f\,(x_0) + \frac{\omega_m(x)}{(m+1)\,!}\,\mathbf{D}^{m+1} f(\xi).$$

§ 10. Generating functions. One of the most useful methods of the Calculus of Finite Differences and of the Calculus of

Probability is that of the Generating Functions, found by ***Laplace*** and first published in his **"Théorie** Analytique des **Probabilités"** [**Courcier,** Paris **1812**].

Given a function f(x) for $x=a,\ a+1,\ a+2, \dots,$ b-l u(t) will be defined as the generating function of f(x) if the coefficient of t^x in the expansion of u(t) is equal to f(x) in the interval *a*, ***b***, and if moreover this coefficient is equal to zero for every other integer value of **x.** Therefore we have

$$u(f) = \sum_{x=a}^{b} f(x) \ t^x.$$

To denote that u(t) is the generating function of *f(x) we* shall write

$$G f(x) = u(t)$$

If ***b*** is infinite, the generating function of *f(x)* will be considered only for values of t for which the series is convergent.

Remark. Another function $u_1(t)$ may be determined so that the coefficient of t^x shall be equal , in certain intervals, to the function f(x) given above.

For instance, denoting by *f,,,(x)* the number of partitions of order *m*, of the number x (with repetition and permutation of 1, 2, ... n), it is easy to see that the generating function cf *f,(x)* is the following:

$$u(f) = (t+t^2+t^3 + \dots + f'')''$$

but for $x \leqq n$ the expansion of the function

$$u_1(t) = (t + t^2 + \dots)^m = \frac{t^m}{(1-t)^m}$$

leads to the same coefficient as the function $u(f)$. Since the expansion of u,(f) is simpler than that of $u(f)$, we will use the former for the determination of *f(x).* We get

$$u_1(f) = t^m \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{-m}{\nu} t^{\nu} = \sum_{\nu=0}^{\infty} \binom{m+\nu-1}{\nu} t^{m+\nu}.$$

Finally we have

$$f_m(x) = \binom{x-1}{m-1}$$

if $x \leqq n$.

To determine the generating functions there are several methods.

First method. In Infinitesimal Calculus a great number of expansions in power series are known. Each function expanded may be considered as a generating function. For instance we found:

1. $\frac{1}{1-t} = \sum_{x=0}^{\infty} t^x$ therefore $\mathsf{G}\,1 = \frac{1}{1-t}$

2. $e^t = \sum_{x=0}^{\infty} \frac{1}{x!} t^x$ $\qquad \mathsf{G}\,\frac{1}{x!} = e^t$

3. $-\log(1-f) = \sum_{x=1}^{\infty} \frac{1}{x} t^x$ $\qquad \mathsf{G}\,\frac{1}{x} = -\log(1\text{-}f)$

4. $(1+t)^n = \sum_{x=0}^{n+1} \binom{n}{x} t^x$ $\qquad \mathsf{G}\binom{n}{x} = (1+t)^n$

5. $f\,(f\text{-}1)\,(f\text{-}2)\ldots(t-n+1) = \sum_{x=1}^{n+1} S_n^x\, t^x$

$$\mathsf{G}\,S_n^x = f(f\text{-}1)\ldots(t-n+1),$$

Where the numbers S_n^x are the *Stirling Numbers* of the first kind. We shall see them later (§ 50).

6. $f(2\nu) = 0\,,$ $\qquad f(2\nu+1) = \frac{2}{2\nu+1}$ $\qquad u(t) = \log\frac{1+t}{1-f}$

7. $f(2\nu) = 0\,,$ $\qquad f(2\nu+1) = \frac{(-1)^\nu}{2\nu+1}$ $\qquad u(f) = \arctan f$

8. $f(2\nu+1) = 0,\ f(2\nu) = \frac{(-1)^\nu}{(2\nu)!}$ $\qquad u(f) = \cos f$

9. $f(2\nu) = 0,$ $\qquad f(2\nu+1) = \frac{(-1)^\nu}{(2\nu+1)!}$ $\qquad u(f) = \sin f$

10. $f(2\nu+1) = 0,$ $\qquad f(2\nu) = \frac{1}{(2\nu)!}$ $\qquad u(f) = \cosh f$

11. $f(0) = 1,\ f(2\nu+1) = 0,\ f(2\nu) = \frac{E_{2\nu}}{(2r)!}$ $\qquad u(t) = \operatorname{sech} f$

The numbers $E_{2\nu}$ are ***Euler*** numbers (§ 105).

12. $$f(1) = -\tfrac{1}{2},\quad f(2\nu+1) = 0,\quad f(2\nu) = \frac{B_{2\nu}}{(2\nu)!},\quad u(t) = \frac{t}{e^t - 1}.$$

The numbers $B_{2\nu}$ are ***Bernoulli*** numbers (§ 78) .

Second method. Starting from the generating functions obtained by the first method we may deduce, by derivation, integration, or other operations, new generating functions. Of course the conditions of convergence needed for these operations must be fulfilled.

For instance starting from formula (1) we remark that the series $\Sigma\, t^x$ is uniformly convergent for every value of t such that $|t| < r < 1$. Therefore all derivative of the series, may be determined by term by term differentiation in the domain $|t| < 1$. The first derivation gives after multiplication by t

13. $$\frac{t}{(1-t)^2} = \sum_{x=0}^{\infty} x\, t^x \qquad \text{so that} \qquad G\,x = \frac{t}{(1-t)^2}.$$

The ν -th derivative obtained from formula (1), multiplied by $t^{\nu}/\nu!$ gives

14. $$\frac{t^{\nu}}{(1-t)^{\nu+1}} = \sum_{x=0}^{\infty} \binom{x}{\nu} t^x \qquad G\binom{x}{\nu} = \frac{t^{\nu}}{(1-t)^{\nu+1}}.$$

Formula (1) may also be integrated term by term and we get

15. $$-\log(1-t) = \sum_{x=1}^{\infty} \frac{t^x}{x} \qquad G\,\frac{1}{x} = -\log(1-t)$$

if $x > 0$.

In this way we obtain again formula (3). A second integration will give

16. $$t + (1-t)\log(1-t) = \sum_{x=2}^{\infty} \frac{t^x}{x(x-1)}$$

$$G\,\frac{1}{x(x-1)} = t + (1-4)\ \log(1-t)$$

if $x > 1$.

Third method. Sometimes it is possible to obtain the generating function of $f(x)$ by performing directly the summation

$$\sum_{x=0}^{\infty} f(x)\ t^x = u(t).$$

First, this is possible if f(x) is &he x -th power of a number a; then we have

$$u(t) = \sum_{x=0}^{\infty} (at)^x = \frac{1}{1-at}$$

Example. Let $f(x) = \cos \vartheta x$. $\cos \vartheta x$ is the real part of $e^{i\vartheta x}$; therefore *a(t)* will be the real part of

$$\frac{1}{1-e^{i\vartheta} t}$$

This is easily determined and we find

17. $$u(t) = \frac{1 - t \cos \vartheta}{1 - 2t \cos \vartheta + t^2}$$

Secondly, the summation may be executed directly if it is possible to express the function *f(x)* by a definite integral in such a way that under the integral sign there figures the x -th power of some quantity independent of x. For instance if we have

$$f(x) = C \int_a^b |\varphi(v)|^x \, dv$$

then we shall have

$$u(t) = C \int_a^b \sum_{x=0}^{\infty} |t\varphi(v)|^x \, dv = C \int_a^b \frac{dv}{1 - t\varphi(v)}$$

In this manner we obtain $u(t)$ in the form of a definite integral whose value may be known.

Example. Let $f(x) = \binom{2x}{x}$. We shall see later that the binomial coefficients are given by *Cauchy's* formula (5), § 22.

$$\binom{2x}{x} = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} [4 \cos^2 v]^x \, dv;$$

from this we conclude

$$u(t) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{dv}{1-4t \cos^2 v}.$$

This is already the required solution; but since in the tables of *Bierens de Haan*[8] we find that this integral is equal to $1/\sqrt{1—4t}$ we may give the generating function this simpler form ;

(18) $$u(f) = \frac{1}{\sqrt{1—4t}} = \mathbf{G}\binom{2x}{x}.$$

Remark 1. Since

$$\binom{2x}{x} = 2 \quad (\$ 7 ; ') \quad = 2 \binom{2x—1}{x},$$

we have

(19) $$\mathbf{G}\binom{2x-1}{x} = \frac{1}{2\sqrt{1—4t}}.$$

Remark II. Starting from the above result we may obtain the generating functions of many other expressions. For instance according to *Cauchy*'s rule of multiplication of series we obtain from (18)

$$[u(t)]^2 = [\sum_{x=0}^{\infty} \binom{2x}{x} t^x]^2 = \sum_{x=0}^{\infty} \sum_{\nu=0}^{x+1} \binom{2x—2\nu}{x—\nu}\binom{2\nu}{\nu} t^x,$$

moreover

$$[u(f)]' = \frac{1}{1—4t} = \sum_{x=0}^{\infty} 4'' \; t^x.$$

Hence noting that the coefficients of t^x are the same in the two expressions we get

$$\sum_{\nu=0}^{x+1} \binom{2x—2\nu}{x—\nu}\binom{2\nu}{\nu} = 4^x.$$

This is a very useful formula.

§ 11. General rules to determine generating functions.

1. The generating function of the sum of functions is equal to the sum of the generating functions

$$\mathbf{G}\,[f,(x) + f_2(x) + \ldots.] = \mathbf{G} f_1(x) + \mathbf{G} f_2(x) + \ldots.$$

[8] *D. Bierens* de *Haan,* Nouvelles Tables d'Intégrales Définies, Leide, 1867. In formula (13) of Table 47 we have to put $p=1$ and $1—q^2 = 41.$

2. If c is a constant, then we have

$$\mathbf{G}\, c\, f(x) = c\, \mathbf{G}\, f(x).$$

Therefore, for instance, if a polynomial of degree n is given by its *Newton* expansion

$$f(x) = f(0) + (f)\ \Delta f(0) + \binom{x}{2} \Delta^2 f(0) + \dots \binom{x}{n} \Delta^n f(0)$$

then according to these rules we shall obtain the generating function of $f(x)$ by aid of formula (14) of § 10. We find

$$(1) \qquad \mathbf{G}\, f(x) = \sum_{\nu=0}^{n+1} \Delta^\nu f(0) \frac{t^\nu}{(1-t)^{\nu+1}}$$

3. Given the generating function of $f(x)$

$$\mathbf{G}\ f(x) = u(t) = \sum_{x=0}^{\infty} f(x)\ t^x$$

we may easily deduce the generating function of $f\,(x+1)$; indeed from the preceding equation we may obtain

$$(2) \qquad \frac{u(f)\ - f(O)}{t} = \sum_{x=0}^{\infty} f(x+1)\ t^x = \mathbf{G}\, f(x+1)$$

Therefore the generating function of the difference of $f(x)$ will be

$$(2') \qquad \mathbf{G}\Delta f(x) = \frac{1}{t}\ [(1-t)u(t)\ - f(o))$$

In the same manner we have

$$(3) \qquad \mathbf{G} f(x+n) = \frac{1}{t^n}\ |u(t) - f(O) - tf(1) - \dots\ t^{n-1} f(n-1)\,|.$$

The last formula enables us to determine the generating functions of $\Delta^m f(x)$ *or* of $\mathbf{M}^m f(x)$ and so on,

Example. We have seen that

$$\mathbf{G} \binom{2x}{x} = \frac{1}{\sqrt{1-4t}}$$

Let us determine $\mathbf{G}\ \Delta \binom{2x}{x}$; according to (2') we have

$$\mathbf{G}\Delta\binom{2x}{x} = \frac{1}{t}\left[\frac{1-t}{\sqrt{1-4t}} - 1\right]$$

Determination of the generating function of t(x) starting from a *difference equation.* If the function f(x) is not directly given, but we know that it satisfies the following equation for x $\geqq 0$

$$(4)\quad a_n f(x+n) + a_{n-1} f(x+n-1) + \ldots + a_1 f(x+1) + a_0 f(x) = V(x)$$

where the a_i are independent of x and V(x) is a given function of x then we shall call the expression (4) "a complete linear difference equation of order *n* with constant coefficients" If we denote the generating function of *f(x)* by *u(t),* the preceding rules (3) permit us to express the generating function of the first member of (4) by aid of *u(t);* if we know moreover the generating function *R(t)* of V(x), then, **equating** the generating functions of both members we obtain an ordinary equation of the first degree in u(t) ; solving this we have finally the generating function of *f(x).*

$$(5)\quad u(t) = \left\{t^n R(t) + \sum_{m=1}^{n+1} a_m t^{n-m} [f(0) + f(1)t + \ldots + f(m-1)t^{m-1}]\right\} \Big/ \sum_{m=0}^{n+1} a_m t^{n-m}.$$

Example. Let us suppose that the generating function of f(x) is required, and that *f(x)* satisfies the following equation for $x \geqq 0$.

$$(6)\quad f(x+1) - f(x) = \frac{1}{x+1}$$

We have seen that $\mathbf{G}\dfrac{1}{x} = -\log(1-t)$; log $1=0$, hence according to (2) we shall have $\mathbf{G}\dfrac{1}{x+1} = -\dfrac{\log(1-t)}{t} = R(t)$,

Therefore from (5) it follows that

$$(7)\quad u(t) = \frac{a_1 f(0) - \log(1-t)}{a_1 + a_0 t} = \frac{f(0) - \log(1-t)}{1-t}.$$

Remark. We shall see that if $f(0) = -$ C (*Euler's* constant) then the solution of the difference equation (6) is the digamma function (§ 19) and then (7) will be its generating function.

4. Starting from the generating function $u(t)$ of f(x) we may obtain immediately the following generating functions

$$\mathsf{G}\ xf(x) = t\mathsf{D}u(t)$$
$$\mathsf{G}\ x(x-1)f(x) = t^2\mathsf{D}^2u(t)$$
$$\mathsf{G}\ x(x-1)\ (x-2), \ldots (x-m+1)f(x) = t^m\mathsf{D}^m u(t)$$

and in the same manner as before

$$\mathsf{G}\ x(x-1) \ldots (x-m+1)f\ (x+n) =$$
$$= t^m\mathsf{D}^m \left[\frac{u(t) - f(0) - tf(1) - \ldots t^{n-1}f\ (n-1)}{t^n}\right].$$

Other simple particular cases **are**

$$\mathsf{G}(x+1)f(x+1) = \mathsf{D}u(t)$$
$$\mathsf{G}(x+2)\ (x+1)f(x+2) = \mathsf{D}^2u(t)$$

and so on.

By aid of these formulae we may determine the generating function of *f(x),* if *f(x)* is given by a linear difference equation whose coefficients are polynomials of x.

We shall call the expression x(x-1) (x-2) . . , (x-m+l) 'factorial of degree m'and denote it by $(x)_m$. If we expand the coefficients of the difference equation into factorials, the equation may be written

$$(8) \qquad \sum_{m=0}^{n+1} f(x+m) \sum_{i=0} a_{mi}\ (x)_i = V(x).$$

By aid of the above relations we deduce from this equation a linear differential equation of $u(t)$ which will determine the required generating function.

Example. Given $(2x+2)f\,(x+1) - (2x+1)f\,(x) = 0$. This we may write

$$2(x+1)f(x+1) - 2xf(x) - f(x) = 0$$

hence

$$2\mathsf{D}u(t) - 2t\mathsf{D}u(t) - u(t) = 0$$

that is $2(1-t)\mathsf{D}u(t) = u(t)$. The solution of this differential equation is easily obtained by integration. We find

$$u[t] = c\ (1-t)^{-1/2}$$

To determine the constant c let us remark that $u(0) = f(0)$ therefore

$$u(t) = f(0)\,[1 - t]^{-\frac{1}{2}}$$

§ 12. Expansion of functions into power series. If the function $f(x)$ is unknown but we have determined its generating function u(t), then to obtain $f(x)$ we have to expand $u(t)$ into a power series. In the Calculus of Finite Differences this occurs for instance when solving difference equations by the method of generating functions. In the Calculus of Probability very often it is much easier to determine the generating function of a quantity than the quantity itself. In these cases also we have to expand the obtained generating function.

The methods for expanding functions into power series are found in the treatises for Infinitesimal Calculus. Here only the most useful will be given.

First method. Expansion by division. Example:

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots + t^x + \ldots$$

Second method. Stirling's method of expansion of $1/(a_0 + a_1 t + a_2 t^2)$ [Methodus Differentialis, 1730. p. 2).

Let us put

$$\frac{1}{a_0 + a_1 t + a_2 t^2} = f(0) + f(1)\,t + f(2)\,t^2 + \ldots + f(x)\,t^x + \ldots$$

multiplying both members by the denominator. We must have

$$1 = a_0 f(0)$$

moreover the coefficient of t^x must be equal to zero for every value of $x > 0$. This gives $a_0 f(1) + a_1 f(0) = 0$ and the following equation

$$a_0 f(x) + a_1 f(x-1) + a_2 f(x-2) = 0.$$

Putting into this equation successively $x = 2, 3, \ldots$ we may determine, starting from $f(0)$ and $f(1)$, step by step $f(2)$, $f(3)$. . ., and so on. Remark: f(x) may be obtained by solving directly the above linear difference equation of the second order with constant coefficients (§ 165).

This method is known as that of indeterminate coefficients.

Third method. Expansion by the binomial theorem. Examples.

(3) $$(1+t)^n = \sum_{x=0}^{n+1} \binom{n}{x} t^x$$

(4) $$\frac{1}{(1-t)^n} = \sum_{x=0}^{\infty} (-1)^x \binom{-n}{x} t^x = \sum_{x=0}^{\infty} \binom{n-1+x}{x} t^x ,$$

Example 1.

$$t(1+t)(1-t)^{-3} = t(1+t) \sum_{\nu=0}^{\infty} \binom{\nu+2}{\nu} t^\nu$$

and finally the coefficient of t^x will he

$$f(x) = \left[\binom{x+1}{x-1} + \binom{x}{x-2}\right] = x^2.$$

Example 2.

(5) $$(1+t+t^2+ \dots + t^m)^n = (1-t^{m+1})^n (1-t)^{-n} =$$

$$= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{n+1} (-1)^\mu \binom{n}{\mu} \binom{n+\nu-1}{\nu} t^{\nu+\mu(m+1)} .$$

To obtain the coefficient $f(x)$ of t^x *we* have to put $x = \nu + \mu(m+1)$. We find

$$f(x) = \sum_{\mu=0}^{n+1} (-1)^\mu \binom{n}{\mu} \binom{x+n-1-m\mu-\mu}{n-1} .$$

Example 3. Given $u = (c_0+c_1t+c_2t^2)^n$. Applying the binomial theorem we have

$$[c_0+(c_1+c_2t)t]^n = \sum_{\nu=0}^{\infty} \binom{n}{\nu} c_0^{n-\nu} t^\nu (c_1+c_2t)^\nu .$$

A second application of the theorem gives

(6) $$u = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu+1} \binom{n}{\nu} \binom{\nu}{\mu} c_0^{n-\nu} c_1^{\nu-\mu} c_2^{\mu} t^{\nu+\mu} .$$

Putting $x=\nu+\mu$, the coefficient f(x) of t^x will be

$$f(x) = \sum_{\nu=0}^{\infty} \binom{n}{\nu} \binom{\nu}{x-\nu} c_0^{n-\nu} c_1^{2\nu-x} c_2^{x-\nu} .$$

Fourth method. Expansion by ***Maclaurin's*** Theorem.

$$u(t) = \sum_{x=0}^{\infty} \frac{D^x u(0)}{x!} t^x.$$

Hence we have to determine $D^x u(t)$ for $t=0$.

***A.* This may be obtained by** successive differentiation, For instance if u is equal **to** $1/(1-t)$, to a^t, to $\sin t$, etc.

***B.* If $u(t)$ is a product of two** factors $u = p(f)$, $\psi(t)$, then we may apply ***Leibnitz*'s theorem**

(7)
$$D^x u(t) = \sum_{\nu=0}^{x+1} \binom{x}{\nu} D^{\nu}\varphi(t)\; D^{x-\nu}\psi(t).$$

C. Derivation of a function of function. [See ***Schlömilch*'s** Compendium II. **p. 4.**] Let us write $u = u(y)$ and $y = y(t)$. *We* have

$$\frac{du}{dt} = \frac{du}{dy}\frac{dy}{dt}$$

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dy^2}\left(\frac{dy}{dt}\right)^2 + \frac{du}{dy}\frac{d^2y}{dt^2}$$

$$\frac{d^3u}{dt^3} = \frac{d^3u}{dy^3}\left(\frac{dy}{dt}\right)^3 + 3\frac{d^2u}{dy^2}\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{du}{dy}\frac{d^3y}{dt^3}$$

$$\cdot \qquad \cdot$$

$$\frac{d^nu}{dt^n} = Y_{n1}\frac{du}{dy} + Y_{n2}\frac{d^2u}{dy^2} + \dots + Y_{nn}\frac{d^nu}{dy^n}$$

where the functions Y_{ni} are independent of u. Therefore we may determine them by choosing the most convenient function $u(y)$,

Let

$$u = e^{\omega y};$$

hence

$$\frac{d^su}{dy^s} = \omega^s e^{\omega y}.$$

Consequently

$$\frac{d^n}{dt^n}(e^{\omega y}) = (\omega Y_{n1} + \omega^2 Y_{n2} + \dots + \omega^n Y_{nn})\, e^{\omega y}.$$

From this we deduce that

(7')
$$Y_{ns} = \frac{1}{s!}\left[\frac{d^s}{d\omega^s}\left(e^{-\omega y}\frac{d^n}{dt^n}e^{\omega y}\right)\right]_{\omega=0}$$

Moreover,

$$\frac{d^s}{dt^s}\varphi(t+h) = \frac{d^s}{dh^s}\varphi(t+h),$$

$$\frac{d^s}{dt^s}\varphi(t) = \left[\frac{d^s}{dh^s}\varphi(t+h)\right]_{h=0};$$

hence

$$\frac{d^n}{dt^n}e^{\omega y(t)} = \left[\frac{d^n}{dh^n}e^{\omega y(t+h)}\right]_{h=0};$$

from this it follows by aid of (7′) that

$$Y_{ns} = \left[\frac{d^{n+s}}{d\omega^s dh^n} e^{\omega y(t+h)-\omega y(t)}\right]_{\omega=0,\ h=0}$$

so that

$$Y_{ns} = \frac{1}{s!}\left\{\frac{d^n}{dh^n}[\underset{h}{\Delta} y(t)]^s\right\}_{h=0}.$$

Finally we have

$$\text{(8)}\qquad \frac{d^n u}{dt^n} = \sum_{s=1}^{n+1}\frac{d^s u}{dy^s}\left\{\frac{1}{s!}\frac{d^n}{dh^n}[\underset{h}{\Delta} y(t)]^s\right\}_{h=0}.$$

For every particular function y we may determine once for all the quantities $Y_{n1}, Y_{n2}, \ldots, Y_{n}$.

Example 1. Let $y = e^t$. We may write

$$[\underset{h}{\Delta} e^t]^s = e^{st}(e^h-1)^s = e^{st}\sum_{i=0}^{s+1}(-1)^i\binom{s}{i}e^{h(s-i)}$$

hence

$$\frac{d^n}{dh^n}[\underset{h}{\Delta} e^t]^s = e^{st}\sum_{i=0}^{s+1}(-1)^i\binom{s}{i}(s-i)^n e^{h(s-i)}.$$

Putting $h = 0$ and dividing by s! we get

$$Y_{ns} = \frac{e^{st}}{s!}\sum_{i=0}^{s+1}(-1)^i\binom{s}{i}(s-i)^n.$$

We shall see in § 58 that putting $t=0$ *the* second member will be equal to $\mathfrak{S}_n^s$, a *Stirling* number of the second kind. So that $Y_{ns} = \mathfrak{S}_n^s$ and

$$\left[\frac{d^n u}{dt^n}\right]_{t=0} = \sum_{m=0}^{n+1}\mathfrak{S}_n^m\left[\frac{d^m u}{dy^m}\right]_{y=1}.$$

Example 2. Let $y = t^2$. We shall have

$$(\Delta_h t^2)^s = (2t+h)^s\, h^s = \sum_{i=0}^{s+1} \binom{s}{i} (2t)^{s-i} h^{s+i}$$

moreover

$$\frac{d^n}{dh^n} (\Delta_h t^2)^s = \sum_{i=0}^{s+1} (s+i)_n \binom{s}{i} (2t)^{s-i} h^{s+i-n}.$$

Putting $h=0$ every term will vanish except that in which $s+i=n$. Therefore we find

$$Y_{ns} = \frac{1}{s!} \left[\frac{d^n}{dh^n} (\Delta_h t^2)^s \right]_{h=0} = \frac{n!}{s!} \binom{s}{n-s} (2t)^{2s-n}.$$

Example *3.* Let $y = 1/t$. We have

$$\Delta_h \frac{1}{t} = -\frac{h}{t(t+h)}.$$

Proceeding in the same manner as in the preceding example we get

$$Y_{nm} = (-1)^n \frac{n!}{m!} \binom{n-1}{m-1} \frac{1}{t^{n+m}}.$$

***Derivation** of a function of function by Faa Bruno's formula.* Given $u=u(y)$ and $y=y(x)$, let us write

$$y = y_0 + (x-x_0) Dy_0 + (x-x_0)^2 \frac{D^2 y_0}{2!} + (x-x_0)^3 \frac{D^3 y_0}{3!} + \dots$$

and

$$u = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} (y-y_0)^\nu \left[\frac{d^\nu u}{dy^\nu} \right]_{y=y_0} =$$

$$= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left[(x-x_0) Dy + (x-x_0)^2 \frac{D^2 y_0}{2!} + \dots \right]^\nu \left[\frac{d^\nu u}{dy^\nu} \right]_{y=y_0}.$$

The coefficient of $(x-x_0)^n$ in the expansion of u is equal first to

$$\left[\frac{1}{n!} \frac{d^n u}{dx^n} \right]_{x=x_0}$$

and secondly in consequence of the above equation to

$$\sum_{\nu=1}^{n+1} \frac{1}{\nu!} \left[\frac{d^\nu u}{dy^\nu}\right]_{y=y_0} \sum \frac{\nu!}{\alpha_1!\,\alpha_2!\dots\alpha_n!} (Dy_0)^{\alpha_1} \left(\frac{D^2y_0}{2!}\right)^{\alpha_2} \dots$$

$$\dots \left(\frac{D^n y_0}{n!}\right)^{\alpha_n}$$

so that finally we may write

$$(9)\quad \left[\frac{d^n u}{dx^n}\right]_{x=x_0} = \sum_{\nu=1}^{n+1} \left[\frac{d^\nu u}{dy^\nu}\right]_{y=y_0} \sum \frac{n!}{\alpha_1!\dots\alpha_n!} (Dy_0)^{\alpha_1} \left(\frac{D^2y_0}{2!}\right)^{\alpha_2} \dots$$

$$\dots \left(\frac{D^n y_0}{n!}\right)^{\alpha_n}$$

where

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = ,$$

and

$$\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n.$$

Remark. Comparing formulae (8) and (9) we find

$$(10)\quad \frac{1}{n!} \left[\frac{d^n}{dh^n} [\Delta y(x)]^\nu\right]_{h=0} = \sum \frac{\nu!}{\alpha_1!\dots\alpha_n!} (Dy_0)^{\alpha_1} \left(\frac{D^2y_0}{2!}\right)^{\alpha_2} \dots$$

$$\dots \left(\frac{D^n y_0}{n!}\right)^{\alpha_n}.$$

§ 13. Expansion of functions by aid of decomposition into partial fractions. Let the following function be given

$$u = \frac{\varphi(t)}{\psi(t)}$$

where $\varphi(t)$ and $\psi(t)$ are polynomials; the degree of $\varphi(t)$ being less than that of $\psi(t)$. We may always suppose that $\varphi(t)$ and $\psi(t)$ have no roots in common; since if they had, it would always be possible to simplify the fraction, dividing by $t-r_m$, if r_m is the root.

A. Let us suppose that the roots $r_1, r_2, \dots, r_n$ of $\psi(t)$ are all real and unequal. We have

$$u = \sum_{i=1}^{n+1} \frac{a_i}{t-r_i}$$

Reducing the fractions to a common denominator we obtain u if for every value of t we have

$$\varphi(t) = \sum_{i=1}^{n+1} a_i \frac{\psi(t)}{t-r_i}$$

therefore this is an identity: so that the coefficients of t^ν, in both members, must be equal, This gives n equations of the first degree which determine the coefficients a_i.

But these may be determined in a shorter way. Indeed

$$\psi(t) = C(t-r_1)(t-r_2)\ldots(t-r_n);$$

therefore putting $t=r_i$ into the above equation every term will vanish except that of a_i and we shall have

$$\varphi(r_i) = a_i \lim \frac{\psi(t)}{t-r_i} = a_i \,\mathrm{D}\psi(r_i);$$

so that

$$a_i = \frac{\varphi(r_i)}{\mathrm{D}\psi(r_i)},$$

and therefore

$$u = \sum_{i=1}^{n+1} \frac{-\varphi(r_i)}{r_i \mathrm{D}\psi(r_i)} \left(1-\frac{t}{r_i}\right)^{-1}.$$

Finally the coefficient f(x) of t^x in the expansion of u will be equal to

$$f(x) = \sum_{i=1}^{n+1} \frac{-\varphi(r_i)}{r_i^{x+1}\mathrm{D}\psi(r_i)}.$$

Example 1. Let $u(t) = 2t / (t^2+t-1)$. From this we deduce $r_1 = \frac{1}{2}(-1+\sqrt{5})$ and $r_2 = \frac{1}{2}(-1-\sqrt{5})$. Hence we have

$$\varphi(r_1) = -1+\sqrt{5} \text{ and } \varphi(r_2) = -1-\sqrt{5}.$$

Since $\mathrm{D}\psi(t) = 2t+1$ it follows that

$$\mathrm{D}\psi(r_1) = \sqrt{5} \text{ and } \mathrm{D}\psi(r_2) = -\sqrt{5}$$

consequently

$$f(x) = \frac{1-\sqrt{5}}{\sqrt{5}}\left(\frac{2}{-1+\sqrt{5}}\right)^{x+1} - \frac{1+\sqrt{5}}{\sqrt{5}}\left(\frac{2}{-1-\sqrt{5}}\right)^{x+1};$$

this gives

$$f(x) = \frac{1}{2^{x-1}\sqrt{5}}\left[(1-\sqrt{5})^x - (1+\sqrt{5})^x\right] \quad ,$$

and finally

$$(1) \qquad f(x) = \frac{-1}{2^{x-2}} \sum_{m=0} \binom{x}{2m+1} 5^m.$$

We could have obtained the same result by solving the difference equation deduced in *Stirling's* method (Method II, § 12).

On the other hand the third method giving the expansion of the trinomial

$$2t(t^2+t-1)^{-1}$$

would lead to

$$-2 \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu+1} \binom{\nu}{\mu} t^{\nu+\mu+1}$$

Putting $x = \nu+\mu+1$ we obtain

$$(2) \qquad f(x) = -2 \sum_{\mu=0} \binom{x-\mu-1}{\mu}$$

Equating this result to (1) gives the interesting relation

$$\sum_{\mu=0} \binom{x}{2\mu+1} 5^{\mu} = 2^{x-1} \sum_{\mu=0} \binom{x-\mu-1}{\mu}.$$

Example 2. Decomposition of the reciprocal factorial

$$u(t) = 1/t(t-1)(t-2)\ldots(t-m),$$

We have

$$a_i = \frac{1}{[Dt(t-1)\ldots(t-m)]_{t=i}} = (-1)^{m+i}\frac{\binom{m}{i}}{m!},$$

therefore

$$u = \sum_{i=0}^{m+1} \frac{(-1)^{m+i}\binom{m}{i}}{m!\,(t-i)}$$

In the same manner we should have

$$u = \frac{1}{t\,(t+1)\,(t+2)\,.\,,\,.\,(t+m)} - \sum_{i=0}^{m+1} \frac{(-1)^i \binom{m}{i}}{m!\,(t+i)}$$

B. The **roots** $r_1, r_2, \ldots, r_n$ *of* $\psi(t)$ are all single, but there are complex roots among **them.**[9] The coefficients of y(t) are real: therefore if $r_1 = \alpha+\beta i$ is a root of y(t) = 0, $r_2 = \alpha-\beta i$ will be a root too. The preceding method is still applicable but the corresponding values of a, 'and a, will be complex conjugate:

$$a_1 = \frac{\varphi(\alpha+\beta i)}{D\psi(\alpha+\beta i)} = G + Hi$$

and

$$a_2 = \frac{\varphi(\alpha-\beta i)}{D\psi(\alpha-\beta i)} = G - Hi.$$

Therefore the decomposition of u will be

$$u = \frac{G+Hi}{(t-\alpha)-\beta i} + \frac{G-Hi}{(t-\alpha)+\beta i} + R(t).$$

This may be written

$$u = \frac{2G(t-\alpha) - 2H\beta}{(t-\alpha)^2 + \beta^2} + R(t).$$

The expansion of the trinomial has been shown in Method III (§ 12); putting into formula (6) $n = -1$, $c_2 = 1$, $c_1 = -2a$ and $c_0 = \alpha^2+\beta^2$, we obtain

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu+1} (-1)^{\mu} \binom{\nu}{\mu} (2a)^{\nu-\mu} (\alpha^2+\beta^2)^{-\nu-1}\, t^{\nu+\mu}$$

This multiplied by the numerator gives

$$2 \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu+1} (-1)^{\mu}\, G \binom{\nu}{\mu} (2\alpha)^{\nu-\mu} (\alpha^2+\beta^2)^{-\nu-1}\, t^{\nu+\mu+1} +$$

$$- \; 2 \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu+1} (-1)^{\mu} (G\alpha+H\beta) \binom{\nu}{\mu} (2a)^{\nu-\mu} (\alpha^2+\beta^2)^{-\nu-1}\, t^{\nu+\mu}$$

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[9] *Ch. Hermite*, **Cours d'Analyse,** 2. p. 261-266, Paris 1873.
Ch. Sturm, **Cours d'Analyse,** Vol. I. p. 331. Paris 1895.

Putting $x=v+\mu+1$ into the first expression and $x=\nu+\mu$ into the second we obtain finally

$$f(x) = 2 \sum_{\mu=0} (-1)^{\mu} (2\alpha)^{x-2\mu-1} (\alpha^2+\beta^2)^{\mu-x}\left[G\binom{x-\mu-1}{\mu} - \right.$$
$$\left. -\frac{2\alpha(G\alpha+H\beta)}{\alpha^2+\beta^2}\binom{x-\mu}{\mu}\right] + \ldots,.$$

C. The roots $r_1, r_2, \ldots, r_n$ of the polynomial $\psi(t)$ are all real, but there are multiple roots. Let us suppose that the multiplicity of the root r_ν is m_ν; where m_ν may be equal to 1, 2, 3, . . . , and so on. The general formula of decomposition into partial fractions is the following:

$$(3) \qquad \frac{\varphi(t)}{\psi(t)} = \sum_{\nu=1}^{n+1}\left[\frac{a_{\nu 1}}{t-r_\nu} + \frac{a_{\nu 2}}{(t-r_\nu)^2} + \ldots + \frac{a_{\nu m}}{(t-r_\nu)^m}\right]$$

If we multiply this expression by $(t-r_\nu)^{m_\nu}$ we get

$$(4) \quad \frac{\varphi(t)}{\psi(t)}(t-r_\nu)^{m_\nu} = a_{\nu 1}(t-r_\nu)^{m_\nu-1} + \ldots a_{\nu m_\nu} + R(t), (t-r_\nu)^{m_\nu}$$

We have

$$\psi(t) = C(t-r_1)^{m_1}(t-r_2)^{m_2}\ldots, (t-r_n)^{m_n}$$

Hence if we denote the first member of (4) by $A_\nu(t)$, $A_\nu(r_\nu)$ will be different from zero:

$$A_\nu(r_\nu) = a_{\nu m_\nu}$$

In the same manner the first $m_\nu - s$ derivatives give

$$(5) \qquad D^{m_\nu-s}A_\nu(r_\nu) = (m_\nu-s)!\, a_{\nu s}.$$

This operation repeated for every value of v will give the required coefficients $a_{\nu s}$, Finally we shall have

$$\frac{\varphi(t)}{\psi(t)} = \sum_{\nu=1}^{n+1}\sum_{s=1}^{m_\nu+1}\frac{D^{m_\nu-s}A_\nu(r_\nu)}{(m_\nu-s)!}\cdot\frac{1}{(t-r_\nu)^s}.$$

This may be written

$$\frac{\varphi(t)}{\psi(t)} = \sum_{\nu=1}^{n+1}\sum_{s=1}^{m_\nu+1}\frac{(-1)^s D^{m_\nu-s}A_\nu(r_\nu)}{(m_\nu-s)!\, r_\nu^s}\left(1-\frac{t}{r_\nu}\right)^{-s}.$$

Therefore the coefficient $f(x)$ of t^x in the expansion of this expression will be

$$f(x) = \sum_{\nu=1}^{n+1} \sum_{s=1}^{m_\nu+1} (-1)^s \frac{D^{m_\nu - s} A_\nu(r_\nu)}{(m_\nu - s)!\, r_\nu^{x+s}} \binom{x+s-1}{x}$$

D. Multiple complex roots. Let us suppose that the roots $\alpha+\beta i$ and $\alpha-\beta i$ are of the multiplicity m. We may write the fraction in the following manner

$$\frac{\varphi(t)}{\psi(t)} = \frac{a_1 t + b_1}{(t-\alpha)^2+\beta^2} + \frac{a_2 t + b_2}{[(t-\alpha)^2+\beta^2]^2} + \dots +$$

$$+ \frac{a_m t + b_m}{[(t-\alpha)^2+\beta^2]^m} + R(t).$$

To obtain the coefficients a_ν and b_ν let us multiply this equation by $[(t-\alpha)^2 + \beta^2]^m$; denoting

$$A(t) = \frac{\varphi(t)}{\psi(t)} [(t-\alpha)^2+\beta^2]^m$$

we shall have

$$(5) \qquad A(t) = R(t)[(t-\alpha)^2+\beta^2]^m + (a_1 t + b_1)[(t-\alpha)^2+\beta^2]^{m-1} + \\ + (a_2 t + b_2)[(t-\alpha)^2+\beta^2]^{m-2} + \dots, + (a_m t + b_m).$$

From this we obtain immediately

$$(6) \qquad A(\alpha+\beta i) = a_m \alpha + b_m + a_m \beta i.$$

We should have also

$$(7) \qquad A(\alpha-\beta i) = a_m \alpha + b_m - a_m \beta i.$$

The last two equations permit us to determine a_m and b_m. But the second one is superfluous; indeed knowing that a_m and b_m are real, by equating first the real parts in both members of (7), and then the imaginary parts we get the two necessary equations.

The first derivative of $A(t)$ for $t=\alpha+\beta i$ is

$$DA(\alpha+\beta i) = a_m + [a_{m-1}(\alpha+\beta i) + b_{m-1}](2\beta i).$$

This gives again two equations. Determining $D^s A(\alpha+\beta i)$ for $s=0$, 1, 2, . . . $(m-1)$ will give $2m$ equations which determine the $2m$ unknowns $a_1, b_1, \dots, a_m, b_m$.

Example 3. Let $u=1 / (t-1)[(t-1)^2+4]^2$. We have

$$A(t) = \frac{1}{t-1} = (a_1t+b_1)\ [(t-1)^2+4] + (a_2t+b_2) + $$
$$+ \frac{c}{t-1} [(t-1)^2+4]^2.$$

Putting $t = 1+2i$ gives

$$A\ (1+2i) = \frac{1}{2i} = a_2 + b_2 + 2a_2i.$$

Hence

$$a_2 + b_2 = 0 \text{ and } 1 = -4a_2$$

or

$$a_2 = -\frac{1}{4}; \quad b_2 = \frac{1}{4}.$$

The first derivative of $A(t)$ gives for $t=1+2i$

$$\Delta A(1+2i) = -\frac{1}{(2i)^2} = (a_1+b_1+2a_1i)\ (4i) + a_2.$$

From this it follows that

$$a_1+b_1=0 \text{ and } \frac{1}{4} = -8a_1 - \frac{1}{4}$$

that is

$$a_1 = -\frac{1}{16} \text{ and } b_1 = \frac{1}{16}.$$

To obtain c we multiply $A(t)$ by (t-1) and put $t=1$, getting

$$1 = 16c.$$

If the coefficients a_s and b_s are calculated we may expand each term

$$\frac{a_st + b_s}{[(t-\alpha)^2+\beta^2]^s}$$

using Method III for the expansion of a trinomial. Here we have to put in formula (6) of § 12 $n= -s$, $c_2=1$, $c_1 = -2\alpha$ and $c_0 = \alpha^2+\beta^2$.

§ 14. Expansion of functions by aid of complex integrals. If the function $u(t)$, of the complex variable t, has no poles or other singularities in the interior or on the boundary of the circle of. radius ϱ, then $f(x)$, the coefficient of t^x in the expansion of u(f), is given, according to a known theorem of *Cauchy*, by the following integral taken round this circle:

$$f(x) = \frac{1}{2\pi i} \int \frac{u(t)}{t^{x+1}} dt.$$

Putting $t = \varrho e^{i\varphi}$, and therefore $dt = i\varrho e^{i\varphi} d\varphi$ we obtain

$$f(x) = \frac{1}{2\pi\varrho^x} \int_0^{2\pi} u(\varrho e^{i\varphi}) e^{-i\varphi x} d\varphi.$$

Example. Let $u(t) = (1+t)^n$. If n is a positive integer then $u(t)$ has no poles in the circle of unit radius. Putting $\varrho=1$, we have

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} (1+e^{i\varphi})^n e^{-i\varphi x} d\varphi.$$

Now writing $\varphi=2\alpha$, we obtain

$$f(x) = \frac{1}{\pi} \int_0^{\pi} (e^{i\alpha} + e^{-i\alpha})^n e^{in\alpha-2i\alpha x} d\alpha =$$

$$= \frac{2^n}{\pi} \int_0^{\pi} \cos^n \alpha \cos (n-2x) \alpha \, d\alpha.$$

Remarking that the coefficient of t^x in the expansion of u(t) is equal, according to *Newton's* rule, to $\binom{n}{x}$, and moreover that the above integral taken between the limits 0 and $\tfrac{1}{2}\pi$ is equal to the half of that taken between the limits 0, π, we conclude that

$$\binom{n}{x} = \frac{2^{n+1}}{\pi} \int_0^{\frac{1}{2}\pi} \cos^n \alpha \cos (n-2x) \alpha \, d\alpha.$$

This is *Cauchy's* formula expressing the binomial coefficient by a definite integral.

§ 15. Expansion of a function by aid of difference equations. If a function of the following form is given, where R(t) is a function expansible into a power series,

$$(1) \qquad u(f) = \frac{R(f)\,.\,t^n + a_{n-1}t^{n-1} + \ldots\ldots + a_1 t + a_0}{a_0 t^n + a_1 t^{n-1} + \ldots, + a_{n-1}t + a_n}$$

then $u(t)$ may be considered as being the generating function of a function f(x) determined by the difference equation

$$(2) \qquad a_n f(x+n) + a_{n-1} f(x+n-1) + \ldots + a_1 f(x+1) + a_0 f(x) = V(x).$$

Indeed, we have seen (formula 5. § 11) that the generating function of f(x) deduced by aid of the difference equation (2) is

$$(3) \qquad u(f) = \frac{t^n R(t) + \sum_{m=1}^{n} \sum_{i=0}^{m} a_m \, f(i) \, t^{n-m+i}}{\sum_{m=0}^{n+1} a, \; t^{n-m}}$$

where $R(f)$ is the generating function of $V(x)$; that is, $V(x)$ is the coefficient of t^x in the expansion of $R(f)$.

Knowing V(x), let us suppose that we are able to solve the difference equation (2) by a method which does not require the expansion of the generating function (3). If we find f(x) = $= \Theta(x, c_1, c_2, \ldots, c_n)$ where the c are arbitrary constants, then we shall determine them by identification of (1) and (3), and thus obtain the *n* equations:

$$(4) \qquad a \, , = \sum_{m=1}^{n+1} a_m f(x+m-n).$$

Putting in the corresponding values of *f(x) we* may determine the *n* arbitrary constants *a,,,* , and obtain in this way the required coefficient f(x) of t^x in the expansion of *u(f).*

Example 1. The function $u(f) = t \;/\; (8t^2-6t+1)$ is to be expanded into a series of powers of t. According to what has been said, if we denote by f(x) the coefficient of t^x in this expansion then *f(x)* will be the solution of the difference equation

$$f(x+2) - 6f(x+1) + 8f(x) = 0.$$

We shall see in § 165 that the solution of this equation is

$$(5) \qquad f(x) = c_1 4^x + c_2 2^x.$$

From (4) we get

$$a_0 = a_2 f(0) \qquad \text{or} \qquad 0 = f(0)$$
$$a_1 = a_1 f(0) + a_2 f(1) \qquad\qquad 1 = -6f(0) + f(1)$$

therefore

$$f(0) = 0 \quad \text{and} \quad f(1) = 1.$$

Putting these values into (5) we may determine the constants c_ν; we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Finally we have

$$f(x) = 2^{2x-1} - 2^{x-1}.$$

If we do not know the solution of the difference equation (2) the method may still be applied. Starting from equation (4) we determine as before the values $f(0)$, $f(1), \ldots, f(n-1)$ and then, by aid of equation (2) we compute step by step the values $f(n)$, $f(n+1)$ and so on.

Example 2. [Leonardo Eulero, Introductio in Analysin Infinitorum 1748. Lausanne, p. 280.] The function $u(t) = t^2/(8t^3-6t+1)$ is to be expanded into a power series. The coefficient $f(x)$ of t^x is given by the difference equation:

$$(6) \qquad f(x+3) - 6f(x+1) + 8f(x) = 0$$

Without solving this equation we get from (4)

$$a_2 = a_1 f(0) + a_2 f(1) + a_3 f(2) \qquad \text{or} \quad 1 = -6f(1) + f(2)$$
$$a_1 = a_2 f(0) + a_3 f(1) \qquad\qquad 0 = f(1)$$
$$a_0 = a_3 f(0) \qquad\qquad 0 = f(0).$$

Hence

$$f(0) = 0, \qquad f(1) = 0, \qquad f(2) = 1.$$

By the aid of these values we obtain from (6) step by step $f(3) = 6$, $f(4) = 36$; $f(5) = 216-8 = 208$, $f(6) = 1248-48 = 1200$ and so on.

Remark. This method is identical with that of Stirling (Method II, § 12). Given the function

$$u(t) = \frac{a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + a_0}{a_0 t^n + a_1 t^{n-1} + \ldots + a_n} =$$
$$= f(0) + tf(1) + t^2 f(2) + \ldots \qquad \text{and}$$

multiplying both members by the denominator we have

$$a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + , \ldots + a_0 = \sum_{i=0}^{\infty} \sum_{s=0}^{n+1} a_{n-s} f(i) \; t^{i+s}.$$

From this we deduce the equations $(\nu=0, 1, 2, \ldots, n-1)$

$$a_{\nu} = \sum_{m=0}^{n+1} a_m \; f(\nu+m-n)$$

which are identical with the equations (4).

CHAPTER II.

FUNCTIONS IMPORTANT IN THE CALCULUS OF FINITE DIFFERENCES.

§ 16. The Factorial. In Infinitesimal Calculus the simplest function is the power. Its derivative is very simple. Inded we have

$$\mathrm{D}\, x^n = n x^{n-1}.$$

On the other hand the difference of a power is complicated; we have

$$\underset{h}{\Delta}\, x^n = \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \ldots + h^n.$$

Therefore powers and power-series will be less useful in the Calculus of Finite Differences than in Infinitesimal Calculus.

But there are other functions whose differences are as simple as the derivatives of x^n. *These* are the *products of equidistant factors,* for instance the function

$$x(x-1)(x-2)\ldots(x-n+1) = (x)_n$$

called *factorial* of degree *n,* which we denoted by $(x)_n$, or the function

$$(1) \qquad x(x-h)(x-2h)\ldots(x-nh+h) = (x)_{n,h}$$

called *the generalised factorial* of degree *n* and denoted by $(x)_{n,h}$.

Of course the above definitions are valid only for positive integer values of *n.* It was *Stirling* who first recognised the importance of the factorial, but he did not use any special notation for it. The first notation is due to *Vandermonde*[10] who was the first to extend the definition of the factorial to any value of n. To do this he proceeded in the following way:

If n and m are positive integers such that $n > m$ then it follows from the definition (1) that

$$(x)_{n,h} = (x)_{m,h}\,(x-mh)_{n-m,h} \tag{2}$$

This formula will be considered valid for any value of n and m. Putting $m=0$, gives

$$(x)_{n,h} = (x)_{0,h}\,(x)_{n,h}$$

that is

$$(x)_{0,h} = 1.$$

Secondly, putting $n = 0$ into (2), it results that

$$1 = (x)_{m,h}\,(x-mh)_{-m,h}$$

so that

$$(x)_{-m,h} = \frac{1}{(x+mh)_{m,h}}$$

Later authors dealing with factorials used different notations:

Kramp's notation in 1799 was

$$x^{n/h} = x(x+h)(x+2h)\ldots(x+nh-h)$$

corresponding to our $(x+nh-h)_{n,h}$. This notation was used till the middle of the XIX. Century. *Gauss* used it, as did *Bierens de Haan,* in his "Tables d'Intégrales Définies" (1867). It is mentioned in *Nielsen,* "Gammafunktionen" 1906, p. 66; and in *Hagen,* "Synopsis der Höheren Mathematik" 1891, Vol. I, p. 118.

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[10] *N. Vandermonde,* Histoire de l'Académie Royale des Sciences, Année 1772 (quarto edition, Imprimerie Royale). Part 1, p. 489—498. "Mémoire sur des Irrationnelles de differens ordres avec application au cercle."

The notation introduced in this paper was

$$[x]^{n} = x(x-1)(x-2)\ldots(x-n+1)$$

and accordingly

$$[x]^{-n} = 1/(x+1)(x+2)\ldots(x+n).$$

This memoir has been republished in a German translation: *N. Vandermonde,* Abhandlungen aus der reinen Mathematik, Berlin 1888, pp. 67—81.

De la Vallée Poussin (loc. cit. 2, p. 332) used the following notation:

$$x^{[n]} = x(x-h)\ (x-2h)\ ,\ ,\ .\ (x-nh+h).$$

Whittaker and Robinson (loc. cit. 1, p. 8) adopt the notation:

$$[x]^n = x(x-1)\ (x-2)\ ,\ .\ .\ .\ (x-n+1)$$

Steffensen (loc. cit. 1, p. 8) introduces three different notations:

$$x^{(n)} = x(x-1)\ (x-2)\ .\ .\ .\ .\ (x-n+1)$$

$$x^{[n]} = (x+\tfrac{1}{2}n-1)\ .\ .\ .\ .\ x\ ,\ ,\ .\ .\ (x-\tfrac{1}{2}n+1)$$

$$x^{(-n)} = x(x+1)\ (x+2)\ .\ .\ .\ (x+n-1)$$

Andoyer, Encyclopédie des Sciences Mathématiques (French edition) I. 21, p. 59, introduce

$$[x]^{(n)} = (x)_{n,\ h} \quad \text{and} \quad [x]^{(-n)} = \frac{1}{(x+nh)_{n,\ h}} = (x)_{-n,\ h}$$

Milne-Thomson's notations [loc. cit. p. 25] are

$$x^{(nh)} = (x)_{n,\ h} \text{ and } x^{(-nh)} = \frac{1}{(x+nh)_{n,\ h}} = (x)_{-n,\ h}$$

Remark. Several authors have introduced special notations for $x(x-h)\ (x-2h)\ ,\ ,\ ,\ (x-nh+h)$ and for $x(x+h)\ (x+2h)\ .\ .\ ,\ (x+nh-h)$ but this is needless, since both factorials can be expressed by the same notation, only the argument being different; for instance, the above quantities would be, in our notation,

$$(x)_{n,\ h} \quad \text{and} \quad (x+nh-h)_{n,\ h}.$$

One should always use the fewest possible notations, since too many of them, especially new ones, make the reading of mathematical texts difficult and disagreeable.

Particular cases. If $h=1$ and $x=0$ we have

$$(0)_{-m} = \frac{1}{1.\ 2\ .3\ .\ .\ m}$$

In this manner the definition is extended to negative integer values of the index and to zero.

It is easy to see that

$$(x+1)_n = (x)_n + n(x)_{n-1}.$$

From this it follows that

$$(x+2)_n = (x)_n + 2n(x)_{n-1} + (n)_2(x)_{n-2}$$

and so on;

$$(x+m)_n = \sum_{i=0}^{m+1} \binom{n}{i}(m)_i(x)_{n-i}.$$

This formula, due to ***Vandermonde***, may be deduced from ***Cauchy's*** formula (14) § 22.

Putting into it $x+n$ instead of x and multiplying both members by (x), we find

$$(x+m+n)_n(x)_{-n} = \sum_{i=0}^{m+1} \binom{n}{i}(m)_i(x)_{-n}(x+n)_{n-i}$$

but in consequence of (2) we have

$$(x)_{-n}(x+n)_{n-i} = (x)_{-i};$$

therefore

$$(x+m+n)_n(x)_{-n} = \sum_{i=0}^{m+1} \binom{n}{i} (m)_i(x)_{-i}.$$

This formula is symmetrical with respect to n and m, and may be written in the following way:

$$(\alpha) \qquad (y)_n(x)_{-n} = \sum_{i=0}^{m+1} \binom{n}{i} (x)_{-i} (y-x-n)_i.$$

The above formula can be considered valid even for fractional values of n.

We may express $(y)_n(x)_{-n}$ by an infinite product in the following manner. Putting y into formula (2) instead of x, and $h=1$ and $m=-\nu$, we get

$$(y)_n = (y)_{-\nu}(y+\nu)_{n+\nu}.$$

Again applying formula (2) we have

$$(y)_n = (y)_{-\nu}(y+\nu)_n(y+\nu-n)_\nu.$$

this may be written

$$(y)_n = (y+\nu)_n \frac{(y)_{-\nu}}{(y-n)_{-\nu}}.$$

In the same manner we should have found

$$(x)_{-n} = (x+\nu)_{-n} \frac{(x)_{-\nu}}{(x+n)_{-\nu}}.$$

From the last two equations we obtain by multiplication

$$(y)_n(x)_{-n} = (y+\nu)_n (x+\nu)_{-n} \frac{(x)_{-\nu}(y)_{-\nu}}{(y-n)_{-\nu}(x+n)_{-\nu}} =$$

$$= \prod_{i=1}^{n+1} \frac{y+\nu-i+1}{x+\nu+i} \prod_{i=1}^{\nu+1} \frac{(x+n+i)\ (y-n+i)}{(x+i)(y+i)}$$

It is easy to see that

$$\lim_{\nu=\infty} \prod_{i=1}^{n+1} \frac{y+\nu-i+1}{x+\nu+i} = 1;$$

therefore it follows that

$$(\beta) \qquad (y)_n(x)_{-n} = \prod_{i=1}^{\infty} \frac{(x+n+i)(y-n+i)}{(x+i)(y+i)}$$

This infinite product may be written as follows:

$$\prod_{i=1}^{\infty} \left(1+\frac{n}{x+i}\right)\left(1-\frac{n}{y+i}\right)$$

There is no difficulty in showing that it is convergent.

Application. 1. Vandermonde determined by aid of the above formulae the value of the following integral:

$$(\gamma) \qquad J = \nu x \int_0^1 t^{\nu x-1}(1-t^\nu)^y\, dt.$$

Using ***Newton's*** formula we find

$$J = \sum_{i=0}^{y+1} \frac{(-1)^i \binom{y}{i} x}{x+i}.$$

Putting $n = -x$ into formula (a) we have

$$(x)_x(y)_{-x} = \sum_{i=0}^{y+1} \binom{-x}{i} (y)_i(x)_{-i} = \sum_{i=0}^{y+1} \frac{(-1)^i\binom{y}{i}}{x+i} x.$$

Finally in consequence of (β) we get

$$(\delta) \qquad J = (x)_x (y)_{-x} = \prod_{i=1}^{\infty} \frac{i(x+y+i)}{(x+i)\ (y+i)}\,,$$

In the particular case of $\nu=2$, $x=\frac{1}{2}$ and $y=-\frac{1}{2}$ equation (γ) gives

$$J = \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \tfrac{1}{2}\pi$$

and according to formula (6)

$$J = (\tfrac{1}{2})_{1/2}(-\tfrac{1}{2})_{-1/2} = \tfrac{1}{2}\pi = \prod_{i=1}^{\infty} \frac{(2i)^2}{(2i-1)\ (2i+1)}$$

This is **Wallis's** well-known formula.

Since from formula (2) it follows that

$$(\tfrac{1}{2})_{1/2} = (\tfrac{1}{2})_1(-\tfrac{1}{2})_{-1/2}$$

hence

$$(-\tfrac{1}{2})_{-1/2} = \sqrt{\pi}\,.$$

We shall see in § 17 that $(x)_x = \Gamma(x+1)$ so that the above quantity is equal to $\Gamma(\frac{1}{2})$, moreover that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Application 2. Putting y-1 instead of y and $\nu=1$ into formula (y) we have according to (1) § 24

$$J = x \int_0^1 t^{x-1}\ (1-t)^{y-1}\ dt = x\, B(x,y).$$

Therefore in consequence of the preceding formulae we may write

$$(\varepsilon) \qquad B(x,y) = \sum_{i=0}^{y} \frac{(-1)^i \binom{y-1}{i}}{x+i} = \frac{(x)_x\ (y-1)_{-x}}{x}$$

Finally in consequence of (d) we have

$$(\zeta) \qquad B(x,y) = \frac{1}{x} \prod_{i=1}^{\infty} \frac{i\ (x+y-1\ +i)}{(x+i)\ (y-1+i)}\,,$$

and in the particular case of $x=\frac{1}{2}$ and $y=\frac{1}{2}$,

$$B(\tfrac{1}{2},\tfrac{1}{2}) = 2 \prod_{i=1}^{\infty} \frac{(2i)^2}{(2i-1)(2i+1)} = \pi.$$

Differences of a factorial. The first difference is

$$\Delta(x)_n = (x+1)_n - (x)_n = [x+1-x+n-1]\,(x)_{n-1} = n(x)_{n-1}\,.$$

The second difference:

$$\Delta^2(x)_n = \Delta[\Delta(x)_n] = n(n-1)\,(x)_{n-2} = (n)_2(x)_{n-2}\,.$$

The higher differences are obtained in the same manner and we get

(3) $$\Delta^m(x)_n = (n)_m(x)_{n-m}\,.$$

Differences of the generalised factorial

$$\underset{h}{\Delta}(x)_{n,h} = [x+h-x+nh-h]\,(x)_{n-1,h} = hn(x)_{n-1,h}$$

and

(4) $$\underset{h}{\Delta}^m(x)_{n,h} = h^m(n)_m(x)_{n-m,h}\,.$$

Differences of the factorial with negative arguments. It has been shown in § 2 that the differences of a function with negative arguments are not of the same form as the differences of the same function with positive arguments, Let us determine for instance the differences of the following generalised factorial: $(-x)_{n,h}$, The simplest way is, first to transform this factorial into another with positive argument and then apply the above rule.

Since

$$(-x)_{n,h} = (-1)^n(x+nh-h)_{n,h}\,.$$

we have

$$\underset{h}{\Delta}^m(-x)_{n,h} = (-1)^n h^m(n)_m(x+nh-h)_{n-m,h}\,.$$

Now the factorial in the second member may be trans-formed into one with a negative argument, so that finally we have

$$\underset{h}{\Delta}^m(-x)_{n,h} = (-1)^m h^m(n)_m(-x-mh)_{n-m,h}\,.$$

That is,in the m -th difference the argument is diminished by mh.

In the particular case of $h=1$, this formula gives

(5) $$\Delta^m(-x)_n = (-1)^m(n)_m(-x-m)_{n-m}\,.$$

Differences of the reciprocal factorial. The differences of $(x)_{-n,h} = 1/(x+nh)_{n,h}$ are determined in the usual manner;

$$\Delta(x)_{-n,h} = \frac{1}{(x+nh+h)_{n,h}} - \frac{1}{(x+nh)_{n,h}} =$$

$$= \frac{1}{(x+nh+h)_{n+1,h}}\left[x+h-x-nh-h\right];$$

therefore

$$\Delta(x)_{-n,h} = -hn(x)_{-n-1,h}.$$

Repeating the operation m times we find

$$\text{(6)} \qquad \Delta^m(x)_{-n,h} = (-1)^m\, h^m(n+m-1)_m(x)_{-n-m,h} =$$

$$= h^m(-n)_m\,(x)_{-n-m,h}.$$

Hence the proceeding is exactly the same as in formula (4) in the case of positive indices.

Particular case. Let $h = 1$ and $n = 1$.

$$\text{(7)} \qquad \Delta^m(x)_{-1} = \Delta^m\frac{1}{x+1} = (-1)^m m!(x)_{-m-1} =$$

$$= \frac{(-1)^m m!}{(x+1)(x+2)\ldots(x+m+1)}.$$

Mean of a factorial. We have

$$\mathbf{M}(x)_n = \tfrac{1}{2}\left[(x+1)_n + (x)_n\right] = \tfrac{1}{2}(x)_{n-1}[2x+2-n].$$

The higher means are complicated.

Computation of factorials. If in an expansion x, $(x)_2$, $(x)_3$, ... $(x)_n$ are needed, to calculate them it is best to multiply first x by $(x-1)$, then to multiply the result $(x)_2$ by $(x-2)$ and so on. If $x, x^2, \ldots, x^n$ were wanted we should proceed also by multiplication. The amount of work is in both cases nearly the same.

If only $(x)_n$ is required, and x is an integer, as it generally will be, then we may obtain

$$\log(x)_n = \log x! - \log(x-n)!$$

using *Duarte*'s Tables;[11] and the computation will not be longer than that of $\log x^n$.

[11] F. *J. Duarte*, Nouvelles Tables de log n! à 33 décimales depuis $n=1$ à $n=3000$. Genève, 1927.

§ 17. The Gamma-Function. The factorial may be considered as a function of the index. Then in (n), the independent variable is x. The differences of this function with respect to x are not simple; since

$$\Delta(n)_x = (n)_x[n-x-1].$$

Hence the higher differences are complicated. This function is seldom considered in the general case; on the other hand in the particular case, if $n=x$ and x is an integer, we get a very important function,

$$(x), = 1.\ 2.\ 3.\ ,\ .\ .\ x.$$

In ***Kramp's*** notation [loc. cit. 10] this function would be $1^{x|1}$.

Considering its importance, shorter notations were introduced. For instance ***Kramp*** used later (in 1808)

$$1 . 2 . 3 \ldots\ldots x = x!$$

We will adopt this notation, which is the most in use to-day. In England in the past the notation $\lfloor x$ was often used for $1 . 2 . 3 . , , x.$

The first difference of $x!$ is simple:

$$Ax! = x . x!$$

but the higher differences are complicated.

The *gamma function* denoted by I’(x) is given for $x > 0$ by the following definite integral:

$$(1) \qquad \Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

It may be considered, we shall see, as a generalisation of $(x-1)_{x-1}$.

From formula (1) it follows that $\Gamma(1)=1$, and by integration by parts we get

$$\Gamma(x) = (x-1) \int_0^\infty e^{-t}\, t^{x-2}\, dt = (x-1)\ \Gamma(x-1).$$

The difference equation thus obtained

$$(2) \qquad \Gamma(x) = (x-1)\ I'(x-1)$$

equivalent to

$$\Delta\Gamma(x) = (x-1)\ I'(x)$$

may be considered as a characteristic property of the gamma function. If x is an integer, then the solution of (2) is given by

$$\Gamma(x) = (x-1)!$$

Therefore $\Gamma(x+1)$ may be considered as a generalisation of $x!$

Another definition of the gamma function may be had starting from the definite integral below (beta-function) in which n is a positive integer. By successive integration by parts we get

$$\int_0^1 u^{x-1}(1-u)^n\,du = \frac{n!}{(x+n)_{n+1}}$$

let us put now $u = t/n$, then

$$\text{(3)}\qquad \int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt = \frac{n!\,n^x}{(x+n)_{n+1}}.$$

If n increases indefinitely, we have

$$\lim_{n=\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}.$$

To determine the limit of an integral containing a parameter n, some caution must be used[12] but here the proceeding is justified, and we shall have

$$\text{(4)}\qquad \Gamma(x) = \lim_{n=\infty}\frac{n!\,n^x}{(x+n)_{n+1}}$$

valid for every value of x.

If x is a integer, then formula (4) may be deduced from formula (2), § 16, indeed from the latter it follows that

$$\frac{n!\,n^x}{(x+n)_{n+1}} = \frac{n^x\,(n)_{n+1-x}\,(x-1)_{x-1}}{(x+n)_x\,(n)_{n+1-x}}$$

since

$$\lim_{n=\infty}\frac{n^x}{(x+n)_x} = 1 \quad\text{and}\quad (x-1)_{x-1} = \Gamma(x).$$

Hence formula (4) is demonstrated.

A third definition of the gamma-function is given by[13]

[12] See **Hobson**, Functions of a Real Variable, Cambridge. 1907, p. 599

[13] The demonstration of this formula may be found. for instance, in E. *Artin*, Einführung in die Theorie der Gammafunktion, 1931. Berlin p 14

$$(5) \qquad \frac{1}{\Gamma(x)} = x\, e^{Cx} \prod_{\nu=1}^{\infty} \left(1+\frac{x}{\nu}\right) e^{-\frac{x}{\nu}}$$

where C is Euler's constant. The formula holds for every real or complex value of x.

Gauss has given a multiplication formula for the gamma function:

$$(6) \qquad \Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\,\Gamma\left(z+\frac{2}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = \frac{(2\pi)^{1/2(n-1)}}{n^{nz-1/2}}\,\Gamma(nz).$$

A demonstration is given in *Arfin* [loc. cit. 13. p. 18].

Owing to ***Euler's*** formula, the values of the gamma function corresponding to negative arguments may be expressed by those corresponding to positive arguments:

$$(7) \qquad \Gamma(-x) = -\frac{\pi}{\sin\pi x\ \Gamma(x+1)}\cdot$$

[*See* ***Artin*** p. 25]. From this formula we easily deduce

$$(8) \qquad \Gamma(x)\ \Gamma(1-x) = \frac{\pi}{\sin \pi x}\cdot$$

Computation of $\Gamma(z)$. Equation (2) gives

$$\Gamma(z) = (z-1)\Gamma(z-1) = (z-1)_2\,\Gamma(z-2) = \ldots$$
$$= (z-1)_n\ \Gamma(z-n).$$

Therefore if z-1 $> n >$ z-2, the argument $z-n=x$ of the gamma function will be such that $1 < x < 2$. Consequently it is sufficient to have tables for these values.

The best tables are those of ***Legendre*** giving $\log\Gamma$ (x) from $x=1{\cdot}000$ to $x=2{\cdot}000$ and the corresponding first three differences, to twelve figures.[14]

Particular values of the gamma function. If x is a positive integer, then we have seen that $\Gamma(x) = (x-1)!$; moreover from (8) it follows that

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[14] *A.* **M. Legendre, Tables of the logarithms of the Complete Γ-function to twelve figures. Tracts for computers. Cambridge University Press. 1921,**

$$\Gamma(-x) = \infty,\ \Gamma(0) = \infty \text{ and } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

The ***definition of the factorial may be extended*** by aid of gamma functions for every value of x and n by the following formula:

$$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} \tag{9}$$

This gives, it n is a positive integer,

$$(x)_n = x(x-1)(x-2) \ldots, (x-n+1).$$

For $n = 0$ (9) will give $(x)_0 = 1$; if n is a negative integer $n = -m$, from (9) we obtain

$$(x)_{-m} = \frac{1}{(x+1)(x+2)\ldots,(x+m)} = \frac{1}{(x+m)_m}$$

conformably to our definitions of § 16.

Moreover, the extension of the definition of the factorial by gamma functions conforms to the extension by formula (2) of § 16; indeed, putting the values (9) into this formula, we find the following identity:

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} \cdot \frac{\Gamma(x-m+1)}{\Gamma(x-n+1)}.$$

§ 18. Incomplete Gamma-Function. The definition of this function is the following:

$$\Gamma_m(p+1) = \int_0^m t^p e^{-t}\, dt. \tag{1}$$

Pearson introduced the function $I(u,p)$, obtained by dividing the incomplete gamma function by the corresponding complete function

$$I(u,p) = \frac{\Gamma_m(p+1)}{\Gamma(p+1)} = \frac{1}{\Gamma(p+1)} \int_0^m t^p e^{-t}\, dt$$

where u **is** an abbreviation for $m/\sqrt{p+1}$.

Pearson published a double-entry table giving the values of the function $I(u,p)$, to seven decimals, from $p = -1$ to $p = 50$

(where $\Delta p = 0{\cdot}2$), for the necessary values of u, (where $\Delta u = 0{\cdot}1$).[15]

The table also contains several formulae enabling us to calculate $I(u,p)$ corresponding to values of u and p outside the range of the table.

Equation (1) gives by integration by parts

$$(2) \qquad I(u,p) = -\frac{e^{-m} m^p}{\Gamma(p+1)} + \int_0^m \frac{e^{-t} t^{p-1}}{\Gamma(p)} dt$$

From this we conclude that

$$(3) \qquad I\left(\frac{m}{\sqrt{p+1}}, p\right) < I\left(\frac{m}{\sqrt{p}}, p-1\right);$$

putting

$$(4) \qquad \psi(m,x) = \frac{e^{-m} m^x}{\Gamma(x+1)}$$

we may write

$$(5) \qquad \Delta_x I\left(\frac{m}{\sqrt{x+1}}, x\right) = -\psi(m,x).$$

The function $\psi(m,x)$ has also been tabulated. There are for instance tables of this function in ***Pearson's*** Tables for Statisticians and Biometricians, [vol. I, pp. **113—121.]** from $m = 0{\cdot}1$ to $m = 15$ (where $\Delta m = 0{\cdot}1$) for all the necessary values of x (if $\Delta x = 1$).

It can be shown by the tables of the function $I(u,p)$ that for $u = \sqrt{p}$ or $m = \sqrt{p(p+1)}$ we have $I(\sqrt{p}, p) < \frac{1}{2}$ and for $u = \sqrt{p+1}$ or $m = p+1$ we have $I(\sqrt{p+1}, p) > \frac{1}{2}$.

If p is an integer, by repeated integration by parts we obtain from (1)

$$(6) \qquad I(u,p) = 1 - \sum_{x=0}^{p+1} \frac{m^x}{x!} e^{-m} = 1 - \sum_{x=0}^{p+1} \psi(m,x).$$

Remark. From this we may deduce the sum of a first section of the series e^m; indeed

$$\sum_{x=0}^{p+1} \frac{m^x}{x!} = e^m [1 - I(u,p)].$$

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[15] **K. Pearson, Tables of the Incomplete Gamma-Function, London, 1922.**

Moreover from (4) we get

$$\int_0^m \psi(t,x)\,dt = -\psi(m,x) + \int_0^m \psi(t,x-1)\,dt.$$

§ 19. The Digamma Function. *Pairman* called the derivative of log $\Gamma(x+1)$ digamma function and denoted it by F(x). (Digamma is an obsolete letter of the Greek alphabet.)[16]

To obtain the derivative of log $\Gamma(x+1)$ we shall start from formula (4) of § 17. It will give

$$\log\Gamma(x+1) = \lim_{n=\infty}\left[\log n! + (x+1)\log n - \sum_{\nu=1}^{n+2}\log(x+\nu)\right].$$

It can be shown that the derivative of the second member may be obtained by derivation term by term, moreover that after derivation the second member tends to a limit if n increases indefinitely; therefore we shall have

$$\text{(2)}\qquad F(x) = \mathrm{D}\log\Gamma(x+1) = \lim_{n=\infty}\left[\log n - \sum_{\nu=1}^{n+2}\frac{1}{x+\nu}\right].$$

From this we deduce

$$\text{(3)}\qquad F(0) = \lim_{n=\infty}\left[\log n - \sum_{\nu=1}^{n+2}\frac{1}{\nu}\right] = -C$$

where C is *Euler's* constant. It may be computed by formula (3) as exactly as required. We find

$$C = 0'57721\ 56649\ 01532\ 86060\ 65120\ 90082.$$

A function denoted by $\psi(x)$, which differs but little from the digamma function, has already been considered, first by *Legendre* in 1809 and later by *Poisson* and *Gauss*. *(See N. Nielsen*, Handbuch der Gammafunktion, p. 15). We have

$$\psi(x) = F(x-1).$$

First difference of the digamma function:

$$\Delta F(x) = \mathrm{D}\Delta\log\Gamma(x+1) = \mathrm{D}\log\frac{\Gamma(x+2)}{\Gamma(x+1)} = \mathrm{D}\log(x+1) = \frac{1}{x+1}.$$

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[16] E. *Pairman*, Tables of the Digamma- and Trigamma-Functions. Tracts for Computers, Cambridge University Press, 1919.

Therefore we have $F(1) = F(0) + 1$ and if n is an integer

$$F(n) = F(0) + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}.$$

Hence $F(x\text{-}1)$ is a function whose difference is equal to $1/x$; it is analogous to logx, whose derivative is $1/x$.

The higher differences of the digamma function are easily obtained by aid of formula (7) § 16; we find

$$(4)\quad \Delta^m F(x) = \Delta^{m-1} \frac{1}{x+1} = \frac{(-1)^{m-1}(m-1)!}{(x+m)_m} = (-1)_{m-1}(x)_{-m}.$$

The values of the digamma function corresponding to negative arguments may be expressed by those of positive arguments. Starting from formula (8) of § 17, we have

$$\log\Gamma(x) + \log\Gamma(1\text{-}x) = \log\pi - \log\sin\pi x;$$

by derivation we obtain

$$(5)\qquad F(-x) = F(x\text{-}4) + \pi\cot\pi x.$$

To deduce a ***Multiplication Formula*** for the digamma function, we start from formula (6) of § 17; taking the logarithms of both members of the equation; we find

$$\sum_{i=0}^{n} \log\Gamma\left(z + \frac{i}{n}\right) = \tfrac{1}{2}(n\text{-}1)\log 2\pi - (nz - \tfrac{1}{2})\log n + \log\Gamma(nz).$$

'Derivation with respect to z gives

$$\sum_{i=0}^{n} D\log\Gamma\left(z + \frac{i}{n}\right) = -n\log n + D\log\Gamma(nz)$$

which gives, putting $z = x + \frac{1}{n}$ and $\mu = n\text{-}i\text{-}l$

$$(6)\qquad F(nx) = \log n + \frac{1}{n}\sum_{\mu=0}^{n} F\left(x - \frac{\mu}{n}\right).$$

Remark. Starting from formula (5) § 17, we may obtain another expression for the digamma function. Indeed, taking the logarithms we have

$$-\log\Gamma(x) = \log x + cx + \sum_{\nu=1}^{\infty}\left[\log\left(1+\frac{x}{\nu}\right) - \frac{x}{\nu}\right].$$

Putting in $x+1$ instead of x, since derivation term by term is justified, we get

$$F(x) = -\frac{1}{x+1} - C + \sum_{\nu=1}^{\infty}\left[\frac{1}{\nu} - \frac{1}{x+1+\nu}\right].$$

This may be written more simply

$$(7) \qquad F(x) = -C + \sum_{\nu=1}^{\infty}\left[\frac{1}{\nu} - \frac{1}{x+\nu}\right].$$

Computation of the digamma function. Pairman's Tables quoted above give the values of the digamma function and its central differences δ^2 and δ^4, to eight decimals, from $x=0{\cdot}00$ to $x=20{\cdot}00$ where $\Delta x=0{\cdot}02$. If $F(z)$ is needed, and $z>20$, then we will use formula (6) and put $z=nx$ so as to have $\frac{z}{n} = x < 20$. It may be useful to remark that the logarithm figuring in this formula is a *Napier's* logarithm.

§ 20. The Trigamma Function. *Pairman* denoted the second derivative of log $\Gamma(x+1)$ by F (x), calling it trigamma function.

$$F(x) = D\,F(x) = D^2 \log \Gamma(x+1).$$

Hence starting from formula (2) or (7) of § 19 we get

$$(1) \qquad F(x) = \sum_{\nu=1}^{\infty} \frac{1}{(x+\nu)^2}.$$

Prom this we deduce

$$F(0) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2}.$$

In the paragraph dealing with *Bernoulli* polynomials we shall see that

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6}.$$

Therefore we have

$$F(0) = \frac{\pi^2}{6} = 1.64493\ 40668\ 48226\ 43647\ 24151\ 66646\ldots$$

First 'difference of the trigamma function:

$$\Delta F(x) = D\Delta F(x) = D\frac{1}{x+1} = -\frac{1}{(x+1)^2}.$$

The higher differences are complicated.

From this we obtain $F(1) = F(0) - 1$ and if n is an integer

$$F(n) = F(0) - 1 - \frac{1}{4} - \frac{1}{9} - \dots - \frac{1}{n^2}.$$

The values of the trigamma function corresponding to *negafive arguments* may be expressed by those of positive arguments. Starting from formula (5) § 19, we obtain by derivation with respect to x

$$(2) \qquad F(-x) = \frac{\pi^2}{\sin^2 \pi x} - F(x-1).$$

Multiplication formula for the trigamma function. Starting from formula (6) § 19, derivation gives

$$(3) \qquad F(nx) = \frac{1}{n^2} \sum_{\mu=0}^{n} F\left(x - \frac{\mu}{n}\right)$$

Compufafion of the frigamma function. In *Pairman's* Tables we find $F(z)$ and its central differences δ^2 and δ^4 to eight decimals from $z = 0\cdot00$ to $z = 20\,00$ (where $\Delta z = 0\cdot02$).

If z > 20 we shall use formula (3) putting $z = nx$ *so* as to have $\frac{z}{n} = x < 20.$

§ 21. Expansion of $\log \Gamma(x+1)$ into a power series.

If $|x| < 1$ then we shall have

$$\log \Gamma(x+1) = \sum_{m=0}^{\infty} \frac{x^m}{m!} [D^m \log \Gamma(x+1)]_{x=0}.$$

We have seen in § 20 that

$$D^2 \log \Gamma(x+1) = \sum_{\nu=1}^{\infty} \frac{1}{(x+\nu)^2}$$

If 1 > I xl this series *is uniformly convergent,* and the derivative of the second member may be obtained term by term, so that we get for $m>1$

$$(1) \qquad D^m \log \Gamma(x+1) = \sum_{\nu=1}^{\infty} \frac{(-1)^m (m-1)}{(x+\nu)^m} 1$$

and therefore

$$[\,D^m \log \Gamma(x+1)\,]_{x=0} = (-1)^m\,(m-1)!\ \sum_{\nu=1}^{\infty} \frac{1}{\nu^m}.$$

Let us Write

$$s_m = \sum_{\nu=1}^{\infty} \frac{1}{\nu^m}.$$

In the paragraph dealing with ***Bernoulli*** polynomials we shall see that

$$s_{2n} = \frac{(2\pi)^{2n}}{2\,(2n)!} B_{2n}$$

where B_{2n} is a ***Bernoulli*** number.

There is no known formula for s_{2n+1} but ***Stieltjes*** computed these sums to 32 decimals for $m = 2, 3, 4, \ldots, 70$.[17]

$\log \Gamma(1) = 0$ and $[\,D \log \Gamma(x+1)\,]_{x=0} = F(0) = -C$.

Hence

$$\text{(2)} \qquad \log \Gamma(x+1) = -Cx + \sum_{m=2}^{\infty} (-1)^m \frac{s_m}{m} x^m$$

if $-1 < x \leq 1$.

Putting $x = 1$ into formula (2) we get

$$C = \sum_{m=2}^{\infty} (-1)^m \frac{s_m}{m}$$

The series is convergent, but the convergence is very slow.

§ 22. The Binomial Coefficient. In Infinitesimal Calculus sequences of functions which satisfy the condition

$$(a) \qquad D\, f_n(x) = f_{n-1}(x)$$

are very important. Such sequences are for instance

$$f_n(x) = \frac{x^n}{n!}$$

and

$$f_n(x) = \varphi_n(x)\ ; \qquad f_n(x) = E_n(x)$$

where $\varphi_n(x)$ is the ***Bernoulli*** polynomial of the first kind, of degree n, **and** $E_n(x)$ the ***Euler*** polynomial.

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[17] *T. J. Stieltjes*, Acta Mathematica, **Vol. 10, 1887. Table des valeurs des sommes**

$$s_m = \sum_{n=1}^{\infty} n^{-m}$$

The sequences satisfying

$$\mathrm{D}\ f_n(x) = f_{n+1}\ (x)$$

a.re also important. Such sequences are for instance

$$f_n(x) = \frac{(-1)^{n-1}\ (n-1)\ !}{x^n}$$

and

$$f_n(x) = e^{-k^2x^2} H_n\ (x)$$

where $H_n(x)$ is the *Hermite* polynomial of degree n (§ 147); and

$$f_n\ (x) = \frac{m^x}{x!}\ e^{-m}\ G_n(x)$$

where $G_n(x)$ is the polynomial of degree n of § 148.

If *f,(x)* is a polynomial of degree n then expanding it in the following manner

$$(\beta) \qquad f''(X) = c_0 \frac{x^n}{n!} + c_1 \frac{x^{n-1}}{(n-1)}, + \ldots\ldots + c_{n-1}\ x + c_n$$

we have in consequence of (α)

$$f_{n-1}\ (x) = c_0 \frac{x^{n-1}}{(n-1)}, + c_1 \frac{x^{n-2}}{(n-2)!} + \ldots\ldots + c_{n-1}$$

from this we conclude that the coefficients c_i of the polynomial (β) are independent of the **degree** n of the polynomial, so that they may be determined once for all. This is a great simplification.

If a function $F(x)$ is expanded into a series of *f,(x)* functions (α)

$$F(x) = \sum_{n=0}^{\infty} c_n f_n(x)$$

then we may easily determine the integral and the derivatives of this function. Indeed we have on the hypothesis that these operations are permited term by term

$$\int F(x)\ dx = k + \sum_{n=0}^{\infty} c_n f_{n+1}(x)$$

and

$$\mathrm{D}^m F(x) = \sum_{n=m}^{\infty} c_n f_{n-m}(x).$$

In the same manner, in the Calculus of Finite Differences the sequences satisfying

$$(\gamma) \qquad \Delta f_n(x) = f_{n-1}(x)$$

are important. Such sequences are, for instance, the binomial coefficients

$$f_n(x) = \frac{x(x-1)(x-2)\ldots(x-n+1)}{1.2.3\ldots\ldots n} = \binom{x}{n}$$

and

$$f_n(x) = \psi_n(x) \quad \text{and} \quad f_n(x) = \zeta_n(x)$$

where $\psi_n(x)$ *is* the *Bernoulli* polynomial of the second kind of degree n, and $\zeta_n(x)$ the ***Boole*** polynomial.

Equally important are the sequences which satisfy the **con-** di tion

$$\Delta f_n(x) = f_{n+1}(x).$$

Such a sequence is for instance

$$f_n(x) = \frac{(-1)^{n-1}\, 1.2.3\ldots\ldots(n-1)}{(x+1)(x+2)(x+3)\ldots(x+n)}.$$

If $f_n(x)$ is a polynomial of degree n then, expanding it in the following manner,

$$(\delta) \qquad f_n(x) = c_0\binom{x}{n} + c_1\binom{x}{n-1} + \ldots\ldots + c_{n-1}\binom{x}{1} + c_n$$

we have in consequence of (y)

$$f_{n-1}(x) = c_0\binom{x}{n-1} + c_1\binom{x}{n-2} + \ldots\ldots + c_{n-1}.$$

Therefore we conclude that the coefficients c_i of the polynomials (δ) are independent of the degree n of the polynomial; they may be determined once for all, which is a great simplification.

If a function F(x) is expanded into a series of such functions

$$F(x) = \sum_{n=0}^{\infty} c_n f_n(x)$$

then we may easily determine the differences of $F(x)$

$$\Delta^m F(x) = \sum_{n=m}^{\infty} c_n f_{n-m}(x)$$

and moreover as we shall see ($32) also the indefinite sum of $F(x)$.

The binomial coefficient is without doubt the most important function of the Calculus of Finite Differences, hence it is necessary to adopt some brief notation for this function. We accepted above the notation of ***J. L. Raabe*** [Journal fiir reine und angewandte Mathematik 1851, Vol. 42, p. 350.] which is most in use now, putting

$$(1) \qquad \binom{x}{n} = \frac{x(x-1)(x-2)\ldots(x-n+1)}{1.2.3\ldots n}$$

The binomial coefficients corresponding to integer values of x and n have been considered long ago. ***Pascal's*** "Triangle Arithmétique", printed in 1654, is formed by these numbers; but they had been published already a century earlier in ***Nicolo Tartaglia's*** "General Trattato di Numeri i Misure" (Vinegia 1556, Parte II, p. 70, 72).

In ***Chu Shih-chieh's*** treatise „Szu-yuen Yü-chien" (The Precious Mirror of the Four Elements), published in 1303, they are indicated as an old invention, ***Omar Khayyam*** of Nishapur (d. 1123) knew them already in the eleventh century; and this is our earliest reference for the subject.

Omar Khayyam's fame as a poet and philosopher seems to have thrown his eminence in mathematics and astronomy into the shade; nevertheless it must be recorded that these sciences owe much to him. [***Woepcke,*** L'Algèbre d'Omar Alkhyâmî, Paris 1851.]

In those early times no mathematical notation was used for these numbers. It seems that the first notation used was that of *Euler.*[18]

Condorcet in the article "Binome" of the Encyclopddie

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[18] *Euler* first used the notation $\left[\frac{x}{n}\right]$ in Acta Acad. Petrop. V. 1781; and then $\left(\frac{x}{n}\right)$ in Nova Acta Acad. Petrop. XV. 1799-1802. *Raabe's* notation $\binom{x}{n}$ is a slight modification of the second. It is used for instance in:

Bierens de Haan, Tables d'Intégrales définies, Leide, 1867.
Hagen, Synopsis Vol. I. p. 57. Leipzig, 1891.
Pascal, Repertorium Vol. I. p. 47, Leipzig, 1910.
Encyclopidie der Math, Wissenschaften, 1898—1930.
L. M. Milne Thomson, Calculus of Finite Differences, London, 1933.
G. H. Hardy, Course of Pure Mathematics, p. 256, London, 1908.

Mdthodique [Tome I, 1784, p. 223] used the notation $(x)^n$ corresponding to our $\binom{x}{n}$; but later, in his "Essai sur l'Application de l'Analyse à la Pluralité des voix" [Paris, 1785, p. 4], he adopted the notation $\frac{x}{n}$ for $\binom{x}{n}$.

Meyer *Hirsch* [Integral Tables, London 1823], uses the Hindenhurgian notation, "well known in Germany" at that time. It was the following:

$$\binom{x}{1} = {}^xA, \qquad \binom{x}{2} = {}^xB, \qquad \binom{x}{3} = {}^xC, \quad \ldots .$$

$$\binom{x}{x} = {}^xM, \qquad \binom{x}{x-1} = {}^xM^{-1}, \quad \binom{x}{x-2} = {}^xM^{-2}, \ldots, .$$

French authors generally use the following notation:

$$\binom{x}{n} = C_x^n.$$

Schlömilch [Compendium, 4 th ed. Vol. 1. p. 35], *Whitfaker* and *Robinson* [loc. cit. 1. p. 43] use the following notation:

$$\binom{x}{n} = (x)_n .$$

Mathews [Encyc. Brit. 11th ed. Vol. I. p. 607] and G. *J. Lidstone* [Journ. Inst. of Actuaries Vol. LX111 p. 59, 1932] use

$$\binom{x}{n} = x_{(n)} ,$$

The following notation has also been proposed:

$$\binom{x}{n} = (x,n).$$

This notation, though seldom used, would be the best; indeed, if x and n are complicated expressions, for instance fractions, then in all other notations the printing is very difficult; moreover, the formulae lose their clearness.

The binomial coefficient $\binom{x}{n}$ was considered first as the coefficient of t^n in the expansion of $(1+t)^x$, for instance by

Tartaglia, and later, as the number of combinations of x elements taken n by n by *Pascal*.

The definition (1) of the binomial coefficient given above may be extended to every value of x, provided that n be a positive integer.

From (1) we get, putting $x=n$,

$$\binom{n}{n} = 1.$$

For $x=0$ we have

$$\binom{0}{n} = 0.$$

Moreover, if x is a positive integer smaller than n, from (1) it follows that

$$\binom{x}{n} = 0.$$

On the other hand, if x is a positive integer larger than n, we deduce

$$\binom{x}{n} = \frac{x!}{n!(n-x)!} = \binom{x}{x-n} \tag{2}$$

Putting into (1) x = $-z$ we obtain

$$\binom{-z}{n} = (-1)^n \binom{z+n-1}{n} \tag{3}$$

Remark. If z is a positive integer, then the absolute value of $\binom{-z}{n}$ is equal to the number of combinations with repetition of z elements taken n by n.

We may extend the definition of the binomial coefficient $\binom{x}{n}$ to every real or complex value of x and n, by writing

$$\binom{x}{n} = \frac{(x)_n}{(n)_n} = \frac{\Gamma(x+1)}{\Gamma(n+1)\,\Gamma(x-n+1)}. \tag{4}$$

It is easy to show that, in the cases considered before, the definition (4) leads to the same results as definition (1). Moreover, putting n = 0 into formula (4) we have

$$\binom{x}{0} = 1.$$

This is true even if x is also equal to zero $\binom{0}{0} = 1.$

If n is a negative integer and if x is not a negative integer, putting $n = -m$ we get

$$\binom{x}{-m} = \frac{\Gamma(x+1)}{\Gamma(1-m)\,\Gamma(x+m+1)} = 0,$$

since then $\Gamma\,(1\text{-}m) = \infty$,

If x and n are negative integers, then putting into (4) $x = -z$ and $n = -m$ we find

$$\binom{-z}{-m} = \frac{\Gamma(1-z)}{\Gamma(1-m)\,\Gamma(m+1-z)}\,.$$

By aid of formula (8) §17, remarking that

$$\sin \pi z = \sin[\pi\,(z\text{-}m) + m\pi] = (-1)^{z-m} \sin m\pi,$$

we get

$$\binom{-z}{-m} = \frac{(-1)^{z+m}\,\Gamma(m)}{\Gamma(z)\,\Gamma(m+1-z)} = (-1)^{z+m}\binom{m-1}{z-1}.$$

From this we conclude, z and m being positive- integers, that if $z>m$ or if $z=0$, then

$$\binom{-z}{-m} = 0, \qquad \binom{0}{-m} = 0$$

and if $z=m$

$$\binom{-m}{-m} = 1.$$

Besides from (4) it follows that $\binom{n}{n} = 1$ for every value of n.

The definition of the binomial **coefficient** may be extended also by ***Cauchy's*** formula. (See § 14, and N. *Nielsen,* **Gamma-**function **p.** 158.) If $x > -1$ then

$$(5)\quad \frac{2^{x+1}}{\pi}\int_0^{\frac{1}{2}\pi} \cos^x\varphi\,\cos(x-2n)\varphi\,d\varphi = \frac{\Gamma(x+1)}{\Gamma(n+1)\,\Gamma(x-n+1)} = \binom{x}{n}$$

Differences of the binomial coefficient with respect to the upper index. If n is a positive integer, we have

(6) $$\Delta\binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1}\left[\frac{x+1-x+n-1}{n}\right] = \binom{x}{n-1}$$

and therefore

$$\Delta^m\binom{x}{n} = \binom{x}{n-m}.$$

Formula (6) may be written

(7) $$\binom{x+1}{n} = \binom{x}{n-1} + \binom{x}{n}.$$

This gives a very good rule for computing step by step a table of the numbers $\binom{x}{n}$,

Equation (7) shows that the numbers $\binom{x}{n}$ form a solution of the homogeneous partial difference equation

(7') $$f(n+1,x+1) - f(n+1,x) - f(n,x) = 0.$$

The general solution of this equation is given later (§ 182 and § 183) where it is shown that starting from the initial conditions $f(0,0) = 1$ and $f(n,0) = 0$ if $n>0$ or $n<0$, we find that $f(n,x)$ is equal to the coefficient of t^n in the expansion of $(1+t)^x$.

Differences of the function with negative argument. We have seen that

$$\binom{-x}{n} = (-1)^n\binom{x+n-1}{n};$$

therefore we have

$$\Delta\binom{-x}{n} = (-1)^n\binom{x+n-1}{n-1} = -\binom{-x-1}{n-1}$$

and moreover

(8) $$\Delta^m\binom{-x}{n} = (-1)^m\binom{-x-m}{n-m}.$$

Let **us remark** that, if the argument of the binomial coefficient **is negative,** it is diminished by one in the difference. This will be useful later.

Differences of the reciprocal binomial coefficient. Since this latter may be written

$$\frac{1}{\binom{x}{n}} = \frac{n!}{(x)_n} = n!\,(x-n)_{-n}$$

we may apply formula (6) § 16, giving the differences of a reciprocal factorial. We find

$$\Delta^m \frac{1}{\binom{x}{n}} = n!\,\Delta^m (x-n)_{-n} = n!\,(-n)_m\,(x-n)_{-n-m} =$$
$$= \frac{(-1)^m\, n}{(n+m)\binom{x+m}{n+m}}\,.$$

***Generalised** binomial coefficients.* We will denote the generalised binomial coefficient of degree n by $\binom{x}{n}_h$; **its definition** is

$$\binom{x}{n}_h = \frac{x(x-h)\,(x-2h)\,\ldots\,(x-nh+h)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,n},$$

To determine the differences of this function in a system in which the increment of x is equal to h, **that** is $\Delta x = h$, we will write

$$\binom{x}{n}_h = \frac{1}{n!}\,(x)_{n,h}$$

and applying our formula giving the difference of the generalised factorial (4) § 16, we find

$$\mathop{\Delta}_h \binom{x}{n}_h = \frac{h}{(n-1)!}\,(x)_{n-1,h} = h\binom{x}{n-1}_h,$$

and in the same manner

$$\mathop{\Delta^m}_h \binom{x}{n}_h = h^m \binom{x}{n-m}_h. \tag{10}$$

Mean of $\binom{x}{n}$ We have

$$\mathbf{M}\binom{x}{n} = \frac{1}{2}\left[\binom{x+1}{n} + \binom{x}{n}\right] = \binom{x}{n-1}\frac{2x+2-n}{2n};$$

therefore the higher means will be complicated.

Generating function of the binomial coefficient with respect to x. We have seen [formula 14, § 10] that this generating function is:

$$\mathbf{G}\binom{x}{n} = \frac{t^n}{(1-t)^{n+1}} = \sum_{x=n}^{\infty}\binom{x}{n} t^x.$$

Starting from this expression we may deduce a forniula analogous to that of ***Cauchy*** given below (14). Obviously we have

$$\sum_{x=n}^{\infty}\sum_{y=m}^{\infty}\binom{x}{n}\binom{y}{m}t^{x+y} = \frac{t^{n+m}}{(1-t)^{n+m+2}}.$$

Putting $x+y=z$ and noting that the coefficient of t^z in both members is the same, *we* have

$$\textbf{(11)}\qquad \sum_{x=n}^{m+1}\binom{x}{n}\binom{z-x}{m} = \binom{z+1}{n+m+1}$$

Formula (11) may be extended to multiple sums. Indeed, starting from (11) we may write

$$\Sigma\binom{z_1}{n_1+n_2+1}\binom{z+1-z_1}{n_3} = \binom{z+2}{n_1+n_2+n_3+2}$$

applying formula (11) to the first factor of the first member of this equation we get

$$\Sigma\,\Sigma\binom{z+1-z_1}{n_3}\binom{x_1}{n_1}\binom{z_1-x_1-1}{n_2}$$

Putting now $z_1-x_1-1 = x_2$ and z-x,-x, $= x_3$ we find:

$$\Sigma\,\Sigma\binom{x_1}{n_1}\binom{x_2}{n_2}\binom{x_3}{n_3} = \binom{z+2}{n_1+n_2+n_3+2}$$

In the sum of the first member every combination of the numbers x_1, x_2 and x_3, with repetition and permutation, must be taken so that

$$x_1 + x_2 + x_3 = z.$$

Continuing in the same way, we should obtain a still more general formula:

$$\Sigma \ldots \Sigma \binom{x_1}{n_1}\binom{x_2}{n_2}\binom{x_3}{n_3}\cdots\binom{x_m}{n_m} = \binom{z+m-1}{n+m-1}$$

where

$$n_1+n_2+,\ldots+n_m = n \quad \text{and} \quad x_1+x_2+,\ldots+x_m = z.$$

In the sum of the first member every combination of the numbers $x_1, x_2, \ldots, x_m$ should be taken, with permutation and repetition, satisfying the above equation.

The derivatives and the integral of the binomial coefficient will be deduced later by aid of the *Stirling* numbers of the first kind. Here only the results are mentioned.

$$\mathsf{D}^m \binom{x}{n} = \frac{1}{n!}\sum_{\nu=m}^{n+1} (\nu)_m \, x^{\nu-m} S_n^\nu$$

$$\int \binom{x}{n} dx = \frac{1}{n!}\sum_{\nu=1}^{n+1} \frac{x^{\nu+1}}{\nu+1} S_n^\nu + k.$$

We shall see later that this integral may be expressed by *the Bernoulli* polynomial of the second kind of degree $n+1$, that is by $\psi_{n+1}(x)$

$$\int \binom{x}{n} dx = \psi_{n+1}(x) + k.$$

The binomial coefficient may be considered as a *function of the lower index.* Let us write it thus: $\binom{n}{x}$.

The difference of this function with respect to x is

$$(12) \quad \Delta\binom{n}{x} = \binom{n}{x+1} - \binom{n}{x} = \binom{n}{x}\left[\frac{n-x}{x+1} - 1\right] =$$

$$= \binom{n}{x}\left[\frac{n-2x-1}{x+1}\right].$$

The higher differences are complicated.

The mean of $\binom{n}{x}$ is

$$\mathsf{M}\binom{n}{x} = \frac{1}{2}\left[\binom{n}{x+1} + \binom{n}{x}\right] = \frac{1}{2}\binom{n}{x}\left[\frac{n+1}{x+1}\right] = \frac{1}{2}\binom{n+1}{x+1}$$

and therefore

(13) $$\mathsf{M}^m \binom{n}{x} = \frac{1}{2^m} \binom{n+m}{x+m}.$$

Hence in this case the higher means are very simple.

The ***generating function*** of $\binom{n}{x}$ with respect to x is

$$\mathsf{G}\binom{n}{x} = (1+t)^n = \sum_{x=0}^{\infty} \binom{n}{x} t^x.$$

From this we may deduce a useful formula found by ***Cauchy.*** Putting

$$(1+t)^n (1+t)^m = (1+t)^{n+m}$$

and noting that in both members of this equation the coefficients of t^z are the same, we get

(14) $$\binom{n+m}{z} = \sum_{x=0}^{z+1} \binom{n}{x} \binom{m}{z-x}.$$

This is ***Cauchy's*** formula. It may be extended for any number of factors. From (14) we deduce

$$\sum_{z_1} \binom{n_1+n_2}{z_1} \binom{n_3}{z-z_1} = \binom{n_1+n_2+n_3}{z} = \sum_{x_1} \sum_{x_2} \binom{n_1}{x_1} \binom{n_2}{x_2} \binom{n_3}{x_3}$$

where $x_2 = z_1 - x_1$ and $x_3 = z - x_1 - x_2$.

Continuing in the same manner we finally obtain ***Cauchy's*** polynomial formula

(15) $$\binom{n_1+n_2+\ldots+n_m}{z} = \sum \binom{n_1}{x_1} \binom{n_2}{x_2} \cdots \binom{n_m}{x_m}.$$

In the sum of the second member every combination with repetition and permutation of the numbers $x_1, x_2, \ldots, x_m$ should be taken so that $x_1+x_2+x_3+\ldots+x_m = z$.

• ***Computation of binomial coefficients.*** If in an expansion $\binom{x}{1}$ $\binom{x}{2}, \ldots, \binom{x}{n}$ **are needed, since** $\binom{x}{n} = (x)_n/n!$**, it is best to compute the factorials** x, $(x)_2, \ldots, (x)_n$**, as has been said** in paragraph 16, and then divide respectively by $1!, 2!, \ldots, n!$.

If only one term $\binom{x}{n_1}$ is required, and if x and n are positive integers, less than 3001, then *we* may use *Dude's* Tables [loc. cit. 11], and the formula

$$\log \binom{x}{n} = \log x! - \log n! - \log (x-n)!$$

There are also small tables giving $\binom{x}{n}$ [19]

To compute a table of the binomial coefficients corresponding to positive integer values of x and *n,* it is best to start from the difference equation (7'), which will be written in the following manner:

$$f(n,x) = f(n,x-1) + f(n-1,x-1),$$

taking account of the initial conditions $f(n,0) = 0$, if n is different from zero and $f(0,0) = 1$. These conditions are necessary and sufficient. Indeed, starting from them we may obtain by the aid of the above equation every value of $f(n,x)$ Putting $x=1$ we find

$$f(n,1) = 0 \text{ if } n>1 \text{ or } n<0$$

$$f(1,1) = 1 \text{ and } f(0,1) = 1.$$

Putting $x=2$ we have

$$f(n,2) = 0 \text{ if } n>2 \text{ or } n<0$$

$$f(2,2) = 1, \; f(1,2) = 2, \; f(0,2) = 1.$$

Continuing in this manner we should find in general

$$f(-n,x) = 0, \; f(x+n,x) = 0 \;\; f(x,x) = 1 \text{ and } f(0,x) = 1.$$

§ 23. Expansion of a function into a series of binomial coefficients. If the function is a polynomial of degree n, and the polynomial $P_i(x)$ of degree i is given for every value of i then, it is easy to show that it may be expanded into a series of *polynomials* $P_i(x)$ in one way only. Indeed, putting

$$f(x) = c_0 + c_1 P_1(x) + \ldots + c_n P_n(x)$$

[19] See *Ch. Jordan,* **Approximation by orthogonal polynomials. Annals of Mathematical Statistics Vol. 3, p. 354. Ann Arbor, Mich. 1932. The table gives the values for $x<111$ and $n<11$.**

and noting that the coefficients of x^ν are in both members identical, we obtain for $\nu=0, 1, \ldots n$, in all, $n+1$ equations determining the $n+1$ unknown coefficients $c_0, c_1, \ldots c_n$.

If the polynomial $f(x)$ is given by

$$(1) \qquad f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

and

$$P_i(x) = \binom{x-a}{i}_h$$

then the required expression is the following:

$$(2) \quad f(x) = b_0 + b_1\binom{x-a}{1}_h + b_2\binom{x-a}{2}_h + \ldots + b_n\binom{x-a}{n}_h.$$

The shortest way to determine the coefficients b_i is to use formula (10) of § 22; to do this, let us write the m-th difference of the expression (2) of $f(x)$; *we* have

$$(3) \qquad \underset{h}{\Delta}^m f(x) = h^m \left[b_m + b_{m+1}\binom{x-a}{1}_h + \ldots + b_n\binom{x-a}{n-m}_h \right]$$

therefore

$$b_m = \underset{h}{\Delta}^m f(a) \,/\, h^m$$

and the expansion will be

$$(4) \qquad f(x) = f(a) + \binom{x-a}{1}_h \frac{\underset{h}{\Delta} f(a)}{h} + \binom{x-a}{2}_h \frac{\underset{h}{\Delta}^2 f(a)}{h^2} + \ldots +$$

$$+ \binom{x-a}{n}_h \frac{\underset{h}{\Delta}^n f(a)}{h^n}.$$

This is the general form of *Newton's. formula* for a polynomial; the function $f(x)$ and its differences for $x=a$ must be known.

From formula (4) we may easily deduce the differences of $f(x)$ for any value whatever of x:

$$(5) \qquad \Delta^m f(x) = \sum_{\nu=m}^{n+1} \binom{x-a}{\nu-m}_h \frac{\underset{h}{\Delta}^\nu f(a)}{h^{\nu-m}}.$$

If f(x) is a polynomial of degree n, then from (3) it follows that the m-th difference of f(x) is of degree $n-m$; the n-th difference is a constant, and the higher differences are equal to zero.

Therefore in the calculus of finite differences it is always advantageous to express polynomials by Newton's formuia.

It *has* been shown elsewhere *[Ch. Jordan* **loc.** cit. **19. p. 257—357]** what importance *Newton's* formula has for the statistician, This is not yet sufficiently **recognised,** since nearly always the statistician expands his polynomials in power series in spite of the fact that he is generally concerned with the differences and sums of his functions, so that he is obliged to calculate these quantities laboriously. In *Newton's* expansion they would be given immediately.

If f(x) corresponding to a given value of x is needed; the computation is not much longer in the case of a *Newton* series. than in that of a power series. In the latter it is necessary to compute $x, x^2, x^3, \ldots, x^n$ and these are obtained most readily by multiplication. In the case of a *Newton* series it is necessary to compute (x-a), (x-a) (x-e--h), (x-u) $(x-a-h)$ (x-a-2h) and so on; these should also be obtained by multiplication; the only additional work is division by 1, 2, 3, and so on.

But if a table of the values $f(a), f(a+h), f(a+2h), \ldots$, and so on, is required, this is obtained by *Newton's* formula with much less work than by a power **series.**[20]

The method used is that of the *addition of differences.* Since *f(x)* is a polynomial of degree n, $\Delta^n f(x) = \& f(u)$ is constant. Putting $x=a+\xi h$ the differences $\underset{h}{\Delta}^{n-1} f(x)$ are then given step by step by

$$\Delta^{n-1} f(a+\xi h) = \Delta^{n-1} f(a) + \xi \Delta^n f(a)$$

for $\xi = 1, 2, 3, \ldots$ The differences $\Delta^{n-2} f(x)$ are given by

$$\Delta^{n-2} f(a+\xi h+h) = \Delta^{n-2} f(a+\xi h) + \Delta^{n-1} f(a+\xi h)$$

and so on:

$$\Delta^{n-\nu} f(a+\xi h+h) = \Delta^{n-\nu} f(a+\xi h) + \Delta^{n-\nu+1} f(a+\xi h).$$

Finally

$$f(a+\xi h+h) = f(a+\xi h) + \Delta f(a+\xi h).$$

In this way we obtain by simple additions not only a table of $f(a+\xi h)$ but also that of the corresponding differences.

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[20] **The method is shown in loc. cit. 19, p. 290, and an example is given on p. 301.**

For instance, given: $\Delta^3 f(x) = 6$, $\Delta^2 f(0) = 12$, $\Delta f(0) = 7$, $f(0) = 1$ we find

x	$\Delta^3 f(x)$	$\Delta^2 f(x)$	$\Delta f(x)$	$f(x)$
0	6	**12**	**7**	1
1		**18**	**19**	8
2		24	**37**	**27**
3		30	**61**	**64**
4		36	91	125

We have seen that *Newton's* formula may be obtained by symbolical methods (§ 6). Indeed from

$$E = 1 + \Delta$$

it follows that

$$E^x = (1+\Delta)^x = \sum_{m=0}^{\infty} \binom{x}{m} \Delta^m.$$

This gives, for $z=a$ if $f(z)$ is a polynomial of degree n

$$f(a+x) = \sum_{m=0}^{n+1} \binom{x}{m} \Delta^m f(a).$$

Putting : $x = \frac{z-b}{h}$ and writing $f(a + \frac{z-b}{h}) = F(z)$ *we* have

$$F(z) = \sum_{m=0}^{n+1} \binom{z-b}{m}_h \frac{\Delta_h^m F(b)}{h^m}. \tag{6}$$

This is the general form of *Newton's* formula.

Application. Expansion of $\overset{x}{\underset{0}{}} f(x)$. Writing in formula (6) x instead of z and v instead of b, and moreover putting $h=1$, we get

$$f(x) = f(\nu) + \binom{x-\nu}{1} \Delta f(\nu) + \binom{x-\nu}{2} \Delta^2 f(\nu) + \dots +$$
$$+ \binom{x-\nu}{n} \Delta^n f(\nu).$$

Since

$$\binom{x}{\nu}\binom{x-\nu}{m} = \binom{m+\nu}{\nu}\binom{x}{m+\nu},$$

we have multiplying the preceding equation by $\binom{x}{\nu}$,

(7) $$\binom{x}{\nu} f(x) = \sum_{m=0}^{n+1} \binom{m+\nu}{\nu} \binom{x}{m+\nu} \Delta^m f(\nu).$$

Example. Expansion of $\binom{x}{\nu}\binom{x}{n}$. We find

$$\binom{x}{\nu}\binom{x}{n} = \sum_{m=0}^{n+1} \binom{m+\nu}{\nu} \binom{x}{n-m} \binom{x}{m+\nu}.$$

The expansion of (x) $_\nu f$ (x) would be made in the same manner. Formula **(7)** is not only useful for determining the differences but as we shall see later also the sum of $\binom{x}{\nu} f(x)$ *

Newton's backward formula. In *Newton's* formula treated above, the argument of each difference is the same. Sometimes it is useful to have decreasing arguments in the expansion,

In § 6 we found by symbolical methods $\mathrm{E} = 1/(1-\frac{\Delta}{\mathrm{E}})$, hence

$$\mathrm{E}^x = \sum_{m=0}^{\infty} \binom{x+m-1}{m} \frac{\Delta^m}{\mathrm{E}^m}.$$

$f(x)$ being a polynomial of degree *n,* if we apply this operation to *f(a) we* have

(8) $$f(a+x) = \sum_{m=0}^{\infty} \binom{x+m-1}{m} \Delta^m f(a-m).$$

Putting into this equation x $=$ *(z-b)* / h and writing

$$f\left(a+\frac{z-b}{h}\right) = F(z)$$

we get

(9) $$F(z) = \sum_{m=0}^{n+1} \binom{z-b+mh-h}{m}_h \frac{\Delta_h^m F(b-mh)}{h^m},$$

This is *Newton's backward formula* in its general form.

Putting $h=1$ and $b = -1$ we obtain an important particular case. Writing x instead of z we have

(10) $$F(x) = \sum_{m=0}^{n+1} \binom{x+m}{m} \Delta^m F(-m-1).$$

It may be useful to remark that

$$(-1)^m [\Delta^m F(x)]_{x=-m-1} = [\Delta^m F(-x)]_{x=1}.$$

Formula (10) may serve in the same manner as (6) for the expansion of $\underset{0^{\nu}}{\overset{x}{}}$ F(x). Indeed, multiplying (10) by $\binom{x}{\nu}$ we obtain

$$(11) \qquad \binom{x}{\nu} F(x) = \sum_{m=0}^{n+1} \binom{m+\nu}{\nu} \binom{x+m}{m+\nu} [\Delta^m F(x)]_{x=-m-1} .$$

Example 1. Given F(x) $= \binom{x}{n}$ Formula (10) will give

$$\binom{x}{n} = \sum_{m=0}^{n+1} \binom{x+m}{m} \binom{-m-1}{n-m} = \sum_{m=0}^{n+1} (-1)^{n+m} \binom{n}{m} \binom{x+m}{m} .$$

Example 2. Let us put into formula (11) $F(x) = \binom{x}{n}$. We find

$$\binom{x}{\nu} \binom{x}{n} = \sum_{m=0}^{n+1} \binom{\nu+m}{\nu} \binom{x+m}{\nu+m} \binom{-m-1}{n-m} .$$

This formula may serve to determine the μ -th difference of the first member; we obtain

$$\Delta^{\mu} \left[\binom{x}{\nu} \binom{x}{n}\right] = \sum_{m=0}^{n+1} \binom{\nu+m}{\nu} \binom{-m-1}{n-m} \binom{x+m}{\nu+m-\mu} .$$

A function f(x) which is not a polynomial may nevertheless in certain cases be expanded nto a *Newton* series (§ 124); but then the series will be infinite. Applying formula (10) we get for instance

$$2^x = \sum_{m=0}^{\infty} \binom{x+m}{m} \frac{1}{2^{m+1}} .$$

An arithmetical progression is a polynomial *f(x)* corresponding to $x=a, a+h, a+2h, \ldots$ whose first difference is constant; therefore it can be expressed by *Newton's* formula in the following way:

$$f(x) = f(Q) + (x-a) \frac{\underset{h}{\Delta} f(a)}{h} .$$

Example 3. Given the series **1, 3, 5,** 7, , . . , Here we must put $h=1$, $a=1$, $f(a)=1$ and $\Delta f(a) = 2$. Therefore we have

$$f(x) = 1 + 2(x-1)$$

The general term of an arithmetical progression of order n is a polynomial whose n-th difference is constant. It may be given by formula (4).

Example 4. Arithmetical progression of the third order. Given: 1, 8, 27, 64, . . . In this case $n=3$, $h=1$, $a=1$ and $f(a)=1$. To determine the differences for $x=1$ let us write the table of the successive values of $f(x)$:

1		8		27		64 . . .
	7		19		37	
		12		18		
			6			

Therefore

$$f(x) = 1 + 7\binom{x-1}{1} + 12\binom{x-1}{2} + 6\binom{x-1}{3}.$$

§ 24. Beta functions. These are functions of two variables denoted by $B(x,y)$. Their definition is

$$(1) \qquad \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = B(x,y).$$

It is easy to show that $B(x,y) = B(y,x)$. The Beta function may be expressed by Gamma functions[20a]:

$$(2) \qquad B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

This follows immediately from formula (E) § 16; indeed, by aid of (9) § 17 we deduce

$$B(x,y) = \frac{(x)_x\,(y-1)_{-x}}{x} = \frac{(x-1)_{x-1}\,(y-1)_{y-1}}{(x+y-1)_{x+y-1}} =$$
$$= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

$B(x,y)$ may be expressed with aid of (2) by a binomial coefficient

$$(3) \qquad B(x,y) = \frac{1}{x\binom{x+y-1}{x}}.$$

[20a] See for instance C. Jordan, Cours d'Analyse, 3. ed. Tome II, p. 222.

This formula may serve to compute $B(x,y)$ if x is a positive integer.

On the other hand, in the general case, the binomial coefficient may be expressed by a Beta function; writing in formula (3) $n+1$ instead of x and $x-n+1$ instead of y we get

$$\binom{x}{n} = \frac{1}{(x+1)\, B(n+1\,,\, x-n+1)}\,.$$

From the expression (2) we may deduce by aid of formula (2) § 17

$$B(x+1,y) = \frac{\Gamma(x+1)\,\Gamma(y)}{\Gamma(x+y+1)} = \frac{x}{x+y}\, B(x,y).$$

This gives the difference of $B(x,y)$ with respect to x

$$\underset{x}{\Delta} B(x,y) = -\frac{y}{x+y}\, B(x,y).$$

In the same manner we should obtain

$$\underset{y}{\Delta} B(x,y) = -\frac{x}{x+y}\, B(x,y).$$

Therefore we conclude that $B(x,y)$ satisfies the following partial difference equation

(4) $$\underset{x}{\mathrm{E}}\, B(x,y) + \underset{y}{\mathrm{E}}\, B(x,y) - B(x,y) = 0.$$

By aid of (3) we get from formula (7) § 16

$$\Delta^m \frac{1}{x+1} = (-1)^m\, B(x+1,\, m+1)$$

and from formula (4) § 19 we obtain the m-th difference of the digamma-function:

$$\Delta^m F(x) = (-1)^{m-1} B(x+1, m).$$

Putting $t = u/(1+u)$ we may deduce from formula (1),

(5) $$\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du = B(x,y).$$

On the other hand, putting $t = \sin^2\varphi$ into (1) we get

$$(6) \qquad B(x,y) = 2\int_0^{1/2\pi} \sin^{2x-1}\varphi \cos^{2y-1}\varphi \, d\varphi .$$

Finally in § 16 we found formula (ζ)

$$(7) \qquad B(x,y) = \frac{1}{x} \prod_{i=1}^{\infty} \frac{i\,(x+y+i-1)}{(x+i)\,(y+i-1)} ,$$

Particular case 1. Let $y=x$. From (6) it follows that

$$B(x,x) = \frac{2}{2^{2x-1}} \int_0^{1/2\pi} \sin^{2x-1}(2\varphi)\, d\varphi$$

but this integral is equal to

$$B(x,x) = \frac{2\,.\,2\,.\,4\,.\,6\,.\,,\,.\,.\,(2x-2)}{2^{2x-1}\,.\,3\,.\,5\,.\,7\,.\,.\,.\,.\,(2x-1)}$$

?

if x is an integer [*Bierens de Haan,* Nouvelles Tables d'Intégrales Définies (2) Table 40].

In the case considered formula (7) will give

$$B(x,x) = \frac{1}{x} \prod_{i=1}^{\infty} \frac{i(2x+i-1)}{(x+i)\,(x+i-1)} ,$$

Since according to (2)

$$B(x,x) = \frac{[\,\Gamma(x)\,]^2}{\Gamma(2x)}$$

and in consequence of the multiplication formula (6) § 17

$$\Gamma(2x) = \Gamma(x)\,\Gamma(x+1/2)\,\frac{2^{2x-1}}{\sqrt{\pi}} .$$

Hence writing $\sqrt{\pi} = \Gamma(1/2)$ we have

$$B(x,x) = \frac{1}{2^{2x-1}}\,\frac{\Gamma(x)\,\Gamma(1/2)}{\Gamma(x+1/2)} = \frac{1}{2^{2x-1}}\,B(x,1/2) .$$

From (6) it follows directly that

$$B(1,1) = 1 \text{ and } B(1/2,1/2) = \pi.$$

Particular case 2. Let $y=1-x$. By aid of formula (6) we *have* [***Bierens de Haan,*** (1) Table 42]

$$B(x,1-x) = 2\int_0^{1/2\pi} \tan^{2x-1}\varphi \, d\varphi = \frac{\pi}{\sin \pi x}.$$

Since from formula (2) it follows that $B(x, 1-x) = \Gamma(x)\,\Gamma(1-x)$, ***Euler's*** formula (8) of § 17 is demonstrated.

Formula (5) gives

$$B(x, 1-x) = \int_0^\infty \frac{u^{x-1}}{1+u}\, du.$$

This integral is, according to ***Bierens de Haan*** [(1) Table 16] equal to $\pi/\sin\pi x$.

Finally from formula (7) we get

$$B(x, 1-x) = \frac{1}{x} \prod_{i=1}^{\infty} \frac{i^2}{i^2-x^2};$$

this is the well-known infinite product giving $\pi/\sin\pi x$.

Particular case 3. Let $y=1$; from (1) we immediately deduce

$$B(x,1) = \frac{1}{x}.$$

§ 25. Incomplete Beta-Function, This function, denoted by $B_x(p, q)$, is the following:

$$B_x(p, q) = \int_0^x t^{p-1}\,(1-t)^{q-1}\, dt. \tag{1}$$

Instead of $B_x(p, q)$ the function $I_x(p, q)$ is generally introduced. This is obtained by dividing the incomplete Beta-function by the corresponding complete function:

$$I_x(p,q) = \frac{B_x(p,q)}{B(p,q)} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^x t^{p-1}(1-t)^{q-1}\, dt. \tag{2}$$

The numerical values of $I_x(p, q)$ may be evaluated by integration by parts or taken from ***Pearson's*** tables, if $p \leqq 50$ and $q \leqq 50$.[21]

[21] *H. E.* ***Soper,*** **Numerical Evaluation of the Incomplete B-function. Tracts for Computers, Cambridge University Press, 1921.**

K. Pearson, Tables of the incomplete Beta-function, Cambridge University Press, 1934.

In the tables, $I_x(p, 9)$ is given only for the values p $\geqq$ q, This is sufficient, since we have

(3) $$I_x(p, q) = 1 - I_{1-x}(q, p).$$

Let us remark moreover that

$$I_x(p, p) = 1 - \frac{1}{2} I_{x_1}(p, \tfrac{1}{2})$$

where $x_1 = \mathrm{I} - (4x - \tfrac{1}{2})^2$.

The tables give the function for p and q from 0'5 to 50. The intervals Δp and Δq are equal to 0'5 from 0'5 to 11, and to 1'0 from 11 to 50. Moreover x is given from 0 to I; and $\Delta x = 0'01$.

Some mafhemafical properties of the function. The function $I_x(p, q)$ satisfies the following difference equation:

(4) $$I_x(p+1, q+1) = x I_x(p, q+1) + (1-x) I_x(p+1, 9\ 1$$

hence the values of $I_x(p, q)$ may be computed for every integer value of p and q, starting from

$$I_x(p, 1) = x^p \quad \text{and} \quad I_x(\mathrm{I}, q) = \mathrm{I} - (1-x)^q.$$

From (2) we obtain, by integration by parts,

(5) $$I_x(p, q) = \frac{\Gamma(p+q)}{\Gamma(p+1)\Gamma(q)} x^p (1-x)^{q-1} + I_x(p+1, q-1).$$

From this it follows immediately that

(6) $$I_x(p, q) > I_x(p+1, q-1).$$

Soper has obtained another formula "***by*** *raising* ***p***" *in* the following way: Starting from

$$D[t^p(1-t)^q] = pt^{p-1}(1-f)^q - qt^p(1-t)^{q-1} =$$
$$= pt^{p-1}(1-t)^{q-1} - (p+q)\, t^p (1-t)^{q-1}$$

integrating both members and multiplying by $\Gamma(p+q)/\Gamma(p+1)\,\Gamma(q)$ gives

(7) $$I_x(p, q) = \frac{\Gamma(p+q)}{\Gamma(p+1)\Gamma(q)} x^p (1-x)^q + I_x(p+1, q).$$

From this it results that

(8) $$I_x(p, q) > I_x(p+1, q);$$

moreover it can be shown in like manner that

$$I_x(p, q+1) > I_x(p, q).$$

Starting from $I_x(p, p)$ we deduce from what **preceds**

$$I_x(p,q) > I_x(p,p) > I_x(q, p) \quad \text{if} \quad q > p$$

and therefore

(9) $$I_x(q, p) + I_{1-x}(q, p) < 1 \quad \text{if} \quad q > p.$$

In the particular case of $p=q$ *we* have

(10) $$I_x(p,p) + \quad I_{1-x}(p,p) = \quad \text{and } I_{1/2}(p, p) = 1/2.$$

moreover if $q > p$, *then*

$$I_{1/2}(q, p) < 1/2.$$

To compute the numerical value of $I_x(p, q)$ if p and q are beyond the range of *Pearson's* table, it is best to follow *Soper*'s method [loc. cit. 21]. First we have to reduce $I_x(p, q)$ by repeated integration by parts to I, $(p+m$, q-m), This gives

(11) $$I_x(p,q) = \Gamma(p+q) \sum_{i=0}^{m} \frac{x^{p+i}(1-x)^{q-1-i}}{\Gamma(p+i+1)\,\Gamma(q-i)} + \\ + I_x(p+m, q-m).$$

We have seen that $I_x(p+m, q-m)$ diminishes with increasing m, hence it is often possible to continue the proceeding till this quantity becomes negligible. To perform the computation *Soper* transformed formula (11) by factoring the first term of the sum, in this way: a, $(1 + \Sigma$, . . then again factor ing the first term of the new sum thus:
$a_1[1 + a_2[1 + \Sigma$. , , , and so on. In the end he obtained

(12) $$I_x(p, q) = \frac{\Gamma(p+q)}{\Gamma(p+1)\,\Gamma(q)} x^p (1-x)^{q-1} [1 + \\ + \frac{q-1}{p+1}\left(\frac{x}{1-x}\right)[1 + \frac{q-2}{p+2}\left(\frac{x}{1-x}\right)[1 + \ldots + \\ + \frac{q-m+1}{p+m-1}\left(\frac{x}{1-x}\right)]]]\ldots] + I_x(p+m, q-m).$$

The computation is continued till
$(q-m+1)x/(p+m-1)(1-x)$ become small enough.

If $xq > (1-x)p$ then it is better to determine $1 - I_{1-x}(q, p)$ instead of $I_x(p, q)$.

It may happen that the desired precision is not attained before $q+1-m$ becomes negative. Should we still continue the proceeding then the absolute value of the terms would begin to increase and the series to diverge. To obviate this inconvenience ***Soper*** advises us then to use the method of raising p.

Let us suppose that by repeated integration by parts $I_x(p, q)$ has been reduced to $I_x(p', q')$ so that $1 > q' > 0$. Now repeating operation (7) n times, we shall have

$$(13) \qquad I_x(p', q') = \frac{\Gamma(p'+q')}{\Gamma(q')} (1-x)^{q'} \sum_{i=1}^{n+1} \frac{x^{p'+i-1}}{\Gamma(p'+i)} + I_x(p'+i, q').$$

To shorten the work of computation ***Soper*** transformed this formula in the following way:

$$(14) \qquad I_x(p', q') = \frac{\Gamma(p'+q')}{\Gamma(p'+1)\Gamma(q')} x^{p'}(1-x)^{q'} \left[1+\frac{p'+q'}{p'+1} x \,.\right.$$

$$\left[1+\frac{p'+q'+1}{p'+2} x_{\,\mathrm{I}} 1+ \ldots + \frac{p'+q'+n-2}{p'+n-1} x\right]\Big] \ldots_{\mathrm{I}} + I_x(p'+n, q').$$

This can be continued till the required precision is obtained. In case p and q are integers we shall have from **(11)**

$$(15) \qquad I_x(p, q) = \Gamma(p+q) \sum_{i=0}^{q} \frac{x^{p+i}(1-x)^{q-1-i}}{\Gamma(p+i+1)\Gamma(q-i)}.$$

Remark 1. If in p+q-2 repeated trials p-l favourable events are obtained, then, according to *Bayed* theorem the probability that the probability of the favourable event does not exceed x is equal to $I_x(p, q)$.

Remark 2. The quantity I_p (x, $n+1-x$) may be considered as the probability that in n trials the number of the **favour**-able events should not be less than x, provided that the probability of the favourable event is equal to p. Indeed, from (15) it follows that

$$(16) \qquad I_p\,(x, n+1-x) = \sum_{\nu=x}^{n+1} \binom{n}{\nu} p^{\nu}(1-p)^{n-\nu}.$$

From this we may deduce

$$(17) \qquad \sum_{\nu=0}^{x} \binom{n}{\nu} a^{\nu} = \frac{1 - I_p(x, n+1-x)}{[1-P]^{n}}$$

where we have to put $p = a/(1+a)$.

Moreover if $p=\frac{1}{2}$ *we* get

$$(18) \qquad \sum_{\nu=0}^{x} \binom{n}{\nu} = 2^n[1 - I_{1/2}(x, n+1-x)].$$

Particular case. If np is an integer, then $\nu=np$ **is** the most probable number of the favourable events in n trials. Putting into (16) $x=np+1$ we get the probability that in n trials the number of the favourable events should be more than np. This will be equal to

$$I_p(np+1, nq).$$

Moreover the probability that v is less than np is

$$1 - I_p(np, nq+1).$$

According to ***Simmons'*** theorem this second probability is greater than the first if $p<q$.

§ 26, Exponential functions. The exponential function is as important in the Calculus of Finite Differences as in Infinitesimal Calculus; the differences and the means of this function are easy to express.

Differences **of** a^x. ***We*** have

$$(1) \qquad \underset{h}{\Delta} a^x = a^{x+h} - a^x = a^x(a^h-1)$$

and therefore

$$(2) \qquad \underset{h}{\Delta^m} a^x = a^x(a^h-1)^m.$$

Particular cases. If $h=1$ and $a=2$, then

$$\Delta 2^x = 2^x.$$

If $h=1$ and $a=\frac{1}{2}$

$$\Delta(\tfrac{1}{2})^x = -(\tfrac{1}{2})^{x+1}$$

$$\Delta^m(\tfrac{1}{2})^x = (-1)^m(\tfrac{1}{2})^{x+m}.$$

Meahs of the function ax. According to what we have seen we have

$$(3) \qquad \underset{h}{\mathrm{M}} a^x = \tfrac{1}{2}a^x(a^h+1)$$

and

$$(4) \qquad \underset{h}{\mathrm{M}^m} a^x = (\tfrac{1}{2})^m a^x (a^h+1)^m.$$

Example. Hyperbolic functions. We have

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

hence the differences and the means of these functions may be expressed by formulae (1) and (3). To avoid mistakes we will write in the formulae below ω instead of h. We find

$$\underset{\omega}{\Delta} \cosh x = \tfrac{1}{2}[e^x(e^\omega - 1) + e^{-x}(e^{-\omega} - 1)] = \tfrac{1}{2}(e^\omega - 1)[e^x - e^{-x-\omega}] =$$

$$= \tfrac{1}{2}(e^{\frac{1}{2}\omega} - e^{-\frac{1}{2}\omega})[e^{x+\frac{1}{2}\omega} - e^{-x-\frac{1}{2}\omega}] = 2 \sinh \tfrac{1}{2}\omega \sinh (x + \tfrac{1}{2}\omega)$$

and in the same manner

$$\underset{\omega}{\Delta} \sinh x = 2 \sinh \tfrac{1}{2}\omega \cosh (x + \tfrac{1}{2}\omega).$$

Therefore

$$\underset{\omega}{\Delta}^{2m} \cosh x = (2 \sinh \tfrac{1}{2}\omega)^{2m} \cosh (x + m\omega)$$

$$\underset{\omega}{\Delta}^{2m+1} \cosh x = (2 \sinh \tfrac{1}{2}\omega)^{2m+1} \sinh (x + m\omega + \tfrac{1}{2}\omega)$$

$$\underset{\omega}{\Delta}^{2m} \sinh x = (2 \sinh \tfrac{1}{2}\omega)^{2m} \sinh (x + m\omega)$$

$$\underset{\omega}{\Delta}^{2m+1} \sinh x = (2 \sinh \tfrac{1}{2}\omega)^{2m+1} \cosh (x + m\omega + \tfrac{1}{2}\omega).$$

The *means of the hyperbolic functions* are obtained in the same manner as the differences. We find

$$\underset{\omega}{\mathrm{M}}^m \cosh x = (\cosh \tfrac{1}{2}\omega)^m \cosh (x + \tfrac{1}{2}m\omega)$$

$$\underset{\omega}{\mathrm{M}}^m \sinh x = (\cosh \tfrac{1}{2}\omega)^m \sinh (x + \tfrac{1}{2}m\omega).$$

The differences and the means of the trigonometric functions could be determined by the above method; but we will determine them directly.

§ 27. Trigonometric Functions, Of these functions only the cosine and the sine functions, whose differences and means are simple enough to calculate, play a role in the Calculus of Finite Differences.

Differences of the trigonometric functions. We have

$$\underset{h}{\Delta}\cos(ax+b) = \cos(ax+b+ah) - \cos(ax+b) =$$
$$= -2\sin\tfrac{1}{2}ah\sin(ax+b+\tfrac{1}{2}ah) =$$
$$= 2\sin\tfrac{1}{2}ah\cos[ax+b+\tfrac{1}{2}(ah+\pi)].$$

Hence

$$(1) \qquad \underset{h}{\Delta^m}\cos(ax+b) = (2\sin\tfrac{1}{2}ah)^m\cos[ax+b+\tfrac{1}{2}m(ah+\pi)].$$

In the same manner we could obtain $\underset{h}{\Delta^m}\sin(ax+b)$, but we may deduce this difference directly from (1) by putting into it $b-\frac{1}{2}\pi$ instead of b; we find

$$(2) \qquad \underset{h}{\Delta^m}\sin(ax+b) = (2\sin\tfrac{1}{2}ah)^m\sin[ax+b+\tfrac{1}{2}m(ah+\pi)].$$

From formulae (1) and (2) it follows that if the period of the trigonometric function is equal to h then its difference in a system of increment h will be equal to zero.

The period of $\cos(ax+b)$ or of $\sin(ax+b)$ will be equal to h if $a = \frac{2\%}{h}$ but then $\sin\frac{1}{2}ah = 0$ and both differences will vanish. Therefore we conclude that

$$\underset{h}{\Delta}\cos\left(\frac{2\pi}{h}x+b\right) = 0 \quad \text{and} \quad \underset{h}{\Delta}\sin\left(\frac{2\pi}{h}x+b\right) = 0.$$

Means of the trigonometric functions. We obtain

$$\underset{h}{\mathbf{M}}\cos(ax+b) = \tfrac{1}{2}[\cos(ax+b+ah) + \cos(ax+b)] =$$
$$= \cos\tfrac{1}{2}ah\cos(ax+b+\tfrac{1}{2}ah).$$

Hence

$$(3) \qquad \underset{h}{\mathbf{M}^m}\cos(ax+b) = (\cos\tfrac{1}{2}ah)^m\cos(ax+b+\tfrac{1}{2}mah).$$

In the same manner we should get

$$(4) \qquad \underset{h}{\mathbf{M}^m}\sin(ax+b) = (\cos\tfrac{1}{2}ah)^m\sin(ax+b+\tfrac{1}{2}mah).$$

From (3) and (4) it follows that if $a = \frac{\pi}{h}$ then the corresponding means in the system of increment h will be equal to zero. Hence

$$\underset{h}{\mathbf{M}}\cos\left(\frac{\pi}{h}x+b\right) = 0 \quad \text{and} \quad \underset{h}{\mathbf{M}}\sin\left(\frac{\pi}{h}x+b\right) = 0.$$

Differences and Weans of $\cos^n x$ ***and*** $\sin^n x$. To determine these quantities, if n is a positive integer, the best way is to transform these powers into sines and cosines of multiples of x by Euler's formula.

Two cases must be distinguished, n odd or even. ***First*** if n is odd: $n=2m+1$ then ***Euler's*** formula will give:

$$2^{2m+1}\cos^{2m+1}x = (e^{ix}+e^{-ix})^{2m+1} = \sum_{\nu=0}^{2m+2}\binom{2m+1}{\nu} e^{ix(2m+1-2\nu)}$$

where $i=\sqrt{-1}$.

Combining together the terms corresponding to v and $2m+1-\nu$ we have

$$\sum_{\nu=0}^{m+1}\binom{2m+1}{\nu}\left[e^{ix(2m+1-2\nu)}+e^{-ix(2m+1-2\nu)}\right] =$$

$$= 2\sum_{\nu=0}^{m+1}\binom{2m+1}{\nu}\cos(2m+1-2\nu)\,x.$$

Therefore

$$\text{(5)}\qquad \cos^{2m+1}x = \frac{1}{2^{2m}}\sum_{\nu=0}^{m+1}\binom{2m+1}{\nu}\cos(2m+1-2\nu)\,x.$$

In the same manner we get by ***Euler's*** formula

$$(-1)^m i\; 2^{2m+1}\sin^{2m+1}x = (e^{ix}-e^{-ix})^{2m+1} =$$

$$= \sum_{\nu=0}^{2m+2}(-1)^\nu\binom{2m+1}{\nu}e^{ix(2m+1-2\nu)}.$$

Combining again the terms corresponding to v and $2m+1-\nu$ we obtain

$$\text{(6)}\qquad \sin^{2m+1}x = \frac{(-1)^m}{2^{2m}}\sum_{\nu=0}^{m+1}(-1)^\nu\binom{2m+1}{\nu}\sin\,(im+l-2v)\,x.$$

Secondly, n is even: $n=2m$. Proceeding as before we get

$$2^{2m}\cos^{2m}x = \sum_{\nu=0}^{2m+1}\binom{2m}{\nu}e^{ix(2m-2\nu)}.$$

The number of the terms in the second member is odd, therefore, combining the terms corresponding to v and $2m-\nu$ there will remain the tern $\binom{2m}{m}$, Hence

$$(7)\qquad \cos^{2m} x = \frac{\binom{2m}{m}}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{\nu=0}^{m} \binom{2m}{\nu} \cos(2m-2\nu)x.$$

In the same manner

$$(-1)^m\, 2^{2m} \sin^{2m} x = \sum_{\nu=0}^{2m+1} (-1)^\nu \binom{2m}{\nu} e^{ix(2m-2\nu)} =$$

$$= (-1)^m \binom{2m}{m} + 2 \sum_{\nu=0}^{m} (-1)^\nu \binom{2m}{\nu} \cos(2m-2\nu)x$$

and finally

$$(8)\qquad \sin^{2m} x = \frac{\binom{2m}{m}}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{\nu=0}^{m} (-1)^\nu \binom{2m}{\nu} \cos(2m-2\nu)x.$$

Now we may write the differences and the means of these quantities, using formulae (I), , . ., (4). We find the differences:

$$(9)\qquad \underset{h}{\Delta} \cos^{2m+1} x = \frac{1}{2^{2m-1}} \sum_{\nu=0}^{m+1} \binom{2m+1}{\nu} \sin[\tfrac{1}{2}h(2m+1-2\nu)]\,.$$
$$. \cos[(2m+1-2\nu)(x+\tfrac{1}{2}h) + \tfrac{1}{2}\pi].$$

$$(10)\qquad \underset{h}{\Delta} \sin^{2m+1} x = \frac{(-1)^m}{2^{2m-1}} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{2m+1}{\nu} \sin[\tfrac{1}{2}h(2m+1-2\nu)]$$
$$. \sin[(2m+1-2\nu)(x+\tfrac{1}{2}h) + \tfrac{1}{2}\pi].$$

$$(11)\qquad \underset{h}{\Delta} \cos^{2m} x = \frac{1}{2^{2m-2}} \sum_{\nu=0}^{m} \binom{2m}{\nu} \sin(m-\nu)h$$
$$\cos[(2m-2\nu)(x+\tfrac{1}{2}h) + \tfrac{1}{2}\pi].$$

$$(12)\qquad \underset{h}{\Delta} \sin^{2m} x = \frac{(-1)^m}{2^{2m-2}} \sum_{\nu=0}^{m} (-1)^\nu \binom{2m}{\nu} \sin(m-\nu)h$$
$$\cos[(2m-2\nu)(x+\tfrac{1}{2}h) + \tfrac{1}{2}\pi]$$

and the means:

$$(13)\qquad \underset{h}{\mathrm{M}} \cos^{2m+1} x = \frac{1}{2^{2m}} \sum_{\nu=0}^{m+1} \binom{2m+1}{\nu} \cos[\tfrac{1}{2}h(2m+1-2\nu)]\,.$$
$$\cos[(2m+1-2\nu)(x+\tfrac{1}{2}h)].$$

$$(14)\qquad \underset{h}{\mathrm{M}} \sin^{2m+1} x = \frac{(-1)^m}{2^{2m}} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{2m+1}{\nu} \cos[\tfrac{1}{2}h(2m+1-2\nu)]$$
$$\sin[(2m+1-2\nu)(x+\tfrac{1}{2}h)].$$

$$(15)\qquad \underset{h}{\mathrm{M}} \cos^{2m} x = \frac{\binom{2m}{m}}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{\nu=0}^{m} \binom{2m}{\nu} \cos (m-\nu) h$$

$$\cos [(2m-2\nu)(x+\tfrac{1}{2}h)].$$

$$(16)\qquad \underset{h}{\mathrm{M}} \sin^{2m} x = \frac{\binom{2m}{m}}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{\nu=0}^{m} (-1)^\nu \binom{2m}{\nu} \cos (m-\nu) h$$

$$\cos [(2m-2\nu)(x+\tfrac{1}{2}h)].$$

§ 28. Alternate functions, The function f(x) is called alternate in a system of increments h if we have

$$f(x) \underset{h}{\mathrm{E}} f(x) < 0.$$

Example 1. The function

$$(1)\qquad f(x) = \cos(\frac{\pi x}{h} + b)\ \varphi(x)$$

is alternate in the system of increments h if q(x) > 0 for every value of x considered.

Differences of the above alternate function. We have

$$\underset{h}{\Delta} f(x) = -\cos(\frac{\pi x}{h} + b)\varphi(x+h) - \cos(\frac{\pi x}{h} + b)\varphi(x) =$$

$$= -2\cos(\frac{\pi x}{h} + b) \underset{h}{\mathrm{M}} \varphi(x)$$

therefore

$$\underset{h}{\Delta^m} [\cos(\frac{\pi x}{h} + b)\varphi(x)] = (-2)^m \cos(\frac{\pi x}{h} + b) \underset{h}{\mathrm{M}^m} \varphi(x).$$

Particular case. If in the forgoing example $b=0$, $h=1$ and for every integer value of x we have $\varphi(x) > 0$, then

$$f(x) = (-1)^x \varphi(x)$$

is an alternate function in the system of increment $h=1$.

The difference of this function will be

$$\Delta[(-1)^x \varphi(x)] = (-1)^{x+1} \varphi(x+1) - (-1)^x \varphi(x) =$$

$$= -2\ (-1)^x \mathrm{M} \varphi(x)$$

and therefore

(2) $$\Delta^m [(-1)^x \varphi(x)] = (-2)^m (-1)^x \mathsf{M}^m \varphi(x).$$

Means of the alternate function (1). In the same manner as before we obtain:

(3) $$\underset{h}{\mathsf{M}}^m [\cos(\frac{\pi x}{h} + b)\, \varphi(x)] = (-\tfrac{1}{2})^m \cos(\frac{\pi x}{h} + b) \underset{h}{\Delta}^m \varphi(x).$$

or if $h=1$ and $b=0$

$$\mathsf{M}^m [(-1)^x \varphi(x)] = (-\tfrac{1}{2})^m (-1)^x \Delta^m \varphi(x).$$

Particular cases. If $\varphi(x)=1$ and then $h=1$ we have

$$\Delta^m (-1)^x = 2^m (-1)^{x+m} \quad \text{and} \quad \mathsf{M}(-1)^x = 0.$$

Expansion of an alternate function into an alternate binomial series. ***Symbolical method.*** We saw in § 6 that $\mathsf{E} = 2\mathsf{M} - 1$, therefore we may write

$$(-1)^x \mathsf{E}^x = (1-2\mathsf{M})^x = \sum_{m=0}^{x+1} (-1)^m \binom{x}{m} 2^m \mathsf{M}^m.$$

Performing this operation on $\varphi(z)$, for $z=0$, we get, if x is a positive integer,

(4) $$(-1)^x \varphi(x) = \sum_{m=0}^{x+1} (-1)^m \binom{x}{m} 2^m \mathsf{M}^m \varphi(0).$$

Example 2. Given $\varphi(x) = a^x$. We have $\mathsf{M}^m a^x = a^x \left(\frac{a+1}{2}\right)^m$

therefore

(5) $$(-1)^x a^x = \sum_{m=0}^{x+1} (-1)^m (a+1)^m \binom{x}{m}.$$

Example 3. Given $f(x) = \binom{n}{x}$. Then

$$\left[\mathsf{M}^m \binom{n}{x}\right]_{x=0} = \binom{n+m}{m} (\tfrac{1}{2})^m$$

therefore

(6) $$(-1)^x \binom{n}{x} = \sum_{m=0}^{x+1} (-1)^m \binom{n+m}{n} \binom{x}{m}$$

This is an expression of the alternate binomial coefficient considered as a function of the lower index, as a function of the binomial coefficient considered as a function of the upper index.

§ 29.. Functions whose differences or means are equal to zero. The difterence oi the function f(x) will he equal to zero if this function satisfies the following difference equation.

$$(1) \qquad f(x+h) - f(x) = 0 \ .$$

It is obvious that if $f(x)$ is equal to a constant, this equation is satisfied. But if w(x) is any periodic function whatever with period equal to h, that is if

$$\omega(x+h) = \omega(x)$$

for every value of x, then $f(x) = \omega(x)$ is also a solution of equation (1).

Example:

$$f(x) = \cos\left(\frac{2\pi x}{h} + b\right) ,$$

The *Mean* of a function will be equal to zero if this function **satisfies** the difference equation

$$(2) \qquad f(x+h) + f(x) = 0 .$$

If we put $f(x) = \cos\left(\frac{\pi x}{h} + 6\right)$ then this equation will obviously be satisfied. But if $\omega(x)$ is any periodic function whatever with period equal to h, then

$$\cos\left(\frac{\pi x}{h} + b\right) \omega(x)$$

will also be a solution of equation (2).

§ 30. Product of two functions. Differences. The operation of displacement performed on a product gives

$$\mathsf{E}^n \left[u(x)v(x)\,w(x)\ldots\right] = u(x+nh)\,v(x+nh)\,w(x+nh)\ldots.$$

or

$$\mathsf{E}^n \left[uvw\ldots\right] = \mathsf{E}^n u\, \mathsf{E}^n v\, \mathsf{E}^n w \ldots$$

Therefore if we want to determine $\varphi(\mathsf{E})\, u\, v\, w, \ldots$ it is sufficient first to expand $\varphi(\mathsf{E})$ into a series of powers of E, and then apply the rule above. For instance

$$\Delta(uv) = (\mathsf{E}-1)\, uv = \mathsf{E} u\, \mathsf{E} v - uv$$

putting $\mathbf{E}\, v = v + \Delta v$ and remarking that $v(\mathbf{E}\, u - u) = v\Delta u$ we have

(1) $$\Delta(uv) = v\Delta u + \Delta u\, \mathbf{E}\, u$$

in the same manner we could have obtained

(2) $$\Delta(uv) = u\Delta v + \Delta u\, \mathbf{E}\, u.$$

The two formulae are not symmetrical with respect to u and v; but we may obtain a symmetrical formula, taking the mean of (1) and (2). We get

(3) $$\Delta(uv) = \mathbf{M}\, u\, \Delta v + \Delta u\, \mathbf{M}\, v.$$

Particular case of $u = v$

$$\Delta(u^2) = 2\, \mathbf{M}\, u\, \Delta u.$$

From (1) we may obtain another symmetrical formula

(4) $$\Delta(uv) = u\Delta v + v\Delta u + \Delta u \Delta v.$$

This formula may be easily generalised for m factors. Putting $u_1 u_2 u_3 \ldots u_m = \omega$ we have

(5) $$\Delta[u_1 u_2 \ldots u_m] = \Sigma\, \omega \frac{\Delta u_i}{u_i} + \Sigma\Sigma\, \omega \frac{\Delta u_i \Delta u_j}{u_i u_j} + $$

$$+ \Sigma\Sigma\Sigma\, \omega \frac{\Delta u_i \Delta u_j \Delta u_k}{u_i u_j u_k} + \ldots .$$

The first sum is to be extended to every combination of the first order of the m elements $u_1, u_2, \ldots, u_m$; the second sum to every combination of the second order of the same elements; and so on. The total number of terms will be $2^m - 1$.

Higher Differences. Starting from formula (3) we may obtain by repeating the operation

$$\Delta^2(uv) = \mathbf{M}^2 u\, \Delta^2 v + 2\, \mathbf{M}\, \Delta u\, \mathbf{M}\, \Delta v + \mathbf{M}^2 v \Delta^2 u$$

and in the same manner

(6) $$\Delta^m(uv) = \sum_{\nu=0}^{m+1} \binom{m}{\nu} \mathbf{M}^{m-\nu} \Delta^\nu u\, \mathbf{M}^\nu \Delta^{m-\nu} v.$$

This formula is not often used, since it presupposes the knowledge both of the differences and means of the functions

u and v; and we have seen that, except for the exponential and. circular functions, whose differences and means are simple, generally, if the differences of a function are easy to obtain, the means are complicated, orcōnversely. Therefore formula (6) is to be transformed so as to contain differences only or means only.

Example for formula (6). Given uv $= a^x$ sinx. Let us put $u = a^x$ and $v = \sin x$. Then

$$\Delta_h^{\nu} a^x = a^x (a^h - 1)^{\nu}$$

moreover

$$\mathbf{M}_h^{m-\nu} \Delta_h^{\nu} a^x = a^x (a^h - 1)^{\nu} \left(\frac{a^h + 1}{2} \right)^{m-\nu}$$

and

$$\Delta_h^{m-\nu} \sin x = (2 \sin \tfrac{1}{2} h)^{m-\nu} \sin \left[x + (m-\nu) \frac{h+\pi}{2} \right]$$

finally

$$\mathbf{M}_h^{\nu} \Delta_h^{m-\nu} \sin x = (2 \sin \tfrac{1}{2} h)^{m-\nu} (\cos \tfrac{1}{2} h)^{\nu},$$
$$\sin [x + \tfrac{1}{2}(m-\nu)\pi + \tfrac{1}{2} mh].$$

We conclude

$$\Delta_h^m (a^x \sin x) = \sum_{\nu=0}^{m+1} \binom{m}{\nu} (a^h - 1)^{\nu} \left(\frac{a^h + 1}{2} \right)^{m-\nu} (2 \sin \tfrac{1}{2} h)^{m-\nu}.$$
$$(\cos \tfrac{1}{2} h)^{\nu} a^x \sin [x + \tfrac{1}{2} (m-\nu)\pi + \tfrac{1}{2} mh].$$

Leibnitr has given a formula to determine the higher derivatives of a product *[see Hardy,* loc. cit. 18, p. 202]

$$\mathbf{D}^n (uv) = \sum_{\nu=0}^{n+1} \binom{n}{\nu} \mathbf{D}^{\nu} u \mathbf{D}^{n-\nu} v.$$

We will deduce an analogous formula for differences. Using the symbolical method we shall write

$$\Delta^n = (\mathbf{E} - 1)^n = \sum_{\nu=0}^{n+1} (-1)^{\nu} \binom{n}{\nu} \mathbf{E}^{n-\nu}$$

This operation performed on uv gives

$$\Delta^n (uv) = \sum_{\nu=0}^{n+1} (-1)^{\nu} \binom{n}{\nu} \mathbf{E}^{n-\nu} u \mathbf{E}^{n-\nu} v. \tag{7}$$

If we put in the place of $\mathsf{E}^{n-\nu}$ its value expressed by differences

(7') $$\mathsf{E}^{n-\nu}u = \sum_{i=0}^{n-\nu+1} \binom{n-\nu}{i} \Delta^i u$$

we shall have

$$\Delta^n(uv) = \sum_{i=0}^{n+1} \Delta^i u \sum_{\nu=0}^{n-i+1} (-1)^\nu \binom{n}{\nu} \binom{n-\nu}{i} \mathsf{E}^{n-\nu}v.$$

Remarking that

(8) $$\binom{n}{\nu}\binom{n-\nu}{i} = \binom{n}{i}\binom{n-i}{\nu}$$

and

(9) $$\sum_{i=0}^{n+1} (-1)^\nu \binom{n-i}{\nu} \mathsf{E}^{n-i-\nu} v = \Delta^{n-i}v;$$

we conclude that

(10) $$\Delta^n(uv) = \sum_{i=0}^{n+1} \binom{n}{i} \Delta^i u \, \Delta^{n-i} \mathsf{E}^i v.$$

This is the formula that corresponds to that of *Leibnitz.* Since it needs only the knowledge of differences, it is more advantageous than formula (6).

Example 1. The n-th difference of $2^x \binom{x}{m}$ is required if $h=1$. Let us put $u=2^x$ and $v=\binom{x}{m}$, formula (10) will give

$$\Delta^n\left[2^x\binom{x}{m}\right] = \sum_{i=0}^{n+1} \binom{n}{i} 2^x \binom{x+i}{m-n+i}.$$

We have seen that there are functions whose difference is a function of the same kind, but the argument is diminished by one:

$$\Delta\binom{-x}{m} = -\binom{-x-1}{m-1}.$$

If we apply formula (10) to the product of such a function, for instance to $\binom{x}{s}\binom{-x}{m}$ putting $u=\binom{x}{s}$ and $v=\binom{-x}{m}$ then the argument of v will be diminished in consequence of Δ^{n-i} by $n-i$ and moreover it will be increased in consequence of E^i by i, so that it will always remain equal to $-x-n$. Often this is a great simplification, as we shall see later.

Example 2.

$$A./\left[\binom{x}{s}\binom{-x}{m}\right] = \sum_{i=0}^{n+1}(-1)^{n-i}\binom{x}{s-i}\binom{-x-n}{m-n+i}.$$

Differences of a product expressed by means. Since there are functions whose means are more easily determined than their differences, it may be useful to determine the differences of **a** product by aid of means.

Putting into formula (7) $E^{n-\nu}$ expressed by means

$$(11)\qquad E^{n-\nu}u = (2M-1)^{n-\nu}u = \sum_{i=0}^{n-\nu+1}(-1)^{n-\nu-i}\binom{n-\nu}{i}(2M)^i u$$

we obtain

$$\Delta^n(uv) = \sum_{i=0}^{n+1}(-1)^{n-i}(2M)^i u \sum_{\nu=0}^{n-i+1}\binom{n}{\nu}\binom{n-\nu}{i}E^{n-\nu}v;$$

applying again formula (8) and remarking that

$$(12)\qquad \sum_{i=0}^{n-i+1}\binom{n-i}{\nu}E^{n-\nu-i}v = 2^{n-i}M^{n-i}v$$

it follows that

$$(13)\qquad \Delta^n(uv) = \sum_{i=0}^{n+1}(-1)^{n-i}2^n\binom{n}{i}M^iuM^{n-i}E^iv.$$

This formula is analogous to **(10)** but means figure in ii instead of differences.

Example 3. The n -th difference of $2^x\binom{n}{x}$ is to be determined if $h=1$. Let us put $u=2^x$ and $v=\binom{n}{x}$. Then we find

$$2^iM^iu = 2^x3^i \quad\text{and}\quad M^{n-i}v = \binom{m+n-i}{x+n-i}$$

therefore

$$\Delta^n\left[2^x\binom{m}{x}\right] = \sum_{i=0}^{n+1}(-1)^{n-i}3^i\binom{n}{i}2^x\binom{n+m-i}{x+n},$$

§ 31. Product of two functions. Means. To obtain the mean of a product, we will apply again the method of displacement used in the preceding paragraph. We have

$$M = \tfrac{1}{2}(1+E),$$

and therefore

$$\mathsf{M}(uv) = \tfrac{1}{2}uv + \tfrac{1}{2}\,\mathsf{E}u\,\mathsf{E}v.$$

Since $\mathsf{E} = 2\mathsf{M}-1$,' we may write

$$2\mathsf{M}(uv) = uv + (2\mathsf{M}u - u)(2\mathsf{M}v - v).$$

This gives

$$\mathsf{M}(uv) = uv - u\mathsf{M}v - v\mathsf{M}u + 2\mathsf{M}u\,\mathsf{M}v.$$

Putting again

$$2\mathsf{M}u - u = \mathsf{E}u \quad \text{we find}$$

$$\mathsf{M}(uv) = uv - v\mathsf{M}u + \mathsf{E}u\,\mathsf{M}v.$$

To determine the higher means we proceed in the same way, writing

$$\mathsf{M}^n = (\tfrac{1}{2})^n (1+\mathsf{E})^n = (\tfrac{1}{2})^n \sum_{\nu=0}^{n+1} \binom{n}{\nu} \mathsf{E}^{n-\nu}$$

and therefore

(1) $$\mathsf{M}^n(uv) = (\tfrac{1}{2})^n \sum_{\nu=0}^{n+1} \binom{n}{\nu} \mathsf{E}^{n-\nu}u\,\mathsf{E}^{n-\nu}v.$$

Putting into this formula $\mathsf{E}^{n-\nu}u$ expressed by means, given by formula (11) of § 30, we get

$$\mathsf{M}^n(uv) = (\tfrac{1}{2})^n \sum_{i=0}^{n+1} (-1)^{n-i} 2^i \mathsf{M}^i u \sum_{\nu=0}^{n-i+1} (-1)^\nu \binom{n}{\nu} \binom{n-\nu}{i} \mathsf{E}^{n-\nu} v.$$

Applying again formulae (8) and (9) of § 30, we finally obtain a formula analogous to that of *Leibnifr:*

(2) $$\mathsf{M}^n(uv) = (-1)^n(\tfrac{1}{2})^n \sum_{i=0}^{n+1} (-1)^i 2^i \binom{n}{i} \mathsf{M}^i u\;\mathsf{E}^i\Delta^{n-i}v.$$

To have a second formula, we put into (1) the expression for $\mathsf{E}^{n-\nu}u$ by differences, given by formula (7') § 30; the result may be written

$$\mathsf{M}^n(uv) = (\tfrac{1}{2})^n \sum_{i=0}^{n+1} \Delta^i u \sum_{\nu=0}^{n-i+1} \binom{n}{\nu} \binom{n-\nu}{i} \mathsf{E}^{n-\nu} v.$$

Applying formulae (8) and (12) of § 30, we obtain the required formula

(3) $$\mathsf{M}^n(uv) = \sum_{i=0}^{n+1} \frac{1}{2^i} \binom{n}{i} \Delta^i u\;\mathsf{E}^i\,\mathsf{M}^{n-i}v.$$

CHAPTER III.

INVERSE OPERATION OF DIFFERENCES AND MEANS. SUMS.

§ 32, Indefinite sums. The operation of differences was defined by

$$\Delta f(x) = f(x+h) - f(X).$$

From the point of view of addition, subtraction, and multiplication, the symbol Δ behaved like an algebraic quantity; but division by Δ or multiplication by Δ^{-1} has not yet been introduced. Let us put

$$(1) \qquad \Delta f(x) = f(x+h) - f(x) = \varphi(x).$$

By symbolical multiplication with Δ^{-1} we should get

$$\Delta^{-1}\varphi(x) = f(x).$$

The significationof theoperation Δ^{-1} is therefore the following: a function f(x) is to be determined, whose difference is equal to a given quantity $\varphi(x)$.

It must be remarked in the first place, that this operation is not univocal. Indeed, if $\omega(x)$ is an arbitrary function whose difference is equal to zero, then

$$\Delta[f(x) + \omega(x)] = \varphi(x)$$

and therefore

$$(2) \qquad \Delta^{-1}\varphi(x) = f(x) + \omega(x).$$

We have seen in § 29 that w(x) may be any arbitrary periodic function with period h.

If the variable x is a discontinuous one, then o(x) is equal to a constant.

From (2) we conclude that the operations Δ^{-1} and Δ are not commutative. Indeed we have

$$\Delta\Delta^{-1} = 1 \quad \text{and} \quad \Delta^{-1}\Delta \neq 1.$$

On the other hand we have seen that the operations A, **M**, **D** and **E** were commutative.

The operation Δ^{-1} being analogous to the inverse operation of derivation which is called indefinite integration, therefore the operation Δ^{-1} has been called *indefinite summation,* and instead of the symbol Δ^{-1} the symbol Σ is generally used. It may be useful to remark that the two symbols must be considered as identical.

Determination of the indefinite sum. $\varphi(x)$ being given, the problem of deducing $\Delta^{-1}\varphi(x) = f(x)$ is identical with the resolution of equation (1).

The variable being considered as a discontinuous one, it is always possible to obtain a system of differences in which the increment is equal to one, and the variable takes only integer values. For this it is sufficient to put $x=a+\xi h$. Then if $x=a+h$, the new variable will be $\xi=1$ and so on. Therefore we may suppose without restriction, that $h=1$ and x is an integer.

Starting from this supposition equation (1) will be

$$f(x+1) - f(x) = \varphi(x). \tag{3}$$

This is a linear difference equation of the first order.

André[22] considered equation (3) as equivalent to the system of x equations:

$$\begin{aligned} f(x) - f(x-1) &= \varphi(x-1) \\ f(x-1) - f(x-2) &= \varphi(x-2) \\ &\ldots \\ f(a+1) - f(a) &= \varphi(a) \end{aligned}$$

where a and x are integers and $f(a)$ is arbitrary. From these x-a equations we may determine the x-a unknowns $f(x)$, $f(x-1)$, . . ., $f(a+1)$ but it is sufficient to determine $f(x)$. For this, let us add together the above equations; we get:

$$f(x) = f(a) + \sum_{i=a}^{x} \varphi(i) = \Delta^{-1}\varphi(x). \tag{4}$$

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[22] *Désiré André,* Terme général d'une série quelconque. Annales de l'Ecole Normale Supérieure. 1878, pp. 375—408.

It is easy to verify that this solution satisfies equation (3). In this manner the operation Δ^{-1} is expressed by a sum and an arbitrary constant f(a). This is one of the reasons why this operation is called indefinite summation.

Formula (4) presents more theoretical than practical interest. Indeed, it is the expression of Δ^{-1} by a sum; and we shall see that generally to evaluate sums we make use of the operation Δ^{-1}. In other cases formula (4) may be used. For instance it may be shown directly that

$$\Delta^{-1} x = f(o) + \sum_{i=1}^{x} i = \binom{x}{2} + f(0).$$

Therefore according to (4) we have $\Delta^{-1} x = \binom{x}{2} + k.$

General *rules.* The symbol Δ^{-1} is distributive, hence

$$\Delta^{-1}(u + v + w + \ldots) = \Delta^{-1} u + \Delta^{-1} v + \Delta^{-1} w + \ldots .$$

and, if c is a constant,

$$A\text{-}{=}c\,\varphi(x) = c\,\Delta^{-1}\varphi(x).$$

Therefore we may determine the indefinite sum of a function *F(x)* by expanding it first into a *Newton* series, and then applying the above rules:

$$\underset{h}{\Delta^{-1}} F(x) = \sum_{\nu=0}^{\infty} \frac{\Delta^{\nu} F(a)}{h^{\nu}} \underset{h}{\Delta^{-1}} \binom{x-a}{\nu}_{h}.$$

We have seen that

$$\underset{h}{\Delta}\binom{x-a}{\nu+1}_{h} = h \binom{x-a}{\nu}_{h}$$

and therefore

$$\underset{h}{\Delta^{-1}} \binom{x-a}{\nu}_{h} = \frac{1}{h} \binom{x-a}{\nu+1}_{h} + k.$$

Finally we obtain

$$\text{(5)} \qquad \underset{h}{\Delta^{-1}} F(x) = \sum_{\nu=0}^{\infty} \frac{\Delta^{\nu} F(a)}{h^{\nu+1}} \binom{x-a}{\nu+1}_{h} + k.$$

In this chapter *k* will signify a quantity whose difference is equal to zero.

From formula (5) it follows that the indefinite sum of a polynomial of degree n may be expressed as a polynomial of degree $n+1$ neglecting k.

Example. Let $F(x)$ be a polynomial of degree n, putting $a=0$ and $h=1$; the indefinite sum of $F(X)$ will be

$$\Delta^{-1}F(x) = k + \binom{x}{1}F(0) + \binom{x}{2}\Delta F(0) + \binom{x}{3}\Delta^2 F(0) + \ldots + \\ + \binom{x}{n+1}\Delta^n F(0).$$

If we have $\Delta^{-1} f(x) = \varphi(x) + k$ then

$$\Delta^{-1} f(-x) = -\varphi(-x+h) + k$$

that is, if in the first case the argument does not change, then in the second the argument $-x$ is increased by h. This will be useful later on.

Remark. The inverse differences may be determined by symbolical methods. We have

$$(6) \qquad \frac{1}{a} = -\frac{1}{1-E} = -(1+E+E^2+\ldots).$$

Taking account of what has been said in § 6 concerning the necessary precautions in the case of infinite series of symbolical expressions we may perform the operations figuring in equation (6), starting from $f(x)$. We find

$$(7) \qquad A\text{-}'f(x) = -\sum_{i=0}^{\infty} f(x+i) = -\sum_{i=x}^{\infty} f(i).$$

This may be easily checked; indeed, the difference of the quantity in the second member is equal to $f(x)$.

Finally we conclude that the inverse difference may be expressed by aid of a sum, if certain conditions of convergence are fulfilled.

§ 33. Indefinite sum obtained by inversion. There are different methods of determining the indefinite sums. The first is that of the inversion of the formulae obtained by calculating the differences. For instance, if we have found

$$\Delta f(x) = c\,\varphi(x-a)$$

then we conclude that

$$\Delta^{-1}\varphi(x) = \frac{1}{c}\, f(x+a) + k\,.$$

In this way we get the following formulae:

(1) $\Delta_h^{-1} c = \frac{c}{h} x + k$

(2) $\Delta_h^{-1} (x)_{n,h} = \frac{(x)_{n+1,h}}{h(n+1)} + k$

(3) $\Delta_h^{-1} \binom{x}{n}_h = \frac{1}{h} \binom{x}{n+1}_h + k$

(4) $\Delta_h^{-1} a^x = \frac{a^x}{a^h - 1} + k$

(5) $\Delta^{-1} \frac{1}{2^x} = -\frac{1}{2^{x-1}} + k$ $(h=1)$

(6) $\Delta_h^{-1} \cos(ax+b) = \frac{\cos(ax+b-\frac{1}{2}ah-\frac{1}{2}\pi)}{2\sin\frac{1}{2}ah} + k$

(7) $\Delta_h^{-1} \sin(ax+b) = \frac{\sin(ax+b-\frac{1}{2}ah-\frac{1}{2}\pi)}{2\sin\frac{1}{2}ah} + k$

(8) $\Delta^{-1} \frac{1}{x} = F(x-1) + k$ $(h=1)$

(9) $\Delta^{-1} \frac{1}{x^2} = -F(x-1) + k$ $(h=1)$

(10) $\Delta_h^{-1} (x)_{-n,h} = -\frac{1}{(n-1)h} (x)_{-n+1,h} + k$ $(n>1)$

(11) $\Delta^{-1} \log x = \log \Gamma(x) + k$ $(h=1)$

(12) $\Delta^{-1} \binom{-x}{n} = (-1)^n \Delta^{-1} \binom{x+n-1}{n} = -\binom{-x+1}{n+1} + k$ $(h=1)$

(13) $\Delta^{-1} (-1)^x \binom{n}{x} = (-1)^{x+1} \binom{n-1}{x-1} + k$ $(h=1)$

(14) $\Delta^{-1} (-1)^x a^x = (-1)^{x+1} \frac{a^x}{a+1} + k$ $(h=1)$

(15) $\Delta^{-1} (-1)^x \cos(ax+b) = (-1)^x \frac{\cos(ax+b-\frac{1}{2}a)}{2\cos\frac{1}{2}a} + k$ $(h=1)$.

Remark. In the formulae **10, 12,** and 13, the argument x is diminished by *h.*

§ 34. Indefinite sum obtained by summation by parts, Starting from the formula of the difference of a product obtained in § 30 (formula 2),

$$\Delta[U(x)\ V_1(x)] = U(x)\ \Delta V_1(x) + V_1(x+h)\ \Delta U(x).$$

Writing $\Delta V_1(x) =$ V,(x), and performing the operation Δ^{-1} on both members of this equation, we have

$$(1)\qquad \Delta^{-1}\,[U(x)\ V_0(x)] = U(x)\ V_1(x) - \Delta^{-1}\,[V_1(x+h)\ \Delta U(x)].$$

This formula, being analogous to that of integration by parts, is called formula of summation by parts. It becomes useful if the indefinite sum of the first member is unknown, while that of the second member may be determined.

Examples. In the following examples we will denote the first factor by $U(x)$ and the second by V,(x). We find

$$(2)\qquad \underset{h}{\Delta^{-1}}\, xa^x = \frac{xa^x}{a^h-1} - \Delta^{-1}\frac{ha^{x+h}}{a^h-1} = \frac{xa^x}{a^h-1} - \frac{ha^{x+h}}{(a^h-1)^2} + k$$

$$(3)\ \underset{h}{\Delta^{-1}}\, x\sin x = \frac{x\sin(x-\tfrac12h-\tfrac12\pi)}{2\sin\tfrac12 h} - \Delta^{-1}\frac{h\sin(x+\tfrac12h-\tfrac12\pi)}{2\sin\tfrac12 h} =$$

$$= \frac{x\sin(x-\tfrac12h-\tfrac12\pi)}{2\sin\tfrac12 h} - \frac{h\sin(x-\pi)}{(2\sin\tfrac12 h)^2} + k$$

(4) A-'(;] $\binom{x}{3} = \binom{x}{1}\binom{x}{4} - \Delta^{-1}$ ("a') =

$$= \binom{x}{1}\binom{x}{4} - \binom{x+1}{5} + k$$

$$(5)\qquad \Delta^{-1}\, x\,.\,\frac{1}{2^x} = -\frac{x}{2^{x-1}} + \Delta^{-1}\frac{1}{2^x} = -\frac{x+1}{2^{x-1}} + k$$

$$(6)\qquad \Delta^{-1}\, F(x)\,.\,1 = F(x)\,.\,x - \Delta^{-1}\frac{x+1}{x+1} = x\,[\,F(x) - 1\,] + k$$

$$(7)\quad \Delta^{-1}\, F(x)\,,\,1 = F(x)\,.\,x + \Delta^{-1}\frac{x+1}{(x+1)^2} = x\,F(x) + F(x) + k$$

If $\omega(x)$ is a periodic function with period *h, we* **have**

$$(8)\qquad \Delta^{-1}\,[\omega(x)\ V_0(x)\,] = \omega(x)\ V_1(x) + k.$$

Repeated summation by parts. If we introduce the following notation

$$\Delta^{-1} V_i(x) = V_{i+1}(x)$$

and start from formula (1), repeating the summation by parts on the last term of the second member, we obtain

$$\Delta^{-1}[V_0(x)U(x)] = V_1(x)U(x) - V_2(x+h)\Delta U(x) + \\ + \Delta^{-1}[V_2(x+2h)\Delta^2 U(x)].$$

Again performing the summation by parts on the last term, we get

$$\Delta^{-1}[V_0(x)U(x)] = V_1(x)U(x) - V_2(x+h)\Delta U(x) + \\ + V_3(x+2h)\Delta^2 U(x) - \Delta^{-1}[V_3(x+3h)\Delta^3 U(x)]$$

and so on; finally we have

$$(10) \quad \Delta^{-1}[V_0(x)U(x)] = V_1(x)U(x) - V_2(x+h)\Delta U(x) + \\ + V_3(x+2h)\Delta^2 U(x) - \ldots + (-1)^{n-1} V_n(x+nh-h)\Delta^{n-1} U(x) + \\ + (-1)^n \Delta^{-1}[V_n(x+nh)\Delta^n U(x)].$$

This formula will be especially useful if $U(x)$ is a polynomial of degree $n-1$; in this case $\Delta^n U(x) = 0$, the last term of (10) vanishes, and the problem is solved. If $U(x)$ is not a polynomial, the last term will be considered as the remainder of the series.

Example 1.

$$(11) \quad \Delta^{-1} \binom{x}{3}\binom{x}{2} = \binom{x}{4}\binom{x}{2} - \binom{x+1}{5}\binom{x}{1} + \binom{x+2}{6} + k,$$

$$(12) \quad \Delta^{-1} \binom{x}{m}\binom{x}{n} = \sum_{s=0}^{n+1} (-1)^s \binom{x+s}{m+1+s}\binom{x}{n-s} + k.$$

We have seen that there are functions whose indefinite sum is a function of the same kind but in which x is diminished by h. For instance if $h = 1$ we have

$$\Delta^{-1}\, 2^{-x} = -2^{-x+1} + k,$$

$$\Delta^{-1} \binom{-x}{n} = -\binom{-x+1}{n+1} + k.$$

Therefore if V_0 is such a function, the argument will remain constant throughout the operations of repeated summation by parts. If moreover $U(x)$ is a polynomial of degree n, then formula (10) will give

$$\Delta^{-1}\left[2^{-x}\,U(x)\right] = -2^{-x+1}\sum_{i=0}^{n+1}\Delta^{i}U(x) + k$$

and

$$\Delta^{-1}\left[\binom{-x}{n}U(x)\right] = -\sum_{i=0}^{n+1}\binom{-x+1}{n+1+i}\Delta^{i}U(x).$$

Formula (10) may also be applied in some cases in which $U(x)$ is not a polynomial, provided that the corresponding series (10) be convergent. Then we have

$$(14)\qquad \Delta^{-1}\left[V_0(x)U(x)\right] = \sum_{m=1}^{\infty}(-1)^{m-1}\,V_m(x+mh-h)\,\Delta^{m-1}U(x).$$

Example 2. Given $x/2^x$; if we put $V_0=x$ and $U=1/2^x$ then formula (14) will give

$$\Delta^{-1}\frac{x}{2^x} = \binom{x}{2}2^{-x} + \binom{x+1}{3}2^{-x-1} + \dots + \binom{x+n}{n+2}2^{-x-n} + \dots.$$

It is easily seen that this is equal to $2-(1+x)/2^{x-1}$ which result could have been obtained directly by putting into (14) $V_0 = 1/2^x$ and $U=x$.

Example 3. Given $2^{-x}/x$. Putting $V_0=2^{-x}$ and $U=1/x$ by aid of formula (14) we find according to formulae 7, § 16 and 6, § 33

$$\Delta^{-1}\frac{2^{-x}}{x} = 2^{-x+1}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)\binom{x+n}{n+1}} + k$$

$$= 2^{-x+1}\sum_{n=0}^{\infty}(-1)^{n+1}B(n+1,x) + k.$$

Remark. Condorcet, in his "Essai sur l'Application de l'Analyse à la Probabilite des Decisions" (Paris, 1785, p. 163) has found a formula of repeated summation by parts, somewhat different from (10), which written in our notation is the following:

$$\Delta^{-1}\left[V_0U\right] = V_1U - (V_2 + V_1)\Delta U + (V_3 + 2V_2 + V_1)\Delta^2U - \dots$$
$$+ (-1)^m\Delta^mU\sum_{i=1}^{m+1}\binom{m}{i}V_{m+1-i},$$

the argument x in every term being the same.

§ 35. Summation by parts of alternate functions. We have **seen that**

$$\Delta(-1)^x = 2(-1)^{x+1}.$$

From this we conclude

(1) A-’ $(-1)^x = \frac{1}{2}(-1)^{x+1}$.

Putting into formula (10) § 34 *V,(X)* = (-1)” and U(x) = f(x) we find

(2) A-’ $(-1)^x$ *f (x)* = $\frac{1}{2}(-1)^{x+1}$ [f(x) — $\frac{1}{2}\Delta f(x) +$

$$+\frac{1}{2^2}\Delta^2 f(x) - \ldots + \text{(-1)”} \frac{1}{2^n}\Delta^n f(x) \,] + k.$$

It must be noted that this formula holds only if *f(x)* is **a** polynomial of degree n; moreover if in this alternate function x is an integer.

Example 1. Let *f(x)* = x; then

(3) $$\Delta^{-1}(-1)^x x = \tfrac{1}{2}(-1)^{x+1}(x-\tfrac{1}{2}) + k.$$

If *f(x)* is not a polynomial, we apply formula (14) of § 34, but the condition of convergence must be satisfied.

Example 2. Given *f(x)* = $1/(x+1)$. We have

(4) $$\Delta^{-1}(-1)^x \frac{1}{x+1} = \tfrac{1}{2}(-1)^{x+1} \sum_{m=0}^{\infty} \frac{m!}{2^m (x+m+1)_{m+1}} + k.$$

It is easy to see that in this case the condition mentioned is satisfied,

Remark. It would be possible to calculate in the same way $\Delta^{-1}(-1)^x a^x$ but it is shorter to put $\Delta^{-1}(-a)^x$ and then apply the known formula of exponential functions:

$$\Delta^{-1}(-a)^x = -\frac{(-a)^x}{a+1}.$$

Summation of alternate functions by aid of inverse means. From formula (3) of § 30 we deduce

A-’ [Mu Δv] = $uv - \Delta^{-1}$ [Mv Au |.

Putting into this equation $v = (-1)^x$ and $u = f(x)$ we find $\mathsf{M}v = 0$, hence we have

$$(5) \qquad \Delta^{-1} [(-1)^x f(x)] = \tfrac{1}{2}(-1)^{x+1} M^{-1} f(x).$$

That is, the indefinite sum of the alternate function figuring in the first member, is expressed by aid of the inverse mean of f(x) (formulae 3, § 38 and 10, § 39).

§ 36. Indefinite sums determined by aid of difference equations, The indefinite sum is according to our definition the solution of the linear difference equation of the first order

$$(1) \qquad f(x+1) - f(x) = \varphi(x)$$

where $\varphi(x)$ is given.

There are several methods for the resolution of these equations, but for our purpose only the methods may serve in which the solution is not obtained by inversion of differences.

One of these methods is that of the generating functions, due to ***Laplace,*** which is applicable if x is an integer.

According to formula (3) § 11, if u is the generating function of *f(x),* then that of f(x+l) will be

$$G\, f(x+1) = \frac{u - f(0)}{t}$$

Let us denote by *R(f)* the generating function of $\varphi(x)$ and not that the corresponding generating functions satisfy the difference equation (1). We have

$$\frac{u - f(0)}{t} - u = R(t);$$

this gives

$$u = \frac{tR(t) + f(0)}{1 - t}.$$

where *f(O)* is an arbitrary constant.

Finally the coefficient of t^x in the expansion of u will be cqual to the required indefinite sum $f(x)$.

Example. Let $\varphi(x) = x^2$. To determine the generating function *R(f)* of x^2 let us remark that $x^2 = 2\binom{x}{2} + \binom{x}{1}$ and the generating function of the second member is, according to formula (14) § 10,

$$Gx^2 = R(f) = \frac{t+t^2}{(1-t)^3};$$

therefore

$$u = \frac{t^2+t^3}{(1-t)^4} + \frac{f(0)}{1-t}.$$

The expansion of u gives

$$(t^2+t^3)(1-t)^{-4} = (t^2+t^3)\,\Sigma\binom{-4}{i}(-1)^i t^i =$$
$$= (t^2+t^3)\,\Sigma\binom{i+3}{i} t^i.$$

Putting into the first term $i=x-2$ and into the second $i=x-3$ we have

$$f(x) = \Delta^{-1} x^2 = \binom{x+1}{x-2} + \binom{x}{x-3} + k = \frac{1}{6}\,x(x-1)(2x-1) + k.$$

§ 37. Differences, sums and means of an infinite series. Let us suppose that the function $f(x)$ is expanded into an infinite series which is convergent in the interval a, b:

(1) $$f(x) = u_0(x) + u_1(x) + \ldots + u_n(x) + \ldots.$$

The function

$$f(x+1) = u_0(x+1) + u_1(x+1) + \ldots + u_n(x+1) + \ldots.$$

will be convergent in the interval a-1, b-1.

It is known that the difference of two convergent series is also a convergent series, whose sum is equal to the difference of the sums of the given series. Therefore

(2) $$\Delta f(x) = f(x+1) - f(x) =$$
$$\Delta u_0(x) + \Delta u_1(x) + \ldots + \Delta u_n(x) + \ldots$$

Consequently if $f(x)$ is given by its expansion into a convergent series, the *difference* of $f(x)$ may be obtained, by taking the difference of the series term by term.

We have $Mf(x) = \tfrac{1}{2}[f(x+1) + f(x)]$; therefore for the same reasons, if f(x) is given by a convergent series, then to obtain the *mean* it is sufficient to determine the mean of the series term by term.

To obtain the *indefinite sum* of $f(x)$ given by formula (1) we determine it term by term.

(3) $\Delta^{-1}f(x) = \Delta^{-1}u_0(x) + \Delta^{-1}u_1(x) + \ldots + \Delta^{-1}u_n(x) + \ldots,$

If this series is convergent, then in consequence of what has been said before, the difference of the second member is equal to $f(x)$ and therefore the required indefinite sum is given by (3).

§ 38. Inverse operation of the mean. The mean has been defined by the following operation

$$\underset{h}{\mathbf{M}} f(x) = \tfrac{1}{2}\,[f(x+h) + f(x)].$$

We have seen that from the point of view of addition, subtraction, and multiplication the symbol $\mathbf{M}$ behaved like an algebraic quantity; but division by $\mathbf{M}$ or multiplication by $\mathbf{M}^{-1}$ has not yet been introduced here. Let us put

(1) $$\underset{h}{\mathbf{M}} f(x) = \tfrac{1}{2}\,[f(x+h) + f(x)] = \varphi(x).$$

Multiplying both members of this quantity by $\mathbf{M}^{-1}$ we shall have

$$f(x) = \mathbf{M}^{-1}\varphi(x).$$

The significance of the operation $\mathbf{M}^{-1}$ is therefore the following: a function $f(x)$ is to be determined so that its mean shall be equal to a given quantity $\varphi(x)$.

Here the same difficulty presents itself as in the case of the operation Δ^{-1}. In § 29 we have seen that there are functions whose mean is equal to zero. We may write these functions in the form

$$\cos\left(\frac{\pi}{h}x + b\right) . \, \omega(x)$$

where $\omega(x)$ is an arbitrary periodical function with period h.

Therefore in consequence of (1) we shall have also

$$\underset{h}{\mathbf{M}} \left[f(x) + \cos\left(\frac{\pi}{h}x + b\right)\omega(x)\right] = \varphi(x)$$

and moreover,

$$\underset{h}{\mathbf{M}}^{-1}\varphi(x) = f(x) + \cos\left(\frac{\pi}{h}x + b\right)\omega(x).$$

From the preceding it follows that

$$\mathbf{M}\mathbf{M}^{-1} = 1$$

but we may have

$$\mathbf{M}^{-1}\mathbf{M} \neq 1.$$

Remark. If the required function f(x) is necessarily a polynomial, then the operation $\mathbf{M}^{-1}$ is univocal and there is only one solution.

Determination of $\mathbf{M}^{-1}\varphi(x)$. In the case of a discontinuous and equidistant variable we may always suppose without restriction that x is an integer and $h=1$, since by introducing a new variable this can always be obtained.

Supposing $h=1$ and x integer, $\mathbf{M}^{-1}\varphi(x) = f(x)$ is the solution of the equation

$$(2) \qquad f(x+1) + f(x) = 2\varphi(x)$$

where $\varphi(x)$ is given.

This is a linear difference equation of the first order: it can be solved by *André's* method [loc. cit. 221, in which equation (2) is considered as being equivalent to the following system of equations

$$\begin{aligned} f(x) + f(x-1) &= 2\varphi(x-1) \\ f(x-1) + f(x-2) &= 2\varphi(x-2) \\ &\dots \\ f(a+1) + f(a) &= 2\varphi(a) \end{aligned}$$

where x and a are integers and $f(a)$ is arbitrary.

Multiplying the first equation by $(-1)^2$, the second by $(-1)^3$ and so on, the n th by $(-1)^{n+1}$; then adding them together we obtain

$$(3) \quad f(x) = (-1)^{x+a} f(a) + 2\sum_{i=a}^{x} (-1)^{x+i+1}\varphi(i) = \mathbf{M}^{-1}\varphi(x).$$

It is easy to verify that $f(x)$ satisfies equation (2) ; indeed we have

$$f(x) = (-1)^{a+x} f(a) + 2[\varphi(x-1) - \varphi(x-2) + \dots + (-1)^{x+a+1}\varphi(a)]$$

and from this

$$f(x+1) = (-1)^{x+a+1} f(a) + 2[\varphi(x) - \varphi(x-1) + \dots + (-1)^{x+a}\varphi(a)];$$

therefore

$$f(x) + f(x+1) = 2\varphi(x).$$

Formula (3) is more important from the theoretical than from the practical point of view, but nevertheless it may be useful in some cases. It is an expression of the inverse mean by alternate sums. If the alternate sum of $\varphi(x)$ is known, formula (3) may be applied directly.

Example. Let $\varphi(x) = c^x$, and $a=0$. It can be shown that

$$\sum_{i=0}^{x} (-1)^i c^i = \sum_{i=0}^{x} (-c)^i = \frac{1-(-c)^x}{c+1}$$

therefore formula (3) will give

$$\mathbf{M}^{-1} c^x = (-1)^x K + \frac{2c^x}{c+1}$$

This may be verified by inversion.

§ 39. **Other methods of obtaining inverse means,** *A. Inversion.* In the preceding chapter we have determined the mean of several functions t(x); now inverting the results we may obtain formulae for $\mathbf{M}^{-1}$. For instance, having found

$$\mathbf{M} f(x) = c\,\varphi(x+a)$$

we deduce

$$\mathbf{M}^{-1} \varphi(x) = \frac{1}{c} f(x-a) + \chi.$$

In this paragraph χ signifies an arbitrary function, whose mean is equal to zero.

By this method we obtain the following formulae:

(1) $$\mathbf{M}^{-1} c = c + \chi$$

(2) $$\mathop{\mathbf{M}}_{h}^{-1} x = x - \tfrac{1}{2}h + \chi$$

(3) $$\mathbf{M}^{-1} \binom{n}{x} = 2\binom{n-1}{x-1} + \chi$$

(4) $$\mathop{\mathbf{M}}_{h}^{-1} a^x = a^x \left(\frac{2}{a^h+1}\right) + \chi$$

(5) $$\mathop{\mathbf{M}}_{\omega}^{-1} \cosh x = \frac{\cosh(x-\tfrac{1}{2}\omega)}{\cosh \tfrac{1}{2}\omega} + \chi$$

(6) $$\mathop{\mathbf{M}}_{\omega}^{-1} \sinh x = \frac{\sinh(x-\tfrac{1}{2}\omega)}{\cosh \tfrac{1}{2}\omega} + \chi$$

(7) $$\mathop{\mathbf{M}}_{h}^{-1} \cos(ax+b) = \frac{\cos(ax+b-\tfrac{1}{2}ah)}{\cos \tfrac{1}{2}ah} + \chi$$

$$\text{(8)}\qquad \underset{h}{\mathbf{M}}^{-1} \sin\,(ax+b) = \frac{\sin\,(ax+b-\frac{1}{2}ah)}{\cos \frac{1}{2}ah} + \chi.$$

B. Symbolical Methods. We have

$$\text{(9)}\qquad \Delta\,[\,(-1)^x F(x)] = (-1)^{x+1}\,[F(x+1)+F(x)] =$$
$$= 2(-1)^{x+1}\,\mathbf{M}\,F(x).$$

Putting into this equation $F(x) = \mathbf{M}^{-1} f(x)$ we get

$$2(-1)^x f(x) = \Delta[\,(-1)^{x+1}\,\mathbf{M}^{-1} f(x)],$$

Performing on both members the operation Δ^{-1} we find

$$\text{(10)}\qquad \mathbf{M}^{-1} f(x) = 2(-1)^{x+1}\,\Delta^{-1}\,[(-1)^x f(x)\,].$$

This is identical with formula (3) of § 38, giving the inverse mean expressed by aid of an alternate sum.

Putting into (10) $f(x) = (-1)^x \varphi(x)$ we obtain an indefinite sum expressed by the inverse mean of the alternate function.

$$\text{(11)}\qquad \Delta^{-1}\,\varphi(x) = \tfrac{1}{2}(-1)^{x+1}\,\mathbf{M}^{-1}\,[(-1)^x\varphi(x)].$$

Examples. To determine $\mathbf{M}^{-1}\binom{x}{n}$ we may use formula (10). It gives

$$\mathbf{M}^{-1}\binom{x}{n} = 2(-1)^{x+1}\,\Delta^{-1}\left[\,(-1)^x\binom{x}{n}\right].$$

By repeated summation by parts performed on the second member we find (formula 10, § 34)

$$\mathbf{M}^{-1}\binom{x}{n} = \sum_{m=0}^{n+1}(-1)^m\frac{1}{2^m}\binom{x}{n-m} + \chi.$$

From formula $\mathbf{M} = 1 + \frac{1}{2}\Delta$ of § 6 we deduce

$$\text{(12)}\qquad \mathbf{M}^{-1} = (1+\tfrac{1}{2}\Delta)^{-1} = \sum_{m=0}^{\infty}(-1)^m\frac{1}{2^m}\,\Delta^m.$$

This formula applied to polynomials presents no difficulty; indeed in this case the series of the second member is finite. Formula (12) would lead in the case of the preceding example directly to the result obtained above.

Formula (12) may be applied to other functions, if certain conditions are satisfied.

C. By Euler's Polynomials. If a function f(x) is expanded into a convergent *Maclautin* series

$$f(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} D^m f(0)$$

then we may obtain the inverse mean of the function, remarking that

$$M^{-1} \frac{x^m}{m!} = E_m(x) + \chi$$

where $E_m(x)$ represents *Euler's* polynomial of degree m (§ 100). The operation performed term-by-term gives

(13) $$M^{-1} f(x) = \sum_{m=0}^{\infty} E_m(x) D^m f(0) + \chi.$$

If this series is convergent, then according to what has been said above, the mean of the second member will be equal to *f(x)* and therefore the inverse mean of *f(x)* is given by (13).

From this we may deduce a formula giving the indefinite sum of an alternate function $(-1)^x$ *f(x).* By aid of formula (10) we get

(14) $$A^{-\prime} [(-1)^x f(x)] = \tfrac{1}{2}(-1)^{x+1} \sum_{m=0}^{\infty} E,(x) D^m f(0) + k.$$

Particular case. Given *f(x)* $= x^n$

(15) $$\Delta^{-1} [(-1)^x x^n] = \tfrac{1}{2}(-1)^{x+1} n! E,(x) + k.$$

D. By aid of β_1 functions. A function *f(x)* being expanded into a convergent reciprocal power series;

(16) $$f(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^n}.$$

To determine its inverse mean we remark that (§ 122)

$$M \beta_1(x-1) = \frac{1}{x}$$

and

$$M D^n \beta_1 (x-1) = \frac{(-1)^n n!}{x^{n+1}}$$

and therefore

$$M^{-1} \frac{1}{x^{n+1}} = \frac{(-1)^n}{n!} D^n \beta_1(x-1).$$

Hence, performing the operation term-by-term we get

$$\mathbf{M}^{-1} f(x) = c_0 + \sum_{n=0}^{\infty} (-1)^n c_{n+1} \frac{D^n \beta_1(x-1)}{n!} + \chi.$$

If this series is convergent it gives the inverse mean of $f(x)$. By aid of **(10)** we may obtain $\Delta^{-1} (-1)^x f(x)$.

§ 40. Sums. It has been mentioned before (§ 2) that in the case of a discontinuous variable with equal intervals, we may always suppose that x is an integer and that $h=1$. Let us consider in this case the indefinite sum:

$$\Delta^{-1} \varphi(x) = f(x) + k.$$

If a and n are integers we may write

$$\Delta^{-1} \varphi(a+n) - \Delta^{-1} \varphi(a) = f(a+n) - f(a).$$

In § 32 we have seen that the operation Δ^{-1} may be expressed by a sum, so that

$$\Delta^{-1} \varphi(x) = k + \varphi(a) + \varphi(a+1) + \ldots + \varphi(x-1) =$$
$$= \sum_{i=a}^{x} \varphi(i) + k.$$

Therefore from the above equation it will follow that

$$(1) \qquad f(a+n) - f(a) = \varphi(a) + \varphi(a+1) + \ldots +$$
$$+ \varphi(a+n-1) = \sum_{i=a}^{a+n} \varphi(i).$$

Hence to calculate the sum of $\varphi(x)$ from $x=a$ to $x=a+n$ it is sufficient to determine $f(x)$, the indefinite sum of $\varphi(x)$, and then obtain $f(a+n)$ by putting x equal to the upper limit, and $f(a)$ by putting x equal to the lower limit. The required sum is equal to the difference of these quantities. The process is exactly the same as that used to determine a definite integral. Moreover we have, as in the case of these integrals,

$$\sum_{x=a}^{b} \varphi(x) + \sum_{x=b}^{c} \varphi(x) = \sum_{x=a}^{c} \varphi(x) = f(c) - f(a)$$

and

$$\sum_{x=a}^{a} \varphi(x) = 0.$$

This harmony and the above simple rule are due to the definition of the sum used in the *Calculus of Finite Differences,* which is somewhat different from the ordinary mathematical definition. Indeed, the definition used here is the following:

$$\varphi(a) + \varphi(a+1) + \ldots \varphi(a+n-1) = \sum_{x=a}^{a+n} \varphi(x)$$

that is, the term $\varphi(a)$ corresponding to the lower limit is included in the sum, but not the term $\varphi(a+n)$ corresponding to the upper limit. In the ordinary notation the sum above would be denoted by

$$\sum_{x=a}^{a+n-1} \varphi(x)$$

but with this notation the concord between the two calculi would cease, and the rules of summation would be complicated.

However, it should be mentioned that our notation is not symmetrical with respect to the limits; this inconvenience is due to the want of symmetry of the notation of forward differences.

Remark. In § 32 formula (7) the inverse difference has been expressed by aid of a sum. Starting from this we obtain the sum of $f(x)$ from $x=a$ to $x=z$, indeed

$$[\Delta^{-1} f(x)]_{x=z} - [\Delta^{-1} f(x)]_{x=a} =$$

$$= -\sum_{i=z}^{\infty} f(i) + \sum_{i=a}^{\infty} f(i) = \sum_{x=a}^{z} f(x).$$

§ 41. Sums determined by indefinite sums.

1. Sum of binomial coefficients. In § 33 we found, if $h=1$, that

$$\Delta^{-1}\binom{x+c}{n} = \binom{x+c}{n+1} + k$$

therefore, a and m being integers

$$\text{(1)} \qquad \sum_{x=a}^{a+m} \binom{x+c}{n} = \binom{a+m+c}{n+1} - \binom{a+c}{n+1},$$

Particular case of $c=n$ and $a=0$

$$\sum_{x=0}^{m} \binom{x+n}{n} = \binom{m+n}{n+1}.$$

Sum of a binomial coefficient with ***negative** argument; we* had (12). § 33,

$$\Delta^{-1}\binom{-x}{n} = -\binom{-x+1}{n+1} + k;$$

therefore

$$\text{(2)} \qquad \sum_{x=0}^{m}\binom{-x}{n} = -\binom{-m+1}{n+1}.$$

To determine the sum of $(-1)^x\binom{x}{m}\binom{-n}{x}$ let us remark that we have

$$(-1)^x\binom{x}{m}\binom{-n}{x} = \binom{n+m-1}{m}\binom{x+n-1}{n+m-1}$$

and therefore

$$\text{(3)} \qquad \sum_{x=0}^{a+1}(-1)^x\binom{x}{m}\binom{-n}{x} = (-1)^m\binom{n+a}{n+m}\binom{-n}{m}.$$

2. *Sum of an arithmetical progression.* We have seen that the general term of an arithmetical progression may be represented by

$$f(x) = a+hx.$$

The corresponding indefinite sum is

$$\Delta^{-1} f(x) = \binom{x}{2} h + \binom{x}{1} a + k.$$

Finally the sum of the n first terms will be

$$\text{(4)} \qquad S = \sum_{x=0}^{n} f(x) = \binom{n}{2} h + \binom{n}{1} a.$$

***Particular** cases.* If $a=1$ and $h=1$ then $S = \binom{n}{2} + n$. If $a=1$ and $h=2$ then $S = 2\binom{n}{2} + n = n^2$.

3. *Sum of an arithmetical progression of order* n. We have seen in § 2 that the general term of this progression is a polynomial $f(x)$ of degree n. To obtain the sum of $f(x)$ it is best to expand it into a *Newton* series, We have

$$f(x) = f(o) + \binom{x}{1}\Delta f(0) + \binom{x}{2}\Delta^2 f(0) + \dots + \binom{x}{n}\Delta^n f(0).$$

The corresponding indefinite sum is (§ 33)

$$\Delta^{-1} f(x) = \sum_{m=0}^{n+1} \binom{x}{m+1} \Delta^m f(0) + k.$$

Finally the sum of first μ terms is

$$\sum_{x=0}^{\mu} f(x) = \sum_{m=0}^{n+1} \binom{\mu}{m+1} \Delta^m f(0).$$

Hence, to obtain the sum required, it is sufficient to know the differences of $f(x)$ for $x=0$.

Example 1. Given $f(x) = (2x+1)^2$. To obtain the differences the simplest way is to write a table of the first values of the function:

1		9		25
	8		16	
		8		

Therefore $f(0) = 1$, $\Delta f(0) = 8$, $\Delta^2 f(0) = 8$. Since the polynomial is of the second degree, the higher differences will be equal to zero.

In this manner it would be possible to determine $\sum_{x=0}^{\mu} x^n$ but later on we shall see shorter methods, by aid of *Stirling* numbers, or by *Bernoulli* polynomials.

If the sum of $f(x)$ is required for $x=a, a+h, \ldots, a+(\mu-1)h$, that is, if the increment is equal to h, this should be indicated in the symbol of the sum; for instance

$$(5) \qquad \sum_{x=a}^{a+\mu h}{}_h \; f(x).$$

To determine this sum we generally introduce a new variable $\xi = (x-a)/h$; then ξ will be an integer, and $\Delta\xi = 1$, and the preceding methods may be applied, We expand $f(a+\xi h)$ into a *Newton* series, determine the indefinite sum, and put into it the limits. We shall have, if $f(x)$ is a polynomial of degree n

$$(6) \qquad \sum_{\xi=0}^{\mu} f(a+\xi h) = \sum_{m=0}^{n+1} \binom{\mu}{m+1} \Delta^m f(a).$$

The significance of $\Delta^m f(a)$ in this formula is $[\Delta^m f(a+\xi h)]_{\xi=0}$.

Example 2. Given $f(x) = x^2$. Introducing the new variable, we get $f(a+\xi h) = (a+\xi h)^2$; determining the differences with respect to ξ we have

$$[\Delta f(a+\xi h)]_{\xi=0} = 2ah + h^2 \text{ and } [\Delta^2 f(a+\xi h)]_{\xi=0} = 2h^2.$$

According to formula (6) we find

$$(7) \qquad \sum_{x=a}^{a+\mu h} {}_h\, f(x) = \sum_{\xi=0}^{\mu} f(a+\xi h) = \mu a^2 + \binom{\mu}{2}(2ah+h^2) + \binom{\mu}{3} 2h^2.$$

On the other hand, it is often possible to obtain the sum (5) directly , by determining

$$\underset{h}{\Delta}^{-1} f(x)$$

and putting the values of the limits into the results.

For this we expand $f(x)$ into a series of generalised binomial coefficients whose increment h is the same as that of x in $f(x)$; if this function is a polynomial of degree n, then we shall have

$$f(x) = \sum_{m=0}^{n+1} \binom{x-a}{m}_h \frac{\underset{h}{\Delta}^m f(a)}{h^m}$$

The corresponding indefinite sum will be

$$\Delta^{-1} f(x) = \sum_{m=0}^{n+1} \binom{x-a}{m+1}_h \frac{\underset{h}{\Delta}^m f(a)}{h^{m+1}} + k$$

and therefore the required sum is

$$(8) \qquad \sum_{x=a}^{a+\mu h} {}_h\, f(x) = \sum_{m=0}^{n+1} \binom{\mu h}{m+1}_h \frac{\underset{h}{\Delta}^m f(a)}{h^{m+1}} = \sum_{m=0}^{n+1} \binom{\mu}{m+1} \Delta^m f(a).$$

The significance $\Delta^m f$ (a) in this formula is $[\underset{h}{\Delta}^m f(x)]_{x=a}$, since

$$[\Delta^m f(a+\xi h)]_{\xi=0} = [\underset{h}{\Delta}^m f(x)]_{x=a} ;$$

therefore formulae (6) and (8) are identical.

Example 3. $f(x) = x^2$. The expansion gives

$$x^2 = a^2 + \binom{x-a}{1}_h (2a+h) + 2\binom{x-a}{2}_h.$$

The indefinite sum is

$$\underset{h}{\Delta}^{-1} x^2 = \binom{x-a}{1}_h \frac{a^2}{h} + \binom{x-a}{2}_h \frac{2a+h}{h} + \binom{x-a}{3}_h \frac{2}{h} + k$$

and finally

$$\sum_{x=a}^{a+\mu h} x^2 = \mu a^2 + \binom{\mu}{2}(2a+h)h + \binom{\mu}{3}2h^2.$$

§ 42. **Sum of reciprocal factorials by indefinite sums.** If the increment of the factorial is equal to that of the variable, then the summation presents no difficulties. For instance if

$$f(x) = (x)_{-n,h} = \frac{1}{(x+nh)_{n,h}}$$

and the sum of f(x) is required for $x=a$, $x=a+h$, $x=a+2h$, $\dots$ $x=a+(\mu-1)h$, then according to formula (10) of **§** 33 we have, if $n>1$

$$\Delta^{-1}(x)_{-n,h} = -\frac{1}{h(n-1)}(x)_{-n+1,h} + k =$$

$$= -\frac{1}{h(n-1)(x+nh-h)_{n-1,h}} + k$$

and therefore

$$\sum_{x=a}^{a+\mu h}(x)_{-n,h} =$$

$$= \frac{1}{h(n-1)}\left[\frac{1}{(a+nh-h)_{n-1,h}} - \frac{1}{(a+\mu h+nh-h)_{n-1,h}}\right]$$

and if $\mu=\infty$

$$\sum_{x=a}^{\infty}(x)_{-n,h} = \frac{1}{h(n-1)(a+nh-h)_{n-1,h}}.$$

Example 1. Let $n=2$, $h=2$, and $a=1$. Then

$$(x)_{-2,2} = \frac{1}{(x+2)(x+4)}$$

and

$$\sum_{x=1}^{\infty}{}_h \frac{1}{(x+2)(x+4)} = \frac{1}{2(1)(1+4-2)} = \frac{1}{6}.$$

Remark. If the increment h of the factorial is not equal to that of the variable, it is advisable to introduce a new variable whose increment is equal to h.

There is another method: to introduce a new variable whose increment is equal to one and expand the function into a series of reciprocal factorials whose increment is also equal to one.

This should be done also in cases when the sum of a fraction is required whose numerator $f(x)$ is a polynomial and the denominator a factorial, the degree of the numerator being less than that of the denominator. For instance, the sum of

$$(2) \qquad \frac{f(x)}{(x+\alpha)(x+\beta)\dots(x+\lambda)}$$

is to be determined, where $\alpha, \beta, \dots, \lambda$ are positive integers in order of increasing magnitude.

This problem has been solved by *Stirling* in his memorable treatise "Methodus Differentialis" (London. 1730).

Let us multiply the numerator and the denominator of this expression by the quantity $x(x+1)$, , , necessary to make the denominator equal to $(x+\lambda)_{\lambda+1}$.

The numerator $F(x)$, whose degree is necessarily less than $\lambda+1$, expanded into a series of factorials gives:

$$F(x) = A_{\lambda+1} + A_\lambda (x+\lambda) + A_{\lambda-1}(x+\lambda)_2 + \dots + A_1 (x+\lambda)_\lambda.$$

Comparing this expression with the *Newton* expansion of $F(x)$ (4, § 23) the coefficient of $(x+\lambda)_m$ is

$$(3) \qquad A_{\lambda+1-m} = \frac{\Delta^m F(-\lambda)}{m!}$$

Therefore we have

$$\frac{f(x)}{(x+\alpha)(x+\beta)\dots(x+\lambda)} = \frac{A_1}{x} + \frac{A_2}{(x+1)_2} + \dots + \frac{A_{\lambda+1}}{(x+\lambda)_{\lambda+1}},$$

the corresponding indefinite sum is 1 be

$$\Delta^{-1}\frac{f(x)}{(x+\alpha)\cdots(x+\lambda)} = A_1 F(x-1) - \frac{A_2}{x} - \frac{A_3}{2(x+1)_2} - \dots - \frac{A_{\lambda+1}}{\lambda(x+\lambda-1)_\lambda} + k.$$

From this the required sum is obtained without difficulty; only the determination of the numbers A_m is needed.

Example 2 (*Stirling,* Meth. Diff. p. 25). If $h=1$ the following sum is required:

$$\sum_{x=1}^{\infty} \frac{1}{x(x+3)}.$$

Here we have $\lambda=3$, and F(x) = $(x+1)$ (x+2). Therefore according to (3) we have

$$A, = F(-3) = 2$$
$$A:, = \Delta F(-3) = -2$$
$$A_2 = \tfrac{1}{2}\Delta^2 F(-3) = 1$$
$$A, = \tfrac{1}{6}\Delta^3 F(-3) = 0.$$

Hence it follows that

$$\Delta^{-1}\frac{1}{(x+3)x} = -\frac{1}{x}+\frac{2}{2(x+1)_2}-\frac{2}{3(x+2),}+k$$

and finally

$$\sum_{x=1}^{\infty}\frac{1}{x(x+3)} = \frac{11}{18}\,.$$

Example 2. The following sum is to be determined *(Stirling* p. 25) if $h=1$,

$$\sum_{x=2}^{\infty}\frac{(x-1)^2}{(x+3)_4}$$

Since here we have $\lambda=3$ and $F(x) = (x-1)^2$,

$$A, = F(-3) = 16$$
$$A:, = \Delta F(-3) = -7$$
$$A, = \tfrac{1}{2}\Delta^2 F\ (-3) = 1$$
$$A_1 = \tfrac{1}{2}\Delta^3 F(-3) = 0$$

The indefinite sum is

$$\Delta^{-1}\frac{(x-1)^2}{(x+3)_4} = -\frac{1}{x}+\frac{7}{2(x+1)_2}-\frac{16}{3(x+2)_3}+k$$

and finally

$$\sum_{x=2}^{\infty}\frac{(x-1)^2}{(x+3)_4} = \frac{5}{36}\,.$$

§ 43. Sums of exponential and of trigonometric functions (same method). The indefinite sum of the exponential function q^x is according to § 33 equal to

$$(1)\qquad \Delta^{-1} q^x = \frac{q^x}{q^h-1} + k.$$

Applications. 1. Sum of a geometrical progression. Given f(x) $= c\, q^x$, the sum of f(x) is to be determined for $x=a$, $a+h$, $a+2h, \ldots, a+(n-1)h$. From formula (1) it follows that

$$\sum_{x=a}^{a+nh}{}_{h}\; c\, q^x = \frac{c\, q^a}{q^h-1}\,[q^{nh}-1].$$

2. We may obtain by formula (1) *the sums of trigonometric or hyperbolic functions.* For instance if hpl, we have

$$\text{(2)}\qquad \Delta^{-1} e^{i\varphi x} = \frac{e^{i\varphi x}}{e^{i\varphi}-1} + k$$

and therefore

$$\sum_{x=0}^{n} e^{i(x+b)\varphi} = e^{ib\varphi}\,\frac{e^{in\varphi}-1}{e^{i\varphi}-1} = \frac{\sin \tfrac{1}{2}n\varphi}{\sin \tfrac{1}{2}\varphi}\, e^{\frac{1}{2}i(n-1+2b)\varphi}.$$

Since $\cos (x+b)\varphi$ is the real part of $e^{i(x+b)\varphi}$,

$$\sum_{x=0}^{n} \cos\,(x+b)\varphi = \frac{\sin \tfrac{1}{2}n\varphi}{\sin \tfrac{1}{2}\varphi}\cos \tfrac{1}{2}(n-1+2b)\,\varphi.$$

This method becomes especially useful in the case of several variables; for instance if we have to determine

$$S = \sum_{x_1=0}^{n_1} \ldots \sum_{x_m=0}^{n_m} \cos\,(x_1+x_2+x_3+\ldots+x_m+b)\varphi.$$

S is the real part of

$$e^{ib\varphi} \prod_{\nu=1}^{m+1} \sum_{x_\nu=0}^{n_\nu} e^{i\varphi x_\nu} = e^{ib\varphi} \prod_{\nu=1}^{m+1} \frac{\sin \tfrac{1}{2}\varphi n_\nu}{\sin \tfrac{1}{2}\varphi}\, e^{\frac{1}{2}i\varphi(n_\nu-1)}$$

and therefore

$$S = \cos \tfrac{1}{2}(\Sigma n_\nu - m + 2b)\varphi \prod_{\nu=1}^{m+1} \frac{\sin \tfrac{1}{2}\varphi n_\nu}{\sin \tfrac{1}{2}\varphi}$$

In the particular case of $n_\nu = n$ for every value, of v, we have

$$S = \cos \tfrac{1}{2}\,(mn-m+2b)\,\varphi \left[\frac{\sin \tfrac{1}{2}n\varphi}{\sin \tfrac{1}{2}\varphi}\right]^m.$$

The *sum of a product* of a polynomial and an exponential function is determined by repeated summation by parts (§ 34).

Example. The sum of xa^x is required from $x=0$ to $x=n+1$ if $\Delta x=1$. The indefinite sum is as we have seen (formula 2, § 34),

$$\Delta^{-1} xa^x = \frac{a^x\ (ax-x-a)}{(a-1)^2} + k$$

therefore

$$\sum_{x=0}^{n+1} x\, a^x = \frac{a^{n+1}\,(an-n-1) + a}{(a-1)^2},$$

In the particular case of $a = ½$ we have

$$\sum_{x=0}^{n+1} \frac{x}{2^x} = 2 - \frac{n+2}{2^n} \qquad \text{and} \qquad \sum_{x=1}^{\infty} \frac{x}{2^x} = 2,$$

The *sum* of *trigonometric functions* may be obtained directly. According to 6, § 33 the indefinite sum of cos ax is, if $h=1$

$$\Delta^{-1} \cos ax = \frac{\sin (ax-\frac{1}{2}a)}{2 \sin \frac{1}{2}a} + k$$

and that of sin ax

$$\Delta^{-1} \sin ax = \frac{-\cos (ax-\frac{1}{2}a)}{2 \sin \frac{1}{2}a} + k,$$

Therefore

(3) $$\sum_{x=0}^{n} \cos ax = \frac{\sin (an-\frac{1}{2}a)}{2 \sin \frac{1}{2} a} + \frac{1}{2}$$

and

(4) $$\sum_{x=0}^{n} \sin ax = \frac{\cos \frac{1}{2}a - \cos (an-\frac{1}{2}a)}{2 \sin \frac{1}{2}a}$$

We will now examine several particular cases of trigonometric functions which play an important part in the trigonometric expansions of functions and in the resolution of difference equations.

1. *First form.* Trigonometric functions occurring in expansions :

$$\sin \frac{2\pi k}{p} x, \quad \cos \frac{2\pi k}{p}.$$

If k and p are integers, and if k is not divisible by p, then in consequence of formulae (3) and (4) it follows that

$$\sum_{x=0}^{p} \cos \frac{2\pi k}{p} x = 0 ; \quad \sum_{x=0}^{p} \sin \frac{2\pi k}{p} x = 0.$$

From these equations we may deduce others. Indeed we have

(5) $$\sum_{x=0}^{p} \cos\frac{2\pi\nu}{p} x \cos\frac{2\pi\mu}{p} x = \tfrac{1}{2} \sum_{x=0}^{p} \cos\frac{2\pi(\nu+\mu)}{p} x + $$
$$+ \tfrac{1}{2} \sum^{p} \cos\frac{2\pi(\nu-\mu)}{p} x.$$

Therefore if v and μ are different and if $\nu+\mu$ and v-p are not divisible by p, then, the sum of the first member will be equal to zero.

If v is equal to μ but different from $\tfrac{1}{2}p$, then we shall have

$$\sum_{x=0}^{p} [\cos\frac{2\pi\nu}{p} x]^2 = \tfrac{1}{2}p.$$

If we have $2\nu=2\mu=p$, it is obvious that this sum is equal to p.

In the same way we could obtain, if v is equal to μ but different from $\tfrac{1}{2}p$,

(6) $$\sum_{x=0}^{p} [\sin\frac{2\pi\nu}{p} x]^2 = \tfrac{1}{2}p.$$

If $2\nu=2\mu=p$ *then* this sum is equal to zero.

Moreover, if v and μ are different, and if $\nu+\mu$ and v-p are not divisible by p, then

(7) $$\sum^{p} \sin\frac{2\pi\nu}{p} x \sin\frac{2\pi\mu}{p} x = 0.$$

If v and μ are integers we always have

(8) $$\sum_{x=0}^{p} \sin\frac{2\pi\nu}{p} x \cos\frac{2\pi\mu}{p} x = 0.$$

Divisibilify. If k is divisible by p, that is if $k=\lambda p$ (λ being an integer) then

$$\sum_{x=0}^{p} \cos\frac{2\pi k}{p} x = p\ ; \quad \sum_{x=0}^{p} [\cos\frac{2\pi b}{p} x]^2 = p; \sum_{x=0}^{p} [\sin\frac{2\pi k}{p} x]^2 = 0.$$

The other formulae will change according to these. For instance, if $\nu+\mu=\lambda p$ and $\nu=\mu$ then we shall have

$$\sum_{x=0}^{p} \cos\frac{2\pi\nu}{p} x \cos\frac{2\pi\mu}{p} x = \tfrac{1}{2}p; \quad \sum_{x=0}^{p} \sin\frac{2\pi\nu}{p} x \sin\frac{2\pi\mu}{p} x = -\tfrac{1}{2}p$$

and so on.

If $2k=p$ then

$$\sum_{x=0}^{p} \cos\frac{2\pi k}{p} x = 0; \qquad \sum_{x=0}^{p} |\cos\frac{2\pi k}{p} x|^2 = p.$$

The corresponding sine values are equal to zero. The above formulae will be used later (§ 146).

2. *Second form.* Trigonometric expressions occurring in certain difference equations.

$$\cos\frac{\pi k}{p} x, \qquad \sin\frac{\pi k}{p} x$$

where k and p are integers. If k is not divisible by p, or if $k=(2\lambda+1)p$, λ being an integer, then

$$\sum_{x=0}^{p} \cos\frac{\pi k}{p} x = \tfrac{1}{2}[1-(-1)^k].$$

On the other hand, if $k=2\lambda p$, then

$$\sum_{x=0}^{p} \cos\frac{\pi k}{p} x = p.$$

If ν and μ differ and are integers, moreover, if v-p and $\nu+\mu$ are not divisible by p, it follows that

$$(9) \qquad S = \sum_{x=0}^{p} \sin\frac{\pi\nu}{p} x \sin\frac{\pi\mu}{p} x =$$

$$= \tfrac{1}{2} \sum_{x=0}^{p} \cos\frac{(\nu-\mu)\pi}{p} x - \tfrac{1}{2} \sum_{x=0}^{p} \cos\frac{(\nu+\mu)\pi}{p} x = 0$$

and

$$(10) \qquad C = \sum_{x=0}^{p} \cos\frac{\pi\nu}{p} x \cos\frac{\pi\mu}{p} x =$$

$$= \tfrac{1}{2} \sum_{x=0}^{p} \cos\frac{(\nu-\mu)\pi}{p} x + \tfrac{1}{2} \sum_{x=0}^{p} \cos\frac{(\nu+\mu)\pi}{p} x = 0.$$

On the other hand, if $\nu=\mu$ but v is different from p and from $\tfrac{1}{2}p$, then

$$s = \tfrac{1}{2}p \qquad \text{and} \qquad c = \tfrac{1}{2}p$$

If $\nu=\mu=p$ then $S=0$ and $C=p$

If $\nu=\mu=\tfrac{1}{2}p$ then $S=p$ and $C=0$.

ν, μ and p being integers, the following expression will always be equal to zero:

$$\sum_{x=0}^{p} \sin \frac{\nu\pi}{p} x \cos \frac{\mu\pi}{p} x = 0 \ .$$

3. ***Third form.*** Trigonometric functions occurring in the resolution of certain difference equations:

$$\cos \frac{\nu\pi}{p}(2x+1)\,; \quad \sin \frac{\nu\pi}{p}(2x+1).$$

a) The indefinite sum of the first expression will be

$$A^{-1} \cos \frac{\pi\nu}{p}(2x+1) = \frac{\sin \frac{2\pi\nu x}{p}}{2 \sin \frac{\pi\nu}{p}} + k.$$

Therefore if v is not divisible by p *we* have

$$\sum_{x=0}^{p} \cos \frac{\pi\nu}{p} (2x+1) = 0$$

and if it is divisible, so that $\nu=\lambda p$, then

(11) $$\sum_{x=0}^{p} \cos 5 \ (2x+1) = (-1)^{\lambda} p.$$

b) We have

$$\Delta^{-1} \sin \frac{\pi\nu}{p} (2x+1) = -\frac{\cos \frac{2nvx}{p}}{2 \sin \frac{\pi\nu}{p}} + k,$$

therefore whether ν is divisible by p or not,

(12) $$\sum_{x=0}^{p} \sin \frac{\pi\nu}{p} (2x+1) = 0.$$

c) If ν is different from μ and if moreover $\nu+\mu$ and $\nu-\mu$ are not divisible by $2p$, *we* have

$$S = \sum_{x=0}^{p} \sin\frac{\pi\nu}{p} (2x+1) \sin \frac{\pi\mu}{p} (2x+1) = 0$$

and

(13) $$C = \sum_{x=0}^{p} \cos \frac{\pi\nu}{p}(2x+1) \cos \frac{\pi\mu}{p}(2x+1) = 0.$$

If $\nu \neq \mu$ and if 2ν is not divisible by p then $S = \frac{1}{2}p$ and $C = \frac{1}{2}p$.
If $\nu = \mu$ and if 2ν is divisible by p then $S = p$ and $C = 0$.

d) In consequence of formula (12) we always have

$$\sum_{x=0}^{p} \sin \frac{\pi\nu}{p}(2x+1) \cos \frac{\pi\mu}{p}(2x+1) = 0.$$

whether 2ν is divisible by p or not.

The sum of a product of a polynomial and a trigonometric function is obtained by repeated summation by parts.

Example. The sum of x sinx, for $x = 0, h, 2h, \ldots, (n-1)h$ is to be determined. The indefinite sum will be, according to formula (3) § 34,

$$\underset{h}{\Delta^{-1}} x \sin x = \frac{2x \sin \frac{1}{2}h \sin (x - \frac{1}{2}h - \frac{1}{2}\pi) + h \sin x}{(2 \sin \frac{1}{2}h)^2} + k$$

and therefore

$$\sum_{x=0}^{nh}{}_{h} \, x \sin x = \frac{2nh \sin \frac{1}{2}h \sin (nh - \frac{1}{2}h - \frac{1}{2}\pi) + h \sin (nh)}{(2 \sin \frac{1}{2}h)^2}.$$

§ 44. Sums of other functions (same method).

1. In paragraph 33 we had (8)

$$\Delta^{-1} \frac{1}{x} = F(x-1) + k$$

therefore

(1) $$\sum_{x=1}^{n+1} \frac{1}{x} = F(n) + C$$

where C is *Euler's* constant (§ 19).

2. Since (9, § 33)

$$\Delta^{-1} \frac{1}{x^2} = -F(x-1) + k$$

we obtain

(2) $$\sum_{x=1}^{n+1} \frac{1}{x^2} = \frac{\pi^2}{6} - F(n).$$

3. Let us mention here two formulae, which we will deduce later. The first is (1, § 83) :

(3) $$\sum_{x=0}^{n} x^m = \frac{(B+n)^{m+1} - B_{m+1}}{m+1}.$$

This is a symbolical formula, in which B_i must be substituted for B^i. The numbers B_i are the ***Bernoulli*** numbers given by the symbolical equation $(B+1)^{m+1} - B_{m+1} = 0$.

The second is given by the symbolical formula (3) of § 108:

$$\sum_{x=0}^{n} (\text{-}1)\text{”} \; x^m = \frac{1}{2^{m+1}} [\mathfrak{C}_m - (-1)^n (\mathfrak{C}+2n)^m].$$

The numbers $\mathfrak{C}_i$ are the ***tangent-coefficients*** of § 104, given by the symbolical relation $(\mathfrak{C}+2)^m + \mathfrak{C}_m = 0$.

4. To determine the sum of $\log x$ from $x=a$ to $x=a+m$ if $\Delta x=1$ we make use of formula (11) § 33:

$$\text{A-’ logx} = \log \Gamma(x) + k$$

from which it follows that

(5) $$\sum_{x=a}^{a+m} \text{l o g x} = \log \frac{\Gamma(a+m)}{\Gamma(a)}.$$

5. To obtain the sum of the digamma function from $x=0$ to $x=n$ if $\Delta x=1$ let us remark that we have found formula (6) § 34:

$$\Delta^{-1} F(x) = xF(x) - x + k.$$

Hence

(6) $$\sum_{x=0}^{n} F(x) = nF(n) - n.$$

6. In § 34 we found (formula (7)) that the indefinite sum of the trigamma function is

$$\Delta^{-1} F(x) = xF(x) + \text{F(x)} + k.$$

Therefore

(7) $$\sum_{x=0}^{n} F(x) = nF(n) + F(n) + C$$

where C is ***Euler’s*** constant.

Example. For $n=20$ ***we*** find

$$\sum_{x=0}^{20} \text{F(x)} = 20F(20) + \text{F(20)} + \text{c} = 4\text{’}57315605.$$

Adding the numbers $F(x)$ from $x=0$ to $x=20$ in *Pairman's* Table we find 4'57315614. The error is equal to 9 units of the eighth decimal.

7. Sum of alternate functions. **Let** *f(x)* be a polynomial **of degree** n; then the indefinite sum of $(-1)^x$ *f(x)* is given by formula (2) of § 35:

$$(8) \qquad \Delta^{-1} (-1)^x f(x) = \tfrac{1}{2}(-1)^{x+1} \sum_{m=0}^{n+1} \frac{(-1)^m}{2^m} \Delta^m f(x).$$

From this formula we deduce, for instance,

$$(9) \qquad \sum_{x=1}^{\mu} (-1)^x\, x = \tfrac{1}{2}(-1)^{\mu+1}(\mu-\tfrac{1}{2})-\tfrac{1}{4}.$$

If the function *f(x)* is not a polynomial, then the series in formula (8) is infinite; but the formula holds if the series is convergent. For instance if

$$f(x) = \frac{1}{x+1}$$

then

$$\Delta^m f(x) = \frac{(-1)^m m!}{(x+m+1)_{m+1}}.$$

The series

$$1/2^m(m+1)\binom{x+m+1}{m+1}$$

is convergent, therefore we shall have

$$(10) \qquad \sum_{x=0}^{\mu} \frac{(-1)^x}{x+1} = \tfrac{1}{2}(-1)^{\mu+1} \sum_{m=0}^{\infty} \frac{m!}{2^m(\mu+m+1)_{m+1}} + \sum_{m=1}^{\infty} \frac{1}{m2^m}.$$

If $\mu=\infty$, in consequence of what has been said before, the first sum of the second member will vanish, and we get

$$(11) \qquad \sum_{x=0}^{\infty} \frac{(-1)^x}{x+1} = \sum_{m=1}^{\infty} \frac{1}{m\,2^m} = -\log(1-\tfrac{1}{2}) = \log 2.$$

§ 45. Determination of sums by symbolical formulae, In § 6 we have expressed the n-th difference of a function by its successive values: we had

$$\Delta^n = \sum_{x=0}^{n+1} (-1)^{x+n} \binom{n}{x} \mathrm{E}^x;$$

from this it follows immediately that

$$(1)\qquad \sum_{x=0}^{n+1} (-1)^x \binom{n}{x} f(x) = (-1)^n \Delta^n f(0).$$

Therefore to have this sum it is sufficient to know the n-th difference of the function $f(x)$, for $x=0$.

Example *1.* Given $f(x) = 1/(2x+1)$. To determine the n-th difference if $\Delta x=1$ this may be written as a reciprocal factorial of the first degree; then we get

$$\tfrac{1}{2}\Delta^n \frac{1}{x+\frac{1}{2}} = \frac{1}{2} \cdot \frac{(-1)^n\, n!}{(x+\frac{1}{2})(x+\frac{3}{2})\ldots\left(x+\frac{2n+1}{2}\right)}.$$

Putting $x=0$,

$$[\Delta^n f(x)]_{x=0} = \frac{(-1)^n\, n!\, 2^n}{1.3.5,\ldots(2n+1)},$$

therefore

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \frac{1}{2x+1} = \frac{n!\, 2^n}{1.3.5\ldots(2n+1)}.$$

Example 2. Given $f(x) = \cos ax$. We have seen in § 27 (formula 1) that if $\Delta x=1$ then

$$\Delta^n \cos ax = (2 \sin \tfrac{1}{2}a)^n \cos [ax+\tfrac{1}{2}n(a+\pi)]$$

hence

$$[\Delta^n \cos ax]_{x=0} = (2 \sin \tfrac{1}{2}a)^n \cos \tfrac{1}{2}n(a+\pi)$$

so that finally

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \cos ax = (-1)^n (2 \sin \tfrac{1}{2}a)^n \cos \tfrac{1}{2}n(a+\pi).$$

Example 3. Given $f(x) = 1/(x+m)_m$. We found the n-th difference of a reciprocal factorial (formula 6, § 16); if $\Delta x=1$

$$\Delta^n \frac{1}{(x+m)_m} = \frac{(-1)^n (m+n-1)_n}{(x+m+n)_{m+n}}$$

hence

$$[\Delta^n f(x)]_{x=0} = \frac{(-1)^n}{(m+n)(m-1)!}$$

and finally

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \frac{1}{(x+m)_m} = \frac{1}{(m+n)(m-1)!}.$$

Example 4. Given f(x) $= \binom{x+a}{m}$ we had *&f(x)* $=$ $= \binom{x+a}{m-n}$. This gives

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \binom{x+a}{m} = (-1)^n \binom{a}{m-n}.$$

Putting into this expression $a=0$ we get a useful formula for the *reversion of series* (see later). Indeed the second member of

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \binom{x}{m} = (-1)^n \binom{0}{m-n}$$

will be equal to zero if m is different from n, and equal to $(-1)^n$ if $n=m$.

Example 5. Given $f(x) = (-1)^x \binom{a-n}{b-x}$. We have

$$\binom{a-n}{b-x} = \binom{a-n}{a-n-b+x}.$$

Moreover formula (2) § 28 gives the n-th difference of an alternate function expressed by means:

$$\Delta^n \left[(-1)^x \varphi(x)\right] = (-2)^n (-1)^x \mathbf{M}^n \varphi(x)$$

since

$$\mathbf{M}^n \binom{a-n}{a-n-b+x} = \frac{1}{2^n} \binom{a}{a-b+x} = \frac{1}{2^n} \binom{a}{b-x}$$

therefore

$$\&f(x) = (-1)^{x+n} \binom{a}{b-x}$$

and finally

$$\sum_{x=0}^{n} \binom{n}{x} \binom{a-n}{b-x} = \binom{a}{b}.$$

This is *Cauchy's* formula, which we have deduced already in another way (formula 14 § 22).

Example 6. Given *f(x)* $=$ F(x). In § 19 we found (formula 4)

$$\Delta^n F(x) = \frac{(-1)^{n-1} (n-1)!}{(x+n)_n} = (-1)^{n-1} B(x+1,n).$$

Therefore

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} F(x) = -\frac{1}{n}.$$

Example 7. Let f(x) $= x^m$. We shall see later that

$$[\Delta^n x^m]_{x=0} = n!\, \mathfrak{S}_m^n$$

where $\mathfrak{S}_m^n$ is a *Stirling* number of the second kind. Hence

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} x^m = (-1)^n n!\, \mathfrak{S}_m^n$$

and in the particular case of $m=n$ *we* have

$$\sum_{x=0}^{n+1} (-1)^{n+x} \binom{n}{x} x^n = n!,$$

Example 8. Given *f(x)* $= \psi_m(x)$, *the Bernoulli* polynomial of the second kind of degree *m.* We shall see later that

$$[\Delta^n \psi_m(x)]_{x=0} = b_{m-n}$$

where b_{m-n} is a coefficient of these polynomials. Therefore

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \psi_m(x) = (-1)^n\ b_{m-n},$$

Particular case: $m=n$

$$\sum_{x=0}^{n+1} (-1)^x \binom{n}{x} \psi_n(x) = (-1)^n.$$

In § 6 we deduced other symbolical formulae. For instance

$$\frac{\Delta^n}{\mathsf{E}^n} = \left(1 - \frac{1}{\mathsf{E}}\right)^n = \sum_{\nu=0}^{n+1} (-1)^\nu \binom{n}{\nu} \mathsf{E}^{-\nu}$$

This operation executed on *f(x)* gives for $x=a$

$$\sum_{\nu=0}^{n+1} (-1)^\nu \binom{n}{\nu} f(a-\nu) = \Delta^n f(a-n). \tag{2}$$

Example. *f(x)* $= \binom{x}{m}$. Then *&f(x)* $= \binom{x}{m-n}$; hence

$$\sum_{\nu=0}^{n+1} (-1)^\nu \binom{n}{\nu} \binom{a-\nu}{m} = \binom{a-n}{m-n}.$$

Particular case of $a=n$:

$$\sum_{\nu=0}^{n+1} (-1)^\nu \binom{n}{\nu} \binom{n-\nu}{m} = \binom{0}{m-n}.$$

This formula is identical with that obtained in Example 4.

A third symbolical formula of § 6 was the following;

$$(1+\mathsf{E})^n = (2\mathsf{M})^n .$$

This operation performed on f(x) gives for $x=0$

$$\sum_{\nu=0}^{n+1}\binom{n}{\nu} f(\nu) = 2^n \mathsf{M}^n f(0). \tag{3}$$

Hence the sum of the first member is known if'we know the n-th mean of $f(x)$ for $x=0$.

Example 1. Given $f(x) = \binom{a}{b+x}$, We get $\mathsf{M}^n f(x) = \binom{a+n}{b+x+n}\frac{1}{2}$, therefore

$$\sum_{\nu=0}^{n+1}\binom{n}{\nu}\binom{a}{a-b-\nu} = \binom{a+n}{a-b}$$

this is again ***Cauchy's*** formula.

Example 2. If n is not very large, we may determine $\mathsf{M}^n f(0)$ numerically. For instance, given *f(x)* $= 1, 9, 25, 49,$ 81,. . , , and $n=3$. Computing a table containing the means we have

1		9		25		49
	5		17		37	
		11		27		
			19			

then from (3) we get

$$\sum_{\nu=0}^{4}\binom{3}{\nu}(2\nu+1)^2 = 2^3 . 19 = 152.$$

Example 3. Given *f(x)* = cos *ax.* Since (formula 3, § 27)

$$\mathsf{M}^n \cos ax = (\cos \tfrac{1}{2}a)^n \cos (ax + \tfrac{1}{2}na)$$

we have

$$\sum_{\nu=0}^{n+1}\binom{n}{\nu}\cos a\nu = 2^n (\cos \tfrac{1}{2}a)^n \cos \tfrac{1}{2}na.$$

A ***fourth*** symbolical formula of § 6 was

$$\left(1+\frac{1}{\mathsf{E}}\right)^n = \frac{2^n \mathsf{M}^n}{\mathsf{E}^n} .$$

The operation performed on *f(x)* gives for $x=a$

(4) $$\sum_{\nu=0}^{n+1} \binom{n}{\nu} f(a-\nu) = 2^n \mathsf{M}^n f(a-n).$$

This formula generally leads to results similar to (3) ; for instance $f(x) = \binom{m}{x}$ gives Cauchy's formula: $f(x) = e^x$ would give $\sum_{\nu=0}^{n+1} \binom{n}{\nu} e^\nu = (1+e)^n$.

Starting from formula (3) of § 6 we may obtain a *fifth* summation formula. Indeed from

$$\frac{1}{\mathsf{E}^n} = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{n+\nu-1}{\nu} \Delta^\nu$$

we get

$$\frac{\mathsf{E}^n-1}{\mathsf{E}^n} = \frac{\Delta[1+\mathsf{E}+\ldots+\mathsf{E}^{n-1}]}{\mathsf{E}^n} = \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \binom{n+\nu-1}{\nu} \Delta^\nu.$$

Performing the operation $\mathsf{E}^n\Delta^{-1}$ on both members we have

$$1+\mathsf{E}+\ldots+\mathsf{E}^{n-1} = C + \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \binom{n+\nu-1}{\nu} \Delta^{\nu-1}\mathsf{E}^n.$$

This operation performed on $f(x)$, if it is a polynomial or if it satisfies certain conditions, gives

$$\sum_{i=0}^{n} f(x+ih) = \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \binom{n+\nu-1}{\nu} \underset{h}{\Delta}^{\nu-1} f(x+nh).$$

Indeed, since for $n=0$ the first member vanishes, we have also $C=0$.

§ 46. Determination of sums by generating functions. It is often possible to determine sums by the method of generating functions. This will be shown in a few examples.

1. If the following sum is required

$$S = \sum_{x=0}^{n+1} \binom{n}{x} x$$

we may start from the generating function

$$u(t) = (1+t)^n = \sum_{x=0}^{n+1} \binom{n}{x} t^x.$$

The operation $t \,.\, \mathsf{D}$ performed on both members of this equation gives

$$t \,.\, \mathbf{D}\, u(t) = tn(1+t)^{n-1} = \sum_{x=0}^{n+1} \binom{n}{x} x\, t^x.$$

Now putting $t = 1$ we obtain

$$S = n\, 2^{n-1}.$$

Remark. According to formula (3) § 45 this sum is

$$s = 2” \, [\mathbf{M}^n x]_{x=0}$$

but $\mathbf{M}x = x + ½$; hence $\mathbf{M}^n x = x + ½n$ and finally $S = n\, 2^{n-1}$.

2. To have

$$S = \sum_{x=0}^{n+1} \frac{(-1)^x}{x+a} \binom{n}{x}$$

we start from

$$u(f) = (1 - f)” = \sum_{x=0}^{n+1} (-1)^x \binom{n}{x} t^x.$$

multiplying both members by t^{a-1} and integrating from 0 to 1 we get

$$S = \int_0^1 t^{a-1} (1-t)^n dt = B(n+1, a).$$

If n and a are integers we have

$$S = \frac{1}{(n+1)\binom{n+a-1}{n+1}} = \frac{n!}{(n+a)_{n+1}}.$$

Remark. According to formula (1) § 45 this sum is

$$S = (-1)' \left[\Delta^n \frac{1}{x+a} \right]_{x=0}$$

but $\Delta^n \dfrac{1}{x+a} = \dfrac{(-1)^n\, n!}{(x+a+n)_{n+1}}$ and therefqre $S = \dfrac{1}{(a+n)_{n+1}}$.

3. To obtain

$$S = \sum_{x=0}^{n+1} x\, a^x$$

let us start from

$$u(f) = \sum_{x=0}^{n+1} (at)” = \frac{(at)^{n+1} - 1}{af - 1}$$

determining $\mathbf{D}\, u(f)$ and putting $t = 1$ we have

$$S=\frac{a^{n+1}\,(an-n-1)+a}{(a-1)^2}.$$

This has already been obtained in § 43 by summation by parts.

4. **We** could easily determine by summation by parts

$$s=\sum_{x=0}^{n+1} x(n-x)$$

but it may also serve as an example for the use of generating functions of two variables, Let

$$u(z,t)=\frac{z^{n+1}-t^{n+1}}{z-t}=\sum_{x=0}^{n+1} z^x\, t^{n-x}$$

from this we conclude that

$$S=\left[\frac{\partial^2 u}{\partial z\,\partial t}\right]_{t=z=1}$$

that is

$$S=\left\{\frac{[(n+1)z^n+(n+1)t^n](z-t)+2(t^{n+1}-z^{n+1})}{(z-t)^3}\right\}_{z=t=1}$$

If we put $z=t$ this gives $0/0$; therefore we must apply *Hôpital's* rule. The derivative with respect to z of the numerator divided by the derivative of the denominator gives again $0/0$ for $z=t$; and so do the second derivatives also. Finally the third derivatives give

$$S=\binom{n+1}{3}.$$

Remark. According to § 34, summation by parts would give

$$\Delta^{-1}\,x(n-x)=\binom{x}{2}(n-x)+\Delta^{-1}\binom{x+1}{2}=$$

$$=\binom{x}{2}(n-x)+\binom{x+1}{3}+k.$$

The required sum is therefore

$$\sum_{x=0}^{n+1} x(n-x)=-\binom{n+1}{2}+\binom{n+2}{3}=\binom{n+1}{3}.$$

§. 47. Determination of sums by geometrical considerations. It occurs sometimes that the result of a geometrical problem is expressed by a sum. If it is possible to solve the problem in another way the result will be equal to the sum obtained previously.

As a simple example let us consider the sum

$$(1) \qquad s = \sum_{\nu=1}^{n+1} \nu(n-\nu)$$

which we have already determined by other methods.

This sum is equal to the number of points whose coordinates are $x = 0, 1, 2, \ldots, (n-2)$; $y = 0, 1, 2, \ldots, (n-2)$; $z = 0, 1, 2, \ldots, (n-2)$ provided that

$$x + y + z \leq n - 1.$$

The points above are situated in a tetrahedron whose summits are the points of coordinates:

$$x=0, y=0, z=0, \qquad x=0, y=0, z=n-2, \qquad x=0, y=n-2, z=0,$$
$$x=n-2, y=0, z=0.$$

The number of the given points contained in the plane

$$x+y = V-1$$

is equal to $\nu(n-\nu)$; and the number of points contained in the planes parallel to this one, and corresponding to $\nu=1, 2, \ldots, (n-l)$ is equal to the required sum S.

On the other hand, if we consider the plane

$$x+y+z = s$$

the number of the points in this plane will be, as is easily seen, equal to

$$1 + 2 + 3 + \ldots + (s+1) = \binom{s+2}{2}.$$

The number of the points contained in the planes parallel to this one, corresponding to $s=0, 1, 2, \ldots, (h-2)$ will be

$$\sum_{s=0}^{n-1} \binom{s+2}{2} = \binom{n+1}{3}.$$

this is the total number of the points, therefore equal to S. So that

$$S = \binom{n+1}{3}.$$

§ 48. Determination of sums by the Calculus of Probability. Sometimes the result of a problem of probability is obtained in the form of a **sum;** if it is possible to solve the problem in another manner, the result will be equal to the sum above.23

Example. Given an urn in which there are m balls marked $1, 2, 3, \ldots, m$. Successively n balls are drawn, putting the ball taken, back into the urn before **the next** drawing.

The probability that the sum of the n numbers obtained shall be less than x is equal, according to *Monmorf's* solution, to

$$P = \frac{1}{m^n} \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \binom{x-m\nu-1}{n}.$$

Since the probability that the obtained sum shall be less than $x = nm+1$ is equal to unity, we have

$$m^n = \sum_{\nu-0}^{\nu_1} (-1)^\nu \binom{n}{\nu} \binom{mn-m\nu}{n}.$$

Finally putting $\mu = n-\nu$ we get

$$(-m)^n = \sum_{\mu-1}^{n+1} (-1)^\mu \binom{n}{\mu} \binom{m\mu}{n}.$$

§ 49, Determination of alternate sums starting from usual sums. Often if we know a sum we may deduce the corresponding alternate sum,

Example. We have, as we shall see in formula (6) § 82,

$$(1)\qquad S = \sum_{x=0}^{\infty} \frac{1}{(x+1)^{2n}} = \tfrac{1}{2}(2\pi)^{2n} \frac{|B_{2n}|}{(2n)!}$$

where B_{2n} **is a** *Bernoulli* number. This multiplied by $1/2^{2n}$ gives

$$(2)\qquad \frac{S}{2^{2n}} = \sum_{x-1}^{\infty} \frac{1}{(2x)^{2n}} = \frac{\pi^{2n}|B_{2n}|}{2(2n)!}$$

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[23] Let us remark that *Crofton* determined by the same method the values of several very interesting definite integrals, which were verified later, by analytical methods, by *Serret.*

Morgan W. *Crofton,* On the Theory of Local Robability, ..., the method used being also extended to the proof of new Theorems in the Integral Calculus. Phil. Trans. Royal Society, London, 1868. Vol. 158, pp. 181-199.

***J. A. Serret,* Sur un problème du calcul intégral, Comptes Rendus de l'Académie des Sciences, Paris, 1869. Vol. 2, p. 1132.**

Sur un problhme de calcul integral, Annales Scientifiques de l'Ecole Normale sup. Paris, 1869, Vol. 6, p, 177.

Since this series is absolutely convergent, it follows that the $2n$ -th power of the odd numbers is

$$(3) \qquad S - \frac{S}{2^{2n}} = \sum_{x=0}^{\infty} \frac{1}{(2x+1)^{2n}} = \tfrac{1}{2}(2^{2n}-1)\ \pi^{2n}\ \frac{|B_{2n}|}{(2n)!}$$

and moreover the alternate sum will be

$$(4) \qquad S - \frac{2S}{2^{2n}} = \sum_{x=0}^{\infty} \frac{(-1)^x}{(x+1)^{2n}} = (2^{2n-1}-1)\ \pi^{2n} \frac{|B_{2n}|}{(2n)!} .$$

CHAPTER IV.

STIRLING'S NUMBERS.

§ 50. Expansion of factorials into power series. Stirling's numbers of the first kind. The expansidn of the factorial into a Maclaurin series gives

$$(1) \qquad (x)_n = \sum_{m=1}^{n+1} x^m \left[\frac{1}{m!} D^m (x)_n\right]_{x=0}$$

Denoting the number in brackets by S_n^m we have

$$(2) \qquad S_n^m = \left[\frac{1}{m!} D^m (x)_n\right]_{x=0}$$

so that equation (1) may be written

$$(3) \qquad (x)_n = S_n^1 x + S_n^2 x^2 + S_n^3 x^3 + \ldots + S_n^n x^n.$$

The numbers S_n^m defined by equation (2) are called *Stirling's Numbers of the first kind.* They were introduced in his Methodus Differentialis, in which there is a table of these numbers (p. 11) up to $n=9$.[24]

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[24] 0. *Schlömilch,* Compendium der Höheren Analysis, Braunschweig. 1895, II, p. 31, gives a table of these numbers, which he calls Facultäten-coefficienten. His notation is different; his

C_m^n corresponds to our $|S_n^{n-m}|$

A larger table is to be found in:

Ch. *Jordan,* On Stirling's Numbers, Tôhoku Mathematical Journal, *Vol. 37, 1933,* p. 255.

J. F. Steffensen, Interpolation, London, 1927, p. 57, introduces numbers which he calls differential coefficients of nothing, denoted by $D^m 0^{(-n)}$, and gives a table of these numbers divided by m!; his

$\dfrac{D^m 0^{(-n)}}{m!}$ corresponds to our $|S_n^m|$

Niels Nielsen, Gammafunktionen, Leipzig, 1906, p. 67, introduces the "Stirlingschen Zahlen Erster Art"; his

C_n^m corresponds to our $|S_m^{n-m}|$

[nearly the same notation as Schlömilch's).

Stirling's Numbers are of the greatest utility in Mathematical Calculus. This however has not been fully recognised; the numbers have been neglected, and are seldom used.

This is especially due to the fact that different authors have reintroduced them under different names and notations, not mentioning that they dealt with the same numbers.

Stirling's numbers are as important or even more so than *Bernoulli's* numbers; they should occupy a central position in the Calculus of Finite Differences.

From (2) we deduce immediately that

$$S_n^0 = 0; \; S_1^1 = 1 \text{ and } S_n^m = 0 \text{ if } m > n.$$

The other numbers could also be calculated by (2), but we will give a shorter way. Let us write the identity

$$(4) \qquad (x)_{n+1} = (x-n)\,(x)_n\,.$$

If we put $v = x-n$ and $u = (x)_n$ then *Leibnifr's* formula giving the higher derivatives of a product (§ 30) is

$$\mathrm{D}^m(uv) = v\mathrm{D}^m u + \binom{m}{1} \mathrm{D}v\, \mathrm{D}^{m-1}u$$

that is.

$$\mathrm{D}^m (x)_{n+1} = (x-n)\,\mathrm{D}^m (x)_n + m\mathrm{D}^{m-1}(x)_n\,.$$

Dividing both members by *m!* and putting $x=0$ we get, in consequence of (2),

$$(5) \qquad S_{n+1}^m = S_n^{m-1} - n \quad S_n^m.$$

This could have been obtained from (4) by expanding the factorials $(x)_{n+1}$ and $(x)_n$ into power series (3), and writing that the coefficients of x^m in both members are the same.

Equation (5) is a partial difference equation of the first order. It is true that the general solution of this equation is unknown; but starting from the initial conditions we may compute by aid of this equation every number S_n^m. According to § 181 one initial condition is sufficient for computing the numbers S_n^m; for instance S_0^m given for every positive or negative integer value of *m*. But from definition (2) it follows directly that

$$S_0^m = 0 \text{ if } m \neq 0 \text{ and } S_0^0 = 1.$$

Therefore, putting into the equation $n=0$ we get $S_1^m = S_0^{m-1}$ and this gives

$$S_1^1 = 1 \text{ and } S_1^m = 0 \text{ if } mjl.$$

Putting into the equation $n=1$ we find

$$S_2^m = S_1^{m-1} - S_1^m.$$

This gives

$$S_2^2 = 1,\ S_2^1 = -1 \text{ and } S_2^m = 0 \text{ if } m > 2 \text{ or } m < 1$$

and so on; we obtain the following table of the *Stirling* numbers:

Stirling's Numbers of the First Kind. S_n^m

$n \backslash m$	1	2	3	4	5
1					
2	—1	1			
3	2	-3	1		
4	—6	11	-6	1	
5	24	-50	35	-10	1
6	-120	274	-225	85	-15
7	720	-1764	1624	-135	175
8	-5040	13068	-13132	6769	—1960
9	40320	-109584	118124	-67284	22449
10	-362880	1026576	-1172700	723680	-269325
11	3628800	-10628640	12753576	-8409500	3416930
12	-39916800	120543840	-150917976	105258076	—45995730

$$S_0^0 = 1$$

$n \backslash m$	6	7	8	9	10	11	12
2							
3							
4							
5							
6	1						
7	-21	1					
8	322	-28	1				
9	4536	546	-36	1			
10	63273	-9450	870	-45	1		
11	-902055	157773	-18150	1320	-55	1	
12	13339535	-2637558	357423	--32670	1925	-66	1

From equation (5) we conclude that the *Stirling* numbers of the first kind are integers.

Putting $x=1$ into equation (3) we get if $n>1$

$$S_n^1+S_n^2+S_n^3+\dots+S_n^n=0. \tag{6}$$

That is, the sum of the numbers **in** each line of the preceding table is equal to zero. This can serve as a check of the table. To obtain a second check, let us put $x = -1$ into equation (3) ; we get

$$(-1)^n n! = \sum_{m=1}^{n+1} (-1)^m \; S_n^m \; .$$

By aid of (5) it can be shown step by that the sign of S_n^m is identical with that of $(-1)^{m-n}$; therefore we have the second check :

$$n! = \sum_{m=1}^{n+1} |S_n^m| \tag{7}$$

viz., the sum of the absolute values of the numbers in the line n is equal to n! .

Generating function of the Stirling numbers of the first kind with respect to ***m.*** From equation (3) we conclude that the generating function of S_n^m with respect to m is

$$u(t,n) = (t)_n, \tag{8}$$

that is, in the expansion of $u(t, n)$ in powers of t the coefficient of t^m is equal to S_n^m.

In fact, the ***Stirling*** numbers, in their definition (2), are given by their generating function.

§ 51, Determination of the Stirling numbers starting from their definition. The expression

$$x(x-1)\;(x-2)\;.\;.\;.\;(x-n+1) = 0$$

may be considered as an equation whose roots are $u_i=i$ for $i=0$, 1, 2, . . . , n. Hence in this equation the coefficient of x^m is equal to

$$S_n^m = (-1)^{n-m} \sum u_1 u_2 \dots \; u_{n-m} \tag{1}$$

where the sum in the second member is extended to every combination of order ***n-m*** of the numbers 1, 2, . , , (n-l), without repetition and without permutation.

Equation **(1)** may be transformed by multiplying the first factor of the second member by **(n-1)!** and by dividing the quantity under the sign Σ by **(n-1) !** . This gives

$$(2) \qquad S_n^m = (-1)^{n-m} (n-1)! \sum \frac{1}{u_1 u_2 \ldots u_{m-1}} .$$

The sum in the second member is extended to every combination of order **m-1** of the numbers $1, 2, 3, \ldots (n-1)$, without repetition and without permutation.

Formulae (1) and (2) show also that the sign of S_n^m is the same as that of $(-1)^{n-m}$.

Later on we shall see that it is possible to express the sum (2) by others having no restriction concerning repetition.

We may obtain a third expression analogous to **(1)** and **(2)**, starting from the following equation, which will be demonstrated later (§ 71). We have

$$(3) \qquad [\log(1+t)]^m = \sum_{n=m}^{\infty} \frac{m!}{n!} S_n^m t^n.$$

On the other hand, from the known series of log $(1+t)$, applying m times ***Cauchy's*** rule of multiplication of infinite series, we get

$$(4) \qquad [\log(1+t)]^m = (-1)^m \sum_{n=m}^{\infty} (-1)^n t^n \sum \frac{1}{u_1 u_2 \ldots u_m} .$$

In the sum contained in the second member, u_i takes every value of 1, 2, 3, , . . with repetition and permutation but satisfying the condition:

$$u_1 + u_2 + u_3 + \ldots + u_m = n.$$

From (3) and (4) it results that

$$(5) \qquad S_n^m = (-1)^{n-m} \frac{n!}{m!} \sum \frac{1}{u_1 u_2 \ldots u_m} .$$

Examples. Let us determine S_5^3 by the formulae obtained above. From (1) we have

$$S_5^3 = 1.2 + 1.3 + 1.4 + 2.3 + 2.4 + 3.4 = 35.$$

Formula (2) gives

$$S_5^3 = 24\left[\frac{1}{1.2}+\frac{1}{1.3}+\frac{1}{1.4}+\frac{1}{2.3}+\frac{1}{2.4}+\frac{1}{3.4}\right] = 35.$$

Formula (5)

$$S_5^3 = 4.5\left[\frac{3}{1.1.3}+\frac{3}{1.2.2}\right] = 35.$$

Starting from equation (3) § 50 by integration we get

$$(6) \qquad \int_0^1 (x)_n \, dx = \sum_{m=1}^{n+1} \frac{S_n^m}{m+1},$$

Later it will be shown that this integral is also equal to $n!\, b_n$ where b_n is a coefficient of the *Bernoulli* polynomial of the second kind. Therefore we have

$$(7) \qquad b_n = \frac{1}{n!}\sum_{m=1}^{n+1} \frac{S_n^m}{m+1}.$$

This is an expression of the coefficient b_n by *Stirling's* numbers.

§ 52. Resolution of the difference equation

$$(1) \qquad S_{n+1}^m = S_n^{m-1} - n S_n^m$$

As has been said, the general solution of this equation of partial differences is unknown; but in some particular cases it can be solved easily.

1. Putting into it $m=1$ we obtain

$$S_{n+1}^1 = -n S_n^1.$$

Taking account of $S_1^1 = 1$ the solution of this linear difference equation with variable coefficients is, as we shall see later,

$$S_n^1 = (-1)^{n-1}(n-1)!.$$

2. Putting $m=2$ into equation (1) we have

$$S_{n+1}^2 + nS_n^2 = S_n^1 = (-1)^{n-1}(n-1)!$$

If we write

$$f(n) = (-1)^n S_n^2/(n-1)!$$

then the above difference equation will be

$$f(n+1) - f(n) = \underset{n}{\Delta} f(n) = \frac{1}{n},$$

and therefore according to § 19

$$f(n) = F(n-1) + k.$$

$S_2^2 = 1$; hence $f(2) = 1$; moreover as $F(1) = 1 - C$, therefore $k = C$, the *Euler's* constant. Finally

$$S_n^2 = (-1)^n (n-1)! \, [F(n-1) + C].$$

Example. Let $n=5$. Replacing the digamma function by the corresponding sum (p. 59) we find

$$S_5^2 = -24[1/1 + 1/2 + 1/3 + 1/4] = -50.$$

Remark. From the above we deduce the sum of 1/x expressed by *Stirling's* numbers:

$$\sum_{x=1}^{n} \frac{1}{x} = \frac{(-1)^n}{(n-1)!} S_n^2$$

3. If $m=3$ *we* have

$$S_{n+1}^3 + n S_n^3 = S_n^2 = (-1)^n (n-1)! \, [F(n-1) + C].$$

Writing again

$$f(n) = (-1)^{n-1} S_n^3 / (n-1)!$$

we get, $\Delta f(n) = [F(n-1) + C]/n$ so that

$$f(n) = \underset{n}{\Delta^{-1}} \frac{1}{n} [F(n-1) + C].$$

By summation by parts we obtain

$$\Delta^{-1} \frac{1}{n} F(n-1) = [F(n-1)]^2 - \Delta^{-1} \frac{1}{n} F(n)$$

since (pp. 58 and 60)

$$F(n) = F(n-1) + \frac{1}{n} \text{ and } \Delta^{-1} - \frac{1}{n^2} = F(n-1) + k.$$

Putting $F(n)$ into the preceding equation we find $\Delta^{-1} F(n-1)/n$ which gives by aid of (2')

$$f(n) = \tfrac{1}{2}[F(n-1)]^2 + \tfrac{1}{2}|F(n-1)| + CF(n-1) + k.$$

Since $f(1) = 0$ and $F(0) = \dfrac{\pi^2}{6}$ it follows that

$$k = \tfrac{1}{2}\left[C^2 - \frac{\pi^2}{6}\right]$$

and finally

$$S_n^3 = (-1)^{n-1} (n-1)! \, \tfrac{1}{2}\left\{[F(n-1) + C]^2 + F(n-1) - \frac{\pi^2}{6}\right\}.$$

Example. Let $n=5$. Then the preceding formula gives, since $F(4)+C = 1+1/2+1/3+1/4$ and $F(4)-\frac{\pi^2}{6} = -(1+1/4+1/9+1/16)$

$$S_5^3 = 35.$$

The formula above gives the sum of $1/x^2$ expressed by *Stirling's* numbers :

$$\sum_{x=1}^{n} \frac{1}{x^2} = \left[\frac{S_n^2}{(n-1)\,!}\right]^2 - \frac{|S_n^3|}{(n-1)\,!}.$$

It would be possible to continue in this manner and determine S_n^4 and so on, but the formulae would be more and more complicated.

4. If we put $m=n+1$ into equation (1) we have

$$S_{n+1}^{n+1} = S_n^n = \ldots\ldots = S_1^1 = 1.$$

5. Putting into equation (1) n+l-m instead of *m we* obtain

$$S_{n+1}^{n+1-m} = S_n^{n-m} - n\, S_n^{n+1-m}$$

therefore it follows that the difference of the Stirling number below, with respect to n, is

$$(2) \qquad \Delta_n S_n^{n-m} = -n\, S_n^{n+1-m}.$$

Particular case of this formula. Making $m=1$ we have

$$\Delta_n S_n^{n-1} = -n$$

and therefore

$$S_n^{n-1} = \Delta_n^{-1}(-n) = -\binom{n}{2} + k.$$

Since $S_1^0 = 0$ therefore $k=0$.

In consequence of formula (2) we have

$$S_n^{n-2} = \Delta_n^{-1}\left[-n\, S_n^{n-1}\right] = \Delta_n^{-1}\left[n\binom{n}{2}\right]$$

and finally

$$S_n^{n-2} = 3\binom{n}{4} + 2\binom{n}{3}.$$

This is a polynomial of the fourth degree. After multiplying it by $-n$, the indefinite sum will give S_n^{n-3}. This is a polynomial of the 6 -th degree; indeed, the degree is increased by one by the multiplication, and by one again by the summation. Continuing in this manner, we obtain S_n^{n-m} which will be a polynomial of the variable n of degree 2m.

Let us write it in the following way

$$(3) \qquad S_n^{n-m} = C_{m,0}\binom{n}{2m} + C_{m,1}\binom{n}{2m-1} + \dots + C_{m,2m}.$$

After multiplication by $-n$ the coefficient of $C_{m,\nu}$ will be

$$-n\binom{n}{2m-\nu} = -(2m-\nu+1)\binom{n}{2m-\nu+1} - (2m-\nu)\binom{n}{2m-\nu}$$

therefore we have

$$S_n^{n-m-1} = \Delta^{-1}[-n\, S_n^{n-m}] = -\sum_{\nu=0}^{2m+1} C_{m,\nu}(2m-\nu+1)\binom{n}{2m-\nu+2} -$$

$$-\sum_{\nu=0}^{2m} C_{m,\nu}(2m-\nu)\binom{n}{2m-\nu+1} + k.$$

Writing in the second sum $\nu-1$ instead of v we obtain

$$S_n^{n-m-1} = -\sum_{\nu=0}^{2m+1}(2m-\nu+1)\binom{n}{2m-\nu+2}[C_{m,\nu} + C_{m,\nu-1}] + k.$$

Here we put $C_{m,-1} = 0$.

On the other hand we have according to (3)

$$S_n^{n-m-1} = \sum_{\nu=0}^{2m+3} C_{m+1,\nu}\binom{n}{2m+2-\nu}.$$

From the last two equations we deduce

$$(4) \quad C_{m+1,\nu} = -(2m-\nu+1)[C_{m,\nu} + C_{m,\nu-1}] \text{ and } k = C_{m+1,2m+2}.$$

The general solution of this partial difference equation is unknown, but in some particular cases the solution is easily obtained.

From $$S_n^{n-1} = -\binom{n}{2}$$

we conclude that $C_{1,0} = -1$ and $C_{1,\nu} = 0$ if v > 0.

Putting $m=1$ into equation (4), it follows that

$$C_{2,\nu} = 0 \text{ if } \nu > 1.$$

From this we conclude, putting $m=2$ into (4), that

$$C_{3,\nu} = 0 \text{ if } \nu > 2$$

and so on; finally we find

$$C_{m,\nu} = 0 \text{ if } \nu > m\text{-}1,$$

Therefore we have also $k = C_{m,\,2m} = 0$; and equation (3) will be

$$(5) \qquad S_n^{n-m} = C_{m,0} \binom{n}{2m} + C_{m,1} \binom{n}{2m-1} + \dots + \\ + C_{m,\,m-1} \binom{n}{m+1}.$$

Equation (4) may be solved in some particular cases.

I. Putting $\nu = m$ it becomes

$$C_{m+1,\,m} = -(m+1)\, C_{m,\,m-1}.$$

This is a homogeneous linear difference equation with variable coefficients; we shall see that its solution is

$$C_{m,\,m-1} = \omega \prod_{i=0}^{m} -(i+1) = (-1)^m\, m!\; \omega.$$

$C_{1,\,0} = -1$; therefore $\omega = 1$.

2. Putting $\nu = 0$ into equation (4) we get

$$C_{m+1,\,0} = -(2m+1)\, C_{m,\,0}.$$

The solution of this equation, which is of the same type as the preceding, will be

$$C_{m,0} = \omega \prod_{i=0}^{m} -(2i+1) = (-1)^m . 1 . 3 . 5 \dots (2m-1) . \omega .$$

Since $C_{1,\,0} = -1$, hence $\omega = 1$.

Multiplying equation (4) by $(-1)^\nu$ and summing from $\nu = 0$ to $\nu = m+1$, we obtain

$$\sum_{\nu=0}^{m+1} (-1)^\nu C_{m+1,\nu} = -\sum_{\nu=0}^{m+1} (-1)^\nu C_{m,\,\nu} - \sum_{\nu=0}^{m+1} (-1)^\nu (2m-\nu) C_{m,\nu} + \\ + \sum_{\nu=0}^{m+1} (-1)^{\nu-1} (2m-\nu+1)\, C_{m,\nu-1}.$$

Remarking that the last two sums in the second member are equal to

$$-\sum_{\nu=0}^{m+1} \underset{\nu}{\Delta} \left[(-1)^{\nu-1} (2m-\nu+1)\, C_{m,\nu-1} \right]$$

moreover that at the limits the quantity in the brackets is equal to zero, as $C_{m,-1} = 0$ and $C_{m,m} = 0$, we have

$$\sum_{\nu=0}^{m+1} (-1)^{\nu} C_{m+1,\nu} = -\sum_{\nu=0}^{m+1} (-1)^{\nu} C_{m,\nu};$$

from this we conclude that

$$(7) \qquad \sum_{\nu=0}^{m} (-1)^{\nu} C_{m,\nu} = (-1)^{m-1} \sum_{\nu=0}^{1} (-1)^{\nu} C_{1,\nu} = (-1)^{m}.$$

Starting from the initial conditions $C_{1,\nu} = 0$ if $\nu \neq 0$ and $C_{1,0} = -1$ we may compute step by step the numbers $C_{m,\nu}$ by aid of equation (4). The results are given in the table below. Equation (7) may be used for checking the numbers.

Table of $C_{m,\nu}$

$m \backslash \nu$	0	1	2	3	4	5	6
1	-1						
2	3	2					
3	-15	-20	-6				
4	105	210	130	24			
5	-945	-2520	-2380	-924	-120		
6	10395	34650	44100	26432	7308	720	

Remark. From formula (5) we may deduce the differences of S_n^{n-m} with respect to n:

$$\Delta_n^{\mu} S_n^{n-m} = \sum_{\nu=0}^{m} C_{m,\nu} \binom{n}{2m-\nu-\mu}$$

and in the particular case of $\mu = 2m$

$$\Delta_n^{2m} S_n^{n-m} = C_{m,0}.$$

From equation (5) we deduce the important formula:

$$(8) \qquad \lim_{n=\infty} \frac{|S_n^{n-m}|}{n^{2m}} = \frac{|C_{m,0}|}{(2m)!} = \frac{1.3.5.7 \ldots (2m-1)}{(2m)!} = \frac{1}{m!\,2^m}$$

permitting us to deduce asymptotic values of S_n^{n-m}.

The difference equation (5), § 50 multiplied by $(-1)^{n+m+1}$ gives

$$|S_{n+1}^{m}| = |S_n^{m-1}| + n\,|S_n^{m}|.$$

Dividing both members of this equation by $n!$ we obtain

(9) $$\underset{n}{\Delta}\left[\frac{|S_n^m|}{(n-1)!}\right] = \frac{|S_n^{m-1}|}{n!}.$$

The difference of the quantity in the brackets is understood to be with respect to n.

From (9) we conclude, $\nu+1$ instead of m

(10) $$\sum_{n=\nu}^{\mu} \frac{|S_n^\nu|}{n!} = \frac{|S_\mu^{\nu+1}|}{(\mu-1)!},$$

Remark 2. Formula (5) is advantageous for the determination of S_n^{n-m} if n is large and m small. For instance we have

$$S_{12}^8 = 105\binom{12}{8} + 210\binom{12}{7} + 130\binom{12}{6} + 24\binom{12}{5} = 357423.$$

§ 53, Transformation of a multiple sum "without repetition" into sums without restriction. Let us consider the following sum

(1) $$\mathcal{S}_m = \sum_{u_1=1}^{n+1} \sum_{u_2=1}^{n+1} \dots \sum_{u_m=1}^{n+1} f(u_1) f(u_2) \dots f(u_m)$$

from which repetitions such as $u_i = u_j$ are excluded. Putting

(2) $$\sum_{u_i=1}^{n+1} [f(u_i)]^k = \sigma_k$$

it is easy to show that $\mathcal{S}_2$ may be expressed by sums without restriction, We have

$$\mathcal{S}_2 = \Sigma f(u_1) \Sigma f(u_2) - \Sigma f(u_1) f(u_1) = \sigma_1^2 - \sigma_2 .$$

Indeed, to obtain $\mathcal{S}_2$ it is necessary to subtract from the first expression the terms in which $u_1 = u_2$.

In the case of $\mathcal{S}_3$ we should proceed in the same manner: from σ_1^3 we must subtract the terms in which $u_1 = u_2$ or $u_1 = u_3$ or $u_2 = u_3$; we obtain $\sigma_1^3 - 3\sigma_1\sigma_2$. But in this manner we have subtracted the terms in which $u_1 = u_2 = u_3$ three times, therefore we must add σ_3 twice. Finally

$$\mathcal{S}_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3 .$$

Continuing in this manner we could show that $\mathcal{S}_m$ may be expressed by a sum of terms of the form

$$a\,\sigma_1^{\lambda_1}\,\sigma_2^{\lambda_2} \dots \sigma_m^{\lambda_m}$$

where λ_i may be equal to 0, $1, 2, 3, \ldots$, but are such that

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots m\lambda_m = m,$$

and a is a numerical constant.

To distinguish these constants, we will introduce first the index $\Sigma\lambda_i$ which is equal to the degree of the term in σ. Since there may be several terms of the same degree, we introduce in these cases a second index μ; $\mu = 1$ will correspond to the term in which λ_1 is the greatest ($\Sigma\lambda_i$ being the same), $\mu = 2$ to the following, and so on. If there are several terms in which λ_1 is the same, then these are ranged in order of magnitude of 1, and so on.

The number of terms in which $\Sigma\lambda_i$ is the same, is equal to the number of partitions of the number m, with repetition but without permutation, into $\Sigma\lambda_i$ parts (i. e. of order $\Sigma\lambda_i$).

Adopting ***Netto*'s** notations (Combinatorik, p. 119) this number will be written:

$$\Gamma^{\Sigma\lambda_i}\left(\int m; 1, 2, \ldots\right) = \Gamma^{\Sigma\lambda_i}\left(\int m\right).$$

It is difficult to determine these numbers in the general case, but there are formulae by aid of which a table containing them may be calculated step by step. For instance

$$\Gamma^h\left(\int m\right) = \Gamma^{h-1}\left(\int m\text{-}1\right) + \Gamma^h\left(\int m - h\right).$$

Starting from $\Gamma^1\left(\int m\right) = 1$ and from $\Gamma^{n+m}\left(\int m\right) = 0$ the following table is rapidly computed by aid of this equation:

Table of $\Gamma^{\Sigma\lambda_i}\left(\int m\right)$.

$m \backslash \Sigma\lambda_i$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	1	1				
4	1	2	1	1			
5	1	2	2	1	1		
6	1	3	3	2	1	1	
7	1	3	4	3	2	1	1
8	1	4	5	5	3	2	1
9	1	4	7	6	5	3	2

The total number of terms in $\mathscr{S}_m$ is equal to the number of all the partitions of m with repetition (but without permutation), of any order whatever. ***Netto*** denoted this number by

$$\Gamma(\textstyle\int m; 1, 2, 3, \ldots) = \Gamma(\textstyle\int m).$$

These numbers are also difficult to determine in the general case; but starting from $\Gamma(\int 1) = 1$, they can be rapidly computed by

$$\Gamma(\textstyle\int m) = \Gamma(\textstyle\int m-1) + \Gamma(\textstyle\int m;\ 2, 3, \ldots).$$

Table of $\Gamma(\int m)$.

m	1	2	3	4	5	6	7	8	9	10	11
$\Gamma(\int m)$	1	2	3	5	7	11	15	22	30	43	58

These numbers are connected with the former ones by ***Euler's*** formula

$$\Gamma^{\Sigma\lambda_i}(\textstyle\int m) = \Gamma(\textstyle\int m - \Sigma\lambda_i; 1, 2, 3, \ldots, \Sigma\lambda_i).$$

Finally our formula will be

$$\mathscr{S}_m = \Sigma\, a_{\Sigma\lambda_i,\,\mu}\, \sigma_1^{\lambda_1} \sigma_2^{\lambda_2}, , , \sigma_m^{\lambda_m} \tag{3}$$

where $\Sigma i\lambda_i = m$ and the sum is extended to $\Sigma\lambda_i = 1, 2, 3, \ldots m$, and μ varies from 1 to $\Gamma^m(\int m)$.

Example. Let $m=4$, **then** the number of the terms will be, according to the second table, equal to five. Moreover two terms will **correspond** to $\Sigma\lambda_i = 2$ (First table). Indeed the partitions are

$$1+1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad 4,$$

therefore $\mathscr{S}_4$ will be

$$\mathscr{S}_4 = a_4\, \sigma_1^{4} + a_3\, \sigma_1^{2}\sigma_2 + a_{2,1}\, \sigma_1\sigma_3 + a_{2,2}\, \sigma_2^{2} + a_1\sigma_4 .$$

Determination of the coefficient $a,_\mu$, First we wiil remark that they are independent of n and of the function f(u) chosen. Therefore we may deduce these numbers by choosing the most convenient function. Let us put first

$$f(u_i) = 1\,;$$

according to formula (2) we have $\sigma_k = n$, for every value of k.

Hence $\mathscr{S}_m$ becomes

$$\mathscr{S}_m = a, n^m + a_{m-1} n^{m-1} + , \ldots , + (a_{\Sigma\lambda_i,1} + a_{\Sigma\lambda_i,2} + , \ldots ,) n^{\Sigma\lambda_i} + \\ + , \ldots + a_1 n$$

but the first member is equal to the number of combinations of order *m*, with permutation but without repitition, of the numbers $1, 2, 3, \ldots, n$, and therefore equal to (n), . From this we conclude, taking account of equation (3) of § 50, that the coefficients may be expressed by ***Stirling's*** numbers of the first kind:

(4) $$a, = S_m^m; \quad a_{m-1} = S_m^{m-1}; \quad a, = S_m^1$$

and

$$a_{\Sigma\lambda_i,1} + a_{\Sigma\lambda_i,2} + , \ldots = S_m^{\Sigma\lambda_i}$$

This gives *m* equations, since the number of the unknown coefficients is equal to $\Gamma(\int m)$ and according to the second table we generally have $\Gamma(\int m) > m$; therefore we need more equations.

To obtain them we will put

$$f(u_i) = \alpha^{u_i}$$

where $u_i = 1, 2, 3, \ldots, n$.

Since the coefficients *a* are independent of *n we* may put $n = \infty$ therefore if a < 1 it follows that:

$$\sigma_k = \frac{\alpha^k}{1 - \alpha^k}.$$

Hence, according to (3), we have

(5) $$\mathscr{S}_m = \Sigma\, a_{\Sigma\lambda_i,\omega}\, \alpha^m (1-\alpha)^{-\lambda_1} (1-\alpha^2)^{-\lambda_2} \ldots (1-\alpha^m)^{-\lambda_m}$$

where, as we have seen, $\Sigma i\lambda_i = m$.

On the other hand, in the case considered

(6) $$\mathscr{S}_m = \Sigma \ldots, \Sigma\, \alpha^{u_1 + u_2 + \ldots + u_m}.$$

The coefficients of α^ω in (5) and (6) are equated, for $\omega = m+1, m+2, m+3, \ldots$ and so on, till the necessary number of equations for determining the unknown number $a_{.,\omega}$ is obtained.

The coefficient of α^ω in (6) is equal to the number of partitions of ω into m parts ($u_i = 1, 2, 3, \ldots$) with permutation but without repetition.

Netto denoted this number in his Combinatorik (p. 119) by

$$V^m(\int \omega; 1, 2, 3, \ldots) = V^m(\int \omega).$$

The general expression for this number is complicated, but we know that

$$(6') \qquad V^m(\int \omega) = 0 \text{ if } \omega < \binom{m+1}{2}$$

and

$$V^m(\int \omega) = m! \text{ if } \omega = \binom{m+1}{2}.$$

Indeed the smallest sum, since there cannot be repetition, is $1+2+\ldots+m = \binom{m+1}{2}$.

The coefficient of α^ω in (5) becomes, after the expansions, equal to

$$(7) \qquad \Sigma\, a_{\Sigma\lambda_i,\mu}\, \Sigma \ldots \Sigma \binom{-\lambda_1}{x_1}\binom{-\lambda_2}{x_2} \cdots \binom{-\lambda_m}{x_m}$$
$$(-1)^{x_1+x_2+\ldots+x_m}\, \alpha^{x_1+2x_2+\ldots+mx_m+m}.$$

In the sums above, x_i takes every value of 0, 1, 2, . . . with repetition and permutation, but so as to have

$$(8) \qquad x_1 + 2x_2 + \ldots + mx_m = \omega - m.$$

Moreover the first sum in (7) is extended to every value of $\Sigma\lambda_i$ and μ such that $\Sigma i\lambda_i = m$.

In this way we get from (6') with the m equations obtained previously, in all $\binom{m+1}{2}$ equations, which will generally be sufficient to determine the $a_{\nu,\mu}$, since according to our table

$$\binom{m+1}{2} > \Gamma(\int m).$$

But if necessary it would be easy to get even more equations.

Example. Let us determine $\mathcal{S}'_4$. Here $\Sigma i\lambda_i = m = 4$; we have seen already that

$$\mathscr{S}_4 = a_4\, \sigma_1^4 + a_3\, \sigma_1^2\sigma_2 + a_{2,1}\, \sigma_1\sigma_3 + a_{2,2}\, \sigma_2^2 + a_1\, \sigma_4 .$$

The equations (4) give

$$a_4 = S_4^4 = 1 \qquad a_3 = S_4^3 = -6$$

$$a_{2,1} + a_{2,2} = S_4^2 = 11 \qquad a_1 = S_4^1 = -6.$$

Since there are five unknowns, we need one equation more. To obtain it, let us remark that since $\omega=5$ is less than $\binom{m+1}{2} = 10$, according to what precedes, the coefficient of a^5 in (6) is for $m=4$ equal to zero. We get an expression for this coefficient starting from (8):

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 1 .$$

The only solution of this equation is $x_1 = 1$ and $x_2=x_3==x_4=0$; therefore from (7) we obtain the required coefficient

$$\Sigma\, a_{\Sigma\lambda_i,\mu}\, \lambda_1 = 4a_4 + 2a_3 + a_{2,1} = 0 .$$

Indeed if $\Sigma\lambda_i = 4$ then $\lambda_1 = 4$; and if $\Sigma\lambda_i = 3$ then $\lambda_1 = 2$; finally if $\Sigma\lambda_i = 2$ then $\lambda_1 = 1$. From the above equations we conclude that

$$a_{2,1} = 8 \quad \text{and} \quad a_{2,2} = 3$$

and finally

$$\mathscr{S}_4 = \sigma_1^4 - 6\, \sigma_1^2\sigma_2 + 8\, \sigma_1\sigma_3 + 3\, \sigma_2^2 - 6\, \sigma_4 .$$

Particular case. $f(u_i) = 1/u_i$ and therefore

$$\mathscr{S}_m = \Sigma\, \frac{1}{u_1 u_2 \ldots u_m}$$

Let $m = 4$ and $u_i = 1, 2, 3, 4$. We shall have

$$\sigma_1 = \frac{25}{12}; \quad \sigma_2 = \frac{205}{144}; \quad \sigma_3 = \frac{2035}{1728}; \quad \sigma_4 = \frac{22369}{20736}$$

we find

$20736(\sigma_1^4) = 390625$ $\qquad 20736(-6\sigma_1^2\sigma_2) = -768750$

$20736(8\sigma_1\sigma_3) = 407000$ $\qquad 20736\ (-6\sigma_4) = -134214$

$20736(3\sigma_2^2) = 126075$ $\qquad$ —————— -902964

—————— 923700

therefore

$$20736\mathscr{S}_4 = 923700 - 902964 = 20736 \quad \text{and} \quad \mathscr{S}_4 = 1.$$

This was obvious, since as there is no repetition the fraction can only be $1/1 \cdot 2 \cdot 3 \cdot 4$, and this in every permutation, that is $4!$ times.

§ 54. Stirling's numbers expressed by sums, Limits. From formula (2) of § 51 we have

$$S_{n+1}^{m+1} = (-1)^{n-m}\, n!\, \Sigma \frac{1}{u_1 u_2 \ldots u_m} \tag{1}$$

where the sum in the second member is to be extended to every combination of order m of the numbers $1, 2, 3, \ldots, n,$ without repetition and without permutation. If there is no repetition the last restriction may he suppressed when dividing the second member by $m!$. In order to suppress the first restriction concerning repetition, let us put

$$\sigma_k = \sum_{u=1}^{n+1} \frac{1}{u^k}$$

According to formula (3) of § 53 the sum in expression (1) may be transformed into ordinary sums, writing

$$S_{n+1}^{m+1} = (-1)^{n-m} \frac{n!}{m!} \Sigma\, a_{\Sigma\lambda_i, \mu}\, \sigma_1^{\lambda_1} \sigma_2^{\lambda_2} \ldots \sigma_m^{\lambda_m} \tag{2}$$

where $\Sigma i\lambda_i = m$.

Particular cases. 1. From $m=0$ it follows that $\lambda_i = 0$ for every value of i, and we have seen (4), § 53 that a, $= S_n^n = 1$; hence

$$S_{n+1}^1 = (-1)^n\, n!$$

2. Putting $m=1$ we have $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \ldots = 0$ since $a_1 = 1$

$$S_{n+1}^2 = (-1)^{n+1}\, n!\, \sigma_1 .$$

3. If $m=2$ *we* may have $\lambda_1 = 2$ or $\lambda_2 = 1$. Moreover we have seen that $a_{m-1} = S_m^{m-1} = S_2^1 = -1$, therefore

$$S_{n+1}^3 = (-1)^n\, n!\, \tfrac{1}{2} (\sigma_1^2 - \sigma_2).$$

4. If $m=3$ we may have $\lambda_1 = 3$, or $\lambda_1 = 1$ and $\lambda_2 = 1$, or $\lambda_3 = 1$ we have $a_{m-1} = S_m^{m-1} = -3$ and a, $= S_m^1 = 2$; therefore

$$S_{n+1}^4 = (-1)^{n+1} \frac{n!}{6} (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3)$$

and so on.

If we have tables giving σ_k we may determine the ***Stirling's*** numbers by these formulae.

Example. $S_{12}^2 = 11!\,\sigma_1$, but according to § 19

$$\begin{aligned} F(11) &= 2'4426\ 6168 \\ C &= 0'5772\ 1566 \\ \hline \sigma_1 &= 3'0198\ 7734 \end{aligned}$$

Hence $S_{12}^2 = 120\ 543839'8$. In this result the error is equal to $-0'2$.

Formula (2) permits us to determine the approximate value of S_{n+1}^{m+1} if n is large. Since according to (14) § 87 for large values of n we have approximately

(3) $$a, = F(n) + C \sim \log (n+l) + C$$

therefore

$$S_{n+1}^2 \sim (-1)^{n+1}\, n!\, [\log (n+l) + C].$$

The logarithm in this formula is a ***Napier's*** logarithm. If for instance $n=11$ then $\log (n+l) + C = 3'062123$. Multiplying it by $11!$ we obtain the result with an error of 1'4% in excess,

Knowing the first values of the coefficients $a_{\Sigma\lambda_i,\mu}$ *we* may deduce from (2) the ***limits*** of certain expressions containing ***Stirling*** numbers, if n increases indefinitely. For instance

$$\lim_{n=\infty} \frac{|S_{n+1}^{m+1}|}{(n+l)!} = \lim_{n=\infty} \frac{1}{m!} \left[\frac{\sigma_1{}^m}{n+1} - \binom{m}{2} \frac{\sigma_1{}^{m-2}\sigma_2}{n+1} + \ldots \right].$$

Since

$$\sigma_k \leq \frac{\pi^2}{6} \quad \text{if} \quad k>1$$

and moreover

$$\lim_{n=\infty} \frac{\sigma_1}{\log (n+1)} = 1\,, \qquad \lim_{n=\infty} \frac{[\log (n+1)]^s}{n} = 0$$

it 'follows that

(4) $$\lim_{n=\infty} \frac{|S_{n+1}^{m+1}|}{(n+1)!} = 0$$

and

(5) $$\lim_{n=\infty} \frac{|S_{m+1}^{n+1}|}{n!\ [\log (n+1)]^m} = \frac{1}{m!}.$$

From this formula asymptotic values of S_{n+1}^{m+1} can be obtained, for instance

$$|S_n^m| \sim \frac{(n-1)!}{(m-1)!}[\log n + C]^{m+1}.$$

Variation of S_n^m in the lines of the table; that is if n remains constant and m increases. Formula (2) permits us to determine approximately the value of m corresponding to the greatest term. We have, writing n instead of n+l

$$\lim_{n=\infty} \frac{1}{\sigma_1}\left|\frac{S_n^{m+1}}{S_n^m}\right| = \lim_{n=\infty} \frac{1}{m}\,\frac{\sigma_1^m - \binom{m}{2}\sigma_1^{m-2}\sigma_2 + \dots}{\sigma_1^{m-1} - \binom{m-1}{2}\sigma_1^{m-3}\sigma_2 + \dots}\,\frac{1}{\sigma_1} = \frac{1}{m}$$

Therefore an asymptotic value of the ratio will be:

$$(6) \qquad \left|\frac{S_n^{m+1}}{S_n^m}\right| \sim \frac{\log n}{m}.$$

If $n>2$ then for $m=1$ this ratio will be greater than one, that is the numbers will increase at the beginning of the rows. This can be verified in the table from $n=3$ upward, S_n^χ will be the greatest term of the row, if

$$\left|\frac{S_n^\chi}{S_n^{\chi-1}}\right| > 1 \quad \text{and} \quad \left|\frac{S_n^{\chi+1}}{S_n^\chi}\right| < 1.$$

From (6) we conclude that we have approximately

$$\chi > \log n > \chi - 1.$$

Examples.

$n=3$	$\log n = 1'099$	greatest term of the row n	S_3^2
$n=8$	$\log n = 2'079$		S_8^3
$n=21$	$\log n = 3'045$		S_{21}^4
$n=55$	$\log n = 4'007$		S_{55}^5
$n=131$	$\log n = 5'003$		S_{131}^6

The first two items can be checked by our table.

Change of S_n^m in the columns of the table; that is if m remains constant and n increases. From formula (2) we may obtain

$$\lim_{n=\infty} \frac{S^m_{n+1}}{nS^m_n} = 1 . \tag{7}$$

This will lead, for instance, to the following asymptotic formula

$$S^m_{n+1} \sim n S^m_n \tag{8}$$

which gives acceptable values if m is not too large. If $m=1$, the values given are exact.

We have seen that, m being given, the limit of $\left|\frac{S^m_n}{n!}\right|$ for $n=\infty$ is equal to zero; now we will determine the differences of this expression with respect to n. Formula (9) § 52 gives

$$\Delta_n \left[\frac{|S^m_n|}{(n-1)!}\right] = \frac{|S^{m-1}_n|}{n!}. \tag{9}$$

Writing

$$\left|\frac{S^m_n}{n!}\right| = \frac{1}{n} \cdot \frac{|S^m_n|}{(n-1)!}$$

and applying to the second member the formula (1) of § 30, which gives the difference of a product, we get

$$\Delta_n \left|\frac{S^m_n}{n!}\right| = - \frac{1}{(n+1)n} \cdot \frac{|S^m_n|}{(n-1)!} + \frac{1}{n+1} \cdot \frac{|S^{m-1}_n|}{n!} = \tag{10}$$

$$= - \frac{1}{(n+1)!} \Delta_m |S^{m-1}_n|.$$

Therefore we conclude that if

$$\Delta_m |S^{m-1}_n| > 0 \quad \text{then we have} \quad \Delta_n \left|\frac{S^m_n}{n!}\right| < 0.$$

Therefore the quantity $\left|\frac{S^m_n}{n!}\right|$ will decrease with increasing n if m is smaller than the index corresponding to the maximum of $|S^{m-1}_n|$ in the row n. We have seen that this maximum is obtained if m-l is equal to the greatest integer contained in $\log n + 1$. Hence if $m-1 > \log n + 1$, $\left|\frac{S^m_n}{n!}\right|$ will increase with increasing n.

For instance, since log $\mathbf{8} = 2'079$, if $n=8$, the quantity $|S_n^{m-1}|$ will be maximum if $m-1=3$ and $\left|\frac{S_n^m}{n}\right|$ will decrease with increasing n if $m < 4$. This can be checked by the table.

Remark. From formula (9) it follows that

$$\Delta^{-1} \frac{|S_n^m|}{n!} = \frac{|S_n^{m+1}|}{(n-1)!} + k.$$

§ 55. Some applications of the Stirling numbers of the first kind.

1. Expansion of factorials and binomial-coefficients into power series. We have seen that

$$(1) \qquad (x)_n = \sum_{m=1}^{n+1} S_n^m x^m$$

and

$$(2) \qquad \binom{x}{n} = \frac{1}{n!} \sum_{m=1}^{n+1} S_n^m x^m.$$

The expansion of the generalised factorial was

$$(3) \qquad (x)_{n,h} = \sum_{m=1}^{n+1} S_n^m x^m h^{n-m}.$$

The above formulae enable us to determine the derivatives or the integral of factorials and binomials. We have

$$(4) \qquad \mathrm{D}^m \binom{x}{n} = \frac{1}{n!} \sum_{\nu=m}^{n+1} S_n^\nu (\nu)_m x^{\nu-m}$$

and

$$(5) \qquad \int \binom{x}{n} dx = \frac{1}{n!} \sum_{m=1}^{n+1} S_n^m \frac{x^{m+1}}{m+1} + k.$$

Moreover we may determine by aid of the preceding formulae the factorial, or the binomial moments of a function f(x) expressed by power-moments, Let us recall the definition of these moments. Denoting the power moment of degree n by $\mathcal{M}_n$, the factorial moment of degree n by $\mathfrak{M}_n$ and the binomial moment of degree n by $\mathcal{B}_n$, we have

$$\mathcal{M}_n = \sum_{x=0}^{\mu} x^n f(x); \qquad \mathfrak{M}_n = \sum_{x=0}^{\mu} (x)_n f(x)$$

$$\mathscr{B}_n = \sum_{x=0}^{\mu} \binom{x}{n} f(x).$$

Therefore if we expand (x), into a power series (1) we have

$$\mathfrak{M}_n = \sum_{m=1}^{n+1} S_n^m \mathscr{M}_m.$$

Expanding $\binom{x}{n}$ into a power series (2) we obtain

$$\mathscr{B}_n = \sum_{m=1}^{n+1} \frac{1}{n!} S_n^m \mathscr{M}_m.$$

Particular *cases.* Expansion of the factorial $(x+n-1)_n$ into a power series. Since we have

$$(x+n-1)_n = (-1)^n (-x)_n$$

it follows that

$$(x+n-1)_n = \sum_{m=1}^{n+1} (-1)^{n+m} S_n^m x^m = \sum_{m=1}^{n+1} | S_n^m | x^m.$$

§ 56, Derivatives expressed by differences. Given f(a), $\Delta f(a)$, $\Delta^2 f(a)$, , , , and so on, it is required to find $D^s f(x)$.

Starting from ***Taylor's*** series

$$f(x) = \sum_{m=0}^{\infty} \frac{(x-a)^m}{m!} D^m f(a)$$

we obtain

$$\text{(1)} \qquad D^s f(x) = \sum_{m=s}^{\infty} \frac{(x-a)^{m-s}}{(m-s)!} D^m f(a).$$

Therefore it is sufficient to express by differences the derivatives of *f(x)* for $x=a$. To obtain them we will expand *f(x)* into a generalised *Newton* series

$$\text{(2)} \qquad f(x) = \sum_{n=0}^{\infty} \binom{x-a}{n}_h \frac{\Delta_h^n f(a)}{h^n}.$$

According to the formulae of the preceding paragraph we have

$$\binom{x-a}{n}_h = \sum_{\nu=n}^{n+1} \frac{1}{n!} (x-a)^\nu h^{n-\nu} S_n^\nu.$$

From this we deduce

$$\mathbf{D}^m \binom{x-a}{n}_h = \sum_{\nu=m}^{n+1} \frac{1}{n!} (\nu)_m (x-a)^{\nu-m} h^{n-\nu} S_n^\nu .$$

For $x=a$ this expression gives

$$\left[\mathbf{D}^m \binom{x-a}{n}_h\right]_{x=a} = \frac{m!}{n!} h^{n-m} S_n^m$$

and finally

(3) $$\mathbf{D}^m f(a) = \sum_{n=m}^{\infty} \frac{m!}{n!} \frac{\Delta_h^n f(a)}{h^m} S_n^m,$$

The value thus obtained put into (1) gives the required formula, If t(x) is a polynomial, there is no difficultyinapplying formulae (1) and (3). If *f(*x*)* is not a polynomial, the *Newton* series (2) must be convergent and the *Taylor series* too.

Example 1. Given $f(x) = 1/x$; if $h = 1$ then we shall have

$$\mathbf{D}^m f(a) = \frac{(-1)^m m!}{a^{m+1}} \quad \text{and} \quad \Delta^n f(a) = \frac{(-1)^n n!}{(a+n)_{n+1}},$$

moreover from (3) it follows that

$$\mathbf{D}^m f(a) = \sum_{n=m}^{\infty} \frac{(-1)^n m!}{(a+n)_{n+1}} S_n^m.$$

Therefore

(4) $$\frac{1}{m!\, a^{m+1}} = \sum_{n=m}^{\infty} \frac{|S_n^m|}{(a+n)!} .$$

If $a=1$ we get

$$1 = \sum_{n=m}^{\infty} \frac{|S_n^m|}{(n+1)!} .$$

Let us remark, that the last sum is independent of *m.* Using formula (10) of § 54 it can be shown that its difference with respect to *m* is equal to zero for every value of *m.*

Example 2. Let $f(x)=2^x$. Putting $a=0$ and $h=1$ we obtain

$$\Delta^n f(x) = 2^x \quad \text{and} \quad \mathbf{D}^m f(x) = 2^x (\log 2)^m;$$

therefore according to (3) it follows that

(5) $$\frac{(\log 2)^m}{m!} = \sum_{n=m}^{\infty} \frac{S_n^m}{n!} .$$

The conditions of convergence are satisfied.

Example 3. Let $f(x) = F(x) = \mathbf{D} \log \Gamma(x+1)$. Putting $a=0$ and $h=1$ we deduce

$$\Delta^n F(x) = \frac{(-1)^{n-1} (n-1)!}{(x+n)_n}.$$

From formula (2) § 21 it follows that

$$\mathbf{D}^m F(0) = (-1)^{m-1} m! \sum_{x=1}^{\infty} \frac{1}{x^{m+1}}.$$

Therefore formula (3) gives

$$\text{(6)} \qquad \sum_{x=1}^{\infty} \frac{1}{x^{m+1}} = \sum_{n=m}^{\infty} \frac{|S_n^m|}{n!\, n}.$$

Remark. Formula (3) becomes especially useful if we deal with functions whose derivatives are complicated, and the differences simple.

§ 57. Stirling's numbers of the **first kind obtained by aid of probability.** Let us consider the following particular case of *Poisson's* problem of repeated trials: n trials are made, and the probability that the i-th trial is favourable is $p_i = i/(n+1)$.

According to the general theorem of repeated trials, the probability that among n trials x shall be favourable is:

$$\text{(1)} \qquad P(x) = \sum_{s=x}^{n+1} (-1)^{x+s} \binom{s}{x} \sum p_{\nu_1 \nu_2 \ldots \nu_s}$$

where $p_{\nu_1 \nu_2 \ldots \nu_s}$ denotes the probability that the ν_1-th, ν_2-th, . . . , and that the ν_s-th trial shall be favourable. Since in the case considered the events are independent, we have

$$p_{\nu_1 \nu_2 \ldots \nu_s} = p_{\nu_1} p_{\nu_2} \ldots p_{\nu_s} = \frac{\nu_1 \nu_2 \ldots \nu_s}{(n+1)^s}.$$

The second sum in formula (1) is to be extended to the sum of the products of the combinations of order s (without repetition and without permutation) of the numbers $1, 2, \ldots n$. But we have seen, in § 51 (formula 1), that this sum is equal to the absolute value of the *Stirling* number of the first kind S_{n+1}^{n+1-s}. Therefore

$$\text{(2)} \qquad P(x) = \sum_{s=x}^{n+1} (-1)^{x+s} \binom{s}{x} \frac{|S_{n+1}^{n+1-s}|}{(n+1)^s}.$$

Multiplying both members of this equation by $\binom{x}{i}$ and summing from $x=i$ to $x=n+1$ we make use of formula (3), § 65:

$$\sum_{x=i}^{n+1} (-1)^{x+s}\binom{x}{i}\binom{s}{x} = \binom{0}{i-s}$$

that is, the sum is equal to zero if $s \neq i$ and equal to one if $s=i$. Consequently we shall have

$$(3) \qquad |S_{n+1}^{n+1-s}| = (n+1)^s \sum_{x=i}^{n+1} \binom{x}{i} P(x) = (n+1)^s \mathcal{B}_s .$$

If we denote by $\Theta(t)$ the generating function of the probability P(x), then $\mathcal{B}_s$ the binomialmoment of order s of the function $P(x)$ is given by

$$\mathcal{B}_s = \frac{1}{s!}\left[\frac{d^s}{dt^s}\Theta(t)\right]_{t=1} .$$

It is easy to show that in the case considered above generating function is the following:

$$(4) \qquad \Theta(t) = \prod_{i=0}^{n+1} (q_i+p_i t)$$

where $q_i = 1-p_i$

From this we conclude that

$$(5) \qquad |S_{n+1}^{n+1-s}| = \frac{(n+1)^s}{s!}\left[\frac{d^s}{dt^s}\Theta(t)\right]_{t=1} .$$

This formula may serve for the determination of the Stirling numbers. From (4) it follows that

$$D@(t) = \Theta(t) \sum_{i=1}^{n+1} \frac{p_i}{(q_i+p_i t)} .$$

Determining the s-l -th derivative of this quantity by *Leibnifr's* theorem, and putting $t=1$ into the result obtained, we get:

$$D^s\,\Theta(1) = \sum_{\nu=0}^{s} \binom{s-1}{\nu} D^{s-1-\nu}\,\Theta(1) \sum_{i=1}^{n+1} (-1)^{\nu}\, \nu!\, p_i^{\nu+1}$$

and in the particular cases:

$$\mathrm{D}\ \Theta(1) = \Sigma p_i$$
$$\mathrm{D}^2\ \Theta(1) = (\Sigma p_i)^2 - \Sigma p_i^2$$
$$\mathrm{D}^3\ \Theta(1) = (\Sigma p_i)^3 - 3\ \Sigma p_i\ \Sigma p_i^2 + 2\ \Sigma p_i^3$$

and so on.

For instance, if $n=6$ and $s=3$, we have

$$S_7^4 = \frac{343}{6}\left[27 - 9 \cdot \frac{91}{49} + 2\,\frac{441}{343}\right] = 735.$$

§ 58. *Stirling's* **numbers** *of* **the second kind.**[25] *Expansion of a power into a factorial series.* **Let us first expand x^n into a** *Newton* **series:**

$$x^n = \sum_{m=1}^{n+1} (x)_m \left[\frac{\Delta^m x^n}{m!}\right]_{x=0}.$$

We will call the number in the brackets, a *Stirling* number of the second kind, and denote it by $\mathfrak{S}_n^m$.

(1) $$\mathfrak{S}_n^m = \left[\frac{\Delta^m x^n}{m!}\right]_{x=0}.$$

Hence we have

(2) $$x^n = \mathfrak{S}_n^1 x + \mathfrak{S}_n^2 (x)_2 + \ldots + \mathfrak{S}_n^m (x)_m + \ldots + \mathfrak{S}_n^n (x)_n.$$

Starting from the definition (1) we conclude immediately that $\mathfrak{S}_n^0 = 0$ and $\mathfrak{S}_n^{n+m} = 0$ if $m>0$. Moreover, putting into equation (2) $x=1$ we get $\mathfrak{S}_n^1 = 1$.

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[25] The first table of these numbers has been published in:

Jacobo Stirling, Methodus Differentialis., . , Londini, 1730, p. 8, up to $n=9$; but the author did not use any notation for them.

There is a table in:

George *Boole,* Calculus of Finite Differences, London, 1860, p. 20, of the "differences of zero", which *Boole* denotes by $\Delta^n 0^m$; this corresponds to

$$\Delta^n 0^m = [\Delta^n x^m]_{x=0} = m!\ \mathfrak{S}_n^m$$

the range of the table is up to $n=10$.

A smaller table of the *Stirling* numbers of the second kind is given by *Oskar Schlömilch,* Compendium der Höheren Analysis, Braunschweig, 1895, p. 31; his notation

$$C_m^{-n} \text{ corresponds to } our\ \mathfrak{S}_{n+m}^{n-m}.$$

Niels Nielsen, Gammafunctionen, Leipzig, 1906, p. 68, calls these numbers, for the first time *Stirling* numbers of the second kind; his

$$\mathfrak{C}_n^m \text{ corresponds to our } (-1)^{n+m}\ \mathfrak{S}_{n+m+1}^{n-1}.$$

In *C. Jordan,* loc. cit. 24, p. 263, there is a table of $\mathfrak{S}_n^m$ up to $n=12$.

From the definition we may deduce a general expression for the *Stirling* numbers of the second kind. Indeed, dealing with symbolical methods in § 6 we found that the operation of the m -th difference is equivalent to the following:

$$\Delta^m = (-1)^m \sum_{i=1}^{m+1} (-1)^i \binom{m}{i} \mathrm{E}^i.$$

If $f(x) = x^n/m!$ the operation gives for $x=0$

$$(3) \qquad \left[\Delta^m \frac{x^n}{m!}\right]_{x=0} = \mathfrak{S}_n^m = \frac{(-1)^m}{m!} \sum_{i=0}^{m+1} (-1)^i \binom{m}{i} i^n .$$

If a few of the *Stirling* numbers are wanted, this formula is a very convenient one to determine them; but if we want to compute a table of these numbers, then there is a better way.

The formula giving the higher differences of a product is the following (§ 30):

$$\Delta^m(uv) = v(x+m)\,\Delta^m u + \binom{m}{1}\Delta v(x+m-1)\,\Delta^{m-1}u + \ldots u\,\Delta^m v(x).$$

Putting $v=x$ and $u=x^n$, gives

$$\Delta^m x^{n+1} = (x+m)\,\Delta^m x^n + m\Delta^{m-1} x^n;$$

dividing both members of this equation by $m!$ and putting $x=0$ we obtain according to (1)

$$(4) \qquad \mathfrak{S}_{n+1}^m = \mathfrak{S}_n^{m-1} + m\,\mathfrak{S}_n^m.$$

We may also obtain this equation starting from $x^{n+1} = x \,.\, x^n$ and expanding x^{n+1} and x^n into series of factorials, then writing $x \,.\, (x)_i = (x)_{i+1} + i \,.\, (x)_i$ and finally equating the coefficients of $(x)_i$ in both members.

We shall see in § 181, dealing with partial difference equations , that the solution of equation (4) leads to formula (3). But if we want a table of the numbers $\mathfrak{S}_n^m$ there is no need to solve this equation. Starting from the initial conditions

$$\mathfrak{S}_0^0 = 1 \text{ and } \mathfrak{S}_0^m = 0 \text{ if } m \neq 0$$

which follow directly from the definition (1), we may obtain these numbers step by step, by aid of the equation (4).

From this equation it follows that the *Stirling* numbers of the second kind are *positive integers.*

Stirling numbers of the second kind $\mathfrak{S}_m^n$

$n \backslash m$	1	2	3	4	5	
	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1
7	1	63	301	350	140	21
8	1	127	966	1701	1050	266
9	1	255	3025	7770	6951	2646
10	1	511	9330	34105	42525	22827
11	1	1023	28501	145750	246730	179487
12	1	2047	86526	611501	1379400	1323652

$n \backslash m$	7	8	9	10	11	12
1						
2						
3						
4						
5						
6						
7	1					
8	28	1				
9	462	36	1			
10	5880	750	45	1		
11	63987	11880	1155	55	1	
12	627396	159027	22275	1705	66	1

If we put $x=-1$ into formula (2) we get

$$(5)\qquad (-1)^n = \sum_{m=1}^{n+1} (-1)^m \, m! \, \mathfrak{S}_n^m .$$

This equation may be used for checking the numbers in the table.

In some particular cases the resolution of the difference equation (4) is simple. The results obtained may shorten the computation of the table. For instance, putting into (4) $m=1$ we have

$$\mathfrak{S}_{n+1}^1 = \mathfrak{S}_n^1 = \mathfrak{S}_1^1 = 1.$$

Putting $m=n+1$ it follows that

$$\mathfrak{S}_{n+1}^{n+1} = \mathfrak{S}_n^n = \mathfrak{S}_1^1 = 1.$$

Determination of $\mathfrak{S}_n^{n-m}$ by' aid of equation **(4).** Putting $m=n$ *we* get

$$\mathfrak{S}_{n+1}^{n} - \mathfrak{S}_n^{n-1} = \Delta\, \mathfrak{S}_n^{n-1} = n.$$

the operation Δ^{-1} gives

$$\mathfrak{S}_n^{n-1} = \binom{n}{2} + k.$$

Since $\mathfrak{S}_2^1 = 1$, $k=0$.

Putting into equation (4) $m=n-\nu+1$ we obtain in the same manner

$$\text{(6)} \qquad \underset{n}{\Delta}\, \mathfrak{S}_n^{n-\nu} = (n-\nu+1)\, \mathfrak{S}_n^{n-\nu+1}.$$

Hence to have $\mathfrak{S}_n^{n-2}$ we multiply $\mathfrak{S}_n^{n-1} = \binom{n}{2}$ by **n-1** and perform the operation Δ^{-1}; consequently $\mathfrak{S}_n^{n-2}$ will be of the fourth degree in *n,* $\mathfrak{S}_n^{n-3}$ of the sixth, and so on. $\mathfrak{S}_n^{n-m}$ will be a polynomial of *n* of degree *2m.* Let us write it as follows:

$$\text{(7)} \qquad \mathfrak{S}_n^{n-m} = \bar{C}_{m,0}\binom{n}{2m} + \bar{C}_{m,1}\binom{n}{2m-1} + \\ + \dots + \bar{C}_{m,m-1}\binom{n}{m+1} + \dots + \bar{C}_{m,2m}.$$

Multiplying both members by *(n-m),* the operation Δ^{-1} gives

$$\mathfrak{S}_n^{n-m-1} = \sum_{s=0}^{2m+3} [(m-s+1)\bar{C}_{m,s-1} + (2m-s+1)\bar{C}_{m,s}]\binom{n}{2m+2-s} + k$$

but in consequence of the definition **(7)** we have also

$$\mathfrak{S}_n^{n-m-1} = \sum_{s=0}^{2m+3} \bar{C}_{m+1,s}\binom{n}{2m+2-s};$$

therefore the 'numbers $\bar{C}_{m,i}$ satisfy the following partial difference equation

$$\text{(8)} \qquad \bar{C}_{m+1,s} = (m-s+1)\bar{C}_{m,s-1} + (2m-s+1)\bar{C}_{m,s}$$

$$\text{and } \bar{C}_{m+1,2m+2} = k.$$

From $\mathfrak{S}_n^{n-1} = \binom{n}{2}$ it follows that $\bar{C}_{1,0} = 1$ and $\bar{C}_{1,s} = 0$ if

$s \neq 0$. Starting from this, we obtain by aid of (8), as in § 52, step by step, that $C_{m,s} = 0$ if s $> m-1$; hence we have also $k=0$.

Therefore, putting $s=m$ into (8) we get:

$$\bar{C}_{m+1,m} = \bar{C}_{m,m-1} = \bar{C}_{1,0} = 1.$$

If $s=0$, from (8) it follows that

$$\bar{C}_{m+1,0} = (2m+1)\bar{C}_{m,0}.$$

The solution of this difference equation is

$$\bar{C}_{m,0} = k \prod_{i=0}^{m} (2i+1).$$

$\bar{C}_{1,0} = 1$; therefore $k=1$, and $\bar{C}_{m,0} = 1.3.5 \ldots\ldots (2m-1)$.

Starting from $\bar{C}_{1,0} = 1$ we may compute by aid of (8) a table of the numbers $\bar{C}_{m,i}$.

Table of the numbers $\bar{C}_{m,i}$.

$m \backslash i$	0	1	2	3	4	5
1	1					
2	3	1				
3	15	10	1			
4	105	105	25	1		
5	945	1260	490	56	1	
6	10395	17325	9450	1918	119	1

From (8) let us deduce an equation which may serve for checking the numbers of the table. Multiplying both members of the equation by $(-1)^s$ and summing from $s=0$ to $s=m+1$ we get

$$\sum_{s=0}^{m+1} (-1)^s \bar{C}_{m+1,s} = \sum_{s=0}^{m} [(-1)^s (m+1-s) \bar{C}_{m,s-1} +$$
$$+ (-1)^s (2m+1-s) C_{m,s}].$$

Putting into the first part of the sum in the second member s instead of s-l and simplifying we obtain

$$\sum_{s=0}^{m+1} (-1)^s \bar{C}_{m+1,s} = (m+1) \sum_{s=0}^{m} (-1)^s \bar{C}_{m,s}.$$

Denoting the sum in the second member by $f(m)$, we have

$$f(m+1) = (m+1)f(m).$$

The solution of this difference equation is $f(m) = m!\,k$. Since $f(1)=C_{1.0}=1$ $k=1$ and finally we have

$$(9) \qquad \sum_{s=0}^{m} (-1)^s \overline{C}_{m,s} = m!$$

This relation may be used for checking the table.

From (7) we conclude that

$$(10) \qquad \underset{n}{\Delta}^{2m} \mathfrak{S}_n^{n-m} = \overline{C}_{m.0}\,.$$

Remark Formula (7) is advantageous for the determination of $\mathfrak{S}_n^m$ if n is large and m small, For instance if $n= 12$ and $m =3$, we have

$$\mathfrak{S}_{12}^9 = 15\binom{12}{6} + 10\binom{12}{5} + \binom{12}{4} = 22275.$$

The computation can be made very short by aid of a table of binomial coefficients.

§ 59. Liits of expressions containing Stirling numbers of the second kind. From equation (3) § 58 it follows that

$$(1) \qquad \frac{\mathfrak{S}_n^m}{m^n} = \frac{(-1)^m}{m!} \sum_{i=1}^{m+1} (-1)^i \binom{m}{i} \left(\frac{i}{m}\right)^n.$$

If n increases indefinitely, every term of the second member will vanish, except that in which $i=m$; therefore we have

$$(2) \qquad \lim_{n=\infty} \frac{\mathfrak{S}_n^m}{m^n} = \frac{1}{m!}\,.$$

This gives an asymptotic value for large n

$$\mathfrak{S}_n^m \sim \frac{m^n}{m!}$$

The values thus obtained are acceptable if m is small. For $m=1$ the value is exact; for $m=2$ it gives 2^{n-1} instead of $2^{n-1}-1$. If $m>2$ then the error increases indefinitely with n.

From (1) we may deduce

$$(3) \qquad \frac{\mathfrak{S}_n^{m+1}}{\mathfrak{S}_n^m} = \left(\frac{m+1}{m}\right)^n \frac{1}{m+1} \left(\frac{1-(m+1)\left(\frac{m}{m+1}\right)^n + \dots}{1-m\left(\frac{m-1}{m}\right)^n + \dots} \right).$$

Neglecting the terms which are not written in this formula we may determine the greatest term of the row n in the table, $\mathfrak{S}_n^\chi$ will be the greatest term if for $m=\chi$ the ratio (3) is smaller than one, and for $m=\chi-1$ greater. According to (3) the conditions for a maximum are

$$(\chi+1)^{n-1} + \chi(\chi-1)^n < 2\chi^n$$

$$\chi^{n-1} + (\chi-1)(\chi-2)^n > 2(\chi-1)^n.$$

By aid of these inequalities we find, for instance, the following greatest terms: $\mathfrak{S}_3^2$, $\mathfrak{S}_5^3$, $\mathfrak{S}_{12}^5$. Our table shows that this is exact.

From formula (3) we may deduce the following limits:

$$(4) \qquad \lim_{n=\infty} \frac{\mathfrak{S}_n^{m+1}}{\mathfrak{S}_n^m \left(\frac{m+1}{m}\right)^n} = \frac{1}{m+1}$$

and from (2)

$$(5) \qquad \lim_{n=\infty} \frac{\mathfrak{S}_{n+1}^m}{\mathfrak{S}_n^m} = m.$$

If m is small in comparison with n, then the asymptotic formula

$$\mathfrak{S}_{n+1}^m \sim m\mathfrak{S}_n^m$$

gives a relatively good approximation. For instance from $\mathfrak{S}_{11}^3$ we should get $\mathfrak{S}_{12}^3 \sim 85503$ instead of 86526.

Starting from the polynomial expression of $\mathfrak{S}_n^{n-m}$, formula (7) § 58, we may deduce:

$$(6) \qquad \lim_{n=\infty} \frac{\mathfrak{S}_n^{n-m}}{n^{2m}} = \frac{C_{m,0}}{(2m)!} = \frac{1.3.5\ldots(2m-1)}{(2m)!} = \frac{1}{m!\,2^m}.$$

We have seen, § 52, formula (8), that the limit of the corresponding *Stirling* number of the first kind is the same.

§ 60. Generating function of the Stirling numbers of the second kind, with respect to the lower index, Starting from the difference equation of these numbers (4, § 58)

$$(1) \qquad \mathfrak{S}_{n+1}^{m+1} - \mathfrak{S}_n^m - (m+1)\,\mathfrak{S}_n^{m+1} = 0$$

we denote the generating function of $\mathfrak{S}_n^m$ with respect to n by $u(m,t)$ and determine it by the method of § 11. We have

$$\mathbf{G}\,\mathfrak{S}_n^m = u(m,t) = \sum_{n=m}^{\infty} \mathfrak{S}_n^m t^n.$$

Obviously it follows that

$$\mathbf{G}\,\mathfrak{S}_n^{m+1} = u(m+1,\ t).$$

Since $\mathfrak{S}_0^{m+1} = 0$,

$$\mathbf{G}\,\mathfrak{S}_{n+1}^{m+1} = \frac{u(m+1,t)}{t}\,.$$

Finally writing that the corresponding generating functions satisfy equation (l), we obtain

$$(1-t-mt)\,u(m+1,t) - t\,u(m,t) = 0.$$

This is a homogeneous linear equation of the first order, with variable coefficients; its solution is, as we shall see later,

$$u(m,\ t) = \omega(t) \prod_{i=0}^{m} \frac{t}{(1-t-it)}$$

where $\omega(t)$ is an arbitrary function of t, which may be determined by the initial conditions. Since $\mathfrak{S}_n^1 = 1$ and therefore $u(1,\ t) = t\ /$ (l-f), we conclude that $\omega(t) = 1$; so that the generating function with respect to n of the numbers $\mathfrak{S}_n^m$ will be

(2) $\quad u = \dfrac{t^m}{\text{(l-f)}\ (1-2t)\ (1-3t)\ ,\ ,\ ,\ \text{(1-mf)}} = \displaystyle\sum_{n=m}^{\infty} \mathfrak{S}_n^m t^n.$

Remark. The expansion of this function in a series of powers of t gives the solution of the difference equation (1). This is ***Laplace's*** method for solving partial difference equations.

To expand the function u let us put $t = 1/z$; then we may write

$$u = z \,/\, z(z-1)\ (z-2)\ \ldots\ (z-m).$$

This decomposed into partial fractions (§ 13, Example 2) gives

$$u = \frac{(-1)^m}{m!}\, z \sum_{i=0}^{m+1} \frac{(-1)^i \binom{m}{i}}{r-i}\,.$$

Putting again $z = 1/t$ we find

$$u = \frac{(-1)^m}{m!} \sum_{i=0}^{m+1} \frac{(-1)^i \binom{m}{i}}{1-it} = \frac{(-1)^m}{m!} \sum_{i=0}^{m+1} (-1)^i \binom{m}{i} \sum_{n=0}^{\infty} i^n t^n.$$

Therefore the coefficient of t^n is the following

$$\text{(3)} \qquad \mathfrak{S}_n^m = (-1)^m \frac{1}{m!} \sum_{i=0}^{m+1} (-1)^i \binom{m}{i} i^n .$$

This formula has already been obtained in § 58.

The generating function (2) may be written in the following form

$$\text{(4)}\ u = (1+t+t^2+\ldots)(1+2t+2^2t^2+\ldots)\ldots(1+mt+m^2t^2+\ldots)t^m.$$

From this we conclude that $\mathfrak{S}_n^m$, the coefficient of t^n in the expansion of this function is equal to the sum of the products of the combinations, with repetition, but without permutation, of order *n-m* of the numbers 1, 2, 3, . . . , *m*.

Example.

$$\mathfrak{S}_5^3 = 1.1 + 1.2 + 1.3 + 2.2 + 2.3 + 3.3 = 25$$

$$\mathfrak{S}_6^2 = 1.1.1.1 + 1.1.1.2 + 1.1.2.2 + 1.2.2.2 + 2.2.2.2 = 31.$$

We may deduce another rule to obtain these numbers.

$$e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots;$$

therefore using *Cauchy's* rule of multiplication of series we get

$$(e^t - 1)^m = \sum_{n=m}^{\infty} t^n \sum \frac{1}{r_1!\, r_2! \ldots r_m!}$$

where the second sum is extended to every value of $r_i > 0$ (with repetition and permutation) such that $r_1 + r_2 + \ldots + r_m = n$.

We shall see later that

$$(e^t - 1)^m = \sum_{n=m}^{\infty} \frac{m!}{n!} \mathfrak{S}_n^m t^n;$$

therefore we conclude that

$$\text{(5)} \qquad \mathfrak{S}_n^m = \frac{n!}{m!} \sum \frac{1}{r_1!\, r_2! \ldots r_m!}$$

where the sum is formed as above.

Examples.

$$\mathfrak{S}_5^3 = 5.4\left[\frac{1}{1!1!3!} + 3\,\frac{1}{1!2!2!}\right] = 25$$

$$\mathfrak{S}_6^2 = 6.5.4.3\left[\frac{2}{1!5!} + \frac{2}{2!4!} + \frac{1}{3!3!}\right] = 31.$$

§ 61. **The Stirling numbers of the second kind obtained by probability.**[35a] Let an urn be given, which contains the numbers $1, 2, 3, \ldots, m$. Let us draw successively n numbers, putting back every time the number drawn, before the next drawing. The probability that in n drawings the number one should occur r_1 times, the number two r_2 times, and so on, finally the number m should occur r_m times, is, according to the generalised *Bernoulli* theorem of repeated trials, equal to

$$\text{(1)} \qquad \frac{n!}{r_1!\,r_2!\ldots r_m!}\left(\frac{1}{m}\right)^n$$

Let us now determine the probability that in n drawings every number out of $1, 2, 3, \ldots, m$, should be drawn at least once. This is given by the theorem of total probabilities

$$\text{(2)} \qquad P = \left(\frac{1}{m}\right)^n \sum \frac{n!}{r_1!\,r_2!\ldots r_m!}$$

where the above sum is extended to every value of $r_i > 0$ with repetition (and permutation), but satisfying the condition

$$r_1 + r_2 + r_3 + \ldots + r_m = n.$$

According to formula (5) § 60 the sum above is equal to $m!\,\mathfrak{S}_n^m$, so that the required probability is

$$\text{(3)} \qquad P = \frac{m!}{m^n}\,\mathfrak{S}_n^m\,.$$

On the other hand this probability can be determined by using the generalised *Poincaré* theorem, which gives the probability that in n trials every one of the m events should occur at least once. According to the theorem, this probability is ob-

[35a] Ch. Jordan, Théorème de la probabilité de Poincaré généralisé. Acta Scientiarum Mathematicarum, Tome VII, p. 103. Szeged, 1934,

tained by subtracting from unity the probability that *one* of the events should not occur; this is

$$\binom{m}{1}\left(\frac{m-1}{m}\right)^n$$

then adding the probability that *two* of the events should not occur

$$\binom{m}{2}\left(\frac{m-2}{m}\right)^n$$

then again subtracting the probability that *three* of the events should not occur

$$\binom{m}{3}\left(\frac{m-3}{m}\right)^n$$

and so on. Finally we have

$$P = \frac{1}{m^n}\sum_{s=0}^{m} (-1)^s \binom{m}{s} (m-s)^n.$$

Equating this value to that obtained above (3), and putting $m-s=i$, we get the expression for the *Stirling* number obtained before (§§ 58 and 60):

$$\mathfrak{S}_n^m = \frac{1}{m!}\sum_{i=1}^{m+1} (-1)^{m-i}\binom{m}{i} i^n$$

Remarks. 1. From formula (1) we may deduce in the same manner the probability that in n trials, out of the m numbers there should be drawn any μ numbers, no more and no less; this is

$$P_\mu = \Sigma \frac{n!\binom{m}{\mu}}{r_1!\,r_2!\ldots r_\mu!}\left(\frac{1}{m}\right)^n$$

where the sum is extended to every value of $r_i > 0$ (with repetition and permutation) such as $r_1 + r_2 + \ldots + r_\mu = n$. Then according to (5) § 60 we shall have

$$\Sigma \frac{1}{r_1!\,r_2!\ldots r_\mu!} = \frac{\mu!}{n!}\mathfrak{S}_n^\mu$$

and finally the required probability will be

(4) $$P_\mu = \frac{(m)_\mu}{m^n}\mathfrak{S}_n^\mu.$$

2. From formula (3) it follows that the number of such combinations with repetition, of order n, of m elements in which each occurs at least once, is equal to $m!\mathfrak{S}_n^m$.

3. Moreover from (3) we may obtain the probability that in n trials every one of the m numbers occurs, but the last of them only at the n -th trial. This is:

$$P_n - P_{n-1} = \frac{m!}{m^n}\left[\mathfrak{S}_n^m - m\,\mathfrak{S}_{n-1}^{m-1}\right] - \frac{m!}{m^n}\,\mathfrak{S}_{n-1}^{m-1}.$$

§ 62. Decomposition of products of prime numbers into factors [see also *Netto,* Combinatorik p. 168]. Given

$$\omega_n = a_1 a_2 a_3 \ldots, . a_n$$

where every a_i is a different prime number, Let us denote by $f(n, \nu)$ the number of ways in which ω_n may be decomposed into ν factors (without permutation). For instance, if $\omega_3 = a_1 a_2 a_3$ we shall have

$f(3,1) = 1$; $(a_1 a_2 a_3)$

$f(3,2) = 3$; $(a_1 a_2) a_3$, $(a_2 a_3) a_1$, $(a_3 a_1) a_2$

$f(3,3) = 1$; $(a_1)(a_2)(a_3)$

From these we may easily deduce the number of ways in which $\omega_4 = a_1 a_2 a_3 a_4$ may be decomposed into factors. For instance, the decompositions of ω_4 into three factors will be obtained:

First by adding a, to each decomposition $f(3, 2)$:

$(a_1 a_2)(a_3)(a_4)$, $(a_2 a_3)(a_1)(a_4)$, $(a_3 a_1)(a_2)(a_4)$.

Secondly by multiplying successively each factor of the decompositions $f(3, 3)$ by a,

$(a_1 a_4)(a_2)(a_3)$, $(a_1)(a_2 a_4)(a_3)$, $(a_1)(a_2)(a_3 a_4)$.

Thus we have obtained every decomposition of ω_4 into three factors, and each only once; therefore we have

$$f(4,3) = f(3,2) + 3f(3,3).$$

Proceeding exactly in the same manner we should obtain $f(n, \nu)$ starting from $f(n\text{-}l, \nu - 1)$ and $f(n\text{-}l, \nu)$; we will obtain them first by adding the factor a, to the decompositions *f* (n-1, r-l), secondly, by multiplying successively every factor of the decompositions *f* (n-l, ν) by a_n ; therefore each of these decompositions will give ν new ones, so that we shall have

$$f(n,\nu) = \mathrm{f(n\text{-}1,\ V\text{-}l)} + \nu f(n-1,\nu).$$

But this is the equation of differences which the *Stirling* numbers of the second kind satisfy; moreover the initial conditions are the same, indeed $f(n,1) = 1$ if $n > 0$ and $f(n,1) = 0$ if n 4 0. Therefore we conclude that

$$f(n,\nu) = \mathfrak{S}_n^\nu. \tag{1}$$

That is: a product of *n* different prime numbers may be decomposed into v factors in $\mathfrak{S}_n^\nu$ different ways (no permutation).

Starting from the number of the decompositions of

$$\omega_{n-2} = a_1 a_2 \ldots . a_{n-2}$$

into v factors, denoted by *f(n-2,v)* we may deduce the number of decompositions of

$$\omega_n = a^2 a_1 a_2 \ldots . a_{n-2}$$

into v factors, which we will denote by *F(n,v).* Let us remark that into ω_n the prime number a enters twice.

The decompositions of ω_n into v factors are obtained:

First, from those of ω_{n-2} into *v-2* factors by adding to each of them the two factors a. a ; the number of decompositions obtained in this way will be equal to *f (n-2,* $\nu - 2)$.

Secondly, from those of ω_{n-2} into v-l factors by adding the factor a and then multiplying each of the ν factors by a; the number obtained in this manner will be equal to *v . f (n-2,* $\nu - 1)$.

Thirdly, from those of ω_{n-2} into v factors by multiplying two of these factors by a; the number of ways in which this may be done is equal to the number of combinations with repetition but without permutation of v numbers taken two by two, that is, $\binom{\nu+1}{2}$. Hence the number of decompositions thus obtained will be equal to $\binom{\nu+1}{2} f(n-2,\nu)$.

Finally **we** have

$$F(n,\nu) = f\ (n\text{-}2,\ v\text{-}2) + v\ f\ (n\text{-}2,\ \nu-1) + \binom{\nu+1}{2} f(n-2,\nu)$$

and in consequence of (1)

$$F(n,\nu) = \mathfrak{S}_{n-2}^{\nu-2} + \nu\,\mathfrak{S}_{n-2}^{\nu-1} + \binom{\nu+1}{2}\,\mathfrak{S}_{n-2}^{\nu}.$$

Example. Let $\omega_n = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$ and $\nu=3$. Since $n=5$, we find

$$F(5,3) = \mathfrak{S}_3^1 + 3\,\mathfrak{S}_3^2 + 6\,\mathfrak{S}_3^3 = 16.$$

This result may easily be verified.

§ 63. Application of the expansion of powers into a series of factorials, The expansion of x^n into a series of generalised factorials is the following:

$$(1) \qquad x^n = \sum_{\nu=1}^{n+1} h^{n-\nu}\,\mathfrak{S}_n^{\nu}(x)_{\nu,h}.$$

1. From this formula we may obtain $\underset{h}{\Delta}^m x^n$ immediately

$\underset{h}{\Delta}^m x^n = \sum_{\nu=1}^{n+1} h^{n-\nu+m} (\nu)_m$ *6; (x)* $_{\nu-m,h}$.

Putting $x=0$ we have

$$[\underset{h}{\Delta}^m x^n]_{x=0} = h^n\, m!\,\mathfrak{S}_n^m.$$

2. Determination *of the* sum *of* x^n. From (1) we obtain

$$(2) \qquad \underset{h}{\Delta}^{-1} x^n = \sum_{\nu=1}^{n+1} \frac{h^{n-\nu-1}}{\nu+1}\,\mathfrak{S}_n^{\nu}(x)_{\nu+1,h} + k.$$

We shall see later, that if $h=1$ the sum may be expressed by the *Bernoulli* polynomial $\varphi_{n+1}(x)$ of degree $n+1$, as follows

$$\Delta^{-1} x^n = n!\,\varphi_{n+1}(x) + k.$$

This polynomial written by aid of *the Bernoulli* numbers B_i is

$$(3) \qquad \varphi_{n+1}(x) = \frac{1}{(n+1)!} \sum_{i=0}^{n+2} B_i \binom{n+1}{i} x^{n+1-i}.$$

Expanding the factorials figuring in the second member of (2), into a series of powers of x we find, if $h=1$,

$$(4) \qquad \Delta^{-1} x^n = \sum_{\nu=1}^{n+1} \frac{\mathfrak{S}_n^{\nu}}{\nu+1} \sum_{\mu=1}^{\nu+2} S_{\nu+1}^{\mu}\, x^{\mu}.$$

Equating the coefficients of x^1 in the expressions (3) and (4) **we get**

$$(5) \qquad B_n = \sum_{\nu=1}^{n+1} \frac{(-1)^\nu \nu!}{\nu+1} \mathfrak{S}_n^\nu .$$

This is an expression for the ***Bernoulli*** numbers by ***Stirling*** numbers of the second kind.

3. Expression of the power moments $\mathcal{M}_n$ by factorial moments $\mathfrak{M}_m$ and by binomial moments $\mathcal{B}_m$. In § 55 we denoted by

$$\mathcal{M}_n = \sum_{x=0}^{N} x^n f(x)$$

$$\mathfrak{M}_n = \sum_{x=0}^{N} x(x-1) \ldots (x-n+1) f(x)$$

$$\mathcal{B}_n = \sum_{x=n}^{N} \binom{x}{n} f(x).$$

From (1) it follows that

$$(6) \qquad \mathcal{M}_n = \sum_{m=1}^{n+1} \mathfrak{S}_n^m \mathfrak{M}_m$$

and

$$(7) \qquad \mathcal{M}_n = \sum_{m=1}^{n+1} m! \, \mathfrak{S}_n^m \mathcal{B}_m .$$

§ 64. Some formulae containing Stirling numbers of both kinds. Let us expand $(x)_n$ into a power series, and then expand the powers into a factorial series again. We get

$$(x)_n = \sum_{i=1}^{n+1} S_n^i x^i = \sum_{i=1}^{n+1} \sum_{m=1}^{i+1} S_n^i \mathfrak{S}_i^m (x)_m .$$

Since the coefficients of $(x)_\mu$ in the first and in the last **member** must be identical, we conclude that

$$(1) \qquad \sum_{i=m}^{n+1} S_n^i \mathfrak{S}_i^m = \binom{0}{n-m}$$

that is, this sum is equal to zero if n is different from m and equal to one if $n=m$; the binomial coefficient $\binom{0}{m}$ being equal to zero for every positive or negative integer value of m according to formula (4) of § 22. Moreover $\binom{0}{0} = 1$.

In the same manner expanding x^n into a factorial series, and then expanding the factorials into a power series again, we find

$$x^n = \sum_{i=1}^{n+1} \mathfrak{S}_n^i \, (x)_i = \sum_{i=1}^{n+1} \sum_{m=1}^{i+1} \mathfrak{S}_n^i \, S_i^m x^m.$$

Since the coefficients of x^μ in the first and in the last member must be the same, we have

$$(2) \qquad \sum_{i=m}^{n+1} \mathfrak{S}_n^i \, S_i^m = \left[{0 \atop n-m} \right].$$

This is also equal to zero if n is different from m and equal to one if $n=m$.

The limits of the sums (1) and (2) can be made independent of n and m, since for $i<m$ or $i>n$ the expressions under the sign Σ are equal to zero. We may sum from $i=0$ to $i=\infty$; this is often very useful. We shall see in the following paragraph that the above formulae may serve for the inversion of series.

§ *65.* Inversion of sums and of series. **Sum** equations. In certain cases it is possible to perform this inversion, by which the following is understood. Given

$$(1) \qquad f(x) = \sum_{i=a}^{\beta} \psi(i) \, \varphi(x,i)$$

where f(x) and $\varphi(x,i)$ are known and $\psi(n)$ is to be determined: Let us suppose that equation (1) holds for every integer value of x in the interval $\delta \geqq x \geqq y$.

If it is possible to find a function $\omega(x,n)$ such that

$$(2) \qquad \sum_{x=\gamma}^{\delta} \varphi(x,i) \, \omega(x,n) = \left[{0 \atop n-i} \right]$$

that, is if the first member is equal to zero for every value of i different from n, and equal to one for $i=n$ (supposing $\beta \geqq n \geqq a$), then multiplying both members of (1) by $\omega(x,n)$ and summing from $x=\gamma$ to $x=\delta$ we obtain

$$\sum_{x=\gamma}^{\delta} f(x) \;\; \omega(x,n) = y(n).$$

Indeed in the second member of (1) each term will vanish except that in which $i=n$.

The only difficulty is to find the function $\omega(x,n)$ corresponding to $\varphi(x,i)$. We have deduced already a few formulae of the form (2). For instance we had in § 45

$$\sum_{x=m}^{n+1} (-1)^{n+x}\binom{n}{x}\binom{x}{m} = \binom{0}{n-m} \tag{3}$$

$$\sum_{x=0}^{n-m+1} (-1)^{x}\binom{n}{x}\binom{n-x}{m} = \binom{0}{n-m} \tag{4}$$

and in § 64

$$\sum_{x=m}^{n+1} S_n^x \mathfrak{S}_x^m = \binom{0}{n-m} \tag{5}$$

$$\sum_{x=m}^{n+1} S_x^m \mathfrak{S}_n^x = \binom{0}{n-m}. \tag{6}$$

In these formulae, the limits may he made independent of n and m since for $m>x\geqq 0$, the first members of (3), (5) and (6) are equal to zero and also for $x>n$. Therefore we may sum from $x=0$ to $x=\infty$.

Example 1. Given the ***Newton*** series

$$f(x) = f(0) + \binom{x}{1}\Delta f(0) + \binom{x}{2}\Delta^2 f(0) + \ldots + \binom{x}{m}\Delta^m f(0) + \ldots$$

multiplying both members by $(-1)^{x+n}\binom{n}{x}$ and summing from $x=0$ to $x=n+1$ we get

$$\sum_{x=0}^{n+1} (-1)^{n+x}\binom{n}{x} f(x) = \Delta^n f(0).$$

This formula has already been obtained in § 6.

Example 2. We found, formula (3) § 52:

$$S_n^{n-m} = \sum_{\nu=0}^{m} C_{m,\nu}\binom{n}{2m-\nu}$$

multiplying both members by $(-1)^{n+i}\binom{2m-i}{n}$ and summing from $n=m+1$ to $n=2m-i+1$ we get, according to formula (3),

$$\sum_{n=m+1}^{2m-i+1} (-1)^{n+i}\binom{2m-i}{n} S_n^{n-m} = C_{m,i}$$

Example 3. We had, formula (7) § 58:

$$\mathfrak{S}_n^{n-m} = \sum_{\nu=0}^{m} \bar{C}_{m,\nu} \binom{n}{2m-\nu};$$

multiplying it by $(-1)^{n+i}\binom{2m-i}{n}$ and proceeding as in Example 2, we obtain

$$\sum_{n=m+1}^{2m-i+1} (-1)^{n+i}\binom{2m-i}{n} \mathfrak{S}_n^{n-m} = \bar{C}_{m,i}.$$

Example 4. In § 50 we found

$$(x)_m = S_m^1 x + S_m^2 x^2 + S_m^3 x^3 + ,\ ,\ , + S_m^i x^i + \ldots + S_m^m x^m$$

multiplying this equation by $\mathfrak{S}_n^m$ and summing from $m=1$ to $m=n+1$ we have, according to (6),

$$\sum_{m=1}^{n+1} (x)_m \mathfrak{S}_n^m = x^n.$$

This formula has already been obtained in § 58.

§ 66, Deduction of certain formulae containing Stirling numbers. 1. Let us write

$$(x+1)_n = (x+1)\,(x)_{n-1}.$$

Expanding both members into power series we obtain

$$\sum_{\nu=1}^{n+1} S_n^\nu \; (x+1)^\nu = (x+1) \sum_{\mu=1}^{n} S_{n-1}^\mu \; x^\mu .$$

Equating the coefficients of x^m in both members of this equation, it results that

$$(1) \qquad \sum_{\nu=m}^{n+1} \binom{\nu}{m} S_n^\nu = S_{n-1}^m + S_{n-1}^{m-1}.$$

Znversion of the sum (1). Let us multiply both members of (1) by $(-1)^{m+k}\binom{m}{k}$ and sum from $m=k$ to $m=n+1$; we get

$$S_n^k = \sum_{m=k}^{n+1} (-1)^{m+k}\binom{m}{k} [S_{n-1}^m + S_{n-1}^{m-1}].$$

If we put $m+1$ instead of m in the second term of the second member, then the terms of the second member may be reunited, and we obtain

(2) $$S_n^k = \sum_{m=k-1}^{n} (-1)^{m+k+1} \binom{m}{k-1} S_{n-1}^m .$$

Repeating the operation, multiplying both members by $(-1)^{k+\nu-1} \binom{k-1}{\nu}$ and summing from $k=\nu+1$ to $k=n+1$ *we* have

(3) $$\sum_{k=\nu+1}^{n+1} (-1)^{k+\nu-1} \binom{k-1}{\nu} S_n^k = S_{n-1}^\nu .$$

Particular cases. Putting $m=1$ into (1) we have

(4) $$\sum_{\nu=1}^{n+1} \nu \, S_n^\nu = S_{n-1}^1 = (-1)^n (n-2)!$$

Putting $k=2$ into formula (2) we find

$$\sum_{m=1}^{n} (-1)^{m-1} \, m \, S_{n-1}^m = S_n^2 .$$

From (3) we get

$$\sum_{m=2}^{n+1} \binom{m}{2} S_{n+1}^{m+1} = S_n^2 .$$

2. Multiplying both members of formula (4) by $\mathfrak{S}_m^n$ and summing from $n=2$ to $n=m+1$ we obtain

(5) $$\sum_{n=2}^{m+1} (-1)^n (n-2)! \, \mathfrak{S}_m^n = \sum_{\nu=1}^{n+1} \sum_{n=2}^{m+1} \nu \, S_n^\nu \, \mathfrak{S}_m^n = m-1 .$$

Indeed, in consequence of formula (2) § 64, the second member summed from $n=1$ to $m+1$ would give *m;* but since the first member is to be summed from $n=2$ to $m+1$ and therefore the second member too, we must subtract from *m* in the second member its value corresponding to $n=1$, that is

$$S_1^1 \, \mathfrak{S}_m^1 = 1$$

so that it will become equal to $m-1$.

3. Formula (2) may be written

$$\sum_{m=\nu}^{n+1} (-1)^{m+\nu} \binom{m}{\nu} S_n^m = S_{n+1}^{\nu+1} .$$

Multiplying both members by $\mathfrak{S}_\mu^n$ and summing from $n=\nu$ to $n=\mu+1$ will give

$$(6) \qquad \sum_{n=\nu}^{\mu+1} S_{n+1}^{\nu+1} \mathfrak{S}_{\mu}^{n} = (-1)^{\mu+\nu} \binom{\mu}{\nu}.$$

Formula (6) multiplied by $\mathfrak{S}_{\nu+1}^{m}$ and summed from $\nu=m-1$ to $\nu=\mu+1$ gives

$$(7) \qquad \mathfrak{S}_{\mu}^{m-1} = \sum_{\nu=m-1}^{\mu+1} (-1)^{\nu+\mu} \binom{\mu}{\nu} \mathfrak{S}_{\nu+1}^{m},$$

4. Formula (3) may be written

$$\sum_{m=\nu}^{n+1} \binom{m}{\nu} S_{n+1}^{m+1} = S_n^{\nu};$$

multiplied by $\mathfrak{S}_{\mu+1}^{n+1}$ and summed from $n=\nu$ to $n=\mu+1$ this gives

$$(8) \qquad \sum_{n=\nu}^{\mu+1} S_n^{\nu} \mathfrak{S}_{\mu+1}^{n+1} = \binom{\mu}{\nu};$$

multiplied again by $\mathfrak{S}_{\nu}^{m}$ and summed from $\nu=m$ to $\nu=\mu+1$ we obtain

$$(9) \qquad \sum_{\nu=m}^{\mu+1} \binom{\mu}{\nu} \mathfrak{S}_{\nu}^{m} = \mathfrak{S}_{\mu+1}^{m+1}.$$

5. Starting from x(x), and expanding the factorial into a power series, and then expanding the powers again into a factorial series, we get

$$x(x)_n = \sum_{\nu=2}^{n+2} S_n^{\nu-1} x^{\nu} = \sum_{\nu=2}^{n+2} \sum_{i=1}^{\nu+1} S_n^{\nu-1} \mathfrak{S}_{\nu}^{i} (x)_i$$

but we have also

$$x(x)_n = (x)_{n+1} + n(x)_n$$

therefore we conclude, writing $\nu+1$ instead of v, that

$$(10) \qquad \sum_{\nu=1}^{n+1} S_n^{\nu} \mathfrak{S}_{\nu+1}^{i} = n \binom{0}{n-i} + \binom{0}{n+1-i},$$

That is, the first member is equal to zero if i is different from n and $n+1$; moreover, it is equal to one if $i=n+1$ and equal to n *if* $i=n$.

6. Starting from x. x^n and expanding x^n into factorials and then the factorials into a power series again, we find

$$x \cdot x^n = \sum_{\nu=1}^{n+1} x(x)_\nu \mathfrak{S}_n^\nu = \sum_{\nu=1}^{n+1} \mathfrak{S}_n^\nu \left[(x)_{\nu+1} + \nu (x)_\nu \right] =$$

$$= \sum_{\nu=1}^{n+1} \mathfrak{S}_n^\nu \sum_{i=1}^{\nu+2} (S_{\nu+1}^i + \nu S_\nu^i) x^i.$$

From this we conclude

(11) $$\sum_{\nu=1}^{n+1} S_{\nu+1}^i \mathfrak{S}_n^\nu + \sum_{\nu=1}^{n+1} \nu S_\nu^i \mathfrak{S}_n^\nu = \binom{0}{n+1-i}.$$

That is, the first member is equal to zero if i is different from $n+1$, and equal to one if $i=n+1$. In consequence of (6) it follows that

(12) $$\sum_{\nu=i}^{n+1} \nu S_\nu^i \mathfrak{S}_n^\nu = (-1)^{n+i} \binom{n}{i-1}$$

if n is different from i-l.

This formula multiplied by $\mathfrak{S}_i^m$ and summed from $i=m$ to $i=n+1$ gives

(13) $$\sum_{i=m}^{n+1} (-1)^{n+i} \binom{n}{i-1} \mathfrak{S}_i^m = m \mathfrak{S}_n^m$$

Equation (12) multiplied by S_m^n and summed from $n=i$ to $n=m+1$ gives

(14) $$\sum_{n=i}^{m+1} (-1)^{n+i} \binom{n}{i-1} S_m^n = m S_m^i .$$

7. Starting from the difference equation (4) § 58

$$\mathfrak{S}_{n+1}^m - \mathfrak{S}_n^{m-1} - m \mathfrak{S}_n^m = 0$$

and summing from $m=1$ to $m=n+2$ we obtain

$$\sum_{m=1}^{n+2} \mathfrak{S}_{n+1}^m = \sum_{m=1}^{n+2} \mathfrak{S}_n^{m-1} + \sum_{m=1}^{n+1} m \mathfrak{S}_n^m ;$$

this gives

(15) $$\sum_{m=1}^{n+2} \mathfrak{S}_{n+1}^m = \sum_{m=1}^{n+1} (1+m) \mathfrak{S}_n^m .$$

Multiplying the difference equation by $(-1)^m$ (m-l)! we get

$$(-1)^m (m-1)! \mathfrak{S}_{n+1}^m = (-1)^m m! \mathfrak{S}_n^m - (-1)^{m-1} (m-1)! \mathfrak{S}_n^{m-1}$$

that is

$$\underset{m}{\Delta}\,[(-1)^{m-1}(m-1)!\,\mathfrak{S}_n^{m-1}] = (-1)^m (m-1)!\,\mathfrak{S}_{n+1}^m$$

hence performing the operation $\underset{m}{\Delta}^{-1}$ we find

(16) $$\underset{m}{\Delta}^{-1}[(-1)^m (m-1)!\,\mathfrak{S}_{n+1}^m] = (-1)^{m-1}(m-1)!\,\mathfrak{S}_n^{m-1} + k.$$

From this we conclude that the sum, m varying from one to $n+2$ is equal to zero, since the quantity in the second member is at both limits equal to zero. That is

(17) $$\sum_{m=1}^{n+1} (-1)^m (m-1)!\,\mathfrak{S}_n^m = 0.$$

Moreover if m varies from $\mu+1$ to $n+2$ formula (16) gives

(18) $$\sum_{m=\mu+1}^{n+2} (-1)^m (m-1)!\,\mathfrak{S}_{n+1}^m = (-1)^{\mu+1}\mu!\,\mathfrak{S}_n^\mu.$$

This equation multiplied by S_ν^{n+1} and summed from $n=\mu$ to $n=\nu$ gives

(19) $$\sum_{n=\mu}^{\nu} S_\nu^{n+1}\,\mathfrak{S}_n^\mu = (-1)^{\nu+\mu+1}\frac{(\nu-1)!}{\mu!}.$$

§ 67. Differences expressed by derivatives, Given $f(a)$, $Df(a), D^2f(a), \ldots$ and so on; it is required to determine $\Delta^s f(x)$.

Starting, from *Newton's series,*

(1) $$f(x) = \sum_{m=0}^{\infty} \binom{x-a}{m}_h \frac{\underset{h}{\Delta^m} f(a)}{h^m},$$

if this series is convergent, we may determine the differences of $f(x)$ term-by-term

(2) $$\underset{h}{\Delta^s} f(x) = \sum_{m=s}^{\infty} \binom{x-a}{m-s}_h h^{s-m}\,\underset{h}{\Delta^m} f(a).$$

Therefore it is sufficient to express, by derivatives, the differences of the function $f(x)$ for $x=a$. To obtain them we will expand $f(x)$ into a *Taylor* series:

$$f(x) = \sum_{n=0}^{\infty} (x-a)^n \frac{D^n f(a)}{n!}.$$

But according to formula (1) § 58 we have

$$[\underset{h}{\Delta^m}(x-a)^n]_{x=a} = m!\,h^n\,\mathfrak{S}_n^m;$$

therefore

(3) $$\Delta_h^m f(a) = \sum_{n=m}^{\infty} \frac{m!}{n!} h^n \mathrm{D}^n f(a) \mathfrak{S}_n^m .$$

The value thus obtained put into (2) gives the required formula. Formula (3) may be deduced by inversion of the series (3) § 56. Indeed, multiplying this series by $h^m \mathfrak{S}_m^k / m!$ and summing from $m=k$ to $m=\infty$, *we* get

$$\sum_{m=k}^{\infty} \frac{h^m}{m!} \mathrm{D}^m f(a) \mathfrak{S}_m^k = \sum_{n=0}^{\infty} \frac{\Delta_h^n f(a)}{n!} \sum_{m=k}^{\infty} S_n^m \mathfrak{S}_m^k .$$

We have seen (5 § 65) that the second sum of the second member is equal to zero if n is different from k and equal to one if $n=k$. Accordingly we have

$$\Delta_h^k f(u) = \sum_{m=k}^{\infty} \frac{k!}{m!} h^m \mathrm{D}^m f(a) \mathfrak{S}_m^k .$$

This formula is identical with (3).

Example 1. Putting into formula (3) $h=1$ and $f(x) = \log x$ we find:

(4) $$\Delta^m \log x = \log e \sum_{n=m}^{\infty} \frac{(-1)^{n+1} m!}{n x^n} \mathfrak{S}_n^m .$$

In the particular case of the even differences $m=2\nu$ it is more advantageous to expand $\Delta^{2\nu}$ logx into a series of inverse powers of $x+\nu$ instead of the inverse powers of x. Writing $z=1/(x+\nu)$ we find $x=(1-\nu z)/z$ and

$$\frac{1}{x^n} = z^n (1-\nu z)^{-n} = \frac{1}{(x+\nu)^n} \sum_{i=0}^{\infty} (-1)^i \binom{-n}{i} \frac{\nu^i}{(x+\nu)^i} ;$$

finally putting $n+i=\mu$ *we* have

(5) $$\Delta^{2\nu} \log x = (2\nu)! \log e \sum_{\mu=2\nu}^{\infty} \frac{\nu^\mu}{\mu (x+\nu)^\mu} \sum_{n=2\nu}^{\mu+1} \frac{(-1)^{n-1} \binom{\mu}{n}}{\nu^n} \mathfrak{S}_n^{2\nu} .$$

From this we conclude according to § 6, that the last sum of the second member is equal to

$$-\left[\Delta_n^\mu \frac{\mathfrak{S}_n^{2\nu}}{\nu^n} \right]_{n=0} .$$

But in consequence of formula (1) , § 59 we have

$$\frac{\mathfrak{S}_n^{2\nu}}{\nu^n} = \frac{1}{(2\nu)!} \sum_{i=1}^{2\nu+1} (-1)^i \binom{2\nu}{i} \left(\frac{i}{\nu}\right)^n;$$

therefore

$$-\left[\Delta^\mu \frac{\mathfrak{S}_n^{2\nu}}{\nu^n}\right]_{n=0} = \frac{1}{(2\nu)!\nu^\mu} \sum_{i=1}^{2\nu+1} (-1)^i \binom{2\nu}{i} (i-\nu)^\mu .$$

We remark that if μ is odd, the terms of this sum corresponding to i and $2\nu-i$ are equal and of opposed sign, moreover the term corresponding to $i=\nu$ is equal to zero. Therefore if μ is odd the sum is equal to zero, so that we may write $2i$ insted of μ in formula (5).

Determining the first two terms of (5) we have

$$\Delta^{2\nu} \log x = \log e \left[-\frac{(2\nu-1)!}{(x+\nu)^{2\nu}} - \frac{\nu(2\nu+1)!}{12\ (x+\nu)^{2\nu+2}} - \right.$$

$$\left. -(2\nu)! \sum_{i=\nu+2}^{\infty} \frac{\nu^{2i}}{2i(x+\nu)^{2i}} \sum_{n=2\nu}^{2i+1} \frac{(-1)^n \binom{2i}{n} \mathfrak{S}_n^{2\nu}}{n\,\nu^n}\right].$$

Particular *cases.*

$$\Delta^2 \log x = \log e \left[-\frac{1}{(x+1)^2} - \frac{1}{2(x+1)4} - \frac{1}{3(x+1)^6} - \dots\right]$$

$$\Delta^4 \log x = \log e \left[-\frac{6}{(x+2)^4} - \frac{20}{(x+2)^6} - \dots\right].$$

Two terms of the first expression and one of the second are sufficient for an exactitude to twenty decimals if $x \geq 10{,}000$.

In the notation of central differences this is

$$\delta^2 \log x = \log e \left[-\frac{1}{x^2} - \frac{1}{2x^4} - \frac{1}{3x^6} - \dots\right]$$

and

$$\delta^4 \log x = \log e \left[-\frac{6}{x^4} - \frac{20}{x^6} - \dots\right].$$

Let us choose tbe most unfavourable example. Putting $x=10{,}000$ the first two terms give

$\delta^2 \log 10{,}000 =$	$-$0'00000	00043	42944	81903	25
	$-$			2171	47
	$-$0'00000	00043	42444	84075	

the first term gives

$$\delta^4 \log 10000 = -0{\cdot}00000\ 00000\ 00000\ 26057\ 64.$$

The results are exact to twenty decimals.

Example 2. Putting $m=1$ into formula (3) and writing x instead of a we get

$$\Delta f(x) = \sum_{i=1}^{\infty} \frac{h^i D^i f(x)}{i!} .$$

Letting $f(x) = 1/x^m$ and $h = 1$, we have

$$\Delta \frac{1}{x^m} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{(m+i-1)_i}{x^{m+i}},$$

Performing the operation Δ^{-1} on both members of this equation we get

$$\frac{1}{x^m} + k = \sum_{i=1}^{\infty} (-1)^i \binom{m+i-1}{m-1} \Delta^{-1} \frac{1}{x^{m+i}}.$$

Let us introduce the notation

$$s_n' = \sum_{x=2}^{\infty} \frac{1}{x^n}.$$

The sum of the above equation from $x=2$ to $x=\infty$ is

$$-\frac{1}{2^m} = \sum_{i=1}^{\infty} (-1)^i \binom{m+i-1}{m-1} s_{m+i}' .$$

Putting $m=1$ we obtain

$$½ = \sum_{i=1}^{\infty} (-1)^{i+1} s_{i+1}'.$$

The series is alternating and $\lim_{n=\infty} s_n' = 0$; therefore the series is convergent.

Verification.

(4) $$\sum_{i=1}^{32} (-1)^{i+1} s_{i+1}' = 0{\cdot}50000\ 00000.$$

The numbers s_n' are taken from *T. J. Stieltjes'* paper, "Table des valeurs des Sommes $\sum_1^{\infty} n^{-k}$." (Acta Mathematica. Vol. 10, p. 299.)

§ 68. Expansion- of a reciprocal factorial into a series of reciprocal powers and vice versa, We found in § 60, that the generating function of the *Stirling* numbers of the second kind $\mathfrak{S}_n^m$, with respect to n, is the following

$$\frac{t^m}{(1-t)(1-2t)\ldots(1-mt)} = \sum_{n=m}^{\infty} \mathfrak{S}_n^m t^n.$$

Putting $t=1/z$ we get

$$(1) \qquad \frac{1}{(z)_{m+1}} = \sum_{n=m}^{\infty} \frac{\mathfrak{S}_n^m}{z^{n+1}} .$$

If $z>m$ this series is convergent. To show it, let us write $u_n = \mathfrak{S}_n^m/z^{n+1}$ and remark. that according to formula (2) § 59 we have

$$\lim_{n=\infty} u_n = \lim_{n=\infty} \frac{\mathfrak{S}_n^m}{m^n} \frac{\left(\frac{m}{z}\right)^n}{z} = \frac{1}{m!\,z} \lim_{n=\infty} \left(\frac{m}{z}\right)^n;$$

therefore if $z>m$ this limit is equal to zero. Moreover from formula *(5) § 59* it follows that

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = \lim_{n=\infty} \frac{\mathfrak{S}_{n+1}^m}{z\mathfrak{S}_n^m} = \frac{m}{z}.$$

Hence if $z>m$ then this limit is smaller than one and the series is convergent.

Formula (1) may be transformed by writing $z=-x$; then we have

$$(2) \qquad (-1)“(x), = \frac{(-1)^m}{(x+m)_m} = \sum_{n=m}^{\infty} (-1)^n \frac{\mathfrak{S}_n^m}{x^n}.$$

In consequence of what has been said, this series is convergent if $x>m$.

The expansion of a reciprocal power into a series of reciprocal factorials may be obtained by the inversion of the series (2). Let us multiply each member of (2) by (-1)’ S_m^ν and sum from $m=0$ to $m=\infty$. According to (5) § 66 every term of the second member will vanish except that in which $n=\nu$. Hence we have

$$(3) \quad \frac{1}{x^\nu} = \sum_{m=\nu}^{\infty} \frac{(-1)^{m+\nu}}{(x+m)_m} S_m^\nu = \sum_{m=\nu}^{\infty} \frac{|S_m^\nu|}{(x+m)_m} = \sum_{m=\nu}^{\infty} |S_m^\nu| (x)_{-m}$$

Multiplying both members by $1/x$ and writing $\nu+1=n$, we obtain a second formula for the expansion of a reciprocal power:

$$(4)\qquad \frac{1}{x^n} = \sum_{m=n-1}^{\infty} \frac{|S_m^{n-1}|}{(x+m)_{m+1}} = \sum_{m=n-1}^{\infty} |S_m^{n-1}|\,(x-1)_{-m-1}\,.$$

This formula was first deduced by *Stirling* (Methodus Differentialis, p. 11).

Application of the above expansions. Formula (1) may serve for the determination of the derivatives of a reciprocal factorial. We find:

$$(5)\qquad \mathrm{D}^s \frac{1}{(z)_{m+1}} = \sum_{n=m}^{\infty} (-1)^s \frac{(n+s)_s}{z^{n+1+s}} \mathfrak{S}_n^m .$$

The integral of the reciprocal factorial is also obtained by aid of (1)

$$(6)\qquad \int \frac{dz}{(z)_{m+1}} = -\sum_{n=m}^{\infty} \frac{\mathfrak{S}_n^m}{n\,z^n} + k\,.$$

Equations (3) and (4) may serve for the determination of the differences, or of the sum, of a reciprocal power. From (3) we deduce

$$(7)\qquad \Delta^s \frac{1}{x^n} = \sum_{m=n}^{\infty} (-m)_s\,|S_m^n|\,(x)_{-m-s}$$

and

$$(8)\qquad \Delta^{-1} \frac{1}{x^n} = -\sum_{m=n}^{\infty} \frac{1}{m-1}\,|S_m^n|\,(x)_{-m+1}\,.$$

Starting from formula (4) we get in the same way

$$(9)\qquad \Delta^s \frac{1}{x^n} = \sum_{m=n-1}^{\infty} (-m-1)_s\,|S_m^{n-1}|\,(x-1)_{-m-1-s}$$

and

$$(10)\qquad \Delta^{-1} \frac{1}{x^n} = -\sum_{m=n-1}^{\infty} \frac{1}{m}\,|S_m^{n-1}|\,(x-1)_{-m}\,.$$

By aid of (8) or (10) we may obtain the sum of the reciprocal powers, x varying from $x=1$ to $x=\mu$. For instance from (10) it follows that

$$\sum_{x=1}^{\mu} \frac{1}{x^n} = \sum_{m=n-1}^{\infty} \frac{1}{m}\,|S_m^{n-1}| \left[\frac{1}{m!} - (\mu-1)_{-m}\right]$$

and if $\mu=\infty$,

$$(11)\qquad \sum_{x=1}^{\infty} \frac{1}{x^n} = \sum_{m=n-1}^{\infty} \frac{|S_m^{n-1}|}{m\,m!}\,.$$

This was the formula that *Stirling* used for the determination of sums of reciprocal power series (p. 28).

In the same manner we should have obtained from (8)

$$(12) \qquad \sum_{x=1}^{\infty} \frac{1}{x^n} = \sum_{m=n}^{\infty} \frac{|S_m^n|}{(m-1)\ m!}.$$

Stirling's example of expansion into reciprocal factorial series. (Meth. Diff. p. 12.) To expand $1/(x^2+ax)$ we write

$$\frac{1}{x^2+ax} = \frac{1}{x^2\left(1+\frac{a}{x}\right)} = \sum_{i=0}^{\infty} (-1)^i \frac{a^i}{x^{i+2}}.$$

Now we may use formula (4) to expand the reciprocal power in the second member; putting $n=i+2$ into (4) we deduce

$$\frac{1}{x^2+ax} = \sum_{i=0}^{\infty} a^i \sum_{m=i+1}^{\infty} \frac{(-1)^{m-1}}{(x+m)_{m+1}} S_m^{i+1} =$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{a(x+m)_{m+1}} \sum_{i=0}^{m} a^{i+1} S_m^{i+1}.$$

But according to § 50 the last sum is equal to $(a)_m$; therefore we have

$$(13) \qquad \frac{1}{x^2+ax} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ (a-1)_{m-1}}{(x+m)_{m+1}}.$$

By aid of this formula we may determine the indefinite sum of this quantity

$$(14) \qquad \Delta^{-1} \frac{1}{x^2+ax} = \sum_{m=1}^{\infty} \frac{(-1)^m\ (a-1)_{m-1}}{m(x+m-1)_m}.$$

Finally the sum, x varying from one to ∞, will be

$$(15) \qquad \sum_{x=1}^{\infty} \frac{1}{x^2+ax} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\ (a-1)_{m-1}}{m\ m!}.$$

§ 69. The operation Θ. In mathematical analysis an operation often occurs which we will denote by Θ, and define by the relation

$$\Theta = x\frac{d}{dx} = x\mathrm{D}.$$

The operation repeated gives

$$\Theta^2 = x\mathrm{D}\,(x\mathrm{D}) = x\mathrm{D} + x^2\mathrm{D}^2$$
$$\Theta^3 = x\mathrm{D}\,\Theta^2 = x\mathrm{D} + 3x^2\mathrm{D}^2 + x^3\mathrm{D}^3$$

and so on. It is easy to see that

$$(1) \qquad \Theta^n = C_1\, x\mathrm{D} + C_2\, x^2\mathrm{D}^2 + \ldots + C_n\, x^n\mathrm{D}^n$$

where the numbers C_i above are independent of the function on which the operation **is performed.For** their determination we can choose the most convenient function. Let $f(x) = x^\lambda$, then we have

$$\Theta\, x^\lambda = \lambda\, x^\lambda \; ; \quad \Theta^n\, x^\lambda = \lambda^n\, x^\lambda \; ; \quad \mathrm{D}^\nu\, x^\lambda = (\text{a}),\; x^{\lambda-\nu}.$$

Putting these values into (1) and dividing by x^λ we get

$$\lambda^n = \sum_{\nu=1}^{n+1} C_\nu (\lambda)_\nu .$$

Taking account of formula (2) of § 58 we conclude that $C_\nu = \mathfrak{S}_n^\nu$ and therefore

$$(2) \qquad \Theta^n = \sum_{\nu=1}^{n+1} \mathfrak{S}_n^\nu\, x^\nu\, \mathrm{D}^\nu .$$

This is the expression of the Θ operation in terms of derivatives.

Particular cases.

$$\Theta\, k = 0 \qquad\qquad \Theta\, x = x.$$
$$\Theta \log x = 1 \qquad\qquad \Theta\, x^\lambda = \lambda\, x^\lambda .$$

Example 1. Let $f(x) = 1/x^\lambda$. We have

$$\Theta\, f(x) =\text{- / } l\, x\text{ - ”} \; ; \; @“f(x) = (-\lambda)^n\, x^{-\lambda}$$

$$\mathrm{D}^\nu f(x) = \frac{(-1)^\nu\, (\lambda+\nu-1)_\nu}{x^{\lambda+\nu}} ;$$

therefore according to (2) it results that

$$(3) \qquad (-\lambda)^n = \sum_{\nu=1}^{n+1} \mathfrak{S}_n^\nu\, (-1)’\, (\lambda+\nu-1)_\nu .$$

Putting $\lambda=1$, this formula gives a formula already **obtained (5 § 58).**

Example 2. Let $f(x) = (x+1)^n$. We obtain

$$(4) \qquad @“f(x) = \sum_{\nu=1}^{n+1} \binom{n}{\nu}\, x^\nu\, \nu^m .$$

On the other hand from (2) it follows that

$$(5) \qquad \Theta^m f(x) = \sum_{\nu=1}^{m+1} \mathfrak{S}_m^\nu \, x^\nu \, (n)_\nu \, (x+1)^{n-\nu}.$$

Equating the second members of (4)and (5) and putting $x=1$ we get

$$(6) \qquad \sum_{\nu=1}^{n+1} \binom{n}{\nu} \nu^m = \sum_{\nu=1}^{m+1} (n)_\nu 2^{n-\nu} \mathfrak{S}_m^\nu .$$

Determination of the power moments of a function. If *u(t)* is the generating function of *f(x),* that is if

$$u(t) = \sum_{x=0}^{\infty} f(x) \, t^x$$

then

$$\Theta^n u(t) = \sum_{x=0}^{\infty} x^n f(x) \, t^x.$$

Therefore, if the Θ^n operation is performed on the generating function of *f(x)* and then we put $t=1$, we obtain the n -th power moment of the function *f(x):*

$$(7) \qquad \mathcal{M}_n = [\Theta^n u(t)]_{t=1}.$$

Example 3. The generating function of the probability of repeated trials

$$(8) \qquad \binom{n}{\nu} p^\nu q^{n-\nu} \qquad \text{(where } q = 1-p)$$

is equal to

$$u(t) = (q+pt)^n.$$

Therefore the m -th power moment of (8) is in consequence of (7) and of (2) equal to

$$\mathcal{M}_m = \sum_{\nu=1}^{n+1} \nu^m \binom{n}{\nu} p^\nu q^{n-\nu} = \sum_{\nu=1}^{m+1} \mathfrak{S}_m^\nu (n)_\nu \, p^\nu .$$

Inversion of the sum (2). Multiplying it by S_m^n and summing from $n=1$ to $n=m+1$ it results that

$$(9) \qquad \mathrm{D}^m = \frac{1}{x^m} \sum_{n=1}^{m+1} S_m^n \, \Theta^n.$$

This formula gives the m -th derivative in terms of the Θ operations.

Example 4. Let f(x) = log x. Since $\Theta \log x = 1$ and $\Theta^n \log x = 0$ if $n > 1$, we have

$$D^m \log x = \frac{S_m^1}{x^m} = (-1)^{m-1}(m-1)!\,\frac{1}{x^m}.$$

Sometimes it is possible to determine both members of the equation (9) directly, and they lead to useful formulae.

Remark A formula analogous to that of *Faa de Bruno* given in § 12 is applicable for the Θ operation performed on a function of function. Given $u=u(y)$ and $y=y(t)$, we have

$$\frac{du}{dt} = \frac{du}{dy}\,\frac{dy}{dt}, \qquad \frac{d^2u}{dt^2} = \frac{d^2u}{dy^2}\left(\frac{dy}{dt}\right)^2 + \frac{du}{dy}\,\frac{d^2y}{dt^2}$$

and so on; on the other hand

$$\Theta u = t\frac{du}{dt} = \frac{du}{dy}\;\Theta y, \quad \Theta^2 u = \frac{d^2u}{dy^2}\,(\Theta y)^2 + \frac{du}{dt}\;\Theta^2 y$$

and so on. It is easy to see that the two sets of formulae will remain similar, so that to obtain $\Theta^n u$ starting from a formula giving $\frac{d^n u}{dt^s}$ it is sufficient to put in it $\Theta^s y$ instead of $\frac{d^s y}{dt^s}$. For *instance* in the case of *Faa de Bruno's* formula (9, § 12) we obtain:

$$(10) \quad [\Theta^s u]_{t=t_0} = \sum_{\nu=1}^{s+1}\left[\frac{d^\nu u}{dy^\nu}\right]_{y=y_0} \sum \frac{s!}{\alpha_1!\,\alpha_2!\ldots\alpha_s!}\left(\frac{\Theta y_0}{1!}\right)^{\alpha_1}\left(\frac{\Theta^2 y_0}{2!}\right)^{\alpha_2}\cdots\left(\frac{\Theta^s y_0}{s!}\right)^{\alpha_s}.$$

The second sum is extended to every value of $\alpha_i = 0, 1, 2, \ldots$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_s = \nu$$

$$\alpha_1 + 2\alpha_2 + \ldots + s\alpha_s = s.$$

Example 5. Determination of the power moments μ_s of *x-np* if $P(x) = \binom{n}{x} p^x q^{n-x}$ and $q=1-p$. We have

$$(q + pt)^n = \Sigma\, P(x)\, t^x;$$

therefore

$$u = (q + pt)^n t^{-np} = (qt^{-p} + pt^q)^n = \Sigma\, P(x) t^{x-np}$$

and

$$\mu_s = [\Theta^s u]_{t=1} = \Sigma\,(x-np)^s P(x).$$

To determine μ_s let us put $y=qt^{-p}+pt^q$; then we have $u=y^n$ and

(11) $$\left[\frac{d^\nu u}{dy^\nu}\right]_{y=1} = (n)_\nu \quad and \quad [\Theta^i y]_{t=1} = pq[q^{i-1} + (-1)^i p^{i-1}].$$

Let us remark that for $i=1$ we have $[\Theta y]_{t=1} = 0$ and therefore $\mu_1=0$; from this we conclude that in the case in hand, in formula (10), a_1 must be always equal to zero. So that $a_2 + , , . + a, = \nu$ and $2a_2 + \ldots + sa, = s$; that is, the partitions of s, formed by 2, 3, 4, . . . only are to be considered.

Formulae (10) and (11) solve the problem.

(12) $$\mu_s = \sum_{\nu=1}^{s+1} (n)_\nu p^\nu q^\nu \;\Sigma \frac{s!}{a_2!a_3!\ldots a_s!}\left(\frac{1}{2}\right)^{a_2}\left(\frac{q-p}{6}\right)^{a_3}\cdot$$
$$\left(\frac{q^3+p^3}{24}\right)^{a_4}\cdot\cdot\cdot\cdot\cdot\left[\frac{q^{s-1}+(-1)^s p^{s-1}}{s!}\right]^{a_s}.$$

§ 70. The operation Ψ. In the Calculus of Finite Differences there occurs an operation analogous to that of Θ, which we will denote by Ψ and define by

$$\Psi = x\Delta.$$

The operation repeated gives, according to the formula for the difference of a product § 30,

$$\Psi^2 = x\Delta\,(x\Delta) = x\Delta + (x+1)_2\,\Delta^2$$
$$\Psi^3 = x\Delta\,\Psi^2 = x\Delta + 3(x+1)_2\,\Delta^2 + (x+2)_3\,\Delta^3$$

and so on. We may write

(1) $$\Psi^n = C_1\,x\Delta + C_2\,(x+1)_2\,\Delta^2 + \ldots + C_n\,(x+n-1)_n\,\Delta^n.$$

The numbers C_i being independent of the function on which the operation is performed, to calculate them we may choose the most convenient function. This is $f(x) = (x+\lambda-1)_\lambda$. Then

$$\Delta^\nu f(x) = (\lambda)_\nu\,(x+\lambda-1)_{\lambda-\nu}$$

and

$$\Psi f(x) = x\Delta f(x) = \lambda(x+\lambda-1)_\lambda\,,\quad \Psi^n f(x) = \lambda^n(x+\lambda-1)_\lambda\,.$$

Putting these values into equation (1) and dividing by $(x+\lambda-1)_\lambda$ we get

$$\lambda^n = \sum_{\nu=1}^{n+1} C_\nu (\lambda)_\nu .$$

Hence according to formula (2) of § 58 we have $C_\nu = \mathfrak{S}_n^\nu$; and finally

$$\text{(2)} \qquad \Psi^n = \sum_{\nu=1}^{n+1} (x+\nu-1)_\nu \mathfrak{S}_n^\nu \Delta^\nu .$$

This gives the Ψ^n operation in terms of differences.

Particular cases:

$$\Psi k = 0 \qquad \Psi x = x$$

$$\Psi F(x-1) = 1 \qquad \Psi (x+\lambda-1)_\lambda = \lambda (x+\lambda-1)_\lambda$$

Example 1. Let $f(x) = 1/(x-1)_\lambda$; it follows that

$$\Psi f(x) = \frac{-\lambda}{(x-1)_\lambda} \text{ and } \Psi^n f(x) = \frac{(-\lambda)^n}{(x-1)_\lambda} .$$

Inversion of formula (2). Multiplying this equation by S_m^n and summing from $n=1$ *to* $n=m+1$ *we* get

$$\text{(3)} \qquad \Delta^m \frac{1}{(x+m-1)_m} \sum_{n=1}^{m+1} S_m^n \Psi^n = (x-1)_{-m} \sum_{n=1}^{m+1} S_m^n \Psi^n .$$

This gives the m-th difference in terms of the Ψ operations.

Example 2. Given $f(x) = F(x-1)$. Since $\Psi F(x-1) = 1$ and $\Psi^n F(x-1) = 0$ if $n > 1$,

$$\Delta^m F(x-1) = \frac{S_m^1}{(x+m-1)_m} = \frac{(-1)^{m-1} (m-1)!}{(x+m-1)_m} .$$

§ 71. **Operations** $\Delta^m D^{-m}$ **and** $D^m \Delta^{-m}$. The operation ΔD^{-1} is univocaf, that is, it leads to one definite solution only. Let us write

$$\int f_i(x)\, dx = f_{i+1}(x) + k ;$$

then we have

$$\Delta D^{-1} f_0(x) = f_1(x+1) - f_1(x) = \Delta f_1(x) = \int_0^1 f_0(x+t_1)\, dt_1 .$$

Repeating the operation we have

(AD-‘)’ *f,,(x)* $= f_2(x+2) - 2f_2(x+1) + f_2(x) = \Delta^2 f_2(x) =$

$$= \int_0^1 \int_0^1 f_0\,(x+t_1+t_2)\,dt_1\,dt_2$$

and so on. Finally we have

$$(1) \qquad (\Delta D^{-1})^m f_0(x) = \Delta^m f_m(x) =$$

$$= \int_0^1 \ldots \int_0^1 f_0(x+t_1+\ldots+t_m)\,dt_1\ldots dt_m.$$

From

(AD-‘)” f,(x) $= \Delta^m f_m(x)$

it follows that

(AD-‘)” f,(x) $= \Delta^m D^{-m} f_0(x).$

That is, the operations $(\Delta D^{-1})^m$ and $\Delta^m D^{-m}$ are equivalent. This is not obvious, since in

(AD-‘) (AD-‘) = A(D-‘A) D^{-1}

the operations D^{-1} and Δ are not commutative.

Particular cases.

AD-’ k $= k$ $\qquad \Delta D^{-1} e^x = e^x(e-1)$

AD-’ $x = x + \frac{1}{2}$ $\qquad$ *AD-’* $2^x = 2^x / \log 2$

AD-’ $\frac{1}{x^2} = \frac{1}{(x+1)_2}$.

Examples. ***1.*** **Let** *f(x)* $= F(x)$. **We have**

$$D^{-1} F(x) = \log \Gamma(x+1) + k;$$

therefore

AD-’ $F(x) = \log\,(x+1).$

To perform the operation $\Delta^m D^{-m}$ on a given function *f(x)* **it is best to expand the function** *f(x)* **into a** ***Maclaurin*** **series**

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} D^n f(0).$$

The operation $\Delta^m D^{-m}$ gives

$$(2) \qquad \Delta^m D^{-m} f(x) = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} D^n f(0)\, \Delta^m \left[\frac{x^{n+m}}{m!}\right].$$

We have seen in **§ 58** that putting $x=0$, the difference in the second member will be equal to $\mathfrak{S}^m_{m+n}$, so that

$$(3) \qquad [\Delta^m \mathrm{D}^{-m} f(x)]_{x=0} = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} \mathrm{D}^n f(0)\, \mathfrak{S}^m_{n+m} .$$

This formula is useful for the determination of the multiple integral

$$(4) \qquad \int_0^1 \ldots \int_0^1 f(t_1+\ldots+t_m)\,dt_1 \ldots dt_m = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} \mathrm{D}^n f(0)\, \mathfrak{S}^m_{n+m}.$$

Sometimes the operation $\Delta^m \mathrm{D}^{-m}$ maybe performed directly on the function *f(x);* then equating the result to the second member of (3) we may obtain interesting relations.

Example 2. Let $f(x) = e^{xt}$. We have

$$\mathrm{D}^n f(0) = t^n, \qquad \mathrm{D}^{-m} f(x) = \frac{e^{xt}}{t^m}, \qquad \text{A m}\ \frac{e^{xt}}{t^m} = \frac{e^{xt}(e^t-1)^m}{t^m};$$

therefore from (3) it follows that

$$(5) \qquad (e^t-1)^m = \sum_{\nu=m}^{\infty} \frac{m!}{\nu!} \mathfrak{S}^m_\nu\, t^\nu.$$

Or putting $e^t - 1 = z$ we have

$$(6) \qquad z^m = \sum_{\nu=m}^{\infty} \frac{m!}{\nu!} \mathfrak{S}^m_\nu\, [\log(z+1)]^\nu .$$

From this formula we may obtain an important expansion by inversion of the series, Let us multiply each member of equation (6) by $S^n_m / m!$ and sum from $m=n$ to $m=\infty$. According to the formulae of **§ 65** we get

$$(7) \qquad [\log(1+z)]^n = \sum_{m=n}^{\infty} \frac{n!}{m!} S^n_m\, z^m .$$

This formula has been applied already in **§** 51.

Operation $\mathrm{D}^m \Delta^{-m}$. This operation is the inverse operation of the preceding. Indeed, from

$$\mathrm{D}^m \Delta^{-m} f(x) = \varphi(x)$$

it follows that

$$f(x) = \Delta^m \mathrm{D}^{-m} \varphi(x).$$

The second operation is univocal, that is, to a given function $\varphi(x)$ there corresponds only one function f(x) ; on the other hand the first operation is not univocal; to a given *f(x)* there correspond several functions $\varphi(x)$. Indeed, we have seen that if $\Delta^{-1}f(x) = f_1(x)$ then the following equation is also true:

$$\Delta^{-1}f(x) = f_1(x) + \omega(x)$$

where $\omega(x)$ is a periodic function with a period equal to one; If $m = 1$ then $\varphi(x) = Df_1(x) + D\omega(x)$. But the derivative of a periodic function is a periodic function with the same period, hence this may be written

$$\varphi(x) = Df_1(x) + \omega_1(x)$$

where $\omega_1(x)$ is an arbitrary periodic function satisfying certain conditions.

In the same manner, if $\Delta^{-m}f(x) = f_m(x)$ then too

$$\Delta^{-m}f(x) = f_m(x) + \binom{x}{m-1}\omega_1(x) + \ldots + \binom{x}{1}\omega_{m-1}(x) + \omega_m(x)$$

The $\omega_i(x)$ are arbitrary periodic functions. From this, taking the m-th derivative, we get

$$\varphi(x) = D^m f_m(x) + \Omega(x)$$

where $\Omega(x) = \binom{x}{m-1}\lambda_1(x) + \binom{x}{m-2}\lambda_2(x) + \ldots + \lambda_m(x)$

the $\lambda_i(x)$ being arbitrary periodic functions.

Remark. For $x=0$ we obtain

$$[D^m\Delta^{-m}f(x)]_{x=0} = D^m f_m(0) + k$$

where the constant k is equal to $\Omega(0) = \lambda_m(0)$.

Particular cases. We have

$$D\Delta^{-1}k = k + \omega_1(x)$$

$$D\Delta^{-1}x = x - \tfrac{1}{2} + \omega_1(x)$$

$$D\Delta^{-1}\frac{1}{(x+1)_2} = \frac{1}{x^2} + \omega_1(x)$$

$$D\Delta^{-1}\frac{1}{x} = DF(x-1) + \omega_1(x) = F(x-1) + \omega_1(x)$$

$$D\Delta^{-1}\log(x+1) = D\log\Gamma(x+1) + \omega_1(x) = F(x) + \omega_1(x).$$

To perform the operation $D^m\Delta^{-m}$ on a function *f(x) we* may expand it first into a *Newton series*

$$f(x) = \ldots \binom{x}{n} \Delta^n f(0);$$

then the operation will give

$$\mathrm{D}^m \Delta^{-m} f(x) = \sum_{n=0}^{\infty} \Delta^n f(0)\, \mathrm{D}^m \binom{x}{n+m} + \Omega_{m-1}(x).$$

The derivative figuring in the second member may be expressed by *Stirling* numbers of the first hind (§ 50); we get, putting $x=0$,

$$[\mathrm{D}^m \Delta^{-m} f(x)]_{x=0} = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} \Delta^n f(0)\, S_{n+m}^m + k.$$

Example. Given $f(x) = (1+t)^x$. We have $\Delta^n f(0) = t^n$ and

$$\Delta^{-m} f(x) = \frac{(1+t)^x}{t^m}$$

moreover

$$\mathrm{D}^m (1+t)^x = (1+t)^x [\log(1+t)]^m$$

therefore

$$(8) \qquad [\log(1+t)]^m = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} S_{m+n}^m t^{m+n} + k.$$

Since for $t=0$ the first member is equal to zero, I $k=0$, and formula (8) becomes identical with our formula (7) obtained above.

§72. Expansion of a function of function by afd of Stirling numbers. Semi-invariants of Thiele. In § 12 *we* found *Schlömilch*'s formula (8) giving the n- th derivative of a function of function. If $u=u(y)$ and $y=y(x)$, then

$$(1) \qquad \frac{d^n u}{dx^n} = \sum_{\nu=1}^{n+1} \frac{1}{\nu!} \frac{d^\nu u}{dy^\nu} \left\{ \frac{d^n}{dh^n} [\mathop{\Delta}_h y(x)]^\nu \right\}_{h=0}.$$

1. Expansion of $u=u(\log x)$. Putting into formula (1) $y=\log x$ we have

$$[\mathop{\Delta}_h \log x]^\nu = \left[\log\left(1+\frac{h}{x}\right)\right]^\nu$$

and according to formula (7) § 71

$$\left[\log\left(1+\frac{h}{x}\right)\right]^\nu = \sum_{m=\nu}^{\infty} \frac{\nu!}{m!} S_m^\nu \frac{h^m}{x^m}$$

the n-th derivative of this quantity with respect to h is

$$\sum_{m=0}^{\infty} \frac{\nu!\,(m)_n\, h^{m-n}}{m!\, x^m}\, S_m^\nu .$$

Putting into it $h=0$ every term will vanish except the term in which $m=n$, that is $\nu!\, S_n^\nu / x^n$; therefore we shall have

$$\text{(2)} \qquad \frac{d^n u}{dx^n} = \sum_{\nu=1}^{n+1} \frac{S_n^\nu}{x^\nu} \frac{d^\nu u}{dy^\nu}.$$

Finally writing in this equation $x=1$, or $y=0$ *Taylor's* theorem will give

$$\text{(3)} \qquad u(\log x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \sum_{\nu=0}^{n+1} S_n^\nu \left[\frac{d^\nu u}{dy^\nu}\right]_{y=0}.$$

This is the required expansion.

2. Expansion of $u=u(e^x)$. Let us put into (1) $y=e^x$. We have

$$[\underset{h}{\Delta}\, e^x]^\nu = e^{\nu x}\,(e^h-1)^\nu .$$

In § 71 we found (formula 5)

$$(e^h-1)^\nu = \sum_{m=\nu}^{\infty} \frac{\nu!\, h^m}{m!}\, \mathfrak{S}_m^\nu .$$

The n-th derivative of this quantity, with respect to h, is

$$\sum_{m=\nu}^{\infty} \frac{\nu!\,(m)_n\, h^{m-n}}{m!}\, \mathfrak{S}_m^\nu .$$

Putting $h=0$ every term will vanish except that corresponding to $m=n$, and this will be $\nu!\,\mathfrak{S}_n^\nu$.

Therefore we shall have

$$\text{(4)} \qquad \frac{d^n u}{dx^n} = \sum_{\nu=1}^{n+1} \frac{d^\nu u}{dy^\nu}\, e^{x\nu}\, \mathfrak{S}_n^\nu .$$

Finally writing in this equation $x=0$, *Maclaurin's* theorem will give the required expansion

$$\text{(5)} \qquad u(e^x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\nu=0}^{n+1} \mathfrak{S}_n^\nu \left[\frac{d^\nu u}{dy^\nu}\right]_{y=1}.$$

Expansion of a function of function by Faa de Bruno's formula. We have seen (formula 9 § 12) that if $u=u(y)$ and $y=y(x)$ are given, the derivatives needed for this expansion are supplied by

$$(6)\quad \left[\frac{d^n u}{dx^n}\right]_{x=x_0} = \sum_{\nu=1}^{n+1}\left[\frac{d^\nu u}{dy^\nu}\right]_{y=y_0} \Sigma \frac{n}{a_1!\ldots a_n!}(\mathrm{D}y_0)^{a_1},,,\left(\frac{\mathrm{D}^n y_0}{n!}\right)^{a_n}$$

Where

$$a, + a_2 + \ldots + a_n = \mathrm{v} \quad \text{and} \quad a, + 2a_2 + \ldots + na, = n.$$

This formula is useful even in cases where $u=f(x)$ is given, but the direct determination of $d^n u/dx^n$ is difficult, provided it is possible to find a function $y=\varphi(u)$ such that the calculation of $d^n y/dx^n$ and of $d^m u/dy^m$ is easy. (See examples 4 and 5).

Example 1. Derivatives of $u=e^{ay}$ and $y=y(x)$. The derivatives for $x=0$ are obtained by aid of formula (6). Remarking that

$$\left[\frac{d^\nu u}{dy^\nu}\right]_{y=y_0} = a^\nu e^{ay_0}.$$

we find

$$(7)\quad \left[\frac{d^n e^{ay}}{dx^n}\right]_{x=0} = e^{ay}\sum_{\nu=1}^{n+1} a^\nu \ \Sigma \frac{n!}{a_1!\ldots a_n!}(\mathrm{D}y_0)^{a_1}\ldots\left(\frac{\mathrm{D}^n y_0}{n!}\right)^{a_n}.$$

The numerical coefficients in this formula are independent of the function y; to determine them we may choose the most convenient function, that is $y=e^x$. Then we shall have $y_0=1$ and $[\mathrm{D}^i y]_{x=0} = 1$; therefore from (7) we get

$$\left[\frac{d^n e^{ay}}{dx^n}\right]_{x=0} = e^a\sum_{\nu=1}^{n+1} a^\nu \ \Sigma \frac{n!}{a,!\ldots, a,!\,(2!)^{a}\ldots,(n!)^{a_n}}.$$

Comparing this result with that obtained in formula (4), we get an expression of *the Stirling* numbers of the second kind:

$$(8)\qquad \mathfrak{S}_n^\nu = \Sigma \frac{n!}{a_1!,\ldots a_n!\,(2!)^{a_2}\ldots(n!)^{a_n}}.$$

where the sum is extended to every value such that $a_1+a_2+ + \ldots + a_n=\nu$ and $a_1+2a_2+\ldots na_n=n$.

From this we conclude that in the derivative (7) the sum of the numerical coefficients of the terms multiplying a^ν is equal to $\mathfrak{S}_n^\nu$.

If $\nu=1$, the coefficient of $a(\mathrm{D}^n y)$ is $\mathfrak{S}_n^1 = 1$; and if $\nu=n$ then the coefficient of $a^n(\mathrm{D}y)^n$ is $\mathfrak{S}_n^n = 1$; moreover if v=n-1 the coefficient of a"' $(\mathsf{D}y)^{n-2}(\mathsf{D}^2y)$ is equal to $\mathfrak{S}_n^{n-1} = \binom{n}{2}$ Finally the sum of all the numerical coefficients in the derivative d^ne^{ay}/dx^n is equal to $\sum_{\nu=1}^{n+1} \mathfrak{S}_n^\nu$; moreover the number of terms in this expression is equal to the number of partitions of n with repetition, but without permutation; this number, $\Gamma(\int n)$ in *Netto's* notation. is given in a table in § 53. For instance in D^6e^{ay} this will be $\Gamma(\int n) = 11$. Moreover we have

$$\mathsf{D}^6e^{ay} = e^{ay} \cdot a\mathsf{D}^6y +$$
$$+ a^2[15(\mathsf{D}^2y)(\mathsf{D}^4y) + 6(\mathsf{D}y)(\mathsf{D}^5y) + 10(\mathsf{D}^3y)^2] +$$
$$+ a^3[15(\mathsf{D}y)^2(\mathsf{D}^4y) + 60(\mathsf{D}y)(\mathsf{D}^2y)(\mathsf{D}^3y) + 15(\mathsf{D}^2y)^3] +$$
$$+ a^4[20(\mathsf{D}y)^3(\mathsf{D}^3y) + 45(\mathsf{D}y)^2(\mathsf{D}^2y)^2] + 15a^5(\mathsf{D}y)^4(\mathsf{D}^2y) +$$
$$+ a^6(\mathsf{D}y)^6$$

in which

$$\mathfrak{S}_6^2 = 15 + 6 + 10 = 31; \quad \mathfrak{S}_6^3 = 15 + 60 + 15 = 90;$$
$$\mathfrak{S}_6^4 = 20 + 45 = 65.$$

Example 2. Expansion of $u = u(y)$, when $y = e^x$. We have

$$[\mathsf{D}^\nu y]_{x=0} = 1.$$

Therefore, according to (6), we shall have

$$\left[\frac{d^nu}{dx^n}\right]_{x=0} = \sum_{\nu=1}^{n+1}\left[\frac{d^\nu u}{dy^\nu}\right]_{y=1} \Sigma \frac{n!}{\alpha_1!\ldots\alpha_n!\,(2!)^{\alpha_2}\ldots(n!)^{\alpha_n}}$$

and in consequence of what has been said in the preceding example we find

$$\left[\frac{d^nu}{dx^n}\right]_{x=0} = \sum_{\nu=1}^{n+1}\left[\frac{d^\nu u}{dy^\nu}\right]_{y=1} \mathfrak{S}_n^\nu.$$

Example 3. Expansion of $u = u(y)$, when $y = \log x$. The derivatives of y are:

$$\mathsf{D}^sy = \frac{(-1)^{s-1}(s-1)!}{x^s} \quad \text{and} \quad [\mathsf{D}^sy]_{y=0} = (-1)^{s-1}(s-1)!$$

since $\Sigma s\alpha_s - \Sigma\alpha_s = n - \nu$. Hence from (6) it follows that

$$\left[\frac{d^n u}{dx^n}\right]_{x=1} = \sum_{\nu=1}^{n+1} \left[\frac{d^\nu u}{dy^\nu}\right]_{y=0} \quad \Sigma \frac{(-1)^{n-\nu}\, n!}{\alpha_1! \ldots \alpha_n!\, 2^{\alpha_2} 3^{\alpha_3} \ldots n^{\alpha_n}}.$$

Comparing the above result with that of formula (2) we find

$$(13) \qquad S_n^\nu = (-1)^{n-\nu}\, \Sigma \frac{n!}{\alpha_1!\, \alpha_2! \ldots \alpha_n!\, 2^{\alpha_2} \ldots n^{\alpha_n}}$$

where $\alpha_1+\alpha_2+\ldots+\alpha_\nu = \nu$ and $\alpha_1+2\alpha_2+\ldots+n\alpha_n = n$. *The* sum is extended to every partition of n, of order v with repetition but without permutation. This formula is different from (5) § 51, in which partitions with permutation are considered.

Example 4. The generating function of the probabilities in ***Poisson's*** problem of repeated trials is

$$u(t) = \prod_{i=1}^{n+1} (q_i + p_i t).$$

To obtain the factorial moments of the probability function the derivatives of u with respect to t are needed for $t=1$. Indeed

$$\mathfrak{M}_n = [D^n u]_{t=1}.$$

Since the derivatives of log u are less complicated than those of u we shall use formula (6) putting $u=e^y$ and $y=\log u$; the derivatives of y with respect to t are

$$D^s y = D^s \log u = \sum_{i=1}^{n+1} \frac{(-1)^{s-1}\,(s-1)!\, p_i^s}{(q_i + p_i t)^s}$$

Since for $y=0$ we have $u=1$ and $t = 1$, and $\frac{d^s u}{dy^s} = 1$ we may write

$$[D^s y]_{t=1} = (-1)^{s-1}\,(s-1)!\ \Sigma p_i^s.$$

Remarking that $\Sigma s\alpha_s - \Sigma \alpha_s = n - \nu$ *we find*

$$[D^n u]_{t=1} = \sum_{\nu=1}^{n+1} (-1)^{n-\nu}\, \Sigma \frac{n!}{\alpha_1! \ldots \alpha_n!} (\Sigma p_i)^{\alpha_1} \left(\frac{\Sigma p_i^2}{2}\right)^{\alpha_2} \left(\frac{\Sigma p_i^3}{3}\right)^{\alpha_3} \ldots$$

$$\ldots \left(\frac{\Sigma p_i^n}{n}\right)^{\alpha_n} = \mathfrak{M}_n.$$

Particular cases:

$$\mathfrak{M}_1 = \mathbf{D}u(1) = \Sigma p_i$$
$$\mathfrak{M}_2 = \mathbf{D}^2u(1) = (\Sigma p_i)^2 - \Sigma p_i^2$$
$$\mathfrak{M}_3 = \mathbf{D}^3u(1) = (\Sigma p_i)^3 - 3\Sigma p_i \Sigma p_i^2 + 2\Sigma p_i^3.$$

Example 5. Expansion of $\Gamma(x+1)$ *in powers of* x. Let us put $u = \Gamma(x+1)$ and $y = \log u$. Starting from the well-known formula (see p. 59)

$$\log \Gamma(x+1) = -(x+1)\, C - \log (x+1) + \sum_{\nu=1}^{\infty} \left(\frac{x+1}{\nu} - \log \frac{x+1+\nu}{\nu} \right)$$

we find

$$\mathbf{D} \log \Gamma(x+1) = -C - \frac{1}{x+1} + \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{x+1+\nu} \right)$$

and if $i > 1$

$$\mathbf{D}^i \log \Gamma(x+1) = \sum_{\nu=0}^{\infty} \frac{(-1)^i (i-1)!}{(x+1+\nu)^i}.$$

Moreover

$$[\mathbf{D} \log \Gamma(x+1)]_{x=0} = -C$$
$$[\mathbf{D}^i \log \Gamma(x+1)]_{x=0} = (-1)^i (i-1)!\, s_i$$

where.

$$s_i = \sum_{\nu=1}^{\infty} \frac{1}{\nu^i}$$

For $x=0$ we get $u=1$ and $y=0$. Moreover since $u=e^y$, for $y=0$ we get $\frac{d^iu}{dy^i} = 1$; therefore remarking that $\Sigma i\alpha_i = n$, from (6) we deduce that

$$(10) \quad [\mathbf{D}^n \Gamma(x+1)]_{x=0} = (-1)^n \sum \frac{n!}{\alpha_1!\alpha_2!\dots\alpha_n!} C^{\alpha_1} \left(\frac{s_2}{2}\right)^{\alpha_2} \left(\frac{s_3}{3}\right)^{\alpha_3} \dots \left(\frac{s_n}{n}\right)^{\alpha_n},$$

We have also

$$[\mathbf{D}^n \Gamma(x+1)]_{x=0} = \int_0^{\infty} e^{-t} (\log t)^n \, dt$$

therefore the second member of (10) will be equal to this definite integral.

Example 6. Semi-invariants of Thiele.* Let x_i and $f(x_i)$ be given for $i = 0, 1, 2, \ldots$, N-1. Denoting the power-moment of order n of $f(x_i)$ by

$$\mathcal{M}_n = \sum_{i=0}^{N} x_i^n f(x_i)$$

and by λ_n the semi-invariant of degree n of $f(x_i)$ with respect to $x_0, x_1, \ldots, x_{N-1}$ the definition of λ_n is

$$(11) \qquad e^{\lambda_1\omega+\frac{\lambda_2}{2!}\omega^2+\frac{\lambda_3}{3!}\omega^3+\cdots} = \frac{1}{\mathcal{M}_0} \sum_{i=0}^{N} e^{\omega x_i} f(x_i).$$

The expansion of the second member gives

$$(12) \qquad \sum_{n=0}^{\infty} \frac{\mathcal{M}_n \omega^n}{\mathcal{M}_0 n!}.$$

To expand the first member of (11) by aid of formula (6) we denote it by u and put $y = \lambda_1\omega + \frac{1}{2}\lambda_2\omega^2 + \frac{\lambda_3}{3!}\omega^3 + \ldots;$ remarking moreover that

$$[y]_{\omega=0} = 0, \quad [D^s y]_{\omega=0} = \lambda_s \text{ and } \left.\frac{d^\nu u}{dy^\nu}\right)_{y=0} = 1;$$

therefore the expansion of the first member of (11) into powers of ω will give according to (6)

$$(13) \qquad \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \sum \frac{n!}{\alpha_1! \ldots \alpha_n!} \lambda_1^{\alpha_1} \left(\frac{\lambda_2}{2!}\right)^{\alpha_2} \cdots \left(\frac{\lambda_n}{n!}\right)^{\alpha_n}.$$

Since the expansions (12) and (13) must be identical, the coefficients of ω^n in both expressions must be the same; therefore

$$(14) \qquad \frac{\mathcal{M}_n}{\mathcal{M}_0} = \sum \frac{n!}{\alpha_1! \ldots \alpha_n!} \lambda_1^{\alpha_1} \left(\frac{\lambda_2}{2!}\right)^{\alpha_2} \cdots \left(\frac{\lambda_n}{n!}\right)^{\alpha_n}$$

giving the power moments in terms of the semi-invariants.

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* *T.* N. *Thiele*, **Theory of Observations, London, 1903.**
Ragnar **Frisch, Semi-invariants et Moments des Distributions Statistiques. Oslo, 1926.**

Particular cases:

$$\mathcal{M}_1 = \lambda_1 \mathcal{M}_0$$

$$\mathcal{M}_2 = [\lambda_2 + \lambda_1^2] \mathcal{M}_0$$

$$\mathcal{M}_3 = [\lambda_3 + 3\lambda_1\lambda_2 + \lambda_1^3] \mathcal{M}_0$$

$$\mathcal{M}_4 = [\lambda_4 + 4\lambda_1\lambda_3 + 3\lambda_2^2 + 6\lambda_1^2\lambda_2 + \lambda_1^4] \mathcal{M}_0$$

The sum of the numerical coefficients in $\mathcal{M}_n$ is equal to $\Sigma \mathfrak{S}_n^\nu$ from $\nu=1$ to $\nu=n+1$.

To deduce the formula which gives the semi-invariants in terms of the moments, let us now denote by y the second member of **(11)**, whose expansion is given by formula (12) :

$$y = \sum_{n=0}^{\infty} \frac{\mathcal{M}_n \, \omega^n}{\mathcal{M}_0 \, n!}$$

and moreover let us write u $=$ logy; then

$$\frac{d^\nu u}{dy^\nu} = \frac{(-1)^{\nu-1} (V-1)!}{Y^{\nu}}$$

Since for $\omega=0$ we have $y=1$, it follows that

$$\left[\frac{d^\nu u}{dy^\nu}\right]_{\omega=0} = (-1)^{\nu-1} (\nu-1)! \text{ and } [D^s y]_{\omega=0} = \frac{\mathcal{M}_s}{\mathcal{M}_0}.$$

On the otherhand the first member of (1 1), equal to y gives u $=$ logy $= \Sigma\lambda_\nu \omega^\nu/\nu!$ Therefore, by aid of (6) we find

$$\textbf{(15)} \qquad [D^n u]_{\omega=0} = \sum_{\nu=1}^{n+1} \frac{(-1)^{\nu-1}(\nu-1)!}{\mathcal{M}_0^\nu} \; \Sigma \; \frac{n!}{\alpha_1! \ldots \alpha_n!} \mathcal{M}_1^{\alpha_1}$$

$$\left(\frac{\mathcal{M}_2}{2!}\right)^{\alpha_2} \cdots \left(\frac{\mathcal{M}_n}{n!}\right)^{\alpha_n} = \lambda_n.$$

Particular cases

$$\lambda_1 = \frac{\mathcal{M}_1}{\mathcal{M}_0} \qquad\qquad \lambda_2 = \frac{1}{\mathcal{M}_0^2} [\mathcal{M}_0\mathcal{M}_2 - \mathcal{M}_1^2]$$

$$\lambda_3 = \frac{1}{\mathcal{M}_0^3} [\mathcal{M}_0^2\mathcal{M}_3 - 3\mathcal{M}_0\mathcal{M}_1\mathcal{M}_2 + 2\mathcal{M}_1^3]$$

$$\lambda_4 = \frac{1}{\mathcal{M}_0^4} [\mathcal{M}_0^3\mathcal{M}_4 - 4\mathcal{M}_0^2\mathcal{M}_1\mathcal{M}_3 - 3\mathcal{M}_0^2\mathcal{M}_2^2 +$$

$$+ 12\mathcal{M}_0\mathcal{M}_1^2\mathcal{M}_2 - 6\mathcal{M}_1^4].$$

Remark. The sum of the numerical coefficients in the derivative (15) is, according to what has been said concerning formula (6), equal to

$$\sum_{\nu=1}^{n+1} (-1)^{\nu-1} (\nu-1)!\, \mathfrak{S}_n^{\nu} \tag{16}$$

but this is, in consequence of formula (17) § 66, equal to zero.

The semi-invariants are useful in the expansion of functions, since if the origin of the variable x is changed, the semi-invariants, except the first, do not change. If the unit is chosen c times greater, the, semi-invariant λ_n becomes equal to $c^n\lambda_n$. Therefore unity and the origin may be chosen so as to have $\lambda_1=0$ and $\lambda_2=1$; this is a great simplification.

The invariance with respect to the origin is a consequence of the fact that the sum of the numerical coefficients in λ_n is equal to zero, according to formula (16).

§ 73, Expansion of a function into reciprocal factorial series, and into reciprocal power series. Given the function $F(x)$ for $x>0$ if the solution $\varphi(t)$ of the integral equation

$$F(x) = \int_0^1 \varphi(t)\, t^{x-1}\, dt \tag{1}$$

is known, then we may expand *F(x)* into a series of reciprocal factorials. For this purpose $\varphi(t)$ is expanded into a series of powers of $(1-t)$:

$$\varphi(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(1-t)^n}{n!} D^n\varphi(1). \tag{2}$$

Since

$$\int_0^1 (1-t)^n\, t^{x-1}\, dt = B(n+1,x)$$

where the second member *is* a *Beta-function* (§ 24), *we* have

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B(n+1,x)\, D^n\varphi(1). \tag{3}$$

If x and n are integers, then

$$B(n+1,x) = \frac{n!\,(x-1)!}{(n+x)!};$$

therefore

$$(4)\qquad F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)_{n+1}} D^n\varphi(1) =$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)_{-n-1} D^n\varphi(1) D^n\varphi(1).$$

This is the required expansion. If the series is convergent (§ 37) then it may serve for the determination of $\Delta^{-1}F(x)$ and $\Delta^m F(x)$. From (4) we get

$$A\text{-}'F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (x\text{-}1)\text{-},\; D^n\varphi(1) + \varphi(1)F(x-1) + k$$

and

$$\Delta^m F(x) = \sum_{n=0}^{\infty} (-1)^n (-n-1)_m (x-1)_{-n-m-1} D\varphi(1).$$

Remark. Formula (2) of **§ 68** may be written

$$(\text{-}1)''\; (x-1)_{-m-1} = \frac{(-1)^m}{(x+m)_m} = \sum_{n=m}^{\infty} \frac{(-1)^n}{x^{n+1}} \mathfrak{S}_n^m;$$

therefore from (4) we may obtain the expansion of $F(x)$ into a *series* of *reciprocal powers*

$$(5)\qquad F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{x^{n+1}} \sum_{m=1}^{n+1} \mathfrak{S}_n^m D^m\varphi(1) + \frac{\varphi(1)}{x}.$$

If this series is uniformly convergent, then we may determine the indefinite integral of $F(x)$ by integrating term by term. We find

$$\int F(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,x^n} \sum_{m=1}^{n+1} \mathfrak{S}_n^m D^m\varphi(1) + \varphi(1)\log x + k.$$

Moreover then the derivatives of $F(x)$ may be determined too:

$$D^s F(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+s}(n+s)_s}{x^{n+s+1}} \sum_{m=1}^{n+1} \mathfrak{S}_n^m D^m\varphi(1) + \frac{(-1)^s s!\varphi(1)}{x^{s+1}}$$

Example 1. By integration by parts we get

$$\int_0^1 (\log t)^m t^{x-1}\,dt = \frac{(-1)^m m!}{x^{m+1}}$$

[See also *Petit Bois,* Tables **d'Intégrales Indéfinies, p. 144.1**

Therefore to obtain the expansion of $1/x^{m+1}$ in a series of reciprocal factorials, we have to put

$$\varphi(t) = \frac{(-1)^m}{m!} (\log t)^m$$

but in consequence of (7) § 71 we obtain;

$$[\mathrm{D}^n (\log t)^m]_{t=1} = m!\, S_n^m$$

and finally according to (4)

$$\frac{1}{x^{m+1}} = \sum_{n=m}^{\infty} \frac{|S_n^m|}{(x+n)_{n+1}} = \sum_{n=m}^{\infty} |S_n^m|\ (x-1)_{-n-1} .$$

This is identical with formula (3) § 68.

Example 2. *Expansion of* $1/x(x+a)$ into a series of inverse factorials. Let us write

$$\frac{1}{x(x+a)} = \sum_{n=2}^{\infty} \frac{a_n}{(x+n-1)_n}$$

Multiplying both members by x it may be written

$$\frac{1}{x+1+(a-1)} = \sum_{n=2}^{\infty} \frac{a_n}{(x+1+n-2)_{n-1}} .$$

We have

$$\frac{1}{x+a} = \int_0^1 t^{a-1}\, t^x\, dt \quad \text{if} \quad x+a > o\,;$$

hence according to (1) $F(x+1)$ gives

$$\varphi(t) = t^{a-1} \quad \text{and} \quad \mathrm{D}^i \varphi(1) = (a-1)_i$$

therefore

$$\frac{1}{x+a} = \sum_{n=2}^{\infty} \frac{(-1)^n (a-1)_{n-2}}{(x+n-1)_{n-1}} ,$$

Finally multiplying by $1/x$ we get

$$(6) \qquad \frac{1}{x(x+a)} = \sum_{n=2}^{\infty} \frac{(-1)^n (a-1)_{n-2}}{(x+n-1)_n} .$$

This problem has been solved by *Stirling* in another way for $a=-\frac{1}{2}$ (loc. cit. 25. p. 27).

Remark. Formula (6) is useful for the summation of $1/x(x+a)$. For instance

$$\sum_{x=1}^{\infty} \frac{1}{x(x+a)} = \sum_{n=1}^{\infty} \frac{(-1)^n (a-1)_{n-1}}{n.\ n!} = \frac{1}{a} \int_0^1 \frac{(1-t)^a - 1}{t}\, dt.$$

For a--1/z the integral gives 4 log 2.

Remark. There is a simple method for the expansion of $\varphi_m(x)/\psi_n(x)$ into a series of inverse factorials, if $\varphi_m(x)$ and $\psi_n(x)$ are polynomials respectively of degree m and n (where $n > m$). Let us write

$$\frac{\varphi_m(x)}{\psi_n(x)} = \sum_{i=0}^{\infty} \frac{a_i}{(x+i-1)_i}.$$

It is easy to show, multiplying both members by $(x+\nu-1)_\nu$ that $a_\nu = 0$ if v < *n-m.* Multiplying by $(x+n-m-1)_{n-m}$ we have

$$\frac{(x+n-m-1)_{n-m}\,\varphi_m(x)}{\psi_n(x)} = a_{n-m} + \sum_{i=n-m+1}^{\infty} \frac{a_i}{(x+i-1)_{i-n+m}}$$

therefore $a_{,,}$ is equal to the constant obtained by the division figuring in the first member, Denoting the rest by $\omega_{n-m}(x)$ which is of degree n-l, we have

$$\frac{\omega_{n-m}(x)}{\psi_n(x)} = \sum_{i=n-m+1}^{\infty} \frac{a_i}{(x+i-1)_{i-n+m}};$$

multiplying both members by $(x+n-m)$ *we* find

$$\frac{(x+n-m)\ \omega_{n-m}(x)}{\psi_n(x)} = a_{n-m+1} + \sum_{i=n-m+2}^{\infty} \frac{a_i}{(x+i-1)_{i-n+m-1}}.$$

a_{n-m+1} is the constant obtained by division. Denoting the rest (of degree *n-l)* by ω_{n-m+1} (x) we continue in the same manner to obtain step by step any coefficient a_i whatever.

Applying this method to the preceding example, the first division gives

$$\frac{x+1}{x+a} = 1 + \frac{1-a}{x+a}$$

the second

$$(1-a)\ \frac{x+2}{x+a} = 1-a + \frac{(1-a)(2-a)}{x+a},$$

the third

$$(1-a)\ (2-a)\ \frac{x+3}{x+a} = (1-a)\ (2-a) + \frac{(1-a)(2-a)(3-a)}{x+a}$$

and so on. $a_2=1$, $a_3=(1-a)$, $a_4=(1-a)(2-a)$.

§ 74. Expansion of the function $1/y^n$ into a series of powers of x. Suppose that for $x=0$ *we have* $y=1$, *Maclaurin's* theorem gives

$$(1)\qquad \frac{1}{y^n} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!}\left[\mathsf{D}^\nu \frac{1}{y^n}\right]_{x=0}.$$

Since

$$(2)\qquad \frac{1}{y^n} = \frac{1}{[1+(y-1)]^n} = \sum_{i=0}^{\infty} \binom{-n}{i} (y-1)^i$$

we have

$$\mathsf{D}^\nu \frac{1}{y^n} = \sum_{i=0}^{\infty} \sum_{m=0}^{i+1} (-1)^{i+m} \binom{-n}{i} \binom{i}{m} \mathsf{D}^\nu y^m.$$

To determine this value corresponding to $x=0$, let us remark that if $i>\nu$ *then we* have

$$[\mathsf{D}^\nu (y-1)^i]_{y=1} = 0;$$

therefore it will be sufficient to make i vary in the first sum from zero to $\nu+1$. Moreover we may put in the second sum also $\nu+1$ for the upper limit, since for $m>i$ *we have* $\binom{i}{m}=0$, *so* that the additional terms are equal to zero. But if tbe upper limit of the second sum is independent of i then the summation with respect to i can be performed. Let us write therefore

$$\left[\mathsf{D}^\nu \frac{1}{y^n}\right]_{x=0} = \sum_{i=0}^{\nu+1} \sum_{m=0}^{\nu+1} (-1)^{i+m} \binom{-n}{i} \binom{i}{m} [\mathsf{D}^\nu y^m]_{x=0}.$$

The part of the second member containing i may be written

$$\sum_{i=0}^{\nu+1} \binom{i}{m} \binom{n+i-1}{n-1} = \binom{n+m-1}{m} \sum_{i=0}^{\nu+1} \binom{n+i-1}{n+m-1} =$$

$$= \binom{n+m-1}{m} \binom{n+\nu}{n+m} = \binom{n+\nu}{\nu} \binom{\nu}{m} \frac{n}{n+m}.$$

Finally we obtain

$$(3)\qquad \left[\mathsf{D}^\nu \frac{1}{y^n}\right]_{x=0} = n \binom{n+\nu}{\nu} \sum_{m=1}^{\nu+1} \frac{(-1)^m}{n+m} \binom{\nu}{m} [\mathsf{D}^\nu y^m]_{x=0}.$$

Therefore, knowing the derivatives of y^m for $x=0$, that is for $y=1$, ***this formula gives the derivatives of*** $1/y^n$ figuring in the expansion (1) so that the problem is solved.

In each particular case $[D^{\nu}\, y^m]_{y=1}$ is to be determined, which is generally much easier than $[D^{\nu}\, 1/y^n]_y\, I_r$.

Example 1. If the expansion of $x^n/(e^x-1)^n$ is required, then we put $y=(e^x-1)/x$ and the derivatives of y^m are to be determined. It is easy to show that for $x=0$ we have $y=1$. Moreover to determine $D^{\nu} y^m$ let us write

$$x^m y^m = (e^x-1)^m. \tag{4}$$

The $\nu+m$ -th derivative of the first member is given by *Leibnitz*'s theorem (§ 30) :

$$\sum_{i=0}^{m+1} \binom{\nu+m}{i} D^{\nu+m-i} y^m\, D^i x^m.$$

For $x=0$ each term of this sum will vanish except that of $i=m$ *so* that *we* shall have

$$m! \binom{\nu+m}{m} [D^{\nu} y^m]_{x=0}.$$

The second member of (4) may be written according to (5) § 71:

$$(e^x-1)^m = \sum_{i=0}^{\infty} \frac{m!}{(i+m)!}\, \mathfrak{S}_{i+m}^{m}\, x^{i+m}.$$

From this expression it follows that for $x=0$ its $\nu+m$ -th derivative is equal to $m!\, \mathfrak{S}_{m+\nu}^{m}$. Equating this to the preceding result we get

$$[D^{\nu} y^m]_{x=0} = \frac{\mathfrak{S}_{m+\nu}^{m}}{\binom{m+\nu}{\nu}}. \tag{5}$$

Therefore from (3) it follows that

$$\left[D^{\nu}\left(\frac{x}{e^x-1}\right)^n\right]_{x=0} = n\binom{n+\nu}{\nu} \sum_{m=1}^{\nu+1} \frac{(-1)^m}{n+m}\, \frac{\binom{\nu}{m}}{\binom{m+\nu}{\nu}}\, \mathfrak{S}_{m+\nu}^{m}. \tag{6}$$

This can also written in the following form:

$$\left[D^{\nu}\left(\frac{x}{e^x-1}\right)^n\right]_{x=0} = \frac{n\binom{n+\nu}{\nu}}{\binom{2\nu}{\nu}} \sum_{m=1}^{\nu+1} \frac{(-1)^m}{n+m} \binom{2\nu}{\nu-m}\, \mathfrak{S}_{m+\nu}^{m}. \tag{7}$$

Finally by aid of (6) we obtain the required expansion:

$$(8)\qquad \left(\frac{x}{e^x-1}\right)^n = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!}\, n \binom{n+\nu}{\nu} \sum_{m=1}^{\nu+1} \frac{(-1)^m}{n+m} \frac{\binom{\nu}{m}}{\binom{\nu+m}{\nu}} \mathfrak{S}_{m+\nu}^{m} .$$

Sometimes it is possible to determine in another way the coefficient of x^ν in the expansion of $1/y^n$; then equating the result to that given by (3), we may thus obtain interesting relations.

For instance in the case of Example 1 we may write:

$$(9)\qquad \left(\frac{x}{e^x-1}\right)^n = \sum_{\nu=0}^{\infty} \frac{A(n,\nu)}{(n-1)_\nu}\, x^\nu .$$

To determine the coefficients $A(n,\nu)$, we will equate the first derivatives of both members of this equation. We shall find

$$\frac{n}{x}\left[\left(\frac{x}{e^x-1}\right)^n (1-x) - \left(\frac{x}{e^x-1}\right)^{n+1}\right] = \sum_{\nu=1}^{\infty} \frac{\nu A(n,\nu)}{(n-1)_\nu}\, x^{\nu-1}$$

Taking account of (9), the coefficient of $x^{\nu-1}$ will be

$$\frac{nA(n,\nu)}{(n-1)_\nu} - \frac{nA(n,\nu-1)}{(n-1)_{\nu-1}} - \frac{nA(n+1,\nu)}{(n)_\nu} = \frac{\nu A(n,\nu)}{(n-1)_\nu} ;$$

this gives after simplification

$$(9')\qquad A(n+1,\nu) = A(n,\nu) - nA(n,\nu-1).$$

The difference equation giving the *Stirling's* Numbers of the first kind (5, § 50) may be written

$$S_{n+1}^{n+1-\nu} = S_n^{n-\nu} - n\, S_n^{n-\nu+1}$$

therefore these numbers satisfy equation (9') and moreover, since the initial values $A(n,0)=1$ and $S_n^n = 1$ are the same, we conclude that

$$A(n,\nu) = S_n^{n-\nu}$$

so that

$$(10)\qquad \left[\mathrm{D}^\nu \left(\frac{x}{e^x-1}\right)^n\right]_{x=0} = \frac{S_n^{n-\nu}}{\binom{n-1}{\nu}} .$$

Finally, equating the second members of (7) and (10) we obtain after simplification

$$(11)\qquad S_n^{n-\nu} = \sum_{m=0}^{\nu+1} (-1)^m \binom{n+\nu}{n+m} \binom{n+m-1}{n-\nu-1} \mathfrak{S}_{m+\nu}^m$$

which gives *the Stirling* number of the first kind in terms of those **of** the second kind. This is the simplest known expression of the *Stirling* numbers of the first kind, It has been found in another way by **Schlömilch** [Compendium 1895, Band II. **p. 28**].

Remark. *Since* $x/(e^x-1)$ is the generating function of $B_\nu/\nu!$ where B_ν is the ν -th *Bernoulli* number, therefore putting into (7) $n=1$ the first will be equal to B_ν; after simplification we get

$$B_\nu = \sum_{m=1}^{\nu+1} (-1)^m \frac{\binom{\nu+1}{m+1}}{\binom{m+\nu}{\nu}} \mathfrak{S}_{m+\nu}^m$$

In the particular case of $\nu=4$, it will give

$$B_4 = -\frac{10}{5} + \frac{10.31}{15} - \frac{5.301}{35} + \frac{1701}{70} = -\frac{1}{30}.$$

In § 63, we found already a much simpler expression (5) giving the *Bernoulli* numbers by aid of the *Stirling* numbers of the second kind,

§ 75. Changing the origin of the intervals. If the values of the function f(x) are given for $x=a,\ a+h,\ a+2h, \ldots$ and so on, then the differences

$$\Delta_h^n f(a) = \sum_{\nu=0}^{n+1} (-1)^{n-\nu} \binom{n}{\nu} f(a+\nu h)$$

are known, and also the following *Newton* expansion of the function

$$(1)\qquad f(x) = \sum_{n=0}^{\infty} \binom{x-a}{n}_h \frac{\Delta_h^n f(a)}{h^n}.$$

Sometimes it is needed to compute the values of *f(x)* corresponding to $x=c,\ c+h,\ c+2h, \ldots$. This problem often occurs in mathematical statistics. To solve it, the best way is to determine the *Newton* expansion of the function *f(x) in the* form given below:

$$(2)\qquad f(x) = \sum_{i=0}^{\infty} \binom{x-c}{i}_h \frac{\Delta_h^i f(c)}{h^i}.$$

Therefore it is necessary to compute the differences $\underset{h}{\Delta^i} f(c)$. They are deduced from formula (1)

$$f(c) = \sum_{n=0}^{\infty} \binom{c-a}{n}_h \frac{\underset{h}{\Delta^n} f(a)}{h^n}$$

and

$$\underset{h}{\Delta^i} f(c) = \sum_{n=i}^{\infty} \binom{c-a}{n-i}_h \frac{\underset{h}{\Delta^n} f(a)}{h^{n-i}} .$$

Particular case. If $c-a=mh$, where m is an integer, then

$$\underset{h}{\Delta^i} f(c) = \underset{h}{\Delta^i} f(a+mh).$$

Therefore if the table of $f(a)$, $f(a+h)$, $f(a+2h)$, . . . contains also the differences of $f(x)$, then the numbers above will already figure in the table.

§ 76, Changing the length of the interval, Sometimes when a function $f(x)$ is given by its differences in a system where the increment is equal to h, it is necessary to express this function in another system of differences in which the increment is k; that is, the differences of $f(x)$ are required in this system for $x=0$, i. e. $\underset{k}{\Delta^m} f(0)$, for $m=1, 2, 3,$ . . .

This **problem** is identical with the following: Given a table of the function $f(x)$ corresponding to equidistant values of x, the interval of x being equal to h; another table is to be computed in which the increment of x is equal to k.

To solve the problem we first expand $f(x)$ into a *Newton* series with increment h (formula 4 § 23)

$$(1) \qquad f(x) = \sum_{n=0}^{\infty} \binom{x}{n}_h \frac{\underset{h}{\Delta^n} f(0)}{h^n} .$$

Hence it is sufficient to expand $\binom{x}{n}_h$, into a series of $(x)_{m,k}$. For this we will write, according to formula (3) § 55:

$$\binom{x}{n}_h = \frac{1}{n!} \sum_{\nu=1}^{n+1} S_n^\nu x^\nu h^{n-\nu} .$$

Formula (1) § 63 gives

$$x^\nu = \sum_{m=1}^{\nu+1} \mathfrak{S}_\nu^m (x)_{m,k} k^{\nu-m} .$$

Therefore we have

$$\binom{x}{n}_h = \frac{1}{n!} \sum_{\nu=1}^{n+1} \sum_{m=1}^{\nu+1} S_n^\nu \mathfrak{S}_\nu^m h^{n-\nu} k^{\nu-m} (x)_{m,k} .$$

Now we may write the differences of $\binom{x}{n}_h$ in a system of increment k:

$$\Delta_k^i \binom{x}{n}_h = \frac{1}{n!} \sum_{\nu=1}^{n+1} \sum_{m=1}^{\nu+1} S_n^\nu \mathfrak{S}_\nu^m h^{n-\nu} k^{\nu-m+i} (m)_i (x)_{m-i,k} .$$

Putting $x=0$ we have

$$\left[\Delta_k^i \binom{x}{n}_h\right]_{x=0} = \frac{i!}{n!} \sum_{\nu=i}^{n+1} h^{n-\nu} k^\nu S_n^\nu \mathfrak{S}_\nu^i .$$

To abbreviate, let us write $k/h=\omega$, and

(2) $$P(n,i) = \sum_{\nu=i}^{n+1} \omega^\nu S_n^\nu \mathfrak{S}_\nu^i ;$$

then we obtain

$$\left[\Delta_k^i \binom{x}{n}_h\right]_{x=0} = \frac{i!\, h^n}{n!} P(n,i)$$

and therefore

(3) $$\binom{x}{n}_h = \sum_{i=1}^{n+1} \binom{x}{i}_k \frac{i!\, h^n}{n!\, k^i} P(n,i).$$

Finally putting this value into (1) we may determine $\Delta_k^m f(0)$. We find

(4) $$\Delta_k^m f(0) = \sum_{n=0}^{\infty} \frac{m!}{n!} P(n,m) \Delta_h^n f(0)$$

and the required expansion will be

$$f(x) = \sum_{n=0}^{\infty} \binom{x}{m}_k \frac{\Delta_k^m f(0)}{k^m} .$$

Determination of the expression $P(n,m)$. *1.* Let $m=1$. Since $\mathfrak{S}_\nu^1 = 1$, it follows that

$$P(n,1) = \sum_{\nu=1}^{\infty} \omega\; S_n^\nu = (\omega)_n .$$

2. Putting $i=n$ into formula (2) we get

$$P(n,n) = \omega^n.$$

3. Putting $i=n-1$ into (2), since $\mathfrak{S}_n^{n-1} = - S_n^{n-1} = \binom{n}{2}$ we find

$$P(n,n-1) = \binom{n}{2} (\omega)_2\ \omega^{n-2}.$$

It is easy to obtain a difference equation corresponding to the quantity $P(n,m)$. Let us start from

$$\underset{k}{\Delta^m} \binom{x}{n+1}_h = \underset{k}{\Delta^m} \left[\binom{x}{n}_h \frac{x-nh}{n+1} \right].$$

The difference figuring in the first member is deduced from (3) ; to have that in the second member we apply the rule which gives the higher differences of a product; we shall find, after putting $x=0$ into the result obtained,

$$\frac{m!\,h^{n+1}}{(n+1)!} P(n+1,m) = \frac{(mk-nh)}{(n+1)} \frac{m!\,h^n}{n!} P(n,m) +$$
$$\frac{mk}{(n+1)} \cdot \frac{(m-1)!\,h^n}{n!} P(n,m-1);$$

simplified this gives

$$(5) \qquad P(n+1,m) = (m\omega-n)\, P(n,m) + \omega P(n,m-1).$$

From (2) it follows that $P(n,m)=0$ if $m>n$; moreover that $P(n,0)=0$. Therefore, putting $m=n+1$ into (5) we get

$$P(n+1,n+1) = \omega P(n,n)$$

According to (2) we have $P(1,1)=\omega$, hence the solution of this equation is $P(n,n)=\omega^n$. This value has been found directly. In the same manner we could deduce from (5) the values of $P(n,n-1)$ and $P(n,1)$ obtained above. This equation is especially useful for the computation of a table of $P(n,m)$.

Application. **1.** Determination of the differences or of the indefinite sum of $\binom{x}{n}_h$ in a system of differences where the increment' is equal to one. Putting $k=1$ into formula (3) we get

$$\binom{x}{n}_h = \sum_{i=1}^{n+1} \binom{x}{i} \frac{i!\,h^n}{n!} P(n,i)$$

and from this

$$\Delta^m \binom{x}{n}_h = \sum_{i=m}^{n+1} \binom{x}{i-m} \frac{i!\,h^n}{n!} P(n,i)$$

moreover

Table of $P(n,m)$.

$n \backslash m$	1	2	3	4
1	ω			
2	$(\omega)_2$	ω^2		
3	$(\omega)_3$	$3(\omega)_2\omega$	ω^3	
4	$(\omega)_4$	$7(\omega)_2\omega^2-11(\omega)_2\omega$	$6(\omega)_2\omega^2$	ω^4
5	$(\omega)_5$	$15(\omega)_3\omega^2-25(\omega)_3\omega$	$25(\omega)_2\omega^3-35(\omega)_2\omega^2$	$10(\omega)_2\omega^3$
6	$(\omega)_6$	$(\omega)_4(31\omega^2-29\omega)+50(\omega)_3\omega$	$(\omega)_3(90\omega^3-105\omega^2)+15(\omega)_2\omega^2$	$(\omega)_2(65\omega^4-50\omega^3)$

$$\Delta^{-1}\binom{x}{n}_h = \sum_{i=1}^{n+1} \binom{x}{i+1} \frac{i!\,h^n}{n!} P(n,i) + K.$$

Let us remark that in this case $\omega = 1/h$.

2. Expansion of the binomial $\binom{ax}{n}$ into a Newton series with increments equal to one.

$$\binom{ax}{n} = \frac{ax(ax-1)\ldots\ldots(ax-n+1)}{n!} = a^n \binom{x}{n}_6$$

where $h=1/a$. Putting $k=1$ and $\omega=a$ into formula (3) we have

$$\binom{ax}{n} = \sum_{i=1}^{n+1} \binom{x}{i} \frac{i!}{n!} P(n,i).$$

Example. Given $\binom{3x}{3}$. Since $\omega=3$, by aid of the table of $P(n,m)$ *we* get

$$\binom{3x}{3} = \binom{x}{1} + 18\binom{x}{2} + 27\binom{x}{3}.$$

3. We shall see later that *Cotes'* numbers may be determined by aid of the $P(n,m)$ and *the* coefficients of the *Bernoulli* polynomials of the second kind too.

§ 77. **Stirling polynomials.** We have seen (§ 52) that the *Stirling* number of the first kind S_x^{x-m} is a polynomial of x of degree $2m$.

$$\text{(1)} \qquad S_x^{x-m} = \sum_{\nu=0}^{m} C_{m,\nu} \binom{x}{2m-\nu}.$$

According to *Nielsen* [Gammafunktionen, p. 71, Leipzig, 1906], let us call the following expression a *Stirling polynomial:*

$$\text{(2)} \qquad \psi_m(x) = \frac{(-1)^{m+1} S_{x+1}^{x-m}}{(x+1)_{m+2}}.$$

If $x > m$ then in consequence of formula (1) the numerator is a polynomial of x of degree $2m+2$ divisible by $(x+1)_{m+2}$; therefore $\psi_m(x)$ is a polynomial of degree m. Indeed we have

$$\text{(3)} \qquad \psi_m(x) = (-1)^{m-1} \sum_{i=0}^{m+1} \frac{C_{m+1,i}}{(2m+2-i)!} (x-m-1)_{m-i} .$$

Particular cases:

1 $\psi_1(x) = \frac{x}{8} + \frac{1}{12}$

$$_2(x) = \frac{1}{48}\, x(x+1).$$

By aid of the numbers C_{mi} (Table in § 52) we may easily *write* down a *Stirling* polynomial of any degree whatever. From c_{m0} it follows that the coefficient of x^m in $\varphi_m(x)$ is equal to $1/(m+1)2^{m+1}$.

Notable particular values. Putting $x=m+1$ into formula (2) we obtain

$$\psi_m(m+1) = \frac{(-1)^{m+1}}{(m+2)!} S^1_{m+2} = \frac{1}{m+2}.$$

If we put $x=m+2$ into formula (2) we get (p. 148)

$$\psi_m(m+2) = \frac{(-1)^{m+1}}{(m+3)!}\; S^2_{m+3} = \frac{1}{m+3} \sum_{\nu=1}^{m+3} \frac{1}{\nu}.$$

If $x<m+1$ then formula (2) gives $0/0$. But we may use formula (3), which gives

$$\psi_m(m) = \sum_{i=0}^{m+1} (-1)^{i+1}\, C_{m+1,i} \frac{(m-i)!}{(2m+2-i)!}$$

$$\psi_m(1) = \sum_{i=0}^{m+1} (-1)^{i+1} \frac{c_{m+1,i}}{(2m+2-i)_3\; (m-1)!}$$

$$\psi_m(0) = \sum_{i=0}^{m+1} (-1)^{i+1} \frac{C_{m+1,i}}{(2m+2-i)_2\; ml}$$

$$\psi_m(-1) = \sum_{i=0}^{m+1} (-1)^{i+1} \frac{C_{m+1,i}}{(2m+2-i)\,(m+1)!}$$

from 7, § 52

$$\psi_m\ (-2) = \sum_{i=0}^{m+1} (-1)^{i+1} \frac{C_{m+1,i}}{(m+2)!} = \frac{(-1)^m}{(m+2)!}.$$

Deduction of the difference equation of the *Stirling* polynomials. We have seen (4) § 52 tbat tbe numbers $C_{m,i}$ satisfy tbe following difference equation:

$$C_{m+1,i} = -\,(2m+1-i)\,(C_{m,i} + C_{m,i-1}).$$

Multiplying both members by $(-1)^{m-1}\,(x-m)_{m-i}\ /\ (2m+1-i)$ I and summing from $i=0$ to $i=m+1$ we obtain

$$(4) \qquad \sum_{i=0}^{m+1} \frac{(-1)^{m+1} C_{m+1,i}}{(2m+1-i)\ !} (x-m)_{m-i} =$$
$$= \sum_{i=0}^{m+1} (-1)^m \frac{C_{m,i} + C_{m,i-1}}{(2m-i)!} (x-m)_{m-i}.$$

The first member of this equation may be written

$$(-1)^{m+1} \sum_{i=0}^{m+1} \frac{C_{m+1,i}}{(2m+2-i)\ !} [(x+2) + (2m-x-i)] (x-m)_{m-i}$$

but this is, according to (3), equal to

$$(5) \qquad (x+2)\ \psi_m(x+1) + (m-x)\ \psi_m(x).$$

The first term of the second member of (4) is

$$\sum_{i=0}^{m+1} \frac{(-1)^m C_{m,i}}{(2m-i)\ !} (x-m)_{m-i} = (x-2m+1+i)\ \psi_{m-1}(x).$$

The second term may be transformed by putting i+l instead of i and we shall have

$$\sum_{i=0}^{m+1} \frac{(-1)^m C_{m,i}}{(2m-1-i)!} (x-m)_{m-1-i} = (2m-i)\ \psi_{m-1}\ (x)\ .$$

We may write $m+1$ for the upper limit, since for $i=m$ this expression is equal to zero. Therefore the second member of (4) will be equal to

$$(x+1)\psi_{m-1}(x).$$

Equating this to the value (5) of the first member obtained before we get the difference equation of the ***Stirling polynomials:***

$$(6) \qquad (x+1)\ \psi_{m-1}\ (x) = (m-x)\ \psi_m(x) + (x+2)\ \psi_m(x+1).$$

Starting from y,,(x) $=\frac{1}{2}$, we may determine by aid of this equation the ***Stirling*** polynomials step by step,

For instance, putting ψ_1 (x) $=a_0+a_1x$ we have

$$\tfrac{1}{2}(x+1) - (1-x)(a_0+a_1x) - (x+2)(a_0+a_1+a_1x) = 0.$$

Since this is an identity, the coefficients of x^i must be equal to zero for every i. This gives $6a_0+4a_1=1$ and $8a_1=1$ so that

$$\psi_1(x) = \frac{x}{8} + \frac{1}{12}.$$

Continuing in the same manner we could obtain the polynomial of any degree.

Equation (6) has been found by ***Nielsen*** in another way. ***Nielsen*** defined ***the Stirling*** polynomials by giving their generating function with respect to m:

$$(7) \qquad \frac{t^x}{(x+1)\,(1-e^{-t})^{x+1}} - \frac{1}{(x+1)t} = \sum_{m=0}^{\infty} \psi_m(x)\; t^m.$$

We will show that this definition leads to the same polynomial $\psi_m(x)$ as our definition (2). Indeed, multiplying both members of equation (7) by $(x+1)t^{-x}$ it becomes:

$$(8) \qquad \frac{1}{(1-e^{-t})^{x+1}} - \frac{1}{t^{x+1}} = (x+1) \sum_{m=0}^{\infty} \psi_m(x)\; t^{m-x}.$$

The derivative of this expression with respect to t divided by $x+1$ will be

$$(9) \qquad \frac{-e^{-t}}{(1-e^{-t})^{x+2}} + \frac{1}{t^{x+2}} = \sum_{m=0}^{\infty} (m-x)\;\; \psi_m(x)\; t^{m-x-1}.$$

Writing into (8) $x+1$ instead of x and adding the result to (9) we find

$$(10) \qquad \frac{1}{(1-e^{-t})^{x+1}} = \sum_{m=0}^{\infty} [(x+2)\psi_m(x+1) + (m-x)\psi_m(x)]\, t^{m-x-1}.$$

Finally equating the coefficients of t^{m-x-1} in the expressions (8) and (10) we find the difference equation (6) of ***the Stirling*** polynomials obtained before. Moreover from (7) it follows directly that

$$\psi_0(x) = \tfrac{1}{2}.$$

Therefore the two definitions lead to the same results.

Putting $x=0$ into the generating function (7) we have

$$\frac{1}{1-e^{-t}} - \frac{1}{t} = \sum_{m=0}^{\infty} \psi_m(0)\; t^m.$$

Multiplying this equation by t, and then writing in it $-t$ instead of t, it will become

$$\frac{t}{e^t-1} = 1 + \sum_{m=0}^{\infty} (-1)^{m+1}\; \psi_m(0)\; t^{m+1}.$$

We shall see later that the first member of this equation is the generating function of $B_m/m!$, where B_m stands for the m-**th** ***Bernoulli*** number; therefore we have

$$\psi_m(0) = (-1)^{m+1}\frac{B_{m+1}}{(m+1)!} = \sum_{i=0}^{m+1} \frac{(-1)^{i+1}\, C_{m+1,i}}{m!\ (2m+2-i)_2}$$

Since $B_{2n+1} = 0$, if $n>0$ hence we conclude,

$$(11) \qquad \psi_{2n}(0) = 0 \quad \text{and} \quad \psi_{2n-1}(0) = \frac{B_{2n}}{(2n)!} = a_{2n}.$$

Writing in the generating function (7) $-t$ instead of t and $-x-1$ instead of x, we obtain

$$\frac{(-1)^x\,(1-e^{-t})^x}{x\ t^{x+1}} - \frac{1}{xt} = \sum_{m=0}^{\infty} (-1)^m\ \psi_m(-x-1)\ t^m.$$

Now multiplying this equation by $x\ t^{x+1}$ *we* get

$$(e^t-1)^x - t^x = x \sum_{m=0}^{\infty} (-1)^m\ \psi_m(-x-1)\ t^{m+x+1}.$$

But we have seen [formula (5) § 71] that

$$(e^t-1)^x = \sum_{n=x}^{\infty} \frac{x!}{n!}\, \mathfrak{S}_n^x\ t^n;$$

therefore

$$\sum_{n=x+1}^{\infty} \frac{x!}{n!}\, \mathfrak{S}_n^x\ t^n = x \sum_{m=0}^{\infty}\ (-1)^m\ \psi_m(-x-1)\ t^{m+x+1}.$$

From this we conclude finally that

$$\frac{x!}{n!}\, \mathfrak{S}_n^x = (-1)^{n-x-1}\ x\psi_{n-x-1}\ (-x-1).$$

This may be transformed by putting $n-x-1=m$; we obtain

$$\frac{(-1)^m}{(n)_{m+2}}\, \mathfrak{S}_n^{n-m-1} = \psi_m(m-n)$$

and writing $x+1$ instead of n it follows for $x>m$ that

$$(12) \qquad \psi_m(m-x-1) = -\frac{(-1)^m\, \mathfrak{S}_{x+1}^{x-m}}{(x+1)_{m+2}}.$$

We have seen that the *Stirling* number $\mathfrak{S}_{x+1}^{x-m}$ is a polynomial of degree $2m+2$ of x [formula (7) § 58]. By aid of this formula we get

$$(13) \qquad \psi_m(m-x-1) = \sum_{i=0}^{m+1} \frac{(-1)^m\ \overline{C}_{m+1,i}}{(2m+2-i)\ !}\,(x-m-1)_{m-i}.$$

If $x \leqq m$ then formula (12) gives O/O but (13) may be used even in these cases.

Particular values. From (13) we deduce

$$\psi_m(0) = \sum_{i=0}^{m+1} \frac{(-1)^i \; \overline{C}_{m+1,i}}{(2m+2-i)\,!}(m+1-i)\,!$$

Hence from (11) it follows if $m>1$,

$$B_m = \sum_{i=0}^{m} \frac{(-1)^i \overline{C}_{m,i}}{\binom{2m-i}{m}} = m \sum_{i=0}^{m} \frac{(-1)^{i+1} C_{m,i}}{(2m-i)_2} \,. \tag{14}$$

This is an expression for the *Bernoulli* numbers in terms of the numbers $\overline{C}_{m,i}$ respectively of $C_{m,i}$.

Putting $x=m+1$ into (13), since $(0)_{m-i}$ is equal to zero for every value of i except for $i=m$, for which it is equal to one, and noting that $\overline{C}_{m+1,m} = 1$, we have $\psi_m(-2) = (-1)^m/(m+2)\,!$ obtained before.

Limits of the Stirling polynomials. Writing in (2) $n+x$ instead of x and n instead of m we have

$$\psi_n(n+x) = \frac{(-1)^{n+1}\,(x-1)\,!}{(n+x+1)\,!}\; S^x_{n+x+1} \,.$$

Since according to formula (4) § 54 we have.

$$\lim_{n=\infty} \frac{S^x_{n+x+1}}{(n+x+1)\,!} = 0$$

if $x>0$

$$\lim_{n=\infty} \psi_n(n+x) = 0.$$

Putting into (12) $x+m$ instead of x we obtain

$$\psi_m(-x-1) = (-1)^m (x-1)\,! \, \frac{\mathfrak{S}^x_{m+x+1}}{(m+x+1)\,!} \,.$$

From formula (2) § 59 it follows that

$$\lim_{m=\infty} \frac{\mathfrak{S}^x_{m+x+1}}{(m+x+1)\,!} = 0 \,.$$

Hence if $x>0$ we conclude that

$$\lim_{m=\infty} \psi_m(-x-1) = 0.$$

CHAPTER V.

BERNOULLI POLYNOMIALS AND NUMBERS.

§ 78. Bernoulli polynomials of the first kind. Denoting the *Bernoulli* polynomial of tbe first kind of degree n by $\varphi_n(x)$, let its definition be[26]

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[26]) Different authors have given different definitions of the Bernoulli polynomials. Our definition is that of Nielsen [Traité des Nombree de Bernoulli, Paris, 1923, p. 40]. He denoted the polynomial by $B_n(x)$.

Seliwanoff [Differenzenrechnung, Leipzig 1904, p. 49 and Encycl. des Sciences Mathem. Vol. I. 20, p 111] gave the following definition for the case of integer values of x:

$$\overline{\varphi}_n(x) = \sum_{m=0}^{x} \frac{m^{n-1}}{(n-1)!}$$

This differs from our definition only by a constant.

In *Saalschütz's* Bernoulli'sche Zahlen [Berlin, 1893, p. 91] the definition is for integer values of x the following

$$\varphi\ (x,n) = \sum_{m=0}^{x} m^{n-1}.$$

This definition differs from ours by a factor equal to (n-l)! and by a constant,

Nörlund [Differenzrechnung, Berlin, 1924, p 19] defined these polynomials by

$$\Delta B_n(x) = nx^{n-1}$$

$$\mathrm{D} B_n(x) = nB_{n-1}(x)$$

so as to have

$$\sum_{m=0}^{x} m^{n-1} = \frac{1}{n}[B_n(x) - B_n]$$

where B_n is the n-th Bernoulli number, This definition differs from ours only by the factor (n-l)! .

Steffensen [Interpolation, London, 1927, p. 119] uses the same definition as *Nörlund;* and so does *Pascal* [Repertorium I., p. 1217].

E. Lindelöf [Calcul des Résidus, Paris, 1905, p. 34] introduces the polynomial "$\varphi_n(x)$" which corresponds to ours by the relation

$$"\varphi_n(x)" = n!\,\varphi_n(x).$$

$$(1) \qquad \Delta\varphi_n(x) = \frac{x^{n-1}}{(n-1)!}$$

From this it follows that

$$\varphi_n(x) = \Delta^{-1}\frac{x^{n-1}}{(n-1)!} + k.$$

Hence the definition (1) is not a complete one; there still remains an arbitrary constant to dispose of. Let *us* write *the Bernoulli* polynomial of degree n in the following way:

$$(2) \qquad \varphi_n(x) = \sum_{m=0}^{n+1} a_m \frac{x^{n-m}}{(n-m)!}.$$

By derivation we obtain from (1)

$$D\Delta\,\varphi_n(x) = \frac{x^{n-2}}{(n-2)!}$$

but the second member is, in consequence of (1), equal to $\Delta\varphi_{n-1}(x)$ so that

$$\text{AD}\ \varphi_n(x) = \Delta\,\varphi_{n-1}(x)\ .$$

Performing the operation Δ^{-1} on both members of this equation, we may dispose of the arbitrary constant, which enters by this operation, so as to have

$$(3) \qquad D\,\varphi_n(x) = \varphi_{n-1}(x);$$

then the *Bernoulli* polynomial φ_n (x) is, by equations (1) and (3), completely defined.

In § 22 a class of functions important in Mathematical Analysis has been mentioned in which

$$D\,F_n(x) = F_{n-1}(x)$$

for $n=1, 2, 3, \ldots$ According to *(3) the Bernoulli* polynomials belong to this class of functions.

Moreover if $F_n(x)$ is a polynomial of degree n belonging to this class

$$F_n(x) = a_0\frac{x^n}{n!} + a_1\frac{x^{n-1}}{(n-1)!} + \ldots + a_n.$$

% have seen that the coefficients a_i are independent of the degree n of the polynomial, so that they may be calculated once for all.

To obtain them we put $n=1$ into formula (1) and get

$$\Delta \varphi_1(x) = 1;$$

therefore $\varphi_1(x) = x + a_1$ hence $a_0 = 1$. Moreover if $n > 1$, then putting into (1) $x=0$ we find

$$\Delta \varphi_n(0) = 0 .$$

On the other hand from (2) we deduce

$$\Delta \varphi_n(0) = \sum_{m=0}^{n} \frac{a_m}{(n-m)!} [\Delta x^{n-m}]_{x=0} .$$

But

$$[(x+1)^{n-m} - x^{n-m}]_{x=0} = 1$$

consequently we find

$$\sum_{m=0}^{n} \frac{a_m}{(n-m)!} = 0. \tag{4}$$

By aid of this equation we may determine, starting from $a_0=1$, step by step, the coefficients a_m. For instance:

$\frac{a_0}{2!} + \frac{a_1}{1!} = 0$ gives $a_1 = -1/2$

$\frac{a_0}{3!} + \frac{a_1}{2!} + \frac{a_2}{1!} = 0$ $a_2 = 1/12$

$\frac{a_0}{4!} + \frac{a_1}{3!} + \frac{a_2}{2!} + \frac{a_3}{1!} = 0$ $a_3 = 0$

and so on.

Table of the numbers a_m.

$a_0 = 1$	$a_6 = 1/30240$
$a_1 = -1/2$	$a_8 = -1/1209600$
$a_2 = 1/12$	$a_{10} = 1/47900160$
$a_4 = -1/720$	$a_{12} = -691/1307674368000$

or to ten decimals,

$a_2 = 0{\cdot}08333\ 33333$	$a_8 = -0{\cdot}00000\ 08267$
$a_4 = -0{\cdot}00138\ 88889$	$a_{10} = 0{\cdot}00000\ 00209$
$a_6 = 0{\cdot}00003\ 30688$	$a_{12} = -0{\cdot}00000\ 00005$

Remark. We found $a_i=0$ for $i=3, 5, 7, 9, 11$. Later we shall see that $a_{2n+1}=0$ if $n>0$.

Writing $a_m=B_m/m!$, equation (2) will be

$$(5) \qquad \varphi_n(x) = \frac{1}{n!} \sum_{m=0}^{n+1} \binom{n}{m} B_m x^{n-m}$$

where the B_m are the Bernoulli numbers.

Equation (5) may be written in a symbolical manner:

$$(6) \qquad \varphi_n(x) = \frac{1}{n!} (x+B)^n.$$

In the expansion of the second member B_m is to be put instead of B^m.

Since $\Delta \varphi_n(0) = 0$ if $n>1$, hence from (6) there follows the symbolical equation giving step by step the *Bernoulli* numbers:

$$(7) \qquad (1+B)^n - B_n = 0.$$

This equation is identical with (4). Starting from $B_0=a_0=1$ it gives:

$1 + 2B_1 = 0$	hence	$BI = -½$
$1 + 3B_1 + 3B_2 = 0$		$B_2 = 1/6$
$1 + 4B_1 + 6B_2 + 4B_3 = 0$		$B_3 = 0$

and so on.

In the Table below we see that the *Bernoulli* numbers increase rapidly with *n* whereas the coefficients $a_{,,,}$ decrease rapidly.

The Bernoulli numbers are very important in Mathematical Analysis, especially when dealing with expansions.[27]

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[27] The *Bernoulli* numbers were first introduced by *Jacob Bernoulli* [Ars Conjectandi, Basileae, 1713, p. 97]. They have been denoted differently by later authors. Our notation is that used by *Nörlund* [Differenzenrechnung, p. 18], *Steffensen* [Interpolation, p. 120], *Pascal* [Repertorium, I. p. 121]. There is another notation in use, in which the *Bernoulli* numbers are considered as being positive, and in which B_n corresponds to our $|B_{2n}|$. This notation is found in *Saalschütz* [Bernoullische Zahlen, p. 4], in *Hagen* [Synopsis *I,* p. 91], *in Seliwanoff* [Differenzenrechnung, p. 45 in and Encycl. des Sciences Mathematiques I. 20. p. 111] in *Nielsen* [Nombres de Bernoulli, p. 42] in *E. Lindelöf* [Calcul des Residue, Paris, 1985, p. 33] and in *G. Peano* [Formulaire Mathématique, Paris, 1901, p. 190]. In the last work we find an extensive table of these numbers, up to B_{60} in our notation,

Table of the Bernoulli Numbers.

B_1 = —½
B_2 = 1/6
$B,$ = —1/30
B_6 = 1/42
B_8 = —1/30
B_{10} = 5/66
B_{12} = —691/2730
B_{14} = 7/6
B_{16} = —3617/510
B_{18} = 43867/798
B_{20} - -17461 1/330
B_{22} = 8545131138
B_{24} = -23636409112730
B_{26} = 855310316
B_{28} = -237494610291870
B_{30} = 8615841276005/14322
B_{32} = —7709321041217/510
B_{34} = 257768785836716
B_{36} = —26315271553053477373/1919190
B_{38} = 292999391384155916
B_{40} = —261082718496449122051/13530
B_{42} = 1520097643918070802691/1806
B_{44} = —27833269579301024235023/690
B_{46} = 596451111593912163277961/282
B_{48} = —5609403368997817686249127547/46410
B_{50} = 495057205241079648212477525/66
B_{52} = —801165718135489957347924991853/1590
B_{54} = 29149963634884862421418123812691/798
B_{56} = —2479392929313226753685415739663229/870
B_{58} = 84483613348880041862046775994036021/354
B_{60} = —1215233140483755572040304994079820246041491/ 56786730

Bernoulli Numbers. Table IL

B_1 =	—0·5
B_2 =	0·16666666 . . .
B_4 =	—0·03333333 . . . . .
B_6 =	0·02389523895 . . . .
B_8 =	-0'03333333. , . ,
B_{10} =	0'075757575 . . , . .
B_{12} =	-0.25311355311355.. . .
B_{14} =	1'166666666.. . .
B_{16} =	-7'0921568627 450980392
B_{18} =	54'9711779448 62155388
B_{20} =	-529'124242424 . . .
B_{22} =	6192'1231884057 898550
B_{24} =	-86580'25311355311355.. ,
B_{26} =	1425517'1666666 , , ,
B_{28} =	-27298231'0678160919 54
B_{30} =	601580873'9006423683 8
B_{32} =	-15116315767'092156862 .
B_{34} =	429614643061·1666 . .
B_{36} =	-13711655205088'332772
B_{33} =	488332318973593'16666. .

Expansion of the Bernoulli polynomial into a series of factor*ials.* **Replacing in formula (I), the power x^{n-1} by factorials, using formula (2) § 58 we obtain**

$$\Delta \varphi_n(x) = \frac{1}{(n-1)!} \sum_{m=0}^{n} \mathfrak{S}_{n-1}^{m} (x)_m .$$

Since

$$\Delta^{-1} (x)_m = \frac{(x)_{m+1}}{m+1} + k$$

we get, performing the operation Δ^{-1} on both members,

$$(8) \qquad \varphi_n(x) = \frac{1}{(n-1)!} \sum_{m=0}^{n} \frac{\mathfrak{S}_{n-1}^{m}}{m+1} (x)_{m+1} + a_n .$$

The constant k, which entered by this operation, has been disposed of, so as to have $\varphi_n(0) = a_n$.

This is the expansion of the *Bernoulli* polynomial into a factorial series. It shows immediately that

$$(9) \qquad \Delta^{\mu}\varphi_n(0) = \frac{(\mu-1)!}{(n-1)!}\mathfrak{S}_{n-1}^{\mu-1}$$

moreover that

$$\varphi_n(0) = \varphi_n(1) = a_n \qquad \varphi_n(2) = a_n + \frac{1}{(n-1)!}$$

and so on.

Formula (8) permits us to determine. the coefficients a,,, and therefore also the numbers B_m in terms of *the Stirling* numbers. For this, let us replace in the second member of (8) the factorial $(x)_{m+1}$ by its expansion into a power series (3, § 50); we get

$$\varphi_n(x) = \frac{1}{(n-1)!}\sum_{m=0}^{n}\frac{\mathfrak{S}_{n-1}^m}{m+1}\sum_{i=1}^{m+2} S_{m+1}^i x^i + a_n .$$

Since the coefficient of x^i in the first member is equal to $a_{n-i}/i!$ and $a_{n-i} = B_{n-i}/(n-i)!$,

$$B_{n-i} = \frac{n}{\binom{n}{i}}\sum_{m=i-1}^{n}\frac{1}{m+1} S_{m+1}^i \mathfrak{S}_{n-1}^m .$$

This may be simplified by putting $i=1$, and writing $n+1$ instead of n. The required expression will be

$$(10) \qquad n!\, a_n = B_n = \sum_{m=1}^{n+1}\frac{(-1)^m m!}{m+1}\mathfrak{S}_n^m .$$

Example. Let us determine B_4 by aid of this formula:

$$B, = -\frac{1}{2} + \frac{2}{3}.7 - \frac{6}{4}.6 + \frac{24}{5} = -\frac{1}{30} .$$

We may transform formula (10) by putting into it the value of $\mathfrak{S}_n^m$ given by formula (3) § 58:

$$B_n = \sum_{m=1}^{n+1}\sum_{i=1}^{m+1}\frac{(-1)^i}{m+1}\binom{m}{i} i^n .$$

§ 79. Particular cases of the Bernoulli polynomials. From the table of the coefficients *a,,,* we deduce

$$\varphi_1(x) = x - \frac{1}{2}$$

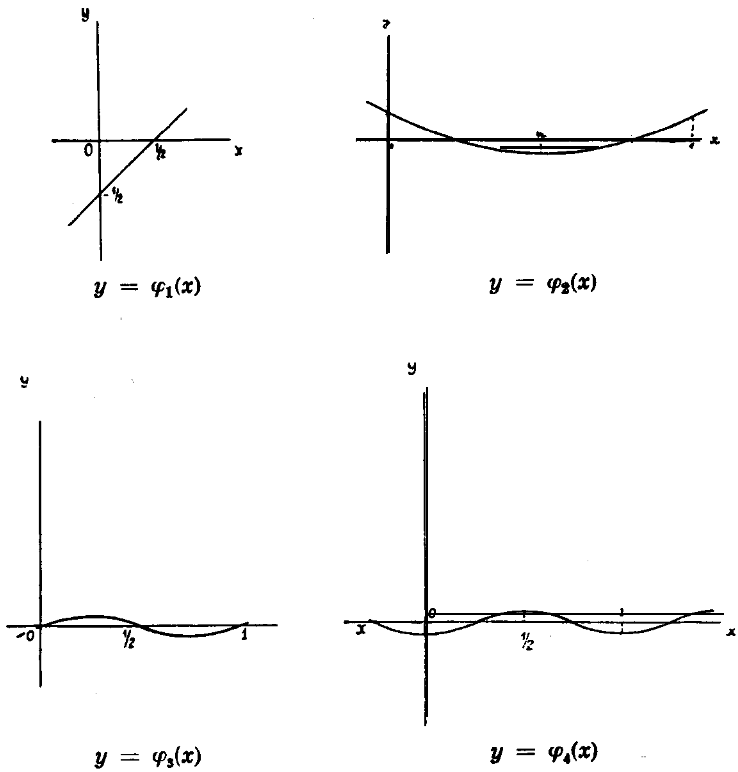
$$\varphi_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$$

$$\varphi_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x$$

$$\varphi_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}.$$

and so on. (See Figure **1.)**

Figure 1.



Particular values of the polynomials. We obtained already if $n > 1$,

$$\varphi_n(0) = a_n \qquad \varphi_{2n-1}(0) = 0$$

$$\varphi_n(1) = a_n \qquad \varphi_n(2) = a_n + \frac{1}{(n-1)!}.$$

Moreover

$$\Delta\varphi_n(-1) = \frac{(-1)^{n-1}}{(n-1)!}; \text{ hence } \quad \varphi_n(-1) = a_n + \frac{(-1)^n}{(n-1)!}.$$

§ 80. Symmetry of the Bernoulli polynomials. From the definition of these polynomials it follows that

$$\Delta\varphi_{2n}(x) = \varphi_{2n}(x+1) - \varphi_{2n}(x) = \frac{x^{2n-1}}{(2n-1)!}.$$

Putting into this equation $-x$ instead of x we get

$$\varphi_{2n}(1-x) - \varphi_{2n}(-x) = -\frac{x^{2n-1}}{(2n-1)!}$$

but the first member is equal to $-\Delta\varphi_{2n}(1-x)$; hence we conclude that

$$\Delta\varphi_{2n}(1-x) = \Delta\varphi_{2n}(x).$$

Let us sum this expression from $x=0$ to $x=z$; since the indefinite sum of the first member is $\varphi_{2n}(1-x)$ and that of the second member $\varphi_{2n}(x)$ we find after putting into them the values of the limits:

$$\varphi_{2n}(1-z) - \varphi_{2n}(1) = \varphi_{2n}(z) - \varphi_{2n}(0).$$

We have seen that $\varphi_m(0) = \varphi_m(1)$, if $m > 1$; hence

$$(1) \qquad \varphi_{2n}(1-z) = \varphi_{2n}(z).$$

Formula (1) has been demonstrated only for integer values of x, but this expression is a polynomial of degree $2n-1$, and it is satisfied for more than $2n-1$ values of x, hence it is an identity, and is therefore satisfied by any value whatever of x.

This is the expression of the symmetry of the polynomials of even degree. This can be written in another way; putting $z = \frac{1}{2} + x$ we get

$$\varphi_{2n}(\tfrac{1}{2}+x) = \varphi_{2n}(\tfrac{1}{2}-x). \quad (2)$$

By derivation this gives, according to (3) § 78:

$$\varphi_{2n-1}(\tfrac{1}{2}+x) = -\varphi_{2n-1}(\tfrac{1}{2}-x). \quad (3)$$

Hence the polynomials of odd degree are symmetrical to the point 'of coordinates $x=\tfrac{1}{2}$, $y=0$. Putting $x=0$ into (3) we have

$$\varphi_{2n-1}(\tfrac{1}{2}) = -\varphi_{2n-1}(\tfrac{1}{2}) = 0.$$

Putting $x=\tfrac{1}{2}$ into (3) we get $\varphi_{2n-1}(1) = -\varphi_{2n-1}(0)$, but we have seen that if $m>1$ then $\varphi_m(1)=\varphi_m(0)$; therefore we conclude that

$$\varphi_{2n-1}(0) = 0 \text{ or } a,,, = 0 \text{ if } n > 1. \quad (4)$$

Roots of the polynomials. We have already seen that the equation $\varphi_{2n+1}(x) = 0$ has three roots if $n>1$, namely $x=0$, $x=\tfrac{1}{2}$ and $x=1$; hence the polynomial of odd degree is divisible by x(x-1) (2x-l) if $n \geqq 1$.

We shall show now that it cannot have more than three roots in the interval 0 6 x $\leqq$ 1; that is, more than one in the interval $0 < x < 1$. If it had at least four in the first interval, then its derivative should have at least three in the interval $0 < x < 1$ and its second derivative at least two. But

$$D^2 \varphi_{2n+1}(x) = \varphi_{2n-1}(x).$$

Hence we conclude that if $\varphi_{2n+1}(x) = 0$ had four roots in the interval 06 x $\leqq$ 1, then $\varphi_{2n-1}(x) = 0$ would have also at least four in this interval, and so on; $\varphi_3(x) = 0$ too, which is impossible, since $\varphi_3(x)$ is only of the third degree.

Summing up, we state that $\varphi_{2n+1}(x) = 0$ has three roots in the interval $0 \leqq x \leqq 1$ and only three (if $n>0$).

$$D\, \varphi_{2n}(x) = \varphi_{2n-1}(x)$$

and the polynomial of the second member has no roots in the interval $0 < x < \tfrac{1}{2}$; therefore $\varphi_{2n}(x)$ cannot have more than one root in the interval $0 \leqq x \leqq \tfrac{1}{2}$. But

$$D\, \varphi_{2n+1}(x) = \varphi_{2n}(x)$$

and we have seen that $\varphi_{2n+1}(x)$ is equal to zero for $x=0$ and $x=\tfrac{1}{2}$; therefore $\varphi_{2n}(x) = 0$ must have at least one root in the interval $0<x<\frac{1}{2}$, therefore $\varphi_{2n}(0)$ and $\varphi_{2n}(\frac{1}{2})$ are both different from zero, so that $\varphi_{2n}(0)=a_{2n} \neq 0$. Finally $\varphi_{2n}(x)=0$ has one root and only one in the interval $0 \leq x \leq \frac{1}{2}$. In consequence of the symmetry, $\varphi_{2n}(x)=0$ has another, root in the interval $\frac{1}{2}<x<1$.

***Extrema** of fhe* **polynomials.** If $n>1$, then $\varphi_{2n-1}(x)$ is equal to zero for $x=0, \frac{1}{2}, 1$; therefore $\varphi_{2n}(x)$ will have extrema at these three places. $\varphi_{2n}(x)$ has only one at $x=\frac{1}{2}$.

If $n>0$ then $\varphi_{2n+1}(x)$ has two extrema, one in the interval $0<x<\frac{1}{2}$ and one in $\frac{1}{2}<x<1$; $\varphi_1(x)$ has none.

Sign of the numbers a,,, and B,. Let us suppose that $a_{2n}>0$; then in consequence of $\varphi_{2n}(0)=a_{2n}$ and of the fact that $\varphi_{2n}(x)=0$ has a root in the interval $0<x<\frac{1}{2}$ without having an extremum in it, we conclude that $\varphi_{2n}(x)$ must be decreasing in the vicinity of $x=0$. Therefore we must have

$$\mathrm{D}\,\varphi_{2n}(x)=\varphi_{2n-1}(x)<0$$

but if x is small, the sign of $\varphi_{2n-1}(x)$ is equal to that of $a_{2n-2}x$ therefore we have

$$a_{2n-2}<0.$$

Had we supposed that $a_{2n}<0$, then we should have found that $\varphi_{2n}(x)$ must be increasing in the vicinity of $x=0$ and consequently we should have

$$a_{2n-2}>0.$$

Finally summing up, we always have

$$a_{2n}\,a_{2n-2}<0$$

that is, the aumbers a_{2n} and also $B_{2n}=(2n)!\,a_{2n}$ are alternately positive and negative with increasing n. Since a, is positive, we have

$$a_{4n}<0 \quad \text{and} \quad a_{4n-2}>0$$

or

$$B_{4n}<0 \quad \text{and} \quad B_{4n-2}>0.$$

§ 81. Operations performed on the Bernoulli polynomials. *Derivatives* of the polynomial. According to (3) § 78, we have

$$\mathrm{D}^m\varphi_n(x)=\varphi_{n-m}(x).$$

From this we deduce the *integral* of the polynomial

$$\int \varphi_n(x)\, dx = \varphi_{n+1}(x) + k$$

and

(1) $$\int_0^1 \varphi_n(x)\, dx = 0 \quad \text{if} \quad n > 0.$$

The difference of $\varphi_n(x)$ may he expressed by a definite integral:

$$\int_z^{z+1} \varphi_n(x) dx = \varphi_{n+1}(z+1) - \varphi_{n+1}(z) = \Delta\, \varphi_{n+1}(z) = \frac{z^n}{n!};$$

therefore if u is an integer $\Sigma\, z^n$ may also be obtained by the integral

$$\int_0^u \varphi_n(x)\, dx = \frac{1}{n!} \sum_{z=0}^{u} z^n = \varphi_{n+1}(u) - \varphi_{n+1}(0).$$

Differences of the polynomial. According to (1) § 78

$$\Delta\, \varphi_n(x) = \frac{x^{n-1}}{(n-1)!},$$

and from this by aid of (2) § 58

$$\Delta^m \varphi_n(x) = \frac{1}{(n-1)!} \sum_{i=m}^{n} (i)_{m-1}\, \mathfrak{S}_{n-1}^{i}\, (x)_{i-m+1};$$

moreover

$$\Delta^m \varphi_n(0) = \frac{(m-l)\,!}{(n-1)!}\, \mathfrak{S}_{n-1}^{m-1}.$$

Sum of the polynomial. Applying the method of summation by parts (*§ 34)* we obtain

$$\text{A-'}\, \varphi_n(x) = x\, \varphi_n(x) - \text{A-'}\, (x+1) \frac{x^{n-1}}{(n-1)!}.$$

Hence

(2) $$\Delta^{-1} \varphi_n(x) = (x-1)\, \varphi_n(x) - n\, \varphi_{n+1}(x) + k.$$

Mean of the polynomial. In § 6 we have seen that $\mathbf{M} = 1 + \frac{1}{2}\Delta;$ therefore

$$\mathbf{M}\, \varphi_n(x) = \varphi_n(x) + \frac{x^{n-1}}{2(n-1)!}.$$

From this we obtain, applying the ***inverse operation of the mean,***

(3) $$\mathbf{M}^{-1} \varphi_n(x) = \varphi_n(x) - \tfrac{1}{2} E_{n-1}(x)$$

where, as we shall see, in § 100, $E_{n-1}(x)$ is the *Euler* polynomial of degree u-l.

§ *82,* Expansion of the Bernoulli polynomial into a Fourier series. Limits. Sum of reciprocal power series,

If y is a periodic function of period equal to one, with limited total fluctuation (or, *a fortiori,* if it has a limited derivative), then it may be expanded into a *Fourier* series (§ 145), such as

$$(1) \qquad y = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos 2\pi mx + \sum_{m=1}^{\infty} \beta_m \sin 2\pi mx$$

where

$$(2) \qquad a_m = 2\int_0^1 y \cos 2\pi mx\, dx\,; \quad \beta_m = 2\int_0^1 y \sin 2\pi mx\, dx.$$

Hence we conclude that $\varphi_{2n}(x)$ may be expanded into a Fourier series between zero and one. By aid of formulae (2) we obtain

$$a_0 = 2\int_0^1 \varphi_{2n}(x)\, dx = \varphi_{2n+1}(1) - \varphi_{2n+1}(0) = 0$$

if $n > 0$. Moreover we find by integration by parts that

$$\tfrac{1}{2}a_m = \int_0^1 \varphi_{2n}(x) \cos 2\pi mx\, dx = \left[\frac{\sin 2\pi mx}{2\pi m}\varphi_{2n}(x)\right]_0^1 -$$

$$- \int_0^1 \frac{\sin 2\pi mx}{2\pi m}\varphi_{2n-1}(x)\, dx.$$

The quantity in the brackets is equal to zero at both limits, and the integral in the second member becomes, after a second integration by parts:

$$\tfrac{1}{2}a_m = \left[\frac{\cos 2\pi mx}{(2\pi m)^2}\varphi_{2n-1}(x)\right]_0^1 - \int_0^1 \frac{\cos 2\pi mx}{(2nm)^2} \varphi_{2n-2}(x)\, dx.$$

The quantity in the brackets has the same value for $x=0$ and $x=1$, if $n>1$, therefore the corresponding difference is equal to zero.

We conclude that the operation of integration by parts performed twice, gives an expression similar to the initial; only the sign has changed, the degree of the polynomial is diminished by two, and it has been divided by the quantity $(2\pi m)^2$. Therefore in *2i* operations we should get

$$\tfrac{1}{2}a_m = (-1)^i \int_0^1 \frac{\cos 2\pi mx}{(2\pi m)^{2i}} \varphi_{2n-2i}(x)\, dx.$$

Writing in it $i=n-1$ we get by integration by parts

$$\tfrac{1}{2}a_m = \left[(-1)^{n-1} \frac{\sin 2\pi mx}{(2\pi m)^{2n-1}} \varphi_2(x) \right]_0^1 + (-1)^n \int_0^1 \frac{\sin 2\pi mx}{(2\pi m)^{2n-1}} \varphi_1(x)\, dx.$$

The quantity in the brackets is equal to zero; further integration by parts gives

$$\tfrac{1}{2}a_m = \left[(-1)^{n-1} \frac{\cos 2\pi mx}{(2\pi m)^{2n}} \varphi_1(x) \right]_0^1 + (-1)^n \int_0^1 \frac{\cos 2\pi mx}{(2\pi m)^{2n}}\, dx.$$

The integral in the second member is equal to zero. Since $\varphi_1(1) = -\varphi_1(0) = \tfrac{1}{2}$ we find

$$a_m = \frac{2(-1)^{n-1}}{(2\pi m)^{2n}}.$$

Determination of the coefficients β_m . It would be possible to determine them by integration by parts in the same way as the coefficients a_m , but it is easy to show by aid of formula (1) § 80 that $\beta_m = 0$. Indeed, putting 1-x into (1) instead of x we should obtain $\varphi_{2n}(1-x) = \varphi_{2n}(x)$ for every value of x.

$$\cos 2\pi m(1-x) = \cos 2\pi mx$$

and

$$\sin 2\pi m(1-x) = -\sin 2\pi mx;$$

hence the coefficients of $\sin 2\pi mx$ in the expansion (1) must all be equal to zero, so that we have $\beta_m = 0$.

Finally, the expansion of $\varphi_{2n}(x)$ will be

(3) $$\varphi_{2n}(x) = 2(-1)^{n-1} \sum_{m=1}^{\infty} \frac{\cos 2\pi mx}{(2\pi m)^{2n}}.$$

To obtain the expansion of $\varphi_{2n-1}(x)$ we determine the derivatives of both members of (3), and find

(4) $$\varphi_{2n-1}(x) = 2(-1)^{n} \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{(2\pi m)^{2n-1}}.$$

Applications of the trigonometrical expansion. 1. If we write $x=0$ in equation (3), we have

$$(5)\qquad \varphi_{2n}(0) = a_{2n} = \sum_{m=1}^{\infty} a\ , \ = \frac{(-1)^{n-1}}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}}\ .$$

This shows again the rule of the signs of a_{2n} obtained in § 80. From formula (5) the sum of the reciprocal power series may be deduced.

$$(6)\qquad \sum_{x=1}^{\infty} \frac{1}{x^{2n}} = \tfrac{1}{2}(2\pi)^{2n}\ |a_{2n}| = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|\ .$$

Particular cases:

$$\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}\ ,\quad \sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}\ .$$

2. Putting $x=\tfrac{1}{2}$ into formula (3) we obtain

$$\varphi_{2n}(\tfrac{1}{2}) = \sum_{m=1}^{\infty} (-1)^m\ a_m = \frac{2(-1)^n}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{2n}}\ .$$

If the central value $\varphi_{2n}(\tfrac{1}{2})$ of the polynomial is known then this expression gives a formula for the sum of the alternating reciprocal power series. We will determine $\varphi_{2n}(\tfrac{1}{2})$ in § 86; but in § 49 we have seen that if we denote the sum (6) by s_{2n} then the sum of the alternatingeries will be

$$s_{2n}\left[1 - \frac{1}{2^{2n-1}}\right]$$

therefore

$$(7)\qquad \sum_{x=1}^{\infty} \frac{(-1)^{x+1}}{x^{2n}} = (2^{2n-1}-1)\ \pi^{2n}\ |\ a_{,,}\ |.$$

Particular cases:

$$\sum_{x=1}^{\infty} \frac{(-1)^{x+1}}{x^2} = \frac{\pi^2}{12}\ ,\quad \sum_{x=1}^{\infty} \frac{(-1)^{x+1}}{x^4} = \frac{7\pi^4}{720}\ .$$

From this we may obtain the value of $\varphi_{2n}(\tfrac{1}{2})$; since the sign of a_{2n} is that of $(-1)^{n-1}$, we get from (7)

$$(8)\qquad \varphi_{2n}(\tfrac{1}{2}) = \left(\frac{1}{2^{2n-1}} - 1\right) a_{2n}\ .$$

We will find this result later by another method.

3. ***Limits. Since*** in the interval zero and one we have

$$\varphi_{2n}(x) = \frac{2(-1)^{n-1}}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \cdot \frac{\cos 2\pi mx}{m^{2n}}$$

it follows that

$$|\varphi_{2n}(x)| \leqq \frac{2}{(2\pi)^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}} = |a_{2n}| .$$

From

$$\varphi_{2n+1}(x) = \frac{2(-1)^{n}}{(2\pi)^{2n+1}} \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m^{2n+1}}$$

we deduce

$$|\varphi_{2n+1}(x)| < \frac{2}{(2\pi)^{2n+1}} \sum_{m=1}^{\infty} \frac{1}{m^{2n+1}} < \frac{2}{(2\pi)^{2n+1}} \sum_{m=1} \frac{1}{m^{2n}} = |a_{2n} 2\pi| .$$

We have seen that

$$1 < \sum_{x=1}^{\infty} \frac{1}{x^{2n}} = \frac{(2\pi)^{2n} |a_{2n}|}{2} \leqq \sum_{x=1}^{\infty} \frac{1}{x^{2}} = \frac{\pi^2}{6} .$$

From this we conclude

(9)
$$\frac{2}{(2\pi)^{2n}} < |a_{2n}| \leqq \frac{1}{12(2\pi)^{2n-2}} ;$$

theiefore

$$\lim_{n=\infty} | a_{2n} | = 0 \qquad \text{and} \qquad \lim_{n=\infty} k^{2n} a_{2n} = 0$$

if I RI $< 2\pi$. Moreover

$$\frac{2\,(2n)\,!}{(2n)^{2n}} < |B_{2n}| \leqq \frac{(2n)\,!}{12\,(2\pi)^{2n-2}} .$$

Hence I B_{2n} I increases indefinitely with n.

$$\sum_{x=1}^{\infty} \frac{1}{x^{2n}} > \sum_{x=1}^{\infty} \frac{1}{x^{2n+2}} ;$$

hence from (6) it follows that

(10)
$$\left| \frac{a_{2n+2}}{a_{2n}} \right| < \frac{1}{(2\pi)^2}$$

and in consequence of this, the series Σ I a_{2n} I is absolutely convergent. On the other hand from (9) it results that

$$\left| \frac{B_{2n+2}}{B_{2n}} \right| > \frac{24(2n+1)(2n+2)}{(2\pi)^4} ;$$

hence this ratio is increasing indefinitely with n. Moreover from (10) we get

$$\left|\frac{B_{2n+2}}{B_{2n}}\right| < \frac{(2n+1)(2n+2)}{(2\pi)^2}.$$

§ 83. Application of the Bernoulli polynomials.

1. *Determination* of *sums of powers.* From formula (1) § 78 we deduce

$$\text{A-'}\; x^n = n!\,\varphi_{n+1}(x) + k = \frac{1}{n+1}\,(x+B)^{n+1} + k;$$

in the last term figures the symbolical expression of the *Bernoulli* polynomial (6, § 78). From this it follows that

$$(1)\qquad \sum_{x=0}^{z} x^n = \frac{1}{n+1}\left[(z+B)^{n+1} - B_{n+1}\right].$$

By aid of this formula we may determine the sum of a function expanded into a *Maclaurin series:* If we have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, D^n f(0)$$

then

$$\text{A-'}\; f(x) = \sum_{n=0}^{\infty} \varphi_{n+1}(x)\, D^n f(0) + k$$

and finally

$$\sum_{x=1}^{z} f(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left[(z+B)^{n+1} - B_{n+1}\right] D^n f(0).$$

Remark. We may. determine $\Delta^{-1}x^n$ by using *Lacroix's* method (Trait6 des Differences et des Series p. 68). The difference of x^{n+1} may be written

$$\Delta x^{n+1} = (x+1)^{n+1} - x^{n+1} = \sum_{i=1}^{n+2} \binom{n+1}{i} x^{n+1-i};$$

the operation Δ^{-1} will give

$$x^{n+1} = (n+1)\,\Delta^{-1}x^n + \sum_{i=2}^{n+2} \binom{n+1}{i} \Delta^{-1}x^{n+1-i}.$$

Putting $\Delta^{-1}x^n = f(n)$ we find

$$(2) \qquad f(n) + \binom{n}{1}\frac{1}{2}f(n-1) + \binom{n}{2}\frac{1}{3}f(n-2) + \ldots$$

$$+ \binom{n}{n}\frac{1}{n+1}f(0) = \frac{x^{n+1}}{n+1}.$$

Starting from $f(0) = x+C$ *Lacroix* determined by aid of this equation step by step $\Delta^{-1}x$, $\Delta^{-1}x^2$ and so on. But we shall solve this difference equation by aid of *André's* method in § 178. Example 3. The result obtained there is the following

$$(3) \qquad f(n) = \Delta^{-1}x^n =$$

$$= \sum_{\mu=0}^{n+1} n! \frac{x^{n+1-\mu}}{(n+1-\mu)!} \sum_{i=1}^{\mu+1} \frac{(-1)^i}{(k_1+1)!(k_2+1)! \ldots (k_i+1)!}.$$

In the second sum $k_1, k_2, \ldots, k_i$ take every integer value with repetition and permutation such that

$$k_1 + k_2 + \ldots + k_i = \mu \text{ and } k_\nu > 0.$$

that is, every partition of μ with repetition and permutation of order i.

But on the other hand from formula (1) it follows that

$$(4) \qquad \Delta^{-1}x^n = \sum_{\mu=1}^{n+1} \binom{n}{\mu-1}\frac{B_\mu}{\mu} x^{n+1-\mu} + C.$$

Equating the coefficients of $x^{n+1-\mu}$ in (3) and (4) we get

$$(5) \qquad \frac{B_\mu}{\mu!} = a_\mu = \sum_{i=1}^{\mu+1} \frac{(-1)^i}{(k_1+1)!(k_2+1)! \ldots (k_i+1)!}$$

where $k_1+k_2+\ldots+k_i=\mu$ and $k_\nu>0$.

This is an interesting expression for the *Bernoulli* numbers, which may be obtained directly, starting from their generating function (p. 251).

Example. Let $\mu=4$. For $i=1$ we have $k_1=4$ and the corresponding term is $-1/120$. For $i=2$ we have $k_1=1$, $k_2=3$ or $k_1=3$, $k_2=1$ or $k_1=2$, $k_2=2$. The corresponding terms are $2/48$ and $1/36$. For $i=3$ we have $k_1=1$, $k_2=1$, $k_3=2$ or $k_1=1$, $k_2=2$, $k_3=1$ or $k_1=2$, $k_2=1$, $k_3=1$ the corresponding term is $-3/24$. For $i=4$ we have $k_1=k_2=k_3=k_4=1$ corresponding to $1/16$. Finally we find

$$\frac{B_4}{24} = a_4 = -\frac{1}{120} + \frac{2}{48} + \frac{1}{36} - \frac{3}{24} + \frac{1}{16} = -\frac{1}{720}.$$

2. One of the most important applications of *the Bernoulli* numbers is the expansion of certain functions by aid of these numbers, as we shall see later.

§ 84, Expansion of a polynomial *f(x) into* **a** *series of* **Bernoulli polynomials.** It is known, that if $\varphi_0(x)$, $\varphi_1(x)$, . . . , $\varphi_n(x)$ are polynomials of degree 0, 1, . . . , n, then it is possible, and only in one manner, to expand any polynomial whatever of degree n, in the following form:

$$(1) \qquad f(x) = \sum_{i=0}^{n+1} c_i \varphi_i(x).$$

If $\varphi_i(x)$ is the *Bernoutti* polynomial of degree i, then the coefficients c_i may be determined in the following way. Integrating both members from zero to one, every term of the second member will vanish in consequence of formula (1) § 81 except that of c_0; since $\varphi_0(x) = 1$, we obtain

$$(2) \qquad c_0 = \int_0^1 f(x)\, dx.$$

According to (3) § 78, the m -th derivative of $f(x)$ will be

$$\mathrm{D}^m f(x) = c_m + c_{m+1} \varphi_1(x) + c_{m+2} \varphi_2(x) + \dots + c_n \varphi_{n-m}(x)$$

integrating both members from zero to one, we find

$$(3) \qquad c_m = \mathrm{D}^{m-1} f(1) - \mathrm{D}^{m-1} f(0) = \Delta \mathrm{D}^{m-1} f(0).$$

Consequently, if we know the integral of a polynomial $f(x)$ from zero to one, and its derivatives for $x=0$ and $x=1$, then the expansion (1) is known.

Formula (1) may serve for the determination of the indefinite integral of $f(x)$, of its derivatives and also of its indefinite sum. This last is obtained by aid of (2) § 81.

Example 1. Given $f(x) = x^n/n!$

$$\int f(x)\, dx = \frac{x^{n+1}}{(n+1)!} + k \quad \text{and} \quad \mathrm{D}^{m-1} f(x) = \frac{x^{n-m+1}}{(n-m+1)!};$$

therefore

$$c_0 = 1/(n+1)! \quad \text{and} \quad c_m = 1/(n-m+1)!$$

and finally

$$\frac{x^n}{n!} = \sum_{m=0}^{n+1} \frac{\varphi_m(x)}{(n+1-m)!}.$$

Example 2. Given t(x) $= \binom{x}{n}$, We shall see later that

$$\int_0^1 \binom{x}{n} dx = b_n$$

where b_n is the general coefficient of the *Bernoulli* polynomial of the second kind. From (2) it follows that $c_0 = b_n$. Moreover since

$$\Delta \binom{x}{n} = \binom{x}{n-1}$$

and according to (4) § 55

$$\mathrm{D}^{m-1} \binom{x}{n-1} = \frac{1}{(n-1)!} \sum_{\mu=1}^{n} (\mu)_{m-1}\, x^{\mu-m+1} S_{n-1}^{\mu};$$

hence putting into it $x=0$ we get

$$c_m = \frac{(m-1)!}{(n-1)!} S_{n-1}^{m-1}.$$

Finally the required expansion will be

$$(4) \qquad \binom{x}{n} = b_n + \frac{1}{(n-1)!} \sum_{m=2}^{n+1} (m-1)!\, \varphi_m(x)\, S_{n-1}^{m-1}.$$

Putting into this expression $x=0$ and writing $\varphi_m(0) = B_m/m!$ we have

$$(5) \qquad b_n = -\frac{1}{(n-1)!} \sum_{m=2}^{n+1} \frac{B_m}{m} S_{n-1}^{m-1}.$$

This is an expression of the coefficient b_n by aid of *Bernoulli* numbers B_m .

According to § 65 we may obtain by inversion an expression of the *Bernoulli* number B_ν. Multiplying both members of (5) by $-(n-1)!\,\mathfrak{S}_{\nu}^{n-1}$ and summing from $n=2$ to $n=\nu+2$, every term of the second member will vanish except that in which $m=\nu+1$; therefore we find

$$(6) \qquad -\sum_{n=2}^{\nu+2} b_n \, (n-1)! \; \mathfrak{S}_{\nu}^{n-1} = \frac{B_{\nu+1}}{\nu+1} .$$

§ 85, **Expansion into** Bernoulli polynomials. Generating functions. If the function to be expanded is not a polynomial, the series will be infinite, and considerations of convergence must be made; but the coefficients c_m will be determined in the same manner as in the preceding paragraph.

Example 1. Let us determine the generating function f(t) of the polynomials $\varphi_n(x)$. In the case of the generating function we must have

$$c_m = \Delta D^{m-1} f(0) = t^m$$

and

$$c_0 = \int^1 f(t)\, dt = 1.$$

Since

$$D^{m-1} e^{xt} = t^{m-1} e^{xt}$$

and

$$\Delta e^{xt} = e^{xt} [e^t - 1]$$

it is easy to see that

$$\left[\Delta D^{m-1} \frac{t\, e^{xt}}{e^t - 1} \right]_{x=0} = t^m ;$$

therefore

$$(1) \qquad f(t) = \frac{t\, e^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} \varphi_m(x) \; t^m$$

is-the ***genetafing function*** of the polynomial $\varphi_m(x)$. From this we may obtain the ***generating function*** of the number a_m by putting $x=0$.

$$(2) \qquad \frac{t}{e^t - 1} = \sum_{m=0}^{\infty} a_m \; t^m .$$

From equation (2) we conclude that Σ a, t^m is the reciprocal value of the function

$$(3) \qquad \frac{e^t - 1}{t} = \sum_{i=1}^{\infty} \frac{t^{i-1}}{i!} .$$

On the other hand, by aid of formula (4) § 78, we could show that the product of the series (2) and (3) is equal to one, and deduce in this way the generating function of the numbers a_m. We have seen that the series $\Sigma\, a_m$ is convergent; therefore, putting into (2) $t = 1$ we obtain the sum of the coefficients a_m

$$\sum_{m=0}^{\infty} a_m = \frac{1}{e-1} = 0'58197\ 67070\ .$$

The convergence is rapid, indeed we have:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -0'5 \\ a_2 &= 0'08333\ 33333 \\ a_4 &= -0'00138\ 88889 \\ a_6 &= 0'00003\ 30688 \\ a_8 &= -0'00000\ 08267 \\ a_{10} &= 0'00000\ 00209 \\ a_{12} &= -0'00000\ 00005. \end{aligned}$$

This gives

$$\sum_{m=0}^{13} a_m = 0'58197\ 67069$$

There is an error only in the tenth decimal.

The generating function (2) may be expanded in the following way writing:

$$\frac{1}{1+\left(\frac{e^t-1}{t}-1\right)} = \sum^{\infty} (-1)^i \left(\frac{e^t-1}{t}-1\right)^i.$$

This will give

$$1 + \sum_{i=1}^{\infty} (-1)^i \left[\sum_{k=1}^{\infty} \frac{t^k}{(k+1)!}\right]^i$$

and after performing the multiplication of the i series the result may be written

$$1 + \sum_{m=1}^{\infty} \sum_{i=1}^{m+1} \frac{(-1)^i\, t^m}{(k_1+1)!\,(k_2+1)!\ldots(k_i+1)!}$$

where $k_1+k_2\ldots,\ +k_i = m$ and $k_\nu > 0$.

Therefore we conclude that

$$a_m = \sum_{i=1}^{m+1} \frac{(-1)^i}{(k_1+1)!\,(k_2+1)!\ldots(k_i+1)!}\ .$$

This expression has been obtained before, in § 83.

§ 86. Raabe's multiplication theorem of the Bernoulli polynomials. Let us start from the expansion of

$$f(x) = p^{n-1} \sum_{\mu=0}^{p} \varphi_n\left(\frac{x+\mu}{p}\right), \quad n>1$$

into a series of Bernoulli polynomials.

To determine c_0 let us put $x=pz$; we shall have

$$c_0 = \int_0^1 f(x)dx = p^n \sum_{\mu=0}^{p} \int_0^{1/p} \varphi_n\left(z+\frac{\mu}{p}\right) dz;$$

therefore

$$c_0 = p^n \sum_{\mu=0}^{p} \left[\varphi_{n+1}\left(\frac{\mu+1}{p}\right) - \varphi_{n+1}\left(\frac{\mu}{p}\right)\right] =$$

$$= p^n[\varphi_{n+1}(1) - \varphi_{n+1}(0)] = 0.$$

Since we have seen that $c_m = \Delta D^{m-1} f(0)$,

$$c_m = \left[\Delta p^{n-m} \sum_{\mu=0}^{p} \varphi_{n-m+1}\left(\frac{x+\mu}{p}\right)\right]_{x=0} =$$

$$= p^{n-m} \sum_{\mu=0}^{p} \left[\varphi_{n-m+1}\left(\frac{1+\mu}{p}\right) - \varphi_{n-m+1}\left(\frac{\mu}{p}\right)\right];$$

from this it follows that

$$c_m = p^{n-m}[\varphi_{n+1-m}(1) - \varphi_{n+1-m}(0)].$$

Therefore if $m \neq n$ then $c_m=0$, **and**

$$c_n = \varphi_1(1) - \varphi_1(0) = 1.$$

Finally we have

$$f(x) = \varphi_n(x).$$

Putting again $x=pz$ **we** obtain *Raabe's* multiplication *theorem*

$$\text{(1)} \qquad \varphi_n(pz) = p^{n-1} \sum_{\mu=0}^{p} \varphi_n\left(z+\frac{\mu}{p}\right).$$

Writing $z=0$ we get

$$\text{(2)} \qquad a_n = p^{n-1} \sum_{\mu=0}^{p} \varphi_n\left(\frac{\mu}{p}\right).$$

Particular cases. For $p=2$ formula (2) gives

$$(3) \qquad \varphi_n(\tfrac{1}{2}) = a_n\left(\frac{1}{2^{n-1}} - 1\right).$$

This result has been obtained in (8) § 82.

We may obtain another expression of $\varphi_n(\tfrac{1}{2})$ starting from $\varphi_n(x) = (x+B)^n/n!$. We get

$$(4) \qquad \varphi_n(\tfrac{1}{2}) = \sum_{i=0}^{n+1} \binom{n}{i} \frac{B_i}{2^{n-i}\, n!}.$$

2. Putting into (2) $n=2m$ and $p=4$, taking account of the preceding result and remarking that in consequence of the symmetry

$$\varphi_{2m}\left(\frac{1}{4}\right) = \varphi_{2m}\left(\frac{3}{4}\right)$$

we find that

$$\varphi_{2m}\left(\frac{1}{4}\right) = \frac{a_{2m}}{2^{2m}}\left(\frac{1}{2^{2m-1}} - 1\right).$$

3. Putting again $n=2m$ and $p=3$, we obtain in the same way

$$\varphi_{2m}\left(\frac{1}{3}\right) = \frac{1}{2}\, a_{2m}\left(\frac{1}{3^{2m-1}} - 1\right).$$

§ 87, The **Bernoulli** series. If certain conditions are satisfied we may expand a function into a series of *Bernoulli* polynomials in the same way as has been done in the case of polynomials (§ 85) ; but the series will be infinite and its convergence must be examined.

Let us start from $f(x+u)$; its expansion will be, in consequence of formulae (2) and (3) § 85,

$$(1) \qquad f(x+u) = \int_u^{u+1} f(t)\,dt + \sum_{m=1}^{\infty} \varphi_m(x)\; \Delta D^{m-1} f(u).$$

Putting into this formula $u=0$ we obtain the expansion of $f(x)$ into a series of *Bernoulli* polynomials found before. On the other hand, putting $x=0$ we have

$$f(u) = \int_u^{u+1} f(t)\,dt + \sum_{m=1}^{\infty} a_m\, \Delta D^{m-1} f(u)$$

or

$$(2) \qquad f(u) = \int_u^{u+1} f(t)dt + \sum_{m=1}^{\infty} \frac{B_m}{m!} \Delta D^{m-1} f(u).$$

This expansion might be termed a *Bernoulli series,* owing to the *Bernoulli* numbers figuring in it, It often leads to very useful formulae. For instance, starting from cos z we may obtain the expansions of cot z, $\tan z$, etc., into power series.

Sometimes these formulae may serve for the computation of the values of the function *f(u)* as for instance in the cases of $F(u)$ and $F(u)$.

Remainder of the Bernoulli series. If the series is infinite, its remainder should be determined. For this we will start from $f(x+u)$ integrating by parts, u varying from zero to unity; proceeding in the following way:

$$(3) \qquad \int_0^1 f(x+u)du = [\varphi_1(u)f(x+u)]_0^1 - \int_0^1 \varphi_1(u) D f(x+u)\, du.$$

To determine the quantity in the brackets, let us remark that

$$\varphi_1(0) = a_1 = -\tfrac{1}{2} \text{ and } \varphi_1(1) = -a_1 = \tfrac{1}{2}$$

therefore we may write this quantity

$$\tfrac{1}{2}[f(x+1) + f(x)] = f(x) + \tfrac{1}{2}\Delta f(x) = f(x) - a_1 \Delta f(x).$$

The integral in the second member of (3) gives by integration by parts

$$-\int_0^1 \varphi_1(u) D f(u+x)\, du = [-\varphi_2(u)\, D f(x+u)]_0^1 +$$

$$+ \int_0^1 \varphi_2(u)\, D^2 f(x+u)\, du,$$

Since we have

$$\varphi_2(1) = \varphi_2(0) = a_2$$

the quantity in the brackets will be equal to $-a_2 \Delta D f(x)$. Continuing in this manner, equation (3) will become

$$(4) \qquad f(x) = \int_x^{x+1} f(t)dt + a_1 \Delta f(x) + a_2 \Delta D f(x) + \ldots, +$$

$$+ a_{2n} \Delta D^{2n-1} f(x) - \int_0^1 \varphi_{2n}(u)\, D^{2n} f(x+u)\, du.$$

This is the expansion of f(x) into a *Bernoulli series,* and the last term of the second member is the *remainder;* it may be transformed as follows:

$$-\int_0^1 [\varphi_{2n}(u) - a_{2n}+a_{2n}]\mathsf{D}^{2n}f(u+x)\,du =$$

$$-a_{2n}\Delta\mathsf{D}^{2n-1}f(x) - \int_0^1 [\varphi_{2n}(u)-a_{2n}]\,,\ \mathsf{D}^{2n}f(x+u)\,du.$$

The quantity in the brackets, in the second member, does not change its sign in the interval 0, 1, therefore the mean value theorem may be applied, and we have

$$(5) \qquad R_{2n} = -\mathsf{D}^{2n}f\ (x+6)\int_0^1 [\varphi_{2n}(u)-a_{2n}]du = a_{2n}\mathsf{D}^{2n}f(x+\vartheta)$$

where $0 < 6 < 1$. Finally the expansion will be

$$(6) \qquad f(x) = \int_x^{x+1} f(t)dt + a_1\Delta f(x) + a_2\Delta\mathsf{D}f(x) + ,\ldots +$$

$$+ a_{2n-2}\Delta\mathsf{D}^{2n-3}f(x) + a_{2n}\mathsf{D}^{2n}f(x+\vartheta)\,.$$

If $\mathsf{D}^{2n}f(x)$ does not change its sign in the interval 0, 1 moreover if $\mathsf{D}^{2n}f(x)\ \mathsf{D}^{2n+2}f\ (x)>0,$ it is possible to obtain a still more simple form of the remainder. Indeed, since $a_{2n}a_{2n+2}<0$ in this case we have

$$R_{2n}\ R_{2n+2} < 0.$$

From (6) it follows that

$$R_{2n} = a_{2n}\ \Delta\mathsf{D}^{2n-1}f\ (x) + R_{2n+2}\ .$$

This equation may be written in the following way; if $A=B+C$ and $AC < 0$; then, from

$$A^2 = AB + AC,$$

it follows that $AB > A^2 > 0$; dividing by AB *we* get

$$1 > \frac{A}{B} > 0$$

so that we may put $A = \xi B$ where $0 < \xi < 1$. Finally we have

$$(7) \qquad R_{2n} = \xi\, a_{2n}\ \Delta\mathsf{D}^{2n-1}f\,(x).$$

Formula (6) may be transformed by putting $\mathsf{D}F(x) = f(x);$ *we* find

$$\text{(8)} \qquad \mathsf{D}F(x) = \Delta F(x) + \sum_{i=1}^{2n-1} \frac{B_i}{i!} \Delta \mathsf{D}^i F(x) + R_{2n}.$$

This formula may be useful first for the determination of $\mathsf{D}F(x)$ if the derivative of F(x) is complicated but the derivatives of $\Delta F(x)$ are simple.

Example 1. If $F(x) = \log\Gamma(x+1)$, then $\mathsf{D}F(x) = F(x)$ and $\Delta F(x) = \log(x+1)$ moreover $\Delta \mathsf{D}^i F(x) = \dfrac{(-1)^{i-1}(i-1)!}{(x+1)^i}$.

Since from formula (1) § 21 we have

$$\mathsf{D}^{2n+1}F(x) = \mathsf{D}^{2n+1}\log\Gamma(x+1) = -(2n)\,I \sum_{m=1}^{\infty} \frac{1}{(x+m)^{2n+1}}$$

therefore $\mathsf{D}^{2n+1}F(x)\,\mathsf{D}^{2n+3}F(x) > 0$ so that the remainder of the form (7) may be used, that is

$$\text{(9)} \qquad R_{2n} = \frac{\xi B_{2n}}{(2n)!} \Delta \mathsf{D}^{2n} F(x)$$

where $0 < \xi < 1$. Finally from (8) we get:

$$\text{(10)} \quad F(x) = \log(x+1) + \sum_{i=1}^{2n-1} (-1)^{i-1} \frac{B_i}{i} \left(\frac{1}{x+1}\right)^i - \frac{\xi B_{2n}}{2n} \left(\frac{1}{x+1}\right)^{2n}.$$

Although the infinite series is divergent, the formula may be used to compute the values of $F(x)$ in *Pairman's* Tables (loc. cit. 16). Indeed at the beginning of the series R_{2n} is diminishing with increasing n. In § 82 we have seen that

$$\left|\frac{B_{2n+2}}{B_{2n}}\right| < \frac{(2n+1)(2n+2)}{(2\pi)^2};$$

neglecting ξ, this gives approximately

$$\left|\frac{R_{2n+2}}{R_{2n}}\right| < \frac{(2n+1)(2n)}{(2\pi)^2(1+x)^2}.$$

So that R_{2n} will diminish till the second member becomes equal to one: or, approximately, till n reaches the value of $3x+3$. Therefore if x is large it will be easy, generally to obtain the prescribed precision. The most unfavourable case is that of $x=0$; this should give *Euler's* constant, Writing $x=0$ we get:

$$C = -F(0) = \tfrac{1}{2} + \sum_{i=1}^{n} \frac{B_{2i}}{2i} + \frac{\xi B_{2n}}{2n}.$$

For $n=1$ this would give $C=0{\cdot}5+\frac{\xi}{12}$; the best value of C is obtained for $n=3$, that is $C=5{\cdot}575+\frac{\xi}{252}$.

Example 2. If F(x) $=$ $\mathrm{D}\log\Gamma(x+1)$, then $\Delta F(x)=f(x)$ and

$$\Delta F(x)=\frac{1}{x+1}; \text{ moreover } \Delta \mathrm{D}^i F(x) = \frac{(-1)^i\, i!}{(x+1)^{i+1}}.$$

From formula (1) § 21 it follows that

$$\mathrm{D}^{2n+1}F(x)=\mathrm{D}^{2n+2}\log\Gamma(x+1)=(2n+1)!\sum_{i=1}^{\infty}\frac{1}{(x+i)^{2n+2}};$$

therefore the remainder will be given by (9) so that we have according to (8):

$$(11)\qquad F(x)=\frac{1}{x+1}+\sum_{i=1}^{2n-1}\frac{(-1)^i B_i}{(x+1)^{i+1}}+\frac{\xi B_{2n}}{(x+1)^{2n+1}}.$$

Though the infinite series is again divergent, the formula is still useful for the computation of $F(x)$; since at the beginning of the series R_{2n} decreases with increasing n. (Approximately till n becomes equal to 3x+3.) The less favourable case is that of $x=0$; then we obtain

$$F(0)=\frac{\pi^2}{6}=1+\tfrac{1}{2}+\sum_{i=1}^{n}B_{2i}+\xi B_{2n}.$$

For $n=1$ this gives $F(0)$ = 1'5 + $\frac{\xi}{6}$. The best value of $F(0)$ is obtained for $n=3$, that is $F(0)=1{\cdot}633+\frac{\xi}{42}$

Secondly formula (8) may 'be useful to determine $\Delta F(x)$ if the derivatives of $\Delta F(x)$

$$\text{———} \left[\frac{1}{(x+1)^i}-\frac{1}{x^i}\right]+$$

$$\frac{\xi B_{2n}}{2n}\left[\frac{1}{(x+1)^{2n}}\right.$$

$$[\Delta \mathbf{D}^i F(x,t)]_{x=0} = c_i t^i \omega(t)$$

where c_i is a numerical constant. From this it follows that

$$[\Delta F(x,t)]_{x=0} = c_0 \omega(t).$$

Writing in (8) $F(x,t)$ instead of F(x), putting $x=0$ and dividing both members of the equation by $c_0 \omega(t)$, *we* find

$$(12) \qquad \frac{[\mathbf{D}F(x,t)]_{x=0}}{c_0 \omega(t)} = \sum_{i=0}^{2n-1} \frac{c_i}{c_0} \frac{B_i}{i!} t^i + \frac{R_{2n}}{c_0 \omega(t)}.$$

This equation will give the expansion of the first member into a power series.

Example 4. If $F(x,t) = e^{xt}$ then $\mathbf{D}F(0) = t$ and $\Delta F(0) = e^t - 1$ moreover $[\Delta \mathbf{D}^i F(x,t)]_{x=0} = t^i (e^t - 1)$ hence $c_i = 1$ and $\omega(t) = e^t - 1$. Since

$$\mathbf{D}^{2n+1} F(x,t)\, \mathbf{D}^{2n+3} F(x,t) > 0$$

it follows that the remainder is given by (9) and from (12) we get

$$(13) \qquad \frac{t}{e^t - 1} = \sum_{i=0}^{2n-1} \frac{B_i}{i!} t^i + \frac{\xi B_{2n}}{(2n)!} t^{2n}.$$

In this manner we again obtained the generating function of the numbers $a_i = B_i / i!$.

Example 5. If $F(x,t) = \sin xt$ then $\mathbf{D}F(0) = t$ and $\Delta F(0) = \sin t$
$\Delta \mathbf{D}F(0) = t(\cos t - 1)$, $\Delta \mathbf{D}^{2i} F(0) = (-1)^i t^{2i} \sin t$.

Since $\mathbf{D}^{2n+1} F(x,t)\, \mathbf{D}^{2n+3} F(x,t) < 0$ the remainder (5) must be used, that is

$$(14) \qquad R_{2n} = \frac{B_{2n}}{(2n)!} \mathbf{D}^{2n+1} F(x+\vartheta)$$

where $0 < \vartheta < 1$. So that the required expansion (12) will be [noting that $c_{2i} = (-1)^i$ and $\omega(t) = \sin t$]

$$\frac{t}{\sin t} = \frac{B_1 t(\cos t - 1)}{\sin t} + \sum_{i=0}^{2n-1} (-1)^i \frac{B_{2i}}{(2i)!} t^{2i} +$$

$$+ (-1)^n \frac{B_{2n}}{(2n)!} t^{2n+1} \frac{\cos \vartheta t}{\sin t}.$$

$(\cos t + 1) / \sin t = \cot \tfrac{1}{2} t$; therefore we may write the preceding equation in the following manner

$$(15) \qquad \tfrac{1}{2}t \ \cot\tfrac{1}{2}t = \sum_{i=0}^{2n-1} (-1)^i \frac{B_{2i}}{(2i)!} t^{2i} + R_{2n}.$$

In § *82* we have seen that $\lim\limits_{*=-} a_{2n} k^{2n} = 0$ if $0 \leq k < 2\pi$; therefore if I t | $< 2\pi$ we shall have $\lim\limits_{n=\infty} R_{2n} = 0$. Moreover, according to formula (10) § 82 we have

$$\frac{a_{2n+2}\, t^{2n+2}}{a_{2n}\, t^{2n}} < \frac{t^2}{(2\pi)^2};$$

hence we conclude that if I t I $< 2\pi$ the first member of this inequality is less than one and the series (15) is convergent.

Writing in (15) $B_{2i}/$ (2i) ! $= a_{2i}$ and subtracting it from $2 + \tfrac{1}{2}t$ we find

$$2 + \tfrac{1}{2}t(1-\cot\tfrac{1}{2}t) = 1 + \tfrac{1}{2}t + \sum_{i=1}^{2n-1} (-1)^{i+1} a_{2i}\, t^{2i} +$$
$$- R_{2n} = \sum_{i=0}^{\infty} |a_i|\, t^i .$$

Hence the first member is the generating function of the absolute values of a_m.

From this formula we may deduce the sum of the absolute values of the a_m, by putting $t=1$:

$$\sum_{m=0}^{\infty} |a_m| = 2'5 - \tfrac{1}{2}\cot\tfrac{1}{2} \doteq 1'5847561$$

(Let us remark that $\text{arc}\tfrac{1}{2} = 28^0$ 38' 52" 40).

The numbers a_m in § 85 would give

$$\sum_{m=0}^{13} |a_m| = 1'58475\ 61391.$$

Writing in formula (15) $t=2z$ and dividing by z we get the well known formula

$$(16) \qquad \cot z = \frac{1}{z} + \sum_{m=1}^{n} (-1)^m 2^{2m} \frac{B_{2m}}{(2m)!} z^{2m-1} +$$
$$+ (-1)^n 2^{2n+1} \frac{B_{2n}}{(2n)!} z^{2n} \frac{\cos 2\vartheta z}{\sin 2z}.$$

We have [illegible] tan [illegible] cot [illegible]

$$\sum_{m=1} \quad \frac{B_{2m}}{(2m)!} z^{2m-1} + R_{2n}.$$

From **(16)** and **(17)** we may obtain several other formulae. For instance by integration

$$-\int_0^z \tan x\,dx = \log \cos z$$

or by derivation

$$\mathsf{D} \tan z = \frac{1}{\cos^2 z}\,, \qquad -\mathsf{D} \cot z = \frac{1}{\sin^2 z}\,.$$

Since

$$\tfrac{1}{2}[\tan\tfrac{1}{2}x + \cot\tfrac{1}{2}x] = \frac{1}{\sin x}.$$

Hence we have

$$(18)\qquad \frac{1}{\sin x} = \frac{1}{x} + \sum_{m=1}^{n} (-1)^m (2-2^{2m}) \frac{B_{2m}}{(2m)!} x^{2m-1} + R_{2n}\,.$$

§ 88. The Maclaurin-Euler Summation Formula. This is a **formula** by aid of which the sum of a function may be expressed by its integral and its derivatives, or the integral by the sum and the derivatives.

Apart from the remainder, the formula may **be** easily deduced by symbolical methods. In § 6, we had formula (3)

$$\Delta = e^{h\mathsf{D}} - 1$$

from this it follows that

$$\Delta^{-1} = \frac{1}{e^{h\mathsf{D}}-1} = \frac{1}{h\mathsf{D}}\left[\frac{h\mathsf{D}}{e^{h\mathsf{D}}-1}\right]$$

But according to formula (2) § 85 the expression in the brackets is equal to the generating function of the numbers a_i, the coefficients figuring in the *Bernoulli* polynomial; since $a_i = B_i/i!$ where B_i is the i-th *Bernoulli* number; hence we may write

$$\Delta^{-1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} (h\mathsf{D})^{i-1} = \frac{1}{h}\mathsf{D}^{-1} + \sum_{i=1}^{\infty} \frac{B_i}{i!} (h\mathsf{D})^{i-1}.$$

This operation performed on f(x) gives, if the sum from $x=a$ to $x=z$ is calculated,

$$\sum_{x=a}^{z} f(x) = \frac{1}{h}\int_a^z f(x)\,dx + \sum_{i=1}^{\infty} \frac{B_i}{i!} h^{i-1} [\mathsf{D}^{i-1}f(z) - \mathsf{D}^{i-1}f(a)].$$

This is the required summation formula.

To obtain the summation formula with its remainder we start from the *Bernoulli series (6)* of § 87. Let us write it in the following way:

$$(1)\qquad f(x) = \int_x^{x+1} f(t)dt + \sum_{m=1}^{2n-1} a_m \,\Delta D^{m-1}f(x) + a_{2n} D^{2n}f(x+\vartheta)$$

where $0 < \vartheta < 1$.

Summing each member of the equation from $x=a$ to $x=z$ we find

$$(2)\qquad \sum_{x=a}^{z} f(x) = \int_a^z f(t)dt + \sum_{m=1}^{2n-1} a_m[D^{m-1}f(z) - D^{m-1}f(a)] +$$

$$+ a_{2n} \sum_{x=a}^{z} D^{2n}f(x+6) \,.$$

Remark. The remainder in this equation may be written:

$$(3)\qquad R_{2n} = a_{2n} \sum_{x=a}^{\infty} D^{2n}f(x+\zeta) - a_{2n} \sum_{x=z}^{\infty} D^{2n}f(x+\zeta).$$

Particular cases. **1.** If we have

$$D^{2n}f(x)\, D^{2n+2}f(x) > 0$$

and moreover if $D^{2n}f(x)$ does not change its sign in the interval (a, z) then, according to § 87, the remainder may be written:

$$(4)\qquad R_{2n} = \xi\, a_{2n}[D^{2n-1}f(z) - D^{2n-1}f(a)]$$

where $0 < \xi < 1$.

2. If $f(x)$ is a function such that for every value of $m=1, 2, 3, \ldots$, we have $D^m f(\infty) = 0$, then we will use form (3) of the remainder; and equation (2) may be written

$$(5)\qquad \sum_{x=a}^{z} f(x) = \int_a^z f(t)dt + \sum_{m=1}^{2n-1} a_m D^{m-1}f(z) -$$

$$- a_{2n} \sum_{x=z}^{\infty} D^{2n}f(x+\zeta) + C_f$$

where C_f denotes the part of $\sum_{x=a}^{z} f(x)$ which is independent of z, that is,

$$C_f = -\sum_{m=1}^{2n-1} a_m\, D^{m-1}f(a) + a_{2n} \sum_{x=a}^{\infty} D^{2n}f(x+\zeta).$$

We may obtain another expression for C_f writing $z=\infty$ in **(5)**

$$\textbf{(6)} \qquad C_f = \sum_{x=a}^{\infty} f(x) - \int_a^{\infty} f(t)dt - a_1 f(\infty).$$

The value of C_f must be determined for each particular function. From (6) we conclude that C_f is independent of n.

Formula (5) will be really advantageous in cases in which one of the expressions $\Sigma\, f(x)$ or $\int f(x)dx$ is unknown,

Example 1. Given $f(x)=1/(x+1)$ and $a=0.$ Here we have $D^m f(\infty)=0$ for $m=1, 2, 3, \ldots$, therefore formula (5) may be used. Since moreover we have $f(\infty)=0,$ it follows that

$$C_f = \sum_{x=0}^{\infty} \frac{1}{x+1} - \int_0^{\infty} \frac{1}{t+1}dt = \lim_{z=\infty} \left[\sum_{x=0}^{z} \frac{1}{x+1} - \log(z+1) \right]$$

but according to formula (3) § 19 the second member is equal to ***Euler's*** constant C; therefore $C_f = C = 0{}^{\cdot}57721$ 56649 01532.. . .

$D^{2n}f$ (x) $D^{2n+2}f(x) > 0$ and $D^{2n}f$ *(x)* does not change its sign in the interval $(0,z)$; therefore according to ***Particular case*** 1, the part of the remainder dependent on z may be written: $a_{2n}\, \xi\, D^{2n-1}f(z).$ For instance, we shall have, stopping at $n=2$

$$\sum_{x=0}^{z} f(x) = C_f + \int_0^{z} f(t)dt + a_1 f(z) + a_2 Df(z) + a_4\, \xi\, D^3 f(z)$$

or

$$\sum_{x=0}^{z} \frac{1}{x+1} = C + \log(z+1) - \frac{1}{2(z+1)} - \frac{1}{12(z+1)^2} + \frac{\xi}{120(z+1)^4}.$$

For instance, if $z=21$ then

C	0'57721 566	
log21	3'04452 244	
$-1/2(21)$	-0'02380 952	
$-1/12(21)^2$	-0.00018 896	
	3'59773 **962**	+ 0'00000 004 ξ.

Comparing this with the exact result, we find, neglecting the remainder, that the error is only -3.10^8.

In *some* cases the *Euler-Maclaurin* formula performed on f(x) will lead to the same result as the expansion of

$$F(z) = \sum_{x=0}^{z} f(x)$$

into a *Bernoulli* series (6) § 87. For instance the expansion of $F(x)+C$ by formula (10) § 87 is identical with the expansion above.

Example 2. Given $f(x) = \log x$, and $\sum_{x=1}^{z} \log x$ is required. We have $D^m \log(\infty) = 0$ for $m = 1, 2, 3, \ldots$ so that formula (5) may be used. Moreover since $D^{2n}f(x)\, D^{2n+2}f(x) > 0$ and $D^{2n}f(x)$ does not change its sign, therefore remainder (4) will be applied. We find by aid of (6):

$$C_f = \lim_{z=\infty} [\log(z-1)! - z \log z + z - 1 - a_1 \log z]$$

but tbis expression may be written

$$C_f + 1 = \log \left[\lim_{z=\infty} \frac{e^z \Gamma(z)}{z^{z-1/2}} \right].$$

Stirling has shown that the limit of the quantity in the brackets is equal to $\sqrt{2\pi}$. Therefore

$$C_f = \log \sqrt{2\pi} - 1.$$

From formula (5) we get

$$\text{(7)} \qquad \sum_{x=1}^{z} \log x = C_f + z \log z - z + 1 + a_1 \log z + \\ + \sum_{m=1}^{n} a_{2m} \frac{(2m-2)!}{z^{2m-1}} + a_{2n}\, \xi \frac{(2n-2)!}{z^{2n-1}}.$$

Finally adding log z to both members of the equation (7) and writing $a_m = B_m/m!$ we obtain

$$\text{(8)} \qquad \log z! = \log \sqrt{2\pi} + (z+\tfrac{1}{2}) \log z - z + \\ + \sum_{m=1}^{n} \frac{B_{2m}}{2m(2m-1)\, z^{2m-1}} + \frac{\xi B_{2n}}{2n(2n-1)\, z^{2n-1}}.$$

This is *Stirling's* celebrated series for log $z!$. According to what we have seen in § 82, the general term increases indefinitely with n and therefore the series obtained by putting $n=\infty$ will be divergent; nevertheless formula (8) is useful for the computation of log $z!$, The best value of n would be approximately $\pi(z+1)$.

Remark. Of course the logarithms figuring in these formulae are Napier's logarithms, To have *Briggs'* logarithms, the result must still be multiplied by log 10 = 2'30258 50929 9.

Example 3. Given $f(x) = 1/(x+1)^2$ and $a=0$. Since $D^m f(\infty)=0$ for $m=1, 2, 3, \ldots$, moreover $D^{2n}f(x)\, D^{2n+2}f(x) > 0$ and $D^{2n}f$ (x) does not change its sign in the interval $(0,z)$ therefore we may use formula (5) and the remainder may be written $\xi\, a_{2n} D^{2n-1} f(x)$.

According to (6) we find

$$C_f = \sum_{x=0}^{\infty} \frac{1}{(x+1)^2} - 1;$$

in consequence of § 82 this is equal to $C_f = \frac{\pi^2}{6} - 1$ and

(9)
$$\sum_{x=0}^{z} \frac{1}{(x+1)^2} = C_f + 1 - \frac{1}{z+1} + \frac{B_1}{(z+1)^2} - $$
$$- \sum_{m=1}^{n} \frac{B_{2m}}{(z+1)^{2m+1}} - \frac{\xi B_{2n}}{(z+1)^{2n+1}}.$$

For instance, stopping at $n=2$, we shall have

$$\sum_{x=0}^{z} \frac{1}{(x+1)^2} = \frac{\pi^2}{6} - \frac{1}{z+1} - \frac{1}{2(z+1)^2} - \frac{1}{6(z+1)^3} + \frac{\xi}{30(z+1)^5}$$

The best value of n would approximately be $\pi(z+1)$.

The series obtained by putting $n=\infty$ would be divergent, but nevertheless the above formula is useful for the computation of $\Sigma\, 1/(x+1)^2$. For instance, let $z=20$:

$\pi^2/6$	1'64493 4067
$-1/(21)$	-0'04761 9048
$-1/2(21)^2$	-0'00113 3787
$-1/6(21)^3$	-0'00001 7997
	1'59616 3235

and

$R_4 = \xi/30(21)^5$	0'00000 0008 ξ.

The expansion (9) is the same as that obtained by expanding

$$\frac{\pi^2}{a} - \sum_{x=0}^{z} \frac{1}{(x+1)^2} = F(z)$$

into a *Bernoulli series* in § 87 (Ex. 2).

§ 89. The Bernoulli polynomials of the second kind. We will denote the polynomial of degree n by $\psi_n(x)$. Its definition is the following:**

$$\text{(1)} \qquad \mathrm{D}\psi_n(x) = \binom{x}{n-1}$$

from this we obtain by integration

$$\psi_n(x) = \int \binom{x}{n-1} dx + k.$$

Hence by the definition (1) the polynomial is not wholly determined, there still remains an arbitrary constant to dispose of. This will be done as follows:

The expansion of the polynomial into a *Newton* series may be written

$$\text{(2)} \qquad \psi_n(x) = b_0\binom{x}{n} + b_1\binom{x}{n-1} + \ldots + b_n.$$

The operation A performed on both members of (1) gives

$$\Delta\mathrm{D}\,\psi_n(x) = \binom{x}{n-2}$$

but this is also equal, in consequence of (1), to $\mathrm{D}\,\psi_{n-1}$ (x). Integrating both members of

$$\Delta\mathrm{D}\psi_n(x) = \mathrm{D}\psi_{n-1}(x)$$

and disposing conveniently of the arbitrary constant figuring in $\psi_{n-1}(x)$ we get

$$\text{(3)} \qquad \Delta\psi_n(x) = \psi_{n-1}(x).$$

The *Bernoulli* polynomial of the second kind is completely determined by equations (1) and (3).

In § 22 it has been mentioned that the sequences of functions *F,,(x)* satisfying

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[28] *Ch.* Jordan, Polynomes analogues aux polynomes de Bernoulli.. . Acta Scientiarum Mathematicarum, Vol. 4., p. 130, 1929, Szeged.

$$\Delta F_n(x) = F_{n-1}(x)$$

for $x=1, 2, 3, \ldots,$ play an important **rôle** in the Calculus of finite differences.

Moreover, if *F,(x)* is a polynomial of degree n:

$$F_n(x) = c_0\binom{x}{n} + c_1\binom{x}{n-1} + \ldots. + c_n$$

then we have seen that the coefficients c_i are independent of the degree n of the polynomial, so that they may be determined once for all.

From (1) we deduce $D\psi_1$ (x) $=1$ and therefore $\psi_1(x)=x+b_1$ so that $b_0=1.$ Moreover if $n>1$ then formula (1) gives $D\psi_n(0)=0.$

On the other hand we have

$$D\psi_n(0) = \sum_{m=0}^{n} b_m \left[D\binom{x}{n-m}\right]_{x=0}$$

but we saw that (§ 50)

$$\left[D\binom{x}{n-m}\right]_{x=0} = \frac{1}{(n-m)!} S^1_{n-m} = \frac{(-1)^{n-m-1}}{n-m}$$

Therefore we get from (2)

$$(4) \qquad \sum_{m=0}^{n} (-1)'' \frac{b_m}{n-m} = 0 .$$

Starting from $b_0=1$ and writing in this equation successively $n=2, 3, \ldots, .,$ we obtain step by step the coefficients $b_,$. **For** instance:

$\frac{1}{2}b_0 - b_1 = 0$	gives	$b_1 = \frac{1}{2}$
$\frac{1}{3}b_0 - \frac{1}{2}b_1 + b_2 = 0$		$b_2 = -\frac{1}{12}$

and so on.

Table of the numbers b_m :

$b_0 = 1$	$b_, = -19/720$	$b_8 = -3395313628800$
$b_1 = \frac{1}{2}$	$b_5 = 3/160$	$b_9 = 57281/7257600$
$b_2 = -1/12$	$b_6 = -863/60480$	$b_{10} = -3250433/479001600.$
$b_3 = 1/24$	$b_7 = 275124192$	

Particular values of the polynomials. From (2) we deduce:

$$\psi_n(0) = b_n$$

$$\psi_n(1) = b_{n-1} + b_n$$

$$\psi_n(-1) = (-1)^n \sum_{m=0}^{n+1} (-1)^m b_m .$$

From the first two it follows that

$$(5) \qquad \int_0^1 \binom{x}{n} dx = \psi_{n+1}(1) - \psi_{n+1}(0) = b_n .$$

It is easy to express the *Bernoulli* polynomial of the second kind by aid of a ***power series. Since*** (§ *50) :*

$$\binom{x}{n} = \frac{1}{n!} \sum_{m=1}^{n+1} S_n^m x^m$$

by integration we get

$$(6) \qquad \psi_{n+1}(x) = \frac{1}{n!} \sum_{m=1}^{n+1} S_n^m \frac{x^{m+1}}{m+1} + b_{n+1}$$

this is the required expression. Putting into it $x=1$ we have

$$(7) \qquad b_n = \frac{1}{n!} \sum_{m=1}^{n+1} \frac{S_n^m}{m+1}.$$

Knowing the Stirling numbers, the coefficients b_m may be computed by aid of this formula. For instance

$$b_4 = \frac{1}{24}\left[-\frac{6}{2} + \frac{11}{3} - \frac{6}{4} + \frac{1}{5}\right] = -\frac{19}{720}$$

Remark. From formula (7) we may obtain another one by inversion (§ 65). Indeed, multiplying both members by $n!\mathfrak{S}_\nu^n$ and summing from $n=1$ to $n=\nu+1$ we obtain:

$$\sum_{n=1}^{\nu+1} n!\, b_n \mathfrak{S}_\nu^n = \frac{1}{\nu+1} .$$

Sign of the numbers b_ν. Let us remark, starting from equation (5), that the sign of $\binom{x}{n}$ in the interval 0, 1 is the same as that of $(-1)^{n-1}$; therefore we shall have

$$b_{2n} < 0 \text{ and } b_{2n+1} > 0.$$

The polynomial $\psi_n(x)$ may be expressed by a definite integral :

$$(8) \qquad \int_0^1 \binom{x+u}{n} du = \psi_{n+1}(x+1) - \psi_{n+1}(x) = \psi_n(x).$$

§ 90. Symmetry of the Bernoulli polynomials of the second kind. *A. Polynomials of an even degree.* Putting $x=n-1+y$ into $\mathrm{D}\psi_{2n}(x)$ we find

$$\mathrm{D}\psi_{2n}(n-1+y) = \frac{1}{(2n-1)!} y(y^2-1)(y^2-2^2)\ldots[y^2-(n-1)^2].$$

Since this is an odd function of y, its integral will be an even function of y; so that

$$(1) \qquad \psi_{2n}(n-1+y) = \psi_{2n}(n-1-y)$$

or

$$\psi_{2n}(x) = \psi_{2n}(2n-2-x).$$

Hence the polynomial $\psi_{2n}(x)$ is symmetrical. The symmetry axis is $x=n-1$.

B. Polynomials of an odd degree. The operation Δ performed on both members of

$$\psi_{2n+2}(n+y) = \psi_{2n+2}(n-y)$$

will give, according to what we have seen (p. 5)

$$(2) \qquad \psi_{2n+1}(n+y) = -\psi_{2n+1}(n-y-1).$$

Writing in this equation $y=n-1-x$, we get

$$\psi_{2n+1}(x) = -\psi_{2n+1}(2n-1-x)$$

or putting $y=x-\tfrac{1}{2}$ into equation (2):

$$(3) \qquad \psi_{2n+1}(n-\tfrac{1}{2}+x) = -\psi_{2n+1}(n-\tfrac{1}{2}-x).$$

This is the expression of the symmetry of the polynomials of odd degree. Putting into it $x=0$ we obtain

$$\psi_{2n+1}(n-\tfrac{1}{2}) = 0.$$

The polynomials of odd degree are symmetrical with respect to the point $x=n-\tfrac{1}{2}$.

General expression of the symmetry of the polynomials:

$$\psi_n(\tfrac{1}{2}n-1+x) = (-1)^n \psi_n(\tfrac{1}{2}n-1-x).$$

Corresponding particular cases:

$$\psi_n(n-2) = (-1)^n \psi_n(0) = (-1)^n b_n$$

$$\psi_n(n-1) = (-1)^n \psi_n(-1) = 1 - \sum_{m=1}^{n+1} |b_m|.$$

§ 91. Extrema *of the* **polynomials,** *A. Polynomials of an odd degree.* From

$$\mathrm{D}\psi_{2n+1}(x) = \frac{1}{(2n)!} x(x-1)(x-2), \ldots (x-2n+1)$$

it follows that the extrema of $\psi_{2n+1}(x)$ correspond to $x=0$, 1, 2, ... $2n-1$. Since between $-\infty$ and; zero the derivative of ψ_{2n+1} (x) is positive,

$$x=0, 2, 4, \ldots, 2n-2$$

correspond to maxima, and

$$x=1, 3, \ldots, 2n-1$$

to minima.

Theorem 1. Every maximum of the polynomial of odd degree is positive and every minimum negative.

To show it, let us write (8, § 89):

$$\int_0^1 \binom{x+u}{m} du = \psi_m(x). \tag{1}$$

Remark 1. x being an integer,' if $x \geqq m + 1$ then every factor of $\binom{x+u}{m}$ is positive, and therefore y,,,(x) > 0.

Remark 2. If $0 \leqq x < m$ then there are $x+1$ positive and m-x-l negative factors in $\binom{x+u}{m}$ so that its sign and that of $\psi_m(x)$ will be the same as the sign of $(-1)^{m-x-1}$.

Remark 3. If x < 0 then every factor of $\binom{x+u}{m}$ is negative and the sign of $\psi_m(x)$ will be that of $(-1)^m$.

From this we conclude that

$$\psi_{2n+1}(2\nu) > 0 \qquad \text{if} \quad \nu \geqq 0. \tag{2}$$

and

(3) $$\psi_{2n+1}(2\nu-1)<0 \quad \text{if} \quad 0<\nu\leqq n.$$

The inequalities (2) and (3) demonstrate the theorem. Moreover in consequence ***of Remark 3 we*** get

$$\psi_{2n+1}(-x)<0 \text{ if } x>0.$$

B. Polynomials of an even degree. Theorem 2. The maxima of the polynomial $\psi_{2n}(x)$ correspond to $x=1, 3, 5, \ldots, 2n-3$ and they are all positive. The minima correspond to $x=0, 2, 4, \ldots, 2n-2$ and they are all negative.

From ***Remark*** 2. we deduce that

(4) $$\psi_{2n}(2\nu-1)>0$$

hence the maxima are all positive. Moreover from 2. it follows that

(5) $$\psi_{2n}(2\nu)<0 \quad \text{if} \quad 0\leqq\nu\leqq n-1$$

hence the minima are all negative. Finally in consequence of 3. we have

$$\psi_{2n}(-x)>0 \quad \text{if} \quad x>0.$$

Theorem 3. If x is an integer, then the absolute value of y,,,(x) decreases from $x=-1$ up to the point or axis of symmetry, and then it increases again. So that we have

(6)
$$|\psi_m(-1)|>|\text{y},(\text{O})|>\ldots>|\psi_m(\tfrac{1}{2}m-1)|<\ldots<|\psi_m(m-1)|.$$

To prove it we shall show that

(7) $$|\text{Y''(X)}|>|\psi_m(x+1)|$$

if -1 $\leqq x$ 6 $\tfrac{1}{2}m-2$. For this we shall determine the mean of y,,,(x). We find

$$\mathbf{M}\,\psi_m(x)=\tfrac{1}{2}\,[\psi_m(x)+\psi_m(x+1)\,1$$

and in consequence of formula (8) § 89 we have:

$$(8) \quad \mathbf{M}\psi_m(x) = \tfrac{1}{2}\int_0^1 \left[\binom{x+1+u}{m} + \binom{x+u}{m}\right] du =$$

$$= \int_0^1 \binom{x+u}{m-1} \frac{x+u+1-\tfrac{1}{2}m}{m}\, du \,.$$

The sign of the mean will be equal to that of $(x+u+1-\tfrac{1}{2}m)$, $(x+u)_{m-1}$. In the interval considered

$$(9) \qquad -1 \leqq x \leqq \frac{m}{2} - 2$$

the first factor is always negative. The number of the negative factors in $(x+u)_{m-1}$ is equal to m-2-x; therefore if m and x are both odd or both even then the mean is negative; that is

$$\psi_{2n}(2\nu) + \psi_{2n}(2\nu+1) < 0$$
$$\psi_{2n+1}(2\nu-1) + \psi_{2n+1}(2\nu) < 0 \,.$$

On the other hand if one of them is odd, the other even, then the mean is positive and we have

$$\psi_{2n}(2\nu-1) + \psi_{2n}(2\nu) > 0$$
$$\psi_{2n+1}(2\nu) + \psi_{2n+1}(2\nu+1) > 0.$$

Hence from (2), (3) and (4) we conclude immediately that in the interyal (9) we have for every integer value of x:

$$|\psi_m(x)| > |\psi_m(x+1)|$$

therefore Theorem 3 is demonstrated.

Since the maxima of $\psi_m(x)$ decrease from $x=0$ to the symmetry point or axis and then increase again, hence the maximum nearest to this point or axis is the ***minimum maximorum. The*** minima of y,,,(x) increase from $x=0$ to the symmetry point or axis, and then they decrease again, so that the minimum nearest to this point is the ***maximum minimorum.***

If $m=2n+1$ and n is odd, the minimum maximorum corresponds to $x=n-1$ and the maximum minimorum to $x=n$. If n is even, then $x=n$ corresponds to the first and $x=n-1$ to the second. In both cases we have

$$\psi_{2n+1}(n-1) = -\psi_{2n+1}(n).$$

Let us remark that the highest maximum corresponds to $x=0$ and is equal to

$$\psi_{2n+1}(0) = b_{2n+1}$$

moreover, that the lowest minimum corresponds to $x=2n-1$ and is equal to

$$\psi_{2n+1}(2n-1) = -\psi_{2n+1}(0) = -b_{2n+1} .$$

If $m=2n$ and if n is even, the minimum maximorum is on the symmetry axis and is equal to $\psi_{2n}(n-1)$; the maximum minimorum is then at $x=n-2$ and we have ψ_{2n} (n-2) = y,.(n). If n is odd, then $\psi_{2n}(n-1)$ is the maximum minimorum and ψ_{2n} (n-2) $=\psi_{2n}$ (n) the minimum maximorum.

Roots of the polynomials. The roots of $\psi_n(x)=0$ are all situated in the interval -1, **n-l** ; indeed in this interval the function changes its sign n times; since y,,(x) is of degree n, hence all roots are real and single.

§ 92. Particular cases of **the polynomials.** From formula (2) § 89 it follows that

$$\psi_1(x) = x + \frac{1}{2}$$

$$\psi_2(x) = \binom{x}{2} + \frac{1}{2}\binom{x}{1} - \frac{1}{12}$$

$$\psi_3(x) = \binom{x}{3} + \frac{1}{2}\binom{x}{2} - \frac{1}{12}\binom{x}{1} + \frac{1}{24}$$

$$\psi_4(x) = \binom{x}{4} + \frac{1}{2}\binom{x}{3} - \frac{1}{12}\binom{x}{2} + \frac{1}{24}\binom{x}{1} - \frac{19}{720}$$

etc. (See Figure 2.)

§ 93. Limits of the numbers b_n and of the polynomials ψ_n (x) . Since

$$b_n = \int_0^1 \binom{x}{n} dx$$

and

$$\binom{x}{n} = \left[x\left(\frac{x-1}{1}\right)\left(\frac{x-2}{2}\right) \cdot \cdot \cdot \cdot \left(\frac{x-n+1}{n-1}\right)\right] \frac{1}{n}$$

then the absolute value of each factor of the second member is less than one if $0 < x < 1$ and we have

$$\left|\binom{x}{n}\right| < \frac{1}{n}$$

and therefore

$$|b_n| < \frac{1}{n}$$

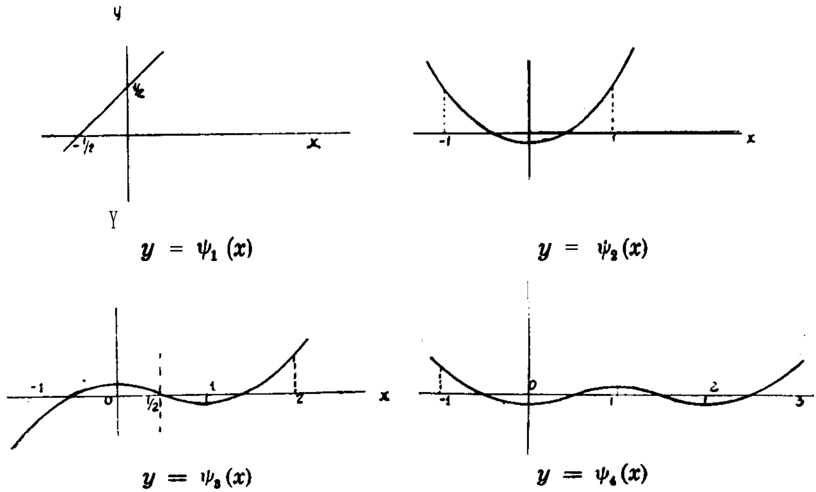
so that

(1) $$\lim_{n=\infty} |b_n| = 0.$$

Moreover we may write

$$b_{n+1} = \int_0^1 \binom{x}{n+1} dx = \int_0^1 \binom{x}{n} \frac{x-n}{n+1} dx.$$

Figure 2.



$\binom{x}{n}$ does not change its sign in the interval $(0,1)$; hence we may apply to **this** integral the mean value theorem. **We** obtain

$$b_{n+1} = \frac{\xi-n}{n+1} b_n \text{ if } 0 < \xi < 1.$$

from this we deduce

$$\frac{n-1}{n+1} |b_n| < |b_{n+1}| < \frac{n}{n+1} |b_n|$$

and

(2) $$\left|\frac{b_{n+1}}{b_n}\right| < \frac{n}{n+1};$$

that is, by increasing *n,* the absolute values of the numbers b_n diminish.

Since the series

$$b_0 + b_1 + b_2 + \ldots\ldots + b_n + \ldots.$$

is alternate, from (1) and (2) it follows that it is convergent.

In the preceding paragraph we have seen that in the interval $0 < x < n-2$

(3) $$|\psi_n(x)| < |b_n|.$$

From this we conclude by aid of (1) that

(4) $$\lim_{n=\infty} \psi_n(x) = 0$$

if $0 < x <$ n-2. But this interval may be somewhat enlarged. Indeed, according to formula (1) of § 89 we have

$$\psi_n(x) = \int_0^1 \binom{x+u}{n} du$$

therefore

$$\psi_n(-1) = \int_0^1 \binom{u-1}{n} du.$$

Starting from this formula *A.* **Szücs** has shown that the limit of $\psi_n(-1)$, for $n=\infty$ is equal to zero. Let us write the quantity under the integral sign in the following way:

$$(-1)^n\left(1-\frac{u}{1}\right)\left(1-\frac{u}{2}\right)\ldots\ldots\left(1-\frac{u}{n}\right).$$

From

$$1 + x \leq e^x,$$

remarking that $0 <$ u $<$ 1, we deduce

$$\left(1-\frac{u}{1}\right)\left(1-\frac{u}{2}\right)\ldots\ldots\left(1-\frac{u}{n}\right) \leq e^{-u(1+\frac{1}{2}+\ldots\frac{1}{n})}$$

and then

$$\int_0^1 \binom{u-1}{n} du < \int_0^1 e^{-u\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)} du = \frac{1-e^{-1-\frac{1}{2}-\ldots-\frac{1}{n}}}{1+\frac{1}{2}+\ldots+\frac{1}{n}}.$$

Since for $n=\infty$ the limit of the last member is equal to zero, we conclude

(5) $$\lim_{n=\infty} \psi_n(-1) = 0.$$

But according to formula (3) § 89 we have

$$\psi_n(-1) = (-1)^n[b_0-b_1+b_2-b_3+\ldots+(-1)^n b_n]$$

or

$$|\psi_n(-1)| = 2-\sum_{m=0}^{n+1} |b_m|$$

and if n increases indefinitely

$$\sum_{m=0}^{\infty} |b_m| = 2.$$

Hence the series Σ I b_m I is convergent.

Moreover, in consequence of (3) the series Σ | y,,,(X) | is convergent too, if n-2 > x ≧ 0.

§ 94, Operations on the Bernoulli polynomials of the second kind. *Differences* of the polynomials. According to formula (3) § 89 we have

(1) $$\Delta^m \psi_n(x) = \psi_{n-m}(x).$$

Hence the *indefinite sum* of the polynomial is

(2) $$\Delta^{-1} \psi_n(x) = \psi_{n+1}(x) + k.$$

From this we deduce the *sum of the polynomial;* for instance, taking account of (3) § 90 we find

$$\sum_{x=0}^{2n-1} \psi_{2n}(x) = \psi_{2n+1}(2n-1) - \psi_{2n+1}(0) = -2\, b_{2n+1}$$

or in consequence of (1) § 90

$$\sum_{x=0}^{2n-2} \psi_{2n-1}(x) = \psi_{2n}(2n-2) - \psi_n(0) = 0.$$

Mean of the polynomial: Since $\mathbf{M} = 1+\frac{1}{2}\Delta$, therefore

(3) MY"(X) $= \psi_n(x) + \frac{1}{2}\psi_{n-1}(x).$

Inverse operation of the mean: According to formula (12) § 39 we have

$$\mathbf{M}^{-1}\ \psi_n(x) = \sum_{m=0}^{n+1} \frac{(-1)^m}{2^m}\ \psi_{n-m}(x). \tag{4}$$

Derivatives of the polynomial:

$$\mathbf{D}\ \psi_n(x) = \binom{x}{n-1}$$

$$\mathbf{D}^m \psi_n(x) = \frac{1}{(n-1)!} \sum_{\nu=m-1}^{n} (\nu)_{m-1}\, S_{n-1}^{\nu}\ x^{\nu-m+1}$$

$$\mathbf{D}^m \psi_n(0) = \frac{(m-1)!}{(n-1)!}\ S_{n-1}^{m-1} \tag{5}$$

Integral of the polynomial. By integration by **parts** we obtain

$$\int \psi_n(x)\, dx = (x-n+1)\ \psi_n(x) - \int (x-n+1) \binom{x}{n-1} dx.$$

Since the quantity under the integral sign of the second member may be written $n\binom{x}{n}$ *we* **have**

$$\int \psi_n(x) dx = (x-n+1)\ \psi_n(x)\ - n\ \psi_{n+1}(x) + k; \tag{6}$$

from this we obtain

$$\int_0^1 \psi_n(x) dx = (2-n) b_{n-1} + (1-n) b_n .$$

§ 95. **Expansion of the Bernoulli polpomials of the second** kind into a series of ***Bernoulli*** polynomials of the first kind. According to **§ 84** the coefficients of this expansion are given by

$$c_0 = \int_0^1 \psi_n\, (x) dx = (2-n) b_{n-1} + (1-n) b_n$$

$$c_1 = \Delta\, \psi_n(0) = \psi_{n-1}(0) = b_{n-1}$$

$$c_m = [\mathbf{D}^{m-1} \Delta \psi_n(x)]_{x=0} = [\mathbf{D}^{m-1} \psi_{n-1}(x)]_{x=0} = \frac{(m-2)!}{(n-2)!}\ S_{n-2}^{m-2} .$$

The values of the above integral and of the derivatives have been obtained in the preceding paragraph.

The required expansion will be

$$\psi_n(x) = (2-n)b_{n-1} + (1-n)b_n + b_{n-1}\varphi_1(x) + $$
$$+ \sum_{m=2}^{n+1} \frac{(m-2)!}{(n-2)!} S_{n-2}^{m-2} \varphi_m(x),$$

Putting $x=0$ we get

$$\left(n-\frac{3}{2}\right) b_{n-1} + nb_n = \sum_{m=2}^{n+1} \frac{(m-2)!}{(n-2)!} S_{n-2}^{m-2} a_m .$$

From this formula we may obtain a_m by inversion (§ 65). Let us multiply both members by (n-2)! $\mathfrak{S}_{i-2}^{n-2}$ and sum from $n=3$ to $n=i+1$. We obtain

$$\frac{B_i}{i(i-1)} = (i-2)!\, a_i = \sum_{n=3}^{i+1} \left[\left(n-\frac{3}{2}\right) b_{n-1} + nb_n\right] (n-2)!\, \mathfrak{S}_{i-2}^{n-2}$$

This equation gives the *Bernoulli* numbers by aid of the coefficients b_n .

§ 96. Application of the **Bernoulli polynomials of the second kind.** *1.* Integration of a function expanded into a *Newton* series, Given

$$f(x) = f(o) + \binom{x}{1} \Delta f(0) + \binom{x}{2} \Delta^2 f(0) + \ldots, + \binom{x}{n} \Delta^n f(0) + \ldots$$

according to formula (1) § 89 we have

$$\int f(x)dx = \psi_1(x)f(0) + \psi_2(x)\Delta f(0) + \ldots, + \psi_{n+1}(x)\Delta^n f(0) +$$
$$+ \ldots + k$$

and moreover

(1) $$\int_0^1 f(x)dx = b_0 f(0) + b_1 \Delta f(0) + \ldots + b_n \Delta^n f(0) + \ldots$$

Example 1.

$$\int_0^1 \frac{dx}{x+1} = \log 2 = \sum_{n=0}^{\infty} \frac{(-1)^n b_n}{n+1} = 1 - \sum_{n=1}^{\infty} \frac{|b_n|}{n+1}$$

§ 97. Expansion of a polynomial into a series of Bernoulli polynomials of the second kind. Let us write

(1) $$f(x) = \sum_{i=0}^{n+1} c_i \psi_i(x).$$

The operation Δ^{-1} executed on both members of this equation gives

$$\Delta^{-1} f(x) = \sum_{i=0}^{n+1} c_i \psi_{i+1}(x) + k;$$

by derivation we obtain

$$D\Delta^{-1} f(x) = \sum_{i=0}^{n+1} c_i \binom{x}{i}.$$

Putting $x=0$ it results that

(2) $$c_0 = D\Delta^{-1} f(0).$$

To obtain c_m let us write

$$\Delta^{m-1} f(x) = c_{m-1} + c_m \psi_1(x) + c_{m+1}\psi_2(x) + \ldots$$

By derivation and subsequently putting $x=0$ we find

(3) $$c_m = D\Delta^{m-1} f(0).$$

Example 1. Given $f(x) = x^n / n!$. We have $\Delta^{-1} f(x) = \varphi_{n+1}(x)$ *the Bernoulli* polynomial of the first kind of degree $n+1$. Hence $D\psi_{n+1}(x) = \psi_n(x)$ and $\psi_n(0) = a_n$. Therefore

$$c_0 = a_n$$

moreover

$$c_1 = Df(0) = 0 \text{ if } n > 1$$

$$\Delta^{m-1} D \frac{x^n}{n!} = \frac{1}{(n-1)!} \Delta^{m-1} x^{n-1}.$$

Hence

$$c_m = \frac{(m-1)!}{(n-1)!} \mathfrak{S}_{n-1}^{m-1}$$

Finally

(4) $$\frac{x^n}{n!} = a_n + \frac{1}{(n-1)!} \sum_{m=2}^{n+1} (m-1)! \, \mathfrak{S}_{n-1}^{m-1} \psi_m(x).$$

Putting $x=0$ we get another **expression** of the *Bernoulli* numbers in terms of the coefficients b_m:

(5) $$\frac{B_n}{n} = (n-1)! \, a_n = -\sum_{m=2}^{n+1} (m-1)! \, b_m \mathfrak{S}_{n-1}^{m-1}$$

From this formula we may **obtain** the number b_i by inversion **(§ 65).** Multiplying both members by S_{i-1}^{n-1} and summing from $n=2$ to $n=i+1$ we get

(6) $$(i-1)! \, b_i = -\sum_{n=2}^{i+1} (n-1)! \, a_n S_{i-1}^{n-1}$$

Remark. To obtain the expansion of the ***Bernoulli*** polynomial of the first kind into a series of the second kind, it is sufficient to sum formula (4) from $x=0$ to $x=z$; we find

$$\text{(7)} \qquad \varphi_{n+1}(z) = a_{n+1} + a_n z + \\ + \frac{1}{(n-1)!} \sum_{m=2}^{n+1} (m-1)!\, \mathfrak{S}_{n-1}^{m-1} [\psi_{m+1}(z) - b_{m+1}].$$

If the function to be expanded is not a polynomial, then the series (1) will be infinite and questions of convergence will arise.

Example 2. Given f(x) $= 2^x$. We find, using the preceding method $c_m =$ log 2 for $m=0, 1, 2, \ldots$; therefore the expansion will be

$$2^x = \log 2\,[1 + \psi_1(x) + \psi_2(x) + \ldots];$$

putting $x=0$, we have

$$\frac{1}{\log 2} = 1 + b_1 + b_2 + \ldots .$$

We have seen in $93 that both series are convergent.

Example 3. By aid of the expansion (1) it is possible to determine the ***generating 'function*** of the ***Bernoulli*** polynomial of the second kind. For this it is sufficient to dispose of *f(x) so* as to have $c_m = t^m$. Since $\Delta^{m-1}(1+t)^x = t^{m-1}(1+t)^x$ and $D(1+t)^x = (1+t)^x \log(1+t)$ it is easy to see that

$$f(x) = \frac{t(1+t)^x}{\log(1+t)}$$

indeed, the coefficients of the expansion of this function will be

$$c_m = [D\Delta^{m-1} f(x)]_{x=0} = t^m;$$

therefore if -1 $< t \leq$ 1 then

$$\text{(8)} \qquad \frac{t(1+t)^x}{\log(1+t)} = \sum_{m=0}^{\infty} \psi_m(x)\, t^m$$

is the generating function of the polynomials y,,,(x).

Putting $x=0$ we obtain the generating function of the numbers b_m,

$$\text{(9)} \qquad \frac{t}{\log(1+t)} = \sum_{m=0}^{\infty} b_m\, t^m .$$

After what we have seen, if $-1 < t \leq 1$ then both series are convergent. Putting into (8) and (9) $t=1$ we obtain the results of Example 2.

Putting into (9) $t = -\frac{1}{2}$ we have

$$\sum_{m=0}^{\infty} \frac{(-1)^m b_m}{2^m} = \frac{1}{2 \log 2}.$$

This may be written in the following manner:

$$\sum_{m=0}^{\infty} \frac{|b_m|}{2^m} = 2 - \frac{1}{2 \log 2}$$

Example 4. Given $f(x) = \log(1+x)$; we have

$$\Delta^{-1} \log(x+1) = \log \Gamma(x+1) + k \text{ and } \quad D\Delta^{-1} \log x = F(x)$$

therefore

$$c_0 = F(0) = -C$$

where C is ***Euler's*** constant. Moreover

$$c_m = [\Delta^{m-1} D \log(x+1)]_{x=0} = \left[\Delta^{m-1} \frac{1}{x+1}\right]_{x=0} = \frac{(-1)^{m-1}}{m}$$

so that finally

$$(10) \qquad \log(x+1) = -C + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \psi_m(x).$$

We have seen that $\Sigma |\psi_m(x)|$ is convergent if $m-2 \geq x \geq -1$, therefore the series (10) is absolutely convergent. Putting $x=0$ we find

$$(11) \qquad C = \sum_{m=1}^{\infty} \frac{|b_m|}{m}$$

This is an expression of ***Euler's*** constant in terms of the numbers b_m, but the convergence is very slow,

§ 98, **The Bernoulli series of the second kind,** If we put

$$\Delta^{-1} 1 = -\psi_1(u-x)$$

then the operation of summation by parts executed on $\Sigma f(x)$ may be written

$$(1) \qquad \Delta^{-1} f(x) = -\psi_1(u-x) f(x) + \Delta^{-1}[\psi_1(u-x-1) \Delta f(x)].$$

Putting again

$$\Delta^{-1} \psi_1 (u-x-1) = -\psi_2(u-x)$$

the operation of summation by partsperformedon the second member of (1) will give

$$-\psi_2(u-x)\,\Delta f(x) + \Delta^{-1}\left[\psi_2(u-x-1)\,\Delta^2 f(x)\right]$$

continuing in the same manner we shall obtain, after having applied the operation of summation by parts $n-1$ times,

$$\Delta^{-1} f(x) = -\sum^{n} \psi_m(u-x)\,\Delta^{m-1} f(x) +$$
$$+ \Delta^{-1}\left[\psi_{n-1}(u-x-1)\,\Delta^{n-1} f(x)\right].$$

As we may add to an indefinite sum any arbitrary constant, the next summation by parts may be written

$$\Delta^{-1}\left[\psi_{n-1}(u-x-1)\,\Delta^{n-1} f(x)\right] = \left[-\psi_n(u-x) + b_n\right]\Delta^{n-1} f(x) +$$
$$+ \Delta^{-1}\left\{\left[\psi_n(u-x-1) - b_n\right]\Delta^n f(x)\right\}$$

therefore we shall have

$$(2) \qquad \Delta^{-1} f(x) = -\sum_{m=1}^{n+1} \psi_m(u-x)\,\Delta^{m-1} f(x) + b_n\Delta^{n-1} f(x) +$$
$$+ \Delta^{-1}\left\{\left[\psi_n(u-x-1) - b_n\right]\Delta^n f(x)\right\},$$

Now let us write the sum of $f(x)$ from $x=u-1$ to $x=z$ in the following manner

$$(3) \quad \sum_{x=u-1}^{z} f(x) = -\sum_{m=1}^{n+1} \psi_m(u-z)\Delta^{m-1}f(z) + b_n\Delta^{n-1}f(z) + \lambda(u) + R_n$$

where $\lambda(u)$ is a function independent of z, obtained by putting into every term of the indefinite sum (2), except the last, $x=u-1$. To abbreviate we have written R_n for the remainder

$$(4) \qquad R_n = \sum_{x=u-1}^{z} \left[\psi_n(u-x-1) - b_n\right]\Delta^n f(x).$$

From (3) we get by derivation with respect to z:

$$D\Delta^{-1}f(z) = -\sum_{m=1}^{n+1} \psi_m(u-z)\,D\Delta^{m-1}f(z) + \sum_{m=1}^{n+1}\binom{u-z}{m-1}\Delta^{m-1}f(z) +$$
$$+ b_n\,D\Delta^{n-1}f(z) + D\,R_n\,.$$

If we put $z=u$ we have

$$(5)\qquad [\underset{z}{D} \sum_{x=u-1}^{z} f(x)]_{z=u} = f(u) - \sum_{m=1}^{n} b_m \text{ DA”-’ } f(u) + [D\, R_n]_{z=u}$$

To determine the remainder let us remark that

$$(6)\qquad \psi_n(u-x-1) - b_n$$

will never be negative if $n=2m$; and it will never be positive if $n=2m+1$ and u-x $\leqq$ *2m.* Therefore, supposing that $u-z<n$ *the* expression (6) will never change its sign in the interval $x=u-1$, $x=z$ so that we may apply the mean **value** theorem We shall find, starting from (4)

$$R_n = \Delta^n f(\xi) \sum_{x=u-1}^{z} [\psi_n(u-x-1) - b_n]$$

where u- 1 $< \xi < z$. We have seen above that the indefinite sum of $\psi_n(u-x-1)$ is $-\psi_{n+1}(u-x)$; therefore the summation in R_n will give

$$R_n = \Delta^n f(\xi)[-\psi_{n+1}(u-z) - b_n z + \psi_{n+1}(1) + b_n(u-1)].$$

Now the derivation with respect to z gives

$$D\, R_n = [-\psi_{n+1}(u-z) - b_n z + \psi_{n+1}(1) + b_n(u-1)]\, D\Delta^n f(\xi) +$$
$$+ \Delta^n f(\xi) \left[\binom{u-z}{n} - b_n\right].$$

If we put into this equation $z=u$, the first term of the second member will vanish, and we shall obtain

$$[D\, R_n]_{z=u} = -b_n \Delta^n f(\eta) \quad \text{where} \quad u\text{-}1 < \eta < u.$$

Putting this value into equation (5) and writing x instead of u *we* obtain the *Bernoulli series* of the second kind:

$$(7)\qquad f(x) = [\underset{z}{D} \sum_{t=x-1}^{z} f(t)]_{z=x} + \sum_{m=1}^{n} b_m D\Delta^{m-1} f(x) + b_n \Delta^n f(\eta)$$

where x-l $< \eta <$ x.

Remark. If $\Delta^n f$ (x) and $\Delta^{n+1} f(x)$ do not change their sign in the interval $(0, 1)$ and if

&f(x) $\Delta^{n+1} f$ *(x)* > 0

and therefore

$$R_n R_{n+1} < 0,$$

then the same ratiocination as that in § 87 leads to

$$(8) \qquad R_n = \xi\, b_n\, \mathrm{D}\Delta^{n-1} f(x) \quad \text{where } 0 < \xi < 1.$$

Formula (7) may be transformed by putting $f(x) = \Delta F(x)$; then we find:

$$(9) \quad \Delta F(x) = \{\underset{z}{\mathrm{D}} [F(z) - F(x-1)]\}_{z=x} + \sum_{m=1}^{n} b_m\, \mathrm{D}\Delta^m F(x) + b_n\, \Delta^{n+1} F(\eta).$$

Remark. The first term of the second member is generally equal to $\mathrm{D}F(x)$, except if $F(x)$ contains a periodic. . function whose period is equal to one, which function vanishes in $F(z)-F(x-1)$; so that then the derivative of this quantity with respect to z is not necessarily equal to that of $F(r)$.

Formula (9) may be useful, first if the derivative of the function is to be determined and the differences of the derivative are simple.

Example 1. If $F(x) = \log \Gamma(x+1)$ then $\Delta F(x) = \log(x+1)$ and $\underset{z}{\mathrm{D}} [\log \Gamma(z+1) - \log \Gamma(x)]_{z=x} = f(x)$ and moreover

$$\mathrm{D}\Delta^m F(x) = \frac{(-1)^{m-1}\,(m-1)!}{(x+m)_m}$$

therefore

$$(10) \quad F(x) = \log(x+1) - \sum_{m=1}^{n} \frac{(-1)^{m-1}\,(m-1)!}{(x+m)_m} b_m - b_n\, \Delta^n \log(\eta+1).$$

This expansion is similar to that of $F(x)$ obtained by aid of the ***Bernoulli series*** of the first kind (formula 10, § 87), but that series was divergent and the present series (10) is convergent. Indeed the absolute value of the general term is smaller than $|b_m/m|$ and we have seen that this series is convergent.

$$\frac{|b_m|}{\binom{x+m}{m} m} < \frac{|b_m|}{m} \qquad \text{if } x > 0,$$

If x is large enough, the convergence is rapid, as we shall see in a numerical example (§ 118).

Putting $x=0$ into (10) we should obtain the expression of the ***Euler's*** constant found before (11, § 97).

Secondly, formula (9) may be useful for the determination of $\Delta F(x)$ if the derivative and its differences are simple.

Example 2. If $F(x)=\log(x+1)$ then $\mathsf{D}F(x)=\frac{1}{x+1}$ and

$$\Delta^m \mathsf{D}F(x) = \frac{(-1)^m m!}{(x+m+1)_{m+1}}$$

and finally

$$\Delta \log(x+1) = \frac{1}{x+1} + \sum_{m=1}^{n} \frac{(-1)^m m!\, b_m}{(x+m+1)_{m+1}} + b_n \Delta^{n+1} \log(\eta+1).$$

Thirdly, formula (9) may be useful still in other cases; for instance if $[\mathsf{D}\Delta^m F(x,t)]_{x=0} = c_m t^m \omega(t)$ where c_m is a numerical constant. From this it follows that $[\mathsf{D}F(x,t)]_{x=0} = c_0$ *o(f).* Dividing both members of equation (9) by $c_0\,\omega(t)$, *we* find

$$(11) \qquad \frac{[\Delta F(x, f)]_{x=0}}{c_0\,\omega(t)} = \sum_{m=0}^{n} \frac{c_m}{c_0} b_m t^m + \frac{R_n}{c_0\,\omega(t)}$$

Example 3. If $F(x)=(1+t)^x$ then $\mathsf{D}F(x)=(1+t)^x \log(t+1)$ and $\Delta^m\mathsf{D}F(x) = (1+t)^x t^m \log(t+1)$ moreover $\Delta^m\mathsf{D}F(0) = t^m \log(t+1)$ and $\Delta F(x) = (1+t)^x\, t$ so that $\Delta F(0) = t$. Consequently we shall have

$$\frac{t}{\log(t+1)} = \sum_{m=0}^{n} b_m t^m + \frac{R_n}{\log(t+1)}$$

To determine the remainder R_n let us remark that in the case considered we have $\Delta^n F(x)\, \Delta^{n+1} F$ (x) > 0; therefore we may use formula (8) so that

$$R_n = \xi\, b_n \mathsf{D}\Delta^n F(x) = \xi\, b_n t^n \log(t+1).$$

The formula above has already been obtained (in 9, § 97), but not the remainder.

§ 99. Gregory's Summation Formula. This is a formula by aid of which the integral of a function may be expressed by its sum and its differences, or the sum by the integral and the differences.

Apart from the remainder, the formula may be easily deduced by aid of the symbolical method. In § 6 we had formula (4) :

$$h\mathsf{D} = \log(1+\underset{h}{\Delta}).$$

From this it follows that

$$\frac{1}{h\mathbf{D}} = \frac{1}{\Delta}\left[\frac{\Delta}{\log(1+\Delta)}\right]$$

Since according to formula (9) § 97 the expression in the brackets is equal to the generating function of the coefficients b_n of the *Bernoulli* polynomial of the second kind, we have

$$\frac{1}{h\mathbf{D}} = \sum_{i=0}^{\infty} b_i \underset{h}{\Delta}^{i-1} = \frac{1}{\underset{h}{\Delta}} + \sum_{i=1}^{\infty} b_i \underset{h}{\Delta}^{i-1}$$

Performing this operation on *f(x),* and calculating the sum corresponding from $x=a$ to $x=z$, we get

$$\frac{1}{h}\int_a^z f(x)\,dx = \sum_{x=a}^{z} f(X) + \sum_{i=1}^{\infty} b_i \left[\underset{h}{\Delta}^{i-1} f(z) - \underset{h}{\Delta}^{i-1} f(a)\right].$$

This is Gregory's formula. Putting $h=1$, $a=0$ and $z=1$ we get formula (1) of § 96.

To obtain the summation formula with its remainder, we start from equation (7) § 98, integrating both members from $x=a$ to $x=z$; the first term of the second member will give

$$\sum_{x=a}^{z} f(x).$$

In consequence of the mean value theorem the remainder will be

$$\int_a^z b_n \Delta^n f(\eta)\,dx = (z-a)\,b_n\,\Delta^n f(\zeta)$$

where $a < \zeta < z$. Finally it follows that

$$(1)\quad \int_a^z f(x)dx = \sum_{x=a}^{z} f(x) + \sum_{m=1}^{n} b_m\left[\Delta^{m-1} f(z) - \Delta^{m-1} f(a)\right] + \\ + b_n (z-a)\,\Delta^n f(\zeta).$$

This formula is more advantageous than *Euler's* summation formula, if we deal with functions whose differences are less complicated than their derivatives. Moreover there are functions which lead to convergent series if we use formula (1), while the corresponding *Euler* formula is divergent. For instance, this may be due to the fact that $\mathbf{D}^n\left(\frac{1}{x}\right)$ increases indefinitely with n and $\Delta^n\left(\frac{1}{x}\right)$ tends to zero if n increases, There are some particular cases in which the remainder may be given in another form.

First, for instance, if $\Delta^n f(x)$ and $\Delta^{n+1} f(x)$ do not change their sign in the interval, a, z, and moreover if

$$\Delta^n f(x)\, \Delta^{n+1} f(x) > 0$$

then we may obtain the remainder by integrating the expression (8) of § 98 from $x=a$ to $x=z$; we find

$$\text{(2)}\qquad \xi\, b_n[\Delta^{n-1} f(z) - \Delta^{n-1} f(a)] \quad \text{where} \quad 0 < \xi < 1.$$

Secondly, let us suppose that $\Delta^m f(\infty) = 0$ for $m=1, 2, 3, \ldots$ and write the remainder of (1) in the following form:

$$R_n = b_n \int_a^\infty \Delta^n f(\zeta) dx - b_n \int_z^\infty \Delta^n f(\zeta) dx.$$

If we denote by C_f the part of the second member of (1) which is independent of z, then

$$\text{(3)}\qquad C_f = b_n \int_a^\infty \Delta^n f(\zeta) dx - \sum_{m=1}^{n} b_m \Delta^{m-1} f(a)$$

and consequently

$$\text{(4)}\qquad \int_a^z f(x) dx = \sum_{x=a}^{z} f(x) + \sum_{m=1}^{n} b_m \Delta^{m-1} f(z) - b_n \int_z^\infty \Delta^n f(\zeta) dx + C_f.$$

Putting $z=\infty$ we obtain an expression which permits us to compute the number C_f. Indeed, in consequence of $\Delta^m f(\infty) = 0$ we obtain from (1)

$$\text{(5)}\qquad C_f = -\sum_{x=a}^{\infty} f(x) - \lim_{z=\infty} b_1 f(z) + \int_a^\infty f(x) dx.$$

Remark. In some cases, when the operation $D\Delta^{-1}$ gives the same result as $\Delta^{-1}D$; then the expansion (1) becomes identical with that of

$$F(z) = \int_a^z f(x)\ dx$$

expanded by aid of formula (7) § 98.

Example 1. Given $f(x) = 1/(x+1)$. We find

$$\int_a^z f(x)dx = \log(z+1) - \log(a+1)$$

and moreover

$$\sum_{x=a}^{z} \frac{1}{x+1} = F(z) - F(a)$$

and

$$\Delta^{m-1} f(x) = \frac{(-1)^{m-1}(m-1)!}{(x+m)_m}.$$

Hence from (1) it follows that

$$\log(z+1) - \log(a+1) = F(z) - F(a) + \sum_{m=1}^{n} b_m (-1)^{m-1}(m-1)!$$

$$\cdot\left[\frac{1}{(z+m)_m} - \frac{1}{(a+m)_m}\right] + b_n(z-a)\frac{(-1)^n n!}{(\zeta+n+1)_{n+1}} .$$

From this we conclude that the remainder is smaller than $b_n(z-a)/(n+1)$.

This formula may serve for the determination of f(z), if a is not too small; otherwise the convergence is too slow. If $a=0$ it is much better to use formula (4). The constant C_f will be determined by (5):

$$C_f = \lim_{z=\infty}\left[\log(z+1) - \sum_{x=0}^{z}\frac{1}{x+1}\right] = -C$$

where C is *Euler's* constant. Since

$$\sum_{x=0}^{z}\frac{1}{x+1} = F(z) + C$$

formula (4) will be

$$\log(z+1) = F(z) + \sum_{m=1}^{n} b_m \Delta^{m-1} f(z) - b_n \int_z^{\infty} \Delta^n f(\zeta) dx.$$

This is the same formula as that we obtained in Example 1, § 98, except for the remainder.

Remark. Formula (1) but without the remainder, has been discovered by ***Gregory*** in 1670 ***[Whittaker and Robinson,*** Calculus of Observations, p. 144]; this was the earliest formula of numerical integration,

CHAPTER VI.

EULER'S AND BOOLE'S POLYNOMIALS. SUMS OF RECIPROCAL POWERS.

§ 100. Euler's polynomials, We shall define the *Euler* polynomial $E_n(x)$ of degree n by the following equation

$$(1) \qquad \mathrm{M}E_n(x) = \frac{x^n}{n!};$$

that is, the mean of this polynomial is equal to $x^n/n!$. From (1) it follows that

$$\mathrm{DM}E_n(x) = \frac{x^{n-1}}{(n-1)!}$$

but according to (1) this is also equal to $\mathrm{M}E_{n-1}(x)$ therefore

$$(2) \qquad \mathrm{DM}E_n(x) = \mathrm{M}E_{n-1}(x)$$

Performingthe operation M^{-1} on both members of this equation we obtain

$$(3) \qquad \mathrm{D}E_n(x) = E_{n-1}(x).$$

Since we are dealing with polynomials only, this solution is, according to § 38, univocal.

From (3) it follows that the *Euler* polynomials belong to the important class of polynomials in which the derivative of the polynomial of degree n is equal to the polynomial of degree $n-1$.

From (1) we deduce

$$E_n(x) = \mathrm{M}^{-1} \frac{x^n}{n!},$$

We have seen in (12) § 39 that

$$\mathbf{M}^{-1} = \sum_{m=0}^{\infty} (-1)^m \frac{\Delta^m}{2^m};$$

hence the above equation gives

$$E_n(x) = \frac{1}{n!} \sum_{m=0}^{n+1} (-1)^m \frac{\Delta^m x^n}{2^m}.$$

From

$$x^n = \sum_{\nu=1}^{n+1} a \; ; \; (x),$$

it follows that

$$\Delta^m x^n = \sum_{\nu=m}^{n+1} (\nu)_m (x)_{\nu-m} \mathfrak{S}_n^{\nu}$$

and finally

$$(4) \qquad E_n(x) = \frac{1}{n!} \sum_{m=0}^{n+1} \sum_{\nu=m}^{n+1} \frac{(-1)^m}{2^m} \nu! \binom{x}{\nu-m} \mathfrak{S}_n^{\nu}.$$

This is the *Newton* expansion of the *Euler* polynomials.

Let us write the expansion of $E,(x)$ in a power series in the following manner:

$$(5) \qquad E,(x) = e_0 \frac{x^n}{n!} + e_1 \frac{x^{n-1}}{(n-1)!} + \ldots + e_{n-1} \frac{x}{1!} + e_n.$$

From (3) it follows that the coefficients e_i are, according to § 22, independent of the degree of the polynomial $E,(x)$. To determine these coefficients we start from **(1)**, writing $\mathbf{M}E_0(x) = 1$ and conclude that $E,(x) = 1$ and $e_0 = 1$.

If $n > 0$ then we have in consequence of (1) $ME,, (0) = 0$; moreover if $i > 0$

$$[\mathbf{M}x^i]_{x=0} = ½.$$

Hence, performing the operation $\mathbf{M}$ on both members of (5) we obtain, if we put subsequently $x=0$,

$$(6) \qquad e_n + \sum_{i=0}^{n+1} \frac{e_i}{(n-i)!} = 0.$$

Starting from $e_0 = 1$ we may determine by aid of this equation step by step any number e_i. For instance we have

$$e_0 + 2e_1 = 0 \qquad \text{which gives} \qquad e_1 = -\tfrac{1}{2}$$

$$\frac{e_0}{2!} + \frac{e_1}{1!} + 2e_2 = 0 \qquad\qquad e_2 = 0$$

$$\frac{e_0}{3!} + \frac{e_1}{2!} + \frac{e_2}{1!} + 2e_3 = 0 \qquad\qquad e_3 = \frac{1}{24}$$

and so on.

Table of the numbers e_i

$$e_{2n} = 0$$

$e_0 = 1$	$e_5 = -1/240$
$e_1 = -\frac{1}{2}$	$e_7 = 17/40320$
$e_3 = 1/24$	$e_9 = -31/362880$

Knowing the numbers e_i we may write the expansion of the *Euler* **polynomials into a power series:**

$$E_0(x) = 1 \qquad\qquad E_3(x) = \frac{x^3}{3!} - \frac{x^2}{2 \cdot 2!} + \frac{1}{24}$$

$$E_1(x) = x - \tfrac{1}{2} \qquad\qquad E_4(x) = \frac{x^4}{4!} - \frac{x^3}{2 \cdot 3!} + \frac{1}{24}x$$

$$E_2(x) = \frac{x^2}{2!} - \frac{x}{2} = \binom{x}{2}$$

and so on.[29] **(See Figure** *3.)*

Particular values **of the** *Euler* **polynomials. From (5) we have**

$$E_n(0) = e_n \,,$$

---

[29] Our definition of the *Euler* polynomial is neariy the same as that used by *Niels Nielsen* Traité Elémentaire des Nombres de Bernoulli, Paris 1923, p. 41, where he introduces the "fonction d'Euler" denoted by "$E_n(x)$". The correspondence of his notation with ours is the following :

$$\text{“}E_n(x)\text{”} = 2E_n(x).$$

That is, our polynomial is the half of that defined by *Nielsen.*

To *Nörlund's* definition of the "Eulersches Polynom", "$E_n(x)$", corresponds in our notation

$$\text{“}E_n(x)\text{”} = n!\ E_n(x).$$

(See his Differenzenrechnung, Berlin 1924, p. 23.)

Ernst Lindelöf, in his "Calcul des Résidus" Paris 1905, introduces the polynomial $\chi_n(x)$ to which in our notation $\frac{1}{2}n!\ E_n(x)$ corresponds.

If $n>0$ then from (1) it follows that $\mathbf{M}E_n(0) = 0$ *and* therefore

$$E_n(0) + E_n(1) = 0$$

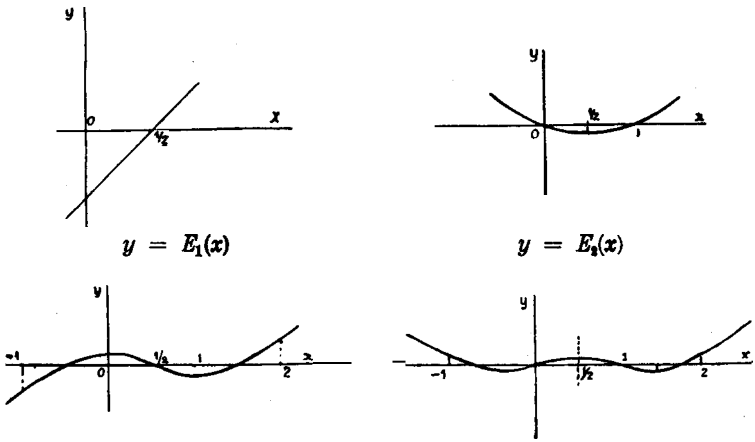
which gives

$$E_n(1) = -e_n.$$

Moreover from (1) we deduce

$$\mathbf{M}E_n(-1) = \tfrac{1}{2}[E_n(-1) + E_n(0)] = \frac{(-1)^n}{n!}.$$

Figure 3.



$y = E_1(x)$ $y = E_2(x)$

$y = E_3(x)$ $y = E_4(x)$

From this we get

$$E_n(-1) = (-1)^n \frac{2}{n!} - e_n.$$

Putting $x=\tfrac{1}{2}$ into equation (5) gives

(7) $$E_n(\tfrac{1}{2}) = \sum_{m=0}^{n+1} \frac{e_m}{2^{n-m}\,(n-m)!}$$

The coefficients e_m may be computed by aid of the *Stirling* numbers of the second kind. Indeed, putting $x=0$ into equation (4) we obtain

(8) $$e_n = \frac{1}{n!} \sum_{m=1}^{n+1} \frac{(-1)^m\, m!}{2^m}\, \mathfrak{S}_n^m.$$

§ 101. Symmetry of the Euler polynomials. Let us start from

$$\mathbf{M}E_{2n+1}(x) = \frac{x^{2n+1}}{(2n+1)!}$$

and put into it $-z$ instead of x; we find

(1) [ME,,,, $(x)]_{x=-z} = -\frac{z^{2n+1}}{(2n+1)!} = -$ ME, ,+, (z) .

On the other hand we have

$$\mathbf{M}E_{2n+1}(x) = \tfrac{1}{2}\,[E_{2n+1}(x) + E_{2n+1}(x+1)];$$

writing $x=-z$, we obtain

(2) [ME,,,, $(x)]_{x=}$ --l = $\tfrac{1}{2}\,[E_{2n+1}(-z) + E_{2n+1}(1-z)] =$

$$= \mathbf{M}E_{2n+1}(1-z).$$

From (1) and (2) follows, if we write again x instead of z,

$$\mathbf{M}E_{2n+1}(x) = -\mathbf{M}E_{2n+1}(1-x).$$

Finally the operation of $\mathbf{M}^{-1}$ leads to the relation of symmetry

(3) $$E_{2n+1}(x) = -E_{2n+1}(1-x)$$

or

$$E_{2n+1}(\tfrac{1}{2}+x) = -E_{2n+1}(\tfrac{1}{2}-x).$$

From this it follows that

$$E_{2n+1}(\tfrac{1}{2}) = 0.$$

Equation (3) gives by derivation

(4) $$E_{2n}(x) = E_{2n}(1-x)$$

or

$$E_{2n}(\tfrac{1}{2}+x) = E_{2n}(\tfrac{1}{2}-x).$$

Putting $x=\tfrac{1}{2}$ we obtain

$$E_{2n}(0) = E_{2n}(1).$$

But we have seen that E, (O) = -E,,(l) if $n>1$; hence we shall have, if $n>0$

(5) $$E_{2n}(0) = E_{2n}(1) = e_{2n} = 0.$$

From this it follows that $E_{2n}(x)$ is divisible by x(x-1). Equation (5) gives by aid of (8) § 100,

$$\sum_{m=1}^{2n+1} \frac{(-1)^m\, m!}{2^m}\, \mathfrak{S}_{2n}^m = 0.$$

Roots of the polynomials between zero and one. According to (5) $E_{2n}(x) = 0$ has roots at $x=0$ and $x=1$. Let us show that there are no roots in the interval $0<x<1$. For if there were we should have at least three roots in the interval $0 \leqq x \leqq 1$, the first derivative should have at least two, and the second $E_{2n-2}(x)$ at least one in the interval $0<x<1$, and therefore at least three in the interval $0 \leqq x \leqq 1$. Continuing in this manner, we should find that *E,(x)* had at least three roots in the interval $0 \leqq x \leqq 1$, which is impossible. Finally we conclude that $E_{2n}(x)$ has no roots in $0<x<1$.

We have seen that *E,,+,(x)* $=0$ has a root at $x=\frac{1}{2}$. We will now show that it has no other roots in the interval $0 \leqq x \leqq 1$. Indeed if it had at least two in this interval, then its derivative $E_{2n}(x)$ should have at least one in $0<x<1$; and we have seen that it has none. Therefore $E_{2n+1}(x) = 0$ has only one root in the interval $0 \leqq x \leqq 1$ so that $e_{2n+1} \neq 0$.

Exttema of the polynomials. From what precedes we conclude that in the interval $0 \leqq x \leqq 1$ the function $y=E_{2n}(x)$ has only one extremum, and this at $x=\frac{1}{2}$; moreover $y=E_{2n+1}(x)$ has only two extrema in this interval, and these correspond to $x=0$ and $x=1$.

If *E,,,(O)* $= e_{2n-1} < 0$ then $E_{2n}(x)$ will decrease at $x=0$ and the extrema of $E_{2n}(x)$ corresponding to $x=\frac{1}{2}$ will be a minimum. On the other hand if $E_{2n-1}(0) = e_{2n-1} > 0$ then $E_{2n}(x)$ will be a maximum.

Let us suppose first that $e_{2n-1} < 0$. Since $E_{2n-1}(\frac{1}{2}) = 0$ its derivative has no roots in this interval; therefore the derivative must be positive in the whole interval, so that

$$E_{2n-2}(x) = \frac{e_0\, x^{2n-2}}{(2n-2)!} + \ldots .\, e_{2n-3}\, x > 0.$$

Since in the vicinity of $x=0$ the sign of *E,,-,(x)* is identical with that of e_{2n-3}, we conclude that

$$e_{2n-3} > 0$$

and therefore

(6) $$e_{2n-1}\, e_{2n-3} < 0.$$

Starting from the supposition that $e_{2n-1} > 0$, we should have also reached the result (6). Consequently from $e_1 = -\frac{1}{2}$ it follows that

(7) $$e_{4n-1} > 0, \quad e_{4n+1} < 0$$

or

$$(-1)^n e_{2n-1} > 0.$$

E. ***Lindelöf*** has shown in a very interesting way **[Calcul** des **Résidus,** p. **37],** using the Theory of **Residues,** discovered by ***Cauchy,*** that the following form may be given to the *Euler* polynomials :

(6) $$E_{2n-1}(x) = \frac{4(-1)^n}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi x}{(2m+1)^{2n}}$$

(7) $$E_{2n}(x) = \frac{4(-1)^n}{\text{pi}} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi x}{(2m+1)^{2n+1}}.$$

These formulae are valid if 0 ≦ x 6 1. From the first it follows that $|E_{2n-1}(x)|$ is maximum in this interval for $x=0$. For this value we find

(8) $$E_{2n-1}(0) = e_{2n-1} = \frac{4(-1)^n}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}}.$$

Prom this it follows, if $0 \leqq x \leqq 1$, that

(9) $$|E_{2n-1}(x)| \leqq |e_{2n-1}|.$$

In § 49 we found (formula 3) that the sum figuring in the second member of (8) may be expressed by ***Bernoulli*** numbers or by the coefficient a_m of the *Bernoulli* polynomials. Indeed we have

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = \tfrac{1}{2}(2^{2n}-1)\,\pi^{2n}\,|a_{2n}| = \tfrac{1}{2}(2^{2n}-1)\,\pi^{2n}\,\frac{|B_{2n}|}{(2n)!};$$

therefore

(10) $$|e_{2n-1}| = 2(2^{2n}-1)\,|a_{2n}|.$$

We have seen that the extremum of $E_{2n}(x)$ is reached if $x=\frac{1}{2}$. Putting this value into (7) we obtain

$$E_{2n}(\tfrac{1}{2}) = \frac{4(-1)^n}{\pi^{2n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}}.$$

This formula is very useful, for it gives the value of the alternate sum, since we have already determined $E_n(\tfrac{1}{2})$ in other ways (7, § 100 and 4, § 102). So that we have

$$(11) \qquad \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+1}} = \frac{(-1)^n}{4} \pi^{2n+1} E_{2n}(\tfrac{1}{2}).$$

This sum is necessarily smaller than

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}};$$

hence from formula (8) we deduce that in the interval $0 \leqq x \leqq 1$

$$(12) \qquad \pi \,|\, E_{2n}(x) \,| \leqq \pi \,|\, E_{2n}(\tfrac{1}{2}) | < | \, e_{2n-1} \, | .$$

§ 102. Expansion of the Euler polynomial into a **series** of **Bernoulli** polynomials of the first **kind.** According to § 84 we have

$$c_0 = \int_0^1 E_n(x)dx = -2\, e_{n+1}, \quad c_1 = [\Delta E_n(x)]_{x=0} = -2\, e_n$$

$$c_m = [\text{AD}''' \; E_n(x)]_{x=0} = E_{n-m+1}(1) - E_{n-m+1}(0) = -2\, e_{n-m+1} .$$

Therefore the required expansion will he

$$(1) \qquad E_n(x) = -2 \sum_{m=0}^{n+1} e_{n-m+1} \; \varphi_m(x).$$

If the *Euler* polynomial is of an even degree, then we shall have, putting $x=0$,

$$(2) \qquad e_{2n} = -2 \sum_{m=0}^{n+1} e_{2n-2m+1}\, a_{2m} = \emptyset$$

where a_{2m} is a coefficient of the *Bernoulli* polynomial (§ 78).

The polynomial being of even degree, if we put into it $x=\tfrac{1}{2}$ we get the central value of $E_{2n}(x)$

$$(3) \qquad E_{2n}(\tfrac{1}{2}) = -2 \sum_{m=0}^{n+1} e_{2n-2m+1}\, \varphi_{2m}(\tfrac{1}{2}).$$

From this, by aid of formula (3) § 86, it follows that

$$E_{2n}(\tfrac{1}{2}) = -2 \sum_{m=0}^{n+1} e_{2n-2m+1}\; a_{2m}\left(\frac{1}{2^{2m-1}} - 1\right);$$

finally in consequence of (2)

(4) $$E_{2n}(1/2) = -\sum_{m=0}^{n+1} e_{2n-2m+1}\, a_{2m} \frac{1}{2^{2m-2}} .$$

The value of $E_n(1/2)$ has already been obtained in another form (7 § 100).

If the pclynomial is of odd degree, then we find

(5) $$E_{2n-1}(x) = -2 \sum_{m=0}^{n+1} e_{2n-2m-1}\, \varphi_{2m+1}(x).$$

§ 103. Operations on the Euler polynomials, From the definition of the polynomials it follows that the derivatives are

(1) $$\mathrm{D}^m E_n(x) = E_{n-m}(x).$$

Therefore the indefinite integral of the polynomial will be

$$\int E_n(x)\, dx = E_{n+1}(x) + k$$

and the integral between zero and one

$$\int_0^1 E_n(x)\, dx = -2\, e_{n+1} .$$

Hence, if the polynomial is of an odd degree this integral will be equal to zero.

The mean of the polynomial follows from its definition:

(2) $$\mathrm{M}E_n(x) = \frac{x^n}{n!} .$$

The differences of the polynomials are more complicated. From the *Newton* expansion of the polynomial, formula (4) § 100, we deduce

$$\Delta^m E_n(x) = \frac{1}{n!} \sum_{\nu=m}^{n+1} \sum_{i=0}^{\nu+1-m} \frac{(-1)^i\, \nu!}{2^i} \binom{x}{\nu-m-i} \mathfrak{S}_n^\nu ;$$

this gives, putting $x=0$,

(3) $$\Delta^m E_,(o) = \frac{1}{n!} \sum_{\nu=m}^{n+1} \frac{(-1)^{\nu-m}\, \nu!}{2^{\nu-m}} \mathfrak{S}_n^\nu$$

From formula (4) § 100 we may determine in the same way the indefinite sum of $E_n(x)$. But we may obtain a more simple formula in another way. Dealing with symbolical methods we saw that

$$\Delta = 2(\mathbf{M}-1)$$

and therefore

$$\Delta E_n(x) = 2\left[\frac{x^n}{n!} - E_n(x)\right]. \tag{4}$$

Let us remark that this again gives Δ *E,(O)* $=$ — $2\, e_n$.

We have

$$\text{A-'}\ \frac{x^n}{n!} = \varphi_{n+1}(x) + k$$

where φ_{n+1} *(x)* is the *Bernoulli* polynomial of degree n+l ; hence from (4) we deduce

$$\text{A-'}\ E_n(x) = \varphi_{n+1}(x) - \tfrac{1}{2} E_n(x) + k. \tag{5}$$

Inverse Mean of the Euler polynomial. In § 6, dealing with symbolical methods, we found

$$\mathbf{M}^{-1} = \frac{1}{1+\frac{1}{2}\Delta} = \sum_{m=0}^{\infty} (-1)^m \frac{\Delta^m}{2^m} \,.$$

This formula may be applied to the *Euler* polynomials; but since their differences are complicated, the formula will not have practical value. We obtain a simpler one, starting from the formula forthe mean of a product deduced in § 31,

$$\mathbf{M}[uv] = uv - v\mathbf{M}u + \mathrm{E}u\mathbf{M}v;$$

performing on this equation the operation $\mathbf{M}^{-1}$ we get

$$uv = \mathbf{M}^{-1}[uv] - \mathbf{M}^{-1}[v\mathbf{M}u] + \mathbf{M}^{-1}[\mathrm{E}u\mathbf{M}v].$$

Putting now $u=x$ and $v=E_n(x)$ *we* find

$$xE_n(x) = \mathbf{M}^{-1}[xE_n(x)] - \mathbf{M}^{-1}[(x+\tfrac{1}{2})E_n(x)] + \mathbf{M}^{-1}\left[(x+1)\frac{x^n}{n!}\right].$$

Remarking that

$$\mathbf{M}^{-1}\left[(x+1)\frac{x^n}{n!}\right] = \mathbf{M}^{-1}\left[(n+1)\frac{x^{n+1}}{(n+1)!} + \frac{x^n}{n!}\right] =$$
$$= (n+1)E_{n+1}(x) + E_n(x)$$

we get after simplification

$$\mathbf{M}^{-1} E_n(x) = 2[(1-x)E_n(x) + (n+1)E_{n+1}(x)] \tag{6}$$

and in the particular case of $x=0$

$$[\mathbf{M}^{-1} E_n(x)]_{x=0} = 2\, e_n + 2(n+1)\, e_{n+1}.$$

But from formula (5) § 100 we obtain immediately

$$\mathbf{M}^{-1} E_n(x) = \sum_{m=0}^{n+1} e_m E_{n-m}(x).$$

Supposing that the polynomial is of an even degree, then, putting into it 2n instead of n, writing $x=0$ and equating the result to that of the preceding equation, will lead to the important relation found by *Euler* in another way:

$$(7) \qquad 2(2n+1)\, e_{2n+1} = \sum_{m=0}^{n} e_{2n-2m-1}\, e_{2m+1}.$$

§ 104. The Tangent-coefficients. Dealing with *Euler* polynomials we could proceed as has been done in the case of the *Bernoulli* polynomials, where we have written

$$n!a_n = B_n$$

and B_n was called a *Bernoulli* number. If we now put

$$n!\, e_n = \mathscr{E}_n$$

the number $\mathscr{E}_n$ would be interesting, and equation (6) § 100 would lead to the symbolical relation

$$[1 + \mathscr{E}]^n + \mathscr{E}_n = 0$$

in the expansion of which $\mathscr{E}_m$ is to be put, instead of $\mathscr{E}^m$. Starting from $\mathscr{E}_0$ it would be possible to determine step by step the numbers $\mathscr{E}_m$.

But it is much better to introduce instead of $\mathscr{E}_m$ the following numbers

$$(1) \qquad \mathfrak{C}_n = 2^n\, n!\, e_n.$$

The number $\mathfrak{C}_n$, which as we shall see is the coefficient of $x^n/n!$ in the expansion of tan x into a power series, has been called **tangent-coefficient.**[30]

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[30] Several authors have introduced the tangent-coefficients, using different definitipns and notations. For instance in *Leonhurdi Euleri,* Opuscula Analytica, Petropolis, 1783-85, p. 372, the number $2^{n-1} e_n$ has been introduced, which is the coefficient of $2x^n$ in the expansion of tan x.

Euler in his work quoted below has given several formulae for the determination of these numbers; one of them is equivalent to our formula (7) § 103; multiplying it by 2^{2n} (2n) I, by aid of (1) it will become

$$(2)\qquad \mathfrak{E}_{2n+1} = \sum_{m=0}^{n} \binom{2n}{2m+1} \mathfrak{E}_{2m+1}\, \mathfrak{E}_{2n-2m-1}$$

or symbolically

$$(3)\qquad \mathfrak{E}_{2n+1} = [\mathfrak{E} + \mathfrak{E}]^{2n}.$$

We may deduce other formulae for the determination of the numbers $\mathfrak{E}_n$. For instance, multiplying both members of equation (6) § 100 by 2^n n! we get the symbolical formula

$$(4)\qquad [2 + \mathfrak{E}]^n + \mathfrak{E}_n = 0.$$

From formula (5) § 100 we obtain in the same way the symbolical expression of the *Euler* polynomials:

$$(5)\qquad E_n(x) = \frac{[2x+\mathfrak{E}]^n}{2^n\, n!}.$$

Finally, equation (8) § 100 gives the number $\mathfrak{E}_n$ by aid of the *Stirling* numbers of the second kind:

$$(6)\qquad \mathfrak{E}_n = \sum_{m=1}^{n+1} (-1)^m\, 2^{n-m}\, m!\, \mathfrak{S}^m_{n}.$$

Since $e_{2m} = 0$, hence from (1) it follows that $\mathfrak{E}_{2m} = 0$. The *Stirling* numbers being integers, from (6) we conclude that the numbers $\mathfrak{E}_m$ are integers too.

Table of the numbers $\mathfrak{E}_n$.

n	$\mathfrak{E}_n$	n	$\mathfrak{E}_n$
0	1	3	2
1	- 1	5	- 16

L. Saalschütz, Vorlesungen iiber Bernoulli'sche Zahlen, Berlin, 1893. p. 22. His β_n corresponds to our $|\mathfrak{E}_{2n-1}|$.

N. Nielsen, Traité des Nombres de Bernoulli, Paris, 1923, p. 178. His T_n corresponds to our $|\mathfrak{E}_{2n-1}|$.

N. E. *Nörlund*, Differenzenrechnung, Berlin. 1924, p. 458. His C_n corresponds to our $\mathfrak{E}_n$.

n	$\mathfrak{E}_n$	n	$\mathfrak{E}_n$
7	272	17	— 209865 342976
9	-7936	19	29 088885 112832
11	353792	21	4951 498053 124096
13	-22 368256	23	1 015423 886506 852352
15	1903 757312	25	-246 921480 190207 983616
		27	70251 601603 943959 887872
		29	-23 119184 187809 597841 473536

§ 105. **Euler Numbers.** If we put $x=\frac{1}{2}$ into the polynomial $E_n(x)$ and multiply the result by $2^n\, n!$ we obtain a number which will be denoted by E_n,

(1) $$E_n = 2^n\, n!\; E_n(\tfrac{1}{2}).$$

This number was first called a secant-coefficient and later an *Euler* number. E_n is, as we shall see, the coefficient of $x^n/n!$ in the expansion of sec x. It would have been better to call the tangent coefficients "*Euler* numbers", because in consequence of (5) § 104 they figure in the coefficients of the *Euler* polynomials, whereas the secant-coefficients are only particular values of the polynomials.[31]

Putting $x=\frac{1}{2}$ into equation (5) § 104, we obtain in consequence of (1)

(2) $$E_n = [1+\mathfrak{E}]^n = \sum_{i=0}^{n+1} \binom{n}{i} \mathfrak{E}_i\,.$$

From this formula it follows that the numbers E_n are inte-

[31] Several authors have dealt with Euler numbers:

Leonhardi ***Euleri***, Institutiones differentiales, 1755, p. 522, or Opera omnia Vol. X, p. 419. Euler denoted the number $|E_0|$ by α, the number $|E_2|$ by β, $|E_4|$ by γ and so on. He gave a table of these numbers up to E_{18}.

Scherk, Mathematische Abhandlungen, Berlin, 1825 p. 7, gives a table of these numbers up to E_{28}, This table is reproduced by ***Saalschütz;*** Vorlesungen über Bernoulli'sche Zahlen, Berlin, 1893, p. 22. His a_n corresponds to our $|E_{2n}|$.

N. *Nielsen*, Traité des Nombres de Bernoulli, Paris, 1923, p. 178, also reproduces Scherk's table. His number E_n corresponds to our $|E_{2n}|$.

Lindelöf, Calcul des Résidus, Paris, 1905, p. 33, Same notation as Nielsen's,

Nörlund, Differenzenrechnung, Berlin, 1924, p. 458. The notation we have adopted here is the same as ***Nörlund's*** notation.

gers. The formula gives the *Euler* numbers in terms of the tangent-coefficients. It is easy to get an inverse formula: indeed, multiplying both members of (2) by $(-1)^{m-n}\binom{m}{n}$ and summing from $n=0$ to $n=m+1$ we obtain, according to § 65 (Inversion),

$$\mathfrak{C}_m = \sum_{n=0}^{m+1} (-1)^{n+m} \binom{m}{n} E_n = [E-1]^m. \tag{3}$$

$E_{2n+1}(\tfrac{1}{2}) = 0$; therefore $E_{2n+1} = 0$. Moreover we have seen that $\mathfrak{C}_{2i} = 0$ if $i > 0$, consequently from (2) we deduce, writing $2n$ instead of n

$$E_{2n} = 1 + \sum_{i=0}^{n} \binom{2n}{2i+1} \mathfrak{C}_{2i+1}$$

and writing in it $2n+1$ instead of n:

$$1 + \sum_{i=0}^{n+1} \binom{2n+1}{2i+1} \mathfrak{C}_{2i+1} = 0.$$

Finally from (3) we obtain in the same manner

$$\sum_{i=0}^{m+1} \binom{2m}{2i} E_{2i} = 0$$

and

$$\mathfrak{C}_{2m+1} = - \sum_{i=0}^{m+1} \binom{2m+1}{2i} E_{2i}.$$

The above formulae were found by *Euler* [Opuscula Analytica, t. II, p. 269-270, Petropolis, 1785].

Table of the numbers E_n.

n	E_n	n	E_n
0	1	14	-199 360981
2	- 1	16	19391 512145
4	5	18	-2 404879 675441
6	-61	20	370 371188 237525
8	1385	22	-69348 874393 137901
10	-50521	24	15 514534 163557 086905
12	2 702765	26	—4087 072509 293123 892361
		28	1 252259 641403 629865 468285

§ 106. Limits of the Euler polynomials and numbers. We found (formula 10, § 101) that

(1) $$|e_{2n-1}| = 2(2^{2n}-1)|a_{2n}|;$$

moreover in § 82 we have seen that (formula 9)

$$\frac{2}{(2\pi)^{2n}} < |a_{2n}| < \frac{1}{12(2\pi)^{2n-2}}.$$

From the formulae above, it follows that

(2) $$4\left(1-\frac{1}{2^{2n}}\right)\frac{1}{\pi^{2n}} < |e_{2n-1}| < \frac{2}{3\pi^{2n-2}}.$$

Therefore

$$\lim_{n=\infty} e_{2n-1} = 0$$

and even

$$\lim_{n=\infty} k^{2n-1} e_{2n-1} = 0 \quad \text{if} \quad |k| < \pi.$$

From equation (1) we deduce

$$\left|\frac{e_{2n-1}}{e_{2n+1}}\right| = \frac{(2^{2n}-1)}{(2^{2n+2}-1)}\left|\frac{a_{2n}}{a_{2n+2}}\right|;$$

according to formula (9) and (10) § 82 we have

(3) $$\frac{2}{3}\pi^4 > \left|\frac{a_{2n}}{a_{2n+2}}\right| > 4\pi^2.$$

Hence in consequence of (1)

(4) $$\frac{\pi^4}{6} > \left|\frac{e_{2n-1}}{e_{2n+1}}\right| > \frac{4}{5}\pi^2.$$

The series Σe_n is absolutely convergent.

In § 101, formulae (9) and (12), we have seen that

$$|E_{2n-1}(x)| \leq |e_{2n-1}| \quad \text{and} \quad \pi|E_{2n}(x)| < |e_{2n-1}| \quad \text{if} \quad 0 \leq x \leq 1;$$

from (2) it follows that

$$|E_{2n-1}(x)| < \frac{2}{3\pi^{2n-2}} \quad \text{and} \quad |E_{2n}(x)| < \frac{2}{3\pi^{2n-1}}.$$

Therefore we conclude that

$$(5) \qquad |E_n(x)| \leqq \frac{2}{3\pi^{n-1}} \quad \text{if} \quad 0 \leqq x \leqq 1$$

and in the same interval

$$\lim_{n=\infty} E_n(x) = 0.$$

The series $\Sigma\, E_n(x)$ is absolutely convergent in the interval $0 \leqq x \leqq 1$.

In § 104, formula **(1)** the tangent-coefficients were given by

$$\mathfrak{E}_{2n-1} = 2^{2n-1}(2n-1)!\, e_{2n-1};$$

from this we deduce by aid of (2)

$$\frac{|\mathfrak{E}_{2n-1}|}{(2n-1)!} < \frac{4}{3}\left(\frac{2}{\pi}\right)^{2n-2}$$

and

$$\lim_{n=\infty} \frac{\mathfrak{E}_{2n-1}}{(2n-1)!} = 0.$$

From these relations we conclude that the series $\Sigma \dfrac{\mathfrak{E}_{2n-1}}{(2n-1)!}$ is absolutely convergent.

In § **105** we defined the *Euler* numbers by

$$E_{2n} = 2^{2n}(2n)!\, E_{2n}(\tfrac{1}{2}).$$

From (3) it follows that

$$|E_{2n}(\tfrac{1}{2})| < \frac{2}{3\pi^{2n-1}}$$

Therefore

$$\frac{|E_{2n}|}{(2n)!} < \frac{4}{3}\left(\frac{2}{\pi}\right)^{2n-1}$$

and

$$\lim_{n=\infty} \frac{E_{2n}}{(2n)!} = 0.$$

The series $\Sigma\, E_{2n}/(2n)!$ is absolutely convergent.

§ 107. Expansion of the Euler Polynomial into a Fourier series. The expansion of the polynomial of even degree in the interval $0 \leqq x \leqq 1$ may be written:

$$(1) \qquad E_{2n}(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos 2\pi mx + \sum_{m=1}^{\infty} \beta_m \sin 2\pi mx.$$

Since

$$E_{2n}(x) = E_{2n}(1-x)$$

we have

$$\beta_m = 0.$$

Moreover we have

$$\tfrac{1}{2}a_0 = \int_0^1 E_{2n}(x)dx = -2\, e_{2n+1}$$

and if $m > 0$

$$(2) \quad \tfrac{1}{2}a_m = \int_0^1 E_{2n}(x)\cos 2\pi mx dx = \left[\frac{\sin 2m\pi x}{2m\pi} E_{2n}(x)\right]_0^1 -$$

$$- \int_0^1 \frac{\sin 2m\pi x}{2m\pi} E_{2n-1}(x)dx.$$

The quantity in the brackets is equal to zero at both limits; a second integration by parts gives:

$$\tfrac{1}{2}a_m = \frac{\cos 2m\pi x}{(2m\pi)^2} E_{2n-1}(x) \Big|_0^1 - \int_0^1 \frac{\cos 2m\pi x}{(2m\pi)^2} E_{2n-2}(x)\, dx.$$

The quantity in the brackets is equal to

$$-\frac{2\, e_{2n-1}}{(2m\pi)^2}$$

moreover the integral is the same as that in (2), only the degree of the polynomial has been diminished by two, and it has been multiplied by $-1/(2m\pi)^2$. Therefore repeating the above operations v times we get:

$$\tfrac{1}{2}\, a_m = \sum_{i=1}^{\nu+1} \frac{(-1)^i\, 2e_{2n+1-2i}}{(2m\pi)^{2i}} + (-1)^\nu \int_0^1 \frac{\cos 2m\pi x}{(2m\pi)^{2\nu}} E_{2n-2\nu}(x)dx.$$

Putting $\nu = n$ the integral in the second member will be equal to zero and we have

$$(3) \qquad a_m = \sum_{i=1}^{n+1} \frac{(-1)^i\, 4\, e_{2n+1-2i}}{(2m\pi)^{2i}}.$$

The expansion of E_{2n-1} (x) will be obtained, in the interval $0 \leqq x \leqq 1$, by determining the derivative of both members of equation (1). We find

$$E\,,\,,\,-\,,\,(\,x\,) = -\sum_{m=1}^{\infty} 2m\pi\alpha_m \sin 2m\pi x = \sum_{m=1}^{\infty} \beta'_m \sin 2m\pi x;$$

from *(3)* it follows that

$$14) \qquad \beta'_m = -\sum_{i=1}^{n+1} \frac{(-1)^i\, 4\, e_{2n+1-2i}}{(2m\pi)^{2i-1}}.$$

Let us remark, that the function $E_{2n-1}(x)$ considered as a periodic function with a period equal to one, is discontinuous at $x=0$, being equal to $\pm e_{2n-1}$; therefore the ***Fourier*** series will give E_{2n-1} *(0)* $=0$.

Remark. Putting $x=0$ into equation (1) we get

$$E,,(O) = 0 = -2\; e_{2n+1} + 2 \sum_{\nu=1}^{n+1} e_{2n+1-2\nu} \sum_{m=1}^{\infty} \frac{2(-1)^{\nu}}{(2m\pi)^{2\nu}}.$$

But we have seen (6), § 82 that the second sum is equal to $-a_{2\nu}$; therefore we have

$$(5) \qquad e_{2n+1} + \sum_{\nu=1}^{n+1} e_{2n+1-2\nu}\; a_{2\nu} = \sum_{\nu=0}^{n+1} a_{2\nu}\; e_{2n+1-2\nu} = 0\,.$$

This equation has been obtained before (2 § 102).

To obtain the central value of $E_{2n}(x)$ expressed by aid of the coefficients of the ***Bernoulli*** polynomials, let us write in equation (1) $x=\frac{1}{2}$. We find

$$E_{2n}(\tfrac{1}{2}) = \tfrac{1}{2}\alpha_0 + \sum_{m=1}^{\infty} (-1)^m \alpha_m;$$

in consequence of (3) this will be

$$E_{2n}(\tfrac{1}{2}) = -2\, e_{2n+1} + 4 \sum_{\nu=1}^{n+1} (-1)^{\nu} e_{2n+1-2\nu} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2\pi m)^{2\nu}}$$

and according to § 49 we have

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m\pi)^{2\nu}} = (-1)^{\nu} a_{2\nu} \left(\frac{1}{2} - \frac{1}{2^{2\nu}} \right).$$

Therefore

$$E_{2n}(\tfrac{1}{2}) = 2 \sum_{\nu=0}^{n+1} a_{2\nu}\; e_{2n+1-2\nu} \left(1 - \frac{1}{2^{2\nu-1}} \right).$$

Since according to (5) the first sum is equal to zero, we find

$$(6)\qquad E_{2n}(\tfrac{1}{2}) = -\sum_{\nu=0}^{n+1} \frac{a_{2\nu}\, e_{2n+1-2\nu}}{2^{2\nu-2}}.$$

This formula is the same as that of 4, § 102.

Multiplying first both members of (6) by $(2n)!\,2^{2n}$ and then the numerator and the denominator of the second member by $(2\nu)!\,(2n+1-2\nu)!$ remarking that $(2\nu)!\,a_{2\nu} = B_{2\nu}$ is the *Bernoulli* number, we get, in consequence of (1) § 104 and (1) § 105,

$$E_{2n} = -(2n)!\,2 \sum_{\nu=0}^{n+1} \frac{B_{2\nu}\, \mathfrak{C}_{2n+1-2\nu}}{(2\nu)!\,(2n+1-2\nu)!}$$

or

$$E_{2n} = -\frac{2}{2n+1} \sum_{\nu=0}^{n+1} \binom{2n+1}{2\nu} \mathfrak{C}_{2n+1-2\nu} B_{2\nu}$$

This may be written symbolically:

$$(7)\qquad E_{2n} = -\frac{2}{2n+1}\,[\mathfrak{C}+B]^{2n+1}.$$

Equation (7) gives the *Euler* numbers in terms of the tangent coefficients and the *Bernoulli* numbers.

Example.

$$E_4 = -\frac{2}{5}\,[\mathfrak{C}_5 + 10\,\mathfrak{C}_3 B_2 + 5\,\mathfrak{C}_1 B_4] = 5.$$

Remark By aid of formula (10) § 101 we may eliminate $a_{2\nu}$ of (6) and get an expression of the *Euler* numbers by tangent coefficients; or we may eliminate $e_{2n+1-2\nu}$ and obtain the *Euler* numbers in term s of the *Bernoulli* numbers.

§ 108. Application of the Euler polynomials, 1. Determination of the inverse mean of f(x) .

Let us expand *f(x)* into a *Maclaurin* series:

$$f(x) = f(0) + xDf(0) + \frac{x^2}{2!}D^2f(0) + \dots + \frac{x^n}{n!}D^nf(0) + \dots$$

The inverse operation of the mean gives

$$(1)\qquad \mathbf{M}^{-1} f(x) = f(0) + E_1(x)\,\mathbf{D}f(0) + E_2(x)\mathbf{D}^2f(0) + \dots$$
$$+ E_n(x)\mathbf{D}^nf(0) + \dots$$

Starting from this formula we may determine moreover the *inverse difference* of the alternate function $(-1)^x f(x)$. According to formula (10) § 39, we have

$$\text{A-'}\ (-1)^x f(x) = \tfrac{1}{2}(-1)^{x+1} \mathsf{M}^{-1} f\ (x).$$

Therefore the indefinite sum of the alternate function will be

(2) $$\text{A-'}\ (-1)^x f(x) = \tfrac{1}{2}(-1)^{x+1}[f(0) + E_1(x)\mathsf{D}f(0) + \ldots + E_n(x)\mathsf{D}^n f(0) + \ldots]$$

and in the particular case of $f(x) = x^m$

$$\text{A-' (-1)''}\ x^m = \tfrac{1}{2} m!\, (-1)^{x+1} E_m(x) + k;$$

moreover

$$\sum_{x=0}^{n} \text{(-1)X}\ x^m = \tfrac{1}{2}\, m!\, [E_m(0) - (-1)^n E_m(n)]$$

or, if we introduce the symbolical formula containing the tangent coefficients,

(3) $$\sum_{x=0}^{n} (-1)^x x^m = \frac{1}{2^{m+1}} [\mathfrak{G}_m - (-1)^n (\mathfrak{G} + 2n)^m].$$

Particular ease.

$$\sum_{x=0}^{n} \text{(-1)''}\ x^2 = (-1)^{n+1} \binom{n}{2}.$$

§ 109. Expansion of a **polynomial** $f(x)$ **into** a series of **Euler** polynomials, Let

(1) $$f(x) = c_0 + c_1 E_1(x) + c_2 E_2(x) + \ldots + c_n E_n(x),$$

The **operation** M gives

$$\mathsf{M} f\ (x) = c_0 + c_1 \frac{x}{1!} + c_2 \frac{x^2}{2!} + \ldots + c_n \frac{x^n}{n!}.$$

Hence

$$c_0 = \mathsf{M} f(0).$$

Moreover

$$\mathsf{D}^m f(x) = c_m + c_{m+1} E_1(x) + c_{m+2} E_2(x) + \ldots + c_n E_{n-m}(x);$$

hence we have in the same manner

(2) $$c_m = \mathsf{M}\mathsf{D}^m f(0).$$

Example 1. Given $f(x) = \varphi_n(x)$, the *Bernoulli* polynomial of degree *n, we* find

$$c_m = \tfrac{1}{2}\left[\varphi_{n-m}(x) + \varphi_{n-m}(x+1)\right]_{x=0}.$$

Hence if $m \neq n-1$

$$c_m = a_{n-m}$$

and

$$c_{n-1} = 0.$$

Finally the expansion will be

$$\varphi_n(x) = \sum_{m=0}^{n+1} a_{n-m} E_m(x) - a_1 E_{n-1}(x); \tag{3}$$

putting $x=0$ we get

$$\sum_{m=1}^{n+1} a_{n-m}\, e_m = a_1\, e_{n-1}.$$

Writing $2n+1$ instead of n *we* find

$$\sum_{m=0}^{n+1} a_{2n-2m}\, e_{2m+1} = 0. \tag{4}$$

This equation has been obtained already in (2) § 102 and (5) § 107.

The formula may be written symbolically,

$$[2B + \mathfrak{E}]^{2n+1} = 0.$$

Example 2. Given $f(x) = x^n/n!$ We deduce

$$\mathrm{D}^m f(x) = \frac{x^{n-m}}{(n-m)!}.$$

Therefore

$$c_m = \mathrm{MD}^m f(0) = \tfrac{1}{2}\left|\frac{x^{n-m} + (x+1)^{n-m}}{(n-m)!}\right|_{x=0}.$$

Hence if $m \neq n$

$$c_m = \frac{1}{2(n-m)!},$$

and if $m=n$

$$c_n = 1.$$

The required expansion will be

$$(5) \qquad \frac{x^n}{n!} = E_n(x) + \sum_{m=0}^{n} \frac{E_m(x)}{2(n-m)!}.$$

If ***f(x) is not a polynomial*** the series becomes infinite and its convergence must be examined, but the coefficients are determined in the same manner.

Example 3. Given f(x) $= e^{xt}$. It follows that

$$D^m e^{xt} = t^m e^{xt}$$

and

$$MD^m e^{xt} = \tfrac{1}{2} t^m (e^{xt+t} + e^{xt});$$

putting $x=0$ we have

$$c_m = \tfrac{1}{2} t^m (e^t+1)$$

and the required expansion is

$$(6) \qquad e^{xt} = \tfrac{1}{2}(e^t+1) \sum_{m=0}^{\infty} E_{,,,}(x) \; t^m.$$

The ***generating function of the polynomial*** $E_n(x)$ ***may*** be obtained immediately from (6) :

$$(7) \qquad GE_n(x) = \frac{2e^{xt}}{e^t+1} = \sum_{m=0}^{\infty} E_{,}(x) \; f''.$$

We have seen in § 106 that, if $0 \leqq x \leqq 1$, we have

$$| E_m(x) | < \frac{2}{3\pi^{m-1}}$$

consequently if I $t | < \pi$, the series (7) is convergent.

Putting into (7) $x=0$ we obtain the generating function of the numbers e_n :

$$(8) \qquad Ge_n = \frac{2}{e^t+1} = \sum_{m=0}^{\infty} e_{,,,} \; t^m = 1 + e_1 t + e_3 t^3 + \ldots.$$

This may be written in another form:

$$(9) \qquad \sum_{ni \; 1}^{\infty} e_m t^m = - \frac{e^t - 1}{e^t + 1} = - \tanh \tfrac{1}{2} t$$

or putting $t=2iz$ we find

$$\tan z = \sum_{m=0}^{\infty} (-1)^{m+1} e_{2m+1} (2z)^{2m+1}.$$

According to § 104 (formula 1) this is equal to

$$(10)\qquad \tan z = \sum_{m=0}^{\infty} (-1)^{m+1}\, \mathfrak{E}_{2m+1} \frac{z^{2m+1}}{(2m+1)!} =$$

$$= \sum_{m=0}^{\infty} |\mathfrak{E}_{2m+1}| \frac{z^{2m+1}}{(2m+1)!}$$

where the numbers $\mathfrak{E}_m$ are the tangent-coefficients. We have

$$[\mathrm{D}^m \tan z|_{z=0} = |\mathfrak{E}_m|.$$

Let us remark that we have already expanded tan z into a power series in § 87 (formula **17**). Writing that the coefficients of z^{2n-1} in the two expansions are identical, we find the important relation

$$(11)\qquad e_{2n-1} = 2(1-2^{2n})\, a_{2n}$$

which has been already obtained in § 101 (formula 10).

Writing $x=\frac{1}{2}$ in *(7)* we get

$$\frac{2e^{t/2}}{1+e^t} = \frac{1}{\cosh \frac{1}{2} t} = \operatorname{sech} \tfrac{1}{2} t = \sum_{m=0}^{\infty} E_m(\tfrac{1}{2})\, t^m.$$

This ***is*** the ***generating function*** of the numbers $E_m(\frac{1}{2})$. Let us write again $t=2iz$; we shall find

$$(12)\qquad \sec z = \sum_{m=0}^{\infty} (-1)^m E_{2m}(\tfrac{1}{2})\, (2z)^{2m};$$

according to formula (1), § 105 this may be written

$$\sec z = \sum_{m=0}^{\infty} (-1)^m E_{2m} \frac{z^{2m}}{(2m)!} = \sum_{m=0}^{\infty} |E_{2m}| \frac{z^{2m}}{(2m)!}$$

where the numbers E_m are ***Euler's*** numbers or the **secant-**coefficients. We have

$$[\mathrm{D}^m \sec z|_{z=0} = |E_m|.$$

Example 4. Given $f(x) = \cos xt$. We have

$$\mathrm{D}^{2m} \cos xt = (-1)^m t^{2m} \cos xt$$

$$\mathrm{D}^{2m-1} \cos xt = (-1)^m t^{2m-1} \sin xt.$$

Hence

$$\mathrm{MD}^{2m} f(0) = \tfrac{1}{2}(-1)^m t^{2m}(1+\cos t)$$

$$\mathrm{MD}^{2m-1} f(0) = \tfrac{1}{2}(-1)^m t^{2m-1} \sin t$$

so that the required expansion will be

$$\text{(13)} \qquad \cos xt = \frac{1+\cos t}{2} + \frac{1}{2} \sum_{m=1}^{\infty} (-1)^m t^{2m-1} \sin t \; E,,-,(x)$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} (-1)^m t^{2m} (1+\cos t) \; E_{2m}(x).$$

Writing $x=0$ and multiplying by $2/\sin t$ we get

$$\text{(14)} \qquad \frac{1-\cos t}{\sin t} = \tan \tfrac{1}{2} t = \sum_{m=1}^{\infty} (-1)^m e_{2m-1} t^{2m-1}.$$

Putting into it $t=2z$ we obtain formula (10) found before. If $|t| < \pi$ then the series (13) is convergent for $0 \leqq x \leqq 1$, in consequence of (5), § 106.

§ 110. Multiplication theorem of the Euler polynomials. To deduce the theorem, let us expand the following polynomial into a series of *Euler* polynomials:

$$F(x) = (2p+1)^n \sum_{m=0}^{2p+1} (-1)^m E_n\left(\frac{x+m}{2p+1}\right).$$

According to § 109 the coefficient of $E_i(x)$ is

$$c_i = \mathbf{M}\mathbf{D}^i F(0)$$

we have

$$\mathbf{D}^i F(x) = (2p+1)^{n-i} \sum_{m=0}^{2p+1} (-1)^m E_{n-i}\left(\frac{x+m}{2p+1}\right.$$

moreover

$$\mathbf{M}\mathbf{D}^i F(x) = \tfrac{1}{2}(2p+1)^{n-i} \sum_{m=0}^{2p+1} (-1)^m \left[E_{n-i}\left(\frac{x+m}{2p+1}\right) + \right.$$

$$\left. + E_{n-i}\left(\frac{x+m+1}{2p+1}\right)\right].$$

If in the second term of the sum we put m instead of m+l then it will become equal to the first term, but with a negative sign; the new limits will be $m=1$ and $m=2p+2$. Therefore we shall have, putting $x=0$:

$$c_i = \mathbf{M}\mathbf{D}^i F(0) = \tfrac{1}{2}(2p+1)^{n-i} \left[E_{n-i}(0) + E_{n-i}(1)\right].$$

Hence $c_n=1$ and $c_i=0$ if $i \neq n$. In consequence $F(x)$ will be equal to *E,(x)*. Finally putting $x=(2p+1)z$ we obtain the first multiplication formula (for odd factors):

$$(1)\qquad E_n[(2p+1)z] = (2p+1)^n \sum_{m=0}^{2p+1} (-1)^m E_n\left(z + \frac{m}{2p+1}\right).$$

Writing in this formula $z=0$ we obtain an expression for e_n. For instance, if $p=1$ we have

$$e_n = 3^n\left[e_n - E_n\left(\frac{1}{3}\right) + E_n\left(\frac{2}{3}\right)\right].$$

To obtain the second theorem of multiplication (for even factors) we shall expand, into a series of *Euler* polynomials, the following polynomial:

$$F(x) = -2(2p)^{n-1} \sum_{m=0}^{2p} (-1)^m \varphi_n\left(\frac{x+m}{2p}\right)$$

where $\varphi_n(\xi)$ is a *Bernoulli* polynomial of the first kind. We obtain

$$\mathrm{D}^i F(x) = -2(2p)^{n-i-1} \sum_{m=0}^{2p} (-1)^m \varphi_{n-i}\left(\frac{x+m}{2p}\right).$$

The operation M gives

$$\mathrm{MD}^i F(x) = -(2p)^{n-i-1} \sum_{m=0}^{2p} (-1)^m\left[\varphi_{n-i}\left(\frac{x+m}{2p}\right) + \varphi_{n-i}\left(\frac{x+m+1}{2p}\right)\right]$$

Putting m instead of $m+1$ in the second term of the sum, it becomes equal to the first term, except the sign and the limits, which will be $m=1$ and $m=2p+1$. Therefore every term will vanish except those corresponding to $m=0$ and $m=2p$; it results, after having put $x=0$, that

$$c_i = (2p)^{n-i-1}\,[\varphi_{n-i}(1) - \varphi_{n-i}(0)].$$

In consequence of what we have seen in § 79, the second member will be equal to zero for every value of i except for $i=n-1$ where $c_{n-1} = 1$, Therefore the expansion will be

$$F(x) = E_{n-1}(x).$$

If we write $x=2pz$, we get the second multiplication formula

$$(2)\qquad E_{n-1}(2pz) = -2(2p)^{n-1} \sum_{m=0}^{2p} (-1)^m \varphi_n\left(z + \frac{m}{2p}\right).$$

Particular case. Let $p=1$, then

$$E_{n-1}(2z) = -2^n [\varphi_n(z) - \varphi_n(z+\tfrac{1}{2})].$$

Putting $z=0$ we have

$$e_{n-1} = -2^n [a_n - \varphi_n(\tfrac{1}{2})].$$

But we have seen that (8, § 82)

$$\varphi_n(\tfrac{1}{2}) = a_n \left(\frac{1}{2^{n-1}} - 1 \right);$$

therefore we find

$$e_{n-1} = 2\, a_n (1-2^n).$$

This equation has already been deduced in § 101 (formula 10), and in § 109 (formula 11).

§ 111. Expansion of a function into an Euler series. Starting from the function $f(x+u)$ and integrating by parts, we may write

$$\int_0^1 f(x+u)\, dx = [E_1(x)\, f(x+u)]_0^1 - \int_0^1 E_1(x)\, \mathsf{D} f(x+u)\, dx.$$

The quantity in the brackets is equal to $-2e_1 \mathsf{M} f(u)$. The integral in the second member gives by integration by parts

$$-[E_2(x)\, \mathsf{D} f(x+u)]_0^1 + \int_0^1 E_2(x)\, \mathsf{D}^2 f(x+u)\, dx.$$

The quantity in the brackets being equal to $+2e_2 \mathsf{M} \mathsf{D} f(u)$ is equal to zero. A further integration by parts gives

$$-2e_3 \mathsf{M} \mathsf{D}^2 f(u) - \int_0 E_3(x)\, \mathsf{D}^3 f(x+u)\, dx$$

and so on. Finally we shall have

$$\int_0^1 f(x+u)\, dx = -2 \sum_{i=1}^{2n+1} e_i \mathsf{M} \mathsf{D}^{i-1} f(u) + \int_0^1 E_{2n}(x)\, \mathsf{D}^{2n} f(x+u)\, dx.$$

By derivation with respect to u we obtain

$$\mathsf{D}_u \int_0^1 f(x+u) dx = \mathsf{D}_u \int_u^{u+1} f(t)\, dt = \Delta f(u);$$

therefore

$$(1) \quad A?(u) = -2 \sum_{i=1}^{2n+1} e_i \mathbf{M} \mathbf{D}^i f(u) + \int_0^1 E_{2n}(x)\, \mathbf{D}^{2n+1} f(x+u)\, dx.$$

Remarking that $\Delta f(u) = 2\,\mathbf{M} f(u) - 2f(u)$, we obtain

$$(2) \quad f(u) = \sum_{i=0}^{2n+1} e_i \mathbf{M} \mathbf{D}^i f(u) - \frac{1}{2} \int_0^1 E_{2n}(x)\, \mathbf{D}^{2n+1} f(x+u)\, dx.$$

The integral in the second member is the remainder; let us denote it by R_{2n}. $E_{2n}(x)$ does not change its sign in the interval $0 \leqq x \leqq 1$ therefore; in consequence of the theorem of mean values, we shall have

$$(3) \quad R_{2n} = -\tfrac{1}{2}\, \mathbf{D}^{2n+1} f(\eta) \int_0^1 E_{2n}(x)\,dx = e_{2n+1}\, \mathbf{D}^{2n+1} f(\eta)$$

where $u < \eta < u+1$.

Particular cases. 1. If $\mathbf{D}^{2n+1} f(u+x)$ and $\mathbf{D}^{2n+3} f(u+x)$ have the same sign and do not change their sign in the interval $0 \leqq x \leqq 1$, then we shall have $R_{2n} R_{2n+2} < 0$; the same ratiocination as in the case of the *Bernoulli* series § 87 will lead to

$$(4) \quad R_{2n} = e_{2n+1}\, \vartheta \mathbf{M} \mathbf{D}^{2n+1} f(u) \quad \text{where} \quad 0 < \vartheta < 1.$$

Example 1. Given $f(u) = e^{ut}$. We have seen that

$$\mathbf{M} \mathbf{D}^i e^{ut} = \tfrac{1}{2}\, t^i (e^t + 1)\, e^{ut}.$$

$\mathbf{D}^{2n+1} f(u)$ and $\mathbf{D}^{2n+3} f(u)$ have the same sign, and do not change it; therefore the remainder (4) may be used, and we have

$$R_{2n} = \tfrac{1}{2}\, \vartheta\, e_{2n+1} (e^t + 1)\, t^{2n+1} e^{ut}.$$

Hence the required series will be, after multiplication by $2/(1+e^t)e^{ut}$,

$$\frac{2}{1+e^t} = 1 + e_1 t + e_3 t^3 + \ldots + e_{2n-1} t^{2n-1} + 6\, e_{2n+1}\, t^{2n+1}.$$

This may be written

$$\frac{1-e^t}{1+e^t} = -\tanh \frac{t}{2} = \sum_{i=1}^{n+1} e_{2i-1}\, t^{2i-1} + \vartheta\, e_{2n+1}\, t^{2n+1}.$$

The preceding series has been obtained before (9, § 109), but not the remainder,

§ 112 Boole's first Summation Formula, It gives the inverse mean of the function f(x) or the sum of the alternate function (-1)X $f(x)$ expressed by aid of the derivatives of $f(x)$.

Apart from the remainder, the formula may be deduced by aid of the symbolical method. In § 6 we found

$$\mathsf{M} = 1 + \tfrac{1}{2}\Delta = \tfrac{1}{2}\,[1 + e^{h\mathsf{D}}].$$

From this it follows that

$$\frac{1}{\mathsf{M}} = \frac{2}{1 + e^{h\mathsf{D}}}$$

but according to formula (8) § 109 the second member is equal to the generating function of the numbers e_i, the coefficients figuring in the *Euler* polynomials. Therefore we have

$$\mathsf{M}^{-1} = \sum_{i=0}^{\infty} e_i \,(h\mathsf{D})^i.$$

This is the first form of the formula.

To obtain the second we remark that in § 39 formula (10) we had

$\mathsf{M}^{-1} = 2\,(-1)^{x+1}$ A-′ (-1)″.

This gives

$$\Delta^{-1}\,(-1)^x = \tfrac{1}{2}(-1)^{x+1} \sum_{i=0}^{\infty} e_i (h\mathsf{D})^i.$$

The above operation performed on $f(x)$ gives, if the sum of the alternate function is calculated from $x=a$ to $x=z$;

$$\sum_{x=a}^{z} (-1)^x f(x) = -\tfrac{1}{2} \sum_{i=0}^{\infty} h^i e_i \,[\,(-1)^z \mathsf{D}^i f\,(z) - (-1)^a\, \mathsf{D}^i f\,(a)].$$

To obtain the formula with its remainder in the particular case of

$$\mathsf{D}^{2n+1}F(u)\;\mathsf{D}^{2n+3}F(u) > 0$$

and if $\mathsf{D}^{2n+1}F(u)$ does not change its sign in the interval $0 \leqq x \leqq 1$ we consider the expansion of $F(u)$ into an *Euler* series according to formulae (2) and (4) of § 111:

$$(1)\quad F(u) = \mathsf{M}F(u) + \sum_{i=0}^{n} e_{2i+1}\mathsf{M}\mathsf{D}^{2i+1}F(u) + \vartheta\, e_{2n+1}\mathsf{M}\mathsf{D}^{2n+1}F(u)$$

where $0 < \vartheta < 1$.

Putting now $F(u) = M^{-1}f(u)$ *we* find

$$M^{-1}f(u) = f(u) + \sum_{i=0}^{n} e_{2i+1} D^{2i+1} f(u) + \vartheta\, e_{2n+1} D^{2n+1} f(u) + (-1)^u k$$

where k is an arbitrary periodic function with period equal to one.

Since according to formula (11) § 39 we have

$$\Delta^{-1} = \tfrac{1}{2}(-1)^{x+1} M^{-1} (-1)^x .$$

the indefinite sum of $(-1)^u f(u)$ will be

$$\Delta^{-1} (-1)^u f(u) = \tfrac{1}{2}(-1)^{u+1} [f(u) + \sum_{i=0}^{n} e_{2i+1} D^{2i+1} f(u) + \vartheta\, e_{2n+1} D^{2n+1} f(u)] + K.$$

Finally the alternate sum from $u=0$ to $u=z$ is

$$(2) \qquad \sum_{u=0}^{z} (-1)^u f(u) = \tfrac{1}{2}[f(0) - (-1)^z f(z)] + \tfrac{1}{2} \sum_{i=0}^{n} e_{2i+1} \cdot [(-1)^{z+1} D^{2i+1} f(z) + D^{2i+1} f(0)] + \tfrac{1}{2}\vartheta\, e_{2n+1} [(-1)^{z+1} D^{2n+1} f(z) + D^{2n+1} f(0)].$$

This is **Boole's** first summation formula for alternate functions, which plays the same rôle in these cases as the ***Maclaurin-Euler*** formula for ordinary functions.[32]

Example 1. Let $f(u) = e^{ut}$. From equation (2) it follows that

$$\sum_{u=0}^{z} (-1)^u e^{ut} = \tfrac{1}{2} [1 + (-1)^{z+1} e^{zt}] [1 + \sum_{i=0}^{n} e_{2i+1} t^{2i+1} + 6\, e_{2n+1} t^{2n+1}].$$

Particular case of *Boole's* formula. Let us suppose that we have $D^m f(\infty) = 0$ for $m = 0, 1, 2, 3, \ldots$, and $D^{2n+1}f(u)\, D^{2n+3}f(u) > 0$ and that $D^{2n+1}f(u+x)$ does not change its sign in the interval $0 < x < 1$; then we may write formula (2) in the following manner:

$$(3) \qquad \sum_{u=0}^{z} (-1)^u f(u) = \tfrac{1}{2}(-1)^{z+1}[f(z) + \sum_{i=0}^{n} e_{2i+1} D^{2i+1} f(z) + \vartheta\, e_{2n+1} D^{2n+1} f(z)] + C_f$$

[32] B. **Boole,** Calculus of Finite Differences, London, 1860, p. 95.

where C_f has been put for

$$(4) \qquad C_f = \tfrac{1}{2} f(0) + \tfrac{1}{2} \sum_{i=0}^{n} e_{2i+1} \mathsf{D}^{2i+1} f(0) + \tfrac{1}{2} \vartheta\, e_{2n+1} \mathsf{D}^{2n+1} f(0).$$

Writing $z = \infty$ we obtain from (3)

$$(5) \qquad \sum_{u=0}^{\infty} (-1)^u f(u) = C_f.$$

The constant C_f must be determined by aid of (4) or (5).

Example 2. **Given** $f(u) = 1/(u+1)$. **We may** determine C easily, using equation (5). Indeed we **have**

$$(6) \qquad \sum_{u=0}^{\infty} (-t)^u = \frac{1}{1+t}.$$

Integrating **both** members of **this** equation from $t=0$ to $t=1$ we get

$$\sum_{u=0}^{\infty} \frac{(-1)^u}{u+1} = \log 2.$$

Hence according to (5) we have $C_f = \log 2$. Since moreover

$$\mathsf{D}^{2i+1} f(u) = \frac{-(2i+1)!}{(u+1)^{2i+2}}$$

the required alternate sum will be

$$(7) \qquad \sum_{u=0}^{z} \frac{(-1)^u}{u+1} = \log 2 + \tfrac{1}{2}(-1)^z \left[\frac{-1}{z+1} + \sum_{i=0}^{n} \frac{(2i+1)!}{(z+1)^{2i+2}} e_{2i+1} + \vartheta\, e_{2n+1} \frac{(2n+1)!}{(z+1)^{2n+2}} \right].$$

The series obtained by putting $n = \infty$ is divergent, **neverthe**less the formula is useful for **the** computation of the alternate sum if z is large enough. On the other band for $z = 0$ the series is practically useless, According to **(3),** § 106 the best value of n is approximately $\tfrac{1}{2}\pi z$.

§ 113. Boole's polynomials, We shall call ***Boole*'s** polynomial of degree n the polynomial $\zeta_n(x)$ satisfying tbe equation

$$(1) \qquad \mathsf{M}\zeta_n(x) = \binom{x}{n}.$$

Performing the operation M^{-1} we obtain

$$\zeta_n(x) = \mathbf{M}^{-1} \binom{x}{n} \tag{2}$$

Since $\zeta_n(x)$ is a polynomial, there is only one solution of equation (1), as we have seen in § 38.

The operation Δ performed on (1) gives

$$\Delta \mathbf{M} \zeta_n(x) = \binom{x}{n-1}.$$

Now executing the operation $\mathbf{M}^{-1}$ we find, according to (2),

$$\Delta \zeta_n(x) = \zeta_{n-1}(x). \tag{3}$$

Therefore *Boole's* polynomials belong to the important class of functions mentioned in § 22 (p. 64).

The coefficients of $\binom{x}{i}$ in the expansion of *Boole's* polynomials into a binomial series, could be determined in the same way as has been done in the case of *the Bernoulli* polynomials of the second kind; but there is a shorter way.

Indeed we have seen, formula (12) § 39, that

$$\mathbf{M}^{-1} = \sum_{m=0}^{\infty} (-1)^m \frac{\Delta^m}{2^m}$$

Applied to equation (2) this gives

$$\zeta_n(x) = \sum_{m=0}^{n+1} \frac{(-1)^m}{2^m} \binom{x}{n-m} \tag{4}$$

This is the expansion of the *Boole* polynomials into a *Newton* series. The coefficients of $\binom{x}{i}$ are in consequence of § 22 independent of the degree of the polynomials; moreover they are very simple,

Particular cases.

$$\zeta_1(x) = x - ½$$

$$\zeta_2(x) = \binom{x}{2} - \frac{1}{2}\binom{x}{1} + \frac{1}{2^2}$$

$$\zeta_3(x) = \binom{x}{3} - \frac{1}{2}\binom{x}{2} + \frac{1}{2^2}\binom{x}{1} - \frac{1}{2^3}$$

and so on. (See Figure 4.)

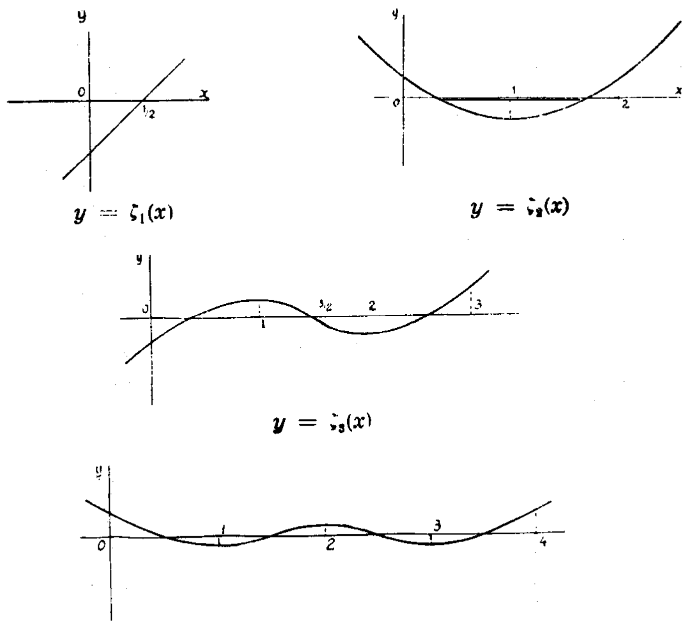
Remark. $(-1)^n\, 2^n\, \zeta_n(x)$ is equal to the first part of the expansion of $(1-2)^x$. Indeed we have

$$(5) \qquad (-1)^n 2^n \zeta_n(x) = \sum_{i=0}^{n+1} (-1)^i \binom{x}{i} 2^i.$$

From this it follows that

$$\zeta_n(0) = \frac{(-1)^n}{2^n}$$

Figure 4.



$y = \zeta_1(x)$ $\qquad$ $y = \zeta_2(x)$

$y = \zeta_3(x)$

$y = \zeta_4(x)$

moreover if x is an integer such that $0 < x \leqq n$ then

$$(5') \qquad \zeta_n(x) = \frac{(-1)^n}{2^n}(1-2)^x = \frac{(-1)^{n+x}}{2^n}.$$

If x=-1 from (5) we obtain

$$\zeta_n(-1) = \frac{(-1)^n}{2^n}[2^{n+1}-1].$$

Roots of the Boole polynomials. According to formula (5’) the polynomial C,(x) changes its sign between 0 and **1, between 1** and 2, and so on, between $n-1$ and n. Hence the roots are all real, single, and situated in the interval $(0, n)$.

Application of the Boole polynomials. 1. Determination of alternate sums. Since according to (10), § 39 the operation $\mathbf{M}^{-1}$ may be expressed by

$$\mathbf{M}^{-1} = 2(-1)x+' \Delta^{-1} (-1)''$$

we have

$$\Delta^{-1}[(-1)^x t(x)] = \tfrac{1}{2}(-1)^{x-1}\mathbf{M}^{-1} t(x) + k.$$

If $f(x)$ is expanded into a ***Newton*** series we have, taking account of (2),

$$(6) \qquad \Delta^{-1} [(-1)'' f(x)] = \tfrac{1}{2} (-1)^{x+1} \sum_{n=0}^{\infty} \Delta^n f(0)\mathbf{M}^{-1}\binom{x}{n} =$$

$$= \tfrac{1}{2}(-1)^{x+1} \sum_{n=0} \zeta_n(x)\, \Delta^n f(0) + k.$$

This is the ***indefinite sum*** of the alternate function.

From formula (6) we deduce immediately the alternate sum for x varying from zero to z,

$$(7) \qquad \sum_{x=0}^{z} (-1)^x f(x) = \tfrac{1}{2} \sum_{n=0}^{\infty} \Delta^n f(0) \left[\frac{(-1)^n}{2^n} - (-1)^z \zeta_n(z)\right].$$

§ 114. Operations on the Boole polynomials. Differences. We have seen that

$$A'' \zeta_n (x) = \zeta_{n-m} (x).$$

From this it follows that the indefinite sum is

$$A-' \zeta_n (x) = \zeta_{n+1} (x) + k$$

and therefore the definite sum

$$\sum^{z} \zeta_n(x) = \zeta_{n+1} (z) - \zeta_{n+1} (0).$$

Derivatives. Starting from the ***Newton*** expansion of the polynomial

$$(1) \qquad \zeta_n(x) = \sum_{m=0}^{n+1} \frac{(-1)^{n-m}}{2^{n-m}} \binom{x}{m}$$

we expand $\binom{x}{m}$ into a power series and determine the derivative term by term. We get

$$\text{(2)} \qquad \mathbf{D}\zeta_n(x) = \sum_{m=1}^{n+1} \frac{(-1)^{n-m}}{m!\,2^{n-m}} \sum_{\nu=1}^{m+1} \nu\, S_m^\nu\, x^{\nu-1}$$

and putting $x=0$

$$\mathbf{D}\,\zeta_n(0) = \frac{(-1)^{n+1}}{2^n} \sum_{m=1}^{n+1} \frac{2^m}{m}.$$

The higher derivatives are easily obtained from (2).

Integral.

$$\int \binom{x}{n} dx = \psi_{n+1}(x) + k$$

where $\psi_{n+1}(x)$ is the *Bernoulli* polynomial of the second kind of § 89; therefore from (1) we obtain:

$$\int \zeta_n(x)\, dx = \sum_{m=0}^{n+1} \frac{(-1)^{n-m}}{2^{n-m}} \psi_{m+1}(x) + k$$

and

$$\int_0^1 \zeta_n(x)\, dx = \sum_{m=0}^{n+1} \frac{(-1)^{n-m}}{2^{n-m}}\, b_m$$

Mean of the polynomial. From the definition of the polynomial it follows directly that

$$\mathbf{M}\zeta_n(x) = \binom{x}{n}$$

Znverse operation of the mean. In consequence of formula (12) § 39 we have

$$\mathbf{M}^{-1}\,\zeta_n(x) = \sum_{m=0}^{n+1} \frac{(-1)^m}{2^m}\, \zeta_{n-m}(x).$$

§ 115. Expansion of the Boole polynomials into a series of Bernoulli polynomials of the second kind. According to formulae (1) and (3) of § 97 we have

$$c_0 = \mathbf{D}\Delta^{-1} f(0) = \mathbf{D}\zeta_{n+1}(0) = \frac{(-1)^n}{2^{n+1}} \sum_{m=1}^{n+2} \frac{2^m}{m}$$

and

$$c_i = \mathbf{D}\Delta^{i-1} f(0) = \mathbf{D}\zeta_{n-i+1}(0) = \frac{(-1)^{n+i}}{2^{n-i+1}} \sum_{m=1}^{n-i+2} \frac{2^m}{m}.$$

The above values of $\mathbf{D}\zeta_m(0)$ are those deduced in the preceding paragraph. Finally we find

$$\zeta_n(x) = \sum_{i=0}^{n+1} \frac{(-1)^{n+i}}{2^{n-i+1}} \psi_i(x) \sum_{m=1}^{n-i+2} \frac{2^m}{m}$$

§ 116. Expansion of a function into a series of Boole polynomials. If f(x) is a polynomial of degree n we may write it as follows:

(1) $$f(x) = c_0 + c_1\zeta_1(x) + c_2\zeta_2(x) + \dots + c_n\zeta_n(x).$$

It is easy to determine the coefficients c_m; indeed, since

$$\mathbf{M}f(x) = c_0 + c_1\binom{x}{1} + c_2\binom{x}{2} + \dots + c_n\binom{x}{n}$$

it follows that

$$c_0 = \mathbf{M}f(0).$$

Moreover

$$\Delta^m f(x) = c_m + c_{m+1}\zeta_1(x) + \dots + c_n\zeta_{n-m}(x)$$

and

$$c_m = \mathbf{M}\Delta^m f(0).$$

If f(x) is not a poynomial, the series (1) will be infinite and considerations of convergence will arise; but the coefficients are determined in the same way as above.

Example 1. Given $f(x) = \binom{x}{n}$, the formulae above give

$$c_{n-1} = 1/2 \text{ and } c_n = 1.$$

The other coefficients c_m are equal to zero. So that we find

$$\binom{x}{n} = \zeta_n(x) + 1/2\, \zeta_{n-1}(x);$$

The second member is equal to $(1+1/2\Delta)\zeta_n(x)$ or to $\mathbf{M}\zeta_n(x)$ therefore this equation would immediately follow from formula (2) § 113.

Example 2. Given $f(x) = \psi_n(x)$, the *Bernoulli* polynomial of the second kind. We have

$$\Delta^m \psi_n(x) = \psi_{n-m}(x);$$

therefore

$$c_m = \mathbf{M}\Delta^m \psi_n(0) = 1/2\,[\psi_{n-m}(0) + \psi_{n-m}(1)] = b_{n-m} + 1/2\, b_{n-m-1}$$

and finally

$$\psi_n(x) = \sum_{m=0}^{n+1} (b_{n-m} + \tfrac{1}{2} b_{n-m-1}) \zeta_m(x).$$

Example 3. Given f(x) $= F(x)$. We have

$$\mathsf{M}F(x) = F(x) + \frac{1}{2(x+1)} \text{ so that } c_0 = \tfrac{1}{2} - C.$$

Since

$$\Delta^m F(x) = \frac{(-1)^{m-1}(m-1)!}{(x+m)_m}$$

it follows that

$$c_m = \mathsf{M}\Delta^m F(0) = \tfrac{1}{2}(-1)^{m-1}\left[\frac{1}{m} + \frac{1}{m(m+1)}\right]$$

so that

$$c_m = \frac{(-1)^{m-1}(m+2)}{2m(m+1)}.$$

Finally the required expansion will be

$$F(x) = \tfrac{1}{2} - C + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(m+2)}{2m(m+1)} \zeta_m(x).$$

(See also the expansion of β_1 (x) in § 122).

The Boole series. The expansion of $f(x+u)$ into a series of *Boole* polynomials gives

$$f(x+u) = c_0 + c_1\zeta_1(x) + c_2\zeta_2(x) + \ldots$$

where as we have seen $c_m = \mathsf{M}\Delta^m f$ *(u)*,

Putting into this equation $u=0$ we obtain the preceding expansion (1) of *f(x)* into ***Boole*** polynomials; and putting into it $x=0$ we obtain the expansion of *f(u)* into a series which we will call a *Boole* series, just as we have done in the cases of the expansion of $f(x+u)$ into series of *Bernoulli* and *Euler* polynomials,

Since ζ_m (0) $= (-1)^m/2^m$ the *Boole* series will be

$$f(u) = \sum_{m=0}^{\infty} \mathsf{M}\Delta^m f(u) \frac{(-1)^m}{2^m}$$

An example is given in § 122.

§ 117. Boole's second summation formula. It gives the inverse mean of the function *f(x)* and the sum of the alternate function $(-1)^x$ *f(x)* expressed by aid of the differences of *f(x).*

The formula may easily be deduced by aid of the symbolical method. In § 6 we found

$$\mathbf{M} = 1 + \tfrac{1}{2}\Delta$$

from this it follows that

$$\frac{1}{\mathbf{M}} = \frac{1}{1+\frac{1}{2}\Delta} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2^i}\Delta^i$$

this leads to the first formula. To obtain the second, we remark that, according to formula (10) § 39.

$$\mathbf{M}^{-1} = 2\,(-1)^{x+1}\,\Delta^{-1}\,(-1)^x$$

so that

$$\Delta^{-1}(-1)^x = \tfrac{1}{2}(-1)^{x+1}\sum_{i=0}^{\infty}\frac{(-1)^i}{2^i}\Delta^i.$$

This operation performed on $f(x)$ gives, if the sum from $x=a$ to $x=z$ is to be calculated,

$$\sum_{x=a}^{z}(-1)^x f(x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{2^{i+1}}\left[(-1)^a\Delta^i f(a) - (-1)^z\Delta^i f(z)\right].$$

We may obtain this formula by aid of the *Boole's* polynomials introduced in the preceding paragraph. The expansion of $f(x+u)$ into a series of *Boole* polynomials may be written

$$f(x+u) = \sum_{m=0}^{\infty}\zeta_n(u)\mathbf{M}\Delta^m f(x).$$

Putting into it $u=0$ we obtain

$$f(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\mathbf{M}\Delta^m f(x).$$

Executing the operation $\mathbf{M}^{-1}$ it will follow that

$$\mathbf{M}^{-1}f(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\Delta^m f(x) + (-1)^x k.$$

Hence by aid of (10), § 39 we deduce the indefinite sum of the alternate function:[33]

$$\Delta^{-1}\left[(-1)^x f(x)\right] = (-1)^{x+1}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{m+1}}\Delta^m f(x) + k.$$

---

[33] G. Rode, Treatise on the Calculus of Finite Differences. London, 1860. p. 95.

This will give the sum from $x=0$ to $x=z$

$$\sum_{x=0}^{z} (-1)^x f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{m+1}} \left[\Delta^m f(0) - (-1)^z \Delta^m f(z)\right].$$

Example 1. Given $f(x) = \frac{1}{x+1}$. We find

$$\sum_{x=0}^{z} \frac{(-1)^x}{1+x} = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \left[\frac{1}{m+1} - \frac{(-1)^z m!}{(z+m+1)_{m+1}}\right].$$

We have seen in (6), § 112 that the first member is equal to log2 if $z=\infty$, so that we have also

$$\sum_{m=1}^{\infty} \frac{1}{m\,2^m} = \log 2.$$

Therefore the preceding sum may be written

$$\sum_{x=0}^{z} \frac{(-1)^x}{1+x} = \log 2 + (-1)^{z+1} \sum_{m=1}^{\infty} \frac{1}{m 2^m \binom{z+m}{m}};$$

this may be useful for the determination of the sum if z is large.

§ 118. Sum of reciprocal powers. Sum of $1/x$ by aid of the **digamma function.** This function has been already treated of in § 19; here some complementary formulae are added.

The definition of the function was the following

$$F(x) = D \log \Gamma(x+1).$$

From this we deduced

$$\Delta F(x) = \frac{1}{x+1}$$

and therefore

$$\Delta^{-1} \frac{1}{x} = F(x-1) + k;$$

moerover if n is a positive integer

$$\sum_{x=1}^{n+1} \frac{1}{x} = F(n) - F(0).$$

We have seen (5, § 19) that the digamma function with negative arguments may be expressed by a digamma function with positive arguments:

$$F(-x) = F(x-1) + \pi \cot \pi x .$$

From formula (2) § 19 we deduce

$$DF(x) = \sum_{m=1}^{\infty} \frac{1}{(x+m)^2} > 0;$$

therefore we conclude that the function F(x) is a monotonously increasing function. Since we have $F(0) < 0$ and $F(1) > 0$ the equation $F(x) = 0$ has only one root, and this in the interval zero to one. Determining it we find $x = 0.46163\ 21$.

The indefinite sum of the digamma function is given by (6) § 34:

$$\Delta^{-1} F(x) = x\,F(x) - x + k$$

from this we obtain, if n is a positive integer:

$$\sum_{x=0}^{n} F(x) = n\,F(n) - n.$$

For instance

$$\sum_{x=0}^{10} f(x) = 10\,F(10) - 10 = 13.51752\ 592$$

This result may be checked by adding the numbers

$$F(0) + F(1) + F(2) + \ldots , + F(9)$$

taken from the tables of the function

We have seen (7, § 19) that the function f(x) may be expressed by the aid of sums. We found:

$$(1) \qquad F(x) = -C + \sum_{\nu=1}^{\infty} \left[\frac{1}{\nu} - \frac{1}{x+\nu} \right] = -C + \sum_{\nu=0}^{\infty} \frac{x}{\nu(x+\nu)} .$$

Let us remark, that this series is uniformly convergent in the interval $0,N$ where N is any finite positive quantity whatever. Indeed, if $0 \leqq x \leqq N$, we have

$$\frac{1}{\nu(x+\nu)} < \frac{1}{\nu^2} .$$

From (1) we may deduce the derivatives of F(x); we obtain

$$D^m F(x) = \sum_{\nu=1}^{\infty} \frac{(-1)^{m+1}\ m!}{(x+\nu)^{m+1}}$$

and

$$\mathbf{D}^m F(0) = \sum_{\nu=1}^{\infty} \frac{(-1)^{m-1}\, m!}{\nu^{m+1}}$$

or if we introduce the notation

$$s_m = \sum_{\nu=1}^{\infty} \frac{1}{\nu^m}$$

then

$$\mathbf{D}^m F(0) = (-1)^{m-1}\, m!\; s_{m+1}\,.$$

Hence the expansion of F(x) into a power series will be

$$\text{(2)} \qquad F(x) = -\,c + \sum_{m=1}^{\infty} (-1)^{m-1}\, s_{m+1}\, x^m.$$

$s_{m+1} \leqq \pi^2/6$; therefore this series is convergent if $|x| < 1$.

The function $F(x)$ may be expressed if x is a positive integer by the definite integral

$$\text{(3)} \qquad F(x) = -\,C + \int_0^1 \frac{1 - t^x}{1-t}\, dt.$$

Indeed the expansion of the quantity under the integral sign will be

$$1 + t + t^2 + \ldots . - [t^x + t^{x+1} + t^{x+2} + \ldots .];$$

integrated from $t = 0$ to $t = 1$ it gives

$$\ldots_{m=1} \left| \overline{m \qquad x+m} \right|,$$

therefore from (1) it follows that the second member of (3) is equal to $F(x)$. Let us remark that this is true even if x is not an integer.

From the above formulae it follows that

$$\text{(4)} \qquad s_n = \frac{(-1)^n}{(n-1)!}\, \mathbf{D}^{n-1} F(0).$$

In formula (6) of § 82 we expressed s_{2n} by aid of the ***Bernoulli*** numbers. Using this formula we obtain from (4)

$$\mathbf{D}^{2n-1} F(0) = \frac{(2\pi)^{2n}}{4n}\, |B_{2n}|\,.$$

Expansion of the digamma function into a Newton ***series.*** Since $\Delta F(x) = 1/(x+1)$, we have

(4') $$\Delta^m F(x) = \frac{(-1)^{m-1}(m-1)!}{(x+m)_m} \quad \text{and} \quad \Delta^m F(0) = \frac{(-1)^{m-1}}{m}$$

so that

(5) $$F(x) = -C + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \binom{x}{m}.$$

If $0 < x < 1$ the absolute value of the general term is smaller than $1/m^2$ therefore the series is uniformly convergent in this interval.

The values of $F(x)$ for $x>1$ may be computed by other series. For instance if x is large the ***Maclaurin-Euler*** series is indicated, We found in § 88 an expansion of the sum of $1/(x+1)$, from which, by remarking that

$$\sum_{x=0}^{z} \frac{1}{x+1} = F(z) + c,$$

we deduce:

(6) $$F(x) = \log(x+1) - \frac{1}{2(x+1)} - \sum_{m=1}^{n} \frac{(2m-1)!}{(x+1)^{2m}} a_{2m} - $$
$$- \xi a_{2n} \frac{(2n-1)!}{(x+1)^{2n}}$$

where $0 < \xi < 1$. The logarithms figuring in these formulae are *Napier*'s logarithms. The best tables for these are *Schultze*'s Recueil de Tables Logarithmiques (Berlin, 1778). If we do not possess such tables, we may obtain the ***Napier*** logarithms by multiplying ***Briggs'*** logarithms by the modulus

$$\log 10 = 2.30258\ 50929\ 94045\ 68402.$$

If in the expansion (6) we put $n=\infty$, the series obtained will be divergent; but nevertheless formula (6) is useful for the computation of ***F(x). E.*** *Pairman* has used it for computing her tables.[34]

Since from formula (6) it follows that the remainder is

$$|R_{2n}| < \frac{|B_{2n}|}{2n(x+1)^{2n}}$$

---

[34] Eleanor *Pairman*, Tables of the Digamma and Trigamma Functions, Tracts for Computers. Cambridge, 1919. They contain $F(x)$ and $F(x)$ from $x=0.00$ to $x=20.00$ $(\Delta x=0.01)$ and the corresponding second and fourth central differences, to eight decimals.

we may increase n in formula (6) advantageously till B_{2n} satisfies the inequalities below

$$(7) \qquad \frac{(x+1)^2\,|B_{2n-2}|}{2n-2} > \frac{|B_{2n}|}{2n} < \frac{|B_{2n+2}|}{(2n+2)\,(x+1)^2} .$$

The greater x is, the closer the approximation that may be obtained. For instance for $x=0$ from Table II of § 78 it follows taking account of (7) that the smallest remainder will be obtained for $n=3$. Then formula (6) gives:

$$F(0) = -0.57517 - \xi \,.\, 0{,}0040398$$

the error will be less then 4 units of the third decimal. But for $x=9$ already a precision of twenty decimal places may be obtained for $n=12$.

Example 1. Determination of F(99) to ten decimal places. We have

$$F(99) = \log 100 - \frac{1}{200} - \frac{a_2}{10^4} - \xi \,\frac{6a_4}{10^8} .$$

We found in § 78 that a, $= 1/12$ and $a_4 = -1/720$; therefore we shall have

$\log\ 100$	$=$	$4.605170\ 1860$
$-1/200$	$=$	$-\ 0.005$
$-1/12.10^4$	$=$	$-\ 0.000008\ 3333$
$F(99)$	$=$	$4.600161\ \ 8527$

According to the remainder the error is less than one unit of the tenth decimal,

If x is large, $F(x)$ may be calculated also by the aid of the series obtained in (10), § 98:

$$(7)\quad F(x) = \log(x+1) - \sum_{m=1}^{n} \frac{(-1)^{m-1}\,(m-1)!}{(x+m)_m} b_m - b_n\,\Delta^n \log(\eta+1)$$

where $x < \eta < x+1$. We have seen that this series is convergent for $x > 0$. This is the expansion of $F(x)$ into a reciprocal factorial series.

The formula applied to the preceding example gives

$$F(99) = \log 100 - \frac{b_1}{100} + \frac{b_2}{10100} - \frac{2b_3}{100.101.102} + \frac{6b_4}{100.101.102.103} .$$

In § 89 we found that $b_1 = \frac{1}{2}$, $b_2 = -1/12$ and $b_3 = 1/24$, $b_4 = 19/720$, therefore

$\log 100$	$=$	4.605170 1860
$-1/200$	$=$	-0.005
$-1/12.100.101$	$=$	-0.000008 2508
$-1/12.100.101.102$	$=$	-0.000000 0809
$-19/120.100.101.102.103$	$=$	-0.000000 0015
$F(99)$	$=$	4.600161 8528

The error is less than one unit of the tenth decimal.

Finally $F(x)$ may be determined by the aid of *Pairman's* tables, and though these tables contain $F(x)$ only up to $x=20$ we may employ them also for $x > 20$ by using the multiplication formula given by (6), § 19.

$$F(nx) = \log n + \frac{1}{n}\sum_{i=0}^{n} F\left(x - \frac{i}{n}\right).$$

Example 2. Determination of F(100). Putting into the preceding formula $x=20$ and $n=5$ *we* get

$$F(100) = \log 5 + \frac{1}{5}[F(20) + F(19{,}8) + F(19{,}6) + F(19{,}4) + F(19{,}2)].$$

By aid of *Pairman's* tables we find

$\log 5$	$=$	1.609437 91
		3.000723 94
$F(100)$	$=$	4,610161 85.

Since $F(100) = 0.01 + F(99)$, the above result is in good accordance with those obtained before.

§ 119, Sum of $1/x^2$ by the aid of the trigamma function. The trigamma function has already been treated of in § 20. Its definition; is the following

$$F(x) = D^2 \log\Gamma(x+1) = DF(x);$$

therefore

$$\Delta F(x) = D\Delta F(x) = D\frac{1}{x+1} = -\frac{1}{(x+1)^2}.$$

From this we obtain the indefinite sum

$$\Delta^{-1}\frac{1}{x^2} = -F(x-1) + k$$

and finally if z is an integer

$$\sum_{x=1}^{z+1} \frac{1}{x^2} = F(0) - F(z).$$

In (7), § 34 we obtained by aid of summation by parts the indefinite sum of $F(x)$

$$\Delta^{-1} F(x) = xF(x) + F(x) + k$$

where k is, as has been said in § 32, an arbitrary periodic function of x with period equal to one.

Fom this it follows that the sum of $F(x)$ is, when x varies from $x=0$ to $x=n$,

$$\sum_{x=0}^{n} f(x) = c + nF(n) + F(n).$$

The trigamma function may be expressed by a sum. Indeed the derivative of $F(x)$ given by formula (1) § 118 is

$$(1) \qquad F(x) = \sum_{m=1}^{\infty} \frac{1}{(x+m)^2}.$$

This series is uniformly convergent if $0 < x$.

It is easy to show that the difference of this expression is equal to $-1/(x+1)^2$.

From (1) it follows that $F(\infty) = 0$; moreover by aid of formula (6) § 82 (1) gives

$$F(0) = \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$$

Since $\Delta F(0) = -1$,

$$F(1) = \frac{\pi^2}{6} - 1,$$

The values of $F(x)$ for negative arguments are deduced by derivation from the formula giving F(x) for negative values (§ 118); we find

$$F(-x) = \frac{\pi^2}{\sin^2 \pi x} - F(x-1).$$

The derivatives of $F(x)$ may be obtained by the relation

$$D^m F(x) = D^{m+1} F(x)$$

which leads by aid of § 118 to

$$\mathbf{D}^m F(x) = \sum_{\nu=1}^{\infty} \frac{(-1)^m\,(m+1)!}{(x+\nu)^{m+2}}$$

and to

$$\mathbf{D}^m F(0) = \sum_{\nu=1}^{\infty} \frac{(-1)^m\,(m+1)!}{\nu^{m+2}} = (-1)^m\,(m+1)!\; s_{m+2}$$

where we introduced the denotation s_m of the preceding paragraph.

Hence the expansion of the trigamma function into a power series will be

$$(2) \qquad F(x) = \sum_{m=0}^{\infty} (-1)^m \text{ (m-t-l) } s_{m+2}\, x^m .$$

The series is convergent if $|x| < 1$.

There are other series useful for the computation of f(x) if x is large. We found in the case of the *Maclaurin-Euler* formula an expansion into a reciprocal power series (9, § 88) from which, by remarking that

$$\sum_{x=0}^{z} \frac{1}{(x+1)^2} = -F(z) + \frac{\pi^2}{6},$$

we deduce

$$(3) \qquad F(x) = \frac{1}{x+1} - \frac{B_1}{(x+1)^2} + \sum_{m=1}^{n} \frac{B_{2m}}{(x+1)^{2m+1}} + \frac{\xi B_{2n}}{(x+1)^{2n+1}} .$$

The series obtained by putting $n=\infty$ is divergent but nevertheless formula (3) is useful for the computation of $F(x)$ if $x>n$. *Pairman's* tables were computed by aid of this formula. In consequence of formula (2) p. 302 the best value of n would approximately be $\pi(x+1)$.

Example 1. The value of f(20) is required to ten decimals; let us write:

$$F(20) = \frac{1}{21} - \frac{B_1}{(21)^2} + \frac{B_2}{(21)^3} + \frac{B_4}{(21)^5} + \frac{\xi B_6}{(21)^7} .$$

Determining the remainder, we see that the error is less than two units of the eleventh decimal. Moreover

$$
\begin{array}{lrr}
1/21 & = 0.047619 & 04762 \\
1/2(21)^2 & = \quad 1133 & 78685 \\
1/6(21)^3 & = \quad 17 & 99661 \\
-1/30(21)^5 & = - & 816 \\
\hline
F(20) & = 0.048770 & 82292
\end{array}
$$

The ***Bernoulli*** numbers B_i have been taken from our table of **§ 78.**

$$B_1 = -\tfrac{1}{2}, \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42.$$

Formula **(4) § 68** gives the expansion of $1/x^2$ into a series of reciprocal factorials:

$$\frac{1}{x^2} = \sum_{n=1}^{\infty} \frac{(n-1)!}{(x+n)_{n+1}}$$

If x $\geqq$ **1** then the general term of this series is less than $1/n^2$ and therefore the series is uniformly convergent; hence the sum of this expression may be calculated term by term; we obtain by the inverse operation of the differences

$$(4) \qquad -F(x-1) = -\sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+n-1)_n} + \omega(x)$$

where w(x) is a periodic function with period equal to one. Therefore the sum from $x=1$ to $x=z$ will be

$$F(0) - F(z-1) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{(n-1)!}{n(z+n-1)_n} + \omega(z) - \omega(1).$$

We remark that, since z is an integer, $\omega(z) = \omega(1)$ moreover that the first sum of second member is equal to $F(0)$, so that this equation may be written

$$F(z-1) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n(z+n-1)_n}$$

or

$$(5) \qquad F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{x+n}{n}}.$$

From this it is easy to see that the general term of (5) is less than $1/n^2$ and that the series is convergent. If x is large, the formula is useful for the computation of $F(x)$.

Remark. Formula (5) has been demonstrated above only for integer values of x.

Example 2. $F(100)$ is to be calculated to nine decimals.[35] Tbe successive terms are:

$$\begin{array}{r} 0.009900\ 9901 \\ 48\ 5343 \\ 6282 \\ 136 \\ 4 \\ \hline \end{array}$$

$$F(100) = 0.009950\ 1666$$

If we calculate by the aid of this formula $F(20)$, it would be an excellent example to show that a convergent series may lead less rapidly to the result than a divergent series. To obtain the same precision as in Example 1, where we calculated four terms, here it would be necessary to calculate more than ten terms; indeed we have

$$\begin{array}{r} 0.047619\ \ 0476 \\ 1082\ \ 2511 \\ 62\ 7392 \\ 5\ 8818 \\ 7529 \\ 1207 \\ 230 \\ 52 \\ 12 \\ 5 \\ \hline \end{array}$$

$$F(20) = 0.048770\ 8232.$$

Comparing this result with that obtained in Example 1, we remark that there is here still an error of three units of the tenth decimal.

Determination of F(x) by the aid of Pairman's tables. If $0 < x \leq 20$ then F(x) may be obtained by interpolation from the tables. If $x > 20$ the tables still give $F(x)$ by using the multiplication formula of the trigamma functions

[35] The binomial coefficients occurring in (5) may be taken out of tables, for instance such tables as are to be found in the "Annals of Mathematical Statistics", Vol. III, p. 364, ff.

$$F(mx) = \frac{1}{m^2} \sum_{i=0}^{m} F\left(x - \frac{i}{m}\right)$$

Example 3. Computation of $F(100)$. We shall put $m = 5$ and $x = 20$; it follows that

$$F(100) = \frac{1}{25}\left[F(20) + F(19{,}8) + F(19{,}6) + F(19{,}4) + F(19{,}2)\right].$$

By aid of the tables we find

$$F(100) = 0{,}009950\ 166$$

in good accordance with the result of Example 2.

§ 120. Sum of a rational fraction. Given a fraction in which the degree of the numerator is at least two less than that of the denominator such as,

(1)

$$F(x) = \frac{a_0 + a_1 x + \ldots + a_{n+2m-2}\, x^{n+2m-2}}{(x+b_1)(x+b_2)\ldots(x+b_n)(x+c_1)^2(x+c_2)^2\ldots(x+c_m)^2}$$

To begin with, we shall decompose this expression into partial fractions, so as to have

$$(2)\qquad F(x) = \sum_{i=1}^{n+1} \frac{A_i}{x+b_i} + \sum_{i=1}^{m+1} \frac{B_i}{x+c_i} + \sum_{i=1}^{m+1} \frac{C_i}{(x+c_i)^2}$$

We have seen in § 13 that if we denote the numerator of the fraction (1) by $\varphi(x)$, and the denominator by $\psi(x)$ then we shall have

$$A_i = \frac{\varphi(-b_i)}{\mathrm{D}\psi(-b_i)}$$

and denoting ,

$$P_i(x) = \frac{\varphi(x)}{\psi(x)}(x+c_i)^2$$

we have (formula 5, § 13)

$$C_i = P_i(-c_i) \quad \text{and} \quad B_i = \mathrm{D}P_i(-c_i)$$

Reducing the fractions (2) to a common denominator, the numerator obtained must be identical with $\varphi(x)$; since the degree of $\varphi(x)$ is at most equal to $n+2m-2$, therefore the coefficient of x^{n+2m-1} obtained must be equal to zero. That is, we must have

$$(3)\qquad A_1 + A_2 + \ldots + A_n + B_1 + B_2 + \ldots + B_m = 0.$$

The operation of Δ^{-1} performed on both members of (2) gives

$$\Delta^{-1}F(x) = \sum_{i=1}^{n+1} A_i F(x+b_i-1) + \sum_{i=1}^{m+1} B_i F(x+c_i-1) - \\ - \sum_{i=1}^{m+1} C_i F(x+c_i-1) + k.$$

From this we deduce without difficulty the **sum** of $F(x)$ from $x=0$ to $x=z$, if z is finite and integer:

$$(4) \qquad \sum_{x=0}^{z} F(x) = \sum_{i=1}^{n+1} A_i [F(z+b_i-1) - F(b_i-1)] + \\ + \sum_{i=1}^{m+1} B_i [F(z+c_i-1) - F(c_i-1)] + \\ + \sum_{i=1}^{m+1} C_i [F(c_i-1) - F(z+c_i-1)].$$

This formula is not applicable if z is infinite, since then the digamma functions in the second member will become infinite too. The following transformation of formula (1) will meet the case:

Let us add to the second member of (2) the quantity

$$-\frac{1}{x+1}[A_1+A_2+\ldots+A_n+B_1+B_2+\ldots+B_n]=0.$$

Since according to formula (7) § 19 we have

$$\sum_{x=0}^{\infty}\left[\frac{1}{x+b_i} - \frac{1}{x+1}\right] = -F(b_i-1) - C$$

the sum of $F(x)$ will be, taking account of (2), equal to

$$(5) \qquad \sum_{x=0}^{\infty} F(x) = -\sum_{i=0}^{n+1} A_i F(b_i-1) - \sum_{i=0}^{m+1} B_i F(c_i-1) + \\ + \sum_{i=0}^{m+1} C_i F(c_i-1).$$

The coefficient of C is in consequence of (3) equal to zero. The last term of the second member has been obtained by aid of the indefinite sum,

Example. To determine by this method the *sum of a reciprocal factorial* let us decompose $1/(x+n)_n$ into partial fractions. Since now $\varphi(x)=1$ and $\psi(x)=(x+n)_n$ we shall have

$$A_i = \frac{1}{\mathrm{D}[(x+n)_n]_{x=-i}} = \frac{(-1)^{i-1}}{(i-1)!\ (n-i)!}.$$

So that

$$(6)\qquad \frac{1}{(x+n)_n} = \sum_{i=1}^{n+1} \frac{A_i}{x+i} = \sum_{i=0}^{n+1} \frac{(-1)^{i-1}}{(i-1)!\,(n-i)!\,(x+i)}$$

Performing on both members the operation Δ^{-1} we get

$$\frac{-1}{(n-1)\,(x+n-1)_{n-1}} = \sum_{i=1}^{n+1} A_i F(x+i-1) + k$$

therefore the sum of (6), x varying from zero to z, will be

$$\frac{-1}{(n-1)}\left[\frac{1}{(z+n-1)_{n-1}} - \frac{1}{(n-1)!}\right] = \sum_{i=1}^{n+1} A_i\,[F(z+i-1)-F(i-1)].$$

To determine the sum, x varying from zero to ∞, we must use formula (5); we get

$$\frac{1}{(n-1)\,(n-1)!} = -\sum_{i=1}^{n+1} A_i F(i-1);$$

putting in the value of A_i obtained above and multiplying by $(n-1)!$ we obtain the formula

$$\frac{1}{n-1} = \sum_{i=1}^{n+1} (-1)^i \binom{n-1}{i-1} F(i-1).$$

Derivatives of the reciprocal factorial. Starting from (6) we may determine the derivatives of $1/(x+n)_n$. We have

$$(7)\qquad \mathrm{D}\frac{1}{(x+n)_n} = \sum_{i=1}^{n+1} \frac{(-1)^i}{(i-1)!\,(n-i)!\,(x+i)^2}$$

and

$$\mathrm{D}^m \frac{1}{(x+n)_n} = \sum_{i=1}^{n+1} \frac{(-1)^{i+m+1}\, m!}{(i-1)!\,(n-i)!\,(x+i)^{m+1}}.$$

From formula (4') of § 118 we deduce the derivative

$$\mathrm{D}\Delta^n F(x) = \Delta^n F(x) = (-1)^{n+1}(n-1)! \sum_{i=1}^{n+1} \frac{(-1)^i}{(i-1)!\,(n-i)!\,(x+i)^2},$$

Putting $x=0$ we get

$$\mathrm{D}\Delta^n F(0) = \Delta^n F(0) = \frac{(-1)^n}{n} \sum_{i=1}^{n+1} \frac{(-1)^{i+1}}{i}\binom{n}{i}.$$

According to formula (5) of § 118 the sum in the second member is equal to F(n) $+C$; therefore we get the interesting formula

$$(8)\quad \mathrm{D}\Delta^n F(0) = \Delta^n F(0) = \frac{(-1)^n}{n}\,[F(n)+C] = \frac{(-1)^n}{n}\sum_{x=0}^{n}\frac{1}{x+1}.$$

Integral *of the reciprocal factorial.* The integral from zero to z is obtained from formula (6):

$$\int_0^z \frac{dx}{(x+n)_n} = \sum_{i=1}^{n+1} A_i \log\frac{z+i}{i}.$$

This formula is not applicable if $z=\infty$, since then the terms of the second member become infinite. To remedy this inconvenience we add to (6) the following quantity (which is equal to zero in consequence of (3)):

$$-\frac{1}{x+1}\,[A_1 + A_2 + \ldots + A_n]$$

then integrating we find

$$\int_0^\infty \frac{dx}{(x+n)_n} = \sum_{i=1}^{n+1} A_i \int_0^\infty \left[\frac{1}{x+i} - \frac{1}{x+1}\right] dx =$$

$$= \sum_{i=1}^{n+1} A_i \left[\log\frac{x+i}{x+1}\right]_0^\infty.$$

Finally

$$\int_0^\infty \frac{dx}{(x+n)_n} - \sum_{i=1}^{n+1} A_i \log i = \frac{1}{(n-1)!}\sum_{i=1}^{n+1}(-1)^i \binom{n-1}{i-1}\log i.$$

From this we conclude:

$$[\Delta^n \log(x+1)]_{x=0} = (-1)^{n+1}\, n! \int_0^\infty \frac{dx}{(x+n+1)_{n+1}}$$

§ 121. Sum of Reciprocal Powers, Sum of $1/x^m$. *A. By aid of Stirling Numbers.*

In § 68 we found that the expansion of a reciprocal power into a series of reciprocal factorials is the following:

$$\frac{1}{x^m} = \sum_{i=m}^{\infty}(-1)^{m+i}\frac{S_i^m}{(x+i)_i} = \sum_{i=m-1}^{\infty}(-1)^{m+i-1}\frac{S_i^{m-1}}{(x+i)_{i+1}},$$

The operation Δ^{-1} gives

$$(1) \quad \Delta^{-1} \frac{1}{x^m} = \sum_{i=m}^{\infty} \frac{(-1)^{m+i+1} S_i^m}{(i-1)(x+i-1)_{i-1}} = \sum_{i=m-1}^{\infty} \frac{(-1)^{m+i} S_i^{m-1}}{i \, . \, (x+i-1)_i} + k.$$

From this we obtain the sum from $x=1$ to $x=\infty$

$$s_m = \sum_{x=1}^{\infty} \frac{1}{x^m} = \sum_{i=m}^{\infty} \frac{(-1)^{m+i} S_i^m}{(i-1)\, . \, i!} = \sum_{i=m-1}^{\infty} \frac{(-1)^{m+i+1} S_i^{m-1}}{i \, . \, i!} \, .$$

We may obtain a third formula for s_m, indeed, remarking that (p. 163)

$$\Delta_i^{-1} \frac{|S_i^m|}{i!} = \frac{|S_i^{m+1}|}{(i-1)!} + k;$$

from the first expression we obtain by summation by parts (p. 105)

$$\left| \frac{S_i^{m+1}}{(i-1)\,!} \cdot \frac{(-1)^{m+i}}{(i-1)} \right|_m^{\infty} - \sum_{i=m}^{\infty} \frac{S_{i+1}^{m+1}}{i!} \frac{(-1)^{m+i+1}}{(i)},$$

Since $S_m^{m+1} = 0$ and the quantity in the brackets is equal to zero for $i=\infty$ (formula 4, p. 160), s_m may be written:

$$s_m = \sum_{i=m+1}^{\infty} \frac{|S_i^{m+1}|}{(i-1), \ (i-1)!}$$

Instead of s, it is more advisable to introduce the quantity

$$s_m' = \sum_{x=2}^{\infty} \frac{1}{x^m} = s_m - 1$$

as the series $\Sigma \, s_m'$ is convergent; moreover there are several interesting formulae concerning s_m'.

From formula (1) we may deduce

$$(2) \quad s_m' = \sum_{k=m-1}^{\infty} \frac{|S_i^{m-1}|}{i \, . \, (i+1)!} = \sum_{i=m}^{\infty} \frac{2\,|S_i^m|}{(i-1)\,(i+1)!} .$$

Since from the first it follows that

$$s_{m+1}' = \sum_{i=m}^{\infty} \frac{|S_i^m|}{i \, . \, (i+1)!}$$

the second relation (2) gives

$$\tfrac{1}{2} s_m' > s_{m+1}'$$

and finally

$$(3) \qquad s_n' < \frac{s_2'}{2^{n-2}} < \frac{1}{3 \,.\, 2^{n-3}} .$$

This may he verified by aid of *Stieltjes'* table.[36]

From (3) it follows that $\lim_{n=\infty} s_n' = 0$ moreover that the series $\Sigma\, s_n'$ is convergent.

Since

$$\sum_{n=2}^{\infty} |S_i^{n-1}| = i!$$

from (2) we get

$$(4) \qquad \sum_{n=2}^{\infty} s_n' = \sum_{i=1}^{\infty} \frac{1}{i\,(i+1)} = -\frac{1}{i}\Big|_1^{\infty} = 1.$$

We have seen moreover that

$$\sum_{\nu=1}^{\infty} |S_i^{2\nu-1}| = \frac{1}{2}\, i! \quad \text{and} \quad \sum_{\nu=1}^{\infty} |S_i^{2\nu}| = \frac{1}{2}\, i! ;$$

hence putting $m=2\nu+1$ into the first equation (2) we obtain

$$(5) \qquad \sum_{\nu=1}^{\infty} s_{2\nu+1}' = \sum_{i=2}^{\infty} \sum_{\nu=1}^{\infty} \frac{S_i^{2\nu}}{i(i+1)!} = \sum_{i=2}^{\infty} \frac{1}{2i(i+1)} = \frac{1}{4}$$

On the other hand, when putting $m=2\nu$, we must also consider the case of $i=1$, so that we have

$$(6) \qquad \sum_{\nu=1}^{\infty} s_{2\nu}' = \sum_{i=2}^{\infty} \sum_{\nu=1}^{\infty} \frac{|S_i^{2\nu-1}|}{i(i+1)!} + \frac{|S_1^1|}{2} = \frac{3}{4}$$

Formulae (5) and (6) are given in *Stieltjes'* paper.

Multiplying both members of the first equation (2) by $(-1)^m \mathfrak{S}_{m-1}^k$ and summing from $m=2$ to $m=\infty$ we obtain, according to formula (5) § 65 (Inversion):

$$(7) \qquad \sum_{m=2}^{\infty} (-1)^m s_m' \,\mathfrak{S}_{m-1}^k = \frac{(-1)^{k+1}}{k(k+1)!}$$

This formula gives in the particular case $k=1$

$$\sum_{m=2}^{\infty} (-1)^m s_m' = \frac{1}{2} .$$

[36] The values of s_m were first calculated by Euler (Institutiones Calculi Differentialis, Acad. Petropolitanae, 1755, p. 456) up to m - 16 to 18 decimals. Later *T. J. Stieltjes'* (Acta Mathematica, Vol. 10, p. 299, 1887) computed these values up to m 70 to 33 decimals.

We have already obtained this result in § **67** (formula 6).

B. Determination of the sum of reciprocal powers by aid of the derivatives of $\log \Gamma(x+1)$.

In § **21** we expanded log $\Gamma(x+1)$ into a series of powers of x; but in consequence of what has been said at the beginning of this paragraph it is better to expand it into a series of powers of x-1.

$$[\log \Gamma(x+1)]_{x=1} = 0 \quad \text{and} \quad [D \log \Gamma(x+1)]_{x=1} = F(1) = 1 - C$$

moreover (1, § 21)

$$[D^m \log \Gamma(x+1)]_{x=1} = \sum_{\nu=2}^{\infty} \frac{(-1)^m (m-1)!}{\nu^m} = (-1)^m (m-1)!\, s_m';$$

therefore we have

$$(8) \qquad \log \Gamma(x+1) = (1-C)(x-1) + \sum_{m=2}^{\infty} \frac{(-1)^m}{m} s_m' (x-1)^m.$$

This gives in the particular case of x=2

$$C = 1 - \log 2 + \sum_{m=2}^{\infty} \frac{(-1)^m}{m} s_m'$$

and if $x = ½$

$$\log \Gamma\left(\frac{3}{2}\right) + ½(1-C) = \sum_{m=2} \frac{s_m'}{m\, 2^m}$$

or if $x = \frac{3}{2}$

$$\log \Gamma\left(\frac{5}{2}\right) - \frac{1}{2}(1-C) = \sum_{m=2}^{\infty} \frac{(-1)^m s_m'}{m\, 2^m}$$

Subtracting the second from the first we get

$$\log \frac{2}{3} + 1 - c = 2 \sum_{m=1}^{\infty} \frac{s_{2m+1}'}{(2m+1)\, 2^{2m+1}}.$$

Finally

$$C = 1 + \log 2 - \log 3 - \sum_{m=1}^{\infty} \frac{s_{2m+1}'}{(2m+1)\, 2^{2m}}.$$

This formula is very useful for the computation of C if the quantities s_n' are known, as *Stieltjes* has shown in his paper,[36] in which s_n is given up to s_{70}, at an exactitude of 33 decimals.

Moreover we have seen in § 82, dealing with the expansion of the *Bernoulli* polynomials into *Fourier* series, that

$$s_{2n} = (2\pi)^{2n} \frac{|B_{2n}|}{2(2n)!} .$$

Let us now introduce the following functions:

$$\text{(10)} \qquad \Psi_m(x) = \mathsf{D}^m \log \Gamma(x+1).$$

These belong to the class of functions mentioned in § 22, indeed we have

$$\text{(11)} \qquad \mathsf{D}\, \Psi_m(x) = \Psi_{m+1}(x).$$

Remark. $\Psi_1(x)$ is equal to the digamma-function and $\Psi_2(x)$ to the trigamma function.

The function $\Psi_m(x)$ may be **expressed by** aid of sums. In § 20 we found formula (1)

$$\mathsf{D}^2 \log \Gamma(x+1) = \sum_{\nu=1}^{\infty} \frac{1}{(x+\nu)^2}$$

and from this it follows according to (10)

$$\text{(12)} \qquad \Psi_m(x) = \sum_{\nu=1}^{\infty} \frac{(-1)^m (m-1)!}{(x+\nu)^m}$$

where $m \geqq 2$. Putting $x=0$ we get

$$\text{(13)} \qquad \Psi_m(0) = \sum_{\nu=1}^{\infty} \frac{(-1)^m (m-1)!}{\nu^m} = (-1)^m (m-1)!\, s_m .$$

Moreover from (12) it follows that $\Psi_m(\infty) = 0$, and therefore from (11) we conclude if $m > 2$,

$$\int_0^{\infty} \Psi_m(x)\,dx = -\Psi_{m-1}(0).$$

From (12) we may deduce

$$\sum_{x=0}^{z-1} \frac{1}{(x+1)^m} = \frac{(-1)^{m-1}}{(m-1)!} [\Psi_m(z) - \Psi_m(0)].$$

In § 68 we have seen formulae (3) and (4)

$$\frac{1}{x^m} = \sum_{i=m}^{\infty} \frac{|S_i^m|}{(x+i)_i} = \sum_{i=m-1}^{\infty} \frac{|S_i^{m-1}|}{(x+i)_{i+1}} .$$

(The second expression may be obtained from the first by summation by parts.) From this it follows, in consequence of (12), after having performed the summation from $\nu=1$ to $\nu=\infty$:

$$\Psi_m(x) = (-1)^m (m-1)! \sum_{i=m-1}^{\infty} \frac{|S_i^{m-1}|}{i(x+i)_i}$$

(14)

$$\Psi_m(x) = (-1)^m (m-1)! \sum_{i=m}^{\infty} \frac{|S_i^m|}{(i-1)(x+i)_{i-1}}$$

From this we conclude that

$$\lim_{x=\infty} (x)_{m-1} \Psi_m(x) = (-1)'' (m-2)! \quad \text{if} \quad m \geqq 2$$

(15)

$$\lim_{x=\infty} (x)_n \Psi_m(x) = 0 \quad \text{if} \quad m > n+1.$$

Since log $\Gamma(x+1)$ may be expressed by aid of a definite integral (Nielsen Gammafunktionen, p. 187)

$$\log \Gamma(x+1) = \int_0^1 \frac{[1-xt-(1-t)^x]dt}{t \quad \log(1-t)} .$$

its m -th derivative $\Psi_m(x)$ will be if $m > 1$

$$\Psi_m(x) = -\int_0^1 \frac{1}{t} (1-t)^x [\log(1-t)]^{m-1} dt$$

and

$$\Psi_m(0) = -\int_0^1 \frac{1}{t} [\log(1-t)]^{m-1} dt.$$

This may also be deduced by starting from formula (7) of § 71, putting there $z=-t$, dividing by t and integrating from $t=0$ to $t=1$. For $m=2$ *we* obtain the trigamma function expressed by a definite integral.

Derivatives of $\Psi_m(x)$. From (11) we get

$$D^n\Psi_m(x) = \Psi_{n+m}(x)$$

and therefore from (13)

$$D^n\Psi_m(0) = (-1)^{n+m} (m+n-1)! \, s_{m+n} .$$

Difference of $\Psi_m(x)$. Remarking that

$$\Delta \Psi_m(x) = D^{m-1}\Delta F(x)$$

it follows that

(16) $$\Delta \Psi_m(x) = D^{m-1} \frac{1}{x+1} = \frac{(-1)^{m-1} (m-1)!}{(x+1)^m}$$

and

$$\Delta\ \Psi_m(0) = (-1)^{m-1}\ (m-1)!\ .$$

This may be obtained directly from (12).

Inverse difference of $\Psi_m(x)$. Summation by parts gives, (p. 105) taking account of (16),

$$A\text{-'}\ \Psi_m(x) = x\ \Psi_m(x) + (m-1)\ \Psi_{m-1}(x) + k.$$

Therefore if $m>2$ in consequence of (15) we have

$$(17)\qquad \sum_{x=0}^{\infty} \Psi_m(x) = -\ (m-1)\ \Psi_{m-1}(0) = (-1)^m\ (m-1)!\ s_{m-1}\ .$$

This formula may also be obtained in the following way: Writing successively $\Psi_m(0)$, $\Psi_m(1)$, $\Psi_m(2)$, and so on, by aid of (12) it is easy to see that in the sum (17) the term $1/x^m$ occurs x times, so that

$$\sum_{x=0}^{\infty} \Psi_m(x) = (-1)^m\ (m-1)!\ \sum_{x=0}^{\infty} \frac{x}{x^m} = (-1)^m\ (m-1)!\ s_{m-1}\ .$$

Moreover from formula (14) we obtain $(m\geqq 2)$

$$\frac{\Psi_m(x)}{(x)_n} = (-1)^m\ (m-1)!\ \sum_{i=m-1}^{\infty} \frac{|S_i^{m-1}|}{i\ .\ (x+i)_{i+n}}$$

and

$$(x+n)_n\ \Psi_m(x) = (-1)^m\ (m-1)!\ \sum_{i=m-1}^{\infty} \frac{|S_i^{m-1}|}{i(x+i)_{i-n}}\ .$$

By summation of the inverse factorial we find

$$\sum_{x=n}^{\infty} \frac{\Psi_m(x)}{(x)_n} = (-1)^m\ (m-1)!\ \sum_{i=m-1}^{\infty} \frac{|S_i^{m-1}|}{i(i+n-1)\,(i+n-1)!}$$

(18)

$$\sum_{x=0}^{\infty} (x+n)_n\ \Psi_m(x) = (-1)^m\ (m-1)!\ \sum_{i=m-1}^{\infty} \frac{|S_i^{m-1}|}{(i-n-1)\,(i)_{i-n}}$$

Putting $m=2$ and $n=1$ into the first formula we obtain

$$\sum_{x=1}^{\infty} \frac{\Psi_2(x)}{x} = s_3\ .$$

Remark. Stieltjes' tables | loc. cit. 36 | give $s_3 = 1.2020569031$ and computing by aid of ***Pairman's*** tables to eight decimals we find

$$\sum_{x=1}^{21} \frac{F(x)}{x} = 1'1538\ 7192.$$

Sum of the alfernafe function $(-1)^x \Psi_m(x)$. This is obtained without difficulty. Indeed putting first into formula (12) $x=0$ and then $x=1$; subtracting the second result from the first we get $(-1)^m$ (m-l) $!/1^m$. Putting now $x=2$ and x=3, proceeding in the same manner as before we find $(-1)^m$ (m-l) $!/3^m$; and so on. Finally we shall have

$$\sum_{x=0}^{\infty} (-1)^x \Psi_m(x) = (-1)^m (m-1)! \sum_{i=0}^{\infty} \frac{1}{(2i+1)^m}.$$

In the same manner as in § 49 it can be shown that the sum in the second member is equal to $(1-2^{-m})\, s_m$, therefore

$$(20) \qquad \sum_{x=0}^{\infty} (-1)^x \Psi_m(x) = \left(1-\frac{1}{2^m}\right) \Psi_m(0).$$

Particular case. For $m=2$ *we* find

$$\sum_{x=0}^{\infty} (-1)^x \Psi_2(x) = \frac{3}{4} \Psi_2(0) = \frac{\pi^2}{8} = 1'2334\ 0055.$$

Remark. By aid of *Pairman's* tables of the trigamma function we get

$$\sum_{x=0}^{20} (-1)^x \Psi_2(x) = 1'2087\ \ 0132 \qquad \text{and}$$

$$\sum_{x=0}^{21} (-1)^x \Psi_2(x) = 1'2574\ \ 7214.$$

C. Sum of the reciprocal powers determined by the Maclaurin-Euler summation formula. Let us put into formula (5) § 88 $f(x)=1/(x+1)^m$ and $a=0$. Since we have

$$\int_0^z \frac{dx}{(x+1)^m} = \frac{1}{m-1} - \frac{1}{(m-1)\ (z+1)^{m-1}}$$

and

$$D^i \frac{1}{(x+1)^m} = \frac{(-1)^i\ (m+i-1)_i}{(x+1)^{m+i}} \text{ and } D^i f(\infty) = 0$$

moreover as $D^{2n}f(x) D^{2n+2}f(x) > 0$, and $D^{2n}f$ (x) does not change its sign in the interval $x=0$ to $x=z$, the remainder (4) of § 88 may be applied and omitting the term independent of z,

$$R_{2n} = \xi a_{2n} \mathbf{D}^{2n-1} f(z) = -\xi a_{2n} \frac{(m+2n-2)_{2n-1}}{(z+1)^{m+2n-1}}$$

so that the expansion will be

$$\sum_{x=0}^{z} \frac{1}{(x+1)^m} = \frac{1}{m-1} - \frac{1}{(m-1)(z+1)^{m-1}} +$$

$$+ \sum_{i=1}^{2n-1} a_i \frac{(-1)^{i-1}(m+i-2)_{i-1}}{(z+1)^{m+i-1}} - \xi a_{2n} \frac{(m+2n-2)_{2n-1}}{(z+1)^{m+2n-1}} + C_f .$$

Writing $z=\infty$ we obtain

$$C_f = \sum_{x=0}^{\infty} \frac{1}{(x+1)^m} - \frac{1}{m-1} = s_m - \frac{1}{m-1}$$

so that we have

$$(27) \qquad \sum_{x=0}^{z} \frac{1}{(x+1)^m} = s_m + \frac{1}{m-1} \left[\sum_{i=0}^{2n-1} (-1)^{i+1} \binom{m+i-2}{i} \cdot \right.$$

$$\left. \frac{B_i}{(z+1)^{m+i-1}} - \xi B_{2n} \binom{m+2n-2}{2n} \frac{1}{(z+1)^{m+2n-1}} \right] .$$

Though the series corresponding to $n=\infty$ is divergent, nevertheless formula (27) may be useful for the computation of the sum, if z is large enough.

The expansion of Y,,,(x) into a *Bernoulli* series leads to a similar result. In § *87* we have seen that the quantities to be determined for this are

$$\int_x^{x+1} \Psi_m(t)\, dt = \Delta \Psi_{m-1}(x) = \frac{(-1)^m (m-2)!}{(x+1)^{m-1}}$$

and

$$\mathbf{D}^{i-1} \Delta \Psi_m(x) = \frac{(-1)^{m+i} (m+i-2)!}{(x+1)^{m+i-1}} .$$

Since we have $\mathbf{D}^{2n}\Psi_m(x)\, \mathbf{D}^{2n+2}\Psi_m(x) > 0$ and $\mathbf{D}^{2n}\Psi_m(x)$ does not change its sign in the interval $(0,x)$ the remainder will be (7, § 87)

$$R_{2n} = \xi a_{2n} \Delta \mathbf{D}^{2n-1} \Psi_m(x)$$

where $0<\xi<1$.

Finally the expansion is:

$$\Psi_m(x) = \sum_{i=0}^{2n-1} \frac{(-1)^{m+i}(m+i-2)!}{(x+1)^{m+i-1}} a_i + \frac{\xi(-1)^m (m+2n-2)!}{(x+1)^{m+2n-1}} a_{2n} .$$

This formula may be useful if x is large enough.

§ 122. Sum of alternate reciprocal powers, by aid of the $\beta_1(x)$ function. Besides the digamma, the **trigamma** and the $\Psi_m(x)$ function, useful for the summation of reciprocal powers, it is advisable to introduce still another function denoted by β_1 (x) serving for the summation of alternate reciprocal powers. It has been considered first by *Stirling* (loc. cit. 25).

Its definition is the following

$$(1) \qquad \beta_1(x) = -2\,\mathsf{D}\log B\left(\frac{x+1}{2},\frac{1}{2}\right).$$

Hence β_1 (x) is the derivative of the logarithm of a particular Beta-function. It may also be expressed by gamma functions:

$$(1') \qquad \beta_1(x) = -2\,\mathsf{D}\log\frac{\Gamma\left(\frac{x+1}{2}\right)\Gamma(1/2)}{\Gamma(1/2x+1)} = -2\,\mathsf{D}\log\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma(1/2x+1)}.$$

Mean of the $\beta_1(x)$ function.

$$(2) \qquad \mathsf{M}\beta_1(x) = -\mathsf{D}\log\frac{\Gamma\left(\frac{x+1}{2}\right)\Gamma(1/2x+1)}{\Gamma(1/2x+1)\,\Gamma\left(\frac{x+3}{2}\right)} =$$

$$= -\mathsf{D}\log\frac{2}{x+1} = \frac{1}{x+1}.$$

Differences of $\beta_1(x)$. Since symbolically we have $\Delta = 2\mathsf{M}-2$ therefore

$$\Delta\beta_1(x) = \frac{2}{x+1} - 2\beta_1(x).$$

From this it is possible to deduce step by step $\Delta^m\beta_1(x)$ and we find

$$\Delta^m\beta_1(x) = (-1)^m 2^m \beta_1(x) + (-1)^{m-1}\sum_{i=0}^{m}\frac{2^{i+1}(m-i-1)!}{(x+m-i)_{m-i}}.$$

Inverse difference of $\beta_1(x)$. We have

$$\mathsf{M}\beta_1(x) = \frac{1}{x+1} = \Delta F(x)$$

and $\mathsf{M} = 1 + 1/2\Delta$, therefore the operation Δ^{-1} performed on the preceding equation gives

$$\Delta^{-1} \beta_1(x) = F(x) - \tfrac{1}{2}\beta_1(x) + k.$$

Finally the sum of $\beta_1(x)$ if x varies from $x=0$ to $x=z$ is

$$\sum_{x=0}^{z} \beta_1(x) = F(z) + C + \tfrac{1}{2}\beta_1(0) - \tfrac{1}{2}\beta_1(z)$$

where C is ***Euler's*** constant.

Expansion of β_1 (x) into a series of ***Boole polynomials.*** According to § 116 the coefficients of this expansion are given by

$$c_m = \mathrm{M}\Delta^m \beta_1(0);$$

therefore

$$c_0 = 1 \text{ and } c_m = \left[\frac{(-1)^m m!}{(x+m+1)_{m+1}}\right]_{x=0} = \frac{(-1)^m}{m+1}$$

Finally

$$(3) \qquad \beta_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} \zeta_m(x).$$

Putting into this formula $x=0$ we obtain (§ 113)

$$\beta_1(0) = \sum_{m=0}^{\infty} \frac{1}{(m+1)\, 2^m}\,.$$

It is easy to see that this is equal to the expansion of -2 log (l--t) into a power series, where $t=\tfrac{1}{2}$; hence

$$\beta_1(0) = 2 \log 2.$$

Expansion of β_1 (x) ***info a Boole series.*** According to § 116 we have

$$(4) \quad \beta_1(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{2^i} \mathrm{M}\Delta^i \beta_1(x) = \sum_{i=0}^{\infty} \frac{i!}{2^i (x+1+i)_{i+1}}$$

From this it follows that $\beta_1(\infty) = 0$; moreover, putting into it $x=0$ we obtain the result above,

Example 1. Computation of β_1 (200) to 12 decimals.

$1/201$	=	0.004975 124378
$1/2(202)_2$	=	12 314664
$1/2(203)_3$	=	60663
$3/4(205)_4$	=	446
$6/4(205)_5$	=	4
		0.004987 500155

Expansion of β_1 (x) into a series of *Euler polynomials.* In § 109 we have seen that the coefficient c_m in this expansion is given by

$$c_m = \mathsf{MD}^m \beta_1(0);$$

therefore

$$c_0 = 1 \text{ and } c_m = (-1)^m m!$$

so that

$$\beta_1(x) = \sum_{m=0}^{\infty} (-1)^m m! \, E_m(x).$$

This series is divergent.

Expansion of $\beta_1(x)$ into an Euler series. According to § 111, since $\mathsf{D}^{2n+1}\beta_1(x)\,\mathsf{D}^{2n+3}\beta_1(x) > 0$ the remainder of the series may be given by (4) § 111; moreover,

$$\mathsf{MD}^i \beta_1(x) = \mathsf{D}^i \frac{1}{x+1} = \frac{(-1)^i i!}{(x+1)^{i+1}}$$

So that the required formula will be

$$(5) \qquad \beta_1(x) = \frac{1}{x+1} - \sum_{\nu=0}^{n} \frac{(2\nu+1)!}{(x+1)^{2\nu+2}} e_{2\nu+1} - \vartheta \frac{(2n+1)!}{(x+1)^{2n+2}} e_{2n+1}.$$

This is the expansion of β_1 (x) into a reciprocal power series, For $n = \infty$ the series is divergent but nevertheless formula (5) is useful for the computation of β_1 (x) especially if x is large. From (4), § 106 it follows that the remainder will decrease until we have approximately $n \sim \frac{3}{2}$ x.

Example 2. Computation of β_1 (200). In § 100 we found $e_1 = -\frac{1}{2}$, $e_3 = 1/24$ and $e_5 = -1/240$.

Hence putting $n=2$ the remainder of (5) will be less than

$$1/2(201)^6 < 1/10^{13}.$$

Therefore we have

$1/201$	=	0.004975 124378
$1/2(201)^2$	=	12 375931
$-1/4(201)^4$	= −	1 5 3
β_1 (200)	=	0.004987 500156

The result is exact up to twelve decimals, *Expression of β_1 (x) by digamma functions.* **From** formula (1') it follows immediately that

$$\beta_1(x) = F(\tfrac{1}{2}x) - F\left(\frac{x-1}{2}\right) \tag{6}$$

Expression of $\beta_1(x)$ by sums. The digamma function has been given (7, § 19) also by formula

$$F(x) = -C + \sum_{m=1}^{\infty}\left[\frac{1}{m} - \frac{1}{x+m}\right].$$

Let us put into this first $\tfrac{1}{2}x$ instead of x and secondly $\tfrac{1}{2}(x-1)$ instead of x; then subtracting the second equation so obtained from the first we get

$$F(\tfrac{1}{2}x) - F\left(\frac{x-1}{2}\right) = 2\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{x+i}$$

and in consequence of (6) we have

$$\beta_1(x) = 2\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{x+\nu+1}. \tag{7}$$

This is an expression for β_1 (x) by aid of an alternate sum. Putting $x=0$ into (6) we find that

$$F(-\tfrac{1}{2}) = -c - 2\log 2.$$

To obtain the sum of the alternate function from zero to z, we may start from

$$\beta_1(x) = M^{-1}\frac{1}{x+1}$$

and apply formula (10) of § 39, which gives

$$\beta_1(x) = 2(-1)^{x+1}\Delta^{-1}\frac{(-1)^x}{x+1} + (-1)^x k.$$

Therefore

$$\sum_{x=0}^{z}\frac{(-1)^x}{x+1} = \tfrac{1}{2}[\beta_1(0) - (-1)^z\beta_1(z)]. \tag{8}$$

On the other hand we may express β_1 (z) by aid of the alternate sum

$$\beta_1(z) = (-1)^z\left[2\log 2 - \sum_{x=0}^{z}\frac{(-1)^x}{x+1}\right.$$

Example 3. Computation of

$$\sum_{u=0}^{200} \frac{(-1)^u}{u+1} = \log 2 - \tfrac{1}{2}\beta_1(200)$$

From Example 1, we have

$$\begin{array}{lrr} -\tfrac{1}{2}\beta_1(200) & = -0.002493 & 750078 \\ \log 2 & = 0.693147 & 180560 \\ \hline & 0{,}690653 & 430482 \end{array}$$

The results is exact to twelve decimals.

Expression of $\beta_1(x)$ by a definite integral. We found in formula (3) § 118 the following expression for the digamma function:

$$f(x) = -C + \int_0^1 \frac{1-t^x}{1-t}\, dt.$$

By aid of formula (6) this gives

$$\beta_1(x) = \int_0^1 \frac{t^{x/2}(1-t^{1/2})}{(1-t)\, t^{1/2}}\, dt$$

and introducing the new variable $u^2=t$ we obtain

(9) $$\beta_1(x) = 2\int_0^1 \frac{u^x}{1+u}\, du.$$

Expanding $1/(1+u)$ into a power series and integrating, we find:

$$\beta_1(x) = 2\sum_{m=0}^{\infty} \int_0^1 (-1)^m u^{x+m}\, du = 2\sum_{m=0}^{\infty} \frac{(-1)^m}{x+m+1}.$$

This is identical with formula (7). It is easy to show that the mean of (9) is equal to $1/x+1$.

Formula for negative values of the argument of $\beta_1(x)$. Putting into (6) first $-x$ instead of x and then x-l instead of x, we obtain the two equations

$$\beta_1(-x) = F(-\tfrac{1}{2}x) - F\left(-\frac{x+1}{2}\right)$$

$$\beta_1(x-1) = F\left(\frac{x-1}{2}\right) - F(\tfrac{1}{2}x-1).$$

Now using the equation deduced in § 19

$$F(x-1) - F(-x) = -\pi \cot \pi x,$$

by the aid of the preceding equations we may eliminate the digamma functions with negative arguments. We find from the first

$$\beta_1(-x) = \pi \cot \tfrac{1}{2}\pi x - \pi \cot \left(\frac{x+1}{2}\right)\pi + F(\tfrac{1}{2}x-1) - F\left(\frac{x-1}{2}\right)$$

and from the second

$$\beta_1(-x) + \beta_1(x-1) = \pi\,[\cot \tfrac{1}{2}\pi x + \tan \tfrac{1}{2}\pi x].$$

Therefore

(10)
$$\beta_1(-x) + \beta_1(x-1) = \frac{2\pi}{\sin \pi x}$$

Particular case. Putting $x = \tfrac{1}{2}$ we find

$$\beta_1(-\tfrac{1}{2}) = \pi .$$

Derivatives of $\beta_1(x)$. $D^m\beta_1(x)$ may be expanded into an *Euler* series in the way used above for $\beta_1(x)$; we find

(11)
$$D^m\beta_1(x) = (-1)^m \left| \frac{m!}{(x+1)^{m+1}} - \sum_{\nu=0}^{n} \frac{(m+2\nu+1)!}{(x+1)^{m+2\nu+2}}\, e_{2\nu+1} + \right.$$
$$\left. - \eta \frac{(m+2n+1)!}{(x+1)^{m+2n+2}}\, e_{2n+1} \right|$$

where $0 \leqq \eta \leqq 1$. The series corresponding to $n=\infty$ is divergent, nevertheless it is useful if x is large. According to formula (4) § 106 the best value of n is approximately

(12)
$$n \sim \tfrac{1}{2}\,(x-c-m) .$$

We may obtain the derivatives of $\beta_1(x)$ expressed by an alternate sum; starting from formula (7) we find

(13)
$$D^m \beta_1(x) = 2\, m! \sum_{\nu=0}^{\infty} \frac{(-1)^{m+\nu}}{(x+\nu+1)^{m+1}}$$

Putting $x=0$ we get

(14)
$$D^m\beta_1(0) = 2\, m! \sum_{\nu=0}^{\infty} \frac{(-1)^{m-\nu}}{(\nu+1)^{m+1}}$$

and if we introduce the notation

$$\sigma_m = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(\nu+1)^{m}}$$

then we find

(15) $$\mathrm{D}^m\beta_1(0) = 2\,m!\,(-1)^m\,\sigma_{m+1}\,.$$

Therefore the expansion of β_1 (x) into a ***power series*** will be

(16) $$\beta_1(x) = 2\sum_{m=0}^{\infty}(-1)^m\,\sigma_{m+1}\,x^m.$$

We may obtain the sum of the alternate function in **(13)**, from zero to x in the following way: Putting $x=0$ into (13) we obtain $\mathrm{D}^m\beta_1$ (0) given by (14) ; now writing in (13) v instead of $x+\nu$ we have

$$(-1)^x\,\mathrm{D}^m\beta_1(x) = 2\,(-1)^m\,m!\sum_{\nu=x}^{\infty}\frac{(-1)^\nu}{(\nu+1)^{m+1}}.$$

Finally

(17) $$\sum_{\nu=0}^{x}\frac{(-1)^\nu}{(\nu+1)^{m+1}} = \frac{(-1)^m}{2\,m!}\left[\mathrm{D}^m\beta_1(0)-(-1)^x\mathrm{D}^m\beta_1(x)\right].$$

In § 49 we have seen that the alternate sum a, may be expressed by the ordinary sum s_m as follows:

(18) $$\text{a,} = s_m\left(1-\frac{1}{2^{m-1}}\right).$$

Since the numbers s_m are given to 33 decimals up to $m=70$ in ***Stieltjes'*** tables [loc. cit. **36**] therefore by aid of (15) we may calculate $\mathrm{D}^m\beta_1$ (0). Moreover $\mathrm{D}^m\beta_1(x)$ may be easily computed by the aid of formula **(11)**, if x is large enough, so that the alternate sum may be obtained by (17).

Example 4. The sum of $(-1)^x/(x+1)^3$ is required to 12 decimals. We have

$$\sum_{x=0}^{100}\frac{(-1)^x}{(x+1)^3} = \frac{1}{4}\left[\mathrm{D}^2\beta_1(0)-\mathrm{D}^2\beta_1(100)\right].$$

According to (18) we have $\sigma_3 = \frac{3}{4}\,s_3$; and in consequence of **(15)**, $\mathrm{D}^2\beta_1(0) = 3s_3$. The tables mentioned give

$$\mathrm{D}^2\beta_1(0) = 3.606170\;709479.$$

$\mathrm{D}^2\beta_1(100)$ is then computed by aid of (11). Stopping at $n=2$, the remainder will be less than

$$\frac{7!\, e^5}{(101)^8} = \frac{21}{(101)^8} < 10^{-14}.$$

Therefore

$2/(101)^3$	$=$	0,000001 941180
$-6e_1/(101)^4$	$=$	28829
$-5!e^3/(101)^6$	$= -$	8
		0,000001 970001

This value subtracted from $\mathsf{D}^2\beta_1$ (0), and the difference divided by four gives the required result

0,901542 184870.

Determination of the alternate sum by Boole's first summation formula (§ 112). From formula (11) it follows that $\mathsf{D}^m\beta_1(\infty) = 0$. Therefore we may apply formula (3) § 112. Putting $f(x) = 1/(x+1)^m$ we have $C_f = \sigma_m$ and

$$\mathsf{D}^{2i+1} f(x) = -\frac{(m+2i)!}{(m-1)!\,(x+1)^{m+2i+1}}$$

so that we have

(19) $$\sum_{x=0}^{z} \frac{(-1)^x}{(x+1)^m} = \tfrac{1}{2}(-1)^{z+1}\left[\frac{1}{(z+1)^m} - \right.$$
$$\left. - \sum_{i=0}^{n} \frac{(m+2i)!\, e_{2i+1}}{(m-1)!\,(z+1)^{m+2i+1}} - \vartheta\, e_{2n+1} \frac{(m+2n)!}{(m-1)!\,(z+1)^{m+2n+1}}\right] + \sigma_m .$$

This is identical with formula (17).

CHAPTER VII.

EXPANSION OF **FUNCTIONS,** INTERPOLATION. CONSTRUCTION OF TABLES.

§ 123. Expansion of a function into a **series of polynomials.** Given the infinite series of polynomials, P_0 , *P,(x), P,(x), , , . ,* $P_n(x), \ldots,$ where we denoted by *P,(x)* a polynomial of degree n.

If certain conditions are satisfied, the function $f(x)$ may he expanded into a series of the polynomials *P,(x)*

$$f(x) = c_0P_0 + c_1P_1(x) + c_2P_2(x) + \ldots + c_nP_n(x) + \ldots$$

But if we stop the expansion at the term P,(x),

$$f(x) = c_0P_0 + c_1P_1(x) + c_2P_2(x) + \ldots + c_nP_n(x) \tag{1}$$

and determine the constants c_i so that the equation (1) shall he satisfied for $x = x_0, x_1, \ldots, x_n$, then for any other value of x the second member will only give an approximation to *f(x).*

Let us suppose that c_n is different from zero, and add to the second member of (1) a term denoted by R_{n+1} and called the remainder of the series. It must be equal to zero for $x = x_0, x_1,$ $\ldots,$ x_n so that it does not change equation **(1)** for these values; therefore we shall write

$$R_{n+1} = (x - x_0)\ (x\text{-}x,)\ \ldots\ (x\text{-}x,)\ \Omega.$$

Now we shall dispose of the function Ω so that

$$f(x) = c_0P_0 + c_1P_1(x) + , , , . + c_nP_n(x) + R_{n+1}$$

shall be exact for a given value $x = z$; hence Ω will be a function of z only. To obtain it we shall write

$$y(x) = f(x) - c_0P_0 - c_1P_1(x) - \ldots - c_nP_n(x) - (x\text{-}x,)\ldots(x\text{-}x,,)!?.$$

Let us suppose now that *f(x)* is a continuous function whose $n+1$ -th derivative has everywhere a determinate value. In consequence of what has been said before, y(x) i's equal to zero for the $n+2$ values

$$x = x_0, x_1, x_2, \ldots , x_n, z .$$

From this we conclude that the $n+1$ -th derivative of $\varphi(x)$ must be equal to zero, at least once in the smallest interval containing these numbers. If this occurs for $x=\xi$ we shall have

$$D^{n+1}\varphi(\xi) = D^{n+1}f(\xi) - (n+1)!\ \Omega = 0;$$

therefore the remainder will be, if $x=z$

$$(2) \qquad R_{n+1} = \frac{(z-x_0)\,(z-x_1)\ldots(z-x_n)}{(n+1)!}\,D^{n+1}f(\xi).$$

Finally writing x instead of z the required expansion will be

$$(3) \qquad f(x) = c_0P_0 + c_1P_1(x) + c_2P_2(x) + \ldots + c_nP_n(x) + \\ + \frac{(x-x_0)\,(x-x_1)\ldots(x-x_n)}{(n+1)!}\,D^{n+1}f(\xi).$$

This is exact for any value whatever of x; ξ being a function of x, whose value is included in the smallest interval containing the numbers

$$x_0, x_1, x_2, \ldots, x_n, x.$$

We conclude that the remainder R_{n+1} depends only on these numbers, and on the derivative of *f(x)* and not on the polynomials *P,,,(x)* chosen.

If a second series of polynomials Q_0 , Q, , . . . Q_n is given and if we expand the function *f(x)* into a series of $Q_m(x)$ polynomials, stoping the expansion at the term $Q_n(x)$ we obtain

$$(4) \qquad f(x) = k_0Q_0 + k_1Q_1(x) + \ldots + k_nQ_n(x)$$

and *if this equation is satisfied for the same values* $x=x_0$, x_1 , ... x_n *as expansion* (1), *then the two expansions are identical and so also will be the remainders of the two series.* and therefore the obtained precision too.

But if the coefficients c_i of formula (3) are tabulated to ν decimals only, so that their error is less than $\varepsilon=5/10^{\nu+1}$ then this will be another cause of error; therefore the absolute value of the error of f(x) will be

$$(5) \qquad |\delta f(x)| < |R_{n+1}| + \varepsilon \sum_{i=0}^{n+1} |P_i(x)|.$$

From this it follows that, finally, the precision will nevertheless depend on the **nature** of the expansion.

Remark 1. If $f(x)$ is a polynomial whose degree does not exceed n, then the remainder is equal to zero and the second member of the expansion (1) will be equal to $f(x)$ for every value of x.

Remark 2. Should c_n be equal to zero, R_{n+1} given by (2) could nevertheless be considered as the remainder of the series (1), but then it would be possible to chose a remainder R_n more simple, in which the range of ξ is smaller.

§ 124. The Newton **series.** We have seen when treating the symbolical methods that the operation $\mathbf{E}$ is equivalent to $1+\Delta$; therefore we have

$$\mathbf{E}^x = (1+\Delta)^x = \sum_{\nu=0}^{\infty} \binom{x}{\nu} \Delta^\nu .$$

Since $\mathbf{E}^x f(a) = f(a+x)$ we may write the above equation in the following manner:

$$(1) \qquad f(a+x) = f(a) + \binom{x}{1} \Delta f(a) + \binom{x}{2} \Delta^2 f(a) + \dots +$$

$$+ \binom{x}{n} \Delta^n f(a) + \dots$$

This is *Newton's* series. It can he transformed in different ways. For instance, putting $x = z - a$ we have

$$(2) \qquad \mathrm{f(z)} = \sum_{m=0}^{\infty} \binom{z-a}{m} \Delta^m f(a)$$

which gives the expansion of $f(z)$ into a series of binomial coefficients.

Expansion into a ***generalised*** *Newton series.* Putting into (1) $x = (z-a)/h$ we find

$$f\left(a + \frac{z-a}{h}\right) = \sum_{m=0}^{\infty} \binom{\frac{z-a}{h}}{m} \Delta^m f(a) = \sum_{m=0}^{\infty} \frac{1}{h^m} \binom{z-a}{m}_h \Delta^m f(a).$$

Indeed we have

$$\binom{\frac{z-a}{h}}{m} = \frac{(z-a)\ \ (z-a-h)\ \ .\ \ .\ \ ,\ \ (z-a-mh+h)}{h^m\, m!} = \frac{1}{h^m}\binom{z-a}{m}_h.$$

The last term is the generalised binomial coefficient of § 22. (p. 70). If we write moreover

$$f\left(a+\frac{z-a}{h}\right)=F(z)$$

then from the above it follows that

$$f(a) = F(a), \quad \mathrm{E}^m f(a) = \underset{h}{\mathrm{E}}^m F(a), \quad \Delta^m f(a) = \underset{h}{\Delta}^m F(a).$$

Finally the expansion of $F(z)$ into a series of generalised binomial coefficients will be:

$$(3) \quad F(s) = F(a) + \binom{z-a}{1}_h \frac{\underset{h}{\Delta} F(a)}{h} + \ldots + \binom{z-a}{m}_h \frac{\underset{h}{\Delta}^m F(a)}{h^m} + \ldots$$

Expansion by aid of Newton's backward formula. This formula has been deduced by symbolical methods in § 6. We had for a polynomial of degree n

$$f(x) = \sum_{m=0}^{n+1} \binom{x+m-1}{m} \Delta^m f(-m)$$

putting again $x=(z-a)/h,$ as above, we get

$$(4) \qquad \mathrm{F}(z) = \sum_{m=0}^{n+1} \binom{z-a+mh-h}{m}_h \frac{\underset{h}{\Delta}^m F\ (a-mh)}{h^m}.$$

Expansion of a function into a Newton series. If the function $f(x)$ is a polynomial of degree $n,$ then we shall have

$$\Delta^{n+m} f(x) = 0 \text{ if } m > 0.$$

Therefore the series will be finite:

$$(5) \qquad f(x) = f(0) + \binom{x}{1}\Delta f(0) + \ldots + \binom{x}{n}\Delta^n f(0).$$

In the case considered $f(x)$ is a polynomial whose degree does not exceed $n;$ therefore according to what we have seen in the preceding paragraph equation (5) is an identity true for any value whatever of x.

Problem. Given $y_0, y_1, \ldots, y_n$ corresponding to $x=0$, 1, 2, ..., n; a polynomial f(x) of degree n is to be determined so that for the given values of x we shall have $f(x) = y_x$.

The solution may be obtained from formula (5), which gives for $x=0, 1, 2, \ldots, n$ the necessary n+l equations for the determination of the n+l unknowns $f(0)$, $\Delta f(0), \ldots, \Delta^n f(0)$. But it is easier to proceed as follows: From $\Delta = E-1$ we deduce

$$\Delta^m = (E-1)^m = \sum_{i=0}^{m+1} (-1)^i \binom{m}{i} E^{m-i} ;$$

this gives

$$\Delta^m f(0) = \sum_{i=0}^{m+1} (-1)^i \binom{m}{i} f(m-i);$$

putting into this $f(m-i) = y_{m-i}$ the problem is solved.

Remark. The problem above is identical with that of determining a curve of degree n passing through given $n+1$ points of coordinates x and y_x.

If f(x) is not a polynomial, the series will be infinite. Stopping at the term $\Delta^n f(0)$ the second member of (5) will give the exact value of $f(x)$ if x is an integer such that

$$0 \leqq x \leqq n. \tag{6}$$

Indeed, for these values of x the terms $\binom{x}{m}\Delta^m f(0)$ will vanish if $m > n$, so that equation (5) will give the same value as the infinite series would give.

If x is an integer satisfying (6), then equation (5) may be written

$$(1+\Delta)^x f(0) = E^x f(0) = f(x).$$

That is, the second member of (5) gives the exact value of $f(x)$ for $x=0, 1, 2, \ldots, n$; but for other values of x it will be only an approximation of $f(x)$. To remedy this inconvenience we may add to the second member the remainder R_{n+1} of (2), § 123. This can be done since, as we have seen above, the conditions enumerated in this paragraph are satisfied.

Finally we shall have

$$(7) \quad f(x) = f(0) + \binom{x}{1} \Delta f(0) + \dots + \binom{x}{n} \Delta^n f(0) + \\ + \binom{x}{n+1} D^{n+1} f(\xi)$$

where ξ is included in the smallest interval containing $0, 1, 2, \dots, n, x$.

Remark. If *f(x)* is a polynomial of higher degree than n and we stop the expansion at the term Δ^n then a remainder must be added as in the general case before.

Putting into this formula $n=0$, we obtain one of *Lagrange's* formulae

$$\frac{f(x) \text{ --r(O)}}{x} = D f(\xi)$$

according to which in the interval 0,x there is at least one point of the curve $y=f(x)$ at which the tangent is parallel to the chord passing through the points of coordinates $0, f$ *(0)* and x, f *(x).*

Putting into formula (7) $x=(z-a)/h$ and $f\left(\frac{z-a}{h}\right) = F(z)$ we obtain the expansion of the function $F(z)$ into a series of generalised binomial coefficients, with its remainder

$$(8) \quad F(z) = F(a) + \binom{z-a}{1}_h \frac{\underset{h}{\Delta} F(a)}{h} + \dots + \binom{z-a}{n}_h \frac{\underset{h}{\Delta^n} F(a)}{h^n} + \\ + \binom{z-a}{n+1}_h D^{n+1} F(a+\xi h)$$

where ξ is included in the smallest interval containing $0, n, \frac{\%\text{--}a}{h}$.

In this formula, by $D^{n+1}F$ we understand $\frac{d^{n+1}F}{dz^{n+1}}$. $\quad \frac{dz}{dx} = h$; therefore in the last term of (8) the number h^{n+1} will vanish from the denominator.

§ 125. Interpolation by aid of Newton's Formula and Construction of Tables. The problem of interpolation consists in the following: Given the values of y_i corresponding to z_i for $i=0, 1, 2, \dots n$; a function *f(z)* of the continuous variable z is to be determined which satisfies the equation

$$(1) \qquad y_i = f(z_i) \text{ for } i = 0, 1, 2, \ldots n$$

and finally *f(z)* corresponding to $z = z'$ is required. The solution of this problem is not univocal; indeed, if we write

$$F(z) = f(2) + (z\text{-}z,,)\ (Z\text{-}2,)\ ,\ .\ .\ .\ (z\text{---}z,)\ \Theta(z)$$

where $\Theta(z)$ is an arbitrary function; then from (1) it will follow that

$$(2) \qquad F(z_i) = y_i$$

for the given values of i.

But if the $n+1$ points of coordinates y_i, z_i are given and a parabola $y=f$ (z) of degree n passing through these points is required, then there is but one solution.

This problem has been solved for equidistant values of z in § 124. If we put $z=a+xh$ and z is given for $x=0, 1, 2, \ldots, n$, then equation (8) of § 124 will become:

$$(3) \qquad F(t) = F(a+xh) = F(a) + x\Delta F(a) + \ldots + \binom{x}{n} \Delta^n F(a) +$$

$$+ \binom{x}{n+1} h^{n+1} \mathrm{D}^{n+1} F(a+\xi h).$$

A second case of interpolation is the following: A function $F(z)$ of the continuous variable z is given in a table for $z=z_0+ih$ where $i=0, 1, 2, \ldots,$ N. A polynomial *f(z)* of degree n is to be determined so as to have

$$(4) \qquad F(z) = f(z) \quad \text{for } z=a, a+h, \ldots, a+nh.$$

Finally the value of *f(z)* is required for $z=z'$ different from $a+ih$. First we must choose the quantity a. This choice is arbitrary, but it *is* considered best to choose *a so* as to have on each side of z' the same number *m* of values for which (4) is satisfied. To obtain this, it is necessary that the number n be odd, that is $n=2m—1$, then a must be chosen so that

$$a+mh—h < z' < a+mh.$$

But sometimes it is not possible to choose a in such a way. This happens for instance at the beginning or at the end of the table, when we are obliged to take respectively the first, and the last *2m* numbers, In some cases it may happen that we have to determine $f(z')$ corresponding to a value of z' outside the range of a table (extrapolation). This is considered as unfavourable. For instance, given the table

$$
\begin{array}{lllll}
f(0) & & & \\
 & \Delta f(0) & & \\
f(1) & & \Delta^2 f(0) & \\
 & \Delta f(1) & & \Delta^3 f(0) \\
f(2) & & \Delta^2 (1) & \vdots \\
 & \Delta f(2) & & \vdots \\
f(3) & \cdot & \cdot & \\
 & & \cdot & \Delta^3 f(n-3) \\
 & \cdot & \Delta^2 f(n-2) & \\
 & \Delta f(n-1) & & \\
f(n) & & &
\end{array}
$$

and $f(n+1)$ is to be calculated. It is best to use ***Newton's backward formula*** (§ 124) which would give

$$f(n+1) = f(n) + \Delta f(n-1) + \Delta^2 f(n-2) + \Delta^3 f(n-3).$$

General case of parabolic interpolation of odd degree by aid of Newton's formula. Let us write $2m-1$ instead of n in formula (3), then the polynomial obtained will give exactly F(z) for $z=a, a+h, \ldots, a+2mh-h$. As has been said, it is best to choose a so as to have

$$a+mh-h < z < a+mh \qquad \text{or} \qquad m-1 < x < m$$

since then the polynomial will give on each side of z the same number m of exact values.

To determine the maximum of the binomial coefficient $\binom{x}{2m}$ figuring in the remainder of (3), if $m-1<x<m$, we shall put $x=\vartheta+m-\tfrac{1}{2}$ so that $-\tfrac{1}{2}<\vartheta<\tfrac{1}{2}$. We have

$$\binom{\vartheta+m-\tfrac{1}{2}}{2m} = \frac{(-1)^m}{(2m)!}\left[\frac{1}{4}-\vartheta^2\right]\left[\frac{9}{4}-\vartheta^2\right]\cdots\left[(m-\tfrac{1}{2})^2-\vartheta^2\right];$$

moreover $\vartheta^2 \leqq 1/4$, hence the maximum of the absolute value of this expression will be obtained if $\vartheta=0$, or $x=m-\tfrac{1}{2}$. So that

$$\left|\binom{x}{2m}\right| \leqq \left|\binom{m-\tfrac{1}{2}}{2m}\right| \qquad \text{if} \qquad m-1 < x < m.$$

For instance

$$\left|\binom{x}{2}\right| \leq \frac{1}{8} \qquad \binom{x}{4} \leq \frac{3}{128} \qquad \left|\binom{x}{6}\right| \leq \frac{5}{1024}$$

and so on.

Finally we shall have if $m-1 < x < m$

$$(5) \qquad |R_{2m}| < h^{2m} \left| \binom{m-\frac{1}{2}}{2m} D^{2m} F(a+\xi h) \right|$$

where ξ is included in the **smallest interval** containing $0, 2m-1, x$.

Construction of *Tables.* A table of a function $F(z)$ should always be computed from the point of view of the interpolation formula to be used for the determination of the values of $F(z)$. Let us suppose that an interpolation of degree n is chosen, the error of F(z), denoted by $\delta F(z)$, will arise from two sources:

First, by putting approximate values of $F(a)$ and $\Delta^i_h F(a)$ into formula (3) instead the exact ones; secondly, by neglecting the remainder of the formula.

If in the table $F(a)$ and $\Delta^i F(a)$ are given exactly up to ν decimals, then the absolute value of their error will not exceed

$$\varepsilon = \frac{6}{10^{\nu+1}} .$$

Hence in the case of *Newton's* formula the corresponding part of the absolute value of the error of $F(a+xh)$ will be less than

$$\left[1 + |x| + \left|\binom{x}{2}\right| + \ldots + \left|\binom{x}{n}\right|\right] \varepsilon = \omega\varepsilon .$$

Since the errors of the function and of its differences are not all of the same sign, the resulting error will generally be much less.

We have

$$|\delta F(a+xh)| < \omega\varepsilon + |R_{n+1}| = \frac{5\omega}{10^{\nu+1}} + |R_{n+1}| .$$

The two parts of the error should be of the same order of magnitude. If the remainder is much greater than $\omega\varepsilon$, then there are superfluous decimals in the table, since with fewer decimals the same precision of $F(z)$ would be obtained by formula (3). If the remainder is much smaller than $\omega\varepsilon$, there are not enough decimals in the table; indeed, increasing them, the same formula would lead to a greater precision.

Therefore we conclude that if the degree of the interpolation formula, the range and the interval of the table, are chosen (the remainder depends on these quantities), they will determine the most favourable number of decimals for the table.

Linear interpolation. Putting into formula (3) $n=1$ or $m=1$ we get

$$(6) \qquad F(z) = F(a+xh) = F(a) + x\Delta F(a) + R_2$$

where according to (5)

$$(7) \qquad |R_2| < \frac{h^2}{8} |D^2 F(a+\xi h)|.$$

Since we chose a so that $a<z<a+h$ or $0<x<1$, we shall have $0<\xi<1$.

If the table contains the numbers $F(a+ih)$ and their differences, the error being the same for each item, for instance less than ε; then, according to what we have seen,

$$|\delta F(a+xh)| < (1+x)\varepsilon + |R_2| < 2\varepsilon + |R_2|.$$

But if the differences are computed by aid of the numbers $F(a+ih)$ of the table, then their error may be much greater.

Parabolic interpolation of the third degree. From formula (3) it follows that

$$(8) \qquad F(z) = F(a+xh) = F(a) + x\Delta F(a) + \binom{x}{2} \Delta^2 F(a) + \\ + \binom{x}{3} \cdot \Delta^3 F(a) + . \, h^4 \binom{x}{4} D^4 F(a+\xi h).$$

Since this curve passes through the points

$$a, F(a); \quad (a+h), \, F(a+h); \, (a+2h), F(a+2h); \, (a+3h), \, F(a+3h)$$

and a must be chosen so as to have two points on each side of I. hence we must have $a+h < z < a+2h$ or $1 < x < 2$, and $0 < \xi < 3$. According to (5) the remainder will be

$$|R_4| < \frac{3h^4}{128} |D^4 F(a+\xi h)|.$$

If the numbers $F(a+ih)$ and their differences are given by the table, to ν decimals then we have

$$|\delta F(a+xh)| < \frac{5}{10^{\nu+1}}\left[1+x+\frac{1}{6}x(x-1)(5-x)\right] + |R_4| \leq \frac{2}{10^{\nu}} + + |R_4|.$$

Parabolic interpolation of the fifth degree. From formula (3) it follows that

$$(8) \qquad F(z) = F(a+xh) = F(a) + x\Delta F(a) + \binom{x}{2}\Delta^2 F(a) + \binom{x}{3}\Delta^3 F(a) + \binom{x}{4}\Delta^4 F(a) + \binom{x}{5}\Delta^5 F(a) + h^6\binom{x}{6}D^6 F(a+\xi h).$$

This formula is of practical use only in tables where the first five differences are given, but there are hardly any such tables. Using this formula it is best, as has been said, to choose a in such a way that $a+2h < z < a+3h$; then as the curve passes through the points corresponding to $z=a$, $a+h, a+2h, a+3h, a+4h, a+5h$, there will be three points on each side of z, and we shall have $2 < x < 3$.

According to (5) the remainder will be

$$|R_6| < \frac{5h^6}{1024}|D^6 F(a+\xi h)|$$

where $0 < \xi < 5$.

It can be shown that interpolating by aid of formula (8) the absolute value of the error of F(z) is

$$|\delta F(z)| < 8\varepsilon + |R_6|.$$

Parabolic interpolation of even degree is seldom used, since in this the curve passes through an odd number of points, and therefore a cannot be chosen so that the position of z shall be symmetrical.

Putting into formula (3) $n=2m$ we get

$$(9) \qquad F(z) = F(a+xh) = F(a) + x\Delta F(a) + \ldots + \binom{x}{2m}\Delta^{2m}F(a) + h^{2m+1}\binom{x}{2m+1}D^{2m+1}F(a+\xi h).$$

In this case the best way is to choose a so as to have

$$a+mh < z < a+mh+h \quad or \quad a+mh-h < z < a+mh$$

but neither of these dispositions is symmetrical. In both cases we have

$$0 < \xi < 2n.$$

If the differences $\Delta, \Delta^2, \Delta^3, \ldots \Delta^n$ are given by the table, then the interpolation by *Newton's* formula is the simplest. The difficulty consists in the differences being printed, which makes the tables bulky and expensive. Moreover to obtain the differences with the same precision as the values of the function, they must be calculated directly, and not by simple subtraction starting from' the values of the function given by the tables, and therefore the determination of the differences is complicated.

§ 126. Inverse interpolation by Newton's formula. If a function $F(z)$ is known for $z = z_0, z_1, \ldots, z_n$, then the *following problems may arise:

a) Interpolation: To determine $F(z)$ if z is given.

b) Inverse Interpolation: To determine z if $F(z)$ is given.

The second problem is not univocal; to a given $F(z)$ there may correspond several solutions; but if we add a condition, for instance that z should satisfy the inequality $a < z < 6$, the interval being small enough, then there will be only one solution.

If $F(z)$ is given by a polynomial of degree n and by a remainder, as in *Newton's* series, then we will have in the second problem an equation of degree n to solve; this is generally done by a somewhat modified method of the *Rule of False Position.* If the polynomial is given by a *Newton,* a *Lagrange* or any other series the procedure will always be the same.

Let us suppose that for the required precision, the function $F(z)$ is given by a *Newton* series of degree n, for $z=a+xh$

$$(1) \qquad F(r) = F(a+xh) = F(Q) + x\Delta F(a) + \ldots, + \binom{x}{n}\Delta^n F(a) + h^{n+1}\binom{x}{n+1} D^{n+1}F(a+\xi h).$$

To obtain a *linear inverse interpolation* we shall start from

$$(2) \qquad F(z) = F(a) + x\Delta F(a) + h^2\binom{x}{2} D^2F(a+\xi h)$$

this gives, neglecting the remainder,

$$x_1 = \frac{F(z) - F(a)}{\Delta F(a)} = \frac{F(z) - F(a)}{F(a+h) - F(a)}$$

and

$$z_1 = a + x_1 h$$

where z_1 is the first approximation to z. If $F(s)$ is considered exact and if the precision of the numbers $F(a)$, $F(a+h)$ in the table is equal to ε, then the absolute value of the error of z, due to this cause, will be less than (§ 133):

$$|\delta z| < \frac{h}{|\Delta F(a)|} \left|2\varepsilon + \frac{h^2}{8} |D^2 F(a+\xi h)|\right| = \delta_1 .$$

From this it follows that if 8, is smaller than the corresponding quantity in linear direct interpolation, then inverse interpolation will give better results than the direct; and if δ_1 is larger, then it will give inferior results.

Example 1. Given a table of $F(z) = \log z$ to seven decimals, of the integers from 1000 to 10000. Here we have $h=1$; the number a is chosen in such a way that

$$F(a) < F(z) < F(a+1)$$

the absolute value of the error of z will be less than

$$|\delta z| < \frac{1}{\Delta \log a} \left| 2\varepsilon + \frac{1}{16a^2} \right| < \frac{1}{\Delta \log a} \left| \frac{10}{10^8} + \frac{7}{100a^2} \right| .$$

It can be easily shown that this is less than $1/10^3$. That is, z will be exact to three decimals, which will give with the four figures of the integer part of z a precision of seven figures.

If a greater precision is wanted it is best to repeat the operation. For this it is necessary to determine $F(z_1)$ with the required precision, by putting into equation (1) $x = x_1$.

If we find $F(z_1) < F(t)$ then we start from the points $z_1, F(z_1)$ and $(a+h), F(a+h)$ and get

$$x_2 = \frac{F(z) - F(z_1)}{F(a+h) - F(z_1)} \quad \text{or } z_2 = z_1 + x_2 h_1$$

where $h_1 = a + h - z_1$.

z_2 will be the second approximation of z. The error of z_2 caused by neglecting the remainder will be less than

$$\frac{h_1^3}{8} \frac{D^2F(z_1+\xi h_1)}{[F(a+h)-F(z_1)]}$$

If a still greater precision is wanted, then we determine $F(z_2)$ by putting into (1) $x = \frac{z_2 - a}{h}$.

Since we shall have $F(z_2) < F(z) < F(a+h)$, we may determine z_3 in the same manner, and continue till the required precision is attained.

If we had $F(a) < F(z) < F(z_1)$ we should have proceeded to determine z_2 in a way similar to the foregoing.

But we may shorten the computation by the following method: having obtained z_1, and shown that the error of z_1 is smaller than δ_1 we may put $h_1 = \delta_1$ and determine $F(z_1+h_1)$. Then we shall have

$$F(z_1) < F(z) < F(z_1+h_1).$$

Starting from these we determine z_2 with a much greater precision, since the interval now considered is much smaller than in the method above. An example will be given in § **133.**

§ 127. Interpolation by the Gauss series. In the following we will give only in a summary way the deductions of the Gauss, ***Bessel*** and ***Stirling*** interpolation formulae, as they are now very seldom used, and are merely of historical interest.

If we stop one of the series at the term of degree $2n-1$, and if the series gives exact values of f(x) for every integer x such that $-n+1 \leqq x \leqq n$, then the error of the approximation of $f(x)$ for other **values** of x will be measured by the remainder given by formula (2) of § 123, that is, by

$$(1) \qquad R_{2n} = \binom{x+n-1}{2n} D^{2n}f(\xi)$$

where ξ is included in the smallest interval containing $-n+1, x, n$.

If the curves pass through the same points, the remainder of every series will be the same; therefore from this point of view there is no advantage whatever in using any one of them.

Gauss's first series may be obtained, starting from the symbolical expression for ***Newton's*** formula:

$$\mathbf{E}^x = \sum_{m=0}^{\infty} \binom{x}{m} \Delta^m.$$

Let us multiply the second member of this equation from the term Δ^2 up by $\frac{\Delta+1}{\mathbf{E}} = \mathbf{1}$. This will not change the value of the series. We find

$$\mathbf{E}^x = 1 + \binom{x}{1}\Delta + \binom{x}{2}\frac{\Delta^2}{\mathbf{E}} + \sum_{m=2}^{\infty} \frac{\Delta^{m+1}}{\mathbf{E}}\left[\binom{x}{m} + \binom{x}{m+1}\right].$$

Remarking that $\binom{x}{m} + \binom{x}{m+1} = \binom{x+1}{m+1}$, the expansion becomes

$$\mathbf{E}^x = 1 + \binom{x}{1}\Delta + \binom{x}{2}\frac{\Delta^2}{\mathbf{E}} + \binom{x+1}{3}\frac{\Delta^3}{\mathbf{E}} + \sum_{m=3}^{\infty} \binom{x+1}{m+1}\frac{\Delta^{m+1}}{\mathbf{E}}.$$

Repeating the operation on this series, from the term Δ^4 up, we obtain

$$\mathbf{E}^x = 1 + \binom{x}{1}\Delta + \binom{x}{2}\frac{\Delta^2}{\mathbf{E}} + \binom{x+1}{3}\frac{\Delta^3}{\mathbf{E}} + \binom{x+1}{4}\frac{\Delta^4}{\mathbf{E}^2} +$$
$$+ \sum_{m=4}^{\infty} \frac{\Delta^{m+1}}{\mathbf{E}^2}\left[\binom{x+1}{m} + \binom{x+1}{m+1}\right].$$

This may be simplified again by remarking that the quantity in the brackets is equal to $\binom{x+2}{m+1}$. Then the above operation is repeated from Δ^6 up, and so on; finally we shall have

$$\text{(2)} \quad \mathbf{E}^x = \sum_{m=0}^{\infty} \left[\binom{x+m-1}{2m}\frac{\Delta^{2m}}{\mathbf{E}^m} + \binom{x+m}{2m+1}\frac{\Delta^{2m+1}}{\mathbf{E}^m}\right]$$

This is the symbolical expression for the first Gauss series. Written in the ordinary way, starting from $f(0)$, we get

$$\text{(3)} \quad f(x) = f(0) + \binom{x}{1}\Delta f(0) + \binom{x}{2}\Delta^2 f(-1) + \binom{x+1}{3}\Delta^3 f(-1) +$$
$$+ \binom{x+1}{4}\Delta^4 f(-2) + \binom{x+2}{5}\Delta^5 f(-2) + \binom{x+2}{6}\Delta^6 f(-3) + \dots$$
$$+ \binom{x+m-1}{2m-1}\Delta^{2m-1} f(-m+1) + \binom{x+m-1}{2m}\Delta^{2m} f(-m) + \dots,$$

If the differences of the function $f(x)$ are known, then its expansion in this series presents no difficulties. If f(x) is a polynomial of degree n, the series is finite and the second member is always exactly equal to $f(x)$.

Example 1. Given the function *f(x)* by its *Newton* expansion

$$f(x) = 8 + 6\binom{x+1}{1} + 4\binom{x+1}{2} + 2\binom{x+1}{3}.$$

Starting from this we may easily construct a table of f(x) and of its differences, in the same manner as in § 2.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-1	8				
		6			
0	14		4		
		10		2	
1	24		6		
		16		2	
2	40		8		
		24			
3	64				

Formula (3) gives by aid of this table

$$f(x) = 14 + 10\binom{x}{1} + 4\binom{x}{2} + 2\binom{x+1}{3}$$

From this we conclude that only the terms of the rows of *f(0)* and Δf (0) are figuring in the formula.

If the function *f(x)* is not a polynomial, then the series is infinite. Stopping the series (3) at the term $\binom{x+n-1}{2n-1}$, we shall obtain

$$f(x) = \sum_{m=0}^{n} \left[\binom{x+m-1}{2m}\Delta^{2m} f(-m) + \binom{x+m}{2m+1}\Delta^{2m+1} f(-m)\right] + \tag{4}$$

$$+ R_{2n}.$$

This will give exact values of f(x) for every integer x such that

$$-n+1 \leq x \leq n.$$

Indeed, in these cases every term of the sum in which $m > n$ will vanish and the limited series will give the same value as the infinite series. From this we conclude that the remainder R_{2n} will be given by **(1).**

Interpolation by the first Gauss formula. Let us put $x=(z-a)/h$ and $f\left(\frac{z-a}{h}\right) = F(z)$, we shall have

$$F(z) = \sum_{m=0}^{n} \left[\binom{x+m-1}{2m} \Delta^{2m}F(a-mh) + \binom{x+m}{2m+1} \Delta^{2m+1}F(a-mh)\right] +$$

$$+\binom{x+n-1}{2n} h^{2n}\mathrm{D}^{2n}F(a+\xi h).$$

It has been said that it is considered advantageous when interpolating to choose a so that the points for which the equation above gives exact values, shall be symmetrical to the point $z,F(z)$. Then

$$a < z < a+h \text{ or } 0 < x < 1 \text{ and } -n+1 < \xi < n.$$

According to (5) of **§ 125** the absolute value of the remainder will be less than

$$(5) \qquad \left| h^{2n}\binom{n-\frac{1}{2}}{2n} \mathrm{D}^{2n}F(a+\xi h) \right|.$$

The second Gauss series may also be obtained, starting from the symbolical form of ***Newton's*** expansion, by first multiplying the terms from Δ up by $\frac{\Delta+1}{\mathrm{E}} = 1.$ We get

$$\mathrm{E}^x = 1 + \binom{x}{1} \frac{\Delta}{\mathrm{E}} + \sum_{m=1}^{\infty} \frac{\Delta^{m+1}}{\mathrm{E}} \left[\binom{x}{m} + \binom{x}{m+1}\right].$$

Noting that the quantity in the brackets is equal to $\binom{x+1}{m+1}$ we repeat the operation from Δ^3 up. This gives

$$\mathrm{E}^x = 1 + \binom{x}{1}\frac{\Delta}{\mathrm{E}} + \binom{x+1}{2}\frac{\Delta^2}{\mathrm{E}} + \binom{x+1}{3}\frac{\Delta^3}{\mathrm{E}^2} +$$

$$+ \sum_{m=2}^{\infty} \frac{\Delta^{m+1}}{\mathrm{E}^2} \left[\binom{x+1}{m} + \binom{x+1}{m+1}\right].$$

We note that the quantity in the brackets is equal to $\binom{x+2}{m+1}$ and repeat the operation from Δ^5 up; and so on. Finally we find

(6) $$\mathsf{E}^x = \sum_{m=0}^{\infty}\left[\binom{x+m}{2m}\frac{\Delta^{2m}}{\mathsf{E}^m} + \binom{x+m}{2m+1}\frac{\Delta^{2m+1}}{\mathsf{E}^{m+1}}\right].$$

This is the symbolical expression of the *second Gauss series:* written in the ordinary way it gives, starting from *f (0):*

(7) $$f(x) = f(o) + \binom{x}{1}\Delta f(-1) + \binom{x+1}{2}\Delta^2 f(-1) +$$

$$+\binom{x+1}{3}\Delta^3 f(-2) + \binom{x+2}{4}\Delta^4 f(-2) + \binom{x+2}{5}\Delta^5 f(-3) +$$

$$+\binom{x+3}{6}\Delta^6 f(-3) + \ldots, \binom{x+m}{2m}\Delta^{2m} f(-m) +$$

$$+\binom{x+m}{2m+1}\Delta^{2m+1} f(-m-1) + , \ldots$$

If *f(x)* is a polynomial, this series is finite and the second member exactly equal to *f(x).*

Example 2. Expansion of the function given in Example 1. By aid of the table computed there, we obtain

$$f(x) = 14 + 6\binom{x}{1} + 4\binom{x+1}{2} + 2\binom{x+1}{3}.$$

The table shows that in the second *Gauss formula* only the numbers of the rows *f(0)* and $\Delta f(-1)$ figure.

If *f(x)* is not a polynomial, then the series is infinite. Stopping at the term Δ^{2n-1} we get a polynomial of degree $2n-1$; which is an approximate value of *f(x).* But for $x=-n, \ldots,$ $0, 1, \ldots, n-l$ it gives exact values. This can be shown by remarking that if x is an integer such that

$$-n \leqq x \leqq n-1$$

then every term of the infinite series vanishes in which $m \geqq n$, and therefore the limited series gives the same value as the infinite series,

For other values of x the error is measured by the remainder (1).

Interpolation by the second Gauss formula. Let us again put $x = (z-a)/h$ and write $f\left(\frac{z-a}{h}\right) = F(z)$. We have

(8) $$F(z) = F(a+xh) = \sum_{m=0}^{n}\left[\binom{x+m}{2m}\Delta^{2m}F(a-mh) +\right.$$

$$\left.+\binom{x+m}{2m+1}\Delta^{2m+1}F(a-mh-h)\right] + h^{2n}\binom{x+n}{2n}\mathsf{D}^{2n}F(a+\xi h).$$

It is best to choose a so that the points, for which the second member of (8) gives exact values, be symmetrical to the point $z, F(z)$. Then

$$a-h < z < a \quad \text{or} \quad -1 < x < 0 \quad \text{and} \quad -n < \xi < n-1$$

and the absolute value of the remainder will be less than (5).

Stopping the first or the second *Gauss* formula at the term Δ^{2n} we get an interpolation formula of *even degree.* The exact values correspond to $x=-n, \ldots, 0, \ldots, n$. The results are the same as in the case of *Newton's* formula of even degree.

§ 128. The Bessel and the Stirling series. To deduce the *Bessel* series, we start from the second *Gauss* formula (6, § 127) writing in it x-l instead of x and multiplying it by E; in this way its value will not change. We get

$$\mathsf{E}^x = \sum_{m=0}^{\infty} \left[\binom{x+m-1}{2m} \frac{\Delta^{2m}}{\mathsf{E}^{m-1}} + \binom{x+m-1}{2m+1} \frac{\Delta^{2m+1}}{\mathsf{E}^m} \right].$$

Now let us determine the mean of this series and of the first *Gauss* series (2, § 127) ; we find , putting $\tfrac{1}{2}(\mathsf{E}+1) = \mathsf{M}$

$$\mathsf{E}^x = \sum_{m=0}^{\infty} \binom{x+m-1}{2m} \frac{\mathsf{M}\Delta^{2m}}{\mathsf{E}^m} + \sum_{m=0}^{\infty} \frac{\Delta^{2m+1}}{2\mathsf{E}^m} \left[\binom{x+m}{2m+1} + \binom{x+m-1}{2m+1} \right]$$

This may still be simplified so that we shall have

$$(1) \qquad \mathsf{E}^x = \sum_{m=0}^{\infty} \binom{x+m-1}{2m} \left[\frac{\mathsf{M}\Delta^{2m}}{\mathsf{E}^m} + \frac{x-\tfrac{1}{2}}{2m+1} \frac{\Delta^{2m+1}}{\mathsf{E}^m} \right].$$

This is the symbolical form of *Bessel's* formula. Fully written, starting from f(0) it will be

$$(2) \qquad f(x) = \mathsf{M}f(0) + (x-\tfrac{1}{2})\Delta f(0) + \binom{x}{2} \mathsf{M}\Delta^2 f(-1) +$$
$$+ \binom{x}{2} \frac{x-\tfrac{1}{2}}{3} \Delta^3 f(-1) + \binom{x+1}{4} \mathsf{M}\Delta^4 f(-2) +$$
$$+ \binom{x+1}{4} \frac{x-\tfrac{1}{2}}{5} \Delta^5 f(-2) + \ldots + \binom{x+m-1}{2m} \left[\mathsf{M}\Delta^{2m} f(-m) + \right.$$
$$\left. + \frac{x-\tfrac{1}{2}}{2m+1} \Delta^{2m+1} f(-m) \right] + \ldots$$

If *f(x)* is a polynomial, then the above series will be finite and the second member will give *f(x)* exactly.

Example 1. Given the function figuring in Example **1 § 126,** we find, using the corresponding table, that

$$f(x) = 19 + 10(x-{}^1{}_2) + 5\binom{x}{2} + 2\binom{x}{2}\frac{x-{}^1{}_2}{3}$$

From this table we conclude that the numbers in *Bessel's* formula are all in the same row of the table, e. i. in that of $\mathsf{M}f(0)$,

$$\mathsf{M}f\,(0) = 19,\ \Delta f(0) = 10,\ \mathsf{M}\Delta^2 f\,(-1) = 5,\ \Delta^3 f(-1) = 2.$$

If $f(x)$ is not a polynomial, and if we stop the series at the term of Δ^{2n-1} then we obtain a polynomial of degree $2n-1$ which is an approximation of $f(x)$ giving for

$$x = -n+1, \ldots, 0, \ldots, n$$

the exact values of $f(x)$. Indeed, formula **(1) shows** that if x is such an integer, every term of the infinite series in which $m \geqq n$ will vanish; therefore the limited series will give the same value as the infinite **series.** Since these numbers are **the** same as those corresponding to the first Gauss **series;** therefore according to **§ 123** the remainder will also be the **same (1, § 127).**

Stirling's series. This series may be obtained by determining **the** mean of the two Gauss series. Of course *Stirling* obtained his series in another way a century before Gauss.

The first Gauss series **(2, § 127)** may be written

$$\mathsf{E}^x = 1 + \sum_{m=1}^{\infty}\left[\binom{x+m-1}{2m-1}\frac{\Delta^{2m-1}}{\mathsf{E}^{m-1}} + \binom{x+m-1}{2m}\frac{\Delta^{2m}}{\mathsf{E}^m}\right]$$

and the second (6, **§ 127)**

$$\mathsf{E}^x = 1 + \sum_{m=1}^{\infty}\left[\binom{x+m-1}{2m-1}\frac{\Delta^{2m-1}}{\mathsf{E}^m} + \binom{x+m}{2m}\frac{\Delta^{2m}}{\mathsf{E}^m}\right]$$

Hence the mean will be

$$\mathsf{E}^x = 1 + \sum_{m=1}^{\infty}\left[\binom{x+m-1}{2m-1}\frac{\Delta^{2m-1}}{\mathsf{E}^m}\left(\frac{\mathsf{E}+1}{2}\right) + \frac{\Delta^{2m}}{2\mathsf{E}^m}\left[\binom{x+m}{2m} + \right.\right.$$

$$\left.\left. + \binom{x+m-1}{2m}\right]\right].$$

After simplification this gives the symbolical expression for *Stirling's* formula

$$(4) \qquad E^x = 1 + \sum_{m=1}^{\infty} \binom{x+m-1}{2m-1} \left[\frac{M\Delta^{2m-1}}{E^m} + \frac{x}{2m} \frac{\Delta^{2m}}{E^m} \right].$$

Or if written in the usual way

$$(5) \qquad f(x) = f(0) + \binom{x}{1} M\Delta f(-1) + \binom{x}{1} \frac{x}{2} \Delta^2 f(-1) +$$

$$+ \binom{x+1}{3} M\Delta^3 f(-2) + \binom{x+1}{3} \frac{x}{4} \Delta^4 f(-2) +$$

$$+ \binom{x+2}{5} M\Delta^5 f(-3) + \binom{x+2}{5} \frac{x}{6} \Delta^6 f(-3) + , \ldots$$

$$+ \binom{x+m-1}{2m-1} \frac{x}{2m} \Delta^{2m} f(-m) + \binom{x+m}{2m+1} M\Delta^{2m+1} f(-m-1) + \ldots$$

If $f(x)$ is a polynomial, the second member of (5) is exactly equal to $f(x)$.

Example 1. From the table of § 126 we obtain

$$f(x) = 14 + 8\binom{x}{1} + 4\binom{x}{1}\frac{x}{2} + 2\binom{x+1}{3}.$$

From this table it is readily seen that the coefficients in *Stirling's* formula are all in the same row, in that of $f(0)$.

If $f(x)$ is not a polynomial, and if we stop the series (5) at the term $M\Delta^{2n-1}$ we obtain a polynomial of degree $2n-1$; by aid of (4) it may be shown that this polynomial gives $f(x)$ exactly for the following $2n$ values $x=-n+1, \ldots, n$. Therefore the remainder will be given by formula (1) of § 127.

Remark. Stirling's formula is in reality a formula of central differences. We have seen in § 8 that .

$$\frac{\Delta^{2m}}{E^m} = \delta^{2m} \quad \text{and} \quad \frac{M\Delta^{2m-1}}{E^m} = \mu\delta^{2m-1}.$$

Therefore we shall have **Stirling's** formula (4) expressed by central difference notation

$$(6) \qquad E^x = 1 + \sum_{m=1}^{\infty} \left[\mu\delta^{2m-1} + \frac{x}{2m} \delta^{2m} \right] \binom{x+m-1}{2m-1}$$

or fully written

$$(7)\quad f(x) = f(0) + \binom{x}{1}\mu\delta f(0) + \binom{x}{1}\frac{x}{2}\delta^2 f(0) + \binom{x+1}{3}\mu\delta^3 f(0) +$$

$$+\binom{x+1}{3}\frac{x}{4}\delta^4 f(0) + \ldots + \binom{x+m-1}{2m-1}\left[\mu\delta^{2m-1}f(0) + \frac{x}{2m}\delta^{2m}f(0)\right] + \ldots$$

Let us remark that in this formula the argument in each term of the second member is the same.

§ 129. Everett's formula. If from *Stirling's* formula expressed by central differences (6, § 128), we eliminate $\mu\delta$ by aid of the following equation (found in § 8),

$$\mu\delta = \mathrm{E} - 1 - \tfrac{1}{2}\delta^2$$

we get

$$\mathrm{E}^x = 1 + \sum_{m=1}^{\infty}\binom{x+m-1}{2m-1}\left[\mathrm{E}\delta^{2m-2} - \delta^{2m-2} - \tfrac{1}{2}\delta^{2m} + \frac{x}{2m}\delta^{2m}\right].$$

Let us write in the first two terms of the preceding sum $m+1$ instead of m; then m must vary in these terms from zero to ∞. Therefore we shall have

$$\sum_{m=0}^{\infty}\binom{x+m}{2m+1}\left[\mathrm{E}\delta^{2m} - \delta^{2m}\right].$$

The third and the fourth terms will give

$$\sum_{m=1}^{\infty}\binom{x+m-1}{2m-1}\frac{x-m}{2m}\delta^{2m} = \sum_{m=1}^{\infty}\binom{x+m-1}{2m}\delta^{2m}.$$

So that the above formula may be written

$$\mathrm{E}^x = \sum_{m=0}^{\infty}\binom{x+m}{2m+1}\mathrm{E}\delta^{2m} + \sum_{m=0}^{\infty}\left[\binom{x+m-1}{2m} - \binom{x+m}{2m+1}\right]\delta^{2m}.$$

From this we obtain after simplification the symbolical expression of *Everett*'s formula

$$(1)\qquad \mathrm{E}^x = \sum_{m=0}^{\infty}\left[\binom{x+m}{2m+1}\mathrm{E}\delta^{2m} - \binom{x+m-1}{2m+1}\delta^{2m}\right]$$

Or written in the usual way, if the operations are performed on $f(0)$ *we* have

$$(2) \qquad f(x) = \binom{x}{1} f(1) + \binom{x+1}{3} \delta^2 f(1) + \binom{x+2}{5} \delta^4 f(1) +$$

$$+ \binom{x+3}{7} \delta^6 f(1) + \ldots - \binom{x-1}{1} f(0) - \binom{x}{3} \delta^2 f(0) -$$

$$- \binom{x+1}{5} \delta^4 f(0) - \binom{x+2}{7} \delta^6 f(0) - \ldots$$

If $f(x)$ is a polynomial, the second member of (2) will be finite and it will give $f(x)$ exactly for every value of x.

Example 1. Given the function of Example 1, § 126. Formula (2) will immediately give the expansion of this function into an *Everett* series by aid of the table (§ 127). We find

$$f(x) = 24 \binom{x}{1} + 6 \binom{x+1}{3} - 14 \binom{x-1}{1} - 4 \binom{x}{3}.$$

The mentioned table shows that in *Everett's* formula there figure only the numbers of the rows $f(0)$ and $f(1)$.

If $f(x)$ is not a polynomial, the series (2) is infinite; stopping it at the term δ^{2n-2} we obtain a polynomial of degree $2n-1$ which gives the exact values of $f(x)$ if

$$x = -n+1, \ldots, 0, \ldots, n,$$

therefore the remainder will be given by formula (1) of § 127.

The precision of **Everett's** formula will be the same as for instance, that of *Newton's* formula of the same degree; but while *Newton's* formula requires a knowledge of the differences Δ, $\Delta^2, \Delta^3, \ldots \Delta^{2n-1}$, *Everett's* formula needs only the even differences $\Delta^2, \Delta^4, \ldots \Delta^{2n-2}$; this is an advantage.

***Interpolation** by Everett's formula.* To obtain the general formula, we put $x=(z-a)/h$ into equation (2) and write $f\left(\frac{z-a}{h}\right) = F(r)$. Stopping at the term δ^{2n-2} we shall have

$$(3) \qquad F(z) = F(a+xh) = \sum_{m=0}^{n} \Bigg[\binom{x+m}{2m+1} \delta^{2m} F(a+h) -$$

$$- \binom{a+m-1}{2m+1} \delta^{2m} F(a) \Bigg] + h^{2n} \binom{x+n-1}{2n} D^{2n} F(a+\xi h).$$

The curve passes through the points corresponding to $z = (a-nh+h), \ldots, (a+nh)$, To determine $F(z)$ it is best to choose, if possible, as has been said, the number a so as to have the same number of points on each side of z, that is

$$a < z < a+h \quad or \quad 0 < x < 1. \tag{4}$$

If we interpolate in a table where the given values correspond to $z_i = z_0 + ih$ (for $i = 0, 1, 2, \ldots, N$) then condition (4) can only be fulfilled if

$$z_{n-1} < z < z_{N-n+1}.$$

Putting into equation (3) x=6 and $1 - x = \varphi$ it will become

$$F(a+\vartheta h) = \sum_{m=0}^{n} \left[\binom{\vartheta+m}{2m+1} \delta^{2m} F(a+h) + \binom{\varphi+m}{2m+1} \delta^{2m} F(a) \right] + \tag{5}$$
$$+ h^{2n} \binom{x+n-1}{2n} D^{2n} F(a+\xi h)$$

where $-n+1 < \xi < n.$

A. J. Thompson has constructed a table which gives to ten decimals the coefficients figuring in this formula,[37]

$$E_{2m}(\vartheta) = (-1)^m \binom{\vartheta+m}{2m+1}$$

for $m = 1, 2, 3$ and $0 \leqq \vartheta \leqq 1$ where $\Delta\vartheta = 0{,}001.$

Formula (5) may be written therefore

$$F(a+\vartheta h) = \vartheta F(a+h) - E_2(\vartheta)\delta^2 F(a+h) + E_4(\vartheta)\delta^4 F(a+h) -$$
$$- E_6(\vartheta)\delta^6 F(a+h) + \varphi F(a) - E_2(\varphi)\delta^2 F(a) + E_4(\varphi)\delta^4 F(a) -$$
$$- E_6(\varphi)\,\delta^6 F(a) + R_8\,. \tag{6}$$

The remainder deduced from formula (5) will be

$$R_{2n} = (-1)^{n-1} h^{2n} \frac{\vartheta - n}{2n} E_{2n-2}(\vartheta)\, D^{2n} F(a+\xi h) \tag{7}$$

where $-n+1 < \xi < n.$

If $z < z_{n-1}$ then we are obliged to put into equation (3) $a = z_0 + (n-1)h$ and we shall have $z - a = \vartheta h < 0$; the formula thus obtained is called by *Pearson,* a first-panel interpolation formula. To compute the value of $F(z)$ it is necessary to calculate

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[37] *A. J. Thompson,* Table of the Coefficients of *Everett*'s Central-Difference Interpolation Formula. Cambridge University Press, 1921.

the binomial coefficients, since 6 is out of the range of ***Thompson's*** table.

If $z > z_{N-n+2}$ we are obliged to put $a = z_0 +$ (N-n+l) h and we shall have $z - a = \vartheta h > h$; the formula obtained is called an **end-**panel interpolation formula. Since 19 is out of the range of the tables, the coefficients must also in this case be calculated.

An interesting particular case of ***Everett's*** formula (5) is that of $\vartheta = \varphi = 1/2$. For this value the formula will become

$$F(a + \tfrac{1}{2}h) = \sum_{m=0}^{n} \binom{\frac{1}{2}+m}{2m+1} 2\mathrm{M}\delta^{2m} F(a) + R_{2n}\,.$$

Stirling already knew this formula, and determined the coefficients up to δ^{10} in his "Methodus Differentialis" (Lond. 1730, p. 111). His formula is, in our notation,

$$F(a + \frac{1}{2}h) = \mathrm{M}F(a) - \frac{1}{8}\mathrm{M}\delta^2 F(a) + \frac{3}{128}\mathrm{M}\delta^4 F(a) -$$

$$- \frac{5}{1024}\mathrm{M}\delta^6 F(a) + \frac{35}{32768}\mathrm{M}\delta^8 F(a) - \frac{63}{262144}\mathrm{M}\delta^{10} F(a) + \ldots$$

Stirling was the first to use interpolation formulae which needed only even differences.

Example 1. Given a seven-figure logarithmic table (***K. Pearson,*** On the Construction of Tables and on Interpolation, Part I. p. 60, Cambridge University Press) where $\log z$ and $\delta^2 \log z$ are given for $10 < z < 100$ to seven decimals, and where $h = 1$.

Let us determine log 35.562. In the table we find

$$F(a) = \log 35 = 1{,}544\ 0680 \qquad \delta^2 F(a) = -3547$$
$$F(a+1) = \log 36 = 1{,}556\ 3025 \qquad \delta^2 F(a+1) = -3352.$$

From ***Thompson's*** Tables we get

$$\vartheta = 0.562 \qquad E_2(\vartheta) = 0.0641$$
$$\varphi = 0.438 \qquad E_2(\varphi) = 0.0590$$

According to (7) the remainder will be

$$|R_4| < \frac{1{,}438}{4} \cdot 0{,}0641 \cdot \frac{3}{34^4} < \frac{1}{10^7}\,.$$

The computation of $F(r)$ gives

$\vartheta F(a+1)$	$= 0.814\ 6420\ 1$
$\varphi F(a)$	$= 0.676\ 3017\ 8$
$-E_2(\vartheta)\delta^2 F(a+1)$	$= \qquad 214\ 9$
$-E_2(\varphi)\delta^2 F(a)$	$= \qquad 209\ 3$
	$1.550\ 9862$

This result is exact up to the last decimal.

Example 2. We have to determine $F(r)$ = antilog 0.9853426 by aid of the antilogarithmic table (Pearson, Interpolation, p. 60) to seven decimals. In this table antilog z and δ^2antilog z are given from zero to one, to seven decimals, and $h=0.01$. We find the following values in the table:

$$F(a) = \text{antilog } 0.98 = 9.549926 \qquad \delta^2 F(a) = 5063$$
$$F(a+h) = \text{antilog } 0.99 = 9.772372 \qquad \delta^2 F(a+h) = 5182$$

Hence 6 = 0.53426 and $\varphi = 0.46574$.

Since *Thompson's* Tables can be entered only with three decimals of 6, then, if there are more, as for instance in this example, E_{2m} may be determined by different methods, depending on the number of figures contained in $\delta^{2m}F(a)$ or $\delta^{2m}F(a+h)$.

If, as in the present example, there are four figures only in $\delta^2 F(a+h)$ then a precision of 6 decimals of $E_2(\vartheta)$ is large enough. This may generally be obtained from *Thompson's* Tables by a rough linear interpolation.

In the present case we have in this manner

$$E_2(\vartheta) = 0{,}063627 \qquad E_2(\varphi) = 0{,}060786.$$

The computation will give

$\vartheta F(a+h)$	$= 5.220\ 987\ 5$
$\varphi F(a)$	$= 4.447\ 782\ 5$
$-E_2(\varphi)\delta^2 F(a)$	$= \qquad 307\ 8$
$-E_2(\vartheta)\delta^2 F(a+h)$	$= \qquad 329\ 7$
	$9.668\ 132\ 5$

If a greater exacitude is required for $E_{2m}(\vartheta)$ and ϑ has more than three decimals and less than seven, it would be possible to determine $E_{2m}(\vartheta)$ by interpolation, using ***Everett's*** formula and ***Thompson's*** Tables; but the computation would be too long, It is far better in these cases to calculate first ϑ^2 and then

$$E_2(\vartheta) = \frac{1}{6}\vartheta(1-\vartheta^2)$$

$$E_4(\vartheta) = E_2(\vartheta)\frac{9-\vartheta^2}{4}$$

$$E_6(\vartheta) = E_4(\vartheta)\frac{25-\vartheta^2}{5}.$$

Thompson has given another method, which we shall see in the next paragraph,

§ 130. Inverse interpolation by Everett's formula. We start from ***Everett's*** formula (3, **§ 129)** in which n is large enough for the required precision.

$$(1) \qquad F(z) = F(a+xh) = xF(a+h) - (x-1)F(a) +$$

$$+ \binom{x+1}{3}\delta^2F(a+h) - \binom{x}{3}\delta^2F(a) + \binom{x+2}{5}\delta^4F(a+h) -$$

$$- \binom{x+1}{5}\delta^4F(a) + \ldots.$$

The function $F(z)$ is given by a table for equidistant values of z, the interval being equal to h and the necessary differences δ^2, δ^4 and so on, are given too. z is to be determined corresponding to a given $F(z)$. We choose a among the values of the table, so as to have

$$F(a) < F(z) < F(a+h)$$

or

$$F(a) > F(t) > F(a+h)$$

the method is similar in both cases. We will suppose that the first inequality is satisfied.

By aid of a linear interpolation we deduce x, the first approximation of x, keeping only three figures in $(z_1-a)/h$.

$$(2) \qquad x_1 = \frac{F(z)-F(a)}{F(a+h) - F(a)} \quad \text{and } z_1 = Q + x_1h.$$

Now we determine $F(z_1)$ by aid of formula (1) with the necessary precision. Let us suppose that $F(z_1) < F(s)$; *then we* determine $F(z_1+h_1)$ also by formula (1), where for instance $h_1 = h/1000$.

If we have

$$F(z_1) < F(z) < F(z_1+h_1)$$

then we may determine x_2 the second approximation of x by formula (2) in which we put respectively z_1 and h_1 instead of a and h. The first member will be equal to x_2, in which we again keep three decimals, and then determine $z_2 = z_1 + x_2 h_1$.

Should $F(z_1+h_1)$ be less than $F(z)$ then we should be obliged to calculate $F(z_1+2h_1)$ and so on, till we have

$$F(z_1+ih_1-h) < F(z) < F(z_1+ih_1).$$

Then starting from this we would determine the second approximation of z.

If we had

$$F(z_1) > F(z) > F(z_1+h_1)$$

then the determination of z_2 would have been similar.

Now we determine by formula (1) the quantity $F(z_2)$ putting into it x_2 instead of x, then x_1 instead of a and h_1 instead of h. Of course the differences δ^2, δ^4 . . . must be given now in the system $\Delta z = h_1$.

It would be possible to calculate these differences by aid of formula (1), but this is complicated, and generally superfluous, since we may nearly always consider the third differences of $F(x)$ in the system h_1 negligible, and therefore the second differences constant. We have

$$\underset{h_1}{\delta^2} F(z_1) = \underset{h_1}{\delta^2} F(z_1+h_1) = \underset{h}{\delta^2} F(a)\left(\frac{h_1}{h}\right)^2 = \underset{h}{\delta^2} F(a+h)\left(\frac{h_1}{h}\right)^2$$

The determination of $F(z_2)$ is simple, since in x_2 there are *only* three figures, *so* that *Thompson's* table is applicable directly. If we have $F(z_2) < F(z)$ then we compute in the same manner $F(z_2+h_2)$ where we have chosen $h_2 = h_1/1000$. If

$$F(z_2) < F(z) < F(z_2+h_2)$$

then by aid of (2) we determine x_3 the third approximation of x to three decimals, and $z_3 = z_2 + x_3 h_2$.

Now we determine the differences $\delta^2_{h_2} F(z_2) = \delta^2_h F(a) \left(\frac{h_2}{h}\right)^2$ in the system $\Delta z = h_2$; if they are small enough to be neglected in determining $F(z_3)$, then the linear determination of z_3 above will give z exact to the required number of decimals, and the problem is solved.

If the differences are not negligible, we compute $F(z_3)$ and $F(z_3+h_3)$, where $h_3 = h_2/1000$, in a way similar to $F(z_2)$. Then we determine z_4 by linear interpolation, and continue in this way till the differences may be neglected.

Remark. It would be possible to determine every value such as $F(z_1)$, $F(z_1+h_1)$, $F(z_2)$, $F(z_2+h_2)$, $F(z_3)$, . . . by aid of the same equation (1); this would make the determination of the differences $\delta^2_{h_1} F(z_1)$, $\delta^2_{h_2} F(z_2)$, . . , superfluous. But in this case the evaluation would be complicated, since then, for instance, $(z_2 - a)/h$ would have a great number of decimals, and *Thompson's* tables of the coefficients would not be directly applicable.

Example 1. Given $F(z) = \log z = \overline{1}.95717$ 32271 83589 39035 and z is to be determined by aid of *Thompson's* Logarithmetica Britannica tables to twenty decimals. There we find

$F(a)$	$= \overline{1}.95717$ 13373 70099 19928	a	= 0.90609
$F(a+h)$	$= \overline{1}.95717$ 61304 04846 19226	$a+h$	= 0.90610

Hence $h = 1/10^5$.

Formula (2) will give, if we determine $(z_1 - a)/h$ to three figures,

$$x_1 = \vartheta = \frac{z_1 - a}{h} = 0.394 \quad or \quad z_1 = 0.90609394.$$

Now we have to determine $F(z_1)$ by formula (1); to obtain it exact to twenty decimals, let us take from the table the differences δ^2 and δ^4:

$\delta^2 F(a)$	$= -$ 52898 29042 . 10^{-20}	$\delta^4 F(a)$	$= -4.10^{-20}$
$\delta^2 F(a+h)$	$= -$ 52897 12282 . 10^{-20}	$\delta^4 F(a+h)$	$= -4.10^{-20}$

By aid of $\vartheta = 0.394$ and $\varphi = 1 - \vartheta = 0.606$, *Thompson's* Tables of the coefficients of ***Everett's*** formula give

$$E_2(0{,}394) = 0.05547\ 28360 \qquad E_4(0{,}394) = 0{,}01066\ldots$$
$$E_2(0{,}606) = 0.06390\ 91640 \qquad E_4(0{,}606) = 0.01160\ldots,$$

We conclude that the fourth differences multiplied by E_4, having no influence on the first 20 decimals, may be neglected.

From formula (1) we get

$$\frac{z_1-a}{h}F(a+h) + \frac{a+h-z_1}{h}F(a) = 1.95717\ 32258\ 25789\ 51451$$
$$-E_2(\vartheta)\delta^2F(a+h) - E_2(\varphi)\delta^2F(a) = \qquad 6315\ 03894$$
$$F(z_1) = 1.95717\ 32258\ 32104\ 55345$$

Remark. In computing this value before performing the multiplication of $F(a+h)$ by $(z_1-a)/h$ and that of $F(a)$ by $(a+h-z_1)/h$ since the sum of these factors is equal to one, therefore the first 6 figures common to $F(a)$ and $F(a+h)$ have been set aside and only added to the result.

The new interval will be $h_1 = 1/10^8$. To determine $F(z_1+h_1)$ let us remark that now $(z_1+h_1-a)/h$ is equal to 6, = 0.395. *Thompson's* tables give $E_2(0.395)$ and $E_2(0.605)$. Finally from formula (1 j we get

$$F(z_1+h_1) = \bar{1}.95717\ 32306\ 25144\ 88111.$$

Since the condition $F(z_1) < F(z) < F(z_1+h_1)$ is fulfilled, we may proceed to determine the second approximation of z by formula (2). We find

$$x_2 = \frac{1}{h_1}(z_2 - z_1) = 0.281 \quad \text{and } z_2 = 0.90609\ 39428\ 1\ .$$

To compute the value of $F(z_2)$ we must start from $F(z_1)$ and $F(z_1+h_1)$ and form the differences

$$\underset{h_1}{\delta^2}F(z_1) = \underset{h_1}{\delta^2}F(z_1+h_1) = \underset{h}{\delta^2}F(a)\,.\,10^{-6} = \underset{h}{\delta^2}F(a+h)\,.\,10^{-6} =$$
$$= -5290\,.\,10^{-20}$$

We determine $F(z_2)$ by *Everett's* formula (1), setting aside the 8 figures common to $F(z_1)$ and $F(z_1+h_1)$. We get

$$F(z_2) = \bar{1}.95717\ 32271\ 78948\ 89087.$$

In the same manner we compute, after choosing $h_2 = 10^{-11}$

$$F(z_2+h_2) = \bar{1}.95717\ 32271\ 83741\ 93121\ .$$

Since the condition

$$F(z_2) < F(z) < F(z_2+h_2)$$

is again satisfied, we may proceed to the evaluation of the third approximation of z; but this time the second differences will be equal to

$$\delta^2 F(a) \cdot 10^{-12} = -5 \cdot 10^{-22}$$

therefore they will not have any influence on the first 20 decimals; hence we conclude that z_3 may be determined by the linear formula, up to the 20 -th decimal. We have

$$z = z_3 = 0.90609\ 39428\ 19681\ 74509.$$

The error is equal to $-3/10^{20}$; z being equal to $e/3$.[38]

§ 131. Lagrange's interpolation formula. This interpolation formula differs from those treated before chiefly by the fact that it does not need the knowledge of the differences of the function and that the abscissae x_i are not necessarily equidistant. This is a great advantage, since the determination of the differences causes much work, their printing makes the tables bulky and expensive. On the other hand the interpolation by aid of this formula is more complicated.

Given n points of coordinates x_i, $f(x_i)$ for $i=0, 1, 2, \ldots, n$ let us put

(1) $$o(x) = (x-x_0)\ (x\text{-}x,)\ (x\text{-}x,)\ \ldots\ (x\text{-}x,)$$

and moreover

(2) $$L_{ni}(x) = \frac{\omega(x)}{(x-x_i)\,[D\omega(x)]_{x=x_i}}\,.$$

It is obvious that we have for $m=0, 1, 2, \ldots$, n,

$L_{ni}(x_i) = L_{ni}(x_m) = 0$ if $i \neq m$ and .

[38] This method is due to Thompson, Logarithmetica Britannica, 1932, Cambridge University Press, Part IX, Introduction p. vii. The example is *Thompson's* example too.

This permits us to deduce immediately a curve of degree R which passes through the given $n+1$ points. We have

$$(3) \qquad f(x) = L_{n0}(x)\, f(x_0) + L_{n1}(x)\, f(x_1) + \ldots + L_{nn}(x)\, f(x_n).$$

This is *Lagrange's* formula.

We may easily determine the sum of the polynomials $L_{ni}(x)$ for $i=0, 1, 2, \ldots$, R; let us remark that l/o(x) decomposed into partial fractions gives (§ 13)

$$\frac{1}{\omega(x)} = \sum_{i=0}^{n+1} \frac{c_i}{x-x_i}$$

where (§ 13)

$$c_i = \frac{1}{[D\omega(x)]_{x=x}}$$

therefore from (2) it follows that

$$\sum_{i=0}^{n+1} L_{ni}(x) = 1.$$

If *f(x)* is a polynomial of degree R it may be represented exactly by formula (3). If $f(x)$ is not a polynomial' or if it is a polynomial of degree higher than n, then the second member of (3) will give *f(x)* exactly for $x=x_0, x_1, \ldots, x_n$ but for the other values of x it will be only an approximation to *f(x)*. The error may be measured by the remainder, which is given according to (2), § 123 by

$$(4) \qquad R_{n+1} = \frac{(x-x_0)\ldots(x-x_n)}{(n+1)!} D^{n+1}f(\xi) = \frac{\omega(x)}{(n+1)!} D^{n+1}f(\xi).$$

Finally we shall have

$$(5) \qquad f(x) = \sum_{i=0}^{n+1} L_{ni}(x)\, f(x_i) + \frac{\omega(x)}{(n+1)!} D^{n+1}f(\xi)$$

where *f(x)* may be any function of x whose $n+1$ derivative has a determi value; and where ξ is in the smallest interval containing the numbers x,,, $x_1, x_2, \ldots x_n$, x.

Lagrange's formula becomes much simpler in some particular cases. Let us suppose for instance that the values of x are equidistant, say that $x_i = \frac{i}{n}$ for $i = 0, 1, \ldots$, R. Then we shall have

$$\omega(x) = x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\cdots\left(x-\frac{n}{n}\right) = (x)_{n+1,h}$$

where $h = \frac{1}{n}$

that is $\omega(x)$ is equal to the generalised factorial of § 16. Moreover

$$[\mathrm{D}\omega(x)]_{x=\frac{i}{n}} = \frac{i}{n}\left(\frac{i-1}{n}\right)\left(\frac{i-2}{n}\right)\cdots\frac{1}{n}\left(-\frac{1}{n}\right)\cdots\left(-\frac{n-i}{n}\right) =$$

$$= \frac{(-1)^{n+i}\, i!\ (n-i)!}{n^n};$$

finally remarking that $nh=1$. we have

$$(6)\qquad L_{ni}(x) = (-1)^{n-i}\, n^n \binom{x}{i}_h \binom{x-ih-h}{n-i}_h = n^n \binom{x}{i}_h \binom{1-x}{n-i}_h$$

Expansion of $L_{ni}(x)$ *into a Newton series.* From (6) it follows that $L_{ni}(0)=0$ (for $i\neq 0$). The values of $\underset{h}{\Delta}^m L_{ni}(0)$ may be deduced from (6), by aid of the formula which gives the higher differences of a product (§ 30) and of the formula giving the differences of a function with negative argument (p. 6). We find

$$\underset{h}{\Delta}^m L_{ni}(x) = n^n \sum_{\mu=0}^{i+1} (-1)^{m-\mu} \binom{m}{i} \binom{x}{i-\mu}_h \binom{1-x-mh}{n-m-i+\mu}_h$$

putting $x=0$, and remembering that $h=1/n$ *we* have for $n \geqq m$

$$(7)\qquad \underset{h}{\Delta}^m\, L_{ni}(0) = (-1)^{m-i}\, n^{n-m} \binom{m}{i} \binom{1-mh}{n-m}_h = (-1)^{m+i} \binom{m}{i}.$$

Let us now express these differences in the system $\Delta x=1$. According to (4), § 76 we have

$$\Delta^{\nu} L_{ni}(0) = \sum_{m=0}^{n+1} \frac{\nu!}{m!} P(m,\nu)\, \underset{h}{\Delta}^m L_{ni}(0).$$

The $P(m,\nu)$ were given by a table in § 76. Let us remark that in the case considered we have to put into the formulae of the table $\omega=n$. $L_{ni}(x)$ is a polynomial of degree n; therefore in the preceding formula, for the upper limit of m we may put $n+1$ instead of ∞. From (7) we get

$$\Delta^{\nu} L_{ni}(0) = \sum_{m=i}^{n+1} (-1)^{m+i} \binom{m}{i} \frac{\nu!}{m!} P(m,\nu)$$

and therefore

$$(8) \qquad L_{ni}(x) = \frac{(-1)^i}{i!} \sum_{\nu=1}^{n+1} \sum_{m=i}^{n+1} \binom{x}{\nu} \frac{(-1)^m \nu!}{(m-i)!} P(m,\nu).$$

This is the *Newton* expansion of $L_{ni}(x)$.

From this formula *we* may deduce the ***Cofes numbers*** defined by

$$C_{ni} = \int_0^1 L_{ni}(x) \, dx.$$

Since we have seen in § 89 that

$$\int_0^1 \binom{x}{\nu} dx = b_\nu$$

where b_i is a coefficient of the *Bernoulli* polynomial of the second kind (§ 89); therefore

$$(9) \qquad C_{ni} = \frac{(-1)^i}{i} \sum_{\nu=1}^{n+1} \nu! \, b_\nu \sum_{m=i}^{n+1} \frac{(-1)^m}{(m-i)!} P(m,\nu).$$

This is the required expression giving the *Cofes* numbers.

Example 1. Computation of C_{32}. According to (9) we have

$$C_{32} = \tfrac{1}{2} b_1 [P(2,1) - P(3,1)] + b_2 [P(2,2) - P(3,2)] - 3 b_3 P(3,3).$$

From our table of $P(m,\nu)$ in § 76 we deduce the following numbers, remarking that we have $\bar{\omega} = \frac{1}{h} = n = 3$,

$$P(2,1) = 6, \qquad P(2,2) = 9$$
$$P(3,1) = 6, \qquad P(3,2) = 54 \qquad P(3,3) = 27.$$

Moreover in § 89 we found:

$$b_1 = \frac{1}{2}, \qquad b_2 = -\frac{1}{12}, \qquad b_3 = \frac{1}{24}.$$

Putting these results into the preceding formula we obtain $C_{32} = 3/8$.

Interpolation by Lagrange's formula. If the quantities $f(x_i)$ are given for $i = 0, 1, 2, , n$ then from formula (5) we obtain f(x) corresponding to a given value of x. If the interpolation is made by aid of a table, it is best to choose the numbers $x_0, x_1, \ldots, x_n$ symmetrical to x.

Linear **interpolation.** Here we have $x_0 < x < x_1$. Formula (5) will give

$$(10) \qquad f(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) + \\ + \tfrac{1}{2}(x-x_0)(x-x_1) D^2 f(\xi).$$

The absolute value of the remainder is

$$(11) \qquad |R_2| < \frac{(x_0-x_1)^2}{8} |D^2 f(\xi)|$$

where ξ is in at least one of the intervals x,, x_1 and x_0, x.

Example 1. x is given, and $f(x) = \log x$ is to be determined by aid of a logarithmic table. If $a < x < a+1$ then we put $x_0 = a$ and $x_1 = a+1$. From (10) it will follow that

$$\log x = (a+1-x) \log a + (x-a) \log(a+1) + \\ + \tfrac{1}{2}(x-a)(a+1-x) \frac{\log e}{\xi^2}.$$

Of course the remainder and the precision are the same as in the linear *Newton* interpolation formula. The formula above may be useful in logarithmic tables where there are no printed differences. The error committed is less than

$$\frac{\log e}{8\xi^2} < \frac{1}{16a^2}.$$

Lagrange's formula is especially useful for interpolation if the values of x_i are not equidistant, since in this case the only other available formula is *Newton's* expansion into a series of divided differences (§ 9). But since in general the divided differences of a function cannot be given in tables, hence they must be computed in each particular case. Therefore this formula is more complicated than that of *Lagrange.*

Lagrange's formula may be useful in some particular cases too, for instance if the x_i are roots of a *Legendre* polynomial of degree *n*, (§§ 138, 157) or the roots of a *Tchebichef* polynomial of degree *n*, (§ 158) and finally if the x_i are equidistant.

Inverse interpolation by Lagrange's formula. This may be done in the same manner as in the cases of *Everett's* or *Newton's* formula if f(x) is given, and we have to determine x by aid of a table giving x_i, $f(x_i)$ for certain values of i (the x_i may be equidistant or not). If

$$f(a) < f(x) < f(b)$$

where $f(a)$ and $f(b)$ are two consecutive numbers of the table; then starting from these values we deduce from (10), putting $x_0 = a$ and $x_1 = b$

$$x = a + \frac{(b-a)\,[f(x) - f(a)]}{f(b) - f(a)} + \frac{(b-x)\,(x-a)\,D^2 f(\xi)}{2[f(b) - f(a)]}\,(b-a).$$

Neglecting the remainder, we obtain the first approximation of x

(12) $$x' = a + \frac{(b-a)\,[f(x) - f(a)]}{f(b) - f(a)}.$$

If the remainder

(13) $$\mathcal{R} = \frac{(b-x)\,(x-a)\,D^2 f(\xi)}{2[f(b)-f(a)]}\,(b-a) \qquad \text{where } a < \xi < b\ .$$

is positive then we have $x' < x$.

Now we compute $f(x')$ by aid of (10) ; if we find

$$f(a) < f(x') < f(x) < f(b)$$

then we determine x'' the second approximation of x, by putting into (12) x' instead of a; the corresponding remainder is obtained by writing in (13) x' instead of a. Then we compute $f(x'')$ and continue in the same maimer till the prescribed precision is reached.

§ 132. Interpolation Formula without Printed Differences, Remarks on the Construction of Tables. We saw that a parabolic interpolation of degree n, by *Newton's* formula presupposes the knowledge of the differences $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$, . . . , , $\Delta^n f(0)$; so that they must be first calculated or given by the table. This last is preferable, but it makes the printing of the tables expensive.

Everett's formula is much better from this point of view, as it needs only the even differences; the calculation is somewhat shorter than that necessary to compute all the differences; and if given by the tables, as the odd differences are omitted, they are smaller and more economical.

Lagrange's formula does not need differences at all; but the interpolation, especially with higher parabolas, necessitates a great amount of computation.

There is another interpolation formula[39] which dispenses with printed differences, reducing the tables to a minimum of size and cost, and requiring no more work than ***Everett's*** formula, but it is applicable only for equidistant values of x.

The reduction thus obtained will generally be more than one-third of the tables. Instead of printing the differences it would be far more useful to publish a table of the inverse function.

To obtain this formula we start from ***Newton's*** formula, which gives the expansion of a function f(x) into a series of divided differences (§ 9).

$$(1)\qquad f(x) = f(x_0) + \sum_{i=1}^{n+1} (x-x_0)(x-x_1)\ldots(x-x_{i-1})\, \mathfrak{D}^i f(x_0) +$$
$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!}\, \mathrm{D}^{n+1} f(\xi).$$

The remainder has been obtained by remarking that the polynomial of degree n of the second member gives exactly $f(x)$ for $x=x_0, x_1, \ldots, x_{n-1}, x_n$; from this it follows, according to § 123, that the remainder is equal to the quantity above, and that ξ is included in the smallest interval containing the numbers:

$$x_0, x_1, \ldots, x_n, x.$$

Let us now consider the remarkable particular case

$$(2)\qquad \begin{array}{l} x_0=0, \quad x_1=1, \qquad x_2=-1, \qquad x_3=2, \ldots, \\ x_{2n-2}=-n+1, \qquad x_{2n-1}=n. \end{array}$$

[39] *Ch. Jordan, Sur* une formule d'interpolation. Atti del Congresso dei Matematici, Tomo VI, Bologna, 1928, pp. 165-177.

Sur une formule d'interpolation dérivée de la formule *d'Everett*, Metron, Roma, 1928.

Putting $2n-1$ instead of n into formula (1) will now give the same results as *Newton's* ordinary formula, if the curve passes through the points corresponding to $x=-(n-1), \ldots,$ $0, \ldots, n$. The remainder will be, according to § 123, equal to

$$(2') \qquad R_{2n} = \binom{x+n-1}{2n} \mathsf{D}^{2n} f(\xi)$$

where ξ is included in the smallest interval containing $-n+1, x, n$.

Determination of the divided differences. The $(2m-1)$ -th divided difference of $f(x)$ for x=x,, , given in § 9 is

$$(3) \qquad \mathfrak{D}^{2m-1} f(x_0) = \sum_{i=0}^{2m} \frac{f(x_i)}{\mathsf{D}\omega_{2m}(x_i)}$$

where ω_{2m} (x) has been written for (x-x,,) (x-x,) . . , $(x-x_{2m-1})$. Putting into it the values (2) we get

$$\omega_{2m}(x) = x(x-1)\,(x+1) \ldots (x+m-1)\,(x-m) = (x+m-1)_{2m} .$$

Therefore equation (3) may be written

$$\mathfrak{D}^{2m-1} f(0) = \sum_{\nu=-m+1}^{m+1} \frac{f(\nu)}{\mathsf{D}\omega_{2m}(\nu)} .$$

Since

$$\mathsf{D}\omega_{2m}(\nu) = (-1)^{m+\nu} (m+\nu-1)!\,(m-\nu)!$$

it follows that

$$\mathfrak{D}^{2m-1} f(0) = \frac{1}{(2m-1)!} \sum_{\nu=-m+1}^{m+1} (-1)^{m+\nu} \binom{2m-1}{m-\nu} f(\nu).$$

This may be simplified by putting $m-\nu=\mu$; we obtain

$$\mathfrak{D}^{2m-1} f(0) = \frac{1}{(2m-1)!} \sum_{\mu=0}^{2m} (-1)^{\mu} \binom{2m-1}{\mu} f(m-\mu).$$

Introducing the symbol E of displacement (§ 3), we may write the above difference in the following way:

$$\mathfrak{D}^{2m-1} f(0) = \frac{1}{(2m-1)!\,\mathsf{E}^{m-1}} \sum_{\mu=0}^{2m} (-1)^{\mu} \binom{2m-1}{\mu} \mathsf{E}^{2m-\mu-1} f\,(0).$$

According to § 6 the sum in the second member is equal to

$$(\mathsf{E}-1)^{2m-1} = \Delta^{2m-1}.$$

Hence

$$(4)\qquad \mathfrak{D}^{2m-1}f(0) = \frac{\Delta^{2m-1}}{(2m-1)!\,\mathrm{E}^{m-1}}\, f(0).$$

By this formula we have expressed the odd divided differences by ordinary differences.

Determination of the *even differences*. We have (§ 9)

$$(5)\qquad \mathfrak{D}^{2m}f(x_0) = \sum_{i=0}^{2m+1} \frac{f(x_i)}{\mathrm{D}\omega_{2m+1}(x_i)}$$

putting the values (2) into $\omega_{2m+1}(x_i)$ we find

$$\omega_{2m+1}(x) = x(x-1)\,(x+1)\,(x-2)\,\ldots\,(x-m)\,(x+m) = (x+m)_{2m-1}.$$

Equation (5) may be written

$$\mathfrak{D}^{2m}f(0) = \sum_{\nu=-m}^{m+1} \frac{f(\nu)}{\mathrm{D}\omega_{2m+1}(\nu)}$$

since

$$\mathrm{D}\omega_{2m+1}(\nu) = (-1)^{m+\nu}\,(m+\nu)!\,(m-\nu)!;$$

therefore putting $m-\nu=\mu$ we obtain

$$\mathfrak{D}^{2m}f(0) = \frac{1}{(2m)!} \sum_{\mu=0}^{2m+1} (-1)^{\mu} \binom{2m}{\mu} f\,(m-\mu).$$

Introducing the symbol E gives

$$\mathfrak{D}^{2m}f(0) = \frac{1}{(2m)!\,\mathrm{E}^m} \sum_{\mu=0}^{2m+1} (-1)^{\mu} \binom{2m}{\mu} \mathrm{E}^{2m-\mu}\, f(o).$$

Finally, this is equal to

$$(6)\qquad \mathfrak{D}^{2m}f(0) = \frac{\Delta^{2m}}{(2m)!\,\mathrm{E}^m} f(0) = \frac{\delta^{2m}}{(2m)!} f(0).$$

Hence $\mathfrak{D}^{2m}f(0)$ may be expressed very simply by aid of central differences.

Putting into equation (1) the values (2) we have

$$f(x) = \sum_{m=0}^{n} \left[(x+m-1)_{2m}\mathfrak{D}^{2m}f(0) + (x+m)_{2m+1}\mathfrak{D}^{2m+1}f(0) \right] + R_{2n}\,.$$

This will give by aid of formulae (4) and (6), writing again $\mathrm{E}-1$ instead of Δ:

$$f(x) = \sum_{m=0}^{n} \left[\binom{x+m-1}{2m} \frac{(\mathsf{E}-1)^{2m}}{\mathsf{E}^m} + \binom{x+m}{2m+1} \frac{(\mathsf{E}-1)^{2m+1}}{\mathsf{E}^m} \right] f(0) + \\ + R_{2n}.$$

Expanding the two terms by the binomial theorem and combining them, we get

$$f(x) = \sum_{m=0}^{n} \sum_{\nu=0}^{2m+2} \left[\binom{x+m}{2m+1} \binom{2m+1}{\nu} - \binom{x+m-1}{2m} \binom{2m}{\nu-1} \right] \cdot \\ \cdot (-1)^{\nu} \mathsf{E}^{m+1-\nu} f(0) + R_{2n}.$$

After simplification it results that

$$(7) \qquad f(x) = \sum_{m=0}^{n} \sum_{\nu=0}^{2m+2} (-1)^{\nu} \binom{x+m-1}{2m} \binom{2m+1}{\nu} \frac{x+m-\nu}{2m+1} \\ \cdot \mathsf{E}^{m+1-\nu} f(0) + R_{2n},$$

To transform this expression, let us determine the sum of the terms corresponding to E^k and E^{-k+1}. If ν varies from 0 to $2m+2$ then k will vary in the sum from 1 to $m+2$. To have the term corresponding to E^k we put into (7) $m+1-\nu=k$ and find

$$\sum_{m=0}^{n} \sum_{k=1}^{m+2} \cdot (-1)^{m+k-1} \binom{x+m-1}{2m} \binom{2m+1}{m+k} \frac{x+k-1}{2m+1} \mathsf{E}^k f(0);$$

to obtain the second we put $m+1-\nu=-k+1$ and get

$$\sum_{m=0}^{n} \sum_{k=1}^{m+2} (-1)^{m+k} \binom{x+m-1}{2m} \binom{2m+1}{m+k} \frac{x-k}{2m+1} \mathsf{E}^{-k+1} f(0).$$

To combine the two last expressions, let us write

$$(8) \qquad I_k = \frac{1}{2k-1} \left[(x+k-1)\, \mathsf{E}^k + (k-x)\, \mathsf{E}^{-k+1} \right] f(0).$$

According to ***Lagrange's*** formula this quantity is the approximate value of $f(x)$ obtained by linear interpolation between $x=-k+1$ and $x=k$. Introducing I_k into formula *(7)* we get

$$(9) \qquad f(x) = \sum_{m=0}^{n} (-1)^m \binom{x+m-1}{2m} \sum_{k=1}^{m+2} (-1)^{k+1} \\ \cdot \binom{2m+1}{m+k} \frac{2k-1}{2m+1} I_k + R_{2n}.$$

To simplify this formula let us introduce the numbers

$$B_{mk} = (-1)^{k+1} \binom{2m+1}{m+k} \frac{2k-1}{2m+1}. \tag{10}$$

The following table gives these numbers, sufficient for parabolic interpolations of the eleventh degree.

Table of B_{mk}.

$m \backslash k$	1	2	3	4	5	6
0	1					
1	1	-1				
2	2	-3	1			
3	5	-9	5	-1		
4	14	-28	20	-7	1	
5	42	-90	75	-35	9	-1

To check the table let us remark that from (10) it follows if $m > 0$ that

$$\sum_{k=1}^{m+2} B_{mk} = 0 \quad \text{and} \quad \sum_{k=1}^{m+2} |B_{mk}| = \binom{2m}{m}. \tag{10'}$$

From the first we conclude for intance, that the sum of the numbers in each row is equal to zero.

We shall put, moreover,

$$C_m(x) = (-1)^m \binom{x+m-1}{2m}. \tag{11}$$

Starting from $C_0(x) = 1$ these numbers are rapidly calculated step by step by aid of the formula

$$C_m(x) = \frac{(x+m-1)(m-x)}{2m(2m-1)} C_{m-1}(x). \tag{12}$$

To shorten the work of computation, a table has been given containing these numbers from $x=0$ to $x=0,5$ (interval $\Delta x = 0.001$) for $m = 1, 2, 3, 4$, sufficient for interpolations up to the ninth degree (loc. cit. 39).

Since we have

$$C_m(1-x) = C_m(x)$$

these tables may be used if $0 < x < 1$.

Introducing the above notations into formula (9) we obtain finally the required interpolation formula

$$\text{(13)} \qquad f(x) = \sum_{m=0}^{n} C_m(x) \sum_{k=1}^{m+2} B_{mk}\, I_k + \binom{x+n-1}{2n} D^{2n} f(\xi)$$

where ξ is included in the smallest interval containing $-n+1,\ n,$ x.

The formula fully written will be

$$\text{(14)} \qquad f(x) = I_1 + C_1[I_1 - I_2] + C_2[2I_1 - 3I_2 + I_3] +$$
$$+ C_3[5I_1 - 9I_2 + 5I_3 - I_4] + C_4[14I_1 - 28I_2 + 20I_3 - 7I_4 + I_5] +$$
$$+ \ldots\ldots + \binom{x+n-1}{2n} D^{2n} f(\xi)$$

where C_i is an abbreviation for $C_i(x)$.

Remark. In the particular case $x = 1/2$ we have:

$$I_k = 1/2\, [f(k) + f(-k+1)].$$

Stopping at the term I_1 we get a linear interpolation; stopping at the term C,(x), we have a parabolic interpolation of degree $2m+1$.

If the absolute value of the error of $F(a+ih)$ in the table is less than ε, then from (8) it follows that the absolute value of the error of I_k is also less than ε, if $0 < x < 1$;

$$|\delta I_k| < \varepsilon$$

moreover according to (10') and to (13) the absolute value of the error of $f(x)$ is

$$|\delta f(x)| < \varepsilon \sum_{m=0}^{n} \binom{2m}{m} |C_m(x)| + R_{2n},$$

Since according to § 125 the maximum of C,(x) in the interval (0, 1) obtained for $x = 1/2$ is equal to $\binom{m-1/2}{2m}$,

$$|\delta f(x)| < \varepsilon \sum_{m=0}^{n} \binom{2m}{m} \left|\binom{m-1/2}{2m}\right| + R_{2n}.$$

Particular** cases. Linear **interpolation:

$$|\delta f(x) < \varepsilon + |R_2|.$$

Interpolation of the third degree:

$$|\delta f(x)| < \frac{5}{4}\varepsilon + |R_4|.$$

Interpolation of the fifth degree:

$$|\delta f(x)| < \frac{89}{64}\varepsilon + |R_6|.$$

From this we conclude that interpolation by aid of formula (14) is much more adventageous than by *Newton's* formula; indeed, it not only dispenses with the calculation and printing of the differences, but moreover the *precision is greater.*

Interpolation in a table in which the interval is equal to h. The table containing the numbers $F(a+ih)$, to obtain $F(z)$ we choose a so as to have $a < z < a+h$. But then an interpolation of degree $2n+1$ is only possible by (14) if the table contains the numbers $F(a-nh+h)$ and $F(a+nh)$, in which case the formula is called a *mid-panel* formula.

We put into (14) $x=(z-a)/h$ and write

$$f(x) = f\left(\frac{z-a}{h}\right) = F(z) = F(a+xh).$$

Hence according to *(8)* I_k may be written

$$(15) \qquad I_k = \frac{1}{2k-1}[(x+k-1)\,F(a+kh) + (k-x)\,F(a-kh+h)].$$

I_k is therefore the result of linear interpolation between the points corresponding to $z=a-kh+h$ and $z=a+kh.$

C_m is given by (11) and the remainder will be

$$(16) \qquad R_{2n} = \binom{x+n-1}{2n} h^{2n}\,\mathrm{D}^{2n}F(a+\xi h)$$

where $-(n-1) < \xi < n.$

In the case of the mid-panel formula we have $0 < x < 1$; hence according to § 125 the remainder will be

$$(17) \qquad |R_{2n}| < h^{2n}\binom{n-\tfrac{1}{2}}{2n}\left|\mathrm{D}^{2n} F(a+\xi h)\right|.$$

The interpolation by formula (14) needs no more work of computation than *Everett's* formula, even a little less; therefore the printing of the even differences in the tables is superfluous.

Linear Interpolation. Stopping at the first term in **(14)**, we put $n=1$, and get

$$\text{(18)} \qquad F(r) = I_1 + \binom{x}{2} h^2 D^2 F(a+\xi h)$$

where $0 < \xi < 1$; moreover $x=(z-a)/h$, and

$$I_1 = xF(a+h) + (1-x)\ F(a).$$

Though the computation of I_1 is easy to make especially if a calculating machine is used, nevertheless it may be useful to indicate the shortest way to follow.

Let us denote by *A* the place on the calculating machine where the result appears, by *B* the place where the number to be multiplied is put in, and finally by C the place where the multiplicator appears when the handle is turning

First $F(a+h)$ is put into *B,* then it is multiplied by x, which number appears in C. Without reading the result in *A, we* cancel $F(a+h)$ in *B* and put in its place *F(a),* leaving the numbers in *A* and C untouched. Then we turn the handle till the number x in C becomes equal to one. The result in *A* will be equal to I_1.

The remainder is the same as that in the *Newton* series. Since $0 < x < 1$ we shall have according to (17)

$$|R_2| < \left| \frac{1}{8} h^2 D^2 F(a+\xi h) \right|.$$

In § 125 it has been shown that the error $\delta F(z)$ of $F(z)$ *is* due to two causes; first to the inexactitude of the numbers $F(a+ih)$ contained in the tables, and then to the neglect of the remainder. If the tables are computed to ν decimals then their error will be less than $\varepsilon=5/10^{\nu+1}$; moreover the resulting error will be

$$|\delta F(z)| < \frac{5}{10^{\nu+1}} + |R_2|.$$

We have seen that ε and R_2 must be of the same order of magnitude (§ 125).

Example 1. Let $F(z) = \log z$; given a logarithmic table to ν decimals from b to c, the interval being $h=1$. We shall have

$$\delta F(z) \mid < \frac{5}{10^{\nu+1}} + \frac{6}{100\, b^2}.$$

f-fence the most favourable number ν of decimals, in the case of linear interpolation, will be

b	c	ν
10	100	3
100	1000	5
1000	10000	7
10000	100000	9

7 - t

The error of $F(z)$ will not exceed one unit of the last decimal.

Example 2. Let $F(z) = 10^z$ Antilogarithmic tables of range O-1 and interval h. Since

$$R_2 \mid < \frac{h^2\ 10^{a+\xi h}}{8\ (\log e)''} \quad 7h^2.$$

From this we conclude that if the interval is equal to h the best value for the number of decimals v is

h	ν
0.01	3
0.001	5
0.0001	7

Remark. The usual seven decimal logarithmic table contains 182 pages, whereas according to what precedes a table giving the same precision by aid of linear interpolation, containing the logarithms from 1000 to 10000 to seven decimals, together with an antilogarithmic table from zero to one ($h=0.0001$) would take only 38 pages that is, hardly more than one fifth.

Example 3. Probability integral, $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt.$

Table beginning at $x=0$; interval h. We have

$$R_2 = \frac{h^2}{8}\frac{z}{\sqrt{2\pi}}e^{-z^2/2} < \frac{h^2}{8\sqrt{2\pi e}} < \frac{3h^2}{100},$$

The interval being equal to h the best number v of decimals is

h	ν
0,01	5
0,001	7

Example 4. Probability function $F(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. Table beginning from $z=0$ interval h. Since

$$|R_z| = \frac{h^2}{8}e^{-z^2/2}\frac{1}{\sqrt{2\pi}} \quad |(z^2-1)| < \frac{5h^2}{100}.$$

The best value of h corresponding to ν is given by the table of Ex. 3.

Example 5. $F(r) = \sin z$. Range of the table 0 — ½π. Interval h. We have

$$|R_2| < \frac{h^2}{8} |\sin(a+\xi h)| < \frac{h^2}{8}.$$

A. Five-decimal table. Determination of the best magnitude of the interval. According to what we have seen we should have

$$\frac{h^2}{8} \sim \frac{5}{10^6} \text{ that is } h=0{,}006324$$

If the circumference is divided into 360 degrees, then $h=0°36$ and if it is divided into 400 grades then $h=0^g4$.

Steinbecher's table (Braunschweig, 1914) in which $h=0''01$ is much too large; if we choose $h=0^g2$ the table would be twenty times as short, giving the same precision. The table would be, as we shall see, too large even for a seven-decimal table.

B. Seven-decimal tables, In the same manner we find $h=0°036$ or $0^g\,04$.

H. Brandenburg's table (Leipzig, 1923) in which $h=10$ seconds or $0°00277$ . . , is twelve times too large.

Example 6, $F(z) = \tan z$. Range of the table 0-$\frac{\pi}{4}$. Interval h. It is sufficient to consider the above range; indeed if we have

$\frac{\pi}{4} < z < \frac{\pi}{2}$ then instead of tanr we put $1/\tan(\frac{1}{2}\pi - z)$. We have

$$|R_1| < \frac{h^2}{8} \frac{2\tan(a+\xi h)}{\cos^2(a+\xi h)} < \frac{h^2}{2}.$$

Five-decimal tables. The best magnitude for h is given by

$$\frac{h^2}{2} = \frac{5}{10^6}.$$

Comparing this with the previous example, we see that now the interval must be twice smaller, that is

$$h = 10' \text{ or } 0^g2$$

and in the seven-decimal table

$$h = 1' .. \quad \text{or} \quad h = 0^g02 ..$$

Therefore we should obtain a rational five-figure trigonometric table by choosing the interval 10' or 0^g2 both for the sine and the tangent function. This would occupy only one page and a half.

The interval chosen in seven-decimal table should be 1' or 0.02 grades. This would take 15 pages

Parabolic interpolation of the third degree. Putting $n=2$ into equation (14) we obtain, if $0 < x < 1$:

$$(19) \quad F(z) = F(a+xh) = I_1 + C_1(x)[I_1 - I_2] + \binom{x+1}{4} h^4 D^4 F(a+\xi h)$$

where $-1 < \xi < 2$. The number C,(x) is given by the table mentioned. Moreover

$$I_2 = \frac{1}{3} [(x+1)\, F(a+2h) + (2-x)\, F(a-h)\,].$$

This is, as has been said, the approached value of $F(z)$ obtained by linear interpolation between $F(a-h)$ and $F(a+2h)$. The remainder is (17) :

$$(20) \qquad |R_4| < \frac{3h^4}{128} |D^4 F(a+\xi h)|.$$

The error of $F(r)$ will be, as we have seen,

$$|\delta F(z)| < \frac{5}{4}\varepsilon + |R_4|.$$

Example 7. Let $F(z) = \log z$ and $h = 1$. The table contains the logarithms of the integers from b to c, then

$$|\delta F(z)| < \frac{7}{10^{\nu+1}} + \frac{7}{100\, b^4}.$$

The most favourable value of v is given by

b	c	ν
10	100	5
32	100	7
100	1000	9
1000	10000	13
10000	100000	17
32000	100000	19

The error of $F(z)$ does not exceed one unit in the last decimal. If the table begins with 32, and z is less, then it is necessary to bring it within the range of the table by multiplying it, for instance by 3 and then log3 must be subtracted from the result.

Example 8. Let $F(z) = 10^z$, the range of the table being 0- 1 and the interval equal to h; then

$$|R_4| < \frac{3h^4}{128} \frac{10^{a+\xi h}}{(\log e)^+} < 7h^4.$$

The best value of ν corresponding to h is

h	ν
0,01	7
0,001	11
0,0001	15
0,00001	19

Remark. A 13 decimal logarithmic table could be constructed from 1000 to 10000 and a 13 decimal antilogarithmic table $(h=0.0001)$ taking together 76 pages and giving 13 exact decimals by the aid of an interpolation of the third degree. This would be the rational logarithmic table for high precision.

Example 9. $F(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt$ is given by ***Sheppard's*** tables (Pearson, Tables for Statisticians and Biometricians) where $h=0{,}01$. ***Linear interpolation.*** Putting $z=x\sqrt{2}$ we get

$$|R_2| = \frac{h^2}{8} \cdot \frac{ze^{-z^2/2}}{\sqrt{2\pi}} = \frac{h^2}{16}\left|\frac{\Phi_2(x)}{2}\right|.$$

The definition of $\Phi_n(x)$ is given by

$$\Phi_n(x) = 2D^{n-1}\frac{e^{-x^2}}{\sqrt{\pi}}.$$

The quantity $\Phi_n(x)/2^{n-1}$ is to be found at the end of ***Czuber's*** Wahrscheinlichkeitsrechnung (Vol. I) and in ***Jahnke's*** Funktionentafeln.

Linear interpolation would give in this table at the beginning 5 decimals: from $z=3.5$ up, 7 decimals; and from $z=5$ up, nine decimals exactly.

Interpolation of the third degree. We have

$$|R_4| < \frac{3h^4}{128}\frac{1}{\sqrt{2\pi}}|(z^3-3z)|\,e^{-z^2/2}.$$

Into this $z=a+\xi h$ must be put. Remarking that

$$\frac{1}{\sqrt{2\pi}}(3z-z^3)\,e^{-z^2/2} = \frac{1}{\sqrt{\pi}}(3x-2x^3)\,e^{-x^2} = \frac{\Phi_4(x)}{8}.$$

we have

$$|R_4| < \frac{h^4}{40}\left|\frac{\Phi_4(x)}{8}\right|.$$

From the tables mentioned it follows that $|\Phi_4(x)/8| < 0{,}55$ and therefore $|R| < 2/10^{10}$.

The error of $F(z)$ will not exceed one unit of the ninth decimal. From the point of view of the third degree approximation the table should contain 9 decimals.

Example 10. Let

$$F(m) = F(a+\mu h) = \psi(m,x) = \frac{m^x}{x!}e^{-m}$$

be considered as a function of *m*. ***Pearson's*** table gives $\psi(m,x)$

to six decimals for every integer value of x; the interval being equal to $\Delta m = h = 0{,}1$.*

According to formula (2) of § 148 we have

$$\underset{m}{\mathrm{D}}^n \psi(m,x) = G_n(m,x)\,\psi(m,x)$$

where $\underset{m}{\mathrm{D}}$ is the symbol of derivation with respect to m. In consequence of formula (2) and (3) § 148 it follows that

$$\underset{m}{\mathrm{D}}^n \psi(m,x) = (-1)^n \underset{x}{\Delta}^n \psi(m,x-n).$$

Therefore according to (17) the remainder of an interpolation of degree $2n-1$ will be, if $0 < x < 1$,

$$|R_{2n}| < h^{2n} \left| \binom{n-\frac{1}{2}}{2n} \underset{x}{\Delta}^{2n} \psi(a+\xi h, x-2n) \right|.$$

Linear interpolation:

$$|R_2| < \frac{h^2}{8} \left| \underset{x}{\Delta}^2 \psi(a+\xi h, x-2) \right|$$

where $0 < \xi < 1$.

R_2 may be simplified by remarking first that if $m < 1$ then $0 > \Delta\psi > -1$ and therefore $|\Delta^2\psi| < 1$. On the other hand, if $m \geqq 1$, then the maximum of ψ is reached for $x = m$; hence according to ***Stirling's*** formula we have

$$0 < \psi < \frac{1}{\sqrt{2nm}} < \frac{0{,}4}{\sqrt{m}}$$

and consequently $|\Delta\psi| < 0{,}4/\sqrt{m}$; moreover $|\Delta^2\psi| < 0{,}8/\sqrt{m}$. So that we always have $|\Delta^2\psi| < 1$.

Of course it is possible to obtain lower limits for $\Delta^2\psi$. For instance, denoting by $\psi(m,i)$ the largest of the quantities $\psi(m,x)$, $\psi(m,x-1)$ and $\psi(m,x-2)$ it follows that

$$|\Delta\psi(m,x-1)| < \psi(m,i) \quad \text{and} \quad |\Delta^2\psi(m,x-2)| < 2\psi(m,i).$$

Finally the remainder may be written, in the case of linear interpolation,

$$|R_2| < \frac{h^2}{4}\psi(m,i).$$

* ***Pearson,*** Tables for Statisticians and Biometricians.

The exactitude will not be much greater than three decimals.

Interpolation of the third degree. We have, if $0 < x < 1$

$$R_4 < \frac{3h^4}{128} \, | \, \Delta^4\psi(a+\xi h, x-4)|$$

where $-1 < \xi < 2$. Starting from $|\Delta^2\psi| < 1$ we get $|\Delta^4\psi| < 4$, and if $m \geqq 1$ from $|\Delta^2\psi| < \frac{0.8}{\sqrt{m}}$ we find $|\Delta^4\psi| < \frac{3.2}{\sqrt{m}}$; or denoting by $\psi(m,i)$ the greatest of the quantities $\psi(m,x)$, $\psi(m,x-1)$, . . . $\psi(m,x-4)$ we have $|\Delta^4\psi| < 8\psi(m,i)$.

From this it follows that

$$|R_4| < \frac{3h^4}{32} \quad \text{or} \quad |R_4| < \frac{3h^4}{40\sqrt{m}} \quad \text{or} \quad |R_4| < \frac{3h^4}{16}\,\psi(m,i).$$

This gives five exact decimals.

Example 11. Let

$$F(m) = F(a+\mu h) = \frac{1}{\Gamma(x+1)} \int_0^m e^{-t} t^x \, dt = I(u,p).$$

Pearson's Tables of the incomplete Gamma-Function give $I(u,p)$ considered as a function of u, to six decimals; the interval being equal to $\Delta u = h = 0{,}1$. Since $u = m/\sqrt{x+1}$ we have $\frac{dm}{du} = \sqrt{x+1}$ and

$$\underset{u}{D} I = \sqrt{x+1}\,\underset{m}{D} I = \sqrt{x+1}\,\psi(m,x)$$

therefore

$$\underset{u}{D}^2 I = (x+1)\,\underset{m}{D}\psi = (x+1)\, G_1\psi = -(x+1)\,\Delta\psi(m,x-1).$$

(The polynomials G_i will be introduced in § 148,)

In consequence of what we have seen in the preceding example, if $m \geqq 1$:

$$|\underset{u}{D}^2 I| < \frac{0.4(x+1)}{\sqrt{m}}$$

or in the general case

$$|\underset{u}{D}^2 I| < (x+1)\psi(m,i)$$

where $\psi(m,i)$ is the larger of the numbers $\psi(m,x)$ and $\psi(m,x-1)$.

Hence the remainder in linear interpolation will be respectively

$$|R_2| < \frac{h^2(x+1)}{20\sqrt{m}} \quad \text{and I } R_2 \text{ I} < \frac{h^2}{8}(x+1)\psi(m,i).$$

The first will be used if m is large or x small, and the second if m is small or x large.

Third degree interpolation. We find

$$\underset{u}{D}^4 I = \text{(x-l-1)"}\ G_3\psi = -(x+1)^2\,\Delta^3\,\psi(m,x-3).$$

We have seen that I $\Delta^3\psi$ | < 2; moreover $|\Delta^3\psi| < 4\psi(m,i)$ where $\psi(m,i)$ is the largest of the quantities $\psi(m,x),\ \psi(m,x-1), \ldots \psi(m,x-3)$ and if $m \geqq 1$ then I $\Delta^3\psi| < \frac{1,6}{\sqrt{m}}$

Therefore:

$$|R_4| < \frac{3h^4}{64}(x+1)^2 \quad \text{o r} \quad I\ R,\ I < \frac{3h^2}{32}(x+1)^2\psi(m,i)$$

moreover if $m \geqq 1$

$$|R_4| < \frac{3h^4(x+1)^2}{80\sqrt{m}}$$

Parabolic interpolation of the third degree will not give, in the case considered, much over five exact decimals.

Parabolic interpolation of the fifth degree by aid of formula (14). We shall have

$$F(a+xh) = I_1 + C_1(x)[I_1 - I_2 I + C_2(x)|2I_1 - 3I_2 + I_3] + R_6$$

where I_3 is, according to (15), equal to

$$I_3 = \frac{1}{5}[(x+2)F(a+3h) + (3-x)F(a-2h)].$$

This is the approached value of $F(r)$ obtained by linear interpolation between $F(a\text{-}2h)$ and $F(a+3h)$. *The* remainder is equal, according to (17) :

$$R_6 = h^6\binom{x+2}{6}D^6F(a+\xi h)$$

and therefore

$$|R_6 I < \frac{5h^6}{1024}|D^6F(a+\xi h)$$

where $0 < x < 1$ and $-2 < \xi < 3$.

The error in $F(a+xh)$ will be, as has been said,

$$|\,\delta F(a+xh)| < \frac{89}{64}\,\varepsilon + |\,R_6\,|$$

Example 12. Let $F(z) = \log z$. The table contains the logarithms of the integers from 10000 to 100000, the interval being equal to one, $h=1$. Therefore the remainder will be

$$|\,R_6\,| < \frac{5}{1024} \cdot \frac{120 \log e}{a^6} < \frac{3}{10^{25}}.$$

If the table contains the logarithms to v decimal places, then the error of $F(a+xh)$ will be

$$|\,\delta F(a+xh)| < \frac{7}{10^{\nu+1}} + \frac{3}{10^{25}}$$

Hence it is best to compute the table to 24 decimals; indeed, then the error will not exceed one unit of the last decimal.

Example 13. Let *F(m)* $= I(u,p)$. (See Example 11.) We have

$$\underset{u}{D}^6 I = (x+1)^3\, G_5\, \psi = -\,(x+1)^3\, \Delta^5 \psi(m,x-5).$$

From the preceding we deduce

$$|\,\Delta^5\psi\,| < 8\,; \text{ moreover, } |\,\Delta^5\psi\,| < 16\,\psi(m,i)$$

where $\psi(m,i)$ is the greatest of the quantities $\psi(m,x), \ldots,$ $\psi(m,x-5)$ and if $m \geqq 1$ then $|\,\Delta^5\psi\,| < \frac{6{,}4}{\sqrt{m}}$. Therefore we have

$$R_6 < \frac{5h^6}{128}\ \text{(x-l-1)''} \quad \text{or} \quad |\,R_6\,| < \frac{5h^6}{64}\,(x+1)^3\,\psi(m,i)$$

moreover if $m \geqq 1$

$$|\,R_6\,| < \frac{h^6(x+1)^3}{32\sqrt{m}}.$$

The third degree interpolation will in most cases be sufficiently exact; that is, *the* fables *should* always *be computed, ii possible either for linear or for third degree* **interpolation.**

Using a calculating machine, the shortest way to obtain a third degree interpolation is the following. First I_1 , will be computed, as has been described above. I_1 will be noted; then we compute I_2. To begin with, we put $F(a+2h)$ in *B* on the machine (p. 398) and multiply it by $(1+x)$; then cancelling $F(a+2h)$ in *B* we put in its place *F(a-h)* without touching the numbers in *A* and C; we turn the handle till $(1+x)$ in C becomes equal to 3. The result in *A* is then equal to $3I_2$. We divide it mentally by three, putting the quotient into *B* and checking it by subtracting it thrice from the number in *A* where now we must ha' : zero.

Turning the handle backwards again once, we have $-I_2$ in *A;* we add to it I_1. The difference I_1-I_2 will now figure in *A.* We remove it and put it into *B.* [There are calculating machines which permit us to transfer the numbers from *A* to *B* by turning a handle. This is useful if we have to calculate a product of three or more factors.] We multiply it by C_1 taken from the table. The product appears in *A.* We cancel the number in *B,* put there I_1 obtained before, and add. In *A we* have the number $F(z)$ desired.

Example 10. To find the value of the probability integral corresponding to $z = 0.6744898$ in *Sheppard's* tables $(h=0.01)$.

Putting $a=0{,}67$ *we* have

$$F(a-h) = 0.745\ 3731 \qquad F(a+h) = 0.751\ 7478$$
$$F(a) = 0.748\ 5711 \qquad F(a+2h) = 0{,}754\ 9029.$$

Of course it is not necessary to copy these numbers out of the table, since they can be transferred directly to the machine when they are needed.

Putting $x=0.44898$ we first determine I_1 by linear interpolation as has been described before. We note the result: $I_1 = 0.749\ 99737$, x contains more than three figures, hence C,(x) cannot be taken from the tables mentioned; it must be computed by multiplication; C, (x) $= \frac{1}{2}x(1-x)$. The result is noted too: C,(x) $= 0.123\ 6985$.

Now we determine I_1-I_2 as has been said above; then without noting it, we multiply it by C,(x) and add I_1 to the product. We find

$$F(z) = 0.750\ 00002.$$

This is exact to seven decimals.

Interpolations by parabolas of higher order than three are performed in a similar way; by calculating first the numbers I,, I,, . . . and noting the results. Secondly, by forming $\Sigma B_{m} I$, and finally multiplying them by $C_m(x)$ and adding the products. | A table giving the numbers C,,(x) is found in loc. cit. 38.|

Let us suppose that a table contains the numbers $z_1, z_2, \ldots, z_N$ and that $F(z)$ is to be determined by a parabola of degree $2n+1$ where .

$$z_m < z < z_{m+1},$$

If moreover

$$n+1 \leqq m \leqq N-n-1$$

that is, if there are $n+1$ points on each side of z; then, putting z, = a and $(z-a)/h=x$ we have $0 < x < 1$. The corresponding formula will be termed, as has been said, a ***mid-panel*** formula.

If

$$m < n+1$$

?hen we put $z_{n+1}=a$ and we shall have $x < 0$. The corresponding formula is a ***first-panel*** formula, which will serve for all interpolations of degree $2n+1$ if $z < z_{n+1}$.

If

$$N-n-1 < m$$

then we put $a=z_{N-n-1}$ and we shall have $x > 1$; then formula is an ***end-panel*** formula; the same will serve for all interpolation of degree $2n+1$ if $z > z_{N-n-1}$.

Remark. In the case of the ***first-panel*** and of the ***end-panel*** formulae the maximum of the remainder (16) will be greater than that given by (17). $x < 0$ or $x > 1$; therefore the corresponding values of C,,(x) are not in the tables mentioned , and they must be calculated in each case.

Conclusions concerning the computation of tables. A rational table should always be calculated taking account of the interpolation formula to be used. For instance:

A ***five-decimal table*** for linear interpolation by (18) should contain the logarithms of the integers from 100 to 1000 and the antilogarithms with an interval 0.01. Four pages altogether.

A ***seven-decimal table*** for linear interpolation by (18) should contain the logarithms of the integers from 1000 to 10000 and the antilogarithms with the interval 0.001 (38 pages).

For greater precision a ***13 decimal table*** intended for interpolation of the third degree by (19) should contain the logarithms of the integers from 1000 to 10000 and the antilogarithms with an interval of 0.0001 (76 pages).

A 19 ***decimal table*** for third degree interpolation containing the logarithms of the integers from 32000 to 100000. ***(Thompson's*** Logarithmica Britannica could serve for this.)

Finally for an extreme precision a ***table of 28 decimals*** could be constructed for interpolation of the fifth degree containing the numbers from 32000 to 100000.

Generally, in order to obtain a given precision of ν decimals, the intervals could be chosen greater at the end of the table than at the beginning The table may be shortened in this way too. Or if the intervals are the same throughout the table, then at the end more decimals can be given than at the beginning. This has been done in ***Sheppard's*** table of the probability function.

Problem. Sometimes if F(u) is determined by aid of a parabola of degree $2n-1$, the tangent to the point of coordinates $u, F(u)$ is also required. For this a knowledge of $\mathsf{D}F(u)$ is necessary. Putting $x=(u-a)/h$ into formula (14)

$$F(u) = \sum_{m=0}^{n} C_m(x) \sum_{\nu=1}^{m+2} B_{m\nu} I_\nu . \tag{21}$$

Since C,(X) $=1$, we get

$$\underset{u}{\mathsf{D}}F(u) = \sum_{m=1}^{n} \frac{1}{h} \underset{x}{\mathsf{D}} C_m(x) \sum_{\nu=1}^{m+2} B_{m\nu} I_\nu +$$

$$+ \sum_{m=0}^{n} C_m(x) \sum_{\nu=1}^{m+2} B_{m\nu} \frac{1}{h} \underset{x}{\mathsf{D}} I_\nu$$

but from formula (15) it follows that

$$\underset{x}{\mathsf{D}} I_\nu = \frac{1}{2\nu-1} [F(a+\nu h) - F(a-\nu h+h)].$$

Taking account of the value of $B_{m\nu}$ given by (10), we may write

$$\sum_{\nu=1}^{m+2} B_{m\nu} \underset{x}{\mathsf{D}} I_\nu = \sum_{\nu=1}^{m+2} (-1)^{\nu+1} \binom{2m+1}{m+\nu} (\mathsf{E}^{\nu} - \mathsf{E}^{-\nu+1}) \frac{F(a)}{2m+1}.$$

It can be shown that

$$\sum_{\nu=1}^{m+2} (-1)^{\nu+1} \binom{2m+1}{m+\nu} (\mathsf{E}^{\nu} - \mathsf{E}^{-\nu+1}) = (-1)^m \mathsf{E}^{-m} \Delta^{2m+1}.$$

Therefore

$$(22) \qquad \sum_{\nu=1}^{m+2} B_{m\nu} \underset{x}{\mathsf{D}} I_{\nu} = \frac{(-1)^m}{2m+1} \Delta^{2m+1} F(a-mh)$$

The derivative of C,,(x) is obtained by the aid of ***Stirling's*** numbers (24).

Parficutar case of the third degree (formula 19). Neglecting the remainder, the derivative of ***F(u)*** will, in consequence of (22), be

$$\underset{u}{\mathsf{D}} F(u) = \frac{1}{h} [(\frac{1}{2} - x)(I_1 - I_2) + \Delta F(a) - \frac{1}{3} C_1(x) \Delta^3 F(a-h)].$$

The μ -th derivative of ***F(u)*** can be obtained by aid of ***Leibnitz's*** formula. Starting from (21) and remarking that $\underset{x}{\mathsf{D}}^2 I_{\nu} = 0$, we have

$$(23) \qquad \underset{u}{\mathsf{D}}^{\mu} F(u) = \frac{1}{h^{\mu}} [\sum_{m=1}^{n} \underset{x}{\mathsf{D}}^{\mu} C_m(x) \sum_{\nu=1}^{m+2} B_{m\nu} I_{\nu} +$$

$$+ \mu \sum_{m=0}^{n} \underset{x}{\mathsf{D}}^{\mu-1} C_m(x) \sum_{\nu=1}^{m+2} B_{m\nu} \underset{x}{\mathsf{D}} I_{\nu}]$$

where

$$(24) \qquad \underset{x}{\mathsf{D}}^{\mu} C_m(x) = (-1)^m \underset{x}{\mathsf{D}}^{\mu} \binom{x+m-1}{2m} =$$

$$= \frac{(-1)^m}{(2m)!} \sum_{i=1}^{2m+1} (i)_{\mu} (x+m-1)^{i-\mu} S^i_{2m} .$$

§ 133, Inverse interpolation by aid of the formula of the preceding paragraph. In § 132 we found

$$(1) \qquad F(z) = F(a+xh) = \sum_{m=0}^{n} C_m \sum_{k=1}^{m+2} B_{mk} I_k +$$

$$+ h^{2n} \binom{x+n-1}{2n} \mathsf{D}^{2n} F(a+\xi h)$$

where $-(n-1) < \xi < n$ or $0 < \xi < x$; and

$$(2) \qquad I_k = \frac{x+k-1}{2k-1} F(a+kh) + \frac{k-x}{2k-1} F(a-kh+h).$$

To begin with, we determine by aid of the remainder in formula (1), the number n of terms necessary to obtain the prescribed precision. Generally $n=2$, corresponding to a third degree parabola, will be sufficient.

If $F(Z)$ is given, and the corresponding value of Z is to be determined by aid of a table containing the numbers $F(z)$ corresponding to equidistant values of z (the increment of z being equal to h) then we choose $F(a)$ in the table, so as to have

$$(3) \qquad F(a) < F(Z) < F(a+h)$$

or

$$F(a) > F(Z) > F(a+h).$$

The method is similar in both cases. We will suppose that the first inequality is satisfied.

Now we compute z_1 the first approximation of Z, by linear inverse interpolation between a and $a+h$, using *Lagrange's* formula (12) of § 131. Putting into it $a+h$ instead of 6, we get

$$(4) \qquad z_1 = a + \frac{h[F(Z)-F(a)]}{F(a+h)-F(a)} + \frac{h(a+h-Z)(Z-a)\,D^2F(a+\xi h)}{2[F(a+h)-F(a)]}$$

where $0 < \xi < 1$.

The maximum of the absolute value of the remainder $\mathcal{R}$ will be

$$|\mathcal{R}| < \left|\frac{h^3 D^2 F(a+\xi h)}{8[F(a+h)-F(a)]}\right| .$$

Denoting by ε the precision of the numbers $F(z)$ contained in the table, for instance $\varepsilon = 5/10^{\nu+1}$ (ν exact decimals), the absolute value of the error of z_1 produced by the inexactitude of the numbers in the table will be:

$$\left|\frac{h\varepsilon}{F(a+h)-F(a)}\right| \left\{1 + 2\left|\frac{F(Z)-F(a)}{F(a+h)-F(a)}\right|\right\} < \left|\frac{3h\varepsilon}{F(a+h)-F(a)}\right|$$

therefore the absolute value of the total error of z_1 is

$$|\delta z_1| < \left|\frac{3h\varepsilon}{F(a+h)-F(a)}\right| + |\mathcal{R}| < \frac{1}{10^{r_1}} = h_1 .$$

Neglecting the remainder in formula (4) we determine z_1 keeping only the ν_1 exact decimals. If Z > 0 we necessarily have

$$a < z_1 < Z < a + h.$$

Then adding one unit of the ν_1 -th decimal, that is $h_1 = 1/10^{\nu_1}$, to z_1 we have

$$z_1 < Z < z_1 + h_1 . \tag{5}$$

Now by aid of (1) we determine the values of $F(z_1)$ and $F(z_1 + h_1)$, putting first into this equation $x = (z_1 - a)/h$ and then $x = (z_1 + h_1 - a)/h$, using the necessary n terms for the required precision, In consequence of (3) and (5) we shall have

$$F(z_1) < F(Z) < F(z_1 + h_1).$$

Then we proceed to the determination of z_2, the second approximation of Z, obtained again by linear inverse interpolation, but between z_1 and $z_1 + h_1$. We find

$$z_2 = z_1 + \frac{h_1 [F(Z) - F(z_1)]}{F(z_1 + h_1) - F(z_1)} + \frac{(z_1 + h_1 - Z)(Z - z_1) D^2 F(a + \xi h)}{2 [F(z_1 + h_1) - F(z_1)]} . \tag{6}$$

The remainder $\mathscr{R}_1$ will be

$$|\mathscr{R}_1| < \left| \frac{h_1^3 D^2 F(a + \xi h)}{8 [F(z_1 + h_1) - F(z_1)]} \right|$$

Moreover the absolute value of the error of $F(z_1)$ or $F(z_1 + h_1)$ [caused by the inexactitude of the numbers $F(z)$ contained in the table, if the interpolation executed was of the third degree], being less than $\frac{5}{4}\varepsilon$, if follows, in consequence of what has been said above, that

$$|\delta z_2| < \left| \frac{3 h_1 \varepsilon}{F(z_1 + h_1) - F(z_1)} \right| + |\mathscr{R}_1| < \frac{1}{10^{\nu_2}} = h_2 . \tag{7}$$

We keep in z_2 only the ν_2 exact decimals. If a greater precision is necessary, then adding one unit of the ν_2 -th decimal, that is $h_2 = 1/10^{\nu_2}$ to z_2, we find

$$z_2 < Z < z_2 + h_2 .$$

Now we determine $F(z_2)$ and $F(z_2 + h_2)$ by (1) in the same way as $F(z_1)$ before. Then starting from

$$F(z_2) < F(Z) < F(z_2+h_2)$$

we determine z_3 the third approximation of Z, by writing into (6) respectively z_3, z_2, h_2 instead of z_2, z_1 , h_1 ; the error δz_3 is given by (7) in the same manner. If δz_3 is negligible compared with the prescribed precision, or if $\mathscr{R}_2$ is smaller than the first part of the error δz_3 due to the inexactitude of the numbers of the table, then the problem is solved; if not, we continue the proceeding as described before.

Example 1. To compare this method of inverse interpolation with that of ***Thompson,*** by aid of **Everett's** formula, using the even differences, we will again choose ***Thompson's*** example. Given

$$F(Z) = \log Z = \bar{1}.95717\ 32271\ 83589\ 39035$$

Z is to be determined by ***Thompson's*** logarithmic table to twenty decimals. In these tables we find

$$F(a) = \bar{1}.95717\ 13373\ 70099\ 19928\ , \qquad a = 0{,}90609$$

$$F(a+h) = \bar{1}.95717\ 61304\ 04846\ 19226\ , \qquad a+h = 0{,}90610$$

therefore $h = 1/10^5$; moreover, $AF(a) > 4/10^6$ and $\varepsilon = 5/10^{21}$.

Starting from these values, we determine first the maximum of the remainder in formula (4). We get

$$|\mathscr{R}| < \frac{h^3}{16a^2\Delta F(a)} < \frac{5}{10^{11}}\,.$$

Hence

$$\delta z_1| < \frac{4}{10^{20}} + \frac{5}{10^{11}} < \frac{1}{10^{10}} = h_1$$

therefore the first ten decimals of z, will be exact. From (4) we obtain, neglecting the remainder:

$$z_1 = 0.90609\ 39428,$$

Since the last decimal is exact, hence $z_1 < Z$ and adding one unit to the tenth decimal of z_1 we have $z_1+h_1 \sim 0.90609\ 39429;$ moreover,

$$z_1 < Z < z_1+h_1\,.$$

Before computing $F(z_1)$ and $F(z_1+h_1)$ by aid of formula (1) we must determine the number n of terms necessary to obtain a precision of 20 decimals. For this we determine the remainder of (1) in the case of $n=2$, and find

$$|R_4| < \frac{3h^4}{128} \frac{6 \log e}{a^4} < \frac{2}{10^{21}},$$

hence a third degree interpolation is sufficient; so that we have only to determine I_1, I_2 and C, (x).

For this we put first into equation (1) $x = \frac{z_1-a}{h} = 0{,}39428$ and $k=1$; we find

$$I_1 = 0.39428\,F(a+h) + 0.60572\,F(a) =$$
$$= \bar{1}.95717\ 32271\ 67839\ 24368.$$

Remark. Before performing the multiplications by $F(a+h)$ and $F(a)$, since the sum of the factors x and (l-x) is equal to unity, the first six figures common to $F(a)$ and $F(a+h)$ *have* been set aside and only added to the result.

In the same way, putting into formula (2) $x=0{,}39428$ and $k=2$ *we* get

$$I_2 = \frac{1.39428}{3} F(a+2h) + \frac{1.60572}{3} F(a-h) =$$
$$= \bar{1}.95717\ 32271\ 14941\ 49591$$

$F(a+2h)$ and $F(a-h)$ were taken out of the logarithmic table; and before the multiplications, again, the five common figures were set aside. From the above results we deduce

$$I_1 - I_2 = 0.00000\ 00000\ 52897\ 74777;$$

moreover

$$C_1\,(x,) = (0.39428)\,(0.30286) = 0.11941\ 16408$$

and

$$C,\,(x,)\,|I_1 - I_2| = 0.00000\ 00000\ 06316\ 60686$$

and finally

$$F(z_1) = \bar{1}.95717\ 32271\ 74155\ 85054.$$

We had $F(a) < F(Z) < F(a+h)$ and $z_1 < Z$; therefore we must have $F(z_1) < F(Z)$, and this is what we really find.

Now to obtain $F(z_1+h_1)$ we put into (1) $x=0.39429$ and get

$$I_1' = 0.39429\, F(a+h) + 0.60571\, F(a) =$$

$$= \bar{1}.95717\ 32272\ 15769\ 59114$$

and

$$I_2' = \frac{1.39429}{3} F(a+2h) + \frac{1.60571}{3} F(a-h) =$$

$$= \bar{1}.95717\ 32271\ 62871\ 84338$$

therefore

$$I_1' - I_2' = 0.00000\ 00000\ 52897\ 74776.$$

Since

$$C_1(0.39429) = 0.11941\ 26979\ 5$$

it follows that

$$C_1(I_1' - I_2') = 0.00000\ 00000\ 06316\ 66278$$

and finally

$$F(z_1 + h_1) = \bar{1}.95717\ 32272\ 22086\ 25392.$$

As was to be expected, we have

$$F(z_1) < F(Z) < F(z_1 + h_1).$$

Now we shall determine the maximum of the remainder corresponding to formula (6). We find

$$|\mathcal{R}| < \frac{h_1^3}{16a^2|F(z_1+h_1) - F(z_1)|} < \frac{2}{10^{21}}.$$

Therefore the error caused by neglecting the remainder will be less than one unit of the 20 -th decimal. The problem is solved. The error due to the inexactitude of the numbers of the table cannot be overcome, and we shall have in consequence of (7)

$$|\delta z_2| < \frac{4}{10^{20}} + \frac{2}{10^{21}}$$

z_2 will be determined by (6), neglecting the remainder. Since we have

$$F(Z) - F(z_1) = 0.00000\ 00000\ 09433\ 53981$$

and

$$F(z_1+h_1) - F(z_1) = 0.00000\ 00000\ 47930\ 40338;$$

therefore

$$z_2 = 0.90609\ 39428\ 19681\ 74509.$$

Remark. Taking account of the fact that $\log Z$ is equal to log (e/3) the error being less than $5/10^{21}$, it follows that the difference between z_2 and $e/3$ should not exceed $4/10^{20}$; and really this difference is equal to $3/10^{20}$.

§ 134. Precision of the **interpolation formulae,** In § 125 we have seen that the precision of an interpolation formula of degree $2n-1$ having the same remainder may be measured by the maximum of the possible error caused by the inexactitude of the numbers in the table, that is by $\omega\varepsilon$, if. the error of the data in the tables is less than $\varepsilon = 5/10^{\nu+1}$, and if ω is the sum of the absolute values of the coefficients figuring in the formula. Therefore, to compare the different formulae we have to determine the corresponding values of ω.

1. In the case of *Newton's* formula (§ 125) we find:

$$\omega_1 = \sum_{m=0}^{2n} \left|\binom{x}{m}\right| = 1 - \sum_{m=1}^{2n} (-1)^m \binom{x}{m};$$

if $0 \leqq x \leqq 1$, the series below is then convergent and

$$\sum_{m=0}^{\infty} (-1)^m \binom{x}{m} = 0$$

therefore tie have $\omega_1 \leqq 2$.

2. In the first Gauss series we had (2, § 127)

$$\omega_2 = \sum_{m=0}^{2n} \left[\left|\binom{x+m-1}{2m}\right| + \left|\binom{x+m}{2m+1}\right|\right] =$$

$$= \sum_{m=0}^{2n} \left|\binom{x+m-1}{2m}\right|\left(1 + \frac{x+m}{2m+1}\right)$$

moreover in the second *Gauss series* (6, § 127)

$$\omega_3 = \sum_{m=0}^{2n} \left[\left|\binom{x+m}{2m}\right| + \left|\binom{x+m}{2m+1}\right|\right]$$

From this we immediately deduce that for every value of x and m we have $\omega_3 > \omega_2$. Indeed from the preceding it follows that

$$x+m > m-x \quad \text{or} \quad x > 0;$$

if this is satisfied, the maximum of the error of the first **Gauss** formula is less than that of the second of the same degree; thus the former is preferable,

3. From Bessel's formula (4, § 128) we obtain

$$\omega_4 = \sum_{m=0}^{2n} \left|\binom{x+m-1}{2m}\right| \left(1 + \left|\frac{x-\frac{1}{2}}{2m+1}\right|\right).$$

It is easy to see that if $x > \frac{1}{2}$ then $\omega_4 < \omega_2$. Indeed for every value of m this gives $x-\frac{1}{2} < x + m$. But even if $\frac{1}{4} < x < \frac{1}{2}$, then we have $\frac{1}{2} - x \leqq x + m$. *So* that if $x > \frac{1}{4}$ then *Bessel's* formula is preferable to the Gauss formulae.

4. *Stirling's* formula gives (§ 128)

$$\omega_5 = \sum_{m=0}^{2n} \left[\left|\binom{+m-1}{2m} \frac{x}{m-x}\right| + \left|\binom{x+m}{2m+1}\right|\right].$$

From what precedes, we may easily deduce that *Stirling's* formula is preferable to the Gauss' formulae, if $0 < x < \frac{1}{2}$.

5. In the case of Everett's formula we get (§ 129) if $0 < x < 1$

$$\omega_6 = \sum_{m=0}^{2n} \left|\binom{m+x-1}{2m}\right|.$$

From this it follows that $\omega_2 > \omega_6$.

7. The interpolation formula dispensing with printed differences of § 132 will give

$$\omega_7 = \sum_{m=0}^{2n} \binom{2m}{m} \left|\binom{m+x-1}{2m}\right|.$$

Other conclusions may be obtained by comparing the inter: polation formulae in some particular cases.

A. Linear interpolation. The *Newton,* the two Gauss and the *Stirling* formulae give

$$0 = 1 + x .$$

To *Bessel's* formula corresponds

$$\omega = 1 + |x - \tfrac{1}{2}|$$

therefore if $x < \frac{1}{4}$ it will lead to results inferior to the former formula, and if $x > \frac{1}{4}$ to better ones.

Everett's formula and that of § 132 give $\omega = 1$; therefore these are the most advantageous.

***B. Interpolation** of fhe third* degree. We obtain the following values:

1. Newton: (if $0 < x < 1$)

$$\omega_1 = 1 + x + \left|\binom{x}{2}\right| + \left|\binom{x}{3}\right| \leqq 2.$$

2. Gauss I:

$$\omega_2 = 1 + x + \left|\binom{x}{2}\right| + \left|\binom{x+1}{3}\right|.$$

3. ***Bessel:***

$$\omega_4 = 1 + x - 1/2 + \left|\binom{x}{2}\right| + \left|\binom{x}{2}\frac{x-1/2}{3}\right|.$$

5. ***Stirling:***

$$\omega_5 = 1 + x + 1/2 x^2 + \left|\binom{x+1}{3}\right|.$$

6. ***Everett:***

$$\omega_6 = 1 + \left|\binom{x}{2}\right|.$$

7. ***Formula of § 132:***

$$\omega_7 = 1 + x(1-x).$$

From the above we conclude that in the case of an interpolation of the third degree the error is generally the smallest when ***using Everett's*** formula.

Moreover if $x > 1/2$ then ***Newton's*** formula is preferable to the Gauss formula I. If $x = 1/2$, then both lead to the same result; and finally, if $x \leqq 1/2$, then Gauss' formula is better.

Comparing ***Newton's*** formula with that of ***Stirling,*** we find exactly the above result.

On the ***other*** hand, comparing ***Newton's*** formula with ***Bessel's,*** we are led to $x^2 - 9x + 2 > 0$; therefore if $x = 0{,}278$ both formulae give the same result; ***Newton's*** formula is superior or inferior to ***Bessel's*** as x is smaller or larger than 0.278.

Comparing ***Stirling's*** formula with ***Bessel's,*** we find that both lead to the same result if $x = 1/4$ moreover that ***Stirling's*** formula is preferable to ***Bessel's*** if $x < 1/4$.

The formula of § 132 gives better results than the Gauss, ***kewton,*** and ***Stirling*** formulae. Comparing it with ***Bessel's*** formula, we find that if $x = 0{,}386$ or if $x = 0{,}613$, both formulae give the same result; and that ***Bessel's*** formula is preferable if $0.386 < x < 0.613$; for the other values of x the formula of § 132 is better.

This is another reason why the tables constructed for interpolation by aid of the formula of § 132 may be shorter than those in which the interpolation is to be made by *Newton's, Gauss'* or *Stirling's* formulae.

§ 135. General Problem of Interpolation. *A. The Function and some of its Differences are given for certain values of the variable.*

We have seen that if a function $f(x)$ and its differences are given for $x=a$ then the function may be expanded into a *Newton* series, Moreover if the function $f(x)$ and its even differences are given for $x=a$ and for $x=a+h$, then the function may be expanded into an *Everett* series.

Now we shall treat the general problem. The function $f(x)$ is given for $x=a_0, a_1, a_2, \ldots, a_i$; moreover some of the differences

$$\Delta f(a_i), \Delta^2 f(a_i), \ldots, \Delta^{\nu_i-1} f(a_i)$$

are given. Let us suppose that

$$\nu_0+\nu_1+\nu_2+\nu_3+\ldots+\nu_m=n.$$

that is, there are given n quantities in all.

We may obtain the required interpolation formula of $f(x)$ satisfying the above conditions by aid of a *Newton* series. Let us write

$$f(x)=f(0)+\binom{x}{1}\Delta f(0)+\binom{x}{2}\Delta^2 f(0)+\ldots+\binom{x}{n-1}\Delta^{n-1} f(0)$$

and

$$\Delta^{\mu} f(x)=\Delta^{\mu} f(0)+\binom{x}{1}\Delta^{\mu+1} f(0)+\ldots+\binom{x}{n-1-\mu}\Delta^{n-1} f(0)$$

Putting into these equations the n given values

$$\Delta^{\mu_i} f(a_i) \text{ for } \mu_i=0, 1, 2, \ldots, (\nu_i-1),$$

we get n equations, which determine the n unknowns:

$$\Delta^k f(0) \quad \text{for} \quad k=0, 1, 2, \ldots, n-1.$$

The remainder of the series will be that of a *Newton* series stopped at the term Δ^{n-1}. Hence the interpolation formula will be

$$f(x)=f(0)+\binom{x}{1}\Delta f(0)+\ldots+\binom{x}{n-1}\Delta^{n-1} f(0)+\binom{x}{n}D^n f(\xi)$$

where $0<\xi<n-1$ or $0<\xi<x$.

B. The Function and some of its Derivatives are given for certain values of the variable.

If a function f(x) and its derivatives are given for $x = a$ then the function may be expanded into a *Taylor series.*

If the function f(x) is given for $x = a_0, a_1, \ldots, a_i$ moreover if $Df(a_i), D^2f(a_i), \ldots, D^{\nu_i - 1}f(a_i)$ are given too, then putting

$$\nu_0 + \nu_1 + \nu_2 + \ldots + \nu_i = n$$

we may expand the function f(x) into a *Taylor* series. Writing

$$f(x) = f(0) + xDf(0) + \frac{x^2}{2!}D^2f(0) + \ldots + \frac{x^{n-1}}{(n-1)!}D^{n-1}f(0)$$

and

$$D^{\mu}f(x) = D^{\mu}f(0) + xD^{\mu+1}f(0) + \ldots + \frac{x^{n-1-\mu}}{(n-1-\mu)!}D^{n-1}f(0).$$

Putting into these equations the n given values

$$D^{\mu_i}f(a_i) \quad \text{for} \quad \mu_i = 0, 1, 2, \ldots, \nu_i - 1$$

we may determine the n unknowns:

$$D^k f(0) \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1,$$

The remainder of this expansion will be that of the *Taylor* series which has been stopped at the term $D^{n-1}f(0)$. Hence the required formula will be

$$f(x) = f(0) + xDf(0) + \ldots + \frac{x^{n-1}}{(n-1)!}D^{n-1}f(0) + \frac{x^n}{n!}D^n f(\xi)$$

where $0 < \xi < x$.

CHAPTER VIII.

APPROXIMATION AND GRADUATION.

§ 136. Approximation according to the principle of moments. When solving the problem of interpolation, f(x) was given for $x = 0, 1, 2, \ldots, n$, and we determined a curve of degree n passing through the points of coordinates $x, f(x)$.

The corresponding problem of approximation is the following: The points of coordinates $x, y = f$ (x) are given for $x = 0, 1, 2, \ldots,$ N-l and a function F(x) satisfying certain conditions is to be determined so that the deviations

$$\varepsilon = F(x) - y$$

shall be, according to some principle, the smallest.

Such a principle is for instance the *principle of least squares,* according to which, a function *F(x)* containing disposable parameters being given, the parameters must be determined so that

$$\mathscr{S} = \sum_{x=0}^{N} \varepsilon^2 = \sum_{x=0}^{N} [F(x) - y]^2$$

shall be a minimum.

A second *principle* of approximation is that of the *moments.* Let us denote by

$$(1) \qquad \mathscr{M}_m = \sum_{x=0}^{N} x^m y.$$

This is the m -th power-moment of y; given a function $F(X)$ containing $n+1$ disposable parameters, these must be determined in such a manner that the moments $\mathscr{M}_0, \mathscr{M}_1, \ldots, \mathscr{M}_n$ of y shall be identical with the corresponding moments of *F(x).*

There are other principles of approximation.

A second case of approximation is the following: a function f(x) of the continuous variable x is given, another function F(x) satisfying certain conditions is to be determined so that the deviations of the two curves shall be the smallest according to some principle.

For instance, according to the principle of least squares, F(x) containing disposable parameters being given, these are to be determined so that

$$\mathcal{S} = \int_{-\infty}^{\infty} [f(x) - F(x)]^2 \, dx$$

shall be a minimum.

On the other hand, according to the principle of moments, if

$$\mathcal{M}_m = \int_{-\infty}^{\infty} x^m f(x) \, dx \tag{2}$$

is given for $m = 0, 1, 2, \ldots,$ n; the parameters of F(x) are to be determined so that the first $n+1$ moments of F(x) shall be equal to the corresponding moments of t(x) given by (2).

The simplest case of approximation is that in which F(x) is a ***polynomial*** of degree n containing n+l disposable coefficients. Let us suppose first that the variable is discontinuous, x = 0, 1, 2, . , . , N-l, and that

$$F(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

According to the method of least squares the equations determining the parameters a_i will be

$$\frac{\partial \mathcal{S}}{\partial a_\nu} = -2 \sum_{x=0}^{N} [f(x) - F(x)] \, x^\nu = 0$$

therefore we shall have

$$\sum_{x=0}^{N} x^\nu f(x) = \sum_{x=0}^{N} x^\nu F(x) \tag{3}$$

for $\nu = 0, 1, 2, \ldots, n.$ But these are also the equations determining the parameters if the principle of moments is applied.

From this we conclude that ***if F(x) is a polynomial the principles of least squares and of moments lead to the same resulf,*** and therefore if the polynomial F(x) is expanded into a series of any polynomials whatever, the method of least squares will always lead to the same result.

Consequently we shall choose the expansion which will require the least work of computation, This will happen if F(x) is expanded into a series of orthogonal polynomials.

Remark. 1. The above results are also true in the case of a continuous variable.

If in order to obtain an approximation the principle of moments is chosen, it is often adventageous to introduce, instead of the power-moments given by formula (1), the *factorial-moments* $\mathfrak{M}_s$ or the *binomial-moments* $\mathcal{B}_s$, given by the following definition:

$$(5)\qquad \mathfrak{M}_s = \sum_{x=0}^{\infty} (x)_s f(x) \qquad \mathcal{B}_s = \sum_{x=0}^{\infty} \binom{x}{s} f(x).$$

The approximation obtained will be the same whatever the chosen moments are, but often the calculus needed is much simpler in the case of binomial-moments than in that of power-moments. Moreover the computation of the binomial-moments is shorter than that of the power moments, as is shown in § 144.

Remurk. 2. If $u(t)$ the generating function of $f(x)$ is known, then we have

$$u(f) = \sum_{x=0}^{\infty} f(x)\ t^x \quad \text{and} \quad \mathbf{D}^s u(t) = \sum_{x=0}^{\infty} (x)_s f(x)\ t^{x-s}$$

so that

$$\mathfrak{M}_s = [\mathbf{D}^s u(t)]_{t=1} \qquad \mathcal{B}_s = \left[\frac{\mathbf{D}^s u(t)}{s!}\right]_{t=1}$$

On the other hand, if the binomial-moments are known we may determine the generating function of $f(x)$ by *Taylor's* formula

$$u(t) = \sum_{s=0}^{\infty} (t-1)^s \mathcal{B}_s.$$

Example. The binomial moments of the probability function $F(x) = \binom{n}{x} p^x (1-p)^{n-x}$ are $\mathcal{B}_s = {}_{0S}^{\,n}\ p^s$ and therefore the generating function of $F(x)$ will be

$$u(t) = \sum_{s=0}^{\infty} \binom{n}{s} [tp-p]^s = (1-p+tp)^n.$$

Remark. 3. The principle of approximation applied is nearly always that of the moments; indeed, the other principles, for instance that of the least squares, or *Fisher's* principle of likelihood, are as a rule used only in cases when they lead to the same result as the principle of moments. The reason for this is not that the principle of moments is more in agreement with *our* idea of approximation than the others (indeed from this point of view, the first place belongs to the principle of least squares), but that the calculus is the simplest in the case of the principle of moments.

If the principle of approximation is chosen, we have still to choose the approximating function $F(x)$ containing disposable parameters. In this, the interval in which x varies, the values of the function f(x) to be approximated, at the beginning and at the end of this interval, finally the maxima and minima of $f(x)$ play the most important parts.

In the case of the principle of least squares, it is the mean-square-deviation that measures the approximation obtained. In the case of the moments, we have not so practical a measure, but we may proceed as follows: If the function $F(x)$ has been determined so that its first $n+1$ power-moments shall respectively be equal to the corresponding moments of $f(x)$, that is to $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n$ then to measure the obtained approximation we have to compare $\Sigma x^{n+1} F(x)$ with $\mathcal{M}_{n+1}$; the less the difference is, the better the approximation may be considered.

Remark. 4. If the function $f(x)$ is expanded into a series

$$c_0 + c_1\varphi_1(x) + c_2\varphi_2(x) + \ldots + c_m\varphi_m(x) + \ldots$$

and if we stop at the term $c_n\varphi_n(x)$ it may happen that the coefficients c_m of this expansion are the same as those we should have obtained by determining them with the aid of the principle of moments, putting the given moments $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n$ of f(x) equal to the corresponding moments of

$$\Phi_n(x) = c_0 + c_1\varphi_1(x) + \ldots + c_n\varphi_n(x).$$

This will occur:

1. If $\varphi_m(x)$ is any polynomial whatever of degree m; for instance a *Legendre* polynomial, if the variable is continuous

(§ 138), or an orthogonal polynomial of §§ 139—141, if the variable is discontinuous ($x = 0, 1, 2, \ldots, N-1$).

2. If the variable x is continuous, and if $\varphi_m(x) = H_m e^{-x^2/2}$ where H_m signifies the Hermite polynomial of degree m (§ 147).

3. If the variable is discontinuous $x = 0, 1, 2, \ldots$ and if $\varphi_m(x) = G_m \dfrac{e^{-m} m^x}{x!}$, where G_m is the polynomial defined in § 148.

This adds to the importance of the principle of moments.

On the other hand it may happen that, stopping the expansion of $f(x)$ at the term $c_n\varphi_n(x)$, the coefficients c_n are the same as those we should have obtained by determining them according to the principle of least squares by making minimum either the sum or the integral of the quantity

$$[f(x) - \Phi_n(x)]^2$$

according as the variable x is discontinuous or continuous.

This will occur:

1. If $\varphi_m(x)$ is any polynomial whatever of degree m.

2. If the variable x is continuous and if $f(x)$ is expanded into a *Fourier* series (§ 145), in the interval (0, 1), and

$$c_m\varphi_m(x) = a_m \cos 2\pi mx + \beta_m \sin 2\pi mx.$$

3. If the variable is discontinuous, and if $f(x)$ is expanded into the trigonometrical series of § 146, where

$$c_m\varphi_m(x) = a, \cos\frac{2\pi mx}{N} + \beta_m \sin\frac{2\pi mx}{N}.$$

This is in favour of the principle of least squares.

§ 137. Examples of the function *F(x)* chosen.

Example 1. Function with two disposable parameters. Continuous variable. The range of x extends from 0 to ∞. If $f(0) = f(\infty) = 0$ and $f(x) \geqq 0$, then we may try to approximate the function $f(x)$ by aid of

$$F(x) = C\frac{e^{-x}x^p}{\Gamma(p+1)}. \tag{1}$$

We have

$$\int_0^\infty F(x)dx = C \qquad \int_0^\infty xF(x)dx = C\frac{\Gamma(p+2)}{\Gamma(p+1)} = (p+1)C$$

Let us write

$$\mathcal{M}_i = \int_0^{\infty} x^i f(x)dx.$$

According to the principle of moments we must put $C = \mathcal{M}_0$ and $(p+1)C = \mathcal{M}_1$; so that the mean of x will be $\mathcal{A}$ (x) $= p+1 = \mathcal{M}_1/\mathcal{M}_0$. The formula (1) presupposes that $p > 0$; therefore we must have $\mathcal{A}(x) > 1$. Writing in (1) $C = \mathcal{M}_0$ and $p = \mathcal{A}(x) - 1$ we get an approximation of $f(x)$.

To measure the obtained approximation let us determine

$$\mathcal{M}_2 - \int_0^{\infty} x^2 F(x)dx = \mathcal{M}_2 - (p+2)(p+1)C = \mathcal{M}_2 - (p+2)\mathcal{M}_1$$

The less this quantity is, the better the approximation,

Let us determine moreover the mean-square deviation σ corresponding to $f(x)$, σ^2 being the mean of $[x - \mathcal{A}(x)]^2$; therefore

$$\sigma^2 = \frac{\mathcal{M}_2}{\mathcal{M}_0} - \left(\frac{\mathcal{M}_1}{\mathcal{M}_0}\right)^2$$

on the other hand, starting from $F(x)$ we get $\sigma^2 \sim p+1 = \mathcal{A}(x)$.

Since the function (1) is maximum for $x = p$, if $f(x)$ is maximum for x=x,, then the approximation will be useful only if approximately $x_m = \mathcal{A}(x) - 1$.

If the approximation of $f(x)$ by $F(x)$ is accepted, then from (1) we conclude that

$$\int_0^x f(t)dt \sim \int_0^x F(t)dt = \mathcal{M}_0 I(u,p).$$

Here $I(u,p)$ represents the **incomplete-gamma-function** of § 18, and $u = x/\sqrt{p+1}$.

The median ϱ of x is obtained from the equation

$$\int_0^{\varrho} f(x)dx = I\left(\frac{\varrho}{\sqrt{p+1}}, p\right) = 1/2.$$

The tables of the incomplete-gamma-function show that

$$p < \varrho < p+1 \quad \text{and therefore} \quad x_m < \varrho < \mathcal{A}(x).$$

Conclusion. The approximation of $f(x)$ by the function (1) may be accepted if x varies from 0 to ∞, if $f(0) = f(\infty) = 0$ and if $f(x) \geqq 0$; moreover if we have approximately

$$\sigma^2 \sim \mathscr{A}(x)$$

$$x_m \sim \mathscr{A}(x) - 1$$

$$\mathscr{A}(x) - 1 < \varrho < \mathscr{A}(x).$$

Remark. 1. In certain circumstances formula (1) may be used also for the approximation of a function $f(x)$, if $f(0) = \infty$ and $f(\infty) = 0$; but then we must have $\mathscr{A}(x) < 1$ and $-1 < p < 0$; moreover there should be no extremum in the interval between $x=0$ and $x=\infty$.

Example 2. Function containing three disposable parameters. If x is a continuous variable and if its range extends from 0 to 1: moreover if $f(0) = f(1) = 0$ and $f(x) \geqq 0$ then we may try to approximate the function $f(x)$ by

$$F(x) = c\,\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} \tag{2}$$

where $p > 1$ and $q > 1$.

Since $\int_0^1 F(x)\,dx = C$,

$$\int_0^1 xF(x)\,dx = C\,\frac{B(p+1,q)}{B(p,q)} = \frac{Cp}{p+q}$$

and

$$\int_0^1 x^2F(x)\,dx = \frac{Cp(p+1)}{(p+q)(p+q+1)}$$

we have to put

$$C = \mathscr{M}_0; \quad \mathscr{M}_1 = \frac{p}{p+q}\mathscr{M}_0 \quad \text{and} \quad \mathscr{M}_2 = \frac{p(p+1)}{(p+q)(p+q+1)}\mathscr{M}_0.$$

From this we conclude that the *mean* of x is given by $\mathscr{A}(x) = p/(p+q)$ and the mean-square deviation by

$$\sigma^2 = \frac{\mathscr{M}_2}{\mathscr{M}_0} - \left(\frac{\mathscr{M}_1}{\mathscr{M}_0}\right)^2 = \frac{pq}{(p+q)^2(p+q+1)}.$$

Starting from the above equations we may determine the parameters p and q; first we get

$$\mathcal{M}_1 q = (\mathcal{M}_0 - \mathcal{M}_1) p \quad \text{and} \quad \mathcal{M}_2 q = (\mathcal{M}_1 - \mathcal{M}_2)(p+1)$$

and finally

$$p = \frac{\mathcal{M}_1(\mathcal{M}_1 - \mathcal{M}_2)}{\mathcal{M}_0\mathcal{M}_2 - \mathcal{M}_1^2} \qquad q = \frac{(\mathcal{M}_0 - \mathcal{M}_1)\ (\mathcal{M}_1 - \mathcal{M}_2)}{\mathcal{M}_0\mathcal{M}_2 - \mathcal{M}_1^2}.$$

To find the ***mode*** x_m we have to put $\mathbf{D}F(x) = 0$, this gives

$$x_m = \frac{p-1}{p+q-2} = \frac{1}{1+\dfrac{q-1}{p-1}}.$$

If $p > q$ then from the above equation it follows that $x_m > \dfrac{\mathcal{M}_1}{\mathcal{M}_0}$, and if $q > p$ then $\dfrac{\mathcal{M}_1}{\mathcal{M}_0} > x_m$.

It is easy to show that the sign of $[\mathbf{D}^2 y]_{x=x_m}$ is the same as that of

$$\frac{(p-1)(1-q)}{p+q-2}$$

Since $p > 1$ and $q > 1$, this is negative, so that a maximum corresponds to x=x,.

If the approximation of $f(x)$ by formula (2) is accepted, then we have also

$$\int_0^x f(t)dt = \int_0^x F(t)dt = I_x(p,q)$$

where $I_x(p,q)$ is the incomplete Beta-function of § 25.

The median ϱ of x given by

$$\int_0^{\varrho} f(x)dx = I_{\varrho}(p,q) = \tfrac{1}{2}$$

is determined by aid of the tables of this function. If $p > q$ then it can be shown that

$$\frac{p-1}{p+q-2} \geqq \varrho \geqq \frac{p}{p+q} \quad \text{or} \quad x_m \geqq \varrho \geqq \mathcal{A}(x)$$

and if $p < q$ then

$$x_m \leqq \varrho \leqq \mathcal{A}(x).$$

If x_m and ϱ are nearly equal to the above values, then the approximation is admissible. To check the obtained precision we determine first $\mathcal{M}_3$, the third power moment of f(x) and then that of F(x); the smaller the difference

$$\mathcal{M}_3 - \frac{(p+2)\mathcal{M}_2}{p+q+2}$$

of the obtained quantities is, the better the approximation should be considered.

Remark. 2. The function (2) may serve as an approximation of f(x) also in the following cases:

1. If $f(0) = f(1) = \infty$; but then we must have $0 < p < 1$, and $0 < q < 1$; moreover x_m will correspond to a minimum.

2. If $f(0) = \infty$ and $f(1) = 0$, then $0 < p < 1$ and $q > 1$. No extrema in the interval (0, 1).

3. If $f(0) = 0$ and $f(1) = \infty$, then $p > 1$ and $0 < q < 1$; no extrema.

Example 3. Function containing three disposable parameters. The variable x is continuous and varies from $-\infty$ to ∞. If $f(\pm\infty) = 0$ and $f(x) \geqq 0$, moreover if there is but one maximum of $f(x)$, it may be approximated by

$$F(x) = \frac{C}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \tag{3}$$

Since

$$\int_{-\infty}^{\infty} F(x)dx = C, \qquad \int_{-\infty}^{\infty} xF(x)dx = Cm$$

and

$$\int_{-\infty}^{\infty} x^2F(x)dx = (\sigma^2+m^2)C$$

moreover denoting

$$\mathcal{M}_i = \int_{-\infty}^{\infty} x^i f(x)dx$$

hence according to the principle of moments we have to put

$$C = \mathcal{M}_0, \qquad m = \frac{\mathcal{M}_1}{\mathcal{M}_0} = \mathcal{A}(x), \qquad \sigma^2 = \frac{\mathcal{M}_2}{\mathcal{M}_0} - \left(\frac{\mathcal{M}_1}{\mathcal{M}_0}\right)^2.$$

Mode. The maximum of $F(x)$ is reached if $x = x_m = m$.

If the approximation by aid of (3) is accepted, then we have also

$$\int_{-\infty}^{z} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2}\, dt$$

where $\lambda = (z-m)/\sigma$.

The *median* is equal to $\varrho = m$.

Conclusion. The function (3) will only be useful as an approximation of f(x) if we have nearly

$$x_m \sim \mathcal{A}(x) \sim \varrho .$$

Since the third moment of the deviation *x-m* with respect to the function (3) is equal to zero, hence the obtained precision may be measured by

$$\mathcal{M}_3 - 3m\,\mathcal{M}_2 + 3m^2\,\mathcal{M}_1 - m^3\,\mathcal{M}_0 = \mathcal{M}_3 - \frac{3\mathcal{M}_1\mathcal{M}_2}{2\mathcal{M}_0} + \frac{2\mathcal{M}_1{}^2}{\mathcal{M}_1{}^2} .$$

The smaller this quantity is, the better the obtained approximation.

Example 4. Discontinuous variable. Two disposable parameters. If x $= 0, 1, 2, \ldots, \infty$ then an approximation may be tried by

(4) $$F(x) = C\,e^{-m}\frac{m^x}{x!} .$$

$$\sum_{x=0}^{\infty} F(x) = C , \quad \sum_{x=0}^{\infty} xF(x) = C\,m , \quad \sum_{x=0}^{\infty} x^2F(x) = C(m^2+m);$$

moreover writing

$$\mathcal{M}_i = \sum_{x=0}^{\infty} x^i F(x)$$

hence according to the principle of moments we have to put

$$C = \mathcal{M}_0 \text{ and } m = \frac{\mathcal{M}_1}{\mathcal{M}_0} = 38(x) .$$

The precision obtained will be measured by

$$\mathcal{M}_2 - (m^2+m)\,\mathcal{M}_0 = \mathcal{M}_2 - \frac{\mathcal{M}_1^2}{\mathcal{M}_0} = \sigma^2\mathcal{M}_0 - \mathcal{M}_1 .$$

Mode. The maximum of (4) is reached for x=x,,, , where

$$m\text{-}l < x_m < m .$$

If the approximation of f(x) by formula (4) is accepted, then we have also

$$\sum_{t=0}^{z+1} f(x) = 1 - I(u,p)$$

where $I(u,p)$ is the incomplete-gamma-function of § 18, and $u = m/\sqrt{z+1}$ and $p = z.$

The median of ϱ is drrermined by aid of

$$\sum_{x=0}^{\varrho} f(x) \sim \sum_{x=\varrho+1}^{\infty} f(x)$$

that is, by

$$1 - I\left(\frac{m}{\sqrt{\varrho}}, \varrho-1\right) \sim I\left(\frac{m}{\sqrt{\varrho+1}}, \varrho\right).$$

Since ϱ must necessarily be an integer, hence this equation will be only approximately satisfied.

From *Example 1* it follows that $m\text{-}l < \varrho < m + 1$.

Conclusion. The approximation by (4) is advisable if we have

$$d(x) - 1 < x_m < \mathscr{M}(x), \quad \mathscr{M}(x) - 1 < \varrho < \mathscr{M}(x) + 1$$

and

$$\sigma^2 \sim \mathscr{M}(x).$$

Example **5.** Discontinuous variable, three disposable parameters $x = 0, 1, 2, \ldots, n.$ The approximation of f(x) may be tried by

$$(5) \qquad F(x) = \frac{C p^x (1-p)^{n-x}}{B(x+1, n+1-x)} = c \binom{n}{x} p^x (1-p)^{n-x}.$$

We have

$$\sum_{x=0}^{n+1} F(x) = C \qquad \sum_{x=0}^{n+1} xF(x) = Cnp$$

and

$$\sum_{x=0}^{n+1} x(x-1)F(x) = Cn(n-1)p^2$$

$$\sum_{x=0}^{n+1} x(x-1)(x-2)F(x) = Cn(n-1)(n-2)p^3$$

Since we denoted by $\mathfrak{M}_i$ the factorial moment of degree *i,* that is

$$\mathfrak{M}_i = \sum_{x=0}^{n+1} (x)_i \, f(x)$$

We have to put

$$C = \mathfrak{M}_0, \quad Cnp = \mathfrak{M}_1, \quad Cn(n-1)p^2 = \mathfrak{M}_2$$

that is

$$n = \frac{\mathfrak{M}_1^2 - \mathfrak{M}_2\mathfrak{M}_0}{\mathfrak{M}_1^2} \qquad p = \frac{\mathfrak{M}_1^2 - \mathfrak{M}_2\mathfrak{M}_0}{\mathfrak{M}_1\mathfrak{M}_0},$$

To check the approximation we determine

$$\mathfrak{M}_3 - Cn(n-1)(n-2)p^3 = \mathfrak{M}_3 - \mathfrak{M}_2\left[\frac{2\mathfrak{M}_2\mathfrak{M}_0 - \mathfrak{M}_1^2}{\mathfrak{M}_1\mathfrak{M}_0}\right].$$

The less this quantity is, the better the obtained approximation.

The *mean* of x is

$$\mathcal{A}(x) = \frac{\mathfrak{M}_1}{\mathfrak{M}_0} = np.$$

The *standard deviation* or *mean square deviation* of x (p. 427)

$$\sigma^2 = \frac{\mathfrak{M}_2}{\mathfrak{M}_0} + \frac{\mathfrak{M}_1}{\mathfrak{M}_0} - \left(\frac{\mathfrak{M}_1}{\mathfrak{M}_0}\right)^2 = np(1-p).$$

Mode. The maximum of F(x) is reached by $x = x_m$ if

$$np < x_m < np+1 \quad \text{or} \quad \mathcal{A}(x) < x_m < \text{d}(x) + 1.$$

If the approximation by (5) is accepted, we have also

$$\sum_{t=0}^{x} f(t) = 1 - I_p(x, n+1-x)$$

where $I_p(x, n+1-x)$ represents the incomplete-beta-function (16), § 25.

The *median* ϱ is obtained from

$$\sum_{x=0}^{\varrho} f(x) \sim \sum_{x=\varrho+1}^{n+1} f(x)$$

or from

$$1 - I_p(\varrho, n+1-\varrho) \sim I_p(\varrho+1, n-\varrho).$$

From example 2 it follows that np-1 $\leqq \varrho \leqq np.$

Conclusion. The approximation by (5) will be useful if we have

$$\text{“R}(x) < x_m < \mathcal{A}(x) + 1$$

moreover if $p > \frac{1}{2}$ or $\frac{1}{2}\mathcal{A}(x) > \sigma^2$ then

$$\mathcal{A}(x) - 1 < \varrho < \mathcal{A}(x)$$

and if $p < \frac{1}{2}$ or $\frac{1}{2}\mathcal{A}(x) < \sigma^2$ then

$$\mathcal{A}(x) < \varrho < \mathcal{A}(x) + 1.$$

§ 138. **Expansion of a function into a series of Legendre's polynomials.** The *Legendre* polynomial of degree n denoted by $X_n(x)$ is defined by

$$(1) \qquad X_n(x) = \frac{1}{n!\,2^n} D^n[(x^2-1)]^n.$$

From this it follows that

$$(2) \qquad X_n(x) = \frac{1}{2^n} \sum_{i=0}^{m+1} (-1)^i \binom{n}{i} \binom{2n-2i}{n} x^{n-2i}$$

where m is the greatest integer in $n/2$.

Important ***particular cases and values*** of the polynomials

$$X_0 = 1 \qquad X_1(x) = x$$

$$X_2(x) = \tfrac{1}{2}(3x^2-1) \qquad X_3(x) = \tfrac{1}{2}(5x^3-3x).$$

From (2) it follows immediately that $X_{2n-1}(0) = 0$ and

$$X_{2n}(0) = \frac{(-1)^n}{2^{2n}}\binom{2n}{n} = \binom{-\frac{1}{2}}{n}.$$

Since (1) may be written:

$$X_n(x) = \frac{n!}{2^n} D^n \left[\frac{(x-1)^n}{n!}\,\frac{(x+1)^n}{n!}\right]$$

hence by aid of ***Leibnitz'*** formula we get

$$(2') \qquad X_n(x) = \frac{1}{2^n} \sum_{i=0}^{n+1} \binom{n}{i}^2 (x-1)^{n-i}(x+1)^i.$$

Therefore

$$X_n(1) = 1 \quad \text{and} \quad X_n(-1) = (-1)^n.$$

Symmetry of the polynomials. From (2') we easily deduce:

$$X_n(-x) = (-1)^n X_n(x),$$

The ***roots*** of the polynomials are all real and single and comprised between -1 and +1.

Roots of $X_n(x) = 0$ for $n = 2, 3, 4, 5$

$n=2$ $\quad x_0 = -0.57735026, \quad x_1 = 0.57735026$

$n=3$ $\quad x_0 = -0.77459666, \quad x_1 = 0. \quad x_2 = 0.77459666$

$n=4$ $\quad x_0 = -0.86113632, \quad x_1 = -0.33998104$

$\qquad\quad x_2 = 0.33998104, \quad x_3 = 0.86113632$

$n=5$ $\quad x_0 = -0.90617994, \quad x_1 = -0.53846922 \quad x_2 = 0$

$\qquad\quad x_3 = 0.53846922, \quad x_4 = 0.90617994$

In consequence of the symmetry of the polynomials, if x_i is a root of $X_n(x) = 0$ then $-x_i$ will also be a root.

In Mathematical Analysis it is shown[40] that

$$\int_{-1}^{1} X_n(x) X_m(x)\, dx = 0 \text{ if } n \neq m$$

(3)

$$\int_{-1}^{1} [X_n(x)]^2\, dx = \frac{2}{2n+1} .$$

That is, the polynomials are orthogonal in the interval $(-1, 1)$.

A function $f(x)$ of limited total fluctuation in the interval $(-1, 1)$ may be expanded into a series of *Legendre's* polynomials. Let us write

$$f(x) = c_0 + c_1 X_1(x) + c_2 X_2(x) + \ldots + c_n X_n(x) + \ldots$$

Multiplying both members of this equation by $X_m(x)$ and integrating from -1 to $+1$ in consequence of the equations (3) we find

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)\, X_m(x)\, dx. \tag{4}$$

Putting into this equation the value of $X_m(x)$ obtained in (2) we have

$$c_m = \tfrac{1}{2}(2m+1) \sum_{i=0}^{\mu} \frac{(-1)^i}{2^m} \binom{m}{i} \binom{2m-2i}{m} \int f(x)\, x^{m-2i}\, dx$$

[40] See for instance C. *Jordan,* Cours d'Analyse, Vol. 2, 3e éd. p. 299. *E. T. Whittaker* and *G. N. Watson,* Modern Analysis (1927), p. 305.

and denoting by $\mathcal{M}_s$ the s -th power moment of f(x) in the interval considered:

$$\mathcal{M}_s = \int_{-1}^{1} x^s f(x)\, dx$$

we finally find that

$$(5) \qquad c_m = \frac{2m+1}{2^{m+1}} \sum_{i=0}^{\mu} (-1)^i \binom{m}{i} \binom{2m-2i}{m} \mathcal{M}_{m-2i}$$

where μ is the greatest integer contained in $\frac{1}{2}m+1$.

The determination of the coefficients c_m is very simple in consequence of the orthogonality of the polynomials; otherwise it would be a laborious task.

Approximation of a function f(x) by a series of *Legendre's* polynomials. Stopping at the term X_n let us write

$$(6) \qquad f_n(x) = \sum_{m=0}^{n+1} c_m X_m(x).$$

The coefficients c_m must be determined according to the principle of least squares so that

$$(7) \qquad \mathcal{S} = \int_{-1}^{1} [f(x) - f_n(x)]^2\, dx$$

shall be a minimum. Putting into it the value of $f_n(x)$ given by (6) the equations determining the minimum will be

$$\frac{\partial \mathcal{S}}{\partial c_m} = 0 \text{ for } m = 0, 1, 2, \ldots, n.$$

In consequence of the orthogonality of the functions we shall find for c_m the value (4) obtained above.

Moreover by aid of (3) and (4) we find

$$\mathcal{S} = \int_{-1}^{1} [f(x)]^2\, dx - \frac{2}{2m+1} \sum_{m=0}^{n+1} c_m^2.$$

§ 139. Orthogonal polynomials with respect to $x=x_0, x_1, \ldots x_{N-1}$. The polynomials $U_m(x)$ of degree m are called orthogonal with respect to $x=x_0, x_1, \ldots, x_{N-1}$, if the equation

$$(1) \qquad \sum_{i=0}^{N} U_m(x_i)\, U_\mu(x_i) = 0$$

is satisfied, if m is different from μ.

These polynomials were first considered by *Tchebichef*[41] and since then several authors have investigated this subject.

In the general case the polynomials are complicated and of little practical use. But if the values of x_i are equidistant, then simple formulae may be obtained.

Determination of the orthogonal polynomials, if the given values of x are equidistant. $x=a+h\xi$ and $\xi=0, 1, 2, \ldots,$ N-l. Instead of starting at (1) we shall employ the following formula:

$$(2) \qquad \sum_{\xi=0}^{N} F_{m-1}(x)\, U_m(x) = 0$$

where *F,-,(x)* is an arbitrary polynomial of degree m-l. If we

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[41]) *Tchebichef,* Sur les fractions continues. Journal de Mathématiques pures et appliquées, 1858, T. III (0euvres, tome I, p. 203). — Sur l'interpolation par la méthode des moindres carrés. Mém. Acad. Imp. de St. Pétersbourg, 1859 (Oeuvres, tome I, p. 473). — Sur l'interpolation des valeurs équidistantes, 1875 (Oeuvres, tome II, p. 219).

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were to expand *F,-,(x)* into a series of U,(x) polynomials we should return to equation (1) again.

Since F_{m-1} (x) is of degree m-l, formula **(10)** of **§** 34, giving the indefinite sum of a product, may be written, putting there

$$U(x) = F_{m-1}(x) \quad \text{and } V_0(x) = U_m(x)$$

in the following way:

$$\Delta^{-1}[F_{m-1}(x)\, U_m(x)] = F_{m-1}(x)\, \Delta^{-1}U_m(x) -$$
$$-\Delta F_{m-1}(x)\, \Delta^{-2}U_m(x+h) + \Delta^2 F_{m-1}(x)\, \Delta^{-3}U_m(x+2h) - \dots +$$
$$+ (-1)^{m-1}\Delta^{m-1}F_{m-1}(x)\, \Delta^{-m}U_m(x+mh-h).$$

Now. since we are considering polynomials only, $\Delta^{-1}U(x)$ **contains** an arbitrary constant, to which may be assigned such a value that $[\Delta^{-1}U_m(x)]_{x=a}$ is equal to zero. But $\Delta^{-2}U_m(x+h)$ contains an additional constant, which may be chosen so that $[\Delta^{-2}U_m(x+h)]_{x=a} = 0$. *C*on**t** inuing after this fashion, we may dispose of all these arbitrary constants in such a way as to have

$$\Delta^{-n}U_m(x+nh-h) = 0$$

for $x=a$, and for every value of n satisfying to $n \leqq m$; therefore

{ A-' IF,,,-, **(x)** *U,(x)* | }$_{x=a}$ *= 0.*

But in order that the definite sum may be equal to zero, it is necessary for the above expression to vanish also for the upper limit $x=a+Nh=b$. But since Δ' *F,-,(b)* is arbitrary for all values of v, it follows that each expression $\Delta^{-\nu-1}\, U_m(x+\nu h)$ obtained for $\nu=0$, 1, 2, . . . , m-l must vanish separately for $x=b$.

From **this** we conclude that (x-a) and *(x-b)* must both be factors of $\Delta^{-1}U_m(x)$. Considering for the moment only the first of these factors, we may therefore write

$$\Delta^{-1}U_m(x) = (x-a)\lambda(x).$$

Applying to this expression the formula for the indefinite sum of a product (10, **§ 34)**, we find

$$\Delta^{-2}U_m(x) = \sum_{\nu=0}^{m+1} (-1)^{\nu} \frac{1}{h^{\nu+1}} \binom{x-a+\nu h}{\nu+2} \Delta^{\nu}\lambda(x).$$

(x-o) must be a factor of $\Delta^{-2}U_m(x)$ too: therefore the additional constant must be equal to zero; it follows that $(x-a)_{2,h}$ is a factor of $\Delta^{-2}U_m(x)$. By successive summation we should find that $(x-a)_{m,h}$ is factor of $\Delta^{-m}U_m(x)$ so that it may be written

$$\Delta^{-m}U_m(x) = \binom{x-a}{m}_h \psi(x).$$

As $(x-b)$, must also be a multiplying factor of $\Delta^{-1}U_m(x)$ we deduce by the same reasoning that

$$\Delta^{-m}U_m(x) = C \binom{x-a}{m}_h \binom{x-b}{m}_h.$$

$\Delta^{-m}U_m(x)$ is of degree *2m;* therefore C is an arbitrary constant, and we conclude that the general formula for the orthogonal polynomials with respect to $x=a+\xi h$, where $\xi=0,1,2,$ . . . , N-l and $b=a+Nh$ is the following:

$$U_m(x) = C\,\Delta^m\left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right]. \tag{3}$$

Starting from this expression, there are two different ways of deducing the expansion of U,(x) into a *Newton* series. First utilising formula (10) § 30 which gives the *m* -th difference of a product; we obtain

$$U,(x) = Ch^m \sum_{i=0}^{m+1} \binom{m}{i} \binom{x-b}{m-i}_h \binom{x-a+ih}{i}_h. \tag{4}$$

Secondly, we can develop $\Delta^{-m}U_m(x)$ into a *Newton* series of generalised binomial coefficients $\binom{x-b}{i}_h$. According to § 22 we shall have

$$\Delta^{-m}U_m(x) = C \sum_{i=0}^{2m+1} \binom{x-b}{i}_h \frac{1}{h^i} \Delta^i\left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right]_{x=b}$$

but

$$\Delta^i\left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right] = \sum_{\nu=0}^{i+1} h^i \binom{i}{\nu} \binom{x-b}{m-\nu}_h \binom{x-a+\nu h}{m-i+\nu}_h;$$

putting $x=b$, we get

$$\Delta^{-m}U_m(x) = C \sum_{i=0}^{2m+1} \binom{i}{m} \binom{b-a+mh}{2m-i}_h \binom{x-b}{i}_h$$

and finally putting $i=m+\nu$, and determining the *m* -th difference of the above expression, we find

$$(5) \qquad U_m(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{b-a+mh}{m-\nu}_h \binom{x-b}{\nu}_h .$$

As $U_m(x)$ is symmetric with respect to a and b, we can get two other formulae from (4) and (5) changing a into b and inversely. For instance, remarking that $b-a=Nh$, from (5) we get

$$(6) \qquad U_m(x) = Ch^{2m} \sum_{\nu=0}^{m+1} h^{-\nu} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \binom{x-a}{\nu}_h .$$

Remark 1. If in this formula we put $m=N$ then every term will vanish except that in which $\nu=m$, and we obtain

$$U_N(x) = Ch^{2N} \binom{2N}{N} \binom{x-a}{N}_h .$$

Therefore, if x is equal to the given numbers $x=a+\xi h$ for $\xi=0, 1, 2, \ldots, N-1$ we shall have

$$U_N(x) = 0 .$$

2. If $m > N$ then every term of (6) will vanish in which $\nu < N$; but if x is equal to $x=a+\xi h$ and $\xi=0, 1, 2, \ldots, N-1$, then the last factor of (6) will be if $\nu > N$

$$\binom{x-a}{\nu}_h = h^\nu \binom{\xi}{\nu} = 0.$$

Conclusion. If $m \geqq N$ and x is one of the given values, then we have

$$U_m(x) = 0.$$

Introducing into (6) the variable $\xi=(x-a)/h$, it will become

$$(7) \qquad U_m(a+\xi h) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \binom{\xi}{\nu} .$$

We have seen that

$$\sum_{\xi=0}^{N} U_m(x)\, U_\mu(x) = 0$$

if $\mu \neq m$. It remains to determine the value of this expression if $\mu=m$. That is

(8) $$\sum_{\xi=0}^{N} [U_m(x)]^2.$$

This may be done by determining the indefinite sum of the product $U_m(x)U_m$ (x). According to formula (10) § 34 we have

(9)
$$\Delta^{-1}[U_m(x)U_m(x)] = U_m(x)\Delta^{-1}U_m(x) - \Delta U_m(x)\Delta^{-2}U_m(x+h) + \ldots$$
$$\ldots, + (-1)^m\Delta^m U_m(x)\Delta^{-m-1}U_m(x+mh).$$

To obtain the sum (8) we have to put in this expression $x=b$ and $x=a$; but we disposed of the arbitrary constants in $U,(X)$ so as to have:

$$\Delta^{-n}U_m(x+nh-h) = 0$$

for $n=1, 2, \ldots, m$ at both limits $x=a$ and $x=b$. Hence it remains to determine the value of the last term at the limits.

For this we start from

$$\Delta^{-m}U_m(x) = C\binom{x-a}{m}_h\binom{x-b}{m}_h;$$

the above quoted formula of the sum of a product will give

$$\Delta^{-m-1}U_m(x) = \frac{C}{h}\left[\binom{x-a}{m}_h\binom{x-b}{m+1}_h - \binom{x-a}{m-1}_h\binom{x-b+h}{m+2}_h + \right.$$
$$\left. + \binom{x-a}{m-2}_h\binom{x-b+2h}{m+3}_h - \ldots + (-1)^m\binom{x-b+mh}{2m+1}_h\right].$$

Putting $x+mh$ instead of x into the preceding expression it will become

$$\Delta^{-m-1}U_m(x+mh) = \frac{C}{h}\left[\sum_{\nu=0}^{m+1}(-1)^\nu\binom{x-a+mh}{m-\nu}_h\binom{x-b+\nu h+mh}{m+1+\nu}_h\right].$$

At the upper limit $x=b$ every term in the second member is equal to zero, so that

(9') $$[\Delta^{-m-1}U_m(x+mh)]_{x=b} = 0.$$

At the lower limit $x=a$ we have (remembering that $b-a=Nh$)

$$[\Delta^{-m-1} U_m (x+mh)]_{x=a} = Ch^{2m} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m}{m-\nu} \binom{m+\nu-N}{m+\nu+1} =$$

$$= (-1)^{m-1} Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m}{\nu} \binom{N}{m+\nu+1}$$

and in consequence of *Cauchy's* theorem (14, § 22) finally we have

$$(10) \qquad [\Delta^{-m-1} U_m (x+mh)]_{x=a} = (-1)^{m+1} Ch^{2m} \binom{N+m}{2m+1}.$$

Moreover from (6) it follows that

$$\Delta^m U_m(x) = Ch^{3m} \sum_{\nu=0}^{m+1} h^{-\nu} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \binom{x-a}{\nu-m}_h ;$$

this gives for $x=a$

$$(11) \qquad \Delta^m U_m(a) = Ch^{2m} \binom{2m}{m}.$$

Finally by aid of (9) we obtain the required sum:

$$(12) \qquad \sum_{\xi=0}^{N} [U_m(x)]^2 = C^2 h^{4m} \binom{2m}{m} \binom{N+m}{2m+1}.$$

It may be useful to remark, that this quantity is independent of the origin of the variable x.

§ 140. Some mathematical properties of the orthogonal polynomials.”

Symmetry of the polynomials. Putting into formula (4) § 139 *afb-h-x* instead of x we find

$$U_m(a+b-h-x) = Ch^m \sum_{=0}^{m+1} \binom{m}{\nu} {}_{11} \binom{b-h+\nu h-x}{\nu}_h \binom{a-h-x}{m-\nu}_h$$

but this is equal to

$$U_m(a+b-h-x) = (-1)^m Ch^m \sum_{\nu=0}^{m+1} \binom{m}{\nu} \binom{x-a+mh-\nu h}{m-\nu}_h \binom{x-b}{\nu}_h.$$

Now putting into it $\mu=m-\nu$ from (4), § 139 it follows that

$$(1) \qquad U_m(a+b-h-x) = (-1)^m U_m(x).$$

This equation shows the symmetry of the orthogonal poly-

. [42] They have been described more fully in loc. cit. 41. *Jordan*, a) pp 314—322 and e) pp. 309—317.

nomials. ***Particular case:*** $x=\frac{1}{2}(a+b-h)$ gives the central value of the polynomial:

$$U_m\left(\frac{a+b-h}{2}\right)=(-1)^m U_m\left(\frac{a+b-h}{2}\right)$$

therefore

$$U_{2m+1}\left(\frac{a+b-h}{2}\right)=0. \tag{2}$$

Functional equation. We may easily deduce a functional equation which is satisfied by the orthogonal polynomials; this can be done by expanding $xU_m(x)$ into a series of orthogonal polynomials. We find

$$\text{(3)}\quad xU_m(x)=A_{m,\,m-1}U_{m-1}(x)+A_{m,m}U_m(x)+A_{m,\,m+1}U_{m+1}(x);$$

as in consequence of the orthogonality of the polynomials the other terms vanish. Indeed

$$\sum_{x=a}^{b} xU_m(x)U_\mu(x)=0$$

if $\mu>m+1$ or $\mu<m-1$ *(2,* § 139).

Hence we have only to determine the above three coefficients, Multiplying by $U_{m+1}(x)$ and summing, we obtain from (3)

$$\sum_{x=a}^{b} xU_m(x)U_{m+1}(x)=A_{m,\,m+1}\sum_{x=a}^{b}[U_{m+1}(x)]^2. \tag{4}$$

We know already the sum in the second member; to determine the first member let us apply the formula giving the indefinite sum of a product (10, § 34).

$$\Delta^{-1}\left[xU_m(x)\cdot U_{m+1}(x)\right]=$$
$$=\left[xU_m(x)\right]\Delta^{-1}U_{m+1}(x)-\Delta\left[xU_m(x)\right]\Delta^{-2}U_{m+1}(x+h)+$$
$$+\dots+(-1)^{m+1}\Delta^{m+1}\left[xU_m(x)\right]\Delta^{-m-2}U_{m+1}(x+mh+h).$$

When determining the polynomials $U_m(x)$ we disposed of the arbitrary constants so as to have

$$\Delta^{-n}U_m(x+nh-h)=0$$

for $n=1,2,3,\dots,m$ at both limits: $x=a$ and $x=b$.

Therefore every term of the preceding series will vanish at these limits except the last term, which according to formulae (9') and (10) of § 139 is equal to zero for $x=b$; and for $x=a$ it is

$$[\Delta^{-m-2}U_{m+1}(x+mh+h)]_{x=a}=(-1)^m C_{m+1}\, h^{2m+2} \binom{N+m+1}{2m+3}.$$

In this formula we have written C_{m+1} instead of C, since this constant may depend upon the degree m+l of the polynomial.

Since formula (11) § 139 gives

$$\Delta^m U_m(a)=C_m\, h^{2m} \binom{2m}{m}$$

we shall have for $x=a$

$$\Delta^{m+1}[xU_m(x)]=h(m+1)\,\Delta^m U_m(x)=C_m\, h^{2m+1}(m+1)\binom{2m}{m}.$$

Finally we find

$$(5)\qquad \sum_{x=a}^{b}[xU_m(x)U_{m+1}(x)]=C_mC_{m+1}\, h^{4m+3}(m+1)\binom{2m}{m}\binom{N+m+1}{2m+3}.$$

Moreover from formula (12) § 139 it follows that

(6) $\sum_{x=a}^{b}$ *[U,+,(x)]”* $= C_{m+1}^2\, h^{4m+4}\binom{2m+2}{m+1}\binom{N+m+1}{2m+3}.$

By aid of the last two equations we get from (4)

$$(7)\qquad A_{m,\ m+1}=\frac{C_m}{hC_{m+1}}\,\frac{(m+1)^2}{2(2m+1)}.$$

Putting into equation (5) m-l instead of m we obtain

$$\sum_{x=a}^{b}[xU_{m-1}(x)]U_m(x)]=C_{m-1}C_m\, h^{4m-1}\, m\binom{2m-2}{m-1}\binom{N+m}{2m+1};$$

moreover putting $m-2$ into (6) instead of m it follows that

$$\sum_{x=a}^{b}[U_{m-1}(x)]^2=C_{m-1}^2\, h^{4m-4}\binom{2m-2}{m-1}\binom{N+m-1}{2m-1}.$$

From (3) we deduce by aid of the last two equations

$$(8)\qquad A_{m.\ m-1}=\frac{C_m h^3}{C_{m-1}}\,\frac{(N^2-m^2)}{2(2m+1)}.$$

To determine $A_{m,m}$ by the preceding method would be more difficult; but, since equation (3) must be true for every value of x, if we put into it values of x for which U,(x) is known, we obtain equations which permit us to determine the coefficients.

Putting into it $x=b-h$ we find

$$(9)\qquad (b-h)U_m(b-h) = A_{m,\ m-1}U_{m-1}(b-h) + A_{m,\ m}U_m(b-h) + A_{m,\ m+1}U_{m+1}(b-h)$$

now if we put into (3) $x=a$ we have

$$(10)\qquad aU_m(a) = A_{m,\ m-1}U_{m-1}(a) + A_{m,m}U_m(a) + A_{m,\ m+1}U_{m+1}(a)$$

remarking that in consequence of the symmetry polynomials U,,(x) it follows' that

$$U_m(a) = (-1)^m U_m\ (b\text{-}h)$$

hence from (9) and (10) we obtain

$$(11)\qquad \begin{aligned} (b+a-h)\,U_m\,(b\text{-}h) &= 2\,A_{,,\ m}\,U_m(b-h) \\ A_{m,m} &= \tfrac{1}{2}(b+a-h) \end{aligned}$$

therefore the function-equation (3) is determined:

$$x\,U_m(x) = \frac{C_m h^3}{C_{m-1}} \cdot \frac{N^2-m^2}{2(2m+1)}\,U_{m-1}(x) + \tfrac{1}{2}(b+a-h)U_m(x) + \frac{C_m}{hC_{m+1}}\,\frac{(m+1)''}{2(2m+1)}\,U_{m+1}\ (x),$$

Application. Let us determine the *central value* of **the func**tion *U,,(x),* that is $U_{2m}\left(\frac{a+b-h}{2}\right) = U_{2m}\left(a+\frac{Nh-h}{2}\right)$. If we **put into** (3) $m=2n+1$ and $x=a+\tfrac{1}{2}(Nh-h)$, **then** in consequence of (2) it will become

$$-\,A_{2n+1,\ 2n}U_{2n}\left(a+\frac{Nh-h}{2}\right) = A_{2n+1,\ 2n+2}U_{2n+2}\left(a+\frac{Nh-h}{2}\right).$$

Writing $m=2n+1$ we obtain $A_{,,,,,\ 2n}$ from equation (8) and $A_{2n+1,\ 2n+2}$ from equation (7). Therefore the preceding formula may be written

$$(12)\qquad F(n+1) = -\frac{N^2-(2n+1)^2}{4(n+1)^2}\,F(n)$$

where, in order to abbreviate it, there has been put

$$F(n) = \frac{U_{2n}\left(a+\dfrac{Nh-h}{2}\right)}{h^{4n}\, c_{2n}}$$

The solution of equation (12), which is a homogeneous linear difference equation of the first order, with variable coefficients, is as, we shall see in § 173, the following:

$$F(n) = \omega \prod_{i=0}^{n} \frac{N^2-(2i+1)^2}{4(i+1)^2} = \omega \prod_{i=0}^{n} \frac{\left(\dfrac{N+1}{2}+i\right)\left(\dfrac{N-1}{2}-i\right)}{(i+1)(i+1)}$$

hence

$$F(n) = \omega\ (-1)^n \binom{\tfrac{1}{2}N-\tfrac{1}{2}+n}{n}\binom{\tfrac{1}{2}N-\tfrac{1}{2}}{n}.$$

Since $F(0)=1$ therefore $\omega=1$, and the required central value will be

$$(13)\quad U_{2n}\left(a+\frac{Nh-h}{2}\right) = C_{2n}h^{4n}(-1)^n\binom{\tfrac{1}{2}N-\tfrac{1}{2}+n}{n}\binom{\tfrac{1}{2}N-\tfrac{1}{2}}{n}$$

this may be written

$$U_{2n}\left(a+\frac{Nh-h}{2}\right) = C_{2n}h^{4n}(-1)^n\binom{2n}{n}\binom{\tfrac{1}{2}N-\tfrac{1}{2}+n}{2n}$$

Difference equafion. It can be shown that the polynomial U,(x) satisfies the difference equation

$$(14)\qquad (x-a+2h)\,(x-b+2h)\Delta^2U_m(x)\ +$$
$$+\ [2x-a-b+3h-m(m+1)h]h\Delta U_m(x)-m\,(m+1)h^2U_m(x) = 0.$$

[See loc. cit[41]. *Jordan* a) p. 315; e) p. 316.1

Roofs of the polynomial. L. Fejér has given [See loc. cit.” *Jordan* a) p. 319] the following theorems concerning these roots:

The roots of U,(x) $=0$ are all real and single, and they are all situated in the interval $a,b-h$.

Whatever ξ may be, in the interval $a+\xi h,\ a+\xi h+h$ there is at most one root of U,(x) $=0$.

Fejér showed moreover that if $P_m(x)$ is a polynomial of degree *m*, and if in its *Newton* expansion the coefficient of $\binom{x}{m}$ is unity, then the polynomial which minimizes the following expression

$$\sum_{x=a}^{b} [P_m(x)]^2$$

is the orthogonal polynomial U,(x) with the constant C suitably chosen.

§ 141. Expansion of a function f(x) into a series of polynomials orthogonal with respect to $x=a+\xi h$, where $\xi=0, 1, 2, \ldots, N-1$.

Supposing first that $f(x)$ is a polynomial of degree n such that $n < N$, then we have

$$f(x)=c_0+c_1U_1(x)+c_2U_2(x)+ \ldots + c_nU_n(x)$$

multiplying both members by U,(x), and summing from $x=a$ to $x=b$ (that is from $\xi=0$ to $\xi=N$, since $a+Nh=b$), we find in consequence of formula **(1)** § 139, that

$$(1) \qquad c_m \sum_{\xi=0}^{N} [U_m(x)]^2 = \sum_{\xi=0}^{N} f(x)U_m(x).$$

Hence the coefficients c_m are easily obtained.

In the general case of t(x) the series will be infinite. Stopping it at the term $U_{N-1}(x)$, the series will nevertheless give exactly the values of f(x), if x is equal to one of the given values $x=a+\xi h$. Indeed, according to what we have seen in the preceding paragraph, every term in which $m \geqq N$ will vanish for these values; so that the limited series will give the same value as the infinite series.

To have f(x) exact for the other values of x we must add the remainder to the limited series; according 'to § 123 this will be

$$(2) \qquad R_N = \frac{1}{N!} (x-a)\ (x-a-h) \ldots, (x-a-Nh+h)\ D^N f\ (a+\zeta h)$$

where $0<\zeta \leqq N-1$ or $a<a+\zeta h< b$.

If $m \geqq N$ then the coefficients of $U_m(x)$ in the expansion of f(x) cannot be determined by formula (1) since in consequence of formula (7) $U_m(x)$ is equal to zero for each of the values of x corresponding to $\xi=0, 1, 2, \ldots$, N-l ; hence both members of equation **(1) will** be equal to zero.

In the preceding paragraph we have already determined $\Sigma[U_m(x)]^2$. Therefore to determine the coefficients c_m by aid of equation (1) it remains still to compute $\Sigma U_m(x) f(x)$. For this, let us start from the expression (7, § 139) of $U_m(x)$; this will give

$$\sum_{x=a}^{b} U_m(x) f(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \sum_{\xi=0}^{N} \binom{\xi}{\nu} f(a+\xi).$$

According to § 136 the last sum in the second member is equal to the binomial moment of order v, denoted by $\mathcal{B}_\nu$, of the function $f(a+\xi h)$; therefore this may be written:

Therefore

$$(3) \qquad \sum_{x=a}^{b} U_m(x) f(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \mathcal{B}_\nu .$$

As will be shown later, there is a far better method for rapidly computing the ***binomial moments*** than is available in the case of power moments. If we operate with equidistant discontinuous variables, it is not advantageous to consider powers; it is much better to express the quantities by binomial coefficients. Indeed, if an expression were given in power series, it would still be advantageous to transform it into a binomial series.

Several statisticians have remarked that it is not advisable to introduce moments of higher order into the calculations, In fact if N is large, these numbers will increase rapidly with the order of the moments, will become very large, and their coefficients in the formulae will necessarily become very small. It is difficult to operate with such numbers, the causes of errors being many.

To remedy this inconvenience, the ***mean binomial moment*** has been introduced. The definition of the mean binomial moment $\mathcal{T}_\nu$ of order ν of the function $f(x+\xi h)$ is the following

$$\mathcal{T}_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} f(a+\xi h) \Big/ \sum_{\xi=0}^{N} \binom{\xi}{\nu}$$

therefore

$$(4) \qquad \mathcal{T}_\nu = \frac{\mathcal{B}_\nu}{\binom{N}{\nu+1}}.$$

The mean binomial moment will remain of the same order of magnitude as ***f(x),*** whatever N or ν may be. For instance, if

$f(x)$ is equal to the constant k then we shall have $\mathcal{T}_\nu = k$ for any value of ν or N. On the other hand the power moment of order v

$$k \sum_{\xi=0}^{N} \xi^\nu$$

will increase rapidly with v and N.

Introducing into formula (3) $\mathcal{T}_\nu$ instead of $\mathcal{B}_\nu$ we shall have

$$\sum_{x=a}^{b} U_m(x)\, f(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \binom{N}{\nu+1} \mathcal{T}_\nu .$$

This may be written in the following form

$$(-1)^m Ch^{2m}\,(m+1) \binom{N}{m+1} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+\nu}{m} \binom{m}{\nu} \frac{\mathcal{T}_\nu}{\nu+1}$$

To simplify the formula we shall write

(5) $$\beta_{m\nu} = (-1)^{m+\nu} \binom{m+\nu}{m} \binom{m}{\nu} \frac{1}{\nu+1} .$$

Since these numbers are very useful they are presented in the following table, which gives all the numbers necessary for parabolas up to the tenth degree.

Table for $\beta_{m\nu}$

$m \backslash \nu$	0	1	2	3	4	5
1	-1	1				
2	1	-3	2			
3	-1	6	—10	5		
4	1	-10	30	-35	14	
5	-1	15	-70	140	-126	42
6	1	-21	140	-420	630	—462
7	-1	28	-252	1050	—2310	2772
8	1	-36	420	-2310	6930	-12012
9	—1	45	—660	4620	-18018	42042
10	1	-55	990	—8580	42042	-126126

$m \backslash \nu$	6	7	8	9	10
6	132				
7	-1716	429			
8	12012	-6435	1430		
9	-60060	51480	-24310	4862	
10	240240	-291720	218790	-92378	16796

The, following relation can be used for checking the numbers:

$$\beta_{m0} + \beta_{m1} + \beta_{m2} + \ldots + \beta_{mm} = 0$$

that is, the sum of the numbers in the rows is equal to **zero.**

Moreover let us put

$$\sum_{\nu=0}^{m+1} \beta_{m\nu} \mathcal{T}_\nu = \Theta_m . \tag{6}$$

If we already know the mean binomial moments, the values of Θ_m may readily be computed with the aid of the table above. Finally we obtain

$$\sum_{x=a}^{b} U_m(x)\, f(x) = Ch^{2m}\,(m+1)\binom{N}{m+1}\Theta_m . \tag{7}$$

As this expression could be termed the orthogonal moment of degree m of f(x), therefore we can consider Θ_m as a certain ***mean orthogonal moment*** of degree m of $f(x)$.

The mean orthogonal moments are independent of the origin, of the interval, and of the constant C. Particular case:

$$\Theta_0 = \mathcal{T}_0 = \mathcal{B}_0/N$$

is equal to the arithmetic mean of the quantities $f(x_i)$.

By aid of equation (7) and of (12), § 139 we deduce from (1) the coefficient c_m:

$$c_m = \frac{(2m+1)\,\Theta_m}{Ch^{2m}\binom{N+m}{m}} . \tag{8}$$

The coefficient c_m is independent of the origin. In particular we have

$$c_0 = \Theta_0 / C.$$

§ 142. Approximation of a function y **given** for $\xi = 0$, 1, 2, . . . , N-1, by aid of a polynomial f(x) of degree n, where $x = a + \xi h$, according to the ***principle of least squares,*** that is, so that the sum of the squares of the deviations $y - f(x)$ for the given values of x

$$\mathscr{S} = \sum_{\xi=0}^{N} [y - f(x)]^2$$

shall be a minimum.

If the polynomial $f(x)$ is expressed by orthogonal polynomials:

$$(1) \qquad f(x) = c_0 + c_1 U_1(x) + c_2 U_2(x) + \ldots + c_n U_n(x)$$

then the conditions of the minimum will be

$$\frac{\partial \mathscr{S}}{\partial c_m} = -2 \sum_{\xi=0}^{N} [y - c_0 - c_1 U_1(x) - \ldots - c_n U_n(x)] U_m(x) = 0$$

for $m = 0$, 1,. , . , n.

In consequence of the orthogonality (1, § 139) most of the terms will vanish and we shall have

$$(2) \qquad \sum_{\xi=0}^{N} y U_m(x) = c_m \sum_{\xi=0}^{N} [U_m(x)]^2.$$

Since this expression is identical with equation (1) of § 141, which gives the coefficient c_m of the expansion of y into a series of orthogonal polynomials: from this we deduce the important result:

To obtain the best approximation possible of a function y, according to the principle of least squares, by aid of a polynomial $f(x)$ of degree n, it is sufficient to expand x into a series of orthogonal polynomials, and to stop the series at the term U,,(x).

Moreover, if the approximation obtained by aid of a polynomial of degree n should not be close enough, then to obtain the best approximation possible by aid of a polynomial of degree $n+1$ it is sufficient to determine only one additional coefficient, c_{n+1} ; the others would not change. This is an important observation, since, if the expansion were not an orthogonal one, then, passing from the approximation of degree

n to that of degree n+l every coefficient would have to be computed anew.

If the approximation of degree $n+l$ is still unsatisfactory, this can be repeated till the required precision is reached.

Since we have seen in the preceding pragraph that the coefficient c_m is given by

$$c_m = \frac{(2m+1)\ \Theta_m}{Ch^{2m}\binom{N+m}{m}} \tag{3}$$

hence if we know the mean orthogonal moments of the y quantities, the problem is solved, It remains but to determine the obtained precision.

Measure of fhe precision. In the method of least squares the precision obtained is measured by the mean square deviation σ_n^2 (or standard deviation), that is by

$$\sigma_n^2 = \mathcal{S}/N.$$

From this it follows that

$$\sigma_n^2 = \frac{1}{N}\sum_{\xi=0}^{N}[y^2 + [f(x)]^2 - 2y\ f(x)]$$

putting into it the above expression of $f(x)$ by orthogonal polynomials, the equation is much simplified in consequence of the orthogonality of U,(x), so that we have

$$\sigma_n^2 = \frac{1}{N}\sum_{\xi=0}^{N} y^2 + \frac{1}{N}\sum_{\xi=0}^{N}[\sum_{m=0}^{n+1} c_m^2[U_m(x)]^2 - 2y\sum_{m=0}^{n+1} c_m U_m(x)].$$

Since from (2) we deduce

$$\sum_{m=0}^{n+1} c_m \sum_{\xi=0}^{N} yU_m(x) = \sum_{m=0}^{n+1} c_m^2 \sum_{\xi=0}^{N}[U_m(x)]^2$$

therefore σ_n^2 will be

$$\sigma_n^2 = \frac{1}{N}\sum_{\xi=0}^{N} y^2 - \frac{1}{N}\sum_{m=0}^{n+1} c_m^2 \sum_{\xi=0}^{N}[U_m(x)]^2.$$

In formula (12) § 139 we found

$$\sum_{\xi=0}^{N}[U_m(x)]^2 = C^2h^{4m}\binom{2m}{m}\binom{N+m}{2m+1}$$

moreover, multiplying it by c_m^2 taken from (3) we obtain after simplification

$$\frac{1}{N} c_m^2 \sum_{\xi=0}^{N} [U_m(x)]^2 = \frac{(2m+1)\binom{N-1}{m}}{\binom{N+m}{m}} \Theta_m^2$$

To abbreviate, let us put

$$(4) \qquad \mathscr{C}_{m0} = (-1)^m (2m+1) \binom{N-1}{m} \Big/ \binom{N+m}{m}.$$

Finally we have

$$(5) \qquad \sigma_n^2 = \frac{1}{N} \sum_{\xi=0}^{N} y^2 - \sum_{m=0}^{n+1} \mathscr{C}_{m0} \, \Theta_m^2.$$

Remark 1. $\mathscr{C}_{m0}$ is easily computed by formula (4) if we have a table of binomial coefficients; moreover there are tables giving this quantity up to $N=100$ and $m=7$, that is, up to a hundred observations, and for polynomials up to the seventh degree. [Loc. cit. 41, *Jordan, e)* p. 336—357.]

Remark 2. All quantities figuring in formula (5) are independent of the origin, of the interval, and of the constant C; consequently this formula is valid for all systems of orthogonal polynomials.

If the approximating parabola is known in its form (1) then the problem is solved; but if it is necessary to compute a table of the values $f(a+\xi h)$ for $\xi = 0, 1, 2, \ldots, N-1$, then the corresponding values of $U_m(x)$ must first be computed by aid of formula (7) § 139. This also seems easy enough, especially when using the tables of the binomial coefficients; yet if N is large, the computation is a tedious one. At all events, the calculation would not be shorter if the $U_m(x)$ were expanded into a power *series.*

The labour will be decreased considerably, however, if tables giving the values of $U_m(a+\xi h)$ are available.[43] But tables with a range large enough would be too voluminous, and we shall see that they are superfluous, as by a transformation of formula (1) into a *Newton* series we can get the required values by the

[43] I adopted this procedure in my paper published in 1921, and later *Essher* and *Lorentz* did the same.

method of ***addition of differences*** § 23; and if an interpolation is necessary for any value whatsoever of x, ***Newton's*** formula will give it in the shortest way. Moreover by this method we shall be independent of the value of the constant C, that is of the orthogonal polynomial chosen.

Transformation of the orthogonal series* (1) *into a Newton's expansion. Since the approximating parabola and the mean square deviation are independent of the constant of the orthogonal polynomial used, it is natural to transform equation (1) so that it shall also be independent of this constant, This can be done by a transformation into a ***Newton series.***

For this it is sufficient to determine the differences of $f(a+\xi h)$ for $\xi=0$ and $\Delta\xi=1$. Starting from (7) § 139 we get

$$\Delta^\mu U_m(a) = Ch^{2m} \binom{m+\mu}{m} \binom{m-N}{m-\mu}.$$

Hence from (1) by aid of (3) it will follow that

$$(5) \qquad \&f(a) = \sum_{m=\mu}^{n+1} (-1)^{m-\mu}(2m+1)\binom{m+\mu}{m}\frac{\binom{N-\mu-1}{m-\mu}}{\binom{N+m}{m}}\Theta_m .$$

To abbreviate let us write

$$(6) \qquad \mathscr{C}_{m\mu} = (-1)^{m-\mu}\,(2m+1)\binom{m+\mu}{m}\frac{\binom{N-\mu-1}{m-\mu}}{\binom{N+m}{m}} .$$

Therefore

$$\Delta^\mu f(a) = \sum_{m=\mu}^{n+1} \mathscr{C}_{m\mu}\,\Theta_m .$$

Knowing the differences for $\xi=0$ the problem is solved. The equation of the approximating parabola is

$$f(a+\xi h) = f(a) + \binom{\xi}{1}\Delta f(a) + \ldots + \binom{\xi}{n}\Delta^n f(a).$$

The numbers $\mathscr{C}_{m\mu}$ may be computed by aid of (6) and a table of binomial coefficients, but there is a table giving them up to $\mu = 7$ (parabolas of the seventh degree] and to $N = 100$ [Loc. cit. 41, ***Jordan e)*** pp. 336-3531.

Remark. Having obtained, by the above method, the ***Newton***

expansion corresponding to the approximating parabola of degree n say $y = f_n(x)$, it may happen that the expansion corresponding to a parabola of degree $n+1$ is desired. Then only the calculation of Θ_{n+1} is necessary, and the coefficients of the new expansion will be

$$\Delta^{\mu} f_{n+1}(a) = \Delta^{\mu} f_n(a) + \mathcal{C}_{n+1,\mu}\, \Theta_{n+1} .$$

The work previously done is therefore not lost.

Summary. Given the points x,y for $x=a+\xi h$ where $\xi = 0, 1, 2, \ldots, N-1$ the equation of the parabola of degree n approximating these points is, according to the principle of least squares, or that of moments,

$$f(a+\xi h) = f(a) + \binom{\xi}{1}\Delta f(a) + \binom{\xi}{2}\Delta^2 f(a) + \cdots + \binom{\xi}{n}\Delta^n f(a)$$

where $\Delta^{\mu} f(a)$ is the μ-th difference of $f(a+\xi h)$ with respect to ξ for $\xi=0$ if $\Delta\xi=1$. We have

$$\Delta^{\mu} f(a) = \sum_{m=\mu}^{n+1} \mathcal{C}_{m\mu}\, \Theta_m .$$

$\mathcal{C}_{m\mu}$ being given by formula (6) and

$$\Theta_m = \sum_{\nu=0}^{m+1} \beta_{m\nu}\, \mathcal{T}_\nu \qquad\qquad \mathcal{T}_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \Big/ \binom{N}{\nu+1} .$$

Moreover $\beta_{m\nu}$ is given by formula (5) or the table in § 141

The precision obtained is measured by σ^2

$$\sigma^2 = \mathcal{M}(y^2) - \sum_{m=0}^{n+1} |\mathcal{C}_{m0}|\, \Theta_m^2$$

where $\mathcal{M}(y^2)$ is the mean value of the y^2, that is $\Sigma y^2/N$.

Remark. If there are two sets of observations y,x and z,x given for $x=a+\xi h$ where $\xi=0, 1, \ldots, N-1$; and if we denote the mean square deviation of z by σ', and the orthogonal moments of z by Θ_m', then the coefficient of correlation between y and z will be

$$R = \frac{1}{\sigma\,\sigma'} \left\{ \mathcal{M}(yz) - \sum_{m=0}^{n+1} |\mathcal{C}_{m0}|\, \Theta_m \Theta_m' \right\}$$

Particular case. Approximation of the values of y,x by a function of the first degree, if y is given for $x=0, 1, 2, \dots,$ N-l. We have

$$\beta_{00} = 1 \qquad \beta_{10} = -1 \qquad \beta_{11} = 1$$

$$\mathscr{C}_{00} = 1 \qquad \mathscr{C}_{10} = -\frac{3(N-1)}{N+1} \qquad \mathscr{C}_{11} = \frac{6}{N+1}$$

$$\mathscr{T}_0 = \Theta_0 = \mathscr{A}(y) \qquad \mathscr{T}_1 = \frac{\Sigma xy}{\binom{N}{2}} \qquad \Theta_1 = \mathscr{T}_1 - \mathscr{T}_0$$

and the required function will be

$$f(x) = f(0) + x\Delta f(0)$$

where

$$f(0) = \mathscr{C}_{00}\Theta_0 + \mathscr{C}_{10}\,\Theta_1 = \mathscr{T}_0 - \frac{3(N-1)}{N+1}(\mathscr{T}_1 - \mathscr{T}_0)$$

$$\Delta f(0) = \mathscr{C}_{11}\,\Theta_1 = \frac{6}{N+1}\,(\mathscr{T}_1 - \mathscr{T}_0).$$

If the coordinates are chosen so that $\mathscr{A}(y) = 0$ then

$$f(x) = \frac{6x\mathscr{T}_1}{N+1} - \frac{3(N-1)\mathscr{T}_1}{N+1}. \tag{7}$$

Finally the mean-square deviation will be

$$\sigma^2 = \mathscr{A}(y^2) - \frac{3(N-1)\mathscr{T}_1^2}{N+1}$$

Therefore to obtain the required approximation by formula (7) it is sufficient to determine $\mathscr{T}_1$, the mean binomial moment of the first degree; so that even in this, the simplest case, the approximation by orthogonal polynomials is preferable to the usual method. But the great advantage of the orthogonal polynomials, in shortening the computations, is shown, if approximations are to be performed by parabolas of higher degrees.

§ 143. Graduation. by the method of least squares. If the observation of a phenomenon has given a set of values $y(x_i)$ where $i = 0, 1, 2, \dots,$ N-1 then the $y(x_i)$ will be affected necessarily by errors of observation; if the quantities $y(x_i)$ are statistical data (frequencies), then they will show accidental irregularities so that the differences $\Delta^m y(x_i)$ will be irregular.

The statistician generally wants to "smooth", to "graduate" or to "adjust" his observations, that is to determine a new sequence with regular differences, which differs as little as possible from the observed series.

The simplest way of smoothing is the graphical method; but this gives no great accuracy. A far better method is that of the means in which, writing f (xi) for the smoothed value of y (x_i), we put for instance

$$f(x_i) = \frac{1}{2k+1}\,[y(x_i-kh) + y(x_i-kh+h) + \ldots + y(x_i+kh)]$$

The best method of smoothing is that of the least squares. According to this method, to obtain the smoothed value of y (xi), let us consider first the points of coordinates $x, y(x)$ where

$$x = x_i-kh,\ x_i-kh+h, \ldots, x_i+kh.$$

Then we determine the parabola $y=f(x)$ of degree n which is the best approximation of these points according to the principle of least squares. Finally $f(x_i)$ will be the required smoothed value of y (xi). This will be repeated for every value of i considered.

Proceeding in the usual way this is complicated, but when using orthogonal polynomials then the parabola of degree n approximating the given points may be written

$$f(x) = c_0 + c_1 U_1(x) + \ldots + c_n U_n(x) \tag{1}$$

where according to the notation of § 139 we have $N=2k+1$;

$$a = x_i - kh \quad \text{and} \quad b = x_i + kh + h$$

and therefore

$$x_i = \tfrac{1}{2}(a+b-h) = a + \tfrac{1}{2}(Nh-h).$$

But we have seen in § 140 (formula 2) that

$$U_{2m+1}\left(\frac{a+b-h}{2}\right) = 0. \tag{2}$$

Moreover, according to formula (13) § 140 we find

$$U_{2m}\left(a+\frac{Nh-h}{2}\right)=C_{2m}h^{4m}(-1)^m\binom{2m}{m}\binom{k+m}{2m}.$$

Since in consequence of formula (3) § 142 the coefficient c_{2m} in the expansion (1) is equal to

$$c_{2m}=\frac{(4m+1)\ \Theta_{2m}}{C_{2m}h^{4m}\binom{N+2m}{2m}}$$

hence $f(x_i)$ the smoothed value of $y(x_i)$ will be

$$f(x_i)=\sum_{m=0}^{n+1}(-1)^m(4m+1)\binom{2m}{m}\frac{\binom{k+m}{2m}}{\binom{2k+1+2m}{2m}}\Theta_{2m}.$$

To abbreviate, let us write

$$\gamma_{2m}=(-1)^m(4m+1)\binom{2m}{m}\binom{k+m}{2m}\Big/\binom{2k+1+2m}{2m}.$$

The number γ_{2m} could be easily computed by aid of a table of binomial coefficients; but the following table gives γ_{2m} up to parabolas of the tenth degree and up to $N=29$ points $(k=14)$. The calculation of $f(x_i)$ is now very simple:

$$f(x_i)=\sum_{m=0}^{n+1}\gamma_{2m}\Theta_{2m} \tag{3}$$

all we need is to compute the mean orthogonal moments Θ_{2m}, and $f(x_i)$ will be the smoothed value of $y(x_i)$ obtained by aid of a parabola of degree 2n approximating $N=2k+1$ points.

Remark. It is useless to consider parabolas of odd degree, indeed in consequence of formula (2) a parabola of degree 2n will give the same smoothed value as would a parabola of degree $2n+1$.

In the next paragraph an example will be given.

Table of the numbers γ_{2m}.

N	γ_2	γ_4	Yfl	γ_8	γ_{10}
3	- 1				
5	-1.42857143	0,428571429			
7	-1.66666667	0.818181818	—0,151515152		
9	-1.81818182	1.13286713	-0.363636364	0.0489510490	
11	-1.92307694	1.38461538	-0‘588235294	0.141700405	—0,0150036437
13	- 2	1.58823529	—0,804953560	0.263157895	-0.0508816799
15	-2.05882353	1.75541796	-1.00619195	0.400457666	—0,106851528
17	-2.10526316	1.89473684	-1.18993135	0.544622426	-0.179405034
19	-2.14285714	2.01242236	-1.35652174	0.689855072	-0.264467766
21	-2.17391304	2.11304348	-1.50724638	0.832583708	-0.358311167
23	-2.2	2,2	-1.64367816	0,970708194	-0.457842047
25	-2.22222222	2.27586207	-1.76739587	1.10307749	-0.560622914
27	-2.24137931	2.34260289	-1.87986652	la22914349	—0,664792712
29	-2.25806451	2,40175953	-1.98240469	1.34873583	-0.768962511

§ 144. Computation of the binomial moments. In the preceding paragraphs we have seen that in the calculus of approximation and also in that of graduation, knowledge of the binomial moments is needed. C. *F. Hardy*[44] gave a very useful method for the determination of the binomial moments, which dispenses with all multiplication and may be executed rapidly by aid of calculating machines.

If y is given for $x=a+\xi h$, where $\xi=0, 1, 2, \ldots, N-1$, then the binomial moment of y of order m is given by

$$\mathscr{B}_m = \sum_{\xi=0}^{N} \binom{\xi}{m} y(\xi).$$

The method consists in the following: Denoting by y(S) the value of y corresponding to ξ; in the first column in a table, the values of y(E) are written in the reverse order of magnitude of ξ, that is

$$y(N-1), y(N-2), \ldots, y(1), y(0).$$

In the first line of every column we write the same number $y(N-1)$. Into the ν-th line of the μ-th column we put the sum of the two numbers figuring in the line $\nu-1$ of column μ, and in the line ν of column $\mu-1$.

Therefore, denoting the number written in the line ν of column μ by $\varphi(\nu,\mu)$, the rule of computation will be

(1) $$\varphi \text{ h-4} = \varphi(\nu-1,\mu) + \varphi(\nu,\mu-1).$$

The solution of this equation of partial differences of the first order, by *Laplace's* method of generating functions (§ 181, Ex. 5) is the following:

(2) $$\varphi(\nu,\mu) = \sum_{i=1}^{\nu+1} \binom{\mu-2+\nu-i}{\mu-2} y(N-i).$$

The initial conditions are satisfied; indeed for $\nu=1$ we get $\varphi(1,\mu)=y(N-1)$. Putting into the formula obtained $\nu=N-\mu+2$ and $i=N-\xi$ we get

[44] G. F. *Hardy,* Theory of construction of Tables of Mortality. 1909, London; p. 59 and onwards.

$$\varphi(N-\mu+2,\mu) = \sum_{\xi=\mu-2}^{N} \binom{\xi}{\mu-2} Y(\xi).$$

Therefore the number figuring in the line $N-\mu+2$ of column μ is equal to the binomial moment of degree $\mu-2$. Hence if we want the binomial moments $\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_n$, then we must compute $n+1$ columns. The results obtained give by aid of formula (4), § 141 the corresponding mean binomial moments $\mathscr{G}_m$.

Remark. If we put $\nu=N$ into formula (2) we get the following moments

$$\sum_{x=a}^{h} \binom{x+\mu-2}{\mu+2} y(x)$$

that is the numbers in the last line of the table arc equal to these quantities.

Example. Given the following observed values:

$x=a$,	$y=2502$	$x=a+5h$,	$y=2904$
$x=a+h$,	$y=2548$	$x=a+6h$,	$y=3064$
$x=a+2h$,	$y=2597$	$x=a+7h$,	$y=3188$
$x=a+3h$,	$y=2675$	$x=a+8h$,	$y=3309$
$x=a+4h$,	$y=2770$		

The graduated values of y corresponding to $x=a+4h$ are to be determined corresponding to nine-point parabolas of the second and of the fourth degree.

To begin with, we shall first determine, by aid of the preceding method, the binomial moments $\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_4$. For this purpose we write in the first column below the values of y in reverse order, and in the first line of every column wc put the last value of y; the other numbers of the table are computed by aid of rule (1).

3309	3309	3309	3309	3309	3309
3188	6497	9806	13115	16424	19733
3064	9561	19367	32482	48906	68630
2904	12465	31832	64314	113220	181859
2770	15235	47067	111381	224601	406460
2675	17910	64973	176358	400959	
2597	20507	85484	261842		
2548	23055	108539			
2502	25557				

The last number in the second column is equal to $\mathcal{B}_0$. hence

$$\mathcal{T}_0 = \mathcal{B}_0 / 9 = 25557 / 9 = 2839.6667.$$

The last number in the third column is equal to $\mathcal{B}_1$, therefore

$$\mathcal{T}_1 = \mathcal{B}_1 / \binom{9}{2} = 108539 / 36 = 3014.9722,$$

The last number in tbe fourth column is $\mathcal{B}_2$, so that

$$\mathcal{T}_2 = \mathcal{B}_2 / \binom{9}{3} = 261842 / 84 = 3117.1667.$$

The last number in the fifth column is $\mathcal{B}_3$, therefore

$$\mathcal{T}_3 = \mathcal{B}_3 / \binom{9}{4} = 400959 / 126 = 3182.2143.$$

Finally the last number in the sixth column is $\mathcal{B}_4$, hence

$$\mathcal{T}_4 = \mathcal{B}_4 / \binom{9}{4} 406460 / 126 = 3225.8730 .$$

Now by aid of formula (6) § 141 we may compute the mean orthogonal moments. We have

$$\Theta_0 = \mathcal{T}_0 = 2839.6667 ; \qquad \Theta_2 = 2\mathcal{T}_2 - 3\mathcal{T}_1 + \mathcal{T}_0 = 29.08335$$

$$0, = 14\mathcal{T}_4 - 35\mathcal{T}_3 + 30\mathcal{T}_2 - 10\mathcal{T}_1 + \mathcal{T}_0 = -10.33148.$$

Tbe required graduated values will be

$$f_2(a+4h) = 0, + \gamma_2\Theta_2 = 2786.78783$$

and

$$f_4(a+4h) = \Theta_0 + \gamma_2\Theta_2 + \gamma_4\Theta_4 = 2775.0837.$$

The constants γ_2 and γ_4 corresponding to $N=9$ (nine-point parabola) were taken from the table of § 143; we found

$$\gamma_2 = -1.81818182 \qquad \text{and} \qquad \gamma_4 = 1.13286713.$$

§ 145. Fourier series. A function f(x) of a continuous variable x, with limited total fluctuation in the interval a, b may be expanded into a *Fourier series.*

Let us write

$$(1) \qquad f(x) = \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} \alpha_m \cos \frac{2\pi m(x-a)}{b-a} + \sum_{m=1}^{\infty} \beta_m \sin \frac{2\pi m(x-a)}{b-a}$$

The determination of the coefficients α_m and β_m is simple in consequence of the orthogonality of the circular functions. Indeed we have, if $m \neq \mu$

$$\int_a^b \cos \frac{2\pi m(x-a)}{b-a} \cos \frac{2\pi \mu (x-a)}{b-a}\, dx =$$

$$= \frac{b-a}{2\pi} \int_0^{2\pi} \cos m\xi \cos \mu\xi \, d\xi = 0$$

where $\xi = 2\pi (x-a)/(b-a)$ has been put, In the same manner we should have

$$\int_0^{2\pi} \sin m\xi \sin \mu\xi \, d\xi = 0 \qquad \text{and} \qquad \int_0^{2\pi} \sin m\xi \cos \mu\xi \, d\xi = 0.$$

The last equation holds for every integer value of m and μ. Moreover if m is different from zero, then

$$\int_a^b \left[\cos \frac{2\pi m(x-a)}{b-a}\right]^2 dx = \frac{b-a}{2\pi} \int_0^{2\pi} \cos^2 m\xi \, d\xi = \tfrac{1}{2}(b-a)$$

and in the same way

$$\int_a^b \left[\sin \frac{2\pi m(x-a)}{b-a}\right]^2 dx = \tfrac{1}{2}(b-a).$$

Therefore, multiplying both members of (1) by $\cos \frac{2\pi m(x-a)}{b-a}$ we obtain, after integration from x-a to $x=b$:

$$(2) \qquad a_m = \frac{2}{b-a} \int_a^b f(x) \cos \frac{2\pi m(x-a)}{b-a}\, dx$$

in the same manner, multiplying by sin $\frac{2\pi m\,(x-a)}{b-a}$ and integrating, we find

$$(3) \qquad \beta_m = \frac{2}{b-a} \int_a^b f(x) \sin \frac{2\pi m(x-a)}{b-a}\, dx.$$

Putting these values into (1) we obtain the expansion of *f(x)* into a *Fourier series.*

Approximation of a function f(x) of a continuous variable by a *Fourier* series of $2n+1$ terms. Let

$$(4) \qquad f_n(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{n+1} a_m \cos \frac{2\pi m(x-a)}{b-a} + \\ + \sum_{m=1}^{n+1} \beta_m \sin \frac{2\pi m\,(x-a)}{b-a}.$$

The coefficients a_m and β_m are to be determined so that according to the principle of least squares

$$(5) \qquad \mathscr{S} = \int_a^b [f(x) - f_n(x)]^2 dx$$

shall be a minimum. Putting into it the value (4) of $f_n(x)$, the equations determining the minimum will be

$$\frac{\partial \mathscr{S}}{\partial \alpha_m} = 0 \quad \text{and} \quad \frac{\partial \mathscr{S}}{\partial \beta_m} = 0.$$

This gives $2n+1$ equations which determine the coefficients a_m and β_m. It is easily seen that in consequence of the **ortho**gonality we obtain the same values as before in (2) and (3).

From this we conclude that, expanding *f(x)* into a *Fourier* series and stopping at the terms a_n, β_n we obtain the best approximation attainable by aid of these terms.

Putting into (5) the value (4) of $f_n(x)$ we get in consequence of the orthogonality and of the equations (2) and (3)

$$\mathscr{S} = \int_a^b [f(x)]^2\, dx - \tfrac{1}{2}(b-a) \sum_{m=1}^{n+1} (\alpha_m^2 + \beta_m^2) - \frac{b-a}{4} a_0^2.$$

Example. In the calculus of probability and in mathematical statistics a function f(x) of a continuous variable is often considered, in which $f(x) = y_\xi$ is constant in the interval $a+\xi h < x < a+\xi h+h$; and this is true for every interval h, though the y_ξ generally differ from one interval to another. For instance, $f(x)$ is the probability that the number of the favourable events does not exceed x.

This function is a discontinuous one; but since the total fluctuation is limited, therefore it is representable by a *Fourier* series. Putting into the formulae above $b=a+Nh$, *we* find

$$\beta_m = \frac{2}{Nh} \sum_{\xi=0}^{N} y_\xi \int_{a+\xi h}^{a+\xi h+h} \sin \frac{2\pi m(x-a)}{Nh}\, dx$$

a n d

$$\beta_m = \frac{1}{\pi m} \sum_{\xi=0}^{N} y_\xi \left| \cos \frac{2\pi m\xi}{N} - \cos \frac{2\pi m(\xi+1)}{N} \right|$$

finally

$$\mathrm{Pm} = \frac{2}{\pi m} \sin \frac{m\pi}{N} \sum_{\xi=0}^{N} y_\xi \sin \frac{\pi m(2\xi+1)}{N} .$$

In the same manner we should have

$$a_m = \frac{2}{m\pi} \sin \frac{m\pi}{N} \sum_{\xi=0}^{N} y_\xi \cos \frac{m\pi(2\xi+1)}{N} .$$

§ 146. Approximation by trigonometric functions of discontinuous variables. Let us suppose that the numbers y(x) are given for $x=a+\xi h$ and $\xi = 0, 1, 2, \ldots$, N-l.

A function $f(x)$ is to be determined which gives for the above values $f(x) = y$ (x). This will be done in the following manner :

Starting from equation

$$\text{(1)} \quad f(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{n+1} \left[\beta_m \sin \frac{2\pi m(x-a)}{Nh} + a_m \cos \frac{2\pi m(x-a)}{Nh} \right]$$

where n is the greatest integer contained in N/2, the coefficients $a_{,,,}$ and β_m are to be determined so as to have

$$f(a+\xi h) = y(a+\xi h) \quad \text{for} \quad \xi = 0, 1, 2, \ldots, \text{N-l}.$$

If N is odd, $N=2n+1$, then this gives N equations of the first **degree** with N unknowns to solve; if N is even, $N=2n$, then the number of the unknowns is also equal to $2n$, since then the term $\beta_{2n}\sin 2\pi\xi$ vanishes for every value of ξ. The resolution is greatly simplified owing to the orthogonality of the introduced circular functions established in § 43.

Remarking that $2m+1 \leqq N$ and $2\mu+1 \leqq N$, the formulae found there will be, writing in order to abbreviate $\xi=(x-a)/h$ if m is an integer different from μ:

$$\sum_{\xi=0}^{N} \sin\frac{2\pi m\xi}{N}\sin\frac{2\pi\mu\xi}{N} = 0 \text{ and } \sum_{\xi=0}^{N}\cos\frac{2\pi m\xi}{N}\cos\frac{2\pi\mu\xi}{N} = 0$$

and for every integer value of m and μ,

$$\sum_{\xi=0}^{N} \sin\frac{2\pi\mu\xi}{N}\cos\frac{2\pi m\xi}{N} = 0.$$

Moreover,

$$(2) \qquad \sum_{\xi=0}^{N}\cos^2\frac{2\pi m\xi}{N} = \sum_{\xi=0}^{N}\sin^2\frac{2\pi m\xi}{N} = \frac{N}{2}.$$

To determine the coefficient a_m, let us multiply both members of (1) by $\cos\dfrac{2\pi m\,(x-u)}{Nh}$ and sum from $x=a$ to $x=a+Nh$; then in consequence of the formulae above we find

$$\sum_{x=a}^{a+Nh} y(x)\cos\frac{2\pi m\,(x-a)}{Nh} = a_m\sum_{x=a}^{a+Nh}\cos^2\frac{2\pi m(x-a)}{Nh} = \frac{N}{2}\alpha_m$$

since the other terms vanish. Finally we have

$$(3) \qquad \alpha_m = \frac{2}{N}\sum_{x=a}^{a+Nh} y(x)\cos\frac{2\pi m\,(x-a)}{Nh}$$

and in the same way

$$(4) \qquad \beta_m = \frac{2}{N}\sum_{x=a}^{a+Nh} y(x)\sin\frac{2\pi m\,(x-a)}{Nh}.$$

Putting (3) and (4) into equation (I) the problem is solved.

Approximation. y(x) is given for $x=a+\xi h$ and $\xi=0, 1, 2, \ldots, N-1$. A function

$$(5) \quad f(x) = \frac{\alpha_0}{2} + \sum_{m=0}^{n+1}\left[\beta_m\sin\frac{2\pi m(x-a)}{Nh} + \alpha_m\cos\frac{2\pi m(x-a)}{Nh}\right]$$

where $n < \frac{1}{2}(N-1)$, is required so that it shall be the best approximation of the given values, according to the principle of the least squares, that is, which makes

$$(6) \qquad \mathscr{S} = \sum_{x=a}^{a+Nh} [f(x) - y(x)]^2$$

a minimum. Therefore the coefficients a,,, and β_m are determined by the following equations

$$\frac{\partial \mathscr{S}}{\partial \alpha_m} = 0 \quad \text{and} \quad \frac{\partial \mathscr{S}}{\partial \beta_m} = 0$$

this gives the necessary $2n+1$ equations to determine the coefficients.

Putting into (6) the value of $f(x)$ given by (5) we obtain for a,,, and β_m the same expressions as before, (3) and (4).

Moreover from (6) we deduce, in consequence of the crthogonality by aid of equations (3) and (4), that

$$(7) \qquad \mathscr{S} = \sum_{x=a}^{a+nh} [f(x)]^2 - \frac{N}{4} a_0{}^2 - \frac{N}{2} \sum_{m=1}^{n+1} (a_m{}^2 + \beta_m{}^2).$$

Formula (5) may be useful for detecting some hidden periodicity of the numbers y(x).

§ 147. **Hermite** polynomials. In the general case this polynomial of degree m could be defined by[45]

$$H_m = e^{k^2x^2} \mathrm{D}^m[e^{-k^2x^2}]$$

but it is better to do it in the following particular case, in which the formulae are the simplest, and which is very suitable for approximation purposes, by aid of the probability function.

$$(1) \qquad H_m = H_m(x) = e^{x^2/2} \mathrm{D}^m[e^{-x^2/2}] .$$

This may be written

$$(2) \qquad H_m = A_{m,m}x^m + A_{m,m-1}x^{m-1} + \ldots\ldots + A_{m,0}$$

where the coefficients $A_{m,i}$ are independent of x; moreover from (1) it follows that

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[45] *Tchebichef,* Sut le dévelop ›pnent des fonctions. 1859. Oeuvres I. p. 505.
Hermite, Sur un nouveau d6veloppement en série. Compte Rendu 1664.
Markoff, Wahrscheinlichkeitsrechnung, p. 259. 1912.
Moser, Jahresbericht, Band 21, p. 9.
CR. *Jordan,* Probabilité des Epreuves répétées, Bulletin, Soc. Math. de France, 1926, pp. 101—137. — Statistique Mathématique, Paris, 1927.

$$\mathbf{D}^m e^{-x^2/2} = \mathbf{D}\,[e^{-x^2/2} H_{m-1}]$$

multiplying both members by $e^{x^2/2}$ this gives

$$H_m = \mathbf{D}H_{m-1} - xH_{m-1}$$

writing by aid of **(2)** that the coefficient of x^i in both members is the same, we get the following equation of partial differences:

(3) $$A_{m,i} = (i+1)A_{m-1,i+1} - A_{m-1,i-1}\,.$$

According to **§ 181** the conditions

$$A_{,,,} = 1 \quad\text{and}\quad A_{m,i} = 0 \text{ if } i > m \text{ or } i < 0.$$

are sufficient for the determination of the coefficients $A_{m,i}$.

Indeed, starting from these values a table of the coefficients may rapidly be computed by aid of (3):

$m \backslash i$	0	1	2	3	4	5
0	1					
1	0	- 1				
2	- 1	0	1			
3	0	3	0	—1		
4	3	0	-6	0	1	
5	0	- 1 5	0	10	0	- 1
		. . . . .				

Therefore we shall have

$$H_0 = 1, \quad H_1 = -x, \quad H_2 = x^2 - 1, \quad H_3 = -x^3 + 3x, \quad H_4 = x^4 - 6x^2 + 3$$

To obtain A_{mi} in a general form it is better to solve **(6')**, and get

$$A_{n,n-\nu} = (-1)^{n-\nu} \binom{n}{\nu} A_{\nu,0}\,.$$

Moreover, putting $x = 0$ into **(7)** to find

$$A_{2\nu+1,0} = 0 \qquad A_{2\nu,0} = (-1)^\nu 1\,.\,3\,.\,5 \ldots (2\nu - 1) = \frac{(-1)^\nu (2\nu)!}{\nu!\,2^\nu}\,.$$

(4) $$H_m = m! \sum_{\nu=0}^{r} \frac{(-1)^{m+\nu} x^{m-2\nu}}{\nu!\,(m-2\nu)!\,2^\nu}$$

where r is the greatest integer contained in $(m+2)/2$.

Mathematical properties of the Hermite polynomials. From (1) we obtained above

(5) $$DH_n = H_{n+1} + xH_n.$$

On the other hand, by aid of (4) it can shown, that

(6) $$DH_n = -nH_{n-1}$$

that is

(6') $$iA_{n,i} = -nA_{n-1,i-1}.$$

This is another difference equation giving the coefficients $A_{m,i}$.

From (5) and (6) we may deduce immediately the function equation of the Hermite polynomials.

(7) $$H_{n+1} + xH_n + nH_{n-1} = 0.$$

The second derivative of H_n obtained from (5) is

$$D^2H_n = DH_{n+1} + H_n + xDH_n$$

finally by aid of (6) *we* get the *differential equation* of the polynomials.

(8) $$D^2H_n - xDH_n + nH_n = 0.$$

From (6) it follows that

(9) $$\int H_n dx = -\frac{H_{n+1}}{n+1} + C$$

and from (1) if $n \geqq 1$

(10) $$\int H_n e^{-x^2/2}\, dx = H_{n-1}\, e^{-x^2/2} + C.$$

Orthogonality of the polynomials. It may be shown by repeated integration by parts, if $n \neq m$, that

$$\int^{\infty} H_n H_m e^{-x^2/2}\, dx = 0$$

(11)

$$\int_{-\infty}^{\infty} (H_m)^2 e^{-x^2/2}\, dx = m! \sqrt{2\pi}.$$

Remark 1. Determining the polynomial Z(x) of degree *n, in* which the coefficient of x^n is equal to $(-1)^n$, and which makes minimum the quantity

$$\int_{-\infty}^{\infty} [Z(x)]^2 e^{-x^2/2}\, dx$$

we find that $Z(x) = H_n(x)$.

Remark 2. The roots of the equation $H_n(x) = 0$ are all real and are included in the interval -2 $\sqrt{n}$ and 2 $\sqrt{n}$.

Expansion of a function into a series of Hermite functions $H_n e^{-x^2/2}$. If a function f(x) and its two first derivatives are continuous and finite from $-\infty$ to ∞, and if

$$f(\pm\infty) = 0 \qquad Df(\pm\infty) = 0, \qquad D^2 f(\pm\infty) = 0$$

then *f(x)* may he expanded into the following convergent series:

$$(12) \qquad f(x) = [c_0 + c_1 H_1 + c_2 H_2 + \ldots] e^{-x^2/2}.$$

Multiplying both members by H_m and integrating from $-\infty$ to ∞ we find, in consequence of the relations of orthogonality (11),

$$(13) \qquad c_m = \frac{1}{m!\sqrt{2\pi}} \int_{-\infty}^{\infty} H_m f(x)\, dx.$$

To obtain the integral contained in the second member, let us introduce the power-moments of the function *f(x)* by

$$\mathcal{M}_n = \int_{-\infty}^{\infty} x^n f(x)\, dx.$$

Replacing H_n in (13) by its value (4) we obtain:

$$(14) \qquad c_m = \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{r} \frac{(-1)^{m+i} \mathcal{M}_{m-2i}}{i!\,(m-2i)!\,2^i}$$

Particular cases:

$$c_0 = \frac{\mathcal{M}_0}{\sqrt{2\pi}} \qquad c_1 = -\frac{\mathcal{M}_1}{\sqrt{2\pi}} \qquad c_2 = \frac{1}{2\sqrt{2\pi}} [\mathcal{M}_2 - \mathcal{M}_0]$$

$$c_3 = \frac{1}{6\sqrt{2\pi}} [3\mathcal{M}_1 - \mathcal{M}_3] \qquad c_4 = \frac{1}{24\sqrt{2\pi}} [3\mathcal{M}_0 - 6\mathcal{M}_2 + \mathcal{M}_4]$$

. . . . .

Example. Expansion of x^n into a series of Hermite polynomials; that is, $x^n e^{-x^2/2}$ is to be expanded into the series (12). Hence the coefficients will be

$$c_m = \frac{1}{m!\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n H_m e^{-x^2/2}\, dx.$$

By m times repeated integration by parts, this gives, taking account of (10),

$$c_m = \frac{(-1)^m \binom{n}{m}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{n-m} e^{-x^2/2} dx.$$

From this it follows immediately that $c_n = (-1)^n$.

Moreover, integration by parts shows that $c_{n-2i-1} = 0$. To obtain c_{n-2i} let us write the above integral in the following way:

$$\int_{-\infty}^{\infty} -x^{2i-1} (-x\, e^{-x^2/2})\, dx = (2i-1) \int_{-\infty}^{\infty} x^{2i-2} e^{-x^2/2} dx.$$

We repeat this operation till we get

$$1 . 3 . 5 \ldots (2i-1) \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1 . 3 . 5 \ldots (2i-1) \sqrt{2\pi}.$$

Finally it results in

$$c_{n-2i} = (-1)^n \binom{n}{2i} 1 . 3 . 5 \ldots (2i-1) = \frac{(-1)^n n!}{i!\,(n-2i)!\,2^i}$$

and

$$\text{(15)} \qquad x^n = (-1)^n n! \sum_{i=0}^{r} \frac{H_{n-2i}}{i!\,(n-2i)!\,2^i}$$

Particular cases:

$$x = -H_1 \qquad x^2 = H_2 + H_0 \qquad x^3 = -H_3 - 3 H_1 .$$

Integration of a function expanded into a series (12). According to (10) we get

$$\int_{-\infty}^{\lambda} f(x)dx = c_0 \int_{-\infty}^{\lambda} e^{-x^2/2} dx + e^{-\lambda^2/2} \sum_{m=1}^{\infty} c_m H_{m-1} (R) .$$

We have seen (formula 14) that in the expansion of a function $f(x)$ into a series of *Hermite* functions, and stopping at H_n,

$$f_n(x) = [c_0 + c_1 H_1 + \ldots c_n H_n]\, e^{-x^2/2}$$

the coefficients c_m are determined by aid of the moments $\mathcal{M}_0$, $\mathcal{M}_1, \ldots, \mathcal{M}_n$ of the function $f(x)$, Let us show now that the first $n+1$ moments of $f_n(x)$ are the same as the corresponding moments of $f(x)$; and therefore $f_n(x)$ is an approximation of $f(x)$ according to the ***principle of moments.***

To obtain the moment of order m of $f_n(x)$ let **us multiply** $f_n(x)$ by x^m given by formula (15) ; we find

$$x^m f_n(x) = \Sigma\Sigma \frac{(-1)^m m!\, c_\nu}{i!\,(m-2i)!\, 2^i} H_{m-2i}\, H_\nu e^{-x^2/2}$$

Integrating from $-\infty$ to ∞, in consequence of the **ortho**gonality (11) we get

$$\Sigma \frac{(-1)^m m!\, c_{m-2i} \sqrt{2\pi}}{i!\, 2^i}$$

Putting into it the value of c_{m-2i} taken from (14), it gives

$$\Sigma\Sigma \frac{(-1)^s\, m!\, \mathcal{M}_{m-2i-2s}}{i!\, s!\, 2^{i+s} (m-2i-2s)!};$$

writing $i+s=\mu$:

$$\sum_{\mu=0}^{1} \frac{m!\, \mathcal{M}_{m-2\mu}}{\mu!\, 2^\mu (m-2\mu)!} \sum_{s=0}^{\mu+1} (-1)^s \binom{\mu}{s}.$$

The second sum is equal to zero for every value of μ except **for** $\mu=0$; but for $\mu=0$ the preceding expression is equal to I,. Consequently, $\mathcal{M}_m$ the moment of order m of $f(x)$, is **also** a moment of $f_n(x)$. Q . E . D.

Numerical evaluation of $H_n(x)e^{-x^2/2}/\sqrt{2\pi}$. Since there are no good tables giving the derivatives of the probability function

$$z = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

we have to compute $H_n(x)$ (formula 4) and multiply the obtained value by z taken from *Sheppard's* Tables.[45a]

Particular values :

$$H_{2m+1}(0) = 0 \qquad H_{2m}(0) = \frac{(-1)^m (2m)!}{m!\, 2^m}$$

[45a] K. Pearson, Tables for Statisticians and Biometricians, Part I. (Table II). London.

§ 148. G. polynomials.[46] Let us denote by $\psi(m,x)$ the following function introduced by *Poisson,* when he deduced **his** formula of probability of repeated trials:

$$\psi = \psi(m,x) = \frac{m^x}{x!}\, e^{-m}. \tag{1}$$

Let the definition of the polynomial $G_n(x)$ of degree n be

$$G_n = G_n(x) = \frac{\underset{m}{D}^n \psi(m,x)}{\psi(m,x)} \tag{2}$$

where by $\underset{m}{D}$ the derivative with respect to m is understood.

***Deduction** of the polynomials.* We have

$$\underset{x}{\Delta}\psi = \left(\frac{m}{x+1} - 1\right)\psi.$$

In § 6 we found the following formula for the n-th difference of a function

$$\Delta^n f(x) = \sum_{i=0}^{n+1} (-1)^i \binom{n}{i} f(x+n-i);$$

therefore,

$$\underset{x}{\Delta}^n \psi(m,x-n) = \sum_{i=0}^{n+1} (-1)^i \binom{n}{i} \psi(m,x-i).$$

Moreover

$$\underset{m}{D}\,\psi = \psi(m,x-1) - \psi(m,x) = -\underset{x}{\Delta}\psi(m,x-1).$$

Hence

$$\underset{m}{D}^n \psi = (-1)^n \underset{x}{\Delta}^n \psi(m,x-n)$$

and finally

$$G_n(x) = \frac{(-1)^n}{\psi(m,x)} \underset{x}{\Delta}^n \psi(m,x-n). \tag{3}$$

In consequence of what precedes this gives, if we introduce ?he variable $\nu = n - i$, this gives

$$G_n(x) = \frac{n!}{m^n} \sum_{\nu=0}^{n+1} (-1)^\nu \binom{x}{n-\nu} \frac{m^\nu}{\nu!}. \tag{4}$$

[46] *Ch. Jordan,* a) Probabilité des Épreuves répétées, Bulletin de la Société Mathématique de France, 1926, pp. 101—137.
b) Statistique Mathématique. Paris, 1927, pp. 36—40, 99—102, etc.

Particular cases:

$$G_0 = 1$$

$$G_1(x) = \frac{1}{m}\left[\binom{x}{1} - m\right]$$

$$G_2(x) = \frac{2!}{m^2}\left[\binom{x}{2} - \frac{m}{1!}\binom{x}{1} + \frac{m^2}{2!}\right]$$

$$G_3(x) = \frac{3!}{m^3}\left[\binom{x}{3} - \frac{m}{1!}\binom{x}{2} + \frac{m^2}{2!}\binom{x}{1} - \frac{m^3}{3!}\right]$$

$$\cdot\ ,\ \cdot\ \cdot\ \cdot$$

Particular values of the polynomial. From (4) it follows immediately that

$$G_n(0) = (-1)^n$$

and

$$G_n(-1) = \frac{(-1)^n n!}{m^n} \sum_{\nu=0}^{n+1} \frac{m^\nu}{\nu!}\,.$$

We get an *important particular value* of $G_n(x)$ by putting into (4) $x=m$:

$$(4') \qquad G_n(m) = \frac{n!}{m^n} \sum_{\mu=0}^{n+1} (-1)^\nu \binom{m}{n-\nu} \frac{m^\nu}{\nu!}$$

or by putting into it $\mu=n-\nu$,

$$G_n(m) = \sum_{\nu=0}^{n+1} (-1)^{n+\mu} \binom{n}{\mu} \frac{(m)_\mu}{m^\mu}$$

therefore according to § 6 we have

$$G_n(m) = \left[\Delta_\mu^n \frac{(m)_\mu}{m^\mu}\right]_{\mu=0}$$

Expanding the factorial into a power series we obtain

$$\frac{(m)_\mu}{m^\mu} = \sum_{i=1}^{\mu+1} S_\mu^i\, m^{i-\mu};$$

writing in it $k=\mu-i$

$$\frac{(m)_\mu}{m^\mu} = \sum_{k=0}^{\mu} S_\mu^{\mu-k}\, m^{-k}.$$

Hence we may write

$$G_n(m) = \Delta_\mu^n \left[\sum_{k=0}^{\infty} m^{-k} S_\mu^{\mu-k}\right]_{\mu=0}.$$

According to formula (5) § 52 we have

$$S_{\mu}^{\mu-k} = \sum_{i=0}^{k} C_{k,i} \binom{\mu}{2k-i};$$

from this we deduce

$$[\underset{\mu}{\Delta^n} S_{\mu}^{\mu-k}]_{\mu=0} = C_{k,\ 2k-n}$$

and finally

$$\text{(4'')} \qquad G,(m) = \sum_{k=r}^{n} C_{k,2k-n} \frac{1}{m^k} \cdot$$

where r is the greatest integer contained in $(n+1)/2$.
The numbers $C_{\nu,m}$ may be computed by aid of the difference equation (4) of § 52, or taken from the table of this paragraph.

Particular cases: $G_1(m) = 0$, $G,(m) = -1/m$, $G,(m) = 2/m^2$, $G_4(m) = 3(m-2)/m^3$, $G,(m) = 4(6-5m)/m^4$ **and so on.**

Mathematical properties of the G polynomials. **Starting** *from (2) we* **deduce, if** $\Delta x = 1$,

$$\underset{x}{\Delta} G_n = \frac{\underset{m}{\mathrm{D}^n}(m\psi) - m\underset{m}{\mathrm{D}^n}\psi}{m\psi}$$

Applying ***Leibnitr's*** **formula giving the** n **-th** derivative of a **product (§ 30), we obtain**

$$\text{(5)} \qquad \underset{x}{\mathrm{AG},} = \frac{n}{m}\ \mathrm{G},-, .$$

In the same manner we may deduce from (2)

$$\underset{m}{\mathrm{D}} G_n = \frac{m\underset{m}{\mathrm{D}^{n+1}}\psi - (x-m)\underset{m}{\mathrm{D}^n}\psi}{m\psi}.$$

that is,

$$\text{(6)} \qquad \underset{m}{\mathrm{D}} G, = G_{n+1} - \frac{x-m}{m} G_n .$$

From *(5)* **it follows that**

$$\text{(7)} \qquad \underset{x}{\Delta^{-1}} G_n = \frac{m}{n+1} G_{n+1} + k.$$

Moreover, by aid of (3) we find

(8) $$\underset{x}{\Delta}^{-1} G_n(x)\psi(m,x) = (-1)^n \underset{x}{\Delta}^{n-1}\psi(m,x-n) + k =$$
$$= - G_{n-1}(x-1)\ \psi(m,x-1) + k.$$

Since

$$\sum_{x=i}^{\infty} \binom{x}{i} \psi(m,x) = \frac{m^i e^{-m}}{i!} \sum_{x=i}^{\infty} \frac{m^{x-i}}{(x-i)!} = \frac{m^i}{i!};$$

therefore from (4)

$$\sum_{x=0}^{\infty} G_n(x)\ \psi(m,x) = \frac{n!}{m^n} \sum_{\nu=0}^{n+1} (-1)^\nu \frac{m^\nu}{\nu!} \sum_{x=0}^{\infty} \binom{x}{n-\nu} \quad \psi(m,x)$$

and finally

$$\sum_{x=0}^{\infty} G_n\psi = \frac{n!}{m^n} \sum_{\nu=0}^{n+1} (-1)^\nu \frac{m^\nu}{\nu!} \frac{m^{n-\nu}}{(n-\nu)!} = \sum_{\nu=0}^{n+1} (-1)^\nu \binom{n}{\nu} = 0$$

so that

(9) $$\sum_{x=0}^{\infty} G_n\psi = 0$$

if $n > 0$.

Orthogonality of the polynomials. The operation of summation by parts gives (§ 35)

$$\Delta^{-1}\left[\binom{x}{i} \cdot G_n(x)\psi(m,x)\right] = -\binom{x}{i} G_{n-1}(x-1)\psi(m,x-1) +$$
$$+ \Delta^{-1}\left[\binom{x}{i-1} G_{n-1}(x)\psi(m,x)\right].$$

It is easy to see that

(10) $$\lim_{x=\infty} \binom{x}{i}\psi(m,x) = 0$$

indeed

$$e^{-m}\binom{x}{i}\frac{m^x}{x!} = \frac{e^{-m}m^i}{i!} \frac{m^{x-i}}{(x-i)!}$$

and

$$\lim_{x=\infty} \frac{m^{x-i}}{(x-i)!} = 0.$$

From (10) it follows in consequence of (4) that

(11) $$\lim_{x=\infty} \binom{x}{i} G_m\psi = 0.$$

Therefore

$$\sum_{x=i}^{\infty}\left[\binom{x}{i} G_n(x)\psi(m,x)\right] = \sum_{x=i-1}^{\infty}\left[\binom{x}{i-1} G_{n-1}(x)\psi(m,x)\right]$$

continuing in this manner the first member will be equal to

$$\text{(12)} \qquad \sum_{x=0}^{\infty} G_{n-i}(x)\psi(m,x).$$

But if $n > i$ then this quantity is equal to zero according to (9). So that we have

$$\text{(13)} \qquad \sum_{x=0}^{\infty} \binom{x}{i} G_n\psi = 0 \qquad \text{if} \qquad n > i.$$

f-fence if P_ν is any polynomial whatever of degree ν and if $n > \nu$ then

$$\text{(14)} \qquad \Sigma\ P_\nu G_n\psi = 0$$

and in particular

$$\text{(15)} \qquad \Sigma \quad G_n G_\nu \psi = 0 \qquad \text{if} \qquad n \neq \nu.$$

This is the relation of orthogonality of the polynomials. Putting into (12) $n=i$ we get

$$\sum_{x=0}^{\infty} \psi(m,x) = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!} = 1.$$

Therefore

$$\text{(16)} \qquad \sum_{x=0}^{\infty} \binom{x}{n} G_n\psi = 1.$$

Finally, if $n < i$ then from the preceding it follows that

$$\sum_{x=0}^{\infty} \binom{x}{i} G_n\psi = \sum_{x=0}^{\infty} \binom{x}{i-n} \psi(m,x)$$

since the second member is equal to

$$\frac{m^{i-n}}{(i-n)!} \sum_{x=i-1}^{\infty} e^{-m} \frac{m^{x-i+n}}{(x-i+n)!} = \frac{m^{i-n}}{(i-n)!}.$$

Therefore

$$\text{(17)} \qquad \sum_{x=0}^{\infty} \binom{x}{i} G_n\psi = \frac{m^{i-n}}{(i-n)!} \qquad \text{if} \qquad i > n.$$

Since, according to (4) the coefficient of $\binom{x}{n}$ in G,(x) is equal to $n!/m^n$ therefore from (16) it follows that

$$(18) \qquad \sum_{x=0}^{\infty} [G_n]^2\psi = \frac{n!}{m^n}.$$

Expansion of a function f(x) of a discontinuous variable $(x=0, 1, 2, \ldots, N-1)$ into a series of functions $G_\nu \psi(m,x)$. That is

$$(19) \qquad f(x) = [c_0 + c_1G_1 + c_2G_2 + \ldots\ldots]\psi(m,x).$$

To obtain the coefficients c_n in (19) let us multiply both members of this equation by G,(x), and sum from $x=0$ to $x=\infty$; this may be done, if we put *f(x)* $=0$ for every integer value of x such as x $\geq$ N. In consequence of the orthogonality (15) we shall have

$$\sum_{x=0}^{\infty} f(x)G_n(x) = c_n \sum_{x=0}^{\infty} [G_n]^2\psi;$$

hence according to (18) the coefficient c_n will be

$$(20) \qquad c_n = \frac{m^n}{n!} \sum_{x=0}^{\infty} G_n(x)f(x).$$

This may be transformed by putting into it the value of G_n given by equation (4). We get

$$c_n = \sum_{i=0}^{n+1} (-1)^i \frac{m^{i}}{i!} \sum_{x=0} \binom{x}{n-i} f(x)$$

finally introducing the binomial moments $\mathscr{B}_\nu$ of *f(x)* we find

$$(21) \qquad c_n = \sum_{i=0}^{n+1} (-1)^i \frac{m^i}{i!} \mathscr{B}_{n-i}.$$

Knowing the coefficients c_n the expansion (19) is determined. But in consequence of (3) this equation may be written in another manner:

$$(22) \qquad f(x) = \sum_{n=0}^{\infty} (-1)^n c_n \underset{x}{\Delta^n}\psi(m,x-n).$$

Remark. We may dispose of *m* so as to have the coefficient c_1 in the expansion (19) equal to zero. For this we shall put

$$c_1 = \mathscr{B}_1 - m\,\mathscr{B}_0 = 0.$$

Then m will be equal to the mean of the quantities x; $m = \mathscr{B}_1/\mathscr{B}_0$.

Example 1. Expansion of $\binom{x}{n}$ into a series of polynomials G,

$$\binom{x}{n} = c_0 + c_1 G_1 + c_2 G_2 + \ldots + c_n G_n. \tag{23}$$

This expansion is identical with that of $\binom{x}{n}\psi(m,x)$ into a series (19). Therefore the coefficients c_i may be determined by aid of (21); but owing to formula (17) there is a shorter way. Indeed, multiplying (23) by G,. $\psi(m,x)$ and summing from $x=0$ to $x=\infty$; in consequence of the orthogonality we find:

$$\sum_{x=0}^{\infty} \binom{x}{n} G_\nu \psi(m,x) = \frac{\nu!}{m^\nu} c_\nu .$$

Equalizing this result to (17) we have

$$c_\nu = \frac{m^n}{n!}\binom{n}{\nu}$$

and finally

$$\binom{x}{n} = \frac{m^n}{n!}\sum_{\nu=0}^{n+1}\binom{n}{\nu} G_\nu . \tag{24}$$

From this we may deduce the binomial moments of a function $f(x)$ expressed by (19). Indeed, multiplying (24) by $f(x)$ and summing from $x=0$ to $x=\infty$ we get by aid of (20)

$$\mathscr{B}_n = \sum_{\nu=0}^{n+1} c_\nu \frac{m^{n-\nu}}{(n-\nu)\,!} . \tag{25}$$

Sum of a function f(x) expanded into a series (19). **Starting** from (19) we may determine $\Delta^{-1} f$ (x) . For this purpose let **us** remark first that for $x=0$ the quantity (8) will be equal to zero. Indeed we have

$$\psi\ (m,-1) = 0.$$

This is readily seen if we write $\psi(m,x)$ in its general form

$$\psi(m,x) = e^{-m} m^x / \Gamma(x+1)$$

the denominator is infinite for **every negative integer value of x.**

Moreover, according to formula (3) § 18 we have

$$\sum_{x=0}^{\lambda+1} \psi(m,x) = 1 - I(u,p)$$

where $I(u,p)$ is the incomplete gamma function; $p = \lambda$ and $u = m/\sqrt{\lambda+1}$.

Finally we shall obtain by aid of (8)

$$(26) \qquad \sum_{x=0}^{\lambda+1} f(x) = c_0[1-I(u,p)] - \psi(m,\lambda) \sum_{n=1}^{\infty} c_n G_{n-1}(\lambda).$$

We should have obtained this formula starting from (22); indeed we have

$$\Delta_x^{-1} f(x) = c_0 \Delta_x^{-1} \psi(m,x) + \sum_{n=1}^{\infty} (-1)^n c_n \Delta_x^{n-1} \psi(m,x-n).$$

This leads to the following result

$$(27) \qquad \sum_{x=0}^{\lambda+1} f(x) = c_0[1-I(u,p)] + \sum_{n=1}^{\infty} (-1)^n c_n \Delta_x^{n-1} \psi(m,\lambda+1-n).$$

The integral with respect to m, of a function expanded into a series (19) may be obtained in the following way:

From (2) we deduce

$$(28) \qquad \underset{m}{\mathrm{D}}[G_n \psi] = G_{n+1} \psi$$

therefore it follows, if $n > 0$, that

$$(29) \qquad \int G_n \psi \, dm = G_{n-1} \psi + k.$$

$$G_{n-1} \psi = \frac{e^{-m} (n-1)!}{x!} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{x}{n-\nu-1} \frac{m^{x-n+1+\nu}}{\nu!};$$

Therefore, if $m=0$ every term of the second member will vanish except that corresponding to $\nu = n-1-x$; so that we shall have

$$\left[G_{n-1}\psi \right]_{m=0} = (-1)^{n-1-x} \binom{n-1}{x}.$$

Id $x > n-1$ this quantity is equal to zero, then from *(29)* it follows that

$$(30) \qquad \int_0^{\lambda} G_n(m,x)\, \psi(m,x)\, dm = G_{n-1}(\lambda,x)\, \psi(\lambda,x)$$

Moreover we have

$$(31) \qquad \int_0^{\lambda} \psi(m,x)\,dm = \int_0^{\lambda} \frac{e^{-m} m^x}{\Gamma(x+1)}\,dm = I(u,p)$$

where $I(u,p)$ is the incomplete gamma function (2, § 18) given in *Pearson's* Tables (loc. cit. 15); $p = x$ and $u = \lambda / \sqrt{x+1}$.

Finally it results from (19) that

$$(32) \qquad \int_0^{\lambda} f(x)\,dm = c_0 I(u,p) + \sum_{n=1}^{\infty} c_n G_{n-1}(\lambda,x)\psi(\lambda,x)$$

if $x > n-1$.

Approximation of the function y(x) given for $x=0, 1, 2, \ldots,$ N-1, ***according to the principle of moments*** by aid of

$$f_r(x) = [c_0 + c_1 G_1 + c_2 G_2 + \ldots + c_n G_n]\psi(m,x).$$

Putting $y(x)=0$ for $x \geqq N$, let us show that the coefficients c_i corresponding to this principle are the same as those obtained by the expansion of y(x) into a series of G $_r\psi(m,x)$ function& when stopping at the term G_n.

According to (25) the binomial moment of $f_r(x)$ of order s is equal to

$$\mathcal{B}_s = \sum_{\nu=0}^{s+1} c_\nu \frac{m^{s-\nu}}{(s-\nu)!};$$

moreover c_ν is given by (21)

$$c_\nu = \sum_{i=0}^{\nu+1} (-1)^i \frac{m^i}{i!} \mathcal{B}_{\nu-i},$$

Where $\mathcal{B}_{\nu-i}$ is the binomial moment of y(x) of order $\nu-i$. From these equation we conclude that

$$\mathcal{B}_s = \Sigma\Sigma \frac{(-1)^i m^{s+i-\nu}}{i!\,(S-\nu)!} \mathcal{B}_{\nu-i}.$$

Writing $s+i-\nu=\mu$ *we* find

$$\mathcal{B}_s = \sum_{\mu=0}^{s+1} \frac{m^\mu}{\mu!} \mathcal{B}_{s-\mu} \sum_{i=0}^{\mu+1} (-1)^i \binom{\mu}{i}.$$

The second sum is equal to zero for every value of μ except for $\mu=0$, and then we have

$$\overline{\mathscr{F}}_s = \mathscr{F}_s$$

for every value of $s=0, 1, 2, \ldots, n$. Therefore the corresponding $n+1$ moments of y(x) and f(x) are the same. Q. E. D.

Remark 1. If we determine the coefficients c_i so that

$$\mathscr{S} = \sum_{x=0}^{N} [f''(X) - y(x)]^2/\psi(m,x)$$

shall be a minimum, we find again the same values of c_i as before. Moreover the minimum of $\mathscr{S}$ will be equal to

$$\mathscr{S} = \sum_{x=0}^{N} [y(x)]^2/\psi(m,x) - \sum_{i=0}^{n+1} \frac{i!}{m^i} c_i^2.$$

Remark 2. Among the polynomials P_n of degree n in which the coefficient of $\binom{x}{n}$ is equal to unity, the polynomial which makes

$$\sum^{\infty} P_n^2\psi\ (m,x)$$

a minimum, is equal to $m^n G_n/n!$.

Computation of the numerical **values.** The function *f(x)* is expanded into a series of $G_n\psi(m,x)$, or into a series of $\Delta^n\psi(m,x-n)$ and *f(x),* corresponding to a given x is to be determined. Since there are no tables of $G_n\psi\ (m,x)$ nor of $\Delta^n\psi(m,x)$ the computation of *f(x)* by aid of the preceding formulae would be complicated. To obviate this difficulty there are two ways:

A. We may start from

$$\psi(m,x)\,G_n = (-1)^n\Delta^n\psi(m,x-n)$$

and express the differences contained in the second member by the successive values of the function (§ 6). We find

$$G_n\psi(m,x) = \sum_{i=0}^{n+1} (-1)^{n+i} \binom{n}{i} \psi(m,x-i).$$

Putting this into formula (19) we get

$$\text{(33)}\qquad f(x) = \sum_{i=0}^{x+1} \psi(m,x-i) \sum_{n=0}^{\infty} (-1)^{n+i}\binom{n}{i} c_n.$$

In the same way from (26) we obtain

$$(34)\qquad \sum_{x=0}^{\lambda+1} f(x) = c_0[1-I(u,p)] +$$

$$+ \sum_{i=0}^{\lambda+1} \psi(m,\lambda-i) \sum_{n=1}^{\infty} (-1)^{n+i} \binom{n-1}{i} c_n .$$

By aid of ***Peurson's*** Tables (§ 18) giving $\psi(m,x)$ the values of $f(x)$ may easily be calculated. $I(u,p)$ is computed by aid of the tables incomplete Γ function (§ 18); but if λ is small and if the value of m is contained in the table of the function ψ, then we may determine

$$1-I(u,p) = \sum_{x=0}^{\lambda+1} \psi\ (m,x)$$

by adding the corresponding values of ψ.

B. There is still a better way to compute the function $f(x)$ corresponding to a given value of x. Indeed we may start from (4) and write

$$G,(x) = \sum_{i=0}^{n+1} (-1)^{n+i} \frac{(n)_i}{m_i} \binom{x}{i}$$

then formula (19) will give

$$(35)\qquad f(x) = \psi(m,x) \sum_{i=0}^{x+1} \frac{(-1)^i}{m^i} \binom{x}{i} \sum_{s=0}^{\infty} (-1)^s (s)_i c_s$$

and (26)

$$(36)\qquad \sum_{x=0}^{\lambda+1} f(x) = c_0[1-I(u,p)] +$$

$$+ \psi(m,\lambda) \sum_{i=0}^{\lambda+1} \frac{(\lambda)_i}{m^i} \sum_{s=1}^{\infty} (-1)^{s+i} (s-1)_i c_s ,$$

The coefficients of $\binom{\lambda}{i}$ are rapidly calculated, and there is only one value of ψ to be determined by the table; this is important in cases when $\psi(m,x)$ is to be determined by interpolation.

Example 2. Let us denote by $P(x)$ the probability that an event shall happen x times in n trials, if the probability of its occurrence is equal to p at each trial. The formula of this probability was given by ***Jacob Bernoulli:***

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

where $q = 1 - p$. This is to be expanded into a series (19). In order to obtain the coefficients c_i we must first determine the binomial-moments of P(x). We have seen in § 136 that if $u(t)$ is the generating function of $P(x)$ then its binomial moment of order i will be equal to

$$\mathcal{B}_i = \left[\frac{D^i u(t)}{i!} \right]_{t=1}$$

And since the generating function of the given probability is $(q+pt)^n$, therefore we shall have

$$\mathcal{B}_i = \binom{n}{i} p^i.$$

Finally the coefficient c_ν of the expansion will be (21)

$$c_\nu = \sum_{i=0}^{\nu+1} (-1)^i \frac{m^i}{i!} \binom{n}{\nu - i} p^{\nu - i}.$$

To simplify we shall dispose of m so as to have $c_1 = 0$; for this we put $m = \mathcal{B}_1 / \mathcal{B}_0$ or $m = np$. This will give in consequence of formula (4'):

$$c_\nu = p^\nu \sum_{i=0}^{\nu+1} (-1)^i \frac{n^i}{i!} \binom{n}{\nu - i} = \frac{n^\nu p^\nu}{\nu!} G_\nu(n,n)$$

where $G_\nu(n,n)$ signifies the particular value of G_ν corresponding to $m = n$ and $x = n$. Finally from formula (4") we get

$$c_\nu = \frac{(np)^\nu}{\nu!} \sum_{k=1}^{\nu} C_{k,2k-\nu} n^{-k}.$$

Particular values. By aid of the table (p. 152) giving the numbers $C_{m\nu}$ we find:

$$c_0 = 1; \quad c_1 = 0; \quad c_2 = -\tfrac{1}{2} np^2; \quad c_3 = np^3/3$$

$$c_4 = (n-2)np^4/8; \quad c_5 = -(5n-6)np^5/30; \text{ and so on.}$$

Writing $m = np$ the approximation of the second degree given by formula (35) will be

$$P(x) = \psi(np,x)[c_0-c_1+c_2 + \frac{x}{np}(c_1-2c_2) + \frac{\binom{x}{2}}{n^2p^2}2c_2]$$

and putting the above values into it we obtain

$$P(x) = \psi(np,x)\left[1-\tfrac{1}{2}np^2 + px - \frac{1}{n}\binom{x}{2}\right].$$

Particular case: $n=10$, $p=1/3$ and $x=5$. **We** find

$$P(5) = \frac{10}{9}\psi\left(\frac{10}{3}, 5\right) = \frac{10}{9}(0.122339) = 0.135932.$$

The exact value would be: 0.1366.

In the same manner an approximation of the fourth degree given by formula (36) is

$$\sum_{x=0}^{\lambda+1} P(x) = c_0[1-I(u,p)] - \psi(np,\lambda)[c_1-c_2+c_3-c_4 +$$

$$+ \frac{\lambda}{m}(c_2-2c_3+3c_4) + \frac{\binom{\lambda}{2}}{m^2}(2c_3-6c_4) + \frac{\binom{\lambda}{3}}{m^3}6c_4].$$

Particular case: $n=10$, $p=1/3$ and $\lambda=5$. The value of $I(u,p)$ is computed by aid of the table of the incomplete Γ function (§ 18) putting $\bar{p}=\lambda=5$ and $\bar{u}=\frac{m}{\sqrt{\lambda+1}}=\frac{10}{3\sqrt{6}}=1.360198$.

Interpolation of the third degree gives

$$I(\bar{u},\bar{p}) = 0.1209823.$$

In ***Pearson's*** Table of the ψ function (§ 18) we find

$$\psi\left(\frac{10}{3}, 5\right) = 0.122339$$

(interpolation of the third degree). Since

$$c_0=1, \quad c_1=0, \quad c_2=-5/9, \quad c_3=10/81, \quad c_4=10/81$$

therefore

$$\sum_{x=0}^{6} f(x) = 0{,}87902 + (0{,}122339)\frac{273}{810} = 0.92025.$$

The exact value is 0.92343.

CHAPTER IX.

NUMERICAL RESOLUTION OF EQUATIONS.

§ 14% Method of False Position, or Regula Falsi. The problem of the numerical resolution of the equation $f(x) = 0$ is identical with the problem of inverse interpolation of $y = f(x)$, when $y = 0$.

The most important method of the resolution of equations is that of the *False Position.* We have already used this method in the inverse interpolations of paragraphs 131, 132 and 134, with slight modifications.

Given the equation $f(x) = 0$; if $f(x)$ is a continuous function in the interval a, b, and if $f(a) f(b) < 0$, then the equation will have at least one root in the interval mentioned. If $b-a$ is chosen small enough, there will be only one root in it.

Let us suppose, moreover, that the function $f(x)$ has a first and a second derivative which do not vanish in the interval. These last conditions serve only for the determination of the error.

The curve $y = f(x)$ will pass through the points of coordinates $a, f(a)$ and $b, f(b)$. Let us consider the chord passing through these points; it will cut the x axis in a point whose absciss is x_1. We have necessarily $a < x_1 < b$; x_1 may be considered as the first approximation of the root.

To obtain x_1 and the corresponding error 8, , we shall write *Lagrange's* linear interpolation formula:

$$f(x) = \frac{x-a}{b-a} f(b) + \frac{x-b}{a-b} f(a) + \tfrac{1}{2}(x-a)(x-b) D^2 f(\xi)$$

where $a < \xi < b$. Putting into this equation $f(x) = 0$ we find

$$(1) \quad x = a - \frac{(b-a)f(a)}{f(b)-f(a)} - \frac{(b-a)}{f(b)-f(a)} \tfrac{1}{2}(x-a)(x-b)\, D^2 f(\xi);$$

neglecting the remainder we obtain x_1 ; the maximum δ_1 , of the remainder will be equal to the maximum of the absolute value of the error. Determining $f(x_1)$, there are two cases to be considered :

First

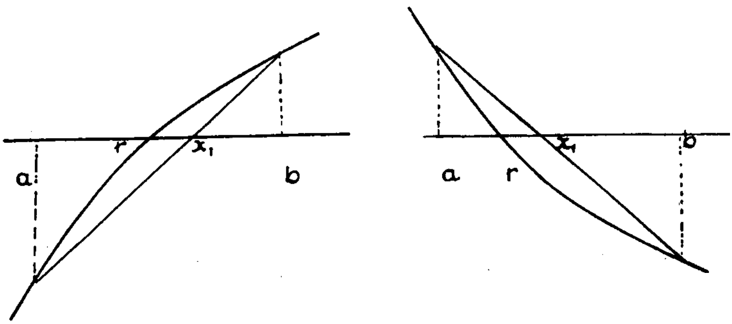
$$f(a) f(x_1) < 0$$

then, denoting the root by r we shall have (Figure 5)

$$a < r < x_1$$

this will occur if $Df(x)\, D^2 f(x) < 0$.

Figure 5.



Putting into formula (1) $b = x_1$, we determine x_2 the second approximation of r and δ_2 the corresponding maximum of the error. Necessarily we have $a < r < x_2$, and continue in the same manner.

Secondly,

$$f(x_1) f(b) < 0$$

then we have (Figure 6)

$$x_1 < r < b$$

this will occur if $Df(x)\, D^2 f(x) > 0$. We put into formula (1) x_1 instead of a and determine x_2 the second approximation of r. We have

$$x_2 < r < b$$

and continue in the same way.

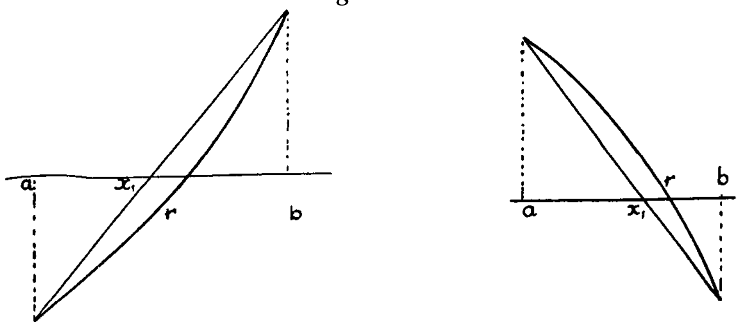
It is possible to shorten the method in the following manner: If we have obtained, for instance, in the first case the approximated value x_1 and δ_1 the maximum of error, then, since in the case considered the error is positive,* we have

$$x_1 - \delta_1 < r < x_1$$

and we may apply the method in this interval. Since 6, is considerably smaller than $x_1 - a$ therefore the second approximation will be far better.

In the second case the error being negative we shall have $x_1 < r_1 < x_1 + 6$, and we may apply the method to this interval.

Figure 6.



Example 1. Let us choose for our first example that of *Newton's* equation given by ***Wallis***

$$f(x) = x^3 - 2x - 5 = 0$$

which has been solved subsequently to **101** decimal places.[47]

Since $f(2) = -1$ and $f(3) = 16$ therefore between $x = 2$ and $x = 3$ there is a root; this must be determined- to twelve decimals. It would be possible to put $a = 2$ and $b = 3$ and apply the method; but remarking that $f(3)$ is very large in comparison with $f(2)$, therefore the root will be near $a = 2$. We may try to put $a = 2$ and $b = 2{,}1$. Since $f(2{,}1) = 0{,}061$ hence $2 < r < 2{,}1$.

* If x_1 is the obtained value for r then by error we understand the difference $x_1 - r$.

[47] *J. Wallis,* Treatise of Algebra, London, 1685, p. 338.
Whittaker and Robinson, Calculus of Observations, p. 106 and p. 86 where 51 decimals are given. $x = 2{,}094\ 551\ 481\ 542\ 326\ 591\ 482, \ldots$.

Putting $a=2$ and $b=2{,}1$ we obtain by aid of formula (1):

$$x_1 = 2 + \frac{0{,}1}{1{,}061} = 2{,}0942\ldots.$$

The corresponding maximum of the error is also given by formula (1); we find $\delta_1 < 6.\ 10^{-4}$. Therefore

$$2.094 < x < 2.095.$$

We repeat the operation, putting $a = 2{,}094$ and $b = 2{,}095$. If follows that

$$f(2{,}094) = -0.00615\ 3416$$

$$f(2{,}095) = 0.00500\ 7375.$$

Therefore from formula (1) we get

$$x_2 = 2.09455\ 134228\ldots,$$

Determining the corresponding maximum of the error we find $\delta_2 < 2\ .\ 10^{-7}$ hence we shall have

$$2.094551 < x < 2.094552.$$

Repeating the operation, starting from these values we obtain

$$f(2{,}094551) = -0.00000\ 53747\ 03234$$

$$f(2{,}094552) = 0{,}00000\ 57867\ 28439.$$

Formula (1) gives

$$x_3 = 2.09455\ 14815\ 4245,\ldots$$

The corresponding maximum of **the error will be** $\delta_3 < 2\ .\ 10^{-13}$; hence the problem is solved and we have

$$x = 2.09455\ 14815\ 42$$

exact to the last decimal.

§ 150, The Newton-Raphson method of numerical solution of equations. Given $f(x)=0$ let us suppose that between $x=a$ and $x=b$ there is a root denoted by *r,* and that

$$f(a)f(b) < 0.$$

We will suppose moreover that *f(x)* is a continuous function whose first two derivatives are different from zero in this interval.

We shall distinguish two different cases. First let

$$Df(x)\, D^2f(x) < 0 \tag{1}$$

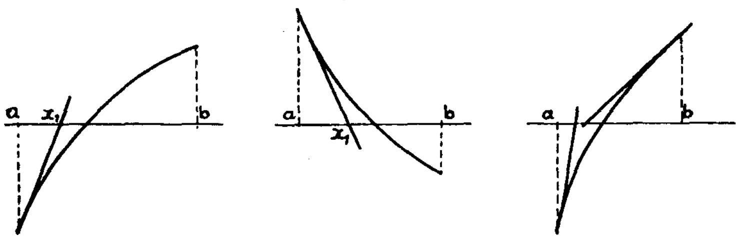
(figure 7) then the tangent to the curve $y=f(x)$ at the point of coordinates a, $f(a)$ will cut the x axis at a point $x=x_1$ such that we have

$$a < x_1 < r < b$$

while on the contrary, the tangent at the point $b, f(b)$ will not necessarily cut the x axis in a point belonging to the interval a, b.

Since we want to consider x_1 as the first approximation of the root r, it is generally advisable to start from the point $a, f(a)$ if inequality (1) is satisfied.

Figure 7.



An exception to this rule is the following. If Df (x) is large, that is, if the curve $y=f(x)$ is steep in the interval, then the tangent to the curve at the point $a, f(a)$ will meet the x axis near $x=a$ and therefore the approximation obtained will he not much better than that of $x=a$; but in this case the tangent at the point $b, f(b)$ may cut the axis nearer to $x=r$, giving thus a more favourable approximation. Hence, although the inequality (1) is satisfied, we may start nevertheless from the point $b, f(b)$ if $Df(x)$ is large.

Secondly (figure 8), if we have

$$Df(x)\, D^2f(x) > 0 \tag{2}$$

then, to obtain certainly a number x_1 belonging to the interval a, b, we must start from the point $b, f(b)$. This we shall generally do **unless** the curve is very steep, in which case we may obtain a better approximation by starting from the point $a, f(a)$.

Having found x_1 we compute $f(x_1)$ and determine the tangent at the point $x_1, f(x_1)$ which will cut the x axis at a point $x=x_2$, serving as a second approximation of the root r. Then we compute $f(x_2)$ and continue in the same manner till the required approximation is reached.

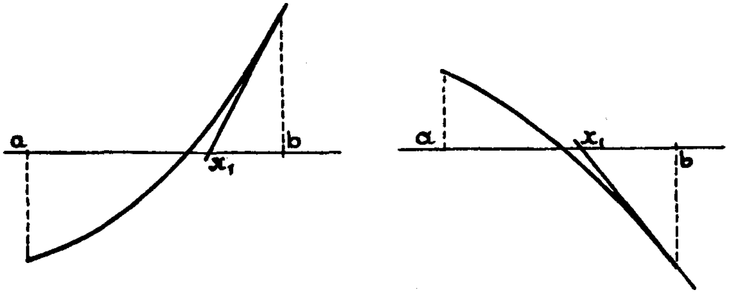
We shall proceed in the following way: First we expand f(x) into a *Taylor* series.

(3) $$f(x) = f(x_0) + (x-x_0)\,Df(x_0) + \tfrac{1}{2}(x-x_0)^2\,D^2f(x_0+\xi)$$

where $0 < \xi <$ x-x,, .

If we start from the point corresponding to $x=a$ then we put into this formula $x_0=a$; in the other case we put $x_0=b$.

Figure 8.



Omitting the remainder in formula (3) we obtain the equation of the tangent at the point $x_0, f(x_0)$; this will cut the x axis in

(4) $$x_1 = x_0 - \frac{f(x_0)}{Df(x_0)}.$$

The corresponding error will be

(5) $$\delta_1 = -\tfrac{1}{2}(x_1-x_0)^2\frac{D^2f(x_0+\xi)}{Df(x_0)}$$

The second approximation and the corresponding error are given by (4) and (5), if we put into these equations x_2 instead of x_1 and x_1 instead of x_0. Having obtained x_2 we continue in the same manner till the required precision is obtained.

Example. We will take again *Newton's* example. Let

$$f(x) = x^3 - 2x - 5 = 0.$$

There is a root between x=2 and x=3; indeed, f(2) = -1 and $f(3) = 16$. We have

$$Df(x) = 3x' - 2$$

and

$$D^2f(x) = 6x.$$

Since in the interval considered we have $Df(x)D^2f(x) > 0$ we should start, as has been said, from the point $3, f(3)$; but since $Df(3) = 25$, the curve is very steep; it is advisable to start from the point $2, f(2)$ where $Df(2) = 10$. Since this is the second case, we shall have $a < r < x_1$.

Starting from $x_0 = 2$, we find $f(x,) = -1$, $Df(x_0) = 10$ and therefore according to (4) and (5)

$$x_1 = 2{,}1, \qquad \delta_1 < 7/1000.$$

Starting from x,=2,1, we have $f(x_0) = 0.061$, $Df(x_0) = 11{,}23$ and therefore

$$x_2 = 2.0945681\,,\ldots \qquad |\delta_2| < 2/10^5.$$

Starting from $x_0 = 2{,}09457$, we obtain $f(x,) = 0.000206694976$ and $Df(x,) = 11{,}16167$ 0455 and therefore

$$x_3 = 2.09455\ 14817\ 26, \qquad |\delta_3| < 3/10^{10}.$$

§ 151. Method of Iteration, Given the equation $f(x) = 0$, let us suppose that it has a root $x = r$ in the interval x_0, $x_0 + h$. If we write

$$f(x) = f,(x) - f_2(x).$$

Then the root of $f(x) = 0$ will be equal to the abscіss of the point in which the two curves $y = f_1(x)$ and $y = f_2(x)$ meet. We will suppose that the decomposition has been made so that the second curve is steeper than the first in the interval considered; that is

$$|Df_2(x_0)| > |Df_1(x_0)|.$$

The method of iteration described below is applicable if it is possible to determine the algebraic solution of $y = f_2(x)$ and obtain $x = \varphi_2(y)$. Then we start from the equations

$$Y = f_1(x) \quad \text{and} \quad x = \varphi_2(y)$$

and put x_0 into the first equation to obtain $y_0 = f\ (x,)$; *then* putting y_0 into the second equation we get $x_1 = \varphi_2(y_0)$. This is the first approximation of r. Since generally we have

$$|r - x_1| < |r - x_0|.$$

The second move is to put x_1 into the first equation and get y_1 , which gives by aid of the second equation x_2, the second approximation of the root. Continuing in this manner we reach the prescribed precision.

Figure 9.

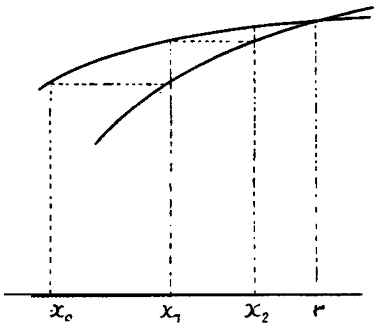
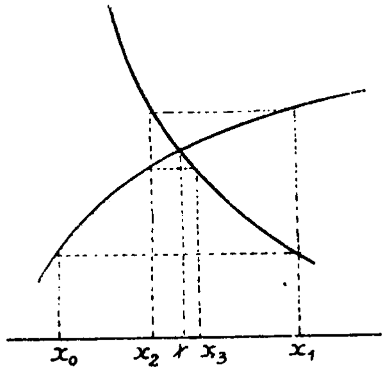


Figure 10.



We may consider two cases; *first,* both curves are increasing (figure 9), or both are decreasing, and therefore

$$Df_1(x)\,Df_2(x) > 0$$

in this case we shall have

$$x_0 < x_1 < x_2 < \ldots\ldots < r.$$

Secondly, one of the curves is increasing and the other decreasing (fig. 10), and therefore

$$Df_1(x)\,Df_2(x) < 0$$

and we shall have

$$x_0 < x_2 < x_4 <, \ldots < r < \ldots, . < x_3 < x_1 .$$

In the first case there is no simple way to estimate the error committed when stopping at the term x_n; in the second case since $x_{2n} < r < x_{2n+1}$, therefore the error is obviously less than

$$|x_{2n} - x_{2n+1}|.$$

Example. The equation

$$f(x) = x^3 - 2x - 5 = 0$$

has a root between 2 and 3, since $f(2)f(3) < 0$. Let us put

$$f_1(x) = 2x+5 \quad \text{and} \quad f_2(x) = x^3.$$

Our condition, that the second curve shall be steeper than the first in the interval 2, 3, is satisfied. Indeed (first case)

$$Df_1(x) = 2 \quad \text{and} \quad Df_2(x) = 3x^2$$

and therefore

$$Df_2(x) > Df_1(x) \qquad \text{in } (2, 3).$$

The second condition was that it must be possible to solve algebraically $y = f_2(x)$. We have

$$x = \varphi_2(y) = \sqrt[3]{y}.$$

Starting from $x_0=2$, the first equation will give $y_0=9$. For this value of y_0 the second equation gives $x_1=2.08\ldots$. The value of x_1 put into the first equation gives $y_1=9.16$, and this by aid of the second: $x_2=2.094$, and so on.

The operations are given in the following table:

i	x_i	$y_i=f_1(x_i)$	$x_{i+1}=\varphi_2(y_i)$
0	2	9	2.08 . . ,
1	2.08	9,16	2.094, , .
2	2.094	9.188	2.0944 , . .
3	2.0944	9.1888	2.09453 . . .

§ 152. Daniel **Bernoulli's method for solving numerical equations.** Given the equation with real coefficients:

$$(1) \qquad f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0.$$

We may consider this as the characteristic equation of the following homogeneous linear difference equation

(2) $\varphi(t+n)+a_{n-1}\varphi(t+n-1)+\ldots+a_1\varphi(t+1)+a_0\varphi(t)=0.$

If the roots of the characteristic equation (1) are all different, say $x_1, x_2, \ldots, x_n$, then we shall see (§ 165) that the solution of (2) is equal to

$$y(f) = c_1 x_1^t + c_2 x_2^t + \ldots + c_n x_n^t .$$

If we denote by x_1 the root whose modulus is the greatest, tben it results that

(3)
$$\lim_{t=\infty} \frac{\varphi(t+1)}{\varphi(t)} =$$

$$= \lim_{t=\infty} \frac{x_1 + \frac{c_2 x_2}{c_1}\left(\frac{x_2}{x_1}\right)^t + \ldots + \frac{c_n x_n}{c_1}\left(\frac{x_n}{x_1}\right)^t}{1 + \frac{c_2}{c_1}\left(\frac{x_2}{x_1}\right)^t + \ldots + \frac{c_n}{c_1}\left(\frac{x_n}{x_1}\right)^t} = x_1 .$$

Hence if we determine $\varphi(t+1)/\varphi(t)$ for t sufficiently great, we shall have the value of the root with the required precision. It is easy to show that equation (3) holds even if the roots of the characteristic equation are multiple roots. If some of the roots of (1) were complex (except the root x_1 with the greatest modulus), we should have the same result; but if x_1 were a complex root, then, as the root conjugate to x_1 would have the same modulus, the ratio (3) would not tend to a limit but would oscillate.

To determine $\varphi(t)$ by the aid of (2) we may start from any n initial values of this function, Let them be

$$y(0) = \varphi(1) = (p(2) = \ldots = \varphi(n-2) = 0 \text{ and } \varphi(n-1) = 1;$$

then the difference equation (2) will give

$$\varphi(n) = -a_{n-1}$$
$$\varphi(n+1) = -a_{n-1}\varphi(n) - a_{n-2}$$
$$\varphi(n+2) = -a_{n-1}\varphi(n+1) - a_{n-2}\varphi(n) - a_{n-3}$$

and so on.

Example. **Given the equation**

$$x^5 + 5x4-5 = 0.$$

The corresponding difference equation is

(4) $\varphi(t+5) + 5\varphi(t+4)$ -559,(f) = 0.

We shall start from

$$\varphi(0) = \varphi(1) = \varphi(2) = \varphi(3) = 0 \quad \text{and} \quad \varphi(4) = 1.$$

From equation (4) we get successively

t	0	1	2	3
$\varphi(t+5)$	- 5	25	-125	625

t	4	5	6	7
$\varphi(t+5)$	-3125	15600	-77875	388750
	5	- 25	125	-625
	-3120	15575	-77750	388125

and from this

$$\varphi(9)\,/\varphi(8) = -4.992\,..$$
$$\varphi(10)/\varphi(9) = -4.991987\,.,$$
$$\varphi(11)/\varphi(10) = -4.99197431\,.,$$
$$\varphi(12)/\varphi(11) = -4.991961415\,..$$

By this method we may also obtain the root whose modulus is the smallest. For this we have to put into (1) $x=1/y$, and apply the method on the equation y(y) $=0$ obtained.

§ 153. The Ch'in-Vieta-Homer method for solving numerical equations, This method was known in China as early as the XIII Century.[48] It was given by ***Ch'in Chiushao*** in his book on the "Nine Sections of Mathematics" published in **1247.**

The method was first used in Europe by ***Vieta*** about **1600.** It is probable that he had not heard of the Chinese method, but that he rediscovered it, since his procedure is more complicated than ***Ch'in's.*** Indeed it was considered as very difficult, was seldom used, and soon superseded in Europe by the Rule of False Position or by other methods.

In 1819 ***Horner*** published **a** numerical process for carrying out the method, which was very convenient and in consequence it became widely known and is now generally considered **as the**

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[48] Y. *Mikami,* The Development of Mathematics in China and Japan, Leipzig, 1913, pp. 74—77.

most practical of the methods for solving numerical equations.

Horner's process differs but very little from that of **Ch'in.** It consists in the following:

Given the equation $f(x) = 0$, let us suppose that by means of a graph or in another way, we know that this equation has a root x_0 between r_0 and r_0+1 ; where r_0 is an integer. Therefore we have

$$f(r_0)f(r_0+1) < 0.$$

The method consists in the following operations:

First we deduce from $f(x) = 0$ another equation, such that the roots of the new equation shall be equal to those of $f(x) = 0$ but each diminished by r_0. This equation will be $f(x+r_0) = 0$, which expanded into a *Taylor* series gives

$$(1) \qquad f(x+r_0) = f(r_,) + xDf(r_0) + , \ldots + \frac{x^n}{n!} D^n f(r_0) = 0.$$

From this, an equation is deduced, such that its roots are each ten times larger than those of (1). This will be

$$f\left(r_0 + \frac{x}{10}\right) = 0 ;$$

expanded into a Taylor series, and multiplied by 10^n it gives:

$$(2) \quad 10^n f\left(r_0 + \frac{x}{10}\right) = 10^n f(r_0) + 10^{n-1} xDf(r_0) + \ldots + \frac{x^n}{n!} D^n f(r_0).$$

Since equation (1) has a root in the interval 0, 1 therefore (2) will, have a root in the interval 0, 10. Let us denote this root by x_1. From (2), by inverse interpolation we obtain approximately

$$x_1 \sim \frac{-10f(r_0)}{Df(r_0)} .$$

If $r_1 < x_1 < r_1 + 1$, where r_1 is an integer, then the approached value of the root obtained by the above operations is

$$x_0 = r_{,,} + \frac{r_1}{10} + , \ldots$$

To have another digit of x_0 we start from (2) and determine an equation (3) whose roots are those of (2) diminished by r_1, and then an equation (4) whose roots are each ten times larger than those of (3). From this last equation we deduce

$$r_2 < x_2 < r_2 + 1$$

and we shall have

$$x_0 = r_0 + \frac{r_1}{10} + \frac{r_2}{100} + \dots .$$

and so on till the prescribed precision is attained.

Let us show *Homer's* method for determining the coefficients of the transformed equations in the case of an equation of the fifth degree.

Given

$$\text{(5)} \qquad f(x) = F + Ex + Dx^2 + Cx^3 + Bx^4 + Ax^5 .$$

The coefficients of the equation whose roots are those of (5) diminished by r_0 are

$$F_1 = f(r_0) = Ar_0^5 + Br_0^4 + Cr_0^3 + Dr_0^2 + Er_0 + F$$

$$E_2 = Df(r_0) = 5Ar_0^4 + 4Br_0^3 + 3Cr_0^2 + 2Dr_0 + E$$

$$D_3 = \frac{1}{2} D^2 f(r_0) = 10Ar_0^3 + 6Br_0^2 + 3Cr_0 + D$$

$$C_4 = \frac{1}{6} D^3 f(r_0) = 10Ar_0^2 + 4Br_0 + C$$

$$B_5 = \frac{1}{24} D^4 f(r_0) = 5Ar_0 + B$$

$$A, = \frac{1}{120} D^5 f(r_0) = A.$$

Homer's process for computing these values is given by the following table

A	B	c	D	E	F
	$+Ar_0$	$B_1 r_0$	$C_1 r_0$	$D_1 r_0$	$E_1 r_0$
	B_1	C_1	D_1	E_1	F_1
	$+Ar_0$	$B_2 r_0$	$C_2 r_0$	$D_2 r_0$	
	B_2	C_2	D_2	E_2	
	$+Ar_0$	$B_3 r_0$	$C_3 r_0$		
	B_3	C_3	D_3		

$$\begin{array}{llll} A & B_3 & C_3 & D_3 \\ & \underline{+Ar_0} & \underline{B_4 r_0} & \\ & B_4 & C_4 & \\ & \underline{+Ar_0} & & \\ & B_5 & & \end{array}$$

, where

$$\begin{aligned} B_i &= B_{i-1} + Ar_0 & E_i &= E_{i-1} + D_i r_0 \\ C_i &= C_{i-1} + B_i r_0 & F_i &= F_{i-1} + E_i r_0 \\ D_i &= D_{i-1} + C_i r_0 & & \end{aligned}$$

In this manner we obtain

$$\text{(6)} \qquad f(x+r_0) = F, + E_2 x + D_3 x^2 + C_4 x^3 + B_5 x^4 + A x^5.$$

The equation, whose roots are each ten times larger than those of (6) will become, if multiplied by 10^5:

$$\text{(7)} \qquad 10^5 f\left(r_0 + \frac{x}{10}\right) = 10^5 F_1 + 10' E_2 x + 10^3 D_3 x^2 + 10^2 C_4 x^3 + \\ + 10 B_5 x^4 + A x^5 = 0.$$

From this we get by linear inverse interpolation

$$x_1 \sim - \frac{10F_1}{E_2}.$$

If we have $r_1 < x_1 < r_1 + 1$, where r_1 is an integer, r_1 will be the second digit of the root. Then the above operations are repeated till the required number of digits is obtained.

The error will always be less than one unit of the last decimal.

Example. Given

$$f(x) = x^3 - 2x - 5 = 0.$$

Since $f(2)f(3) < 0$ therefore this equation has a root x_0 in the interval 2, 3; so that we shall put $r_0 = 2$.

To obtain an equation whose roots are those of $f(x) = 0$ but diminished by 2, and multiplied by ten, we will perform *Horner's* computations:

1	0	- 2	- 5
	2	4	4
	2	2	- 1
	2	8	
	4	10	
	2		
	6		

The transformed equation will **be**

$$x^3 + 60x^2 + 1000x - 1000 = 0.$$

From the last two terms we conclude that $r_1 = 0$; so that there is no need to diminish the roots by r_1. We make the **roots** ten times larger and get

$$x^3 + 600x^2 + 100000x - 1000000 = 0. \tag{8}$$

From the last two terms we deduce $r_2 = 9$. To determine the coefficients of an equation whose roots are the same as those of (8) but diminished by nine and multiplied by ten; we follow ***Horner's*** method and find

1	600	100000	—1000000
	9	5481	949329
	609	105481	-5067 1
	9	5562	
	618	111043	
	9		
	627		

so that the fourth equation will be

$$x^3 + 6270x^2 + 11104300x - 50671000 = 0.$$

The last two terms give $r_3 = 4$. Now we diminish the roots of this equation by **4** and multiply by **10,** using the same method as before. We have

1	6270	11104309	-50671000
	4	25096	44517584
	6274	11129396	-6153416
	4	, 25112	
	6278	11154508	
	4		
	6282		

Therefore the fifth equation will be

$$x^3 + 62820x^2 + 1115450800x - 6153416000 = 0.$$

This will give $r_4 = 5$; and we could proceed in the same manner as before; but let us suppose that the root is only required to six decimals exact; then we may obtain the last three decimals by simple division. In fact we may write

$$x = \frac{6153416000 - 62820x^2 - x^3}{1115450800},$$

The term in x^3 is less than 216 (the root being less than 6), hence if we neglect this term the error in x will be less than $2/10^7$ and that of the root x,, less than $2/10^{11}$. Moreover the term in x^2 is less than (62820) (36) $= 2261520$; therefore neglecting this term too, the error of x will be less than $3/10^3$, and that of x_0 less than $3/10^7$. Accordingly we have

$$r_4 = 6153416000/1115450800 = 5.5165$$

and finally

$$x_0 = 2.094551\ 65.$$

The equation of this example has been solved by this method, in 1850, to 101 decimals.

One of the advantages of the method is the following: If for instance the root is exactly equal to two figures, then we obtain these in two operations, whereas by the other methods a great number of operations is necessary to show that the root is approximately equal to these two figures.

If the root to be determined is greater than ten, it is advisable to deduce first an equation whose roots are the same as those of the given equation, but divided by 10^k; where k is chosen so that the root under discussion should be in the interval 0, 10. For instance, equation

$$a_n x^n + \ldots . + a_1 x + a_0 = 0$$

will be transformed into

$$a_n x^n + \frac{a_{n-1}}{10^k} x^{n-1} + \ldots + \frac{a_1}{10^{kn-k}} x + \frac{a_0}{10^{kn}} = 0$$

and then we proceed as before.

Ch'in Chiushao's method differs but little from the preceding. Indeed the only difference is that if the first digit r_0 of the root of f(x) $=0$ is found, then ***Ch'in*** first deduces an equation whose roots are ten times larger than those of f(x) $=0$, and secondly deduces from this another equation whose roots are diminished by $10r_0$.

The result will be the same as in the method described above, moreover, ***Ch'in's*** computation is exactly the same as that of **Horner.** Indeed, starting from the equation

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

we shall have

$$\begin{array}{lllll}
A & 10B & 10^2C & 10^3D & 10^4E \\
 & \underline{10Ar_0} & \underline{10B_1r_0} & \underline{10C_1r_0} & \underline{10D_1r_0} \\
 & B_1 & Cl & D_1 & E_1 \\
 & \underline{10Ar_0} & \underline{10B_2r_0} & \underline{10C_2r_0} & \\
 & B_2 & C_2 & D_2 & \\
 & \underline{10Ar_0} & \underline{10B_3r_0} & & \\
 & B_3 & C_3 & & \\
 & \underline{10Ar_0} & & & \\
 & B_4 & & &
\end{array}$$

and the new equation will be

$$Ax^4 + B_4x^3 + C_3x^2 + D_2x + E_1 = 0.$$

Example. Ch'in solves the following equation

$$x^4 - 763200x'' + 40642560000 = 0.$$

It is easily seen that $100 < x_0 < 1000$ therefore we deduce another equation whose roots are those of (1) but each **divided** by 100; so that the root will be between 0 and ten. This equation is

$$(8') \qquad x^4 - 76{,}32\, x^2 + 406{,}4256 = 0$$

since $f(8)f(9) < 0$ therefore there is a root between 8 and 9.

Ch'in's computation for multiplying the roots by ten and diminishing them afterwards by 80, will give

1	0	-7632	0	4064256
	80	6400	-98560	-7884800
	80	-1232	-98560	-3820544
	80	12800	925440	
	160	11568	826880	
	80	19200		
	240	30768		
	80			
	320			

Therefore the transformed equation will be

$$(9) \qquad x^4 + 320x^3 + 30768x^2 + 826880x - 3820544 = 0.$$

From this we deduce by linear inverse interpolation $r_1 = 4$. The roots multiplied by ten and dinimished by 40 will give

1	3200	3076800	826880000	—38205440000
	40	129600	128256000	38205440000
	3240	3206400	955136000	0

Since the last term vanishes, hence 4 will be the last digit of the root, so that the root of equation (8') will be **8,4** and that of the given equation $x_0 = 840$.*)

§ 154. **Root-squaring** Method of **Dandelin, Lobatchevsky** and **Graeffe.** Given the equation:

$$(1) \qquad a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

*) *Mikami's* interpretations of *Ch'in's* processes, in the book quoted above, are not quite correct, and in consequence of this the coefficients of the equation corresponding to (9), given on page 77, are erroneous. Indeed the coefficients of x and x^3 are ten times smaller and the absolute term is 10000 larger than they should be.

We shall deduce from this equation another whose roots are $x_1^2, x_2^2, \ldots x_n^2$. Since equation (1) is identical with

$$a_n(x-x_1)\ (\text{x-x,})\ \ldots\ (\text{x-x,}) = 0$$

it is obvious that multiplied by

$$a_n(x+x_1)\,(x+x_2)\ldots(x+x_n) = 0$$

it will give the required equation, if we put into the product obtained x instead of x^2.

But the last equation is identical with

$$(2)\qquad a_n x^n - a_{n-1} x^{n-1} + \ldots + (-1)^n a_0 = 0$$

therefore if we write the new equation in the following way:

$$(3)\qquad b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0 = 0$$

then we shall have

$$b_\nu = (-1)^{n-\nu} a_\nu^2 + \sum_{i=1} 2(-1)^{n+\nu-i} a_{\nu+i} a_{\nu-i}.$$

The upper limit of the sum in the second member is equal to the smaller of the numbers $\nu+1$ and $n-\nu+1$.

Starting from (3) we may obtain, repeating the same operation, an equation whose roots are $x_1^4, x_2^4, \ldots, x_n^4$; and repeating it again several times we get an equation whose roots are $x_1^{2^m}\ x_2^{2^m}, \ldots, x_n^{2^m}$.

Let this equation be

$$(4)\qquad \omega_n x^n + \omega_{n-1} x^{n-1} + \ldots + \omega_1 x + \omega_0 = 0.$$

Moreover we will suppose now that the moduli of the roots are unequal, so that we have in descending order of magnitude

$$|x_1| > |x_2| > \ldots > |x_n|.$$

Since

$$\frac{\omega_{n-i}}{\omega_n} = (-1)^i \sum x_1^{2^m} x_2^{2^m} \ldots x_i^{2^m}.$$

Therefore we shall have

$$\lim_{m=\infty} \frac{\sum x_i^{2^m}}{x_1^{2^m}} = \lim_{m=\infty} \frac{-\omega_{n-1}}{\omega_n x_1^{2^m}} = 1$$

and approximately

$$x_1^{2^m} \sim -\frac{\omega_{n-1}}{\omega_n}$$

Let us suppose first that the roots of equation (1) are all real; then we must have $\omega_{n-1}/\omega_n < 0.$

In the same way we have

$$\lim_{m=\infty} \frac{\Sigma x_i^{2^m} x_j^{2^m}}{x_1^{2^m} x_2^{2^m}} = \lim_{m=\infty} \frac{\omega_{n-2}}{\omega_n x_1^{2^m} x_2^{2^m}} = 1$$

and therefore

$$x_1^{2^m} x_2^{2^m} \sim \frac{\omega_{n-2}}{\omega_n}$$

or

$$x_2^{2^m} \sim -\frac{\omega_{n-2}}{\omega_{n-1}}$$

and finally

$$x_\nu^{2^m} \sim -\frac{\omega_{n-\nu}}{\omega_{n-\nu-1}}$$

starting from these equations the roots are easily calculated by aid of logarithms. In this way we get in the same computation every root of equation (1) without needing any preliminary approximate determination. This is one of the advantages of the method.

There is but one small difficulty; the method gives only the absolute values of the roots. The signs must he found out by putting the obtained values into the equation. Often Descartes' rule giving the number of the positive and negative roots is useful, and sometimes the relation $\Sigma x_i = - a_{n-1}/a_n$ gives also some indications,

It is advantageous to make the computations in the following tabular form:

a_n	a_{n-1}	a_{n-2}	a_{n-3}	a_{n-4}	a_{n-5}
	$-a_{n-1}$	a_{n-2}	$-a_{n-3}$	a_{n-4}	$-a_{n-5}$
a_n^2	$-a_{n-1}^2$	a_{n-2}^2	$-a_{n-3}^2$	a_{n-4}^2	$-a_{n-5}^2$
	$2a_na_{n-2}$	$-2a_{n-1}a_{n-3}$	$2a_{n-2}a_{n-4}$	$-2a_{n-3}a_{n-5}$	$2a_{n-4}a_{n-6}$
		$2a_na_{n-4}$	$-2a_{n-1}a_{n-5}$	$2a_{n-2}a_{n-6}$	$-2a_{n-3}a_{n-7}$
			$2a_na_{n-6}$	$-2a_{n-1}a_{n-7}$	$2a_{n-2}a_{n-8}$
				$2a_na_{n-8}$	$-2a_{n-1}a_{n-9}$
					$2a_na_{n-10}$
b_n	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	b_{n-5}

Example 1. **Given the equation**

$$x^3 + 9x^2 + 23x + 14 = 0.$$

The roots are to be determined *to four significant digits.* **According to the preceding table we will write**

1	9	23	14	First powers
	-81	529		
	46	-252		
1	-35	277	-196	Second powers
	-1225	76729		
	554	-13720		
1	-671	63009	-38416	4 th powers
	450241	$3{,}9701.10^9$		
	126018	$-0{,}0516$		
1	$-3{,}2422.10^5$	$3{,}9185.10^9$	$-1{,}4758.10^9$	8 th powers
	$-1{,}0512.10^{11}$			
	785			
1	$-9{,}728.10^{10}$	$1{,}5355.10^{19}$		16 th powers
	$-9{,}4634.10^{21}$			
	307			
1	$-9{,}4327.10^{21}$	$2{,}3578.10^{38}$		32 nd powers
	$-8{,}8976.10^{43}$			
	5			
1	$-8{,}8971.10^{43}$			64 th powers.

If the coefficient corresponding to $\Sigma\, x_i^{2^m}$ is only the square of the coefficient corresponding to $\Sigma\, x_i^{2^{m+1}}$ then we stop the computation, since by continuing we should get the same value for x_1. In the present case this happens at the 64 -th power.

For the determination of $\Sigma\, x_i^{2^m}\, x_j^{2^m}$ the computation could have been stopped at the 8 th power; therefore to minimise the error we will choose this value for the determination of x_2. To have x_3 we start, for the same reason, from $x_1x_2x_3 = 14$. We have

$$\log x_1 = \frac{1}{64} \log 8.8971.10^{43} = 43.94924/64 = 0.68671$$

$$\log x_1x_2 = \frac{1}{8} \log 3.9185.10^{9} = 9.59312/8 = 1.19914$$

$$\log x_2 = \log x_1x_2 - \log x_1 = 0.51243$$

$$\log x_1x_2x_3 = \log 14 = 1.14613$$

and therefore

$$\log x_3 = \log x_1x_2x_3 - \log x_1x_2 = \bar{1}.94699.$$

Since according to ***Descartes'*** rule the given equation has only negative roots, therefore we have

$$x_1 = -4.8608, \qquad x_2 = -3.2541, \qquad x_3 = -0.8851.$$

General *case.* Let us suppose that the equation $f(x)=0$ *with* real coefficients has both real and complex roots. We shall denote the real roots by x and the pairs of complex roots by

$$\varrho_\nu(\cos\varphi_\nu + i\sin\varphi_\nu) \qquad \text{and} \qquad \varrho_\nu(\cos\varphi_\nu - i\sin\varphi_\nu).$$

The indices will correspond as before to the moduli in decreasing order of magnitude. We shall have

$$\frac{a_{n-1}}{a_n} = \Sigma\, x_\nu + 2\,\Sigma\, \varrho_\nu \cos\varphi_\nu$$

$$\frac{a_{n-2}}{a_n} = \Sigma\, x_\nu x_\mu + \Sigma\, \varrho_\nu^2 + 2\,\Sigma\, \varrho_\nu x_\mu \cos\varphi_\nu + 2\,\Sigma\, \varrho_\nu \varrho_\mu \cos(\varphi_\nu \pm \varphi_\mu)$$

$$\frac{a_{n-3}}{a_n} = \Sigma\, x_\nu x_\mu x_\lambda + \Sigma\, x_\nu \varrho_\mu^2 + 2\,\Sigma\, \varrho_\nu \varrho_\mu^2 \cos\varphi_\nu + 2\,\Sigma\, x_\lambda x_\mu \varrho_\nu \cos\varphi_\nu +$$

$$+ 2\,\Sigma\, x_\nu \varrho_\mu \varrho_\lambda \cos(\varphi_\mu \pm \varphi_\lambda) + 2\,\Sigma\, \varrho_\lambda \varrho_\nu \varrho_\mu \cos(\varphi_\lambda \pm \varphi_\nu \pm \varphi_\mu)$$

and so on.

Particular case. I. Let us suppose that we have for instance an equation of the fourth degree, the roots corresponding to the **moduli:**

$$\varrho_1 > |x_2| > |x_3|.$$

From the preceding formulae we deduce

$$-\frac{\omega_{n-1}}{\omega_n} \sim 2\varrho_1^{2^m} \cos 2^m\varphi_1 \qquad \frac{\omega_{n-2}}{\omega_n} \sim \varrho_1^{2^{m+1}}$$

$$-\frac{\omega_{n-3}}{\omega_n} \sim x_2^{2^m} \varrho_1^{2^{m+1}} \qquad \frac{\omega_{n-4}}{\omega_n} \sim x_2^{2^m} x_3^{2^m} \varrho_1^{2^{m+1}}.$$

By aid of these four equations we determine ϱ_1, x_2, x_3, and $\frac{2\pi}{2^m} k \pm \varphi_1$.

Particular case ZZ. $x_1 > \varrho_2 > |x_3|$. We have

$$-\omega_{n-1}/\omega_n \sim x_1^{2^m} \qquad \omega_{n-2}/\omega_n \sim x_1^{2^m} \varrho_2^{2^m} \cos 2^m\varphi_2$$

$$-\omega_{n-3}/\omega_n \sim x_1^{2^m} \varrho_2^{2^{m+1}} \qquad \omega_{n-4}/\omega_n \sim x_1^{2^m} x_3^{2^m} \varrho_2^{2^{m+1}}.$$

Particular case III. Let $\varrho_1 > \varrho_2 > x_3$. We have

$$-\omega_{n-1}/\omega_n \sim 2\varrho_1^{2^m} \cos 2^m\varphi_1 \qquad \omega_{n-2}/\omega_n \sim \varrho_1^{2^{m+1}}$$

$$\omega_{n-3}/\omega_n \sim \varrho_1^{2^{m+1}} \varrho_2^{2^m} \cos 2^m\varphi_2 \qquad \omega_{n-4}/\omega_n \sim \varrho_1^{2^{m+1}} \varrho_2^{2^{m+1}}$$

$$\omega_{n-5}/\omega_n \sim x_3^{2^m} \varrho_1^{2^{m+1}} \varrho_2^{2^{m+1}}.$$

Example 2. Given the equation

$$x^3 - 6x^2 + 9x + 50 = 0.$$

The roots are to be determined to ***four significant*** figures accurately. The computation by the method considered will give:

1	- 6	9	50	First powers
	-36	81		
	18	600		
1	-18	681	-2500	Second powers
	-324	4,63761.10^5		
	1362	-90000		
1	1038	3.73761.10"	—6,25.10^6	4 thpowers

1	1038	$3.73761.10^5$	$-6.25.10^6$	
	$-1.07744.10^6$	$1.39697.10^{11}$		
	74752	- 12975		
1	$-0.32992.10^6$	$1.52672.10^{11}$	-3.90625.10's	8 thpowers
	$-1.08847.10^{11}$	$2.33087.10^{22}$		
	3.05344	-258		
1	$1.96497.10^{11}$	$2.32829.10^{22}$		16 th powers.

Since the numbers of the first column oscillate between positive and negative values, therefore we conclude that the root with the greatest modulus is a complex one. So that we have according to Case I,

$$-\frac{\omega_{n-1}}{\omega_n} = 2\varrho_1^{16}\cos 16\varphi_1 = -1.96497.10^{11}$$

$$\frac{\omega_{n-2}}{\omega_n} = \varrho_1^{32} = 2.32829.10'''$$

$$\varrho_1^2 x_3 = 50.$$

From the second we obtain $\varrho_1 = 5.0000$. The third equation divided by ϱ_1^2 gives $|x_3| = 2$.

Finally from the first we deduce

$$\cos 16\varphi_1 = -\frac{1.96497.10^{10}}{515} \quad \text{or} \quad \varphi_1 = \frac{k\pi}{8} \pm 8^0\ 7'48''$$

where k is equal to one of the values $0, 1, 2, \ldots, 7$. To determine k we choose one of these values, determine the corresponding root, and put it into the given equation: this we repeat till the equation is verified. But generally there is a shorter way. For instance, in the present example we know that the only real root x_3 must be negative, since for $x=-\infty$ the first member of the equation is negative and for $x=0$ it is positive. Therefore $x_3=-2$. Moreover the sum of the roots must be equal to 6, that is,

$$x_3 + 2\varrho_1\cos\varphi_1 = -2 + 10\cos\varphi_1 = 6$$

this gives $\varphi_1 = 36^0\ 52'\ 12''$ corresponding to $k=1$.

Finally the roots are

$$x_1 = 4 + 3i, \qquad x_2 = 4 - 3i, \qquad x_3 = -2.$$

Multiple roots. If the given equation has multiple roots or roots whose moduli are the same, the treatment of the problem is somewhat different. Let us suppose for instance that we have

$$|x_1| = |x_2| > |x_3| .$$

From our computation table we conclude that

$$-\frac{\omega_{n-1}}{\omega_n} = 2x_1^{2^m}; \quad \frac{\omega_{n-2}}{\omega_n} = x_1^{2^{m+1}}; \quad -\frac{\omega_{n-3}}{\omega_n} = x_3^{2^m} x_1^{2^{m+1}}.$$

On the other hand if we have

$$|x_1| > |x_2| = |x_3|$$

then

$$\frac{\omega_{n-1}}{\omega_n} = x_1^{2^m}; \quad \frac{\omega_{n-2}}{\omega_n} = 2x_1^{2^m} x_2^{2^m}; \quad -\frac{\omega_{n-3}}{\omega_n} = x_1^{2^m} x_2^{2^{m+1}}.$$

Therefore if in a column of the table of computation the number corresponding to the 2^m-th powers does not tend to the square of the number corresponding to 2^{m+1} but to half the square of this number, then we conclude that there are two roots corresponding to this column whose moduli are the same.

There is no difficulty in deducing, in the same way, the rule corresponding to any other case of multiple roots.

Example 3. Given the equation

$$x^3 - 2x^2 - 9x + 18 = 0.$$

The table of computation will be:

1	- 2	- 9	18	First powers
	4	81		
	-18	72		
1	-22	153	-324	Second powers
	484	2.3409.10’		
	306	-1.4256		
1	-178	$9.153 \cdot 10^3$	$-1.0498 \cdot 10^5$	4 th powers

1	-178	9.153-10s	$-1.0498 \cdot 10^5$	
	$-3.1684 \cdot 10^4$	$8.3777 \cdot 10^7$		
	1.8306	-3.7373		
1	$-1.3378 \cdot 10^4$	$4.6404 \cdot 10^7$	-1.1021~10'0	8 th powers
	$-1.7897 \cdot 10^8$	2.153510'5		
	9281	-2949		
1	$-8.616 \cdot 10^7$	$1.8586 \cdot 10^{15}$	$-1.2146 \cdot 10^{20}$	16 th powers
	$-7.4235 \cdot 10^{15}$	$3.4544 \cdot 10^{30}$		
	3.7172	-209		
1	$-3.7063 \cdot 10^{15}$	$3,4335 \cdot 10^{30}$	$-1,4751 \cdot 10^{40}$	32 nd powers
	$-1.3737 \cdot 10^{31}$	$1.1789 \cdot 10^{61}$		
	6867			
1	$-0.6870 \cdot 10^{31}$			64 th powers.

Since the *number* of the first column corresponding to the 64 **th** powers is approximately equal to half the square of the number corresponding to the 32 nd powers, we may stop the computation, concluding that there are two roots having the greatest modulus.

$$|x_1| = |x_2| > |x_3|.$$

Hence we have

$$2x_1^{64} = 6{,}870 \cdot 10^{30}.$$

This gives $|x_1| = 3{,}000$. In the second column we should get the same number from

$$x_1^{32} = 3{,}4335 \cdot 10^{30}.$$

Finally $x_1x_2x_3=18$ gives $|x_3| = 2$. From $x_1+x_2+x_3 = 2$ we conclude that $x_1=3$, $x_2=-3$ and $x_3=2$.

We have seen that in the ***Ch'in-Vieta-Horner*** method the fewer figures there are in the root, the less is the work of computation. The above example shows that the computation in ***Graeffe's*** method is independent of the number of figures by which the root is expressed, and this is a drawback; but the method is nevertheless very useful, especially for the determination of complex roots.

§ 155. Numerical integration. Given y (x,,) y $(x_1), \ldots, y(x_n)$, the integral of a function f(x) satisfying to

$$f(x_i) = y(x_i) \quad \text{for} \quad i = 0, 1, 2, \ldots, n.$$

is to be determined from $x = a$ to $x = b$.

The problem becomes univocal if the condition is added that the function *f(x)* shall be a polynomial of degree *n*.

Then by aid of ***Langrange's*** interpolation formula (§ 132) we obtain this function

$$(1) \qquad f(x) = \sum_{i=0}^{n+1} L_i(x) y_i(x)$$

where $L_i(x)$ is given by formula (2) § 132. It may be written as follows:

$$(2) \qquad L_i(x) = \frac{(x-x_0)\ (x\text{-}x,)\ \ldots\ (x\text{-}x,)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_n)}.$$

In this expression the terms: x-xi in the numerator and $x_i - x_i$ in the denominator are suppressed.

The required integral will be

$$(3) \qquad I = \int_a^b f(x)dx = \sum_{i=0}^{n+1} y(x_i) \int_a^b L_i(x)dx.$$

To determine this integral we will introduce a new variable t so that the limits of the integral shall be zero and ~~one~~ n. We put

$$x = a + \frac{(b-a)}{n} t \quad \text{and} \quad x_i = a + \frac{(b-a)}{n} t_i.$$

The transformation gives

$$L_i(x) = \frac{(t-t_0)\ (f\text{-}f,)\ ,\ \ldots\ (f\text{-}f,)}{(t_i-t_0)(t_i-t_1)\ldots(t_i-t_n)}$$

This is of the same form as the corresponding expression of x; we shall denote it by $\mathscr{L}_{ni}(t)$. As *we* have $dx = (b-a)dt/n$ therefore

$$Z = (b-a) \sum_{i=0}^{n+1} y(x_i) \frac{1}{n} \int_0^n \mathscr{L}_{ni}(t)dt.$$

If the same values of $t_0, t_1, \ldots, t_n$ occur often, it is useful to construct a table of the integrals

$$\frac{1}{n}\int_0^n \mathscr{L}_{ni}(t)dt \quad \text{f o r} \quad i = 0, 1, \ldots, n$$

which are independent of the function *f(x).* Having these values, the computation of the integral I is reduced to a few multiplications and additions. The above integral has been determined for several systems of t_m , as we shall see.

Formula in the case of equidistant values of x. Putting $b—a=nh$ and $x_i=a+ih$ *we have* $t_i=\dfrac{ihn}{b-a}=i.$

The integrals corresponding to these values of t have been denoted by

$$C_{ni} = \frac{1}{n}\int_0^n \mathscr{L}_{ni}(t)dt.$$

The numbers C_{ni} are the celebrated Cotes numbers; they are given in the table below[49]

Table of the C_{ni}.

n / i	*0*	1	2	3	4	5	
1	½	½					
2	1/6	4/6	1/6				
3	1/8	3/8	318	1/8			
4	7/90	32/90	12/90	32/90	7/90		
5	19/288	75/288	50/288	50/288	75/288	191288	
6	41/840	216/840	27/840	272/840	27/840	2161840	41/840

Remark. The Cotes numbers are not always positive, for instance in the cases of $n=8$ and $n=10$ some of them are negative.

The integral Z expressed by aid of these numbers will be

$$(4) \qquad \int_a^b f(x)dx = (b—a)[C_{n0}y(x_0) + C_{n1}y(x_1) + \ldots + C_{nn}y(x_n)].$$

[49] *E.* ***Pascal,*** **Repertorium Vol. I., p. 524.**
Cotes, **Harmonia Mensurarum, 1722.**

Putting into this $a=0, b=1, f(x)=1$ and $y(x_i)=1$ we obtain an interesting relation between the **Cotes** numbers:

$$C_{n0} + C_{n1} + \dots + C_{nn} = 1.$$

It can be shown moreover that

$$C_{n,n-i} = C_{ni}.$$

Determination of the Cotes numbers by aid of Stirling numbers. If $t_i = i$, then Lagrange's polynomial $\mathcal{L}_{ni}(t)$ will become

$$\mathcal{L}_{ni}(t) = \binom{t}{i}\binom{n-t}{n-i}$$

Replacing the binomials figuring in this formula by their expansion into a power series (§ 55) we obtain if, $n > i$:

$$\frac{1}{n}\,\mathcal{L}_{ni}(t) = \frac{n^n}{n\,(i)!\,(n-i)!}\sum_{\nu=1}^{i+1}\sum_{\mu=1}^{n-i+1} S_i^\nu\, S_{n-i}^\mu\, t^\nu\,(n-t)^\mu .$$

The integration, if t varies from zero to n, gives, according to § 24,

$$(5)\qquad C_{ni} = \frac{1}{n!}\binom{n}{i}\sum_{\nu=1}^{i+1}\sum_{\mu=1}^{n-i+1} n^{\nu+\mu}\,\frac{1}{(\mu+1)\binom{\nu+\mu+1}{\mu+1}}\,S_i^\nu\, S_{n-i}^\mu .$$

In the case of $i=n$ *we have*

$$(6)\qquad \frac{1}{n}\,\mathcal{L}_{nn}(t) = \frac{1}{n}\binom{t}{n} = \frac{t}{n\,.\,n!}\sum_{\nu=1}^{n+1} t^\nu S_n^\nu .$$

Integration from $t=0$ to $t=n$ gives:

$$(7)\qquad C_{nn} = \frac{1}{n!}\sum_{\nu=1}^{x+1}\frac{n^\nu S_n^\nu}{\nu+1}.$$

If the *Stirling* numbers of the first kind S_n^ν are known then the *Cotes* number C_{nn} may be computed by aid of this formula; but we may obtain them. still in another way. By aid of formula (5) of § 89 from (6) we get

$$(8)\qquad C_{nn} = \frac{1}{n}\left[\psi_{n+1}(n) - \psi_{n+1}(0)\right]$$

where $\psi_{n+1}(x)$ is the **Bernoulli** polynomial of the second kind (§ 89) of degree $n+1$.

In consequence of the formulae of p. 269 we may write

$$(9) \qquad C_{nn} = \frac{1}{n}\left\{1 - \sum_{i=1}^{n+1} |b_i| - b_{n+1}[1+(-1)^n]\right\}.$$

On the other hand if the Cotes numbers C_{nn} are known the sum of the coefficients b_i in the *Bernoulli* polynomial of the second kind may be obtained by

$$(10) \qquad \sum_{i=0}^{2n} | b_i | = 2 - (2n-1)\, C_{2n-1,\, 2n-1}\,.$$

Examples. If C_{33}, from (7) it follows that

$$C_{33} = \frac{1}{6} \sum_{\nu=1}^{4} \frac{3^\nu}{\nu+1} S_3^\nu = \frac{1}{8}.$$

If C_{32}, then $n=3$ and $i=2$, and every term will vanish except those in which $\mu = 1$; since $S_1^1 = 1$, hence we shall have

$$C_{32} = \frac{1}{2} \sum_{\nu=1}^{3} \frac{3^{\nu+1} S_2^\nu}{(\nu+2)(\nu+1)} = \frac{3}{8}.$$

and from (9) by aid of the table of p. 266 we find

$$C_{33} = \frac{1}{3}\left[1 - \frac{1}{2} - \frac{1}{12} - \frac{1}{24}\right] = \frac{1}{8}.$$

In § 131 we have seen another method for obtaining these numbers (Formula 9).

Application of the Cotes numbers. Trapezoidal rule. If $n=1$ then $C_{10}=C_{11} = \frac{1}{2}$ and the integral (4) will be

$$I = \tfrac{1}{2}(b-a)[f(x_0) + f(x_1)].$$

If we have m+l equidistant points $x_{i+1} - x_i = h$, then applying this rule between each two points we get

$$I = h[\tfrac{1}{2}f(x_0) + f(x_1) + \ldots + f(x_{m-1}) + \tfrac{1}{2}f(x_m)].$$

Simpson's rule. If $n=2$ then $C_{20}=C_{22}=\frac{1}{6}$ and $C_{21}=\frac{4}{6}$ and the integral will be

$$I = \frac{h}{6}[f(x_0) + 4f(x_1) + f(x_2)].$$

If there are $2m+1$ equidistant points, then applying the rule between each three points we have

$$I = \frac{1}{6}\left[y_0 + y_{2m} + 2(y_2+y_4+\cdots y_{2m-2}) + 4(y_1+y_3+\cdots+y_{2m-1})\right].$$

Boole's formula. If $n=4$, by aid of the Cotes numbers we deduce

$$I = \frac{h}{90}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right].$$

§ 156. Hardy and Weddle's formulae, *Hardy's* proceeding for numerical integration is the following. We start from the, points of coordinates:

$$a-3, y(a-3), \quad a-2, y(a-2), \quad a, y(a), \quad a+2, y(a+2), \quad a+3, y(a+3)$$

and determine the parabola of the fourth degree passing through these five points. Let it be

$$(1) \qquad f(a+x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4.$$

From this it follows that

$$\begin{aligned} f(u) &= y(a) = C_0 \\ f(a+2) &= y(a+2) = C_0 + 2C_1 + 4C_2 + 8C_3 + 16C_4 \\ f(a-2) &= y(a-2) = C_0 - 2C_1 + 4C_2 - 8C_3 + 16C_4 \\ f(a+3) &= y(a+3) = C_0 + 3C_1 + 9C_2 + 27C_3 + 81C_4 \\ f(a-3) &= y(a-3) = C_0 - 3C_1 + 9C_2 - 27C_3 + 81C_4. \end{aligned}$$

By the aid of the above five equations we may determine the coefficients C_i. For instance we find

$$C_2 = \frac{1}{360}\{81[y(a+2)+y(a-2)] - 16[y(a+3)+y(a-3)] - 130y(a)\}$$

$$C_4 = \frac{1}{360}\{4[y(a+3)+y(a-3)] - 9[y(a+2)+y(a-2)] + 10y(a)\}$$

Now we may determine the integral of $f(a+x)$ from a-3 to $a+3$. It will give

$$I = \int_{-3}^{3} f(a+x)\,dx = 6C_0 + 18C_2 + \frac{486}{5}C_4 .$$

Putting into it the obtained values of C_0, C_2 and C_4 we get

$$(2) \qquad I = \frac{1}{50}\left\{110y(a) + 81[y(a+2) + y(a-2)] + 14[y(a+3) + \right.$$
$$\left. + y(a-3)]\right\}.$$

This is *Hardy's* formula of numerical integration.

Weddle's formula is obtained from (2) by adding to it a quantity equal to zero, therefore it will give the same values as *Hardy's* formula.

The quantity added is the sixth difference of $f(a+x)$ divided by 50; since $f(a+x)$ is of the fourth degree, obviously the sixth difference is equal to zero, According to § 6 this quantity is equal to

$$\frac{1}{50}\Delta^6 f(a-3) = \frac{1}{50}[f(a-3) - 6f(a-2) + 15f(a-1) - 20f(a) +$$
$$+ 15f(a+1) - 6f(a+2) + f(a-l-3)] = 0.$$

We shall have

$$(3) \qquad I = \frac{3}{10}[y(a-3) + 5y(a-2) + 6y(a) + 5y(a+2) +$$
$$+ y(a+3) + f(a-1) + f(a+1)].$$

Now $f(a+1)$ and $f(a-1)$ should be determined by aid of equation (1), but if we do this, then the obtained formula becomes identical with *Hardy's* formula.

Weddle's formula is obtained from (3) by putting simply $f(a+1) = y(a+1)$ and $f(a-1) = y(a-1)$ which is inexact.

§ 157. The Gauss-Legendre method of numerical integration. This method differs from the preceding methods especially in the following: The given $n+1$ points through which the parabola of degree n shall pass do not correspond to equidistant abscissae but to the roots of *Legendte's* polynomial of degree $n+1$.

The roots of the polynomials of the first five degrees are given in § 138.

The equation of the parabola is given by *Lagrange's* interpolation formula § 132:

$$(1) \qquad f(x) = \sum_{i=0}^{n+1} L_{ni}(x) f(x_i)$$

where

$$(2) \qquad L_{ni}(x) = \frac{X_{n+1}(x)}{(x-x_i)\,\mathrm{D}X_{n+1}(x_i)} \qquad \text{(where } 1=0, 1, 2, \dots, n).$$

If we add to (1) the remainder

$$R_{n+1} = \frac{X_{n+1}(x)}{(n+1)!}\,\mathrm{D}^{n+1}f\,(\xi) \qquad \text{(where } -1 < \xi < 1)$$

then it will represent any curve passing through the given $n+1$ points.

If the integral of (1) is required from $x=-1$ to $x=1$, then we must determine the integral of $L_{ni}(x)$. We shall have

$$(3) \qquad A_{ni} = \int_{-1}^{1} L_{ni}(x)\,dx = \frac{2}{(1-x^2)\,[\mathrm{D}X_{n+1}(x_i)]^2}.$$

These numbers may be determined once for all and given by a table. They may be considered as *Cotes* numbers corresponding to the *Gauss-Legendre* method of numerical integration.

The numbers A_{ni} are always positive; this is an advantage over the numbers corresponding to equidistant abscissae, which as has been mentioned may be negative.

Table of the numbers A_{ni}.

	0	1	2	3 .	4
1	1	1			
2	0.5555556	0.8888889	0,5555556		
3	0.3478558	0'6521452	0.6521452	0.3478558	
4	0.2369268	0.4786286	0.5688889	0,4786286	0.2369268

Remark. It can be shown that $A_{ni}=A_{n,n-i}$ and

$$\sum_{i=0}^{n+1} A_{ni} = 2.$$

Finally the integral of $f(x)$ will be

$$(4) \qquad \int_{-1}^{1} f(x)\,dx = A_{n0}y(x_0) + A_{n1}y(x_1) + \dots + A_{nn}y(x_n).$$

Gauss introduced this method because the value of the integral (4) is the same as that corresponding to a parabola of degree $2n+1$ passing through the given n+l points and through any other n+l chosen points.

Indeed, if we add to (1) the following remainder

$$\mathscr{R}_{n+1} = \frac{X_{n+1}(x)}{(n+1)!} Q_n(x)$$

where Q,,(X) is a polynomial of degree *n,* then, first the curve will still pass through the given n+l points; as $\mathscr{R}_{n+1}$ is equal to zero at these values; secondly, we may dispose of the coefficients in $Q_n(x)$ so that the curve shall pass also through the other $n+1$ chosen points. Moreover, in consequence of the orthogonality of the polynomial X,(x) we have

$$\int_{-1}^{1} X_{n+1}(x) Q_n(x)\, dx = 0.$$

From this we conclude that the integral corresponding to the new curve will be equal to (4).

L. Fejér's step parabola. [50] **Fejér** disposes of the polynomial $Q_n(x)$ so that the tangents at the points of coordinates x_i, $y(x_i)$ are parallel to the x axis. He found

$$(5)\qquad f(x) = \sum_{i=0}^{n+1} \frac{1-2xx_i+x_i^2}{1-x_i^2}\left[\frac{X_{n+1}(x)}{(x-x_i)DX_{n+1}(x_i)}\right]^2 y(x_i)$$

where the x_i are the roots of $X_{n+1}(x)=0$; and $y(x_i)$ the corresponding given values.

The advantage of this formula over the other is the following. If a given continuous function *F(x),* is approximated by the step-parabola (5) then if *n* is increasing indefinitely the curve (5) tends to *F(x).* This is not so in the general case.

§ 158. Tchebichef's formula for numerical integration. Given x_i, y (x_i) for i $=$ 0, 1, 2, . . . , n, if a polynomial *f(x) is* determined so as to have

$$f(x_i) = y(x_i)$$

---

[50] *L. Fejér:* Az interpolációról. Mathematikai és Természettudományi Értesítő, Vol. 34. 1916. pp. 209 and 229,

Über Interpolation, Nachrichten der Gesellschaft der Wissenschaften. Göttingen, 1916.

for the given values, then according to *Lagrange's* formula (§ 132) we have

$$f(x) = \sum_{i=0}^{n+1} L_{ni}(x)\ y(x_i)$$

and the integral of $f(x)$ from $x=-1$ to $x=1$ will be

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n+1} y(x_i) \int_{-1}^{1} L_{ni}(x)dx$$

or if we denote the integral in the second member by B_{ni} (the Cotes numbers corresponding to this formula of numerical integration), then we have

$$(1) \qquad \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n+1} B_{ni}\, y(x_i).$$

Since the numbers B_{ni} are independent of $y(x_i)$ therefore putting $y(x_i)=1$ and $f(x)=1$ we find

$$\sum_{i=1}^{n+1} B_{ni} = 2$$

whatever the system of values $x_0, \ldots, x_n$ may be; but the B_{ni} themselves depend on these values.

If the quantities $y(x_i)$ are results of observation, then owing to the errors of observation, which necessarily occur, they will not be absolutely exact, and consequently the integral (1) will be affected by the errors.

Denoting the error of $y(x_i)$ by ε_i, the error E of the integral will be

$$E = B_{n0}\varepsilon_0 + B_{n1}\varepsilon_1 + \ldots + B_{nn}\varepsilon_n .$$

Supposing that the probability of the error ε_i is independent of i, and denoting by σ the standard error of $y(x_i)$, then the probability $P(\lambda)$ of having $|E| < \lambda$, as shown in the Calculus of Probability is given approximately by

$$P(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\xi} e^{-t^2/2}\, dt$$

where we have

$$\xi = \frac{\lambda}{\sigma\sqrt{\Sigma B_{ni}^2}} .$$

Now Tchebichef wanted to dispose of the numbers B_{ni} *so* that the probability P(1) should be maximum. This problem is identical with the following. Given $\Sigma B_{ni} = 2$, to dispose of the B_{ni} so that ΣB_{ni}^2 shall be minimum. This will be obtained when the equations

$$\frac{\partial}{\partial B_{ni}} [\Sigma B_{ni}^2 - \lambda(\Sigma B_{ni} - 2)] = 0$$

are satisfied for every value of i. We get

$$2B_{ni} - \lambda = 0$$

or $B_{ni} = \frac{1}{2}\lambda$. The numbers B_{ni} are all equal and therefore $B_{ni} = 2/(n+1)$. Finally the integral will be

$$I = \frac{2}{n+1} [y(x_0) + y(x_1) + \ldots + y(x_n)].$$

Therefore it is necessary to dispose of the values x_0 , x, , $x_2, \ldots, x_n$ so that

$$\int_{-1}^{1} L_{ni}\, dx = \frac{2}{n+1}. \tag{2}$$

Tchebichef has introduced a certain polynomial obtained from

$$x^n e^{-\sum_{\nu=1}^{\infty} \frac{n}{2\nu(2\nu+1)x^{2\nu}}}. \tag{3}$$

by expanding the second factor into a series of inverse powers; neglecting, after multiplication by x^n, the fractional parts. Let us denote the polynomial of degree n by T,.

Tchebichef has shown that the roots of the polynomial T_{n+1} satisfy the conditions (2) .[51]

If y(x) is a function of x such that y (O)=O then the expansion of e^y will be

$$e^y = 1 + \frac{x}{1!} [De^y]_{y=0} + \frac{x^2}{2!} [D^2e^y]_{y=0} + \frac{x^3}{3!} [D^3e^y]_{y=0} + \ldots.$$

The expressions $D^m e^y$ are given by formulae (6) of § 72.

If we put

[51] *Tchebichef,* Oeuvres, Vol. II, pp. 165—180.

$$y = -\sum_{\nu=1}^{\infty} \frac{n\, z^{2\nu}}{2\nu(2\nu+1)}\,.$$

Writing in the exponential of (3) $1/x = z$ we shall have

$$D^m y = -\sum_{\nu=r}^{\infty} \frac{n(2\nu-1)_{m-1}}{2\nu+1}\, z^{2\nu-m}$$

where r is the greatest integer in $(m+1)/2$.

From this we conclude that

$$[D^{2m+1}y]_{z=0} = 0 \quad \text{and} \quad [D^{2m}y]_{z=0} = -\frac{n(2m-1)!}{2m+1}$$

Therefore from the above-mentioned formulae it follows that

$$[D^{2m+1}e^y]_{y=0} = 0$$

and

$$[D^2e^y]_{y=0} = [D^2y]_{z=0} = -\frac{n}{3}$$

$$[D^4e^y]_{y=0} = [3(D^2y)^2 + D^4y]_{z=0} = \frac{n^2}{3} - \frac{6n}{5}$$

$$[D^6e^y]_{y=0} = [15(D^2y)^3 + 15D^2y\, D^4y + D^6y]_{z=0} =$$

$$= 6n^2 - \frac{5}{9}\, n^3 - \frac{120}{7}\, n\,.$$

Finally the expansion of e^y will be

$$e^y = 1 - \frac{n}{6x^2} + \frac{n}{24x^4}\left(\frac{n}{3} - \frac{6}{5}\right) + \frac{n}{720x^6}\left(6n - \frac{5}{9}\, n^2 - \frac{120}{7}\right) + \\ + \ldots$$

and the polynomial of degree n

$$T_n = x^n - \frac{nx^{n-2}}{6} + \frac{nx^{n-4}}{24}\left(\frac{n}{3} - \frac{6}{5}\right) + \\ + \frac{nx^{n-6}}{720}\left[(6n - \frac{5n^2}{9} - \frac{120}{7}\right] + \ldots .$$

Particular ***cases:***

$$T_2 = x^2 - \frac{1}{3} \qquad T_5 = x^5 - \frac{5}{6}\, x^3 + \frac{7}{72}\, x$$

$$T_3 = x^3 - \frac{1}{2}\, x \qquad T_6 = x^6 - x^4 + \frac{1}{5}\, x^2 - \frac{1}{105}$$

$$T_4 = x^4 - \frac{2}{3}\, x^2 + \frac{1}{45} \qquad T_7 = x^7 - \frac{7}{6}\, x^5 + \frac{119}{360}\, x^3 - \frac{149}{6480}$$

The roots of these polynomials are given in the table below. Let us remark that we have

$$x_i = - x_{n-i-1} .$$

Table.

$n = 2$	$x_0 = -0{,}577350$	$n = 6$	$x_0 = -0.866247$
	$x_1 = 0{,}577350$		$x_1 = -0{,}422519$
$n = 3$	$x_0 = -0.707107$		$x_2 = -0.266635$
	$x_1 = 0$		$x_3 = 0.266635$
	$x_2 = 0.707107$		$x_4 = 0.422519$
			$x_5 = 0.866247$
$n = 4$	$x_0 = -0.794654$		
	$x_1 = -0.187592$	$n = 7$	$x_0 = -0.883862$
	$x_2 = 0.187592$		$x_1 = 0{,}529657$
	$x_3 = 0.794654$		$x_2 = -0{,}323912$
$n = 5$	$x_0 = -0.832497$		$x_3 = 0$
	$x_1 = -0.374541$		$x_4 = 0.323912$
	$x_2 = 0$		$x_5 = 01529657$
	$x_3 = 0{,}374541$		$x_6 = 0.883862$
	$x_4 = 0.832497$		

This method is only useful if the roots of the polynomial used are all real; it has been shown that in certain cases some of the roots may be complex; for instance if $n=8$ *or* if $n=10.$

Let us mention here the numerical integration corresponding ***to the "Tchebichef abscissae"*** given by

$$\cos n\vartheta = 0 \quad \text{and} \quad \cos\vartheta = x$$

and that corresponding to the abscissae

$$\frac{\sin(n+1)\vartheta}{\sin\vartheta} = 0 \text{ and } \cos\vartheta = x.$$

Fejér* has shown that in both cases the **corresponding *Cotes*** numbers are always positive, like those in the ***Gauss-Legendre***

* *L. Fejér,* **Mechanische Quadraturen mit positiven Cotesschen Zahlen. Mathematische Zeitschrift, Berlin, 1933.**

method (§ 157) ; moreover that if the abscissae are chosen so that the corresponding Cotes numbers are positive, then the integral obtained by numerical integration will converge to

$$\int^{1} F(x)\, dx.$$

§ 159. Numerical f&ration of functions expanded into a series **of their differences.** In the preceding paragraphs we have integrated *Lagrange's* formula in different particular cases; now we shall proceed to the integration of series expressed by differences.

1. Newton's formula. Given $f(a)$, $\Delta f(a)$, . . , $\Delta^n f(a)$, *the* integral of the corresponding polynomial of degree n is to be determined. We have

$$(1) \qquad f(x) = \sum_{m=0}^{n+1} \binom{x-a}{m}_h \frac{\Delta^m f(a)}{h^m} .$$

To determine $\int f(x)dx$ we need the integrals of the generalized binomial **coefficients.** These we may obtain by expanding the coefficients into a series of powers of (X-U) by aid of the *Stirling* numbers of the first kind. (§ 55).

$$\binom{x-a}{m}_h = \frac{1}{m!} \sum_{\nu=0}^{m+1} S_m^\nu (x-a)^\nu h^{m-\nu}$$

therefore

$$(2) \qquad \int f(x)dx = \sum_{m=0}^{n+1} \frac{\Delta^m f(a)}{m!} \sum_{\nu=0}^{m+1} S_m^\nu \frac{(x-a)^{\nu+1}}{(\nu+1)\, h^\nu} + k .$$

Particular cases. The most important particular case is that in which the integral is taken between a and $a+nh$. Then we obtain

$$(3) \qquad \int_a^{a+nh} f(x)dx = h \sum_{m=0}^{n+1} \frac{\Delta^m f(a)}{m!} \sum_{\nu=0}^{m+1} S_m^\nu \frac{n^{\nu+1}}{\nu+1} .$$

If we put in this formula $n=2$, it becomes identical with *Simpson's* formula; and for $n=4$ with *Boole's* formula (§ 155).

We get a more simple expression if the limits are a and $a+h$. Since according to formula (7) § 89 we have

$$\frac{1}{m!} \sum_{\nu=1}^{m+1} \frac{S_m^\nu}{\nu+1} = b_m$$

where b_m is a coefficient of the *Bernoulli* polynomial of the second kind, hence from (2) it follows that

$$\int_a^{a+h} f(x)dx = h \sum_{m=0}^{n+1} b_m \Delta^m f(a). \tag{4}$$

This is a particular case of Gregory's formula.

2. Everett's formula. Given $f(0)$, $f(1)$ and the corresponding central differences $\delta^2 f(0), \delta^2 f(1), \ldots, \delta^{2n} f(0), \delta^{2n} f(1)$, the integral of $f(x)$ is to be calculated. In § 130 we had (1)

$$f(x) = \sum_{\nu=0}^{n+1} \binom{x+\nu}{2\nu+1} \delta^{2\nu} f(1) - \sum_{\nu=0}^{n+1} \binom{x+\nu-1}{2\nu+1} \delta^{2\nu} f(0). \tag{5}$$

In § 89 we have seen that

$$\int \binom{x}{m} dx = \psi_{m+1}(x) + k$$

where $\psi_{m+1}(x)$ is the *Bernoulli* polynomial of the second kind of degree $m+1$. Therefore from (5) we deduce

$$\int f(x)dx = \sum_{\nu=0}^{n+1} \psi_{2\nu+2}(x+\nu)\delta^{2\nu} f(1) - \tag{6}$$

$$- \sum_{\nu=0}^{n+1} \psi_{2\nu+2}(x+\nu-1)\delta^{2\nu} f(0) + k.$$

Let us consider first the particular case of this integral if the limits are 0 and 1. Since $\Delta\psi_i(x) = \psi_{i-1}(x)$, we have

$$\int_0^1 f(x)dx = \sum_{\nu=0}^{n+1} \psi_{2\nu+1}(\nu)\, \delta^{2\nu} f(1) - \sum_{\nu=0}^{n+1} \psi_{2\nu+1}(\nu-1)\, \delta^{2\nu} f(0);$$

remarking moreover that

$$\psi_{2\nu+1}(\nu) = -\psi_{2\nu+1}(\nu-1)$$

finally we get

$$\int_0^1 f(x)dx = \sum_{\nu=0}^{n+1} \psi_{2\nu+1}(\nu)[\delta^{2\nu} f(1) + \delta^{2\nu} f(0)]. \tag{7}$$

3. Numerical integration by the Euler-Maclaurin formula. According to § 88 this formula may be written (the interval being *h*):

$$(8)\quad \int_a^b f(u)du = h \sum_{x=a}^{h} f(x) - \sum_{n=1}^{2m} \frac{h^n B_n}{n!} [D^{n-1}f(b) - D^{n-1}f(a)]$$

$$- \frac{B_{2m}}{(2m)!} h^{2m+1} \sum_{x=a}^{b} D^{2m} f(x+\vartheta h)$$

where $\Delta x = h$, and $b = a + zh$, z being an integer, and $0 < 6 < 1$.

The numbers B_n are the *Bernoulli* numbers (§ 78). Given are the values of $f\,(a+ih)$ and the derivatives of $f(x)$ corresponding to $x=a$ and to $x=b$.

Since $B_{2n+1} = 0$ if $n > 0$, therefore formula (8) may be written:

$$(9)\ \int_a^b f(u)du = h\,[\tfrac{1}{2}f\,(a) + f(a+h) + \ldots + f(b-h) + \tfrac{1}{2}f(b)] -$$

$$- \sum_{n=1}^{m} h^{2n} \frac{B_{2n}}{(2n)!} [D^{2n-1}f\,(b) - D^{2n-1}f\,(a)] - R_{2m},$$

4. Numerical integration by aid of Gregory's formula. Let us start from formula (1) § 99, putting into it $x = a + \frac{u-a}{h}$ and $f\left(a + \frac{u-a}{h}\right) = F(u)$, moreover writing $z - a = y$. Since $dx = \frac{du}{h}$ hence we shall have

$$(10)\qquad \frac{1}{h} \int_a^{a+yh} F(u)du = \sum_{u=a}^{a+yh} F(u) +$$

$$+ \sum_{m=1}^{n} b_m [\Delta_h^{m-1} F\,(a+yh) - \Delta_h^{m-1} F(a)] + y b_n \Delta_h^n F(a+\xi h)$$

where $\Delta u = h$ and $0 < \xi h < z$ or $0 < \xi < n$. The numbers b_i **are** ***the*** coefficients of the *Bernoulli* polynomial of the second kind **(§ 89).**

Example 1. *[Whittaker* and ***Robinson,*** Calculus of Observations, **p. 145].** We have to determine

$$\int_{100}^{105} \frac{dx}{x}.$$

Since

$$A" \frac{1}{x} = \frac{(-1)^m m!}{(x+m)_{m+1}}.$$

Therefore from formula (10) we deduce

$$\int_{100}^{105} \frac{dx}{x} = \left[\frac{1}{2}\cdot\frac{1}{100}+\frac{1}{101}+\frac{1}{102}+\frac{1}{103}+\frac{1}{104}+\frac{1}{2}\cdot\frac{1}{105}\right] -$$

$$-\frac{1}{12}\left[\frac{-1}{105.106}+\frac{1}{100.101}\right]+\frac{1}{24}\left[\frac{2}{105.106.107}-\frac{2}{100.101.102}\right].$$

The terms in the three brackets will give

$$\int_{100}^{105} \frac{dx}{x} = 0.08790\ 95 - 0.000000\ 76 - 0.000000\ 01 =$$

$$= 0.048790\ 18.$$

There is an error of one unit of the last decimal, since the integral is equal to

$$\log 105 - \log 100 = 0.048790\ 169432\ldots$$

§ 160. Numerical resolution of differential equations. The best method is that of *I. C. Adams*. Let us apply it to a differential equation of the first order. Given the quantity y_0 corresponding to x_0 and the equation

(1) $$DY = f_1(x,y).$$

A suite of numbers $y_1, y_2, y_3, \ldots$ corresponding to x_1, $x_2, x_3, \ldots$, is to be determined.

From (1) we deduce by derivation

$$D^2y \; 3 \; \frac{\partial f_1(x,y)}{\partial x} + \frac{\partial f_1(x,y)}{\partial y} Dy = \frac{\partial f_1}{\partial x} + f_1 \frac{\partial f_1}{\partial y} = f_2(x,y).$$

From **this** we obtain in the same manner $D^3y = f_3(x,y)$, and so on: $D^m y$.

First, putting x_0 and y_0 into the obtained formulae we compute the values of Dy_0, D^2y_0, D^3y_0 and D^4y_0.

Then we may obtain y(x) by the aid of *Taylor's* series:

(2) $$y(x) = \sum_{m=0}^{\infty} \frac{(x-x_0)^m}{m!} D^m y_0 .$$

But it must be remarked that x-x,, should be chosen small enough, so that stopping formula (2) at the term $m=4$, this latter should be negligible; hence putting $x = x_0 + \xi h$ the quantity h must be small, and moreover let us say: $\xi < 5$.

Now we may compute y_1, y_2, y_3, y_4 by aid of formula **(2)**, and then $\mathsf{D}y_1$, Dy, , Dy, , $\mathsf{D}y_4$ by aid of (1). Finally let us form by simple subtractions the following table of the differences of $h\mathsf{D}y_i$: —

$$
\begin{array}{lllll}
h\mathsf{D}y_0 & & & & \\
 & \underset{h}{\Delta} h\mathsf{D}y_0 & & & \\
h\mathsf{D}y_1 & & \underset{h}{\Delta^2} h\mathsf{D}y_0 & & \\
 & \underset{h}{\Delta} h\mathsf{D}y_1 & & \underset{h}{\Delta^3} h\mathsf{D}y_0 & \\
h\mathsf{D}y_2 & & \underset{h}{\Delta^2} h\mathsf{D}y_1 & & \underset{h}{\Delta^4} h\mathsf{D}y_0 \\
 & \underset{h}{\Delta} h\mathsf{D}y_2 & & \underset{h}{\Delta^3} h\mathsf{D}y_1 & \\
h\mathsf{D}y_3 & & \underset{h}{\Delta^2} h\mathsf{D}y_2 & & \\
 & \underset{h}{\Delta} h\mathsf{D}y_3 & & & \\
h\mathsf{D}y_4 & & & &
\end{array}
$$

Consequently h should be small enough, so that $\Delta^4 h\mathsf{D}y_i$ shall also be negligible.

The expansion of $\mathsf{D}y(a+\xi h)$ by *Newfon's* backward formula **(8), § 23, gives**

$$\mathsf{D}y(a+\xi h) = \sum_{n=0}^{\infty} \binom{\xi+n-1}{n} \underset{h}{\Delta^n} \mathsf{D}y\ (a-nh).$$

Multiplying it by $dx = hd\xi$, the integration of this expression will give, if x varies from a to $a+h$, or ξ from 0 to 1:

$$\int_a^{a+h} \mathsf{D}y(x)dx = \underset{h}{\Delta} y(a) = h \sum_{n=0}^{\infty} \underset{h}{\Delta^n} \mathsf{D}y(a-nh) \int_0^1 \binom{\xi+n-1}{n} d\xi .$$

According to **§** 89 we have

$$\int_a^b \binom{x}{n}\ dx = \psi_{n+1}(b) - \psi_{n+1}(a)$$

where $\psi_{n+1}(x)$ is the ***Bernoulli*** polynomial of the second kind of degree $n+1$. Remarking that $\Delta\psi_{n+1}(x) = \psi_n(x)$, it follows that

$$\int_0^1 \binom{E+n-1}{n} d\xi = \psi_{n+1}(n) - \psi_{n+1}(n-l) = \psi_n(n-1)$$

but in consequence of the symmetry of these polynomials (§ 90) we have

$$\psi_n(n-1) = (-1)^n \psi_n(-1).$$

Moreover we found (§ 89) that

$$\psi_n(-1) = (-1)^n [1 - \sum_{m=1}^{n+1} |b_m|].$$

This will give

n	$\psi_n(n-1)$
1	1/2
2	5/12
3	3/8
4	251/720
5	233/720

Finally we have

$$(3) \qquad \underset{h}{\Delta} y\,(a) = h \sum_{n=0}^{\infty} \psi_n\,(n-1)\, \Delta^n \mathrm{D}\,(a-nh)$$

that is

$$(4) \qquad \underset{h}{\Delta} y(a) = h\mathrm{D}y(a) + \frac{1}{2}\, h\underset{h}{\Delta}\mathrm{D}y(a-h) + \frac{5}{12} h\underset{h}{\Delta^2}\mathrm{D}y(a-2h) +$$

$$\frac{3}{8}\, h\underset{h}{\Delta^3}\mathrm{D}y\,(a\text{-}3h) + \frac{251}{720}\, h\underset{h}{\Delta^4}\mathrm{D}y(a-4h).$$

Putting into this equation $a = x_0 + 4h = x_4$ *we* get

$$(5) \qquad \underset{h}{\Delta} y_4 = h\mathrm{D}y_4 + \frac{1}{2}\,\underset{h}{\Delta} h\mathrm{D}y_3 + \frac{5}{12}\,\underset{h}{\Delta^2} h\mathrm{D}y_2 + \frac{3}{8}\,\underset{h}{\Delta^3} h\mathrm{D}y_1 +$$

$$+ \frac{251}{720}\,\underset{h}{\Delta^4} h\mathrm{D}y_0\,.$$

Thirdly, by aid of (5) we compute $y_5 = y_4 + \Delta y_4$ and $\mathrm{D}y_5 = f_1(x_5, y_5)$.

Then we add the number $h\mathrm{D}y_5$ to the table above, and compute the differences $\underset{h}{\Delta} h\mathrm{D}y_4$, $\underset{h}{\Delta^2} h\mathrm{D}y_4$, $\underset{h}{\Delta^3} h\mathrm{D}y_4$ and $\underset{h}{\Delta^4} h\mathrm{D}y_4$.

After this we may put $a = x_0 + 5h = x_5$ into equation (4) and proceed to the determination of y_6 in the same way; and so on. In this way we may obtain y_ν generally with sufficient exactitude for any value of ν.

CHAPTER X.

FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES.

§ 161. Functions of two independent variables. Given a function $z=f(x,y)$ we may apply the methods already established for determining the differences of z with respect to the variable x, considering y constant. The first difference will be denoted as follows:

$$\Delta_x f(x,y) = f(x+h,y) - f(x,y).$$

Moreover we may determine the difference of z with respect to y, considering x constant:

$$\Delta_y f(x,y) = f(x,y+k) - f(x,y).$$

The above differences are called *partial differences* of the function z. The second partial differences are obtained in the same way:

$$\Delta_x^2 f(x,y) = f(x+2h,y) - 2f(x+h,y) + f(x,y)$$

$$\Delta_x \Delta_y f(x,y) = f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y)$$

$$\Delta_y^2 f(x,y) = f(x,y+2k) - 2f(x,y+k) + f(x,y).$$

The other operations will be defined in the same manner. For instance the operation of *displacement* with respect to x is

$$\mathrm{E}_x^n f(x,y) = f(x+nh,y)$$

and that of the displacement with respect to y

$$\mathrm{E}_y^m f(x,y) = f(x,y+mk).$$

An *equation of partial differences* will be written symbolically

$$\Phi(\underset{x}{\mathrm{E}}, \underset{y}{\mathrm{E}})\, f(x,y) = V(x,y).$$

where Φ is a polynomial.

The operation of the mean with respect to x will be

$$\underset{x}{\mathrm{M}}\, f(x,y) = \tfrac{1}{2}[f(x+h,y) + f(x,y)]$$

and with respect to y

$$\underset{y}{\mathrm{M}}\, f(x,y) = \tfrac{1}{2}[f(x,y+k) + f(x,y)].$$

Strictly in the symbols $\underset{x}{\mathrm{A}}$, $\underset{y}{\mathrm{A}}$, . . , the increments of x and of y should also be indicated, for instance thus:

$$\underset{x,h}{\mathrm{A}}, \quad \underset{y,k}{\mathrm{A}}$$

but if in the following formulae we always have $\Delta x = h$ and $\Delta y = k$ then, to simplify, the increments. may be omitted in the notations.

The operations will be executed exactly as has been described in the foregoing paragraphs. For instance:

Expansion of a function of two variables by Newton's theorem. Given the function $z = f(x,y)$, first we expand it with respect to x considering y constant; we shall have

$$f(x,y) = \sum_{n=0}^{\infty} \binom{x-a}{n}_h \frac{\underset{x}{\Delta^n} f(a,y)}{h^n}.$$

Now considering x constant we will give y an increment equal to k and obtain

$$(1) \qquad f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{x-a}{n}_h \binom{y-b}{m}_k \frac{\underset{x}{\Delta^n} \underset{y}{\Delta^m} f(a,b)}{h^n k^m}.$$

If $f(x,y)$ is a polynomial the expansion **(1)** is **always** legitimate, since then the series is finite. If not, formula **(1)** is applicable only if certain conditions are satisfied.

Putting into equation (1) $x = a + \xi h$ and $y = b + \eta k$ we find

$$f(a+\xi h, b+\eta k) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{\xi}{n} \binom{\eta}{m} \underset{x}{\Delta^n} \underset{y}{\Delta^m} f(a,b).$$

§ 162. Interpolation in a double entry table. If the values of $f(u_i,v_j)$ corresponding to $i = 0, 1, 2, \ldots$ and to $j = 0, 1, 2, 3, \ldots$ are given, it is possible to range them into a double entry table:

$i \diagdown j$	0	1	2	3	4
0	$f(u_0,v_0)$	$f(u_0,v_1)$	$f(u_0,v_2)$	$f(u_0,v_3)$	
1	$f(u_1,v_0)$	$f(u_1,v_1)$	$f(u_1,v_2)$	$f(u_1,v_3)$	....
2	$f(u_2,v_0)$	$f(u_2,v_1)$	$f(u_2,v_2)$	$f(u_2,v_3)$	....
3	$f(u_3,v_0)$	$f(u_3,v_1)$	$f(u_3,v_2)$	$f(u_3,v_3)$	....

....

Let us suppose that such a table is given, where the values $f(u_i,v_j)$ are equidistant, that is, where

$$u_i = u_0 + ih \quad \text{and} \quad v_j = v_0 + jk.$$

The following problem occurs very often: the value of $f(u,v)$ is to be determined corresponding to given values of u and v which do not figure in the table. We choose from the numbers $u_0, u_1, u_2, \ldots$ and from $v_0, v_1, v_2, \ldots$ the numbers a and b so as to have respectively

$$a < u < a + h \quad \text{and} \quad b < v < b + k.$$

If linear interpolation is considered as sufficiently exact, and if the table contains, besides the values of $f(u_i,v_j)$ also the first differences of this function with respect to u and to v then, using *Newton's* formula, we have

$$(1) \qquad f(u,v) = f(a,b) + (u-a)\frac{\Delta_u f(a,b)}{h} + (u-b)\frac{\Delta_v f(a,b)}{k}.$$

In this way the first approximation of the point u,v is obtained by the plane (1) passing through the three points corresponding to a,b; $a+h,b$ and $a,b+k$.; but these are not symmetrical with respect to u,v (Figure 11).

If the differences are given, the determination of $f(u,v)$ by aid of this formula presents no difficulty.

But if greater precision is wanted, then the application of *Newton's* formula requires the knowledge of the following five differences:

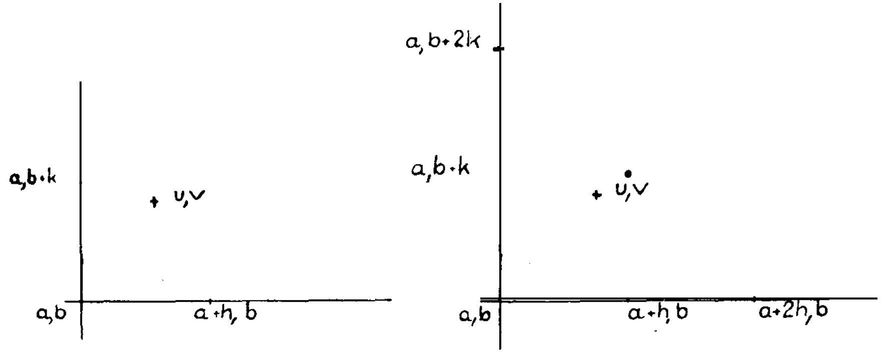
$$\underset{u}{\Delta} f, \quad \underset{v}{\mathrm{A}}\mathrm{f}, \quad \underset{u}{\Delta^2} f, \quad \underset{v}{\Delta^2} f, \quad \underset{u}{\Delta}\underset{v}{\Delta} f$$

indeed

$$(2) \qquad f(u,v) = f(a,b) + (u\text{-}a)\,\frac{\underset{u}{\mathrm{Af}}(a,b)}{h} + (o\text{-}b)\,\frac{\underset{v}{\mathrm{Af}}(a,b)}{k} +$$

$$+ \binom{u-a}{2}_h \frac{\underset{u}{\Delta^2} f(a,b)}{h^2} + (u\text{-}a)\,(u\text{-}b)\,\frac{\underset{u}{\Delta}\underset{v}{\Delta} f(a,b)}{hk} + \binom{v-b}{2}_k \frac{\underset{v}{\Delta'}\mathrm{f}\,(a,b)}{k^2}$$

Figure 11.



Here $w = f\,(u,v)$ is an equation of a second degree hyperboloid which passes through the six points corresponding to ***a,b*; *a,b+k*; *a,b+2k*;** ***a+h,b;*** ***a+h,b+k*; *a+2h,k*.** Since these points are not symmetrical with respect to the point corresponding to ***u,v*,** to be determined, therefore the interpolation formula is not advantageous (Figure 11) .

There are hardly any tables containing the five differences mentioned above. A third degree approximation by ***Newton's*** formula would require the knowledge of ten differences; these it would be nearly impossible to give in a table, and it would be superfluous, since by the aid of ***Everett's*** formula for two variables, the same approximation may be obtained by the two central differences

$$\underset{u}{\delta^2} f \text{ (a,b)} \quad \text{and} \quad \underset{v}{\delta^2} f \text{ (a,b)}.$$

Moreover, using ***Everett's*** formula it would be possible to obtain an approximation corresponding to that of a fifth degree Newton formula if the following five differences were given:

$$\underset{u}{\delta^2} f(a,b) \quad \underset{v}{\delta^2} f(a,b) \quad \underset{u}{\delta^4} f(a,b) \quad \underset{v}{\delta^4} f(a,b) \quad \underset{u}{\delta^2}\underset{v}{\delta^2} f(a,b).$$

In fact, in ***Pearson's*** celebrated Tables of the Incomplete Gamma-Function [§ 18, loc. cit. 15] the first four are given; to apply ***Everett's*** formula only the last is to be calculated.

We will not describe the proceedings of interpolation of a function of two variables by ***Everett's*** formula, since there is another formula which permits us to interpolate as rapidly with the same amount of work, and has the advantage that no printed differences at all are necessary, and therefore the table may be simplified by suppressing the differences. The reduction thus obtained will generally be more than one-third of the table.[52]

Let us suppose that a table contains the values of $f(u,v)$ corresponding to $u = u_0, u_0+h, u_0+2h, \ldots$ and to $v = v_0, v_0+k, v_0+2k, \ldots$ If $f(u,v)$ is to be determined for a given value of u and v, then we will choose from the table the numbers $a = u_0+nh$ and $b = v_0+mk$ *so* as to have

$$a < u < a+h \quad \text{and} \quad b < v < b+k.$$

Putting moreover

$$x = (u-a)/h \quad \text{and} \quad y = (v-b)/k$$

we shall have

$$0 < x < 1 \quad \text{and} \quad 0 < y < 1.$$

To begin with we start from the formula (16) of the third degree of § 133

$$F(z) = I_1 + C_1[I_1 - I_2] + \binom{x+1}{4} \mathrm{D}^4 F(a+\xi h) \tag{3}$$

where $-1 < \xi < 2$, and $C_1 = \frac{1}{2}x(1-x)$. Moreover according to (15), § 133, we have

$$I_\nu = \frac{1}{2\nu-1} \left[(x+\nu-1)F(a+\nu h) + (\nu-x)F(a-\nu h+h) \right]. \tag{4}$$

If we apply formula (3) to $F(u,v)$, considering u variable and v constant, we get

* *Ch.* **Jordan, Interpolation without printed differences in the case of two or three variables. London Mathematical Society, Journal, Vo. 8, 1933.**

$$F(u,v) = I_1(v) + C_1(x)[I_1(v) - I_2(v)] \tag{5}$$

where according to *(4)*

$$I_1(v) = xF(a+h,v) + (1-x)F(a,v)$$

and

$$I_2(v) = \frac{1}{3}[(x+1)F(a+2h,v) + (2-x)F(a-h,v)].$$

Let us again apply formula (3) to the above quantities, but considering now v variable and u constant. We find

$$I_1 = S_{11} + C_1(y)[S_{11} - S_{12}]$$
$$I_2 = S_{21} + C_1(y)[S_{21} - S_{22}]$$

where S_{11} is the result of interpolation of Z,(v), between the points corresponding to $v=b$ and $v=b+k$; or that of $F(u,v)$ interpolated between the four points corresponding to $u=a,a+h$ and $v=b,b+k$ (Fig. 12). Therefore S_{11} is equal to

$$\begin{aligned} S_{11} = {} & xy \cdot F(a+h,b+k) + \\ & + x(1-y) \cdot F(a+h,b) + \\ & + (1-x)y \cdot F(a,b+k) + \\ & + (1-x)(1-y) \cdot F(a,b). \end{aligned}$$

S_{12} is obtained by interpolating Z,(v) between the points corresponding to $v=b-k$ and $v=b+2k$, or $F(u,v)$ between the four points corresponding *to* $u=a,a+h$ and $v=b-k,b+2k$. Hence we shall have

$$\begin{aligned} 3S_{12} = {} & x(2-y) \cdot F(a+h,b-k) + \\ & + x(1+y) \cdot F(a+h,b+2k) + \\ & + (1-x)(2-y) \cdot F(a,b-k) + \\ & + (1-x)(1+y) \cdot F(a,b+2k) \end{aligned}$$

S_{21} is the result of interpolation of $I_2(v)$ between the points corresponding to $v=b$ and $v=b+k$; or that of $F(u,v)$ between the four points corresponding to $u=a-h,a+2h$ and $v=b,b+k$.

$$\begin{aligned} 3S_{21} = {} & (1+x)(1-y) \cdot F(a+2h,b) + \\ & + (1+x)y \cdot F(a+2h,b+k) + \\ & + (2-x)(1-y) \cdot F(a-h,b) + \\ & + (2-x)y \cdot F(a-h,b+k). \end{aligned}$$

And finally S_{22} is obtained by interpolation of Z,(u) between the two points corresponding to $v=b—k$ and $v=b+2k$; or that of $F(u,v)$ between the four points corresponding to $u=a—h,a+2h$ and $v=b—k,b=2k$. Therefore

$$
\begin{aligned}
9S_{22} = \; & (1+x)(2—y)\; F(a+2h,b—k) + \\
+ \; & (1+x)(1+y)\; F(a+2h,b+2k) + \\
+ \; & (2—x)\,(2-y)\; F(a—h,b—k) + \\
+ \; & (2—x)(1+y)\; F(a—h,b+2k)
\end{aligned}
$$

Consequently the numbers S_{11} , S_{12} , S_{21} and S_{22} can be easily calculated, especially when a calculating machine is used.

In the end, putting the obtained values of Z,(u) and Z,(u) into equation (5), the interpolation formula for two independent variables will be

$$
\begin{aligned}
(6) \quad F(u,v) = S_{11} + C_1(x)[S_{11}—S_{21}] + C_1(y)[S_{11}—S_{12}] + \\
C_1(x)C_1(y)[S_{11} + S_{22} — S_{12} — S_{21}].
\end{aligned}
$$

The numbers C,(x) and C,(y) may be taken from the table mentioned in § 133. If we had first interpolated with respect to v and after obtaining I_1 (u) and Z,(u) interpolated with respect to u, we should have found the same formula (6). This is an advantage over some of the other methods.

The term S_{11} may be considered as the ***first approximation*** of $f(u,v)$; in which the hyperboloid $z=S_{11}$ of the second degree passing through the mentioned four points is substituted for the surface $z=f(u,v)$.

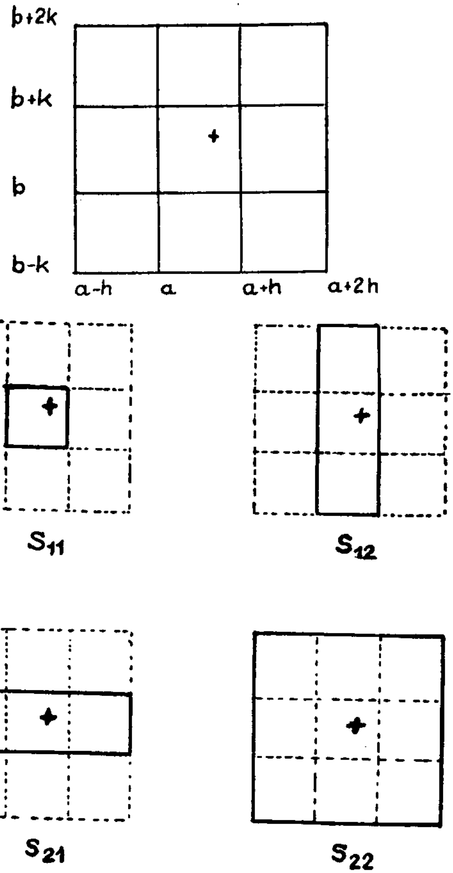
Formula (6) may be considered as the ***second approximation*** of $f\,(u,v)$ in which we substitute for the surface $z=F(u,v)$ *an* hyperboloid of the sixth degree which passes through the sixteen points corresponding to $u = a — h,a,a + h,\; a + 2h,$ and $v=b—k,b,b+k,b+2k.$

Remark. Equation (6) of the hyperboloid does not contain terms with x^4, x^5, x^6, y^4, y^5, y^6, or terms of the first degree or a constant.

Formula (6) is identical with the ***Newton*** expansion of a function of two variables, in which the partial differences of higher order than three are neglected and in which the first term is $F\,(a—h,b—k)$.

The sixteen points through which the hyperboloid passes are represented in *Figure 12.* As has been said, we must choose if possible a, and b so that the point u,v should be in the inner square. The figure shows also the four points corresponding respectively to the interpolation of $S_{\nu\mu}$.

Figure 12.



If the square of the sixteen points lies on one of the boundaries of the table, then the point u,v may be situated in one of the side squares; formula (6) is still valid, but then x and y do not necessarily satisfy the inequalities

$$0 < x < 1 \quad \text{and} \quad 0 < y < 1.$$

In these cases C,(x) and C,(y) must be calculated by

$$C_1(x) = \tfrac{1}{2}(1-x)x \quad \text{and} \quad C_1(y) = \tfrac{1}{2}(1-y)y$$

as the corresponding values of x and y are outside the range of the mentioned tables.

Example. Let us take Pearson's example of the Incomplete Gamma Function [Loc. cit. 15 p. XXXIII]. The table contains the numbers $F(u,v)$ corresponding to $u=u_0,u_0+h,u_0+2h,\ldots$ and $v=v_0,v_0+k,v_0+2k,\ldots$ where $h=0.1$ and $k=0.2$. The value of $F(4.025;7.05)$ is to be computed by interpolation. We shall have $a=4.0$; $b=7.0$ and

$$x = \frac{u-a}{h} = 0.25\,; \qquad y = \frac{v-b}{k} = 0.25.$$

Computation of S_{11}

xy	$= 0.0625$	$F(a+h,b+k)$	$= 0.8861276$
4 1 - Y)	$= 0.1875$	$F(a+h,b)$	$= 0.8913309$
$(1-x)y$	$= 0.1875$	$F(a,b+k)$	$= 0.8700917$
(l-x) (l-y)	$= 0.5625$	$F(a,b)$	$= 0.8759367$

The sum of the products is equal to $S_{11}=0.878364106$, this may be considered as the first approximation of the required value of $F(u,v)$.

Computation of S_{21}

$(1+x)y$	$= 0.3125$	$F(a+2h,b+k)$	$= 0.9004831$
$(1+x)(1-y)$	$= 0.9375$	$F(a+2h,b)$	$= 0.9050932$
$(2-x)y$	$= 0.4375$	$F(a-h,b+k)$	$= 0.8522574$
$(2-x)(1-y)$	$= 1.3125$	$F(a,b)$	$= 0.8759367$

One-third of the product is equal to S_{21} ~0.876650329.

Computation of S_{12}

$x(1+y)$	$= 0.3125$	$F(a+h,b+2k)$	$= 0.8807594$
X(2-Y)	$= 0.4375$	$F(a+h,b-k)$	$= 0.8963699$
(l-x) (1 $+y$)	$= 0.9375$	$F(a,b+2k)$	$= 0.8640723$
(l-x) (2-y)	$= 1.3125$	$F(a,b-k)$	$= 0.8816070$

One third of the sum of the product is equal to $S_{12}=0.878192038$.

Computation of S_{22}

$(1+x)\ (1+y)$	$= 1.5625$	$F(a+2h,b+2k)$	$= 0.8957190$
$(1+x)\ (2-y)$	$= 2{,}1875$	$F(a+2h,b-k)$	$= 0{,}9095504$
$(2-x)\ (1+y)$	$= 2{,}1875$	$F(a-h,b+2h)$	$= 0.8455420$
$(2-x)\ (2-y)$	$= 3.0625$	$F(a-h,b-k)$	$= 0.8651402$

One-ninth of the sum of the products is equal to $S_{22}=0{,}876479714$.

Since we have $C_1(x) = C_1(y) = 0{,}09375$ and $C_1(x)C_1(y) = = 0{,}008789$ therefore

$$C_1(x)\ [S_{11}-S_{21}] = 0.000160667$$

$$C_1(y)\ [S_{11}-S_{12}] = 0{,}000016149$$

$$C_1(x)C_1(y)\ [S_{11}+S_{22}-S_{12}-S_{21}] = 0{,}000000127$$

and finally

$$f(u,v) = 0.8785410.$$

This result is exactly the same as that obtained by *Pearson* using *Everett's* formula and requiring the second central differences. Our computation by formula (6) is in no way longer than that by *Everett's* formula, perhaps even somewhat shorter. It is not necessary to copy out of the table the values contained in the second columns; they can be transferred directly to the machine.

Should the precision of the result obtained by formula (6) be insufficient, it would be possible to make a further step and obtain a *third approximation* (of the tenth degree) by starting from

$$F(u,v) = I_1(v) + C_1(x)[I_1(v)-I_2(v)\,] + \\ + C_2(x)[2I_1(v)-3I_2(v)+I_3(v)\,]$$

obtained by interpolation of the fifth degree with respect to u; this followed by an interpolation of the fifth degree with respect to v will give an hyperboloid of the tenth degree passing through the 36 points corresponding to

$$u = a-2h, a-h, a, a+h, a+2h, a+3h$$

and

$$v = b-2k, b-k, b, b+k, b+2k, b+3k.$$

We find

$$(7) \quad F(u,v) = S_{11} + C_1(x)[S_{11}-S_{21}] + C_1(y)[S_{11}-S_{12}] +$$
$$+ C_1(x)C_1(y)[S_{11}-S_{12}-S_{21}+S_{22}]+C_2(x)[2S_{11}-3S_{21}+S_{31}] +$$
$$+ C_2(y)[2S_{11}-3S_{12}+S_{13}] + C_1(x)C_2(y)[2S_{11}+3S_{22}-S_{23}-$$
$$-2S_{21}-3S_{12}+S_{13}] + C_2(x)C_1(y)[2S_{11}+3S_{22}+S_{31}-2S_{12}-$$
$$-3S_{21}-S_{32}] + C_2(x)C_2(y)[4S_{11}+2S_{13}+S_{33}+9S_{22}+2S_{31}-6S_{12}-$$
$$-6S_{21}-3S_{23}-3S_{32}].$$

Where the nine values of $S_{\nu\mu}$ are obtained by nine interpolations, each between four points, we shall have

$$
\begin{aligned}
(2\nu-1)(2\mu-1)S_{\nu\mu} &= (\nu-1+x)(\mu-1+y)\,F(a+\nu h, b+\mu k) + \\
(8) \qquad &+ (\nu-1+x)(\mu-y)\,F(a+\nu h, b-\mu k+k) + \\
&+ \text{b - 4}\,(\mu-1+y)\,F(a-\nu h, b+\mu k) + \\
&+ (\nu-x)(\mu-y)\,F(a-\nu h+h, b-\mu k+k)
\end{aligned}
$$

In formula (6) we had four such values to determine, therefore the work of computation would be now somewhat more than the double of the preceeding example. If a machine is used, this may be undertaken in important cases. Nevertheless, ***constructing a table of a function*** $f(u,v)$, ***the intervals should be chosen so that the second approximation by formula (6) should be sufficiently exact.*** This may generally be done.

Remark. If the tangent plane is wanted at the point corresponding to u,v we have to determine

$$\partial F(u,v)/\partial u \quad \text{and} \quad \partial F(u,v)/\partial v.$$

In the case of the first approximation we shall have

$$\frac{\partial F(u,v)}{\partial u} = \frac{1}{h}\frac{\partial S_{11}}{\partial x} \quad \text{and} \quad \frac{\partial F(u,v)}{\partial v} = \frac{1}{k}\frac{\partial S_{11}}{\partial y}.$$

$\partial S_{11}/\partial x$ and $\partial S_{11}/\partial y$ may be expressed by the differences of the function $F(u,v)$. Indeed we have

$$\frac{\partial S_{11}}{\partial x} = y \mathop{\Delta}_{u}\mathop{\Delta}_{v} F(a,b) + \mathop{\Delta}_{u} F(a,b)$$

and

$$\frac{\partial S_{11}}{\partial y} = x \mathop{\Delta}_{u}\mathop{\Delta}_{v} F(a,b) + \mathop{\Delta}_{v} F(a,b).$$

In the case of the interpolation of the sixth degree (6) we should find

$$\frac{\partial F(u,v)}{\partial u} = \frac{1}{h}\left\{\left[\frac{\partial S_{11}}{\partial x} + C_1(x)\left[\frac{\partial S_{11}}{\partial x} - \frac{\partial S_{21}}{\partial x}\right] + \right.\right.$$

$$+ C_1(y)\left[\frac{\partial S_{11}}{\partial x} - \frac{\partial S_{12}}{\partial x}\right] + C_1(x)C_1(y)\left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{22}}{\partial x} - \frac{\partial S_{12}}{\partial x} - \right.$$

$$\left.\left. - \frac{\partial S_{21}}{\partial x}\right]\right\} - (x-½)\,[S_{11}-S_{21}] -$$

$$- (x-½)C_1(y)\,[S_{11}+S_{22}-S_{12}-S_{21}].$$

Therefore it will be necessary to compute the numbers $\partial S_{\nu\mu}/\partial x$. To obtain them we have only to change in formula (8) of $S_{\nu\mu}$ the first factors. For instance in the case of $\partial S_{11}/\partial x$ they will be y, *(l-y),* $-y$, -(l-y), The term $\partial F(u,v)/\partial v$ is obtained in the same manner.

§ 163. Functions of three independent variables, It is easy to extend the methods of the preceding paragraphs to functions of *three variables.* Interpolating between the 8 points corresponding to $u=a,a+h$, $v=b,b+k$ and $w=c,c+j$ we obtain a hyper-surface of the third degree $z = Q_{111}$ passing through these points and giving the *first approximation* of $f(u,v,w)$; $Q_{,,}\,_{\iota}$ is given by formula (9) below. Its computation is easy enough but generally the precision obtained will be insufficient. To remedy this inconvenience we shall determine a hypersurface of the ninth degree passing through the 64 points corresponding to

$$u=a-h,a,a+h,a+2h\,; \quad v=b-k,b,b+k,b+2k;$$
$$w=c-j,c,c+j,c+2j\,.$$

To obtain it we shall start from formula (6) considering the quantities $S_{\nu\mu}$ as functions of w. For instance

$$S_{,,}(w) = xyF(a+h,b+k,w) + x(1-y)\;F(a+h,b,w) +$$
$$(1-x)y\;F(a,b+k,w) + (l-x)\;(1-y)\;F(a,b,w)$$

and so on; then we interpolate with respect to w by formula (3).

Denoting by $Q_{\nu\mu\lambda}$ the result of interpolation of $S_{\nu\mu}$ (w) between the two points $c-\lambda j+j$ and $c+\lambda j$, or that of $F(u,v,w)$ between the 8 points corresponding to

$$u=a-\nu h+h,a+\nu h\;;\; v=b-\mu k+k,\; b+\mu k\;; \quad w=c-\lambda j+j,\, c+\lambda j.$$

Therefore we shall have

$$(1) \qquad (2\nu-1)(2\mu-1)(2\lambda-1)\,Q_{\nu\mu\lambda} =$$

$$
\begin{aligned}
&= (x+\nu-1)(y+\mu-1)(z+\lambda-1) && F(a-\nu h+h, b-\mu k+k, c-\lambda j+j)\\
&+ (x+\nu-1)(y+\mu-1)\ \text{u-4} && F(a-\nu h+h, b-\mu k+k, c+\lambda j)\\
&+ (x+\nu-1)\ \text{(P-Y)}\ (z+\lambda-1) && F(a-\nu h+h, b+\mu k, c-\lambda j+j)\\
&+ (x+\nu-1)(\mu-y)(\lambda-z) && F(a-\nu h+h, b+\mu k, c+\lambda j)\\
&+ (\nu-x)(y+\mu-1)(z+\lambda-1) && F(a+\nu h, b-\mu k+k, c-\lambda j+j)\\
&+ (\nu-x)(y+\mu-1)(\lambda-z) && F(a+\nu h, b-\mu k+k, c+\lambda j)\\
&+ (\nu-x)(\mu-y)(z+\lambda-1) && F(a+\nu h, b+\mu k, c-\lambda j+j)\\
&+ (\nu-x)\ \text{(P-Y)}\ (\lambda-z) && F(a+\nu h, b+\mu k, c+\lambda j).
\end{aligned}
$$

Finally the interpolation formula giving the *second approximation* of $f(u,v,w)$ will be

$$
\begin{aligned}
(2)\quad F(u,v,w) = {} & Q_{111} + C_1(x)[Q_{111}-Q_{211}] + C_1(y[Q_{111}-Q_{121}] + \\
& + C_1(z)[Q_{111}-Q_{112}] + C_1(x)C_1(y)[Q_{111}+Q_{221}- \\
& - Q_{121}-Q_{211}] + C_1(x)C_1(z)[Q_{111}-Q_{112}-Q_{211}+ \\
& + Q_{212}] + C_1(y)C_1(z)[Q_{111}-Q_{112}-Q_{121}+Q_{122}] + \\
& + C_1(x)C_1(yC_1(z)[Q_{111}-Q_{112}+Q_{221}-Q_{222}- \\
& - Q_{121}+Q_{122}-Q_{211}+Q_{212}]
\end{aligned}
$$

where $z = (w-c)/j$. Since we have to compute nine values of $Q_{\nu\mu\lambda}$ and each is twice as long as that of $S_{\nu\mu}$ hence the interpolation will give a little more than four times as much work as the example in § 162.

CHAPTER XI.

DIFFERENCE EQUATIONS.

§ 164. Genesis of the difference equations. Let y be a function of the variable x, which is considered as a discontinuous one, taking only integer values. If we had a function of the variable z taking the values of z_0+ih (where i is an integer) then we should introduce a new variable $x=(z-z_0)/h$, which would take only integer values, and $\Delta x=1$. Given

$$(1) \qquad \psi(x,y,a) = 0$$

where a is a constant parameter. If x is increased by one, the increment of y will be Δy. As $y+\Delta y=\mathrm{E}y$, we get

$$(2) \qquad \psi(x+1,\mathrm{E}y,a) = 0.$$

Eliminating a from (1) and (2) we obtain

$$F(x,y,\mathrm{E}y) = 0.$$

As in this expression one is the highest exponent of E, this is called a difference equation of the first order.

If we start from a function containing two parameters a and b, we have

$$\psi(x,y,a,b) = 0$$

and deduce in the same manner

$$\psi(x+1,\mathrm{E}y,a,b) = 0$$

and repeating the operation,

$$\psi(x+2,\mathrm{E}^2y,a,b) = 0$$

then eliminating from the last three equations a and b, we find

$$F(x,y,\mathrm{E}y,\mathrm{E}^2y) = 0$$

which is a difference equation of the second order.

If the function (1) contained n arbitrary constants c_1 , c_2, ..., c_n

$$\psi(x,y,c_1,c_2,\ldots\ldots, c_n) = 0 \tag{3}$$

then we should be led in the same way to the difference equation of the n-th order:

$$F(x,y,\mathrm{E}y,\mathrm{E}^2y,\ldots\ldots, \mathrm{E}^ny) = 0. \tag{4}$$

If the quantities c_i instead of being constants should be periodic functions with period equal to one, then we should. be led to the same equation.

Example 1. Given $y=c_1+c_2x$; it follows that

$$\mathrm{E}y = c_1 + c_2(x+1)$$

and

$$\mathrm{E}^2y = c_1 + c_2(x+2)$$

after elimination of c_1 and c_2 from the three equations we obtain a difference equation of the second order

$$\mathrm{E}^2y - 2\mathrm{E}y + y = 0.$$

Generally, instead of deducing the difference equation of order n, starting from a function containing n constants, the inverse problem is to be solved; that is, the difference equation of order n being given, a function y containing n arbitrary periodic functions with period equal to one is to be determined, which satisfies the given difference equation.

If starting from a difference equation of the n- th order the function obtained contains n arbitrary periodic functions, then it is called a general solution; if it contains fewer, it is a particular solution.

Remark. If the highest power of E in the difference equation (4) is equal to n and the lowest power to m (where y is considered as being equal to E^0y), then the equation is of order n-m only, and there will be only n-m arbitrary constants in the solution,

In the particular case of $n=m$ equation, (4) becomes an ordinary equation and its solution $\mathrm{E}^ny=f(x)$ or $y=f(x-n)$ does not contain arbitrary constants.

Equations of the form:

$$(5) \qquad F(x,y,\Delta y,\Delta^2 y, \ldots \Delta^n y) = 0$$

$$(6) \qquad F(x,y,\mathsf{M}y,\mathsf{M}^2 y, \ldots, \mathsf{M}^n y) = 0$$

are also difference equations according to the above definition, indeed if we eliminate from them $\Delta^m y$ or $\mathsf{M}^m y$ by aid of

$$\Delta^m = (\mathsf{E}-1)^m \quad \text{o r} \quad \mathsf{M}^m = (1+\mathsf{E})^m \frac{1}{2^m}.$$

we obtain an equation of the form (4).

There are some cases in which it is easier to solve equation (5) or (6) than (4).

In the foregoing Chapters we have already solved several such difference equations. For instance, when $\Delta y = f(x)$ was given, and we determined the indefinite sum $y = \Delta^{-1} f(x)$; moreover when solving $\mathsf{M}y = \varphi(x)$ by aid of $y = \mathsf{M}^{-1}\varphi(x)$; and also in other cases.

Difference equations are classed in the same manner as differential equations. The equation

$$(7) \qquad a_n \mathsf{E}^n y + a_{n-1}\mathsf{E}^{n-1} y + , \ldots + a_1 \mathsf{E} y + a_0 y = V(x)$$

is called a linear difference equation, provided that the coefficients a_i are independent of y and $\mathsf{E}^m y$.

If $V(x)=0$ then the equation is termed homogeneous; if not it is a complete equation.

The coefficients a_i may be constants or functions of x; we shall consider the two cases separately,

Remark. Sometimes, though the increment of x is equal to cne, the difference equation may concern a function of a continuous variable. Then for instance

$$f(x+1) - f(x) = V(x)$$

is true for every value of x. These equations will be solved by the same methods as those of a discontinuous variable; hut as we shall see some precautions will be necessary.

§ 165. Homogeneous linear difference equations with constant coefficients. Before solving the equation

$$(1) \qquad a_n \mathsf{E}^n y + , \ldots + a_1 \mathsf{E} y + a_0 y = 0$$

we will establish a preliminary theorem. It is easy to see that

$$E[r^x] = r \cdot r^x; \qquad E^s[r^x] = r^s \cdot r^x; \qquad E^{-s}[r^x] = r^{-s} r^x.$$

From this we conclude that if $\psi(E)$ is a polynomial of E then

$$\psi(E)[r^x] = r^x \psi(r). \tag{2}$$

Moreover if $\varphi(E)$ and $\psi(E)$ are polynomials, then we may give the following definition of the operation $\varphi(E)/\psi(E)$ performed on f(x): The equation

$$\left[\frac{\varphi(E)}{\psi(E)}\right] f(x) = y(x)$$

means that y(x) is a particular solution of the difference equation

$$\psi(E) y(x) = \varphi(E) f(x).$$

This definition is, in agreement with that given of

A-’ $= 1/$ (E-1) (6, § 32)

$M^{-1} = 2/(E+1)$ (2, § 38).

In the particular case of $f(x) = r^x$ the equation

$$\left[\frac{\varphi(E)}{\psi(E)}\right] r^x = y(x)$$

implies that

$$\psi(E) y(x) = \varphi(E) r^x = r^x \varphi(r)$$

but according to (2) this equation is satisfied by

$$y(x) = \frac{r^x \varphi(r)}{\psi(r)}$$

therefore

$$\frac{\varphi(E)}{\psi(E)}[r^x] = r^x \frac{\varphi(r)}{\psi(r)}.$$

In this way the operation F(E) has been performed on r^x if F(E) is a rational fraction.

$$F(E)[r^x] = r^x F(r). \tag{3}$$

Remark. In the particular case when $F(E) = 1/(E-1) = \Delta^{-1}$ we find the formula $\Delta^{-1} r^x = r^x /$ (r-l) and if $F(E) = 2/(E+1) =$

$= M^{-1}$ then we have $M^{-1} r^x = 2r^x/(r+1)$ as it has been obtained before.

Now if r_1 is equal to one of the roots (real or complex) of the equation

$$\text{(4)} \qquad \psi(r) = a_n r^n + a_{n-1} r^{n-1} + . , . . + a_1 r + a, = 0$$

which. is called the characteristic equation of (1) then it follows from (2) that cr_1^x is a particular solution of (1). If it has n unequal roots, we Shall have n solutions of the form $y_i = c_i r_i^x$.

It is easy to see that if $c_1 y_1$, $c_2 y_2$, . . . , $c_n y_n$ are particular solutions of the difference equation (1); then their sum.

$$y = c_1 r_1^x + c_2 r_2^x + ... + c_n r_n^x$$

will also be a solution; since this function contains n arbitrary periodic functions c_i, it is the general solution.

In the Calculus of Finite Differences the variable is discontinuous and takes only equidistant values; the particular solutions $y_1, y_2, . . ., y_n$, will be called *independent* if starting from any n initial values such as $y, Ey, ... E^{n-1}y$, the corresponding constants c_i may be determined by aid of the equations

$$Y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$$
$$EY = c_1 Ey_1 + c_2 Ey_2 + ... + c_n Ey_n$$
$$E^{n-1}y = c_1 E^{n-1}y_1 + c_2 E^{n-1}y_2 + , , . + c_n E^{n-1}y_n$$

that is if the determinant

$$\begin{vmatrix} y_1 & y_2 & \cdots\cdots & y^n \\ Ey_1 & Ey_2 & \cdots\cdots & EY'' \\ & \cdots\cdots & \cdots\cdots & \\ E^{n-1}y_1 & E^{n-1}y_2 & \cdots\cdots & E^{n-1}y_n \end{vmatrix}$$

is different from zero.

On the other hand if it is equal to zero for every value of a then the constants c_i corresponding to given initial values E'y cannot be determined; and the particular solutions are *not independent.* It can be shown that in this case one of them may 'be expressed by aid of the others. For instance if $y_2, y_3, . . . y_n$ are given, then y_1 may be determined by aid of the linear equation of differences of order n-l, obtained by writing that the determinant is equal to zero.

From the above it follows that an equation of differences of order n has only n independent particular solutions; moreover that if the n initial values y(a), $y(a+1)$, . . . y(a+n-1) are given, then every value of $y(a+\xi)$ may be computed step by step by aid of the equation of differences (1).

Example 1. Given the difference equation

$$f(x+2) - 3f(x+1) + 2f(x) = 0 \tag{5}$$

the corresponding characteristic equation is

$$r^2 - 3r + 2 = 0$$

therefore $r_1 = 1$ and $r_2 = 2$. So that the general solution is

$$f(x) = c_1 + c_2 2^x.$$

Putting this value into (5) it is easy to see that it verifies this equation.

Example 2. Fibonacci's numbers are the following

$$0,1,1,2,3,5,8,13,21, \ldots$$

each of which is the sum of the two numbers immediately preceding it. If we denote the general term of this series by $f(x)$, we shall have

$$f(x) + f(x+1) = f(x+2).$$

The corresponding characteristic equation is

$$r^2 - r - 1 = 0$$

and therefore

$$r_1 = \tfrac{1}{2}(1+\sqrt{5}), \quad r_2 = \tfrac{1}{2}(1-\sqrt{5})$$

so that the general solution will be

$$f(x) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^x + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^x. \tag{6}$$

To obtain the general term of the *Fibonacci* numbers, the arbitrary constants c_1 and c_2 should be determined by the inital conditions. Let us put $f(0) = 0$ and $f(1) = 1$; we find

$$c_1 + c_2 = 0 \qquad (c_1 - c_2)\sqrt{5} = 2$$

therefore

$$c_1 = -c_2 = \frac{1}{\sqrt{5}}.$$

Finally expanding the powers in (6) by Newton's formula we have

$$f(x) = \frac{1}{2^{x-1}}\left[\binom{x}{1} + \binom{x}{3}5 + \binom{x}{5}5^2 + \dots + \binom{x}{2m+1}5^m + \dots\right]$$

§ 166. Characteristic equation with multiple roots, If the characteristic equation y(r) = 0 has multiple roots, then, proceeding as before, the solution will contain less than n arbitrary constants or periodic functions, and therefore it will be only a particular solution. To obviate this we must find new solutions of the difference equation $\psi(\mathrm{E})y = 0$.

Let us show the method, first in the case of *double roots.* If r_1 is a double root of the characteristic equation, then we shall suppose that r_1 and $r_1+\varepsilon$ are roots of this equation, and later on we will put $\varepsilon=0$.

If r_1 and $r_1+\varepsilon$ are roots of the characteristic equation, ψ (t) = 0, then obviously

$$(1) \qquad y = \frac{(r_1+\varepsilon)^x - r_1^{\,x}}{\varepsilon} = \frac{\Delta r_1^{\,x}}{\varepsilon}$$

will also be a solution of y(r) = 0.

The expression (1) will really be a solution, if putting $\varepsilon=0$; but

$$\lim_{\varepsilon=0} \frac{\Delta r_1^{\,x}}{\varepsilon} = \mathrm{D} r_1^{\,x} = x r_1^{\,x-1}.$$

Where D denotes the derivative with respect to r_1.

Therefore $a_2 x r_1^{\,x-1}$ is a new solution, putting $a_2 = c_2 r$ this solution will be $c_2 x r_1^{\,x}$. If r_1 is the only double root of the characteristic equation, then the general solution will be

$$y = (c_1 + c_2 x) r_1^{\,x} + c_3 r_3^{\,x} + , . , + c_n r_n^{\,x} .$$

If there were other double roots we should proceed in the same way.

It is easy to verify that xr_1^x is a solution of the differenue equation $\psi(E)y=0$ if r_1 is at least a double root of the characteristic equation y(r) $=0$. Indeed, putting $y=x\,r_1^x$ the operation E^s gives

$$a_s E^s x r_1^x = a_s\,(x+s) r_1^{x+s} = a_s x r_1^x (r_1^s) + a_s r_1^{x+1} (s r_1^{s-1})$$

from this we conclude, summing from $s=0$ to $s=n+1$, that

$$\psi(E) x r_1^x = x r_1^x\, \psi(r_1) + r_1^{x+1}\, D\psi(r_1).$$

Since r_1 is a root of $\psi(r)=0$ therefore the first term will vanish and $y=xr_1^x$ will be a solution if we have $D\psi(r_1)=0$ that is, if r_1 is at least a double root of the characteristic equation.

Example 1. Given the difference equation

$$f(x+2) - 6f(x+1) + 9f(x) = 0.$$

The corresponding characteristic equation is

$$r^2 - 6r + 9 = 0$$

so that $r_1 = r_2 = 3$. According to what we have seen the complete solution of the difference equation is

$$f(x) = (c_1 + c_2 x) 3^x.$$

Example 2. Calculus of Probabilify. The first player has a shillings, the second b shillings. The probability of winning one shilling in each game is ½ for each player; play is finished if one of them has won all the money of his adversary. The probability is required that the first player shall win, Let us denote this probability by f(x) if he possesses x shillings.

He may *win in* two ways. *First,* by winning the next game; the probability of this event is ½; then his fortune will be $x+1$ and the probability of winning will become $f(x+1)$. *Secondly,* by losing the next game, the probability of this event is also ½ ; then his fortune will be x-l, and the probability of winning will become f (x-l), Hence applying the theorem of total probabilities we have

$$f(x) = \tfrac{1}{2} f(x+1) + \tfrac{1}{2} f(x-1)$$

that is f(x) will satisfy the difference equation

$$f(x+2) - 2f(x+1) + f(x) = 0.$$

The corresponding characteristic equation is

$$r^2 - 2r + 1 = 0 \quad \text{hence} \quad r_1 = r_2 = 1.$$

Therefore the general solution is f(x) $= c_1 + c_2 x$.

The arbitrary constants are determined by the initial conditions, If $x=0$ the first player has lost and $f(0)=0$; if $x=a+b$ he has won, $f(a+b)=1$. From this we deduce

$$c_1 = 0 \quad \text{and} \quad c_2 = 1/(a+b).$$

So that the required probability is $f(x) = x/(a+b)$. *This gives* at the beginning when $x=a$, $f(a) = a/(a+b)$.

Multiple roots. Let us suppose now that r_1 is a root of multiplicity m of the equation $y(r) = 0$. To obtain the general solution of the difference equation $\psi(\mathrm{E})y=0$ we may proceed as before considering first the m roots of the characteristic equation as being different, and equal to

$$r_1, r_1+\varepsilon, r_1+2\varepsilon, \ldots, r_1+(m-1)\varepsilon .$$

Supposing $i < m$, then a solution of the difference equation is given by

$$\frac{1}{i!\,\varepsilon^i}\left[(r_1+i\varepsilon)^x - \binom{i}{1}(r_1+i\varepsilon-\varepsilon)^x + \ldots + (-1)^i r_1^x\right] = \frac{\Delta^i r_1^x}{i!\,\varepsilon^i}$$

Putting $\varepsilon=0$ this solution will become

$$y = a_i \frac{1}{i!} \mathrm{D}^i r_1^x = \binom{x}{i} r_1^{x-i} a_i$$

or

$$Y = c_i \binom{x}{i} r_1^x \quad \text{if} \quad i < m.$$

Therefore the solution corresponding to r_1 containing m arbitrary constants will be

$$(2) \qquad y = \left[c_1 + c_2\binom{x}{1} + c_3\binom{x}{2} + \ldots + c_m\binom{x}{m-1}\right] r_1^x .$$

This formula holds for all real or complex values of r_1.

Verification. r_1 being a root of $\psi(r) = 0$ let us determine the conditions so that $\binom{x}{i} r_1^x$ shall be a solution of $\psi(\mathrm{E})y=0$. We have

$$a_s \mathsf{E}^s \left[\binom{x}{i} r_1^{\,x} \right] = a_s \binom{x+s}{i} r_1^{\,x+s}$$

In consequence of ***Cauchy*'s** rule (14, § 22) this may be written

$$a_s r_1^{\,x+s} \sum_{\nu=0}^{i+1} \binom{x}{i-\nu} \binom{s}{\nu} = \sum_{\nu=0}^{i+1} \frac{r_1^{\,x+\nu}}{\nu!} \binom{x}{i-\nu} a_s (s)_\nu r_1^{\,s-\nu}.$$

Hence, summing from $s=0$ to $s=n+1$ we must have

$$\psi(\mathsf{E}) \left[\binom{x}{i} r_1^{\,x} \right] = \sum_{\nu=0}^{i+1} \binom{x}{i-\nu} \frac{r_1^{\,x+\nu}}{\nu!} \mathsf{D}^\nu \psi(r_1) = 0.$$

Therefore $\binom{x}{i} r_1^{\,x}$ is a solution of the difference equation if

$$\mathsf{D}^\nu \psi(r_1) = 0 \qquad \text{for} \qquad \nu = 0, 1, 2, \ldots, i.$$

That is, r_1 must be at least a root of multiplicity $i+1$ of $\psi(r)=0$; hence, if r_1 is of multiplicity m, then a particular solution with m arbitrary constants is given by (2).

§ 167. Negative **roots.** We have seen that the obtained solutions of the difference equation $\psi(\mathsf{E})y=0$ were applicable also in cases when the roots of the characteristic equation y(r) =0 were negative or complex. But in these cases the solutions may appear in a complex form, and generally the real solutions are required; therefore they must be isolated.

If the function considered is one of a discontinuous variable the increment of x being equal to one, and the root r_1 is negative, then the preceding methods may be applied without modification.

Example 1. Let us consider the series

$$0,\ 1,\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{8},\ \frac{11}{16},\ \frac{21}{32},\ \ldots.$$

obtained in the following manner: each number is equal to the arithmetical mean of the two numbers immediately preceding it. If we denote the general term of the series by f(x), we have

$$\tfrac{1}{2}[f(x) + f(x+1)] = f(x+2)$$

this is a homogeneous linear difference equation with constant coefficients

$$2f(x+2) - f(x+1) - f(x) = 0.$$

The corresponding characteristic equation is

$$2r^2 - r - 1 = 0, \text{ hence } r_1 = 1 \text{ and } r_2 = -\tfrac{1}{2}.$$

Therefore the general solution will he

$$Y = c_1 + c_2(-\tfrac{1}{2})^x.$$

The arbitrary constants are determined by the initial conditions. We have

$$y(0) = 0 \text{ and } y(1) = 1$$

therefore

$$c_1 + c_2 = 0 \text{ and } c_1 - \tfrac{1}{2}c_2 = 1.$$

This gives

$$c_1 = -c_2 = \frac{2}{3}.$$

Finally the general term of the series will be $y = \dfrac{2}{3} - \dfrac{(-1)^x}{3.2^{x-1}}$.

Remark:

$$\lim_{x=\infty} y = \frac{2}{3}.$$

Secondly, if the function considered is one of a continuous variable and if r_1 is negative, then the solution $y = r_1^x$ is complex.

Let us write the general expression of a number r_1

$$(1) \qquad r_1 = a_1 + i\beta_1 = \varrho_1(\cos\varphi_1 + i\sin\varphi_1) = \varrho_1 e^{i\varphi_1}$$

where $i = \sqrt{-1}$; $\varrho_1 > 0$; $\varrho_1^2 = a_1^2 + \beta_1^2$ and $\tan\varphi_1 = \beta_1/a_1$.

From (1) it follows that

$$(2) \qquad r_1^x = \varrho_1^x(\cos x\varphi_1 + i \sin x\varphi_1) = \varrho_1^x e^{ix\varphi_1}$$

If r_1 is a negative number, then from (1) we deduce $\varphi_1 = \pi$ and $\varrho_1 = -r_1$. Therefore

$$r_1^x = \varrho_1^x(\cos\pi x + i \sin\pi x) = \varrho_1^x e^{i\pi x}.$$

The real part of the solution is

$$y = c\varrho_1^x \cos\pi x .$$

According to what we have seen, if r_1 is a double root of $\psi(r) = 0$ then we have the particular solution corresponding to r_1:

$$y = (c_1 + c_2 x)\varrho_1^x \cos\pi x .$$

If the coefficients in the difference equation are all real, then the real and the complex part of r_1^x must each satisfy separately the difference equation; so that the solution may be written

$$y = \varrho_1^x [c_1 \text{ cosnx} + c_2 \sin \pi x].$$

But since the difference equation is of the first order, the two particular solutions cannot be independent; and indeed the corresponding determinant is equal to zero. Moreover since x takes only the values $x = a + \xi$ (where ξ is an integer) hence we have

$$\sin \pi x = \sin \pi \text{ (a+\$)} = (-1)^{\xi} \sin \pi a, \qquad \cos \pi x = (-1)^{\xi} \cos \pi a$$

therefore

$$\text{sinnx} = \tan \pi a \text{ cosnx}.$$

Finally the above solution will be

$$y = \varrho_1^x [c_1 + c_2 \tan \pi a] \cos \pi x = c \varrho_1^x \text{ cosnx}.$$

Example. 2. Given the homogeneous linear difference equation

$$F(x+1) + F(x) = 0.$$

The corresponding characteristic equation $r+1=0$ gives $r_1 = -1$. Therefore the complete solution will be

$$F(x) = c_1 \text{ cosnx} .$$

§ 168. Complex roots. Given a homogeneous difference equation with real, constant coefficients $\psi(E)y=0$. If the root r_1 of the characteristic equation $\psi(r) = 0$ is a complex one,

$$r_1 = \varrho_1 (\cos\varphi_1 + i \sin\varphi_1)$$

then, since the coefficients of the difference equation are real, it follows that

$$r_2 = \varrho_1 (\cos\varphi_1 - i \sin\varphi_1)$$

will also be a root of the equation. Therefore if the characteristic equation has no multiple roots, the general solution may be written

$$y = k_1 r_1^x + k_2 r_2^x + \ldots = (k_1 + k_2)\varrho_1^x \cos x\varphi_1 +$$
$$+ i(k_1 - k_2)\varrho_1^x \sin x\varphi_1 + \ldots,$$

If we put

$$k_1 = \tfrac{1}{2}(c_1 - ic_2) \quad \text{and} \quad k_2 = \tfrac{1}{2}(c_1 + ic_2)$$

we have

$$y = c_1 \varrho_1^x \cos x\varphi_1 + c_2 \varrho_1^x \sin x\varphi_1 + \ldots .$$

Remark. If the complex root is a double one, the solution is of the following form:

$$y = [c_1 + c_2 x]\,\varrho_1^x \cos x\varphi_1 + [b_1 + b_2 x]\,\varrho_1^x \sin x\varphi_1 .$$

Example 1. Given the difference equation

$$f(x+2) + 2f(x+1) + 4f(x) = 0$$

the corresponding characteristic equation is

$$r^2 + 2r + 4 = 0$$

therefore

$$r_1 = -1 + i\sqrt{3}, \qquad r_2 = -1 - i\sqrt{3}$$

this gives, according to (1), § 167:

$$\varrho_1 = 2, \quad \tan\varphi_1 = -\sqrt{3} \quad \text{and} \quad \varphi_1 = 2\pi/3.$$

The complete solution will be

$$y = c_1 2^x \cos\frac{2\pi x}{3} + c_2 2^x \sin\frac{2\pi x}{3} .$$

Example 2. Given the difference equation

$$f(x+4) - f(x) = 0.$$

The corresponding characteristic equation is

$$r^4 - 1 = 0.$$

It is necessary therefore to determine the fourth roots of unity. Let us write the general expression of a number a in the following way

$$a = \varrho[\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi)].$$

Its n th roots are given by

$$(1) \qquad \sqrt[n]{a} = \sqrt[n]{\varrho}\left[\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right]$$

putting for k the values 0, 1, 2, . . . (n-i).

Therefore writing $a = 1$, $\varrho = 1$, $\varphi = 0$ and $n = 4$ we shall have

$$\sqrt[4]{1} = \cos\tfrac{1}{2}k\pi + i\sin\tfrac{1}{2}k\pi$$

for $k = 0$, 1, 2, 3. So that

$$r_1 = 1, \quad r_2 = -1, \quad r_3 = i, \quad r_4 = -i.$$

The general solution will be

$$y = c_1 + c_2\cos\pi x + c_3\cos\tfrac{1}{2}\pi x + c_4\sin\tfrac{1}{2}\pi x .$$

Example 3. Given the equation

$$f(x+4) + f(x) = 0.$$

The characteristic equation is

$$r^4 + 1 = 0.$$

Hence the fourth roots of -1 are needed. Putting $a = -1$, $\varrho = 1$, $\varphi = \pi$ and $n = 4$ into (1) we get

$$r_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \qquad r_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$

$$r_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \qquad r_4 = \cos\frac{7\pi}{4} + i\sin\frac{7x}{4}$$

Remarking that

$$\cos\frac{7\pi}{4} = \cos\frac{\pi}{4} \qquad \cos\frac{5\pi}{4} = \cos\frac{3\pi}{4}$$

$$\sin\frac{7\pi}{4} = -\sin\frac{\pi}{4} \qquad \sin\frac{5\pi}{4} = -\sin\frac{3\pi}{4}$$

the general solution will be

$$y = c_1\cos\frac{\pi x}{4} + c_2\sin\frac{\pi x}{4} + c_3\cos\frac{3\pi x}{4} + c_4\sin\frac{3\pi x}{4}.$$

Example 4. Given the difference equation

$$f(x+2) - 2\cos\varphi\, f(x+1) + f(x) = 0.$$

The characteristic equation is

$$r^2 - 2r\cos\varphi + 1 = 0$$

hence $r = \cos\varphi \pm i\sin\varphi$ and therefore

$$f(x) = c_1 \cos\varphi x + c_2 \sin\varphi x. \tag{2}$$

The constants must be determined by the aid of the initial conditions. Let for instance $f(0) = 0$ and $f(n) = 0$, then equation (2) will give

$$c_1 = 0 \quad \text{and} \quad c_2 \sin\varphi n = 0.$$

From this we conclude that if φ is not equal to $\pi\nu/n$, where $\nu = 1, 2, 3, \ldots, n-1$ then $c_2 = 0$ and the only solution is $f(x) = 0$.

On the other hand if $\varphi = \pi\nu/n$ then we shall have

$$f(x) = c_2 \sin \frac{\pi\nu x}{n}.$$

Since this equation still contains an arbitrary constant, therefore we may impose upon $f(x)$ a further condition.

Recapitulation. Given the homogeneous linear difference equation, with real, constant coefficients:

$$a_n f(x+n) + a_{n-1} f(x+n-1) + \ldots + a_1 f(x+1) + a_0 f(x) = 0$$

if

$$r_\nu = \varrho_\nu (\cos\varphi_\nu + i \sin\varphi_\nu)$$

is a real or complex root of the characteristic equation, of multiplicity m, then the solution of the difference equation is

$$y = \ldots + \left[c_1 + c_2\binom{x}{1} + \ldots + c_m\binom{x}{m-1}\right] \cdot$$

$$\cdot\, \varrho_\nu^x (\cos x\varphi_\nu + i \sin x\varphi_\nu) + \ldots$$

§ 169. Complete linear equation of differences with constant coefficients. Given the equation

$$a_n f(x+n) + \ldots + a_1 f(x+1) + a_0 f(x) = V(x) \tag{1}$$

or written symbolically

$$\psi(\mathrm{E}) f(x) = V(x).$$

Let us suppose that the function u is a particular solution of equation (1), so that we have

$$\psi(\mathrm{E}) u = V(x).$$

Subtracting this equation from the preceding we obtain

$$\psi(\mathrm{E})[f(x) - u] = 0.$$

Hence $y = f(x) - u$ is a solution of the homogeneous equation

$$\psi(\mathrm{E})y = 0.$$

From this we conclude that $f(x) = u + y$ is the solution of equation (1). Therefore first we have to determine the general solution of the homogeneous equation; then we must find one particular solution of the complete equation. This last may be attained in different ways; sometimes a particular solution may be obtained by simple reasoning. Often the symbolical methods lead to it easily enough, or even the direct method of determination.

Symbolical methods. They may be useful in some particular cases, A, First let $V(x) = Ca^x$. If the difference equation

$$(1) \qquad \psi(\mathrm{E})f(x) = Ca^x$$

is given, where a is not a root of the characteristic equation $\psi(r) = 0$ that is $\psi(a) \neq 0$; (the equation $\psi(r)$ may have *multiple root8* or not).

Dividing formula (1) by p(E) and using formula (3) of § 165 we have

$$(2) \qquad f(x) = \frac{1}{\psi(\mathrm{E})} Ca^x = \frac{Ca^x}{\psi(a)}$$

which is a particular solution of equation (1). Denoting again by y the general solution of the homogeneous equation corresponding to (1), the solution of the complete equation will be

$$(3) \qquad f(x) = \frac{Ca^x}{\psi(a)} + y.$$

Particular case. $a = 1$, $V(x) = C$. If $r = 1$ is not a root of $\psi(r) = 0$ then according to (3) the general solution will be

$$(4) \qquad f(x) = \frac{C}{\psi(1)} + y.$$

Example 1. Given the equation

$$f(x+2) - 5f(x+1) + 6f(x) = 2.$$

The corresponding characteristic equation is

$$y(r) = r^2 - 5r + 6 = 0$$

so that $r_1 = 2$ and $r_2 = 3$; since moreover $a = 1$ and $\psi(1) = 2$, the general solution of the complete equation will be given by (4)

$$f(x) = 1 + c_1 2^x + c_2 3^x .$$

Example 2. Given the difference equation

$$f(x+3) - 7f(x+2) + 16f(x+1) - 12f(x) = Ca^x.$$

The corresponding characteristic equation is

$$\psi(r) = r^3 - 7r^2 + 16r - 12 = 0.$$

Hence $r_1 = 2$, $r_2 = 2$ and $r_3 = 3$.

Therefore if a is different from 2 and from 3, then the general solution of the complete equation is given by (3)

$$f(x) = \frac{Ca^x}{a^3 - 7a^2 + 16a - 12} + (c_1 + c_2 x) 2^x + c_3 3^x .$$

Secondly, given the difference equation

$$\psi(E) f(x) = Ca^x$$

where a is a simple root of the equation $\psi(r) = 0$. To solve the difference equation we put

$$\psi(E) f(x) = C(a+\varepsilon)^x$$

then the particular solution will be as we have seen:

$$u = \frac{C(a+\varepsilon)^x}{\psi(a+\varepsilon)} .$$

The general solution of the homogeneous equation is

$$y = k_1 a^x + c_2 r_2^x + c_3 r_3^x + \dots$$

If we put

$$k_1 = c \quad , \quad \frac{C}{\psi(a+\varepsilon)}$$

then the general solution of the complete equation may be written

$$f(x) = C\,\frac{(a+\varepsilon)^x - a^x}{\psi(a+\varepsilon)} + c_1 a^x + c_2 r_2^x + c_3 r_3^x + \dots .$$

$\psi(a)=0$; hence we have

$$\lim_{\varepsilon=0} \frac{(a+\varepsilon)^x - a^x}{\psi(a+\varepsilon)} = \frac{xa^{x-1}}{D\psi(a)};$$

moreover a being a simple root, therefore $D\psi(a) \neq 0$ **and the** required solution will be:

$$f(x) = \frac{Cxa^{x-1}}{D\psi(a)} + c_1 a^x + c_2 r_2{}^x + \ldots \tag{5}$$

Particular case. Let $a=1$, $\psi(1)=0$ and $D\psi(1) \neq 0$. The solution of

$$\psi(E)f(x) = C$$

will be, according to (5),

$$f(x) = \frac{c\,x}{D\psi(1)} + c_1 + c_2 r_2{}^x + \ldots \tag{6}$$

Example 3. Given the equation of differences

$$f(x+2) - 4f(x+1) + 3f(x) = 3^x.$$

The characteristic equation is

$$r^2 - 4r + 3 = 0 \quad \text{hence} \quad r_1 = 1 \quad \text{and} \quad r_2 = 3.$$

Since 3 is a simple root of this equation, the solution is given by (5)

$$f(x) = \tfrac{1}{2}x3^{x-1} + c_1 + c_2 3^x.$$

Example 4. Given the equation of differences

$$f(x+2) - 4f(x+1) + 3f(x) = 2.$$

The characteristic equation is

$$r^2 - 4r + 3 = 0 \quad \text{hence} \quad r_2 = 1,\ r_2 = 3;$$

moreover we have $D\psi(r) = 2r-4$. The solution will be, according to (6):

$$f(x) = -x + c_1 + c_2 3^x.$$

B. Let $V(x) = a^x\varphi(x)$ where $\varphi(x)$ is a polynomial of degree n. Before solving the equation of differences

$$\psi(E)f(x) = a^x\varphi(x) \tag{7}$$

we will establish the following auxiliary theorem. It is easy to see that

$$\mathbf{E}^m[a^x\varphi(x)] = [\mathbf{E}^m a^x][\mathbf{E}^m\varphi(x)] = a^x[a\mathbf{E}]^m\varphi(x).$$

This holds for every positive or negative integer value of ***m.*** Therefore if y(E) is a polynomial of $\mathbf{E}$ and $\varphi(x)$ is any function whatever, we have

$$(8) \qquad \psi(\mathbf{E})[a^x\varphi(x)] = a^x\psi(a\mathbf{E})\varphi(x).$$

Let us show now that, if $\psi(\mathbf{E})$ is a rational fraction, $\psi(\mathbf{E}) = U(\mathbf{E})/V(\mathbf{E})$ where U(E) and V(E) are polynomials, we have

$$(9) \qquad \left[\frac{U(\mathbf{E})}{V(\mathbf{E})}\right][a^x\varphi(x)] = a^x\left[\frac{U(a\mathbf{E})}{V(a\mathbf{E})}\right]\varphi(x).$$

lndeed, multiplying both members by V(E) we find:

$$U(\mathbf{E})\,[a^x\varphi(x)] = V(\mathbf{E})\,[a^x\,\frac{U(a\mathbf{E})}{V(a\mathbf{E})}\,\varphi(x)]$$

but according to (8) the second member will be equal to

$$a^xV(a\mathbf{E})\,\frac{U(a\mathbf{E})}{V(a\mathbf{E})}\,\varphi(x)$$

that is, it will be identical with the first member. Therefore formula (9) is demonstrated; so that if F(E) is a rational fraction we have

$$(10) \qquad F(\mathbf{E})a^x\varphi(x) = a^xF(a\mathbf{E})\varphi(x).$$

Let us execute the operation $1/\psi(a\mathbf{E})$ performed on a polynomial $\varphi(x)$ of degree s; if $\psi(a\mathbf{E})$ is a polynomial of E of degree n; and if a is not a root of the characteristic equation $\psi(r) = 0$. (The roots of this equation may be multiple or not.) From the equation of differences *(7)* it follows by aid of (10) that

$$(11) \qquad f(x) = a^x\,\frac{1}{\psi(a\mathbf{E})}\,\varphi(x).$$

Eliminating E from $\psi(a\mathbf{E})$ by aid of $\mathbf{E} = \Delta + 1$ we get a polynomial @(A) of degree n. Let us suppose first that the coefficient of Δ^0 in @(A) is different from zero. Then $1/\Phi(\Delta)$ may be expanded into a power series of A:

$$c_0 + c_1\Delta + c_2\Delta^2 + \ldots + c_s\Delta^s$$

$\varphi(x)$ being of degree s the series may be stopped at the term $c_s\Delta^s$, since $\Delta^{s+i}\varphi(x) = 0$ (if $i > 0$).

Finally we have

$$(12) \qquad \frac{1}{\psi(aE)}\,\varphi(x) = [c_0 + c_1\Delta + \ldots, + c_s\Delta^s]\,\varphi(x)$$

The coefficients c_i are easily determined. For instance, ***Taylor's*** series would give

$$(13) \qquad \frac{1}{\psi(a+a\Delta)}\,\varphi(x) = \sum_{i=0}^{s+1} \frac{a_i}{i!}\left[D^i \,\frac{1}{\psi(a+a\Delta)}\right]_{\Delta=0} \Delta^i\varphi(x)$$

and the required particular solution will be given by (11).

Since the higher derivatives of $1/\psi$ are necessarily complicated, it is better, if $n \geqq 2$, to determine the coefficients in another way. If we have

$$\psi(a+a\Delta) = b_0 + b_1\Delta + , \ldots + b_s\Delta^n$$

then the coefficients c_i of the expansion (12) are given by the following equations:

$$b_0c_0 = 1$$
$$b_1c_0 + b_0c_1 = 0$$
$$b_2c_0 + b_1c_1 + b_0c_2 = 0$$
$$\cdots\cdots$$
$$b_ic_0 + b_{i-1}c_1 + \ldots + b_0c_i = 0.$$

On the other hand if for instance $b_0 = b_1 = b_2 = 0$, then we may write

$$\frac{1}{\Delta^3[b_3 + b_4\Delta + \ldots + b_n\Delta^{n-3}]} = \frac{1}{\Delta^3}[c_0 + \text{c, A} + \ldots + c_{s+3}\Delta^{s+3}]$$

and the coefficients c_i are determined in a similar way, as before.

Example 5. Given the equation

$$f(x+2) - 4f(x+1) + 3f(x) = x4^x.$$

The roots of the characteristic equation $r^2 - 4r + 3 = 0$ are $r_1 = 1$

and $r_2=3$. Moreover $D\psi(r) = 2r - 4$ and $D\frac{1}{\psi} = -\frac{D\psi}{\psi^2}$ therefore $\frac{1}{\psi(a)} = \frac{1}{3}$ and $-\frac{aD\psi(a)}{\psi^2(a)} = -\frac{16}{9}$ so that the particular solution according to (11) and (13) will be

$$u = 4^x.\frac{3x-16}{9}.$$

Using the second method, we have

$$\psi(a+a\Delta) = a^2(1+\Delta)^2 - 4a(1+\Delta) + 3 = 3 + 16\Delta + 16\Delta^2$$

therefore the equations will be

$$3c_0 = 1 \qquad 3c_1 + 16c_0 = 0$$

so that

$$u = 4^x\left(\frac{1}{3} - \frac{16}{9}\Delta\right) x = 4^x\frac{3x-16}{9}$$

Example 6. Given the equation

$$f(x+2) - 4f(x+1) + Y(x) = x.$$

The roots of $\psi(r) = r^2-4r+4=0$ are $r_1=r_2=2$. Moreover $a=1$ and $\varphi(x) = x$. Therefore

$$y(a) = a^2 - 4a + 4 = 1$$
$$D\psi(a) = 2a - 4 = -2$$

from (13) it follows that

$$u = (1+2\Delta)x = x+2.$$

The second method would give

$$\psi(a+a\Delta) = a^2(1+\Delta)^2 - 4a(1+\Delta) + 4 = 1 - 2\Delta + \Delta^2.$$

Therefore

$$c_0 = 1 \quad \text{and} \quad c_1 - 2c_0 = 0$$

and

$$u = (1 + 2\Delta)x = x+2.$$

Example 7. Given

$$f(x+2) + f(x) = xa^x .$$

We have $\psi(r) = r^2+1 = 0$; the roots of the equation are complex. The particular solution will be, according to (11) :

$$u = a^x \left[\frac{x}{a^2+1} - \frac{2a^2}{(a^2+1)^2} \right].$$

Particular case of the difference equation of the first order. In the second case (B) the difference equation of the first order may he written

$$(14) \qquad f(x+1) - r_1 f(x) = a^x \varphi(x)$$

where $\varphi(x)$ is a polynomial of degree s.

Let us suppose that $a \neq r_1$, then we shall have

$$\mathbf{D}^m \frac{1}{r-r_1} = \frac{(-1)^m\, m!}{(r-r_1)^{m+1}}.$$

Therefore the particular solution given by (13) will be

$$(15) \qquad u = a^x \sum_{m=0}^{s+1} (-1)^m \frac{a^m \Delta^m \varphi(x)}{(a-r_1)^{m+1}}.$$

Example 8. Given the equation

$$f(x+1) - 2f(x) = \binom{x}{3}.$$

Hence $r_1=2$, $a=1$, $\psi(1)=-1$ and $\mathbf{D}\psi(1)=1$, therefore

$$u = -\binom{x}{3} - \binom{x}{2} - \binom{x}{1} - 1.$$

Formulae could be deduced for cases in which a is a single or a multiple root of $\psi(r)=0$; but it is simpler then to use the method given in the next paragraph.

Summing up we conclude that this method is independent of the order n of the difference equation, and therefore especially useful for equations of high order. It does not matter if the roots of $\psi(r)=0$ are single. or multiple, real or complex; to obtain the particular solution it is not even necessary to determine the roots of the characteristic equation; but the formulae are different if a is a root of $\psi(r)=0$. Since the number of terms in the solution given by (11) increases with the degree s of the polynomial $\varphi(x)$, the method is not advantageous if it is of a high degree.

§ 170. Determination of the particular solution in the general case. Given the difference equation

$$\psi(\mathbf{E})f(x) = V(x).$$

The particular solution obtained by the symbolical method is

$$u = \frac{1}{\psi(\mathrm{E})} \mathrm{V}(x).$$

To determine the second member we shall decompose l/y(E) into partial fractions. Let us denote the roots of the characteristic equation y(r) =0 by $r_1, r_2, \ldots, r_n$; first we suppose that these roots are all real and single.

If we write

$$\frac{1}{\psi(\mathrm{E})} = \frac{b_1}{\mathrm{E}-r_1} + \frac{b_2}{\mathrm{E}-r_2} + \ldots + \frac{b_n}{\mathrm{E}-r_n}$$

then according to § 13 we have

(2) $$b_m = 1/[\mathrm{D}\psi(r)]_{r=r_m}.$$

Let us apply now to each term $\frac{b_m}{\mathrm{E}-r_m}$ V(x) the following transformation. It is easy to see that

$$(\mathrm{E}-a)[a^{x-1}F(x)] = a^x \Delta F(x).$$

Putting the second member equal to *V(x),* it gives

$$\Delta^{-1}[a^{-x}V(x)] = F(x)$$

moreover from the preceding equation we get

$$\frac{1}{\mathrm{E}-a} V(x) = a^{x-1}F(x).$$

Eliminating F(x) between the last two equations we obtain the important formula

(3) $$\frac{1}{\mathrm{E}-a} V(x) = a^{x-1}\Delta^{-1}[a^{-x}V(x)]$$

this gives the required particular solution in the form of indefinite sums:[53]

(4) $$u = b_1 r_1^{x-1} \Delta^{-1}[r_1^{-x}V(x)] + \ldots + b_n r_n^{x-1} \Delta^{-1}[r_n^{-x}V(x)].$$

The formula supposes that $r=0$ is not a root of $\psi(r)=0$ but this may always be attained, For instance, starting from

[53] This formula is identical with that given by G. *Wahlenberg* und *A. Guldberg,* Theorie der linearen Differenzengleichungen, Berlin 1911, p. 175.

$$f(x+3) - 2f(x+2) + f(x+1) = V(x)$$

we get by putting x-l instead of x

$$f(x+2) - 2f(x+1) + f(x) = V(x-1).$$

If $\psi(r)=0$ has multiple roots, for instance if m is the multiplicity of the root r_ν, then the decomposition of $1/\psi(\mathrm{E})$ into partial fractions will contains terms of the form:

$$\frac{b_{\nu\mu}}{(\mathrm{E}-r_\nu)^\mu}, \quad \text{for } \mu = 1, 2, 3, \ldots, m.$$

Denoting

$$A(r) = \frac{(r-r_\nu)^m}{\psi(r)}$$

then the numbers $b_{\nu\mu}$ are given according to (4) § 13 by

$$b_{\nu\mu} = \left[\frac{\mathrm{D}^{m-\mu} A(r)}{(m-\mu)!}\right]_{r=r_\nu}.$$

Now we have to perform the operations

$$\frac{1}{(\mathrm{E}-r_\nu)^\mu} V(x).$$

For this let us remark that, according to (3), the operation $1/(\mathrm{E}-a)$ performed on

$$\frac{1}{\mathrm{E}-a} V(x)$$

will give

$$\frac{1}{(\mathrm{E}-a)^2} V(x) = a^{x-1} \Delta^{-1} \{a^{-1} \text{A-'} [a^{-x} V(x)]\} = \text{a"-" A-' [a-" v(x)]}.$$

Continuing in this manner we get

$$\frac{1}{(\mathrm{E}-r_\nu)^\mu} V(x) = r_\nu^{x-\mu} \Delta^{-\mu} [r_\nu^{-x} V(x)].$$

Finally the corresponding particular solution will be

$$(5) \qquad u = \sum_{\mu=1}^{m+1} b_{\nu\mu} r_\nu^{x-\mu} \Delta^{-\mu} [r_\nu^{-x} V(x)] + \ldots,$$

This method may also be applied in the particular cases of

§ 169. Sometimes it is even more advantageous than the **previous** methods. For instance, given the equation

$$\psi(E)f(x) = a^x\varphi(x)$$

where $\varphi(x)$ is a polynomial, then in this method it does not matter if a is a root of $\psi(r)=0$ or not. The number of terms in formula (5) is equal to the order of the difference equation, therefore the method is especially useful for equations of low order. For instance, in the case of an equation of the first order we have only one term. The degree of $\varphi(x)$ does not increase the number of terms.

If the roots of $\psi(r)=0$ are multiple or complex, if the equation is of a high order, then the method is more complicated than that of § 169. The following example will serve as a comparison:

Example 1. Given the equation of differences

$$f(x+3) - 7f(x+2) + 16f(x+1) - 12f(x) = Ca^x .$$

We have y(r) $= r^3 - 7r^2 + 16r-12 = 0$, hence $r_1=3$, $r_2=r_3=2$. Moreover $D\psi(r) = 3r^2-14r + 16$ and $A(r) = 1/(r-3)$.

By aid of the preceding formulae we find

$$b_1 = 1,\ b_{21} = -1 \text{ and } b_{22} = -1$$

therefore the required particular solution will be

$$u = 3''' \Delta^{-1}[3^{-x} Ca^x] - 2^{x-1}\Delta^{-1}[2^{-x} Ca^x] - 2^{x-2}\Delta^{-2}[2^{-x} Ca^x]$$

determining the indefinite sums in the second member we obtain

$$u = Ca^x \,/\, (u\text{-}2)^2 (a\text{-}3).$$

This result would have been given directly by formula (3) of § 169.

Exampte 2. Given the equation

$$f(x+1) - 2f(x) = 2^x\binom{x}{3}.$$

Since we have y(r) $=r-2$ and $r_1=2$, hence the particular solution is given by (4):

$$u = 2^{x-1}\Delta^{-1}\left[2^{-x}\,2^x\binom{x}{3}\right] = 2^{x-1}\binom{x}{4}.$$

The method of § 169 would have been much more complicated. Indeed, since $a=2$ is a root of $\psi(r)=0$, hence after executing the derivations in the second member of formula (9), putting $a+\varepsilon$ instead of a and determining the limits for $\varepsilon=0$ we should have had

$$u = x\,2^{x-1}\binom{x}{3}-\binom{x+1}{2}2^{x-1}\binom{x}{2}+\binom{x+2}{3}2^{x-1}\binom{x}{1}-\binom{x+3}{4}2^{x-1}$$

It can be shown that this is equal to $2^{x-1}\binom{x}{4}$.

Negatiue roofs. If the root r of $\psi(r)=0$ is negative, then we may put

$$r_\nu^{\,x} = \varrho_\nu^{\,x} \cos \pi x$$

and formula (4) will give

$$u = \ldots . + b_\nu \varrho_\nu^{\,x-1} \cos(x-1)\pi\, \Delta^{-1} [\varrho_\nu^{\,-x} \cos \pi x\, V(x)] + . , . .$$

Complex roofs. If the root r_1 of the characteristics equation is complex

$$r_1 = \varrho(\cos\varphi + i\,\sin\varphi)$$

then

$$r_2 = \varrho(\cos\varphi - i\,\sin\varphi)$$

is also a root of this equation, since the coefficients of the given equation are supposed to be real. The decomposition of $1/\psi(\mathsf{E})$ into partial fractions will give

$$\frac{b_1}{\mathsf{E}-r_1}+\frac{b_2}{\mathsf{E}-r_2}+\ldots . \qquad \text{where} \qquad b_\nu = \frac{1}{\mathsf{D}\psi(r_\nu)}$$

Putting $\mathsf{D}\psi(r_1)=M+iN$ and therefore $\mathsf{D}\psi(r_2)=M-iN$, *we* have

$$b_1 = \frac{M - iN}{M^2+N^2}, \qquad b_2 = \frac{M+iN}{M^2+N^2}$$

consequently formula (4) will give

$$u = b_1\, \varrho^{x-1}[\cos(x-1)\varphi+i\sin(x-1)\varphi]\cdot$$
$$\cdot\, \Delta^{-1}\{\varrho^{-x}[\cos x\varphi - i\,\sin x\varphi]\, V(x)\} +$$
$$+\, b_2\, \varrho^{x-1}[\cos(x-1)\varphi-i\sin(x-1)\varphi]\Delta^{-1}\{\varrho^{-x}[\cos x\varphi+i\,\sin x\varphi]\, V(x)\}$$

and after simplification

$$u = \frac{2M}{M^2+N^2} \varrho^{x-1} \cos (x-1) \varphi\Delta^{-1} \{\varrho^{-x} \cos x\varphi\, V(x) \} +$$

$$+ \frac{2N}{M^2+N^2} \varrho^{x-1} \sin(x-1)\varphi\Delta^{-1} \{\varrho^{-x} \sin x\varphi\, V(x) \}$$

§ 171. Method of the arbitrary constants. Given the equation of differences

$$\text{(1)} \qquad \psi(\mathsf{E})\, f(x) = V(x).$$

We will suppose that the coefficient a_n of $f(x+n)$ in this equation is equal to one, and that V(x) is of the following form

$$\text{(2)} \qquad V(x) = a^x \left[\alpha_0 + \alpha_1 \binom{x}{1} + \alpha_2 \binom{x}{2} + \ldots + \alpha_m \binom{x}{m} \right].$$

We shall consider a as a root of multiplicity k of the characteristic equation $y(r) = 0$. If a is not a root of this equation then $k=0$.

According to what we have seen, the particular solution of this equation may be obtained also by the method of § 169, or by that of § 170.

Now we will try to write a particular solution in the following way

$$\text{(3)} \qquad u = a^x \left[\beta_0 \binom{x}{k} + \beta_1 \binom{x}{k+1} + \ldots + \beta_m \binom{x}{k+m} \right]$$

and dispose of the coefficients β_ν in such a manner as to make this expression satisfy the difference equation (1). Putting there $f(x) = u$ we shall have

$$\sum_{\nu=0}^{m+1} \beta_\nu \psi(\mathsf{E}) \left\{ a^x \binom{x}{k+\nu} \right\} = V(x).$$

In consequence of formula (8) § 169 we get

$$\sum_{\nu=0}^{m+1} \beta_\nu\, a^x \psi(a\mathsf{E}) \binom{x}{k+\nu} = V(x)$$

on the other hand we have

$$\psi(a\mathsf{E}) = \psi(a+a\Delta) = \sum_{\mu=k}^{n+1} \mathsf{D}^\mu \psi(a) \frac{(a\Delta)^\mu}{\mu!}.$$

Indeed, y(r) being of degree n, hence $D^{n+1}\psi(a) = 0$; moreover a is a root of multiplicity k of $\psi(r)=0$, therefore $D^{\mu}\psi(a)=0$ if $\mu < k$.

Finally we shall have

$$a^x \sum_{\nu=0}^{m+1} \beta_\nu \sum_{\mu=k}^{k+\nu+1} \frac{a^\mu D^\mu\psi(a)}{\mu!} \binom{x}{k+\nu-\mu} = V(x)$$

writing $k+\nu-\mu=i$ we find

$$a^x \sum_{i=0}^{m+1} \binom{x}{i} \sum_{\nu=1}^{m+1} \beta_\nu \frac{a^{k+\nu-i} D^{k+\nu-i}\psi(a)}{(k+\nu-i)!} = V(x).$$

NOW we shall dispose of the coefficients β_ν so as to make the first member identical with V(x) given by (2); we obtain

$$(4) \quad \alpha_i = \sum_{\nu=i}^{m+1} \beta_\nu \frac{a^{k+\nu-i} D^{k+\nu-i} \psi(a)}{(k+\nu-i)!} = \sum_{\mu=0}^{\infty} \beta_{\mu+1-k} \frac{a^\mu}{\mu!} D^\mu\psi(a)$$

for $i=0, 1, 2, \ldots, m$.

If $D^k\psi(a)$ and a are different from *zero* then the last of these equations will give β_m, the last but one β_{m-1}, and so on; since a is a root of the equation $\psi(x)=0$ of multiplicity k, therefore the first condition is satisfied. The second is satisfied too; indeed, if we had $a=0$ then equation (1) would be homogeneous.

The β_ν being determined, the problem is solved. This method is applicable whatever the roots of $\psi(r)=0$ may be, and it is not even necessary to determine these.

***Particular** case of the equation of the first order.*

$$(5) \quad f(x+1) - r_1 f(x) = a^x\left[\alpha_0 + \alpha_1\binom{x}{i} + \ldots + \alpha_m\binom{x}{m}\right].$$

From (4) we deduce

$$(6) \quad \alpha_i = \beta_{i-k}\psi(a) + \beta_{i+1-k}\, a D\psi(a).$$

Here we have y(r) $= r - r_1$, $D\psi(a)=1$ and $D^2\psi(a)=0$. Let us suppose first that $a \neq r_1$ and therefore $k=0$, *we* find

$$\alpha_i = \beta_i\psi(a) + \beta_{i+1} a$$

for $i = 0, 1, 2, \ldots$ (m-l), moreover since $\beta_{m+1} = 0$, therefore

$$a_m = \beta_m \psi(a).$$

Example 1. Given

$$f(x+1) - 2f(x) = \binom{x}{3}.$$

We have $\psi(r) = r-2$, $D\psi(r) = 1$ and $a = 1$; therefore $k=0$ and formulae (6) are applicable.

$$\begin{aligned} a_3 &= 1 = -\beta_3 \\ a_2 &= 0 = -\beta_2 + \beta_3 \\ a_1 &= 0 = -\beta_1 + \beta_2 \\ a_0 &= 0 = -\beta_0 + \beta_1 \end{aligned}$$

therefore $\beta_0 = \beta_1 = \beta_2 = \beta_3 = -1$; and according to (3)

$$u = -1 - \binom{x}{1} - \binom{x}{2} - \binom{x}{3}.$$

Secondly let $a = r_1$ and therefore $k = 1$; since $\psi(a) = 0$ formula (6) will give,

(7) $$a_i = \beta_i a D\psi\,(a) = a\beta_i.$$

The particular solution given by (3) is

(8) $$u = a^{x-1} \sum_{i=0}^{m+1} a_i \binom{x}{i+1}.$$

Example 2. Given

$$f(x+1) - 2f(x) = 2^x \binom{x}{2}.$$

Since $\psi(r) = r-2$, $a=2$ and moreover $a_0=0$, $a_1=0$ and $a_2=1$ therefore the particular solution is given by (8)

$$u = 2^{x-1}{}_0\binom{x}{3}.$$

Example 3. Given

$$f(x+2) - 3f(x+1) + 2f(x) = \binom{x}{2}.$$

Therefore

$y(r) = r^2 - 3r + 2$; $D\psi\,(t) = 2r-3$ and $D^2\psi\,(r) = 2$.

Moreover $r_1=1$, $r_2=2$. Since $a=1$, it follows that $k=1$. **Putting** into (4) $m=2$, $a_0=0$, $a_1=0$, $a_2=1$ *we* find

$$\alpha_0 = o = \beta_0 a\mathbf{D}\psi(a) + \beta_1 \tfrac{1}{2} a^2 \mathbf{D}^2\psi(a) = -\beta_0 + \beta_1$$
$$\alpha_1 = 0 = \beta_1 a\mathbf{D}\psi(a) + \beta_2 \tfrac{1}{2} a^2 \mathbf{D}^2\psi(a) = -\beta_1 + \beta_2$$
$$a, = 1 = \beta_2 a\mathbf{D}\psi(a) = -\beta_2$$

so that

$$\beta_0 = \beta_1 = \beta_2 = -1$$

and according to formula (3):

$$u = -\binom{x}{1} - \binom{x}{2} - \binom{x}{3}.$$

Example 4. Given

$$f(x+3) - 2f(x+2) + f(x+1) - 2f(x) = x3”.$$

Hence

$$y(r) = r^3 - 2r^2 + r - 2 \text{ and } \mathbf{D}\psi(r) = 3r^2 - 4r + 1;\ a = 3.$$

Since y(3) $\neq$ 0 therefore $k=0$. Moreover putting into (4) $m=1$, $\alpha_0=0$ and $\alpha_1=1$ we find

$$\alpha_0 = 0 = \beta_0\psi(a) + \beta_1 a\mathbf{D}\psi(a)$$
$$a, = 1 = \beta_1\psi(a).$$

Since $\psi(a)=10$ and $\mathbf{D}\psi(a)=16$ therefore

$$10\beta_0 + 48\beta_1 = 0$$
$$10\beta_1 = 1$$

so that $\beta_1=1/10$ and $\beta_0=-12/25$. From (3) we obtain the required particular solution

$$u = 3^x\left[\frac{1}{10} - \frac{12x}{25}\right].$$

§ 172. Resolution of linear equations of differences by aid of generating functions. $u=u(t)$ is the generating function of *f(x)* if in the expansion of $u(t)$ into a series of powers of t, ths coefficient of t^x is equal to *f(x);* that is if

$$(1) \qquad u = f(0) + f(1)t + f(2)t^2 + \ldots + f(x)t^x + \ldots$$

The generating function of *f(x)* was denoted in § 10 by $\mathbf{G}f(x)$. Starting from this generating function we deduced in § 11 the generating function of *f (x+m) :*

(2) $$G f(x+m) = \frac{1}{t^m}[u(t) - f(0) - f(1)t - \dots - t^{m-1}f(m-1)].$$

Therefore if the following difference equation is given

(3) $$a_n f(x+n) + a_{n-1} f(x+n-1) + \dots + a_1 f(x+1) + a_0 = V(x)$$

then, denoting the generating function of $V(x)$ by $R(t)$ we obtained the following relation between the corresponding generating functions: .

$$\sum_{m=0}^{n+1} \frac{a_m}{t^m} [u(t) - f(0) - tf(1) - \dots - t^{m-1}f(m-1)] = R(t).$$

From this we deduced

(4) $$u(f) = \{t^n R(t) + \sum_{m=1}^{n+1} a_m t^{n-m} [f(0) + f(1)t + \dots + f(m-1)\, t^{m-1}]\} \Big/ \sum_{m=0}^{n+1} a_m t^{n-m}.$$

Remark 1. The characteristic equation of the difference equation (3) is the following

$$a_n r^n + \dots + a_1 r + a_0 = 0$$

hence the denominator of $u(f)$ is obtained from this equation by multiplying it by t^n and putting into it $r=1/t$.

Remark 2. Knowing $u(f)$ the generating function of $f(x)$, it is easy to deduce $w(t)$ the generating function of the indefinite sum of $f(x)$, that is of $\Delta^{-1}f(x) = F(x)$. Indeed we have

$$F(x+1) - F(x) = f(x)$$

we get

$$\frac{w(t) - F(0)}{t} - w(f) = u(f)$$

that is

$$w(t) = \frac{tu(t) + F(0)}{1 - f}.$$

Having determined $u(t)$ by aid of (4), the expansion of this function into a series of powers of t will give the required function,

In the general solution of a difference equation of order n we must have n arbitrary constants; in fact the generating function (4) contains the n constants f(O), $f(1), \ldots, , , f(n-1)$. Hence it will lead to the general solution. Finally the constants will be disposed of so that the initial conditions shall be satisfied.

Formula (4) gives the generating function of the difference equation of the *first order* if we put into it $n = 1$:

$$(5) \qquad u = \frac{a_1 f(0) + tR(t)}{a_1 + a_0 t}.$$

This expression expanded into a series of powers of t will become

$$u = [f(0) + \frac{1}{a_1} \sum_{\mu=0}^{\infty} V(\mu) t^{\mu+1}] \sum_{\nu=0}^{\infty} \left(-\frac{a_0}{a_1}\right)^{\nu} t^{\nu}$$

therefore, putting $-a_0/a_1 = r_1$, we deduce

$$(6) \qquad f(x) = r_1^x f(0) + \frac{1}{a_1} \sum_{\mu=0}^{x} V(\mu) r_1^{x-1-\mu}.$$

This formula is identical with that corresponding to (4) § 170.

Example 1. Given the equation

$$f(x+1) - 2f(x) = xa^x.$$

In consequence of (6) we have

$$f(x) = 2^x f(0) + 2^{x-1} \sum_{\mu=0}^{x} \mu(\tfrac{1}{2}a)^{\mu}.$$

According to formula (2) § 34 the second member is equal to

$$f(x) = 2^x f(0) + \frac{a^{x+1}(x-1) - 2xa^x + a2^x}{(a-2)^2}.$$

From formula (4) we may obtain the generating function of the difference equation of the *second order.* Putting into it $n=2$ we find

$$(7) \qquad u = \frac{a_2 f(0) + [a_1 f(0) + a_2 f(1)]t + t^2 R(t)}{a_2 + a_1 t + a_0 t^2}.$$

Example 2. Given the equation

$$f(x+2) - 3f(x+1) + 2f(x) = 0.$$

Since $r_1 = 1$ and $r_2 = 2$, hence the denominator of u is equal to (l-f) (1-2t) and the expansion of u will give

$$u = \{f(0) + [f(1)-3f(0)]t\} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} 2^\nu t^{\nu+\mu}.$$

Therefore

$$f(x) = f(0) \sum_{\nu=0}^{x+1} 2^\nu + [f(1)-3f(0)] \sum_{\nu=0}^{x} 2^\nu$$

and

$$f(x) = 2f(0) - f(1) + [f(1)-f(0)]\, 2^x.$$

Example 3. Calculus of Probability. A player possessing x shillings plays a game repeatedly; he stakes each time one shilling, if he wins he gets two. The probability of winning is equal to p. It is required to find the probability $f(x)$ that the player will lose his money before winning $m-x$ shillings. This problem is a little more general than that of Example 2, § 166. Applying the theorem of total probability as in this example, we shall get

$$f(x) = pf(x+1) + qf(x-1)$$

where $q=1-p$. Hence we have to solve the difference equation

$$pf(x+2) - f(x+1) + qf(x) = 0.$$

Applying formula (7) we get

$$u = \frac{pf(0) + [pf(1) - f(0)]t}{p - t + qt^2}.$$

Since the roots of the characteristic equation are $r_1=1$ and $r_2=q/p$, hence the denominator will be $p(1-t)\,(1-\frac{q}{p}t)$ and the expansion of u will give

$$u = \left\{f(0) + [f(1) - \frac{1}{p} f(0)]t\right\} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \left(\frac{q}{p}\right)^\nu t^{\nu+\mu}$$

so that

$$f(x) = f(0) \sum_{\nu=0}^{x+1} \left(\frac{q}{p}\right)^\nu + [f(1) - \frac{1}{p} f(0)] \sum_{\nu=0}^{x} \left(\frac{q}{p}\right)^\nu$$

performing the summations we find

$$(8) \qquad f(x) = \frac{qf(0) - pf(1)}{q-p} + \frac{p[f(1)-f(0)]}{q-p} \left(\frac{q}{p}\right)^x.$$

To determine the arbitrary constants let us remark that if $x=0$ the player has lost so that $f(0)=1$; and if $x=m$ he cannot lose before winning *m-x,* as he has already done so; therefore $f(m)=0$.

Putting into (8): $f(0)=1$, $x=m$ and $f(m)=0$ we find

$$f(1)=\frac{q(p^{m-1}-q^{m-1})}{p^m-q^m}$$

and finally

$$f(x)=\left[\, q^m-p^m\left(\frac{q}{p}\right)^x\right]\Big/\,(q^m-p^m).$$

Remark. In the method considered above we supposed that x is a positive integer, but putting the obtained result, in order of verification, into the given equation of differences, it is easy to see that the result holds for any value whatever of x, the operation E being always the same.

Since the method does not presuppose the resolution of the characteristic equation, it may be useful if this equation cannot be solved and if we are able to expand the generating function without the knowledge of the roots of the characteristic equation.

Generally this method leads to the same formulae as the methods applied previously, so that its real advantage will show only when applied to functions of several independent variables, when the other methods fail.

§ 173. Homogeneous linear equations of differences of the first order with variable coefficients. We may write these equations in the following way

$$(1)\qquad f(x+1)-p(x)f(x)=0.$$

Let us suppose that x takes only integer values such as $x\geqq a$. Then we shall introduce a function y(x) so as to have $y(a)=1$ and for $x>a$:

$$y(x)=p(a)\,p(a+1)\,p(a+2)\ldots p(x-1).$$

Now dividing both members of equation **(1)** by $y(x+1)$ and putting

$$u(x)=f(x)\,/\,y(x)$$

we obtain

$$u(x+1)-u(x)=0\qquad\text{or}\qquad \Delta u(x)=0$$

therefore

$$u(x) = C \quad \text{and} \quad f(x) = Cy(x)$$

moreover $f(a)=C$; hence from $x > a$ it follows that

$$(2) \qquad f(x) = f(a) \prod_{i=a}^{x} p(i).$$

Example 1. The general term f(x) of the series given by the following equation of differences is to he determined if the initial value is $f(0) = 1$. [Stirling, Methodus Differentialis, p. 108.1

$$f(x+1) - \frac{2x+1}{2x+2} f(x) = 0.$$

According to (2) we have

$$f(x) = f(0) \prod_{i=0}^{x} \frac{2i+1}{2i+2} = \binom{2x}{x} \frac{1}{2^{2x}}.$$

Example 2. Let us denote by *f(x)* the number of permutations of x elements. Starting from the permutations of x elements we may obtain those of x-i-l elements, if we insert the $x+1$ th element into every permutation, successively in every place. For instance, if x=2 the permutations are *ab* and *ba.* This gives for $x=3$

cab, acb, abc, and ***cba,*** *bca,* ***bac***

therefore we have

$$f(x+1) - (x+1)f(x) = 0.$$

The solution of this equation of differences is, according to formula (2),

$$f(x) = f(1) \prod_{i=1}^{x} (i+1) = f(1)x!.$$

Since $f(1) = 1$ hence the solution is $f(x) = x!$.

Example 3. Let us denote by *f(x)* the number of the possible combinations with permutation of y elements taken x by x, *i* . e. of order x.

Starting from the combinations of order x we may obtain those of order $x+1$, if we add at the end of each combination successively one of the still disposable y-x elements. In this way we obtain from each combination of order x a number of y-x different combinations of order $x+1$.

For instance, if $y=4$ and $x=1$ we have the combinations a, *b, c, d;* this gives for $x=2$: ***ab, ac, ad, ba, bc, bd,*** and so on.

Hence we shall have

$$f(x+1) - (y-x)f(x) = 0.$$

The solution is according to (2),

$$f(x) = f(1) \prod_{i=1}^{x} (y-i) = f(1)\,(y-1)_{x-1}\,.$$

Since $f(1)=y$ therefore $f(x) = (y)_x$.

Example 4. Let us denote by $f(x)$ the number of combinations ***without*** permutation of y elements of order x. Starting from the combinations of order x we may obtain those of order $x+1$ by adding to each combination one of the still disposable y-x elements. But proceeding in this way we obtain every combination $x+1$ times.

For instance if $y=4$, $x=1$, we have the combinations *a,b,c,d.* From this we obtain the combinations of order x=2:

ab, ac, ad, ***ba, bc, bd, ca, cb, cd, da, db, dc***

every combination has been obtained twice.

Therefore we have the difference equation

$$f(x+1) = \frac{y-x}{x+1} f(x)$$

whose solution is according to (2):

$$f(x) = f(1) \prod_{i=1}^{x} \frac{y-i}{i+1} = f(1)\,(y-1)_{x-1}\,/\,x!\,.$$

Since $f(1)=y$, hence

$$f(x) = \binom{y}{x}.$$

Example 5. Let us denote by $f(x)$ the number of combinations with repetition but without permutation of y elements of order x. Starting from the combinations of order x we obtain those of order $x+1$, by adding first to every combination successively every element of y. In this way we should obtain every combination of order $x+1$ but not each the same **number** of times; to obviate this, we add secondly to every combination of

order x successively each element which figures already in the combination. Then we get each combination $x+1$ times.

For instance, if $y=4$ and $x=1$ we have a, b, c, d. First we have

aa, ab, ac, ad
ba. bb, bc, bd
ca, cb, cc, cd,
da, db, dc, dd.

The combination *ab* has been obtained twice, but *aa* only once. Secondly we get

aa, bb, cc, dd

now we have obtained each combination twice.

Therefore the difference equation will be

$$f(x+1) - \frac{y+x}{x+1} f(x) = 0$$

and its solution

$$f(x) = f(1) \prod_{i=1}^{x} \frac{y+i}{i+1} = \frac{(y+1)(y+2)\dots(y+x-1)}{2.3\dots.x}$$

Since $f(1) = y$ therefore

$$f(x) = \binom{y+x-1}{x}.$$

§ 174. Laplace's method for solving linear homogeneous difference equations with **variable coefficients.** Given the equation

$$(1) \qquad a_n f(x+n) + \dots, + a_1 f(x+1) + a_0 f(x) = 0$$

where the a_i are polynomials of x. Let us put

$$(2) \qquad f(x) = \int_a^b t^{x-1} v(t) dt.$$

The function $v(t)$ is still to be disposed of, and so are the limits of the integral. From (2) it follows that

$$f(x+n) = \int_a t^{x-1+n} v(t) dt$$

therefore we get $f(x+n)$ starting from $f(x)$, if we write in (2) $t^n v(t)$ instead of $v(t)$.

Multiplying equation (2) by x and integrating by parts we find

$$xf(x) = \int_a^b x t^{x-1} v(t) dt = [t^x v(t)]_a^b - \int_a^b t^x \mathrm{D} v(t) dt.$$

Multiplying this by $x+1$ and repeating the integration by parts we have

$$(x+1)_2 f(x) = [(x+1) t^x v(t) - t^{x+1} \mathrm{D} v(t)]_a^b + \int_a^b t^{x+1} \mathrm{D}^2 v(t) dt.$$

Multiplying this by (x+2) and repeating the operation, and so on, in the end we obtain

$$(x+m-1)_m f(x) = \left[\sum_{i=1}^{m+1} (-1)^{i+1} (x+m-1)_{m-i} t^{x-1+i} \mathrm{D}^{i-1} v(t) \right]_a^b +$$
$$+ (-1)^m \int_a^b t^{x+m-1} \mathrm{D}^m v(t) dt.$$

Now writing into it $t^n v(t)$ instead of $v(t)$ we get the general formula

$$(x+m-1)_m f(x+n) = \tag{3}$$
$$= \left\{ \sum_{i=1}^{m+1} (-1)^{i+1} (x+m-1)_{m-i} t^{x-1+i} \mathrm{D}^{i-1} [t^n v(t)] \right\}_a^b +$$
$$+ (-1)^m \int_a^b t^{x+m-1} \mathrm{D}^m [t^n v(t)] dt.$$

Since we may suppose that the coefficients a,. of (1) are expanded into a series of factorials $(x+m-1)_m$ therefore formula (3) permits us to write the difference equation (1) in the following form:

$$[\varphi(x,v,t)]_a^b + \int_a^b t^{x-1} \psi(v,t) dt = 0.$$

First we dispose of $v=v(t)$ so as to have $\psi(v,t)=0$. Generally this gives a homogeneous linear differential equation whose solution is equal to u(f) . Having obtained this, we dispose secondly of the limits of the integral, so that $\varphi(x,v,t)$ shall be equal to zero at both limits. The limits must therefore be roots of the equation in t:

$$\varphi(x,v,t) = 0.$$

Let us suppose that they are, in order of magnitude, $t_1, t_2, \ldots$; each combination of these roots, two by two, will give by aid of (2) a particular solution of the difference equation. The sum of n independent particular solutions each multiplied by an arbitrary constant will give the general solution.

Remark. Having determined $f(x)$ by equation (2) it is easy to deduce $\Delta^m f(x)$ and $\Delta^{-1} f(x)$; we find

$$\Delta^m f(x) = \int_a^b t^{x-1}(t-1)^m v(t)\, dt$$

$$\Delta^{-1} f(x) = \int_a^b t^{x-1} \frac{v(t)}{t-1}\, dt.$$

Example 1. Given

$$f(x+1) - xf(x) = 0.$$

From (3) it follows that

$$\left[- t^x v(t)\right]_a^b + \int_a^b t^x [v(t) + \mathrm{D}v(t)]dt = 0.$$

Putting first

$$v(t) + \mathrm{D}v(t) = 0$$

we obtain

$$v = Ce^{-t}.$$

Secondly, the limits of the integral are the roots of

$$t^x e^{-t} = 0$$

therefore $t_1 = 0$ and $t_2 = \infty$.

Finally we shall have

$$f(x) = c \int_0^\infty t^{x-1} e^{-t} dt = C\Gamma(x).$$

Starting from the difference equation of $\Gamma(x)$, we have got its expression by a definite integral.

Particular case of formula (1). The coefficients a_i are all of the first degree in x; so that we may write equation (1) in the following manner:

$$\sum_{\nu=0}^{n+1} (\alpha_\nu x+\beta_\nu) f(x+\nu) = 0$$

where a,. and β_ν are numerical coefficients. From (2) we obtain

$$\sum_{\nu=0}^{n+1} \beta_\nu f(x+\nu) = \sum_{\nu=0}^{n+1} \beta_\nu \int_a^b t^{x-1+\nu} v\, dt$$

and from (3)

$$\sum_{\nu=0}^{n+1} a_\nu x f(x+\nu) = [t^x v \sum_{\nu=0}^{n+1} a_\nu t^\nu]_a^b - \sum_{\nu=0}^{n+1} a_\nu \int_a^b t^x \mathrm{D}[t^\nu v]\, dt.$$

Hence v will be the solution of the equation

$$\sum_{\nu=0}^{n+1} (\beta_\nu - \nu a_\nu)\ t^{x-1+\nu} v - \sum_{\nu=0}^{n+1} a_\nu t^{x+\nu} \mathrm{D}v = 0$$

which can be written

$$\frac{\mathrm{D}v}{v} = \frac{\Sigma\, t^\nu (\beta_\nu - \nu a_\nu)}{\Sigma\, a_\nu t^{\nu+1}} . \tag{4}$$

The limits of the integral are chosen from the roots of the equation

$$t^x v \sum_{\nu=0}^{n+1} a_\nu t^\nu = 0 .$$

First we have the root $t=0$; moreover, if the roots of

$$a_n t^n + \ldots + a_1 t + a_0 = 0$$

are all real, different from zero and unequal, say $t_1, t_2, \ldots,$ t_n, then the general solution will be:[54]

$$f(x) = c_1 \int_0^{t_1} t^{x-1} v\, dt + \ldots + c_n \int_0^{t_n} t^{x-1} v\, dt.$$

Example 2. Given the difference equation corresponding to Siding's series (Ex. 1, § 173):

$$(2x+2) f(x+1) - (2x+1) f(x) = 0.$$

[54] See Schlesinger, Handbuch der linearen Differenzialgleichungen, Vol. 1.. p. 409.

In consequence of (4) we have

$$\frac{Dv}{v} = \frac{1}{2t(1-t)}$$

whence

$$v = \left(\frac{t}{1-t}\right)^{1/2}$$

therefore the limits are determined by

$$t^x(2t-2)v = -2t^{x+1/2}(1-t)^{1/2} = 0$$

so that we have $t_1 = 0$ and $t_2 = 1$, and finally

$$f(x) = \int_0^1 \frac{t^{x-1/2}}{\sqrt{1-t}}\, dt.$$

Verification. This integral is equal, according to a formula in **Bierens** *de* **Haan** [**Nouvelles** Tables **d'Intégrales définies,** Table **8,** F. **2.]** to

$$f(x) = \frac{1.3.5.\ ,\ \ldots\ (2x-1)}{2.4.6\ldots.2x} \left.\frac{2^{2x}}{I\ x}\right\} \frac{1}{2^{2x}}.$$

This result has been found **before** (Ex. 1. § 173).

§ 175. Complete linear equation of differences of the first order with variable coefficients,

We will suppose again as in § 173 that x takes only integer values, such that $x \geqq a$. Given the equation

$$(1) \qquad f(x+1) - p(x)f(x) = V(x),$$

we introduce the function y(x) as in § 173; putting $y(a) = 1$ and $y(x) = p(a)p(a+1)\ldots p(x-1)$, for $x > 4$; moreover

$$u(x) = f(x)\ /\ y(x).$$

Now dividing both members of (1) by $y(x+1)$ we get

$$\Delta u(x) = \frac{V(x)}{y(x+1)}$$

therefore

$$u(x) = u(a) + \sum_{i=a}^{x} \frac{V(i)}{y(i+1)}$$

and finally

$$(2) \qquad f(x) = y(x)\left[f(a) + \sum_{i=a}^{x} \frac{V(i)}{y(i+1)} \right].$$

Example 1. Given the equation

$$f(x+1) - xf(x) = (x+1)!$$

where x $\geqq$ 1, therefore we have to put $a=1$ and y(x) $=$ (x-l) ! . Formula (2) will give

$$(3) \qquad f(x) = (x-1)! \left[f(1) + \sum_{i=1}^{x} \frac{(i+1)!}{i!} \right] =$$

$$= (x-1)! \left[f(1) + \binom{x+1}{2} - 1 \right]$$

Remark 1. In the particular case, if $a=0$ and *p(x)* **is** constant, say $p(x) = r_1$, then we shall have $y(x)=r_1{}^x$ and formula (2) will be the same as that corresponding to (4) of § 170.

Remark 2. As has been said at the end of § 172, though we supposed x to be an integer, nevertheless, if the result may be written in such a way that it has a meaning for every value of x, for instance in the case of (3), writing

$$f(x) = \Gamma(x) \left[f(1) + \binom{x+1}{2} - 1 \right]$$

then this will necessarily satisfy the given equation, Indeed the operation $\mathbf{E}$ is the same whatever the value of x may be.

§ 176. Reducible linear equations of differences with variable coefficients. In some cases the equations with variable coefficients can be reduced to equations with constant coefficients.

A. If the following equation is given

$$a_n f(x+n) + a_{n-1} p(x) f(x+n-1) + a_{n-2} p(x) p(x-1) f(x+n-2) +$$

$$(1) \qquad + \ldots + a_0\, p(x) p(x-1) \ldots p(x-n+1) f(x) = V(x)$$

where the a_i are numerical coefficients, then the equation may be reduced by putting

$$f(x) = p(x-n) p(x-n-1) \ldots p(a)\, \varphi(x).$$

Supposing that x $\geqq n + a$.

Dividing both members of equation (1) by $p(x)p(x-1) \ldots$ *p(a)* we find

$$a_n\varphi(x+n) + \ldots + a_1\varphi(x+1) + a_0\varphi(x) = \frac{V(x)}{p(x)p(x-1)\ldots p(a)}$$

B, Given the equation where the a_i and c are constants,

$$(2) \qquad a_n f(x+n) + a_{n-1}c^x f(x+n-1) + a_{n-2}c^{2x} f(x+n-2) + \\ + \ldots + a_0 c^{nx} f(x) = V(x),$$

If we write $c^{\nu x}$ in the following manner

$$c^{\nu x} = c^x c^{x-1} \ldots c^{x-\nu+1} c^{1/2\nu(\nu-1)}.$$

Then putting $c^x = p(x)$ equation (2) will become of the same type as equation (1) and may be reduced therefore to constant coefficients.

In the particular case of the equation of differences of the first order

$$f(x+1) - P(X) f(x) = V(x)$$

putting $f(x) = p(x-1)p(x-2) \ldots p(a)\, y(x)$ we find

$$\varphi(x+1) - \varphi(x) = \Delta\varphi(x) = \frac{V(x)}{p(x)p(x-1) \ldots p(a)}.$$

The solution of this is the same′ as that in § 170.

Example 1. Given the difference equation.

$$f(x+2) - 3xf(x+1) + 2x(x-1) f(x) = 0$$

let us put

$$f(x) = (x-2)(x-3) \ldots 2 \cdot 1 \cdot \varphi(x) = (x-2)!\,\varphi(x)$$

where $x \geqq z$ and $a = 1$.

Dividing both members of the given equation by $x!$ we find

$$\varphi(x+2) - 3\varphi(x+1) + 2\varphi(x) = 0$$

the equation is reduced to one of constant coefficients. The corresponding characteristic equation is

$$r^2 - 3r + 2 = 0$$

therefore $r_1 = 1$ and $r_2 = 2$, so that

$$\varphi(x) = c_1 + c_2 2^x$$

and finally

$$f(x) = (x-2)!\,(c_1 + c_2 2^x).$$

§ 177. Linear equations of differences whose coefficients are polynomials of x, solved by the method **of generating fllnctions.5"**

We have seen in § 172 that if the generating function of $f(x)$ is u then the generating function of $f(x+m)$ is

$$\mathsf{G}\,.\,f(x+m) = \frac{1}{t^m}\left[u-f(0) - tf(1) - \ldots - t^{m-1}f(m-1)\right].$$

The first derivative of this function with respect to t multiplied by t gives the generating function of $xf(x+m)$. The second derivative multiplied by $t^2/2!$ will give that of $\binom{x}{2}f(x+m)$ and so on.

$$(1)\quad \mathsf{G}\left[\binom{x}{k} f(x+m)\right] = \frac{t^k}{k!}\, \mathsf{D}^k \frac{u-f(0)-tf(1)-\ldots-t^{m-1}f(m-1)}{t^m}$$

Let us consider now the equation of differences

$$(2)\qquad a_n f(x+n) + \ldots + a_1 f(x+1) + a_0 f(x) = V(x)$$

where the coefficients a_i are polynomials of x. If the generating function of $V(x)$ is known, we may write that the generating functions corresponding to both members of the equation (2) are equal. This will give a linear differential equation:

$$\Phi(\ldots, \mathsf{D}^k u, \ldots, \mathsf{D}u, u, t) = 0$$

whose solution gives the generating function u. Developing it into a series of powers of t we get the required function $f(x)$.

If the coefficients in equation (2) are of the first degree, then the differential equation will be of the first order.

Example 1. We have seen that if we denote by $f(x)$ the numbers of the combinations of n elements of order x, then this number is given by the equation of differences

$$(x+1)f(x+1) - (n-x)f(x) = 0.$$

Let us solve this equation by the method of generating functions. Noting that

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[55] ***Laplace,*** Théorie analytique des Probabilités, p, 80.

Selivanof-Andoyer, Calcul des Differences et Interpolation, Encycl. des Sciences Mathimatiques, I. 21,. p. 76.

$$G[(x+1)f(x+1)] = Du$$

from the given equation it follows that

$$Du - nu + tDu = 0$$

and therefore

$$\frac{du}{u} = \frac{n\,dt}{1+t}.$$

Hence

$$u = C(1+t)^n.$$

Developing this into a series of powers of t this gives

$$f(x) = c\binom{n}{x}.$$

As $f(1)=n$, therefore it follows that $C=1$ and finally $f(x)=\binom{n}{x}$

Example 2. Given the equation

$$(2x+2)f(x+1) - (2x+1)f(x) = 0$$

according to formula (2) we have

$$2Du - 2tDu - u = 0.$$

Hence

$$\frac{Du}{u} = \frac{1}{2(1-t)}$$

whose solution is

$$u = C(1-t)^{-1/2}$$

and finallv

$$f(x) = c\binom{2x}{x}\frac{1}{2^{2x}}.$$

§ 178. André's method for solving difference equations.[56]

If x is a positive integer, the method consists in considering the solution of a linear equation of differences as identical with that of a system of x linear equations, with x unknowns.

Given for instance the complete linear equation of differences of the n-th order, written in the following way:

(1)

$$f(x) - a_1(x)f(x-1) - a_2(x)f(x-2) - \ldots - a_n(x)f(x-n) = V(x)$$

[56] *Désiré André,* Terme général d'une série quelconque. Annales de l'Ecole Normale Supérieure. 1878, pp. 375—408.

and the corresponding initial values $f(1), f(2), \dots, f(n)$; then the solution will be that of the following system of equations:

$$f(x) - a_1(x)f(x-1) - a_2(x)f(x-2) - \dots - a_n(x)f(x-n) = V(x)$$

$$f(x-1) - a_1(x-1)f(x-2) - \dots - a_n(x-1)f(x-n-1) = V(x-1)$$

$$\dots\dots$$

$$f(n+1) - a_1(n+1)f(n) - \dots - a_n(n+1)f(1) = V(n+1)$$

$$f(n) - a_1(n)f(n-1) - \dots - a_{n-1}(n)f(1) = y_n$$

$$\dots\dots$$

$$f(2) - a_1(2)f(1) = y_2$$

$$f(1) = y_1 .$$

If instead of the $f(1), f(2), \dots, f(n)$, the quantities $y_1, y_2, \dots, y_n$, are considered as known, then from the above system of x equations, the x unknowns, $f(1), f(2), \dots, f(x)$ may be determined; but it is sufficient to determine only one of them, *viz.* $f(x)$.

It is easy to see that the determinant figuring in the denominator of $f(x)$ is equal to unity. Indeed it is equal to

$$\begin{vmatrix} 1 & -a_1(x) & -a_2(x) & -a_3(x) & \dots \\ 0 & 1 & -a_1(x-1) & -a_2(x-1) & \dots \\ 0 & 0 & 1 & -a_1(x-2) & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots\ 1 \end{vmatrix} = 1$$

The numerator of $f(x)$ is equal to the following determinant of order x

$$\begin{vmatrix} V(x) & -a_1(x) & -a_2(x) & -a_3(x) & \dots & \\ V(x-1) & 1 & -a_1(x-1) & -a_2(x-1) & \dots & \\ V(x-2) & 0 & 1 & -a_1(x-2) & \dots & \\ V(x-3) & 0 & 0 & 1 & \dots & \\ \dots & \dots & \dots & \dots & & \\ y_2 & 0 & 0 & 0 & \dots\ 1 & -a_1(2) \\ y_1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Let us determine the coefficient of $V(x-m)$ in the expansion of this determinant.

The coefficient of V(x) is obviously equal to one. The coefficient of $V(x-1)$ is equal to u,(x); that of $V(x-2)$ is

$$\begin{vmatrix} -a_1(x) & -a_2(x) & -a_3(x) & \dots \\ 1 & -a_1(x-1) & -a_2(x-1) & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

therefore equal to

$$\begin{vmatrix} -a_1(x) & -a_2(x) \\ 1 & -a_1(x-1) \end{vmatrix} = a_1(x)a_1(x-1)+a_2(x)$$

In the same manner we should have the coefficient of V(x-3)

$$(-1)^3 \begin{vmatrix} -a_1(x) & -a_2(x) & -a_3(x) \\ 1 & -a_1(x-1) & -a_2(x-1) \\ 0 & 1 & -a_1(x-2) \end{vmatrix} =$$

$$= a_1(x)a_1(x-1)a_1(x-2)+a_3(x)+a_1(x)a_2(x-1)+a_2(x)a_1(x-2)$$

and so on; the coefficient of $V(x-m)$ will be

$$(-1)^m \begin{vmatrix} -a_1(x) & -a_2(x) & -a_3(x) & \dots & -a_m(x) \\ 1 & -a_1(x-1) & -a_2(x-1) & \dots & -a_{m-1}(x-1) \\ 0 & 1 & -a_1(x-2) & \dots & -a_{m-2}(x-2) \\ 0 & 0 & 1 & \dots & -a_{m-3}(x-3) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -a_1(x-m-1), \end{vmatrix}$$

Let us denote the general term of the expansion of this determinant by

$$a_{k_1}(x_1)a_{k_2}(x_2)\dots a_{n_i}(x_i).$$

We shall show that these expressions satisfy the following conditions :

1. The terms are homogeneous with respect to the indices of a, and we have

$$k_1 + k_2 + \ldots + k_i = m.$$

2. The first argument is always equal to x, so that $x_1 = x$; the others are: $x_2 = x - k_1$, $x_3 = x - k_1 - k_2$, ...

$$x_i = x - k_1 - k_2 - \ldots - k_{i-1} = x - m + k_i .$$

Therefore the coefficient of $V(x-m)$ will be equal to

$$\Sigma\, a_{k_1}(x_1)\, a_{k_2}(x_2) \ldots a_{k_i}(x_i).$$

This **sum** is extended to every partition of the number m with repetition and permutation. The number of the terms being equal to

$$\Gamma\left(\int m\right) = 2^{m-1} .$$

(See ***Netto*'s Combinatorik,** p. 120.)

This may be verified in the cases considered above of $V(x-1)$, $V(x-2)$ and $V(x-3)$. For instance in the last the terms correspond to $1+1+1$, $1+2$, $2+1$, 3.

We shall show now that if the sum above is equal to the coefficient of $V(x-m)$, then it follows a similar formula for $V(x-m-1)$ and therefore the formula is true for every m such that $x > m$.

The coefficient of $V(x-m-1)$ in the numerator of $f(x)$ being equal to the determinant

$$(-1)^{m-1}\begin{vmatrix} -a_1(x) & -a_2(x) & \ldots & -a_m(x) & -a_{m+1}(x) \\ 1 & -a_1(x-1) & \ldots & -a_{m-1}(x-1) & -a_m(x-1) \\ 0 & 1 & \ldots & -a_{m-2}(x-2) & -a_{m-1}(x-2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -a_1(x-m-1) & -a_2(x-m+1) \\ 0 & 0 & \ldots & 1 & -a_1(x-m) \end{vmatrix}$$

We shall examine each term of the expansion of this determinant to see whether our conditions are satisfied.

The coefficient of the term a,+,(x) is equal to one, hence it satisfies both conditions.

The coefficient of $a_m(x-1)$ is equal to a,(x) therefore the term $a_1(x)a_m(x-1)$ satisfies the conditions.

The coefficient of $a_{m-1}(x-2)$ is equal to that of $V(x-2)$, Since the sum of the indices in this coefficient is equal to 2, therefore $2+m-1=m+1$ satisfies condition (1). Moreover the arguments in the last factors of the terms in this coefficient being equal to $(x-2-k_i)$ so that the terms

$$\ldots\ldots a_{k_i}(-2+k_i)a_{m-1}(x\text{-}2)$$

also satisfy condition (2).

The coefficient of $a_{m-2}(x-3)$ is equal to that of $V(x-3)$. For the same reasons this coefficient satisfies both conditions, and so on; finally, the coefficient of $a_1(x-m)$ is equal to that of *V(x-m).* The sum of the indices which has been equal to *m,* has become now equal to $m+1$; moreover the argument in the last factor has been $(x-m+k_i)$, therefore

$$\ldots\ldots a_{k_i}(x-m+k_i)a_1(x\text{-}m)$$

satisfies the second condition too.

Finally putting

$$(2) \qquad \Sigma\, a_{k_1}(x_1)a_{k_2}(x_2)\ldots a_{k_i}(x_i)=\psi(x,m)$$

we have

$$(3) \qquad f(x)\sum_{m=0}^{x-n}\psi(x,m)\,V(x-m)+\sum_{m=x-n}^{x}\psi(x,m)y_{x-m}.$$

We have already used *Andrh's* method in simple cases, so, for instance, in § 32 determining the indefinite sum of V(x), that is, solving the difference equation

$$f(x+1)\ \text{--r(X)} = V(x)$$

or in § 38 by the inverse operation of the mean, solving the equation

$$f(x+1)+f(x) = 2V(x).$$

Now we shall apply it to more complicated problems.

Particular case. *1.* Difference equation of the first order with constant coefficients. We have

$$\begin{aligned}
f(x) - a_1 f(x-1) &= V(x)\\
f(x-1) - a_1 f(x-2) &= V(x-1)\\
&\cdots\\
f(2) - a_1 f(1) &= V(2)\\
f(1) &= y_1 .
\end{aligned}$$

Multiplying the second equation by a_1, the third by a_1^2 the fourth by a_1^3, and so on; after addition we get

$$(4) \qquad f(x) = \sum_{m=0}^{r-1} a_1^m V(x-m) + a_1^{x-1} y_1 .$$

This result could have been obtained directly by formula (3). Indeed the sum of the indices must be equal to m, therefore we have a_1^m; moreover, since a_1 is constant, there is no need to consider the arguments. Finally we shall have $\Sigma a_1^m V(x-m)$ and $a_1^{x-1} y_1$.

Particular case 2. If the given equation is homogeneous, then we have $V(x) = 0$ and the solution (3) will become

$$(5) \qquad f(x) = \sum_{m=x-n}^{x} \psi(x,m) y_{x-m} .$$

Particular case 3. Linear equations with constant coefficients of order n. If the coefficients a_ν are constants, then formula (2) will become

$$(6) \qquad \psi(x,m) = \Sigma a_{k_1} a_{k_2} \ldots a_{k_i} .$$

In this sum every combination with repetition and permutation of the numbers $k_\nu = 1, 2, \ldots, n$ will occur, in which

$$k_1 + k_2 + \ldots + k_i = m.$$

If, in the product above, the coefficient a_ν occurs α_ν times for $\nu = 1, 2, \ldots, n$, then we may write this product as follows

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}$$

where

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n\alpha_n = m.$$

But in the sum (6) this product will occur

$$\frac{\lambda!}{\alpha_1! \alpha_2! \ldots \alpha_n!}$$

times, where $\alpha_1+\alpha_2+\ldots+\alpha_n=\lambda$ has been put. Indeed this expression is equal to the number of permutations of λ elements among which there are α_1 elements a_1, moreover $a,$ elements a_2 and so on, u,, elements being equal to $a,$.

Therefore formula (6) may be written

$$(7) \qquad \varphi(x,m) = \Sigma \frac{\lambda!}{\alpha_1!\alpha_2!\ldots\alpha_n!} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}$$

where the sum is extended first to every combination with repetition of the numbers $a, = 0, 1, 2, \ldots$, satisfying the equations:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \lambda \quad \text{and} \quad \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = m$$

and secondly extended to every value of λ from the smallest integer not less than m/n to m (inclusive).

Finally f(x) is given by (5).

In this way we obtain the solution of the difference equation without determining the roots of the characteristic equation. This is an advantage of the method; but the result is obtained in the form of a sum generally very complicated.

If we solve the problem in both ways, then, equalizing the results, we may obtain interesting formulae.

Example 1. | *Waring,* Meditationes Algebraicae, Cantabrigiae, 1782] *Newton* has deduced a difference equation, one of whose particular solutions f(x) is equal to the sum of the x th powers of the roots of a given equation. Writing this equation in the following way:

$$(8) \qquad r^n - a_1 r^{n-1} - a_2 r^{n-2} - \ldots - a_{n-1} r - a_n = 0$$

he found

$$f(1) = a,$$
$$f(2) - a_1 f(1) = 2a_2$$
$$f(3) - a_1 f(2) - a_2 f(1) = 3a_3$$
$$\ldots\ldots$$
$$f(n) - a_1 f(n-1) - \ldots - a_{n-1} f(1) = na_n$$

and if $x > n$, then

$$(9) \qquad f(x) - a_1 f(x-1) - a_2 f(x-2) - \ldots - a_n f(x-n) = 0.$$

The last equation is a linear equation of differences of order n whose characteristic equation is given by (8). Therefore, denoting the roots of (8) by $r_1, r_2, \ldots . r_n$, the solution of (9) **will** be

$$f(x) = C_1 r_1^x + C_2 r_2^x + \ldots . + C_n r_n^x .$$

If we dispose of the constants so as to have $C_1 = C_2 = \ldots = C_n = 1$, then we get the particular solution

$$f(x) = \sum_{i=1}^{n+1} r_i^x$$

that is, f(x) obtained from Newton's difference equation (9) will be really equal to the sum of the x th powers of the roots of equation (8).

Waring has determined the generating function of $f(x)$; this we will obtain by ***André's*** method of solving equation (9). Remarking that in the problem considered the n given values figuring in this method are

$$y_1 = a_1, \quad y_2 = 2a_2, \ldots ., \quad y_n = na_n .$$

Equation (9) being homogeneous therefore, according to formula (5), its solution will be

$$f(x) = \sum_{m=x-n}^{x} \psi(x,m)(x-m)a_{x-m} .$$

Putting into this equation the value of $\psi(x,m)$ obtained in the case of equations with constant coefficients (7) we find

$$(10) \qquad f(x) = \Sigma (x-m) \frac{\lambda!}{\alpha_1! \ldots \alpha_n!} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n} a_{x-m} .$$

This sum is first to be extended to every value of $\alpha_i = 0, 1, 2, \ldots$ satisfying the following equations

$$(11) \qquad \begin{aligned} \alpha_1 + \alpha_2 + \ldots . + \alpha_n &= \lambda \\ \alpha_1 + 2\alpha_2 + \ldots . + n\alpha_n &= m \end{aligned}$$

secondly to every value above mentioned of λ and finally to every value of m from $m=x-n$ to $m=x$ (inclusive).

We may write (10) in the following way, putting $x-m=i$

$$f(x) = \Sigma \frac{i(\alpha_1 + \ldots + \alpha_n)!}{\alpha_1!\alpha_2! \ldots \alpha_i!} a_1^{\alpha_1} \ldots a_i^{\alpha_i+1} \ldots a_n^{\alpha_n}$$

where we have

$$a_1+2a_2+\ldots+na_n = x-i.$$

Writing moreover a_i instead of a_i+1 we get

$$f(x) = \Sigma\, ia_i\, \Sigma\, \frac{(a_1+\ldots+a_n-1)!}{a_1!\ldots a_n!}\, a_1^{\alpha_1}\cdots a_n^{\alpha_n}$$

and

$$a, + 2a_2 + \ldots + na, = x.$$

Now the second sum is independent of *i,* therefore we may execute the summation with respect to *i.* Since $\Sigma\, ia_i = x$ hence finally we shall have

$$f(x) = x\, \Sigma\, \frac{(a_1+\ldots+a_n-1)!}{a,!\ldots a,!}\, a_1^{\alpha_1}\ldots a_n^{\alpha_n}$$

This sum is to be extended to every combination with permutation and repetition of the numbers $a,.$ satisfying $a_1+2a_2+na_n=x.$ This is *Waring's* formula. It is easy to see that *f(x)* is the coefficient of t^x in the expansion of the following expression

$$x \sum_{\nu=0}^{x+1} \frac{1}{\nu} (a_1 t + a_2 t^2 + \ldots + a_n t^n)^\nu$$

that is in

$$\log(1-a_1 t-a_2 t^2-\ldots-a_n t^n)^{-x}$$

therefore this is the generating function of *f(x).*

Example 2. Equation with variable coefficients. Problem of Coincidences. (Rencontre.) From an urn containing the numbers 1, 2, 3, . . . , x these are all drawn one after another. If at the *m* th drawing the number *m* is found, then it is said that there is coincidence at the *m* th drawing.

The number of the possible cases is equal to the number of permutations of x elements, that is to $x!$. If we denote by *f(x)* the number of permutations in which there are no coincidences at all; then the number of permutations in which there is one coincidence will be $\binom{x}{1} f(x-1)$; the number of those in which there are two coincidences is equal to $\binom{x}{2} f(x-2)$ and so on; finally, there is only one permutation in which there are x coincidences.

Therefore we may write the following difference equation

$$f(x) + \binom{x}{1} f(x-1) + \binom{x}{2} f(x-2) + \ldots + \binom{x}{x-1} f(1) = x\,! - 1\,.$$

From (2) we may deduce

$$\psi(x,m) = \Sigma\,(-1)^i \binom{x_1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_i}{k_i}$$

Since $x_1 = x$, $x_2 = x - k_1$, $x_3 = x - k_1 - k_2$ and so on, therefore putting

$$\binom{x_1}{k_1} = \frac{x\,!}{k_1!\,(x-k_1)\,!}\,;\quad \binom{x_2}{k_2} = \frac{(x-k_1)\ !}{k_1!\ (x-k_1-k_2)\ !}\,;\ \ldots\ldots$$

we obtain

$$\psi(x,m) = \Sigma\,(-1)^i \frac{x\,!}{k_1!\,k_2!\ldots k_i!\,(x-m)\,!}\,.$$

In this sum we have

(12) $$k_1 + k_2 + \ldots k_i = m.$$

Therefore if i is given and the $k_i > 0$ then according to (5) § 60 we have

$$\Sigma\ \frac{1}{k_1!\,k_2!\ldots k_i!} = \frac{i!}{m!}\,\mathfrak{S}_m^i$$

where the sum is extended to every value of k, different from zero and satisfying equation (12). The number $\mathfrak{S}_m^i$ is a *Stirling*'s number of the second kind. In consequence of (5) § 58 we have

$$\sum_{i=1}^{m+1} (-1)^i\, i!\ \mathfrak{S}_m^i = (-1)^m$$

the expression of $\psi(x,m)$ will be equal to $(-1)^m \binom{x}{m}$ and finally according to (3)

$$f(x) = \sum_{m=0}^{x} (-1)^m \binom{x}{m} [\,(x-m)\ ! - 1\,] = x!\ \sum_{m=2}^{x+1} (-1)^m \frac{1}{m\,!}$$

Example 3. In § 83 we deduced *Lacroix's* difference equation giving $f(x) = \Delta_u^{-1} u^{x-1}$; we found:

$$(13)\quad f(x)+\binom{x-1}{1}\frac{1}{2}f(x-1)+\binom{x-1}{2}\frac{1}{3}f(x-2)+\ldots+$$
$$+\binom{x-1}{\nu}\frac{1}{\nu+1}f(x-\nu)+\ldots+\frac{1}{x}f(1)=\frac{u^x}{x}.$$

The initial condition being

$$f(1)=u+c.$$

Let us solve this equation by aid of the preceding method. We have

$$(14)\qquad a_{\nu}=\binom{x-1}{\nu}\frac{1}{\nu+1}$$

and

$$V(x-m)=\frac{u^{x-m}}{x-m}.$$

According to formula (2) we have

$$\psi(x,m)=\Sigma\binom{x_1}{k_1}\binom{x_2}{k_2}\cdots\binom{x_i}{k_i}\frac{(-1)^i}{(k_1+1)(k_2+1)\ldots(k_i+1)}$$

where the sum is extended first to every combination of order i with repetition and permutation of the numbers $k_\nu=1,2,3,\ldots$ such that

$$k_1+k_2+\ldots+k_i=m$$

then to every value of i. Moreover,

$$x_1=x,\quad x_2=x-k_1,\quad\ldots\quad x_\nu=x-k_1-k_2-\ldots-k_{\nu-1};$$

therefore $x_i=x-m+k_i$. After simplification we have

$$(15)\qquad \psi(x,m)=m!\binom{x-1}{m}\Sigma\frac{(-1)^i}{(k_1+1)!(k_2+1)!\ldots(k_i+1)!}$$

and $\psi(x,0)=1$. Finally the solution is given by formula (3):

$$(16)\quad f(x)=\frac{1}{x}\sum_{m=0}^{x-1}m!\binom{x}{m}u^{x-m}\,\Sigma\frac{(-1)^i}{(k_1+1)!(k_2+1)!\ldots(k_i+1)!}+$$
$$+(u+C)(x-1)!\,\Sigma\frac{(-1)^i}{(k_1+1)!(k_2+1)!\ldots(k_i+1)!}.$$

The last sum is extended to $k_1+k_2+\ldots+k_i=x-1$.

Remark. It is possible to express the preceding sums in another form. In § 60 we found formula (5):

$$(17) \qquad \Sigma \frac{1}{k_1!\,k_2!\ldots k_i!} = \frac{i!}{m!}\,\mathfrak{S}_m^i$$

the sum being extended to the same values of k_ν as the sums figuring in (16). If in the latter sum k_ν could also he equal to zero, then its value would be, in consequence of (17),

$$\frac{i!}{(m+i)!}\,\mathfrak{S}_{m+i}^i$$

In order to obtain from this the required sum, we have to subtract from this the terms in which $k_\nu = 0$, and that for every value of $\nu = 1, 2, 3, \ldots, i$. That is:

$$\binom{i}{1}\frac{(i-1)!}{(m+i-1)!}\,\mathfrak{S}_{m+i-1}^{i-1}\,.$$

Proceeding in this way, we have twice subtracted the terms in which we had $k_{\nu_1} = k_{\nu_2} = 0$. Hence we must add these again for every combination of ν_1, ν_2, that is $\binom{i}{2}$ times:

$$\binom{i}{2}\frac{(i-2)!}{(m+i-2)!}\,\mathfrak{S}_{m+i-2}^{i-2},$$

But in this manner we have added twice the terms in which $k_{\nu_1} = k_{\nu_2} = k_{\nu_3} = 0$; so that we must now subtract

$$\binom{i}{3}\frac{(i-3)!}{(m+i-3)!}\,\mathfrak{S}_{m+i-3}^{i-3}.$$

and so on; finally we shall have:

$$\sum_{\nu=0}^{i+1}\frac{(-1)^\nu (i-\nu)!}{(m+i-\nu)!}\binom{i}{\nu}\,\mathfrak{S}_{m+i-\nu}^{i-\nu}\,.$$

putting now $m+i-\nu = n$, we get

$$\frac{i!}{(m+i)!}\sum_{n=m+1}^{m+i+1}(-1)^{m+i-n}\binom{m+i}{n}\,\mathfrak{S}_n^{n-m}$$

but we have seen in § 65 (p. 185) that this sum is equal to $\overline{C}_{m,m-i}$ (see Table p. 172); therefore

$$(18) \qquad \Sigma \frac{1}{(k_1+1)!\,(k_2+1)!\ldots(k_i+1)!} = \frac{i!}{(m+i)!}\,\overline{C}_{m,m-i}\,.$$

The sum being extended to every combination of order i with repetition and permutation of the numbers $k_\nu = 1, 2, 3, \ldots$ such that

$$k_1+k_2+\ldots+k_i=m, \qquad (k_\nu>0).$$

To obtain $f(x)$ we have still to sum the expression (18) from $i=1$ to $i=m+1$.

In § 83 we found the symbolical formula

$$f(x) = \frac{1}{x}\,[u+B]^x + k.$$

Equating the coefficient of u^{x-m} in this expression and in (16) we have

$$(19) \qquad B_m = m!\,\Sigma\,\frac{(-1)^i}{(k_1+1)!\,(k_2+1)!\ldots(k_i+1)!}$$

where $k_1+k_2+\ldots+k_i=m$ and $k_\nu>0$ and the sum is extended for every value of i from one to m. Moreover in consequence of (18)

$$(20) \qquad B_m = \sum_{i=1}^{m} \frac{(-1)^i}{\binom{m+i}{m}}\,\bar{C}_{m,m-i}\,.$$

§ 179. **Sum equations which are reducible to equations of differences.** Besides the forms of difference equations (4), (5) and (6) considered in § 164 there is another form in which sums and differences figure. The simplest case is:

$$(1) \qquad \Phi[x, f(x), \Delta f(x), \Delta^2 f(x), \ldots, \sum_{i=a}^{x} f(i)] = 0.$$

In certain cases this equation may be solved; for instance if it may be written in the following way:

$$(2) \qquad \sum_{i=a}^{x} f(i) = \Phi_1[x, f(x), \Delta f(x), \ldots]$$

then performing on both members the operation Δ we find

$$f(x) = \Delta\Phi_1[x, f(x), \Delta f(x), \ldots]$$

an ordinary equation of differences.

Example 1. Probability of repeated trials. In a game the player gets n times the amount of his stake, if he wins. If he loses he plays again, staking each time anew, and continues till he wins in the end.

Let us denote by $f(i)$ his stake in the i th game. This quantity is to be determined in such a manner that if the player wins, say the x th game, he gets back not only the stakes he lost previously but moreover a certain sum s fixed in advance. Therefore we shall have

$$n f(x) = s + \sum_{i=1}^{x+1} f(i)$$

performing the operation of differences we obtain

$$n \Delta f(x) = f(x+1)$$

or

$$f(x+1) - \frac{n}{n-1} f(x) = o.$$

This is a linear equation with constant coefficients, whose solution is

$$f(x) = c \left(\frac{n}{n-1} \right)^x$$

To determine C let us remark, that we must have (n-l)f(l) $= s$, therefore $C = s/n$ and

$$f(x) = \frac{s}{n} \left(\frac{n}{n-1} \right)^x$$

Particular case. If $n = 2$ (for instance in the roulette), then

$$f(x) = s2^{x-1}$$

that is, the stakes must always be doubled.

The given equation may be somewhat more general than (2) for instance:

$$\sum_{i=a}^{x} p(i) f(i) = \Phi[x, f(x), \Delta f(x), \ldots, .] \tag{3}$$

where $p(i)$ is a given function. Proceeding in the same way as before we perform the operation of differences and get

$$p(x) f(x) = \Delta \Phi[x, f(x), \Delta f(x), \ldots].$$

§ 180. Simultaneous linear equations of differences with constant coefficients. Given two such equations:

$$\begin{aligned} \varphi(\mathsf{E})u + \psi(\mathsf{E})v &= F(x) \\ \varphi_1(\mathsf{E})u + \psi_1(\mathsf{E})v &= F_1(x) \end{aligned} \tag{1}$$

where the unknown functions u and v are to be determined,

Starting from these equations we may deduce a difference equation containing only one function. Indeed, executing the operation $\varphi_1(\mathsf{E})$ on both members of the first equation, and the operation $\varphi(\mathsf{E})$ on both members of the second, we get, after subtracting the second result from the first:

$$[\varphi_1(\mathsf{E})\psi(\mathsf{E}) - \varphi(\mathsf{E})\psi_1(\mathsf{E})]v = \psi_1(\mathsf{E})F(x) - \varphi(\mathsf{E})F_1(x). \tag{2}$$

This is a complete linear equation of differences with constant coefficients whose order n is equal to the highest power of E in the first member of (2); n will also be equal to the number of the arbitrary constants in the solution of v.

Denoting by V(x) a particular solution of (2), then if the characteristic equation has no multiple roots, the general solution of (2) will be

$$v = c_1 r_1^x + \dots + c_n r_n^x + V(x). \tag{3}$$

In the same manner we could get a difference equation of order n determining the function u. Denoting by $U(x)$ a particular solution of

$$[\varphi_1(\mathsf{E})\psi(\mathsf{E}) - \varphi(\mathsf{E})\psi_1(\mathsf{E})]u = \psi(\mathsf{E})F_1(x) - \psi_1(\mathsf{E})F(x) \tag{4}$$

then its general solution will be

$$u = c_1' r_1^x + \dots + c_n' r_n^x + U(x). \tag{5}$$

It may be shown that if V(x) is a particular solution of (2) and $U(x)$ a particular solution of (4), then it follows that V(x) and $U(x)$ are also solutions of equation (1).

u contains also n arbitrary constants, but they are not independent of those figuring in v. Indeed, from the first of the equations (1) it follows, in consequence of formula (2) § 165, that

$$\sum_{i=1}^{n+1} [c_i \psi(r_i) + c_i' \varphi(r_i)] r_i^x = 0.$$

Since this is to be satisfied for all values of x, therefore the coefficients of each exponential must be separately equal to zero, so that we have

$$c_i' = -\frac{c_i\psi(r_i)}{\varphi(r_i)} .$$

Remark. If the equations **(1)** are homogeneous, the functions u and v will differ only by the values of the arbitrary constants.

Example 1. Given

$$(4E - 17)u + (E-4)v = 0$$
$$(2E - 1)u + (E-2)v = 0$$

from these it follows that

$$[E^2 - 8E + 15]v = 0$$

and therefore

$$v = c_1 3" + c_2 5^x$$

moreover

$$c_1' = -\frac{1}{5}c_1 \quad \text{and} \quad c_2' = -\frac{1}{3}c_2$$

so that

$$u = -\frac{1}{5}c_1 3^x - \frac{1}{3}c_2 5^x .$$

If the characteristic equation of **(2)** has double roots, the calculus is a little more complicated. We have seen that in this case the solution is of the form

$$v = (c, + c_2 x)r_1^x + \dots .$$
$$u = (c,' + c_2'x)r_1^x + , \dots$$

Before putting these values into the first equation **(1)**, let us deduce the following formula. In § **165** we had

$$\varphi(E)r^x = r^x\varphi(r).$$

Derivation with respect to r followed by multiplication by r gives

$$\text{(6)} \qquad \varphi(E)xr^x = xr^x\psi(r) + r^x r D\varphi(r)$$

therefore from (1) we get

$$c_1'\varphi(r_1) + c_2' r_1 D\varphi(r_1) + c_2'\varphi(r)x + c_1\psi(r_1) + c_2 r_1 D\psi(r_1) +$$
$$+ c_2\psi(r_1)x = 0$$

for every, value of x, and therefore

(7)
$$c_2' = -\frac{\psi(r_1)}{\varphi(r_1)} c_2$$

$$c_1' = -\frac{c_1 \psi(r_1) + c_2 r_1 \mathrm{D}\psi(r_1) + c_2' r_1 \mathrm{D}\varphi(r_1)}{\varphi(r_1)}$$

Example 2. Given

$$(\mathrm{E}^2 - 8)u + 2\mathrm{E}v = 0 \quad \text{and} \quad \mathrm{E}u - (\mathrm{E}^2 - 2)v = 0.$$

From this we deduce

$$(\mathrm{E}^2 - 4)^2 v = 0.$$

Hence the solution will be

$$v = (c_1 + c_2 x)2^x + (c_3 + c_4 x)(-2)^x$$

finally by aid of (7) we deduce

$$u = (c_1 + 3c_2 + c_2 x)2^x - (c_3 + 3c_4 + c_4 x)(-2)^x.$$

CHAPTER XII.

EQUATIONS OF PARTIAL DIFFERENCES.

§ 181. Introduction. If z is a function of the two independent variables x and y, so that $z=f(x,y)$, then the equation

$$(1) \qquad F(x,y,z,\Delta_x z, \Delta_y z, \Delta_x^2 z, \ldots, \Delta_x^n z, \Delta_y^m z) = 0$$

is called an equation of partial differences, or a difference equation with two independent variables.

Eliminating from equation **(1)** the symbols Δ_x and Δ_y by aid of $E_x = 1 + \Delta_x$ and $E_y = 1 + \Delta_y$ we obtain a **second form of** these equations

$$(2) \qquad F_1(x,y,z,E_x z, E_y z, E_x^2 z, E_y^2 z, \ldots, E_x^n z, E_y^m z) = 0.$$

The function z may be determined by aid of (2) if certain initial conditions are given. But in these cases a few particular values of z are not sufficient, here particular functions must be given. For instance, if equation (2) is a linear equation of the first order with respect to x and also with respect to y, and if the equation contains each of the four possible terms, so that we have

$$(3) \qquad a_{11} E_x E_y z + a_{01} E_y z + a_{10} E_x z + a_{00} z = V(x,y)$$

then, to enable us to compute $z=f(x,y)$ for every integer value of x and y, two functions must be given as initial conditions. For instance:

$$f(x,0) = \varphi(x) \quad \text{and} \quad f(0,y) = \psi(y)$$

for every positive or negative integer value of x and y.

If the two functions are given, then putting $x=0$ and $y=0$

into (3) the only unknown quantity in this equation will be $f(1,1)$. This being calculated we put into (3) $x=1$ and $y=0$ and get $f(2,1)$, and so on. Moreover putting into (3) $x=-1$ and $y=0$ we obtain $f(-1,1)$; and so on step by step we may find any value $f(x,1)$ whatever. [Figure 13,)

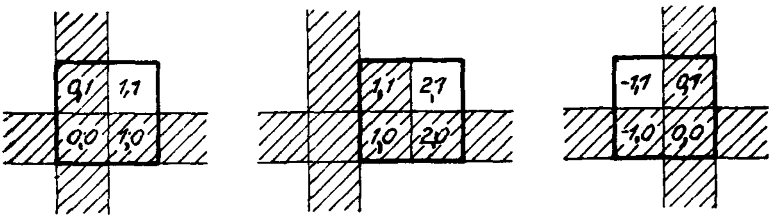
In the same way, starting from $x=0$ and $y=1$ we now obtain $f(x,2)$ and starting from $x=0$ and $y=2$ we get $f(x,3)$. Continuing we may find finally any value of $f(x,y)$ whatever.

In the general case of a difference equation of order n with respect to x and of order m with respect to y, the number of the possible terms

$$E_x^{\nu} E_y^{\mu} \text{ for } \nu = 0, 1, 2, \ldots n \text{ and } \mu = 0, 1, \ldots m$$

will be equal to $(n+1)(m+1)$.

Figure 13.



If the equation contains all these terms, then the initial conditions necessary to compute $f(x,y)$ are $m+n$ given independent functions. For instance the following:

$$f(x,i) = \varphi_i(x) \quad \text{for} \quad i = 0, 1, 2, \ldots, (n-1)$$

$$f(j,y) = \psi_j(y) \quad \text{for} \quad j = 0, 1, 2, \ldots, (m-1).$$

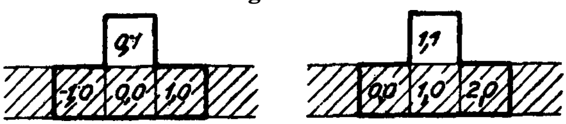
Indeed, after having put $x=0$ and $y=0$ into (2), the above equations will give $mn+m+n$ of the quantities figuring in (2), so that there will remain in this equation only one unknown, f (n,m). Having determined this we may proceed to the determination of $f(n+1,m)$, and so on.

But if from the $(n+1)(m+1)$ possible terms in equation (2) some are missing and if x and y are positive, then the number of the necessary conditions may often be reduced. For instance, if the equation is of order n with respect to x and of the first order with respect to y; moreover if there is but one term E_y in the equation, and if y ≥ 0, then instead of the $n+1$ necessary conditions, one condition will be sufficient. For instance if we have

$$a_{20}f(x+2,y) + a_{11}f(x+1,y+1)+a_{10}f(x+1,y) + a_{00}f(x,y)=V(x,y)$$

then $f(x,0)=\varphi(x)$ given for every positive and negative value of x will be sufficient for computing step by step every number $f(x,y)$. Indeed, putting $y=0$ and $x=-1$ into this equation, the only unknown will be $f(0,1)$; this being calculated we put into the equation $y=0$, $x=0$ and obtain $f(1,1)$, and so on. (Figure 14.)

Figure 14.



We may determine in every particular case the number of the necessary and sufficient conditions for the computation of $f(x,y)$. We have to dispose of them so, that putting the corresponding values of $f(x_i,y_j)$ into equation (2), there shall remain only one unknown in it, for instance $f(n,m)$. But this must be done in such a way that having determined $f(n,m)$ *we* may proceed in the same manner to the determination of $f(n+1,m)$ and PO on.

The conditions must be independent; that is, no condition shall be obtainable starting from the other conditions, by aid of the equation of differences.

Let us suppose that the necessary initial conditions corresponding to an equation of partial differences are the following functions:

$$f(x,y_0),\ f(x,y_1),\ \ldots,\ f(x,y_j);\quad f(x_0,y),\ f(x_1,y)\ \ldots,\ f(x_i,y)$$

given for every value of x and y; if the arbitrary functions contained in the solution are such that they may be disposed of so as to satisfy the above initial conditions, then this solution may be considered as the general solution.

§ 182. Resolution of linear equations of partial differences with constant coefficients, by Laplace's method of generating functions.

If the given equation is of order n in x and of order m in y then the complete difference equation may be written in the following manner:

$$(1) \qquad \sum_{\nu=0}^{n+1} \sum_{\mu=0}^{m+1} a_{\nu\mu} \, f(x+\nu, y+\mu) = V(x,y).$$

Let us call $u(t,t_1)$ the "generating function of $f(x,y)$ with respect to x and y" if in the expansion of $u(t,t_1)$ into a double series of powers of t and t_1 the coefficient of $t^x t_1^y$ is equal to $f(x,y)$; if x and y vary from zero to ∞, then

$$(2) \qquad u(t,t_1) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} f(x,y) \, t^x t_1^y$$

[should y vary for instance from one to ∞, we should simply have to put $f(x,0)=0$].

This generating function is also denoted by $\underset{x,y}{\mathbf{G}} . f(x,y)$. If first we expand $u(t,t_1)$ into a series of powers of t and have

$$(3) \qquad u(t,t_1) = \sum_{x=0}^{\infty} w(x,t_1) \, t^x$$

then we say that $u(t,t_1)$ is the generating function of $w(x,t_1)$ with respect to x, that is

$$\underset{x}{\mathbf{G}} . w(x,t_1) = u(t,t_1)$$

t_1 is in this formula only a parameter.

Expanding $w(x,t_1)$ into a series of powers of t_1 we get

$$(4) \qquad w(x,t_1) = \sum_{y=0}^{\infty} f(x,y) \, t_1^y = \underset{y}{\mathbf{G}} . f(x,y).$$

Here $w(x,t_1)$ is the generating function of $f(x,y)$ with respect to y; in this formula x is a parameter.

In this manner, instead of a generating function of two variables we have obtained two functions of one variable.

From (4) we deduce directly

$$\underset{y}{\mathbf{G}} . f(x+\nu,y) = w(x+\nu,t)$$

and according to formula (3) § 11

$$\underset{y}{\mathbf{G}}.\, f(x+\nu,y+\mu) = \frac{1}{t_1^{\mu}} [w(x+\nu,t_1) - f(x+\nu,0) - f(x+\nu,1)t_1 - \dots$$
$$- f(x+\nu,\mu-1)t_1^{\mu-1}] .$$

Let us denote by $R(x,t_1)$ the gknerating function of $V(x,y)$ with respect to y, that is

$$\underset{y}{\mathbf{G}}.\, V(x,y) = R(x,t_1).$$

Now we may write that the generating functions with respect to y corresponding to the terms of equation (1) satisfy this difference equation. Therefore we have

$$(5)\quad \sum_{\nu=0}^{n+1} \sum_{\mu=0}^{m+1} a_{\nu\mu} t_1^{m-\mu} [w(x+\nu,t_1) - f(x+\nu,0) - t_1 f(x+\nu,1) - \dots,$$
$$- t_1^{\mu-1} f(x+\nu,\mu-1)] = t_1^{m} R(x,t_1).$$

This is a linear difference equation with constant coefficients of the variable x, of order n; it contains already m arbitrary functions of x

$$f(x+\nu,\mu-1) \text{ where } \mu = 1, 2, 3, \dots, m.$$

Into the solution $w(x,t_1)$ there will enter moreover n arbitrary functions of t_1, The expansion of $w(x,t_1)$ into a series of powers of t_1 will give $f(x,y)$; the arbitrary functions of t_1 expanded will give n arbitrary functions of y.

Finally the $m+n$ arbitrary functions of x or of y are determined by the aid of the initial conditions.

Example 1. *Problem of poinfs. In* a game the first player needs x points to win the stakes; the second needs y points. The probability that the first player shall win the stakes, is denoted by $f(x,y)$. Let us suppose that the probability of winning a point is p for the first player and $1-p=q$ for the second.

To determine $f(x,y)$ let us remark, that the first player may win the stakes in two different manners; first by winning the next point, the probability of which event is p; then, as he needs now only $x-1$ points to win and his adversary y points, the probability of his winning the stakes will be $f(x-1,y)$; and the compound event, viz. of winning the next point and afterwards the stakes too is $pf(x-1,y)$.

Secondly, the first player may win the stakes by losing the next point, the probability of which is q; then for winning the stakes he needs still x points and his adversary y-l points, therefore the probability of the compound event will be $qf(x,y-1)$.

According to the theorem of total probabilities, the probability $f(x,y)$ that the first player shall win in one of the two ways is equal to the sum of the two probabilities obtained above, that is, to

$$f(x,y) = pf(x-1,y) + qf(x,y-1).$$

We shall write this equation of differences in the following manner:

$$f(x+1,y+1) - pf(x,y+1) - qf(x+1,y) = 0. \tag{6}$$

To solve this equation we have to put into (5) $n=m=1$; we find

$$(1-qt_1)w(x+1,t_1) - pw(x,t_1) = f(x+1,0) - pf(x,0). \tag{7}$$

According to § 181, it is easy to see that, knowing the values of $f(x,0)$ and $f(0,y)$ *we* may compute $f(x,y)$ by aid of the equation of differences, step by step for every integer value of x.

But $f(x,O)$ is the probability that the first player wins the stakes if he needs x points and his adversary none. This probability is obviously equal to zero, Therefore $f(x,0)=0$ for every positive value of x; $f(0,0)$ cannot occur.

$f(0,y)$ is the probability that the first player wins the stakes if he needs no points at all; if y $>$ 0 this probability is equal to one, since he has won already. Therefore $f(0,y)=1$.

Since $f(x,0)=0$ if $x > 0$, therefore from (7) it follows that

$$w(x+1,t_1) - \frac{p}{1-qt_1} w(x,t_1) = 0.$$

If $x=0$ then we have

$$w(0,t_1) = f(0,1)t_1 + f(0,2)t_1^2 + \ldots\ldots$$

Since $f(0,y)=1$, hence

$$w(0,t_1) = \frac{t_1}{1-t_1}.$$

The equation above is a homogeneous linear equation of differences of the first order, with constant coefficients; hence its soiution will be

$$w(x,t_1) = \varphi(t_1)\left(\frac{p}{1-qt_1}\right)^x$$

$\varphi(t_1)$ must be determined by aid of the initial condition corresponding to $x=0$, therefore

$$\varphi(t_1)=w(0,t_1).$$

Hence

$$w(x,t_1) = \frac{t_1}{1-t_1}\left(\frac{p}{1-qt_1}\right)^x.$$

The expansion of $w(x,t_1)$ into a series of powers of t_1 will give $f(x,y)$

$$w(x,t_1) = \Sigma\,\Sigma\, t_1 t_1^{k} p^{x}(-1)^{i}\binom{-x}{i} q^{i} t_1^{i}.$$

If we put $k+1+i=y$, then the coefficient of t^{y_1}, that is, the required probability, will be

$$f(x,y) = p^{x}\sum_{i=0}^{y}\binom{x+i-1}{i} q^{i}.$$

Example 2. The problem of coincidence considered in § 178 (Ex. 2), somewhat generalised, will lead to a partial equation of differences. From an urn containing the numbers 1, 2, 3, . . . , . , y these latter are all drawn one after another. The probability is required of not having coincidence in x given drawings.

The number of the different ways of drawing y numbers is equal to the number of permutations of y elements, that is, to y! If we denote by $f(x,y)$ the number of the favourable cases, that is, those in which there is no coincidence in x given drawings; then the required probability will be $f(x,y)/y!$

But $f(x,y)$ may be considered as equal to the number of ways of drawing y numbers so as to have no coincidences at x-l given places, that is to $f(x-1,y)$ less the number of ways of drawing the y numbers; so that there is coincidence at the place x, and no coincidences at the other given x-l places, This last number is obviously equal to f (x-l,y-1). Consequently we have

$$f(x,y) = f(x-1,y) - f(x-1,y-1).$$

Let us write this equation of partial differences in the following manner :

(8) $$f(x+1,y+1) - f(x,y+1) + f(x,y) = 0.$$

Since this equation contains only one term of $x+1$, moreover, since y ≧ xl 0, therefore one initial condition will be sufficient for the computation of $f(x,y)$. Such a condition is for instance $f(0,y) = \varphi(y)$. But $f(0,y)$ is equal to the number of ways of drawing y numbers without any restriction at all; therefore $f(0,y)=y!$ for $y=0$, 1, 2,. . . . Moreover, since there is only one way of not drawing numbers, we have $f(x,0)=1$.

Starting from the conditions $f(x,0)=1$ and $f(0,y)=y!$ we may compute step by step every value of $f(x,y)$.

In consequence of (5) the generating function of $f(x,y)$ with respect to y, denoted by $w(x,t_1)$, will be given by

$$w(x+1,t_1) - (1-t_1)w(x,t_1) = f(x+1,0) - f(x,0) = 0$$

therefore, if $x > 0$, the solution is

$$w(x,t_1) = \varphi(t_1)(1-t_1)^x$$

moreover if $x=0$, then

$$w(0,t_1) = f(0,0) + f(0,1)\, t_1 + f(0,2)t_1^2 + \ldots$$

and in consequence of $f(0,y) = y!$ we have:

$$w(0,t_1) = \sum_{\nu=0}^{\infty} \nu!\, t_1^{\nu}$$

this is equal to the arbitrary function $\varphi(t_1)$, and we get

$$w(x,t_1) = (1-t_1)^x \sum \nu! t_1^{\nu}$$

The expansion of this function will give

$$w(x,t_1) = \sum \nu!\, t_1^{\nu} \sum (-1)^{\mu} \binom{x}{\mu} t_1^{\mu}$$

Finally, putting $\nu+\mu=y$, the coefficient of t_1^y will be equal to

$$f(x,y) = \sum_{\mu=0}^{x+1} (-1)^{\mu} \binom{x}{\mu} (y-\mu)!$$

From this there follows the interesting formula (p. 8)

$$f(x,y) = (-1)^x \left[\Delta_{\mu}^{x} (y-\mu)!\right]_{\mu=0} \,.$$

In the particular case of $y=x$ we obtain a formula already deduced in § 178:

$$f(x,x) = x! \sum_{m=0}^{x+1} \frac{(-1)^m}{m!}$$

From this the probability of not having **coicidences** at all in x drawings is obtained by dividing $f(x,x)$ by x! , If x increases indefinitely this probability tends to e^{-1} .

Example 3. The rule of computation of the numbers in *Pascal's* arithmetical triangle is given by the following equation of partial differences:

$$f(x+1,y+1) = f(x+1,y) + f(x,y).$$

Since this equation contains only one term of $y+1$, and since $y \geqq x \geqq 0$ therefore the condition, which follows from the definition of these numbers:

$$f(x,0) = 0 \text{ if } x \gtrless 0 \text{ and } f(0,0) = 1$$

will be sufficient for the computation of $f(x,y)$.

According to (5) $\underset{x}{\mathbf{G}} f(x,y)$ will be given by

$$u(y+1,t) - (1+t)\, u(y,t) = f(0,y+1) - f(0,y).$$

Starting from the initial conditions it is easy to show that $f(0,y) = 1$, therefore we have

$$u(y+1,t) = (1+t)\, u(y,t).$$

The resolution of this equation gives

$$u(y,t) = \varphi(t)\, (1+t)^y$$

for $y=0$ we get $u(0,t) = \varphi(t)$; but in consequence of the initial condition it follows that

$$u(0,t) = f(O,O) + f(1,0)t + f(2,0)t^2 + \ldots, = 1$$

so that finally

$$u(y,t) = (1+t)^y$$

and

$$f(x,y) = \binom{y}{x}.$$

Example 4. We have seen in § 58 that, denoting the *Stirling* numbers of the second kind $\mathfrak{S}_y^x$ by $f(x,y)$, they satisfy the following equation of partial differences:

(9) $$f(x+1,y+1) - (x+1)\,f(x+1,y) - f\,(x,y) = 0.$$

Since this equation contains only one term in $y+1$, and $y \geqq x \geqq 0$ hence the condition

$$f(x,O) = 0 \quad \text{if} \quad x \neq 0 \quad \text{and} \quad f(O,O) = 1$$

which follows from the definition of the *Stirling* numbers, is sufficient for the computation of $f\,(x,y)$.

According to (5) $w(x,t_1)$ *the* generating function of $f(x,y)$ with respect to y is given by the difference equation:

$$w(x+1,t_1) - f(x+1,0) - (1+x)t_1\,w(x+1,t_1) - t_1 w(x,t_1) = 0.$$

Since $f\,(x+1,0) = 0$ therefore we have

(10) $$w(x+1,t_1) - \frac{t_1}{1-(1+x)t_1}\,w(x,t_1) = 0.$$

This is a homogeneous equation of the first order with variable coefficients, whose solution is (§ 173):

$$w(x,t_1) = w(0,t_1) \prod_{i=0}^{x} \frac{t_1}{1-(1+i)t_1}\,.$$

To determine $w(0,t_1)$ let us remark that starting from the initial condition it is easy to show that $f(O,y) = 0$ if $y > 0$; therefore

$$w(0,t_1) = f(0,0) + f(0,1)t_1 + f(0,2)t_1^2 + \ldots = 1$$

so that finally

$$w(x,t_1) = \frac{t_1^x}{(1-t_1)(1-2t_1)\ldots,(1-xt_1)}\,.$$

To expand this generating function into a series of powers of t_1 it is best to decompose it into partial fractions and then expand. This has been done in § 60, where we found the coefficient of t_1^y equal to

$$f(x,y) = \frac{(-1)^x}{x!} \sum_{i=0}^{x+1} (-1)^i \binom{x}{i} i^y = \mathfrak{S}_y^x.$$

Example 5. In § 144 we denoted by F(E) the frequency of ξ which was given for $\xi = 0, 1, 2, \ldots, \ldots, N\text{-}1$. There in a table, we put into the first line of every column the same number

F(JV-1) ; moreover into the first column we put $F(N-1)$, $F(N-2)$, $F(N-3)$, . . . , F(l), $F(0)$.

Denoting by $f(x,y)$ the number figuring in the line x and in the column y, we have

$$f(1,y) = F(N-1)$$
$$f(x,1) = F(N-x)$$

where x $\geqq$ 1 and y $\geqq$ 1. These are the initial conditions. The other numbers $f(x,y)$ of the table are computed by aid of the equation

$$(12) \qquad f(x+1,y+1) - f(x,y+1) - f(x+1,y) = 0.$$

Since in this equation there are two terms of $y+1$ and two of $x+1$, hence, according to § 181, two equations of condition are necessary to compute $f(x,y)$. The two equations given above are such.

To solve equation (12) we shall denote by $w(y,t)$ the generating function of $f(x,y)$ with respect to x; since in our problem $x > 0$ therefore

$$w(y,t) = f(1,y)t + f(2,y)t^2 + \ldots + f(x,y)t^x + \ldots$$

Starting from equation (12) we obtain

$$w(y+1,t) - \frac{1}{1-t}\, w(y,t) = 0.$$

The solution of this linear equation with constant coefficients is

$$w(y,t) = \frac{1}{(1-t)^y}\, \varphi(t)$$

where $\varphi(t)$ is an arbitrary function to be determined by aid of the initial conditions. Putting into this equation $y=1$ we find

$$\frac{\varphi(t)}{1-f} = w(1,t) = \sum_{\nu=1}^{\infty} f(\nu,1)\, t^{\nu} = \sum_{\nu=0}^{N+1} F(N-\nu)\, t^{\nu}$$

therefore

$$w(y,t) = \frac{1}{(1-t)^{y-1}} \sum_{\nu=1}^{N+1} F(N-\nu)\, t^{\nu}$$

and expanded into powers of t

$$w(y,t) = \sum_{\mu=0}^{\infty} \binom{y+\mu-2}{y-2} t^{\mu} \sum_{\nu=1}^{N+1} F(N-\nu)\, t^{\nu}$$

Finally putting $\nu+\mu=x$ we obtain the solution

$$f(x,y) = \sum_{\nu=1}^{x+1} \binom{y+x-\nu-2}{y-2} F(N-\nu).$$

This will give, in the particular case, the number of the line $x=N-y+2$ and of the column y:

$$f(N-y+2,y) = \sum_{\nu=1}^{N-y+1} \binom{N-\nu}{y-2} F(N-\nu).$$

Therefore this number is equal to the binomial moment of degree y-2, of the function $F(\xi)$:

$$f(N-y+2,y) = \mathscr{B}_{y-2}.$$

Computing the table mentioned above the required binomial moments are obtained all at the same time by simple additions: this is the shortest way, since no multiplications are necessary.

Example 6. Bernoulli's formula of the probability of repeated trials, Let the probability of an event be equal to p at each trial. The probability is required that the event shall occur x times in n trials. Denoting this-probability by $P(n,x)$ *we* obtain by aid of the theorems of compound and total probabilities

$$(1) \qquad P(n+1,x+1) = pP(n,x) + qP(n,x+1).$$

Indeed the probability that the event shall occur $x+1$ times in n+l trials is eqtial to the sum of the two probabilities: first, the probability that it shall happen x times in n trials, and moreover **also** at the n+l th trial; and secondly, the probability that the event shall occur $x+1$ times in n trials and fail at the last trial.

Equation (1) is a linear homogeneous equation of partial differences of the first order with respect to both variables. To solve it, since $n \geqq x \geqq 0$, according to § 181 one equation of initial conditions is sufficient, For instance, if $P(0,x)$ is given for every value of x; but $P(0,x)=0$ unless $x=0$ and then $P(O,O)=1$; indeed, if the number of trials is equal to zero, then the event can only occur zero times; hence the probability of it is equal to one.

If now we denote by $u(n,t)$ the generating function of $P(n,x)$ with respect to x, that is

$$\underset{x}{\mathbf{G}}\, P(n,x) = u(n,t)$$

then we have (p. 608)

$$\underset{x}{\mathbf{G}} P(n+1,x+1) = \frac{1}{t}\,[u(n+1,t) - P(n+1,0)]$$

and

$$\mathbf{G}P(n,x+1) = \frac{1}{t}\,[u(n,t) - P(n,0)]$$

hence from (1) it follows that

$$\text{(2)} \qquad u(n+1,t) - (pt+q)\,u(n,t) = P(n+1,0) - qP(n,0) = 0.$$

Indeed, from the theorem of compound probability it follows immediately that

$$P(n+1,0) - qP(n,0) = 0$$

hence the solution of equation (2) will be

$$u(n,t) = C(pt+q)^n.$$

In consequence of the initial condition $u(0,t)=1$, we have $C= 1$, and $P\,(n,x)$ will be equal to the coefficient of t^x in the expansion of $u(n,t)=(pt+q)^n$, that is $P(n,x) = \binom{n}{x} p^x q^{n-x}$.

§ 183. Boole's symbolical method for solving partial difference equations. This method is applicable *to partial difference equations in which one of the variables (e. g. the variable y) does not figure in an explicit manner:

$$\text{(1)} \qquad \psi(x,\underset{x}{\mathbf{E}},\underset{y}{\mathbf{E}})f(x,y) = 0.$$

We write k instead of the symbol $\mathbf{E}$, and considering it as a constant solve the equation (1) and f&d

$$f(x,y) = \psi_1(x,k)\,\varphi(y) = \psi_1(x,\underset{y}{\mathbf{E}})\,\varphi(y)$$

where $\varphi(y)$ is an arbitrary function of y.

Finally we obtain $f(x,y)$ by performing the operation $\psi_1(x,\underset{y}{\mathbf{E}})$ on the function $\varphi(y)$.

As in every symbolical method, the result obtained must be verified by putting it into equation (1).

The function $\varphi(y)$ will be determined in particular cases by aid of the initial conditions.

Example 1. Given the difference equation

$$\mathop{\mathrm{E}}_{x} f(x,y) - \mathop{\mathrm{E}}_{y} f(x,y) = 0$$

let us put $\mathop{\mathrm{E}}_{y} = k$ and have

$$\mathop{\mathrm{E}}_{x} f\,(x,y) = kf\,(x,y)$$

the solution of this homogeneous equation with constant coefficients is

$$f(x,y) = k^x \varphi(y) = \mathop{\mathrm{E}}_{y}^{x} \varphi(y) = \varphi(x+y).$$

Where $\varphi(y)$ is an arbitrary function. Since the equation contains only one term of $x+1$, so that to compute $f(x,y)$ for $x \geq 0$ only one equation of condition is necessary, and this is for instance $f(0,y)$ given for every integer value of y. Putting $x=0$ into the above equation, we get $f\,(0,y) = \varphi(y)$ and therefore

$$f(x,y) = f(0,y+x).$$

It is easy to verify that this result satisfies the given equation.

Example 2. Given the equation

$$\mathop{\mathrm{E}}_{x} \mathop{\mathrm{E}}_{y} f(x,y) - \mathop{\mathrm{E}}_{y} f(x,y) - f(x,y) = 0$$

putting $\mathop{\mathrm{E}}_{y} = k$ we have

$$k \mathop{\mathrm{E}}_{x} f(x,y) - (k+1) f\,(x,y) = 0$$

the solution of the equation with constant coefficients is

$$f(x,y) = \left(\frac{k+1}{k}\right)^x \varphi(y).$$

Finally the expansion gives

$$f(x,y) = \sum_{i=0}^{x+1} \binom{x}{i} \frac{1}{k^i} \varphi(y) = \sum_{i=0}^{x+1} \binom{x}{i} \varphi(y-i).$$

This result satisfies the given equation. In this case too, one initial condition is sufficient for the computation of any value of $f(x,y)$. For instance if $x \geqq 0$ and if $f(0,y)$ is given for every integer value of y, we find $f(0,y) = \varphi(y)$ and

$$f(x,y) = \sum_{i=0}^{x+1} \binom{x}{i} f(0,y-i).$$

Example 3. Given the equation

$$\underset{x}{\mathrm{E}}\,\underset{y}{\mathrm{E}}\, f(x,y) - xf(x,y) = 0$$

the transformation gives

$$k\,\underset{x}{\mathrm{E}}\, f(x,y) - xf(x,y) = 0.$$

The solution of this equation with variable coefficients is

$$f(x,y) = \prod_{i=1}^{x} \frac{i}{k}\, y_1(y) = (x-1)\,!\,\underset{y}{\mathrm{E}}^{-x+1}\, \varphi(y) = (x-1)\,!\,\varphi(y-x+1).$$

Since the given equation contains only one term of $x+1$, hence if $x \geqq 0$ one initial condition, for instance $f(1,y)$ given for every integer value of y, is sufficient for the computation of $f(x,y)$. We find $f(1,y) = \varphi(y)$ and

$$f(x,y) = (x-1)!\; f(1,y-x+1).$$

Example 4. The difference equation giving the numbers in *Pascal's* arithmetical triangle is

$$f(x+1,y+1) - f(x+1,y) - f(x,y) = 0$$

or written in the symbolical way

$$[\underset{x}{\mathrm{E}}\,\underset{y}{\mathrm{E}} - \underset{x}{\mathrm{E}} - 1]\, f(x,y) = 0$$

the transformation gives

$$(k-1)\,\underset{x}{\mathrm{E}}\, f(x,y) - f(x,y) = 0$$

whose solution is

$$f(x,y) = \left(\frac{1}{k-1}\right)^x \varphi(y) = \left(\frac{1}{\underset{y}{\Delta}}\right)^x \varphi(y).$$

The function y(y) is determined by aid of the initial values.

Since the given equation contains only one term of y-l-1 and y $\geqq$ 0 hence one initial condition $f(x,0) = 0$ if x $\neq 0$ and $f(0,0) = 1$ is sufficient for the computation of $f(x,y)$.

Putting into the above result $x=0$ we get

$$\varphi(y) = f(0,y).$$

From the initial condition it follows that $f(0,y) = 1$. Therefore $\varphi(y)=1$ and

$$f(x,y) = \underset{y}{\Delta}^{-x} 1 = \binom{y}{x} + F(y)$$

where $F(y)$ is a polynomial of y; but putting $x=0$ into this equation we find $F(y) = 0$. Finally

$$f(x,y) = \binom{y}{x}$$

that is, the numbers in the arithmetical triangle are the binomial coefficients.

§ 184. Method of Fourier, Lagrange and Ellis, for solving equations of partial differences, Given the linear homogeneous equation of partial differences

$$(1) \qquad \psi(\underset{x}{\mathrm{E}},\underset{y}{\mathrm{E}}) f(x,y) = 0$$

and likewise the necessary initial conditions corresponding to a problem, the method consists in determining a certain number of particular solutions of equation (1), multiplying them by arbitrary constants, forming the sum of the products obtained, and finally disposing of the arbitrary constants in such *a* manner that the initial conditions may be satisfied.[57]

The method was first applied in the case of partial differential equations; but there the difficulties were much greater; indeed, to satisfy the initial conditions, the number of arbitrary constants and therefore that of the particular solutions to be determined is infinite. On the other hand difference equations are generally valid only for a finite number of values of the variables, and therefore it will suffice to dispose of a finite number of arbitrary constants, in order to satisfy the initial conditions.

Of course, the number of these constants will be very great, especially in the case of three, four, or more variables; so that it

[57] *R. L. Ellis,* On the solution of equations in finite differencer. Cambridge Mathematical Journal, Vol. 4, p, 182, 1844; or in his Mathematical and other Writings, Cambridge, 1863, p. 202.

would be impossible to carry out the calculations in the usual way; this is only possible, as we shall see, by means of the orthogonal properties of certain trigonometric functions given in § 43.

Example 1. "Third Problem of *Play."* Two players have between them a number a of counters; they play a game in which the first player has a chance p of winning one counter from the second in each game. At the beginning, the first player has x counters. Required the probability that after a number y of games the first player shall have z counters, neither of the players having previously lost all his counters (ruin).

Let us denote this probability by $f(x,y,z)$. If the first player wins the next game, of which the probability is p, then the required probability becomes equal to $f(x+1,y-1,z)$. If he loses it, the probability of which is $1-p=q$, then it becomes equal to $f(x-1,y-1,z)$. Therefore, applying the theorem of total probabilities, we get

$$(2) \qquad f(x,y,z) = pf(x+1,y-1,z) + qf(x-1,y-1,z).$$

This is an equation of partial differences of two variables, which may be written

$$(3) \qquad pf(x+2,y,z) - f(x+1,y+1,z) + qf(x,y,z) = 0$$

where z is merely a parameter,

Since there is only one term of $y+1$ in this equation, and moreover since $y \geqq 0$, hence, according to § 181, one initial condition will be sufficient for computing the values of $f(x,y,z)$ by aid of equation (3). Such a function is, for instance, $f(x,0,z)$, given for every integer value of x from $-\infty$ to $+\infty$, We shall suppose that $0 < z < a$.

Obviously we have

$$(4) \qquad f(x,0,z) = 0 \text{ if } x \neq z \text{ and } f(z,0,z) = 1$$

this will give for $x = 1, 2, \ldots, a\text{-}l$ in all $a\text{-}l$ conditions.

Starting from (4) we may compute step by step any value of $f(x,y,z)$. There is but one difficulty; in our problem there are two supplementary conditions, viz. :

$$(5) \qquad f(0,y,z) = 0 \quad \text{and } f(a,y,z) = 0$$

that is, the probability of having z counters after y games is equal to zero if one of the players has already lost all his counters, since then the play is over.

But the values (5) are necessarily incompatible with those corresponding to the difference equation (2). To obviate this inconvenience, we will restrict the validity of (2) to the interval $0 < x < a$; then there can be no contradiction.

In the end we shall have a-l condition to satisfy, therefore the number of the arbitrary constants in the particular solution must also be equal to that number.

It is easy to find a particular solution of (3), by putting

$$f(x,y,z) = \alpha^y F(x)$$

and disposing of a in such a way that equation (3) shall be satisfied. From (3) it follows that:

$$(6) \qquad p F(x+2) - \alpha F(x+1) + q F(x) = 0.$$

If we write $\alpha = 2 \cos\varphi \sqrt{pq}$ then the roots of the characteristic equation corresponding to (6) will be

$$r = (\cos\varphi \pm i \sin\varphi) \left(\frac{q}{p}\right)^{1/2}.$$

In consequence of the sign ± this gives, according to § 165, two different solutions of (6), so that we obtain the general solution of (6)

$$F(x) = \left(\frac{q}{p}\right)^{x/2} [A_1 \cos\varphi x + A_2 \sin\varphi x]$$

and therefore a solution of (3) containing the three disposable parameters A_1, A_2, φ, will be

$$(7) \qquad f(x,y,z) = (4pq)^{y/2} \left(\frac{q}{p}\right)^{x/2} (\cos\varphi)^y [A_1 \cos\varphi x + A_2 \sin\varphi x].$$

To satisfy the condition $f(0,y,z) = 0$ we have to put in (7) $A_1 = 0$. Moreover, to satisfy $f(a,y,z) = 0$ we put $\varphi = \nu\pi/a$, where v is an integer such that $0 < \nu < a$. To each value of ν there corresponds one particular solution of (3) ; multiplying it by the constant C_ν and summing from $\nu = 0$ to $\nu = a$ we get a particular solution containing a-l arbitrary constants.

$$\text{(8)} \qquad f(x,y,z) = (4pq)^{y/2}\left(\frac{q}{p}\right)^{x/2}\sum_{\nu=0}^{a} C_\nu \sin\frac{\nu\pi x}{a}\left(\cos\frac{\nu\pi}{a}\right)^{y}.$$

(There is no objection to beginning the summation with $\nu=0$ instead of $\nu=1$, since the term corresponding to $\nu=0$ is equal to zero.)

The number of the conditions (4) still to be satisfied is also equal to a-l, therefore we may attain this result by disposing of the constants C_ν in the following manner:

Putting $y=0$ into equation (8); after division by $\left(\frac{q}{p}\right)^{x/2}$ we get for $x=1,2,\ldots,$ a-l the equations

$$\left(\frac{p}{4}\right)^{x/2} f(x,0,z) = \sum_{\nu=0}^{a} C_\nu \sin\frac{\nu\pi x}{a}.$$

Multiplying the first of these equations by $\sin\frac{\mu\pi}{a}$, the second by $\sin\frac{2\mu\pi}{a}$, . . , . the x th by $\sin\frac{\mu\pi x}{a}$ and adding up the results, we find

$$\sum_{x=1}^{a}\left(\frac{p}{q}\right)^{x/2} f(x,0,z)\sin\frac{\mu\pi x}{a} = \sum_{\nu=0}^{a} C_\nu \sin\frac{\nu\pi x}{a}\sin\frac{\mu\pi x}{a}.$$

The first member of this equation is equal, in consequence of (4), to

$$\left(\frac{p}{q}\right)^{z/2}\sin\frac{\mu\pi z}{a}$$

The sum in the second member, according to formula (9) § 43, is equal to zero if $\nu\neq\mu$ and to $\frac{1}{2}aC_\mu$ if $\nu=\mu$. Therefore we shall have

$$C_\mu = \frac{2}{a}\left(\frac{p}{q}\right)^{z/2}\sin\frac{\mu\pi z}{a}.$$

Finally the required probability obtained from (8) will be

$$\text{(9)} \qquad f(x,y,z) = \frac{2}{a}(4pq)^{y/2}\left(\frac{q}{p}\right)^{(x-z)/2}\sum_{\nu=0}^{a}\sin\frac{\nu\pi x}{a}\sin\frac{\nu\pi z}{a}\left(\cos\frac{\nu\pi}{a}\right)^{y}$$

This formula has been found by *Ellis* (loc. cit. 57, p. 210), it may be transformed in the following way; the product of sines is expressed by the corresponding difference of cosines, and the power of cosine by cosines of multiples.

Let us suppose first that y is odd; $y=2n-1$; then the sum in the second member may be written:

$$\sum_{\nu=0}^{a} \frac{1}{2^{2n-1}}\left[\cos\frac{x-z}{a}\nu\pi-\cos\frac{x+z}{a}\nu\pi\right]\sum_{i=1}^{n+1}\binom{2n-1}{n-i}\cos\frac{2i-1}{a}\nu\pi .$$

The products of cosines occurring in this formula are again expressed by sums of cosines. We find

$$\sum_{i=1}^{n+1}\frac{1}{2^{2n}}\sum_{\nu=0}^{a}\binom{2n-1}{n-i}\left[\cos\frac{2i-1+x-z}{a}\nu\pi+\cos\frac{2i-1-x+z}{a}\nu\pi-\right.$$

(10)

$$\left.-\cos\frac{2i-1+x+z}{a}\nu\pi-\cos\frac{2i-1-x-z}{a}\nu\pi\right]$$

In § 43 we have seen that

(11) $$\sum_{\nu=0}^{a}\cos\frac{\alpha\nu\pi}{a}=0$$

if a is an integer not divisible by a. Therefore the first term of **(10)** will be equal to zero unless we have

$$2i+x-z-1=\lambda a \quad \text{or} \quad i=\tfrac{1}{2}(\lambda a-x+z+1).$$

Since the number y of the games is odd, hence z-x is also odd; moreover, since i must be an integer, therefore ?.a must be even. If λ is odd, we have

(12) $$\sum_{\nu=0}^{a}\cos\lambda\nu\pi=1-(-1)^a.$$

But if 1 is odd, then a is necessarily even, and the above expression is equal to zero. Hence it is sufficient to consider the even values of λ. If λ is even, then

(13) $$\sum_{\nu=0}^{a}\cos\lambda\nu\pi=a.$$

If we put $\lambda=2\gamma$, then in the first term of **(10)** we shall have $i=a\gamma+\tfrac{1}{2}(z-x+1)$, and the sum of this term will be

$$a\sum\binom{2n-1}{n-a\gamma-\tfrac{1}{2}z+\tfrac{1}{2}x-\tfrac{1}{2}} .$$

Since we must have $i \geqq 1$, hence if $z > x$ then γ may be equal to $0, 1, 2, \dots$ and if $z < x$, then $\gamma = 1, 2, 3, \dots$,

In the second term we have $i=a\gamma+\frac{1}{2}(x-z+1)$, therefore it gives

$$a\sum \binom{2n-1}{n-a\gamma-\frac{1}{2}x+\frac{1}{2}z-\frac{1}{2}}.$$

If $x > z$, then in this sum we have $\gamma=0, 1, 2, \dots$, if not, then $\gamma=1, 2, 3, \dots$ Since

$$\binom{2n-1}{n-\frac{1}{2}z+\frac{1}{2}x-\frac{1}{2}} = \binom{2n-1}{n+\frac{1}{2}z-\frac{1}{2}x-\frac{1}{2}}$$

therefore we conclude that the first two terms of (10) will give

$$a\binom{2n-1}{n-\frac{1}{2}x+\frac{1}{2}z-\frac{1}{2}} +$$

$$+\ a\sum_{\gamma=1}\left[\binom{2n-1}{n-a\gamma-\frac{1}{2}z+\frac{1}{2}x-\frac{1}{2}}+\binom{2n-1}{n-a\gamma+\frac{1}{2}z-\frac{1}{2}x-\frac{1}{2}}\right].$$

In the third term of (10) we have $i=a\gamma-\frac{1}{2}(x+z-1)$, hence we get

$$-a\sum_{\gamma=1}\binom{2n-1}{n-a\gamma+\frac{1}{2}x+\frac{1}{2}z-\frac{1}{2}}$$

In the fourth term we have $i=a\gamma+\frac{1}{2}(x+z+1)$, therefore

$$-\ a\sum_{\gamma=0}\binom{2n-1}{n-a\gamma-\frac{1}{2}x-\frac{1}{2}z-\frac{1}{2}}.$$

Finally the required probability will be

$$(14)\qquad f(x,2n-1,z) = (pq)^{n-\frac{1}{2}}\left(\frac{q}{p}\right)^{(x-z)/2}$$

$$\left\{\sum_{\gamma=0}\left[\binom{2n-1}{n-\frac{1}{2}-a\gamma-\frac{1}{2}x+\frac{1}{2}z}-\binom{2n-1}{n-\frac{1}{2}-a\gamma-\frac{1}{2}x-\frac{1}{2}z}\right]+\right.$$

$$\left.+\sum_{\gamma=1}\left[\binom{2n-1}{n-\frac{1}{2}-a\gamma+\frac{1}{2}x-\frac{1}{2}z}-\binom{2n-1}{n-\frac{1}{2}-a\gamma+\frac{1}{2}x+\frac{1}{2}z}\right]\right\}.$$

In the same manner we could obtain the formula for $f(x,2n,z)$; but there is a shorter way; indeed, we have

$$f(x,2n,z) = pf(x+1,2n-1,z) + qf(x-1,2n-1,z).$$

By aid of (14) this will give

$$f(x,2n,z) = p(pq)^{n-1/2}\left(\frac{q}{p}\right)^{(x-z+1)/2}\left\{\Sigma \ldots .\right\} +$$

$$+\, q(pq)^{n-1/2}\left(\frac{q}{p}\right)^{(x-z-1)/2}\left\{\Sigma \ldots .\right\}.$$

Let us remark that both factors preceding the brackets are equal to

$$(pq)^n\left(\frac{q}{p}\right)^{(x-z)/2}.$$

Moreover, the two first terms under the Σ signs corresponding to x-l and x-l-l, are

$$\binom{2n-1}{n-1-a\gamma-\frac{1}{2}x+\frac{1}{2}z} + \binom{2n-1}{n-a\gamma-\frac{1}{2}x+\frac{1}{2}z}.$$

We have seen in § 22, formula (13) :

$$\binom{m}{u}+\binom{m}{u+1} = 2M\binom{m}{u} = \binom{m+1}{u+1}$$

therefore the sum of the two terms above will be equal to

$$\binom{2n}{n-a\gamma-\frac{1}{2}x+\frac{1}{2}z}.$$

Combining in the same way the other terms two by two, we find

$$(15) \qquad f(x,2n,z) = (pq)^n\left(\frac{q}{p}\right)^{x/2}.$$

$$\cdot\left\{\sum_{\gamma=0}\left[\binom{2n}{n-a\gamma-\frac{1}{2}x+\frac{1}{2}z}-\binom{2n}{n-a\gamma-\frac{1}{2}x-\frac{1}{2}z}\right]+\right.$$

$$\left.+\sum_{\gamma=1}\left[\binom{2n}{n-a\gamma+\frac{1}{2}x-\frac{1}{2}z}-\binom{2n}{n-a\gamma+\frac{1}{2}x+\frac{1}{2}z}\right]\right\}.$$

From (14) and (15) it follows that for y even or odd we have

$$f(x,y,z) = (pq)^{1/2y}\left(\frac{q}{p}\right)^{(x-z)/2}.$$

$$\cdot\left\{\sum_{\gamma=0}\left[\binom{y}{\frac{1}{2}y-\frac{1}{2}x+\frac{1}{2}z-a\gamma}-\binom{y}{\frac{1}{2}y-\frac{1}{2}x-\frac{1}{2}z-a\gamma}\right]+\right.$$

$$\left.+\sum_{\gamma=1}\left[\binom{y}{\frac{1}{2}y+\frac{1}{2}x-\frac{1}{2}z-a\gamma}-\binom{y}{\frac{1}{2}y+\frac{1}{2}x+\frac{1}{2}z-a\gamma}\right]\right\}.$$

The probability above is necessarily equal to zero if $y-z+x$ is odd; therefore, to simplify, we may put

$$2\omega = y - z + x.$$

We find

$$(16) \qquad f(x,y,z) = (pq)^{y/2}\left(\frac{q}{p}\right)^{(x-z)/2} \cdot$$

$$\cdot \left\{ \sum_{\gamma=0} \left[\binom{y}{\omega+z-x-a\gamma} - \binom{y}{\omega-x-a\gamma}\right] + \right.$$

$$\left. + \sum_{\gamma=1} \left[\binom{y}{\omega-a\gamma} - \binom{y}{\omega+z-a\gamma}\right]\right\}$$

This formula is advantageous for the computation of the required probability.

Particular Cases. *1.* If the two players have the same chance for winning, then $p=q=\frac{1}{2}$, and the term preceding &he sign Σ in formula (9) reduces to $2/a$; moreover the term preceding the brackets in (16) will be equal to $1/2^y$. In this case the formulae will be symmetrical with respect to x and z.

2. In the particular case of $x=z$, formulae (9) and (16) give the probability that the players have the same number of counters at the beginning and at the end of the play. The formulae are much simplified by this substitution.

3. An important particular case is that in which the number of counters the second player has, may be considered infinite; this occurs if he has more counters than the number y of games he will play, that is if $a-x>y$. But in this case we have to put into formula (16) $\gamma=0$; it will become

$$(17) \qquad f(x,y,z) = (pq)^{y/2}\left(\frac{q}{p}\right)^{(x-z)/2}\left[\binom{y}{\omega+z-x} - \binom{y}{\omega-x}\right].$$

This formula was found by *D. Arany.*[58]

We may obtain another formula for the required probability, by starting from (9) and putting into it $\varphi=\nu\pi/a$ and therefore $\Delta\varphi=\pi/a$. Now if a increases indefinitely, then the sum in (9) will become a definite integral:

[58] Considerations sur le Problème de la Durée du Jeu. Tôhoku Mathematical Journal, 1926, Vol. 30, p. 160. What in our notation is $f(x,y,z)$, is in *Arany*'s notation, in the case of $a=\infty$: $y^{x}_{z-x,x-z}$; and in the general case: $y^{x,a-x}_{z-x,x-z}$

(9') $$f(x,y,z) = 2(4pq)^{y/2}\left(\frac{q}{p}\right)^{(x-z)/2} \frac{1}{\pi}\int_0^{\pi} \sin x\varphi \, \sin z\varphi \, (\cos\varphi)^y \, d\varphi .$$

From (9') and (16) it follows that

(18) $$\frac{1}{\pi}\int_0^{\pi} \sin x\varphi \, \sin z\varphi \, (\cos\varphi)^y \, d\varphi =$$

$$= \frac{1}{2^{y+1}}\left[\binom{y}{\tfrac{1}{2}y-\tfrac{1}{2}x-\tfrac{1}{2}z} - \binom{y}{\tfrac{1}{2}y-\tfrac{1}{2}x+\tfrac{1}{2}z}\right].$$

In the case of $z=x$ this may be simplified as has been mentioned before.

Problem of ruin. Let us suppose now that play continues until one of the players has won all the counters. What is the probability that the first player shall lose his last counter at the y th game? This would be $f(x,y,0)$; but this number cannot be obtained by putting $z=0$ into formula (9) or into (16). Indeed, in establishing these formulae we supposed z to be different from zero. Nevertheless we may derive this probability from these formulae. First, by determining the probability that the first player has only one counter left after y-l games; this is equal to $f(x,y-1,1)$ and may be obtained by the formulae (9) or (16); and secondly by writing that he lost the y th game; the probability of this is equal to q, so that the required probability will be: $qf(x,y-1,1)$.

Putting y-l instead of y and $z=1$ into formula (9), we find after multiplication by 9

(19) $$f(x,y,0) =$$

$$= \frac{1}{a}(4pq)^{y/2}\left(\frac{q}{p}\right)^{x/2} \sum_{\nu=0}^{a} \sin\frac{\nu\pi}{a} \sin\frac{\nu\pi x}{a}\left(\cos\frac{\nu\pi}{a}\right)^{y-1} .$$

This formula has also been obtained by *Ellis,* but starting anew from the difference equation (2).

We may obtain $qf(x,y-1,1)$ also by aid of equation (16), putting into it y-l instead of y and $z=1$; after multiplication by 9 we find

$$f(x,y,0)=(pq)^{y/2}\left(\frac{q}{p}\right)^{x/2}\left\{\sum_{\gamma=0}\left[\binom{y-1}{\omega+1-x-a\gamma}-\binom{y-1}{\omega-x-a\gamma}\right]+\right.$$

(20)

$$\left.+\sum_{\gamma=1}\left[\binom{y-1}{\omega-a\gamma}-\binom{y-1}{\omega+1-a\gamma}\right]\right\}$$

where $2\omega=y+x-2.$

If moreover we put $a=\infty$, then the above formula will be

$$f(x,y,0)=(pq)^{y/2}\left(\frac{q}{p}\right)^{x/2}\left[\binom{y-1}{\omega+1-x}-\binom{y-1}{\omega-x}\right]$$

This may be simplified; indeed in § 22 we found formula (12) :

$$\Delta\binom{m}{u}=\binom{m}{u+1}-\binom{m}{u}=\binom{m}{u}\left(\frac{m-2u-1}{u+1}\right)$$

therefore the difference in the brackets is equal to

$$\frac{2x}{y-x}\binom{y-1}{\omega-x}$$

so that the required probability will be

$$f(x,y,0)=(pq)^{y/2}\left(\frac{q}{p}\right)^{x/2}\frac{2x}{y-x}\binom{y-1}{\omega-x}. \tag{21}$$

This formula is identical with that found by *Ampère.*[59]

We may obtain a corresponding formula by starting from (9) and putting $\varphi=\nu\pi/a$ and therefore $\Delta\varphi=\pi/a.$ *Now* if a increases indefinitely, then the sum in (9) will become a definite integral

$$f(x,y,0)=(4pq)^{y/2}\left(\frac{q}{p}\right)^{x/2}\frac{1}{\pi}\int_0^\pi \sin\varphi\,\sin x\varphi\,(\cos\varphi)^{y-1}\,d\varphi. \tag{22}$$

From (21) and (22) it follows if $p=\frac12$ that

$$\frac{1}{\pi}\int_0^\pi \sin\varphi\,\sin x\varphi\,(\cos\varphi)^{y-1}\,d\varphi=\frac{1}{2^y}\,\frac{x}{y-x}\binom{y-1}{\frac12 y-\frac12 x-1}.$$

Remark. Starting from (19) we may moreover deduce the probability that the first player shall be ruined during the games $1,2,3,\ldots,\omega$. This probability will be

[59] Considérations sur la Théorie Mathématique du Jeu. Lyon, 1802. In *Ampere's* notation $A(m+2p)$ corresponds to *our* $f(m,m+2p,0)$. *See* p. 9.

$$\sum_{y=1}^{\omega+1} f(x,y,0) = \frac{1}{a} 2\sqrt{pq} \left(\frac{q}{p}\right)^{x/2} \sum_{\nu=0}^{a} \sin\frac{\nu\pi x}{a} \sin\frac{\nu\pi}{a} \cdot$$
$$\cdot \sum_{y=1}^{\omega+1} \left(2\sqrt{pq}\cos\frac{\nu\pi}{a}\right)^{y-1} .$$

the second sum being equal to

$$\frac{1-\left(2\sqrt{pq}\cos\frac{\nu\pi}{a}\right)^{\omega}}{1-2\sqrt{pq}\cos\frac{\nu\pi}{a}} .$$

From this it follows the probability $\varphi(x)$ that the first player shall be ruined at all, by putting $\omega = \infty$. We find

(23)

$$\varphi(x) = \sum_{y=1}^{\infty} f(x,y,0) = \frac{1}{a} 2\sqrt{pq}\left(\frac{q}{p}\right)^{x/2} \sum_{\nu=0}^{a} \frac{\sin\frac{\nu\pi}{a}\sin\frac{\nu\pi x}{a}}{1-2\sqrt{pq}\cos\frac{\nu\pi}{a}}$$

This probability can be determined directly by aid of the difference equation

$$\varphi(x) = p\varphi(x+1) + q\varphi(x-1)$$

the initial values being equal to $\varphi(0)=1$ and $\varphi(a)=0$. According to § 165, the solution will be

$$\varphi(x) = c_1 + c_2\left(\frac{q}{p}\right)^x$$

and taking account of the initial conditions we find

(24)
$$\varphi(x) = \frac{q^x\,[p^{a-x}-q^{a-x}]}{p^a-q^a} .$$

Now from (23) and (24) we deduce the value of the definite sum

(25)
$$\sum_{\nu=0}^{a} \frac{\sin\frac{\nu\pi}{a}\sin\frac{\nu\pi x}{a}}{1-k\cos\frac{\nu\pi}{a}} = \frac{ak^{x-1}}{2^x}\left[\frac{p^{a-x}-q^{a-x}}{p^a-q^a}\right]$$

where we put $2\sqrt{pq} = k$. To have the definite sum, we must still determine p and q by aid of k. Starting from $2\sqrt{p(1-p)}=k$, we find $p=\frac{1}{2}(1-\sqrt{1-k^2})$ and $q=\frac{1}{2}(1+\sqrt{1-k^2})$ or vice versa; the formula being symmetrical with respect to p and q.

In the particular case, if a may be considered as infinite, we shall put into the first member of (25) $\varphi=\nu\pi/a$, then φ will become a continuous variable whose range is $0,\pi$. In the second member, if q has been chosen for the larger root, we have

$$\lim_{a=\infty}\left[\frac{p^{a-x}-q^{a-x}}{p^a-q^a}\right] = \frac{1}{q^x} = 2^x\left[\frac{1-\sqrt{1-k^2}}{k^2}\right]^x$$

Finally we shall have

$$(26)\qquad \int_0^{\pi}\frac{\sin\varphi\ \sin x\varphi}{1-k\cos\varphi}\,d\varphi = \frac{\pi}{k}\left[\frac{1-\sqrt{1-k^2}}{k}\right]^x .$$

The formula may be verified by aid of formula 12, Table 64, *in Bietens de Haan's* book quoted above.

Example 2. Problem of **Parcours.** A point is moving on the x axis; starting from x, it may advance one step to $x+1$ or it may go back one step to x-1, the probabilities of both events being the same, that is, equal to $\frac{1}{2}$. Having moved, it can again take one step in one of the two directions, under the same conditions. The probability is required that in n steps the point shall be at x_1 without having touched in its movement the points x-0 and $x=2a$.

If at the first move the point has advanced one step (probability $\frac{1}{2}$), then it has still to cover a distance of x_1-x-1 in *n-1* steps. If we denote the required probability by $\varphi(x,n)$, then the probability of both the above events is $\frac{1}{2}\varphi(x+1,n-1)$.

If at the first move the point has gone back to x-l (probability $\frac{1}{2}$), then it has to cover a distance of x_1-x+1 in n-l steps; the probability of both events is $\frac{1}{2}\varphi(x-1,n-1)$. Finally, according to the theorem of total probabilities φ *(x,n)* will be equal to the sum of the two probabilities mentioned.

$$\varphi(x,n) = \tfrac{1}{2}\varphi(x+1,n-1) + \tfrac{1}{2}\varphi(x-1,n-1).$$

This equation of partial differences is the same as that of Example 1.; moreover, the initial conditions are also the same, except that in (9) 2a is to be written instead of *a*, moreover n instead of y and $p=q=\frac{1}{2}$. We find

$$(27)\qquad \varphi(x,n)=\frac{1}{a}\sum_{\nu=1}^{2a}\sin\frac{\nu\pi x}{2a}\sin\frac{\nu\pi x_1}{2a}\left(\cos\frac{\nu\pi}{2a}\right)^n.$$

Let us consider the *particular* case in which $x=a$. Since $\sin\frac{1}{2}\nu\pi$ is equal to zero for the even values of v, hence we may put $\nu=2i+1$. Then $\sin\frac{1}{2}(2i+1)\pi=(-1)^i$. Introducing moreover the new variable $\xi=x-a$ (so that $\xi_1=x_1-a$), the above formula will give:

$$(28)\qquad F(\xi_1,n)=\frac{1}{a}\sum_{i=0}^{a}\cos\frac{2i+1}{2a}\pi\xi_1\left|\cos\frac{2i+1}{2a}\pi\right|^n.$$

This is the probability that the point starting from $\xi=0$ shall reach in n steps the point $\xi=\xi_1$, without having touched in its movement the point $\xi=\pm$ a.

The formula has been found by *Courant* and *Arany*[60] in different ways, but in the first paper there are certain mistakes.

From formula (16) we obtain by putting into it *2a* instead of *a*, moreover $x=a$ and $p=q=\frac{1}{2}$, $y=n$ and $z-a=\xi_1$

$$F(\xi_1,n)=\frac{1}{2^n}\left\{\sum_{\gamma=0}\left[\binom{n}{\omega+\xi_1-2a\gamma}-\binom{n}{\omega-a-2a\gamma}\right]+\right.$$

$$\left.+\sum_{\gamma=1}\left[\binom{n}{\omega-2a\gamma}-\binom{n}{\omega+\xi_1+a-2a\gamma}\right]\right\}$$

where $2\omega=n-\xi_1$. The formula may be still further simplified by writing

$$(29)\quad F(\xi_1,n)=\frac{1}{2^n}\left\{\binom{n}{\omega+\xi_1}+\sum_{i=1}(-1)^i\left[\binom{n}{\omega+\xi_1-ai}+\binom{n}{\omega-ai}\right]\right\}$$

If $a=\infty$, then the sum vanishes, and we have

$$(30)\qquad F(\xi_1,n)=\frac{1}{2^n}\binom{n}{\omega+\xi_1}=\frac{1}{2^n}\binom{n}{\omega}.$$

The problem may be solved directly in this case by aid of Combinatorial Analysis. The total number of ways possible in

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[60] *R. Courant*, Über partielle Differenzengleichungen, Atti del Congresso Matematico di Bologna, 1928, Vol. 3., p. 83.

D. Arany, Le Problème des Parcours, Tôhoku Mathematical Journal, Vol. 37, p. 17.

n steps is 2^n. To obtain the number of ways starting from $\xi=0$ and ending in ξ_1, let us remark that to reach the point ξ_1, in n steps, starting from $\xi=0$ the number of steps in positive direction must be equal to $\omega+\xi_1$ and in negative direction equal to ω. Since the number of steps is n, therefore

$$2\omega = n - \xi_1 .$$

Hence the number of ways ending in ξ_1 in n steps is equal to the number of permutations of n elements, among which there are $\omega+\xi_1$ elements equal to $+1$ and ω elements equal to -1. That is

$$\frac{n!}{(\omega+\xi_1)!(\omega)!} = \binom{n}{\omega}$$

and the required probability will be $P = (½)^n \binom{n}{\omega}$.

§ 185. Homogeneous linear equations of mixed differences. If such an equation $F(\underset{x}{\Delta}, \underset{y}{\mathrm{D}}, x, z) = 0$ is given; where z is an unknown function of **the two** variables x and y, and F is a polynomial with respect to the symbols Δ and D, moreover the variable y does not figure explicitly in F, then ***Boole*'s** symbolical method is the following: Instead of the symbol $\underset{y}{\mathrm{D}}$ we have to write k, and considering it as a constant, to **solve** the ordinary equation of differences

$$F(\underset{x}{\Delta}, k, x, z) = 0.$$

Supposing that its solution **is**:

$$z = \psi_1(x,k)\, C_1 + \psi_2(x,k)\, C_2 + \ldots\ldots$$

Now we put

$$k = \underset{y}{\mathrm{D}} \quad \text{and} \quad C_i = \varphi_i(y)$$

where the $\varphi_i(y)$ are arbitrary functions of **y. The equation**

$$z = \psi_1(x,\underset{y}{\mathrm{D}})\varphi_1(y) + \psi_2(x,\underset{y}{\mathrm{D}})\varphi_2(y) + \ldots\ldots$$

will give the function z.

Example 1. Given the equation

$$\underset{x}{\Delta} z - \underset{y}{\mathrm{D}} z = \underset{x}{\mathrm{E}} z - z - \underset{y}{\mathrm{D}} z = 0$$

writing $\underset{Y}{\mathsf{D}}=k$ we have

$$\underset{y}{\mathsf{E}}z-(1+k)z=0.$$

The solution of this equation is

$$z=(1+k)^xC=(1+\underset{Y}{\mathsf{D}})^x\varphi(y)$$

that is

$$z=\sum_{\nu=0}^{x+1}\binom{x}{\nu}\underset{y}{\mathsf{D}}^{\nu}\varphi(y). \qquad (1)$$

If for instance the initial conditions are for $x=0$ to have $z=y^n/n!$, then $\varphi(y)=y^n/n!$ and

$$z=\sum_{\nu=0}^{x+1}\overset{x}{0^{\nu}}\frac{y^{n-\nu}}{(n-\nu)!}.$$

As has been said; when solving difference equations by symbolical methods, it is always necessary to verify the results by putting the obtained functions into the given equation. In the case considered, it is easily seen that the general solution (1) satisfies the given equation.

§ 186. Difference equations of three independent variables. Sometimes it is possible to solve a linear homogeneous equation of differences of three independent variables by using the method of § *184* due to *Fourier, Lagrange* and *Ellis.*

Example. Problem of parcours in two dimensions. A mobile starting from the point of coordinates x,y may advance one step to the point $x+1,y$, or to the point $x,y+1$; it may go back to the point $x-1,y$ or to the point $x,y-1$. The probability of either of the four events is equal to $1/4$. Having moved, it may again take one step in one of the four directions, and so on. The probability is required that the mobile reaches in n steps the point of coordinates x_1,y_1, without touching the four lines,

$$x=0,\quad y=0,\quad x=2a,\quad y=2b.$$

The same ratiocination as that employed in the case of the problem in one dimension (§ 184) shows that this probability satisfies the following difference equation:

$$(1) \qquad f(x,y,n) = \frac{1}{4} f(x-1,y,n-1) + \frac{1}{4} f(x+1,y,n-1) + $$
$$+ \frac{1}{4} f(x,y-1,n-1) + \frac{1}{4} f(x,y+1,n-1).$$

The values of the function $f(x,y,n)$ may be computed step by step by aid of (1) starting from $f(x,y,0)$. In the problem considered this latter is known, Indeed, for every value of x,y except for x_1,y_1 we have

$$(2) \qquad f(x,y,0) = 0 \quad \text{and} \quad f(x_1,y_1,0) = 1.$$

Putting $n=1$ into (1) we obtain $f(x,y,1)$; and having determined in the same manner $f(x-1,y,1)$, $f(x+1,y,1)$, $f(x,y-1,1)$ and $f(x,y+1,1)$ we may proceed to the computation of $f(x,y,2)$ and so on; finally we should obtain any value of $f(x,y,n)$.

In this problem, as in that of § 185, there are also supplementary conditions to satisfy:

$$(3) \qquad \begin{array}{ll} f(0,y,n) = 0, & f(x,0,n) = 0 \\ f(2a,y,n) = 0, & f(x,2b,n) = 0. \end{array}$$

They are also necessarily incompatible with the results given by (1); to obviate this contradiction we are again obliged to restrict the validity of (1) to $0 < x < 2a$ and $0 < y < 2b$.

To determine a particular solution of (1), let us write:

$$(4) \qquad f(x,y,n) = \alpha^n \beta^x \gamma^y .$$

Now we dispose of a, β, γ so as to satisfy equation (1) ; putting the expression (4) into equation (1), we get after simplification:

$$(5) \qquad \gamma\beta^2 + (\gamma^2 - 4\alpha\gamma + 1)\beta + y = 0.$$

If we put

$$(6) \qquad \cos\varphi = -\frac{\gamma^2 - 4\alpha\gamma + 1}{2\gamma}$$

then from (5) it follows that

$$\beta = \cos\varphi \pm i \, \mathrm{sing7}$$

$(i = \sqrt{-1})$ and from (6)

$$\gamma^2 - 2(2\alpha - \cos\varphi)\gamma + 1 = 0;$$

putting into it

$$\cos\psi = 2a - \cos\varphi$$

we have

$$\gamma = \cos\psi \pm i \sin\psi$$

and moreover

$$a = \tfrac{1}{2}(\cos\varphi + \cos\psi).$$

Finally, by aid of these expressions, from (4) we get

$$f(x,y,n) = \frac{1}{2^n}(\cos\varphi+\cos\psi)^n(\cos\varphi x \pm i \sin\varphi x)(\text{cosyy} \pm i \text{ sinyy}).$$

In consequence of the signs $\pm$, this gives four different particular solutions with two arbitrary parameters φ and ψ. Multiplying them by arbitrary constants and summing, we shall have the following particular solution:

$$f(x,y,n) = \frac{1}{2^n}(\cos\varphi+\cos\psi)^n (A, \sin\varphi x \sin\psi y + A_2 \cos\varphi x \cos\psi y +$$

$$+ A, \sin\varphi x \cos\psi y + A, \cos\varphi x \text{ sinyy}).$$

To satisfy the first two of the conditions (3) that is $f(0,y,n)=0$ and $f(x,0,n)=0$ we have to put $A_2=A_3=A_4=0$. To satisfy the other two conditions (3) viz. $f(2a,y,n)=0$ and $f(x,2b,n)=0$, we put

$$\varphi = \frac{\nu\pi}{2a} \quad \text{and} \quad \psi = \frac{\mu\pi}{2b}$$

where v and μ are arbitrary positive integers.

Finally, writing $C_{\nu\mu}$ instead of A, and summing from $\nu=0$ to $\nu=2a$, and also from $\mu=0$ to $\mu=2b$, we obtain a solution of (1), which is sufficiently general for our problem, as will be seen.

(7) $$f(x,y,n) = \sum_{\nu=0}^{2a}\sum_{\mu=0}^{2b} C_{\nu\mu} (\$17 \cos\frac{\nu\pi}{2a} + \cos\frac{\mu\pi}{2b}\Big)^n \sin\frac{\nu\pi x}{2a}\sin\frac{\mu\pi y}{2b}$$

Putting $n=0$ into equation (7) we obtain for x $= 1, 2, \ldots,$ $2a-1$ and $y=1, 2, \ldots, 2b-1$ the equations:

$$f(x,y,0) = \sum_{\nu=0}^{2a}\sum_{\mu=0}^{2b} C_{\nu\mu} \sin\frac{\nu\pi x}{2a}\sin\frac{\mu\pi y}{2b}.$$

Let us first consider the equations corresponding to $x=1, 2, \ldots, 2a-1$. Multiplying the first equation by $\sin\frac{\nu_1\pi}{2a}$, the second by $\sin\frac{2\nu_1\pi}{2a}$ and so on, the x th by $\sin\frac{\nu_1\pi x}{2a}$, and summing up the results, we find:

$$\sum_{x=1}^{2a} f(x,y,0)\sin\frac{\nu_1\pi x}{2a} = \sum_{\nu=0}^{2a}\sum_{\mu=0}^{2b} C_{\nu\mu}\sin\frac{\mu\pi y}{2b}\sum_{x=1}^{2a}\sin\frac{\nu\pi x}{2a}\sin\frac{\nu_1\pi x}{2a}$$

We have seen in § 43 that the last sum is equal to zero if $\nu \neq \nu_1$ and to a if $\nu=\nu_1$. Therefore the second member will be

$$a\sum_{\mu=0}^{2b} C_{\nu_1\mu}\sin\frac{\mu\pi y}{2b}.$$

Now we consider the equations obtained for $y=1, 2, \ldots, 2b-1$, and multiply the first equation by $\sin\frac{\mu_1\pi}{2b}$, the second by $\sin\frac{2\mu_1\pi}{2b}$, and so on, the y th by $\sin\frac{\mu_1\pi y}{2b}$. Summing up the results obtained we have:

$$\sum_{x=1}^{2a}\sum_{y=1}^{2b} f(x,y,0)\sin\frac{\nu_1\pi x}{2a}\sin\frac{\mu_1\pi y}{2b} =$$

$$= a\sum_{\mu=0}^{2b} C_{\nu_1\mu}\sum_{y=1}^{2b}\sin\frac{\mu\pi y}{2b}\sin\frac{\mu_1\pi y}{2b}.$$

The last sum is equal to zero if $\mu \neq \mu_1$ and equal to b if $\mu=\mu_1$, Therefore the second member will be $ab\,C_{\nu_1\mu_1}$. In consequence of (2) the first member becomes equal to

$$\sin\frac{\nu_1\pi x_1}{2a}\sin\frac{\mu_1\pi y_1}{2b}.$$

Finally we shall have writing v and μ instead of ν_1 and μ_1:

$$C_{\nu\mu} = \frac{1}{ab}\sin\frac{\nu\pi x_1}{2a}\sin\frac{\mu\pi y_1}{2b}$$

This gives, by aid of (7), the required probability:

$$(8)\qquad f(x,y,n) = \sum_{\nu=0}^{2a}\sum_{\mu=0}^{2b}\frac{1}{2^n ab}\left(\cos\frac{\nu\pi}{2a}+\cos\frac{\mu\pi}{2b}\right)^n\cdot$$

$$\cdot\sin\frac{\nu\pi x}{2a}\sin\frac{\nu\pi x_1}{2a}\sin\frac{\mu\pi y}{2b}\sin\frac{\mu\pi y_1}{2b}.$$

Particular case. The mobile starts from $x=a$, $y=b$ (centre);

putting these values into (8), the terms $\sin \frac{\nu\pi}{2}$ and $\sin \frac{\mu\pi}{2}$ in which ν or μ is even, are equal to zero, so that we may write

$$\nu = 2i+1 \quad \text{and} \quad \mu = 2j+1.$$

Moreover, introducing the new variables $\xi = x - a$ and $\eta = y - b$ we obtain the probability that the mobile starting from $x = a$, $y = b$ that is from $\xi = 0$, $\eta = 0$ reaches the point $\xi_1 = x_1 - a$, $\eta_1 = y_1 - b$ in n steps:

$$(9) \quad P = \sum_{i=0}^{a} \sum_{j=0}^{b} \frac{1}{2^n ab} \left(\cos \frac{2i+1}{2a} \pi + \cos \frac{2j+1}{2b} \pi \right)^n \cdot$$

$$\cdot \cos \frac{2i+1}{2a} \pi\xi_1 \cos \frac{2j+1}{2b} \pi\eta_1 .$$

If a and b may be considered infinite, then the expression of the probability (9) is transformed into a double integral; putting $u = (2i+1)\pi/2a$ and $v = (2j+1)\pi/2b$ and therefore $\Delta u = \pi/a$ and $\Delta v = \pi/b$, we get

$$(10) \qquad P = \frac{1}{2^n \pi^2} \int_0^\pi \int_0^\pi (\cos u + \cos v)^n \cos u\xi_1 \cos v\eta_1 \, du \, dv.$$

But if $a = \infty$ and $b = \infty$ then the probability P may be obtained directly by aid of combinatorial analysis. The total number of ways possible in n steps is, under the conditions of the problem, equal to 4^n. To determine the number of ways starting from the point $\xi = 0, \eta = 0$ and ending in ξ_1, η_1 let us remark, that the number of steps taken in the positive direction of the ξ axis is equal to $i + \xi_1$ and that in the negative direction to i; moreover, the number of steps in the positive direction of the η axis is equal to $\omega - i + \eta_1$ and in the negative direction to $\omega - i$. Since the number of steps is equal to n, hence we have

$$2\omega = n - \xi_1 - \eta_1 .$$

Therefore the required number of ways ending in ξ_1, η_1 is given by the number of permutations of n elements, among which there are $i + \xi_1$ elements equal to a and i elements equal to $-a$, moreover $(\omega - i + \eta_1)$ elements equal to β and $(o-i)$ elements to

—β. **But this number of permutations** must still be summed for every value of i from $i=0$ to $i=\omega+1$. Hence we have

$$\sum_{i=0}^{\omega+1} \frac{n!}{(i+\xi_1)!\; i!\; (\omega-i+\eta_1)\,!\; (o\text{-}i)\;!}$$

This may be expressed by a product of binomial coefficients:

$$\binom{n}{\omega} \sum_{i=0}^{\omega+1} (5) \binom{n-\omega}{n-\omega-\xi_1-i}.$$

Applying ***Cauchy's*** formula (14) § 22, we find that the last sum is equal to

$$\binom{n}{n-\omega-\xi_1} = \binom{n}{\omega+\xi_1}.$$

Finally the required probability will be

$$P = \frac{1}{4^n}\binom{n}{\omega}\binom{n}{\omega+\xi_1}. \tag{11}$$

The formulae found by *Courant* and *Arany* (loc. cit. 60) are particular cases of formula (9).

Equating the quantities (10) and (11), we obtain an evaluation of the double integral (10).

§ 187. Difference equations of four independent variables. The method of § 184 is applicable in the case of four or more independent **variables.**

Example: Problem of Parcours in three dimensions. The mobile starts from the point x,y,z and may take one step parallel to the axes in one of the six directions, the probability of either direction being equal to 1/6. Having moved, it can take again one step in one of the six directions, and so on. The probability is required that the mobile reaches in n steps the point of coordinates x_1, y_1, z_1 without touching the six planes

$$x=0,\quad y=0,\quad z=0,\quad x=2a,\quad y=2b,\quad z=2c.$$

It may be shown, in the same way as in the case of one or two dimensions, that this probability satisfies the following equation:

$$(1) \quad f(x,y,z,n) = \frac{1}{6} f(x+1,y,z,n-1) + \frac{1}{6} f(x-1,y,z,n-1) + \frac{1}{6} f(x,y+1,z,n-1) + \frac{1}{6} f(x,y-1,z,n-1) + \frac{1}{6} f(x,y,z+1,n-1) + \frac{1}{6} f(x,y,z-1,n-1).$$

To compute $f(x,y,z,n)$ by aid of this equation, it is sufficient to know the values of the function $f(x,y,z,0)$. In the problem considered we know that this function is equal to zero for every value of x,y,z except x_1,y_1,z_1, so that we have:

$$(2) \qquad f(x,y,z,0) = 0 \quad \text{and} \quad f(x_1,y_1,z_1,0) = 1.$$

Here too we have supplementary conditions:

$$(3) \qquad \begin{array}{lll} f(0,y,z,n) = 0 & f(x,0,z,n) = 0 & f(x,y,0,n) = 0 \\ f(2a,y,z,n) = 0 & f(x,2b,z,n) = 0 & f(x,y,2c,n) = 0. \end{array}$$

Since the conditions (3) are necessarily incompatible with the general results given by equation (1), hence we must restrict the validity of this equation to

$$0 < x < 2a \qquad 0 < y < 2b \qquad 0 < z < 2c.$$

To determine a particular solution of (1), let us write

$$(4) \qquad f(x,y,z,n) = \alpha^n \beta^x \gamma^y \delta^z .$$

This gives by aid of (1), after simplification,

$$6\alpha\beta\gamma\delta = \beta^2\gamma\delta + \gamma\delta + \gamma^2\beta\delta + \beta\delta + \delta^2\beta\gamma + \beta\gamma$$

that is

$$\beta^2\gamma\delta + \beta(\gamma^2\delta+\delta^2\gamma+\gamma+\delta-6\alpha\gamma\delta)+\gamma\delta = 0$$

therefore, putting

$$(5) \qquad \cos\varphi = -\frac{\gamma^2\delta+\delta^2\gamma+\gamma+\delta-6\alpha\gamma\delta}{2\gamma\delta}$$

so that we obtain

$$\beta = \cos\varphi \pm i \sin\varphi$$

$(i = \sqrt{-1})$ from (5) it follows that

$$\gamma^2\delta+\gamma(2\delta\ \cos\varphi+\delta^2+1-6a\delta)+\delta=0.$$

Putting again

$$\text{(6)}\qquad \cos\psi=-\frac{2\delta\cos\varphi+\delta^2+1-6a\delta}{26}$$

we find

$$\gamma=\cos\psi\pm i\ \text{siny}$$

and from *(6)*

$$\delta^2+2\delta\ (\cos\varphi+\cos\psi-3a)+1=0.$$

Putting

$$\cos\chi=-(\cos\varphi+\cos\psi-3a)$$

we have

$$6=\cos\chi\pm i\sin\chi$$

and

$$a=\frac{\cos\varphi+\cos\psi+\cos\chi}{3}$$

The above values give by aid of (4), the solution

$$f(x,y,z,n)=\frac{1}{3^n}(\cos\varphi+\cos\psi+\cos\chi)^n(\cos x\varphi\pm i\sin x\varphi)\,.$$
$$.\,(\text{cosyy}\pm i\sin y\psi)(\cos z\chi\pm i\sin z\chi)\,.$$

In consequence of the signs $\pm$, this gives 8 different particular solutions; multiplying them by arbitrary constants and summing, we obtain a more general solution:

$$\text{f}\ (x,y,z,n)=\frac{1}{3^n}(\cos\varphi+\cos\psi+\cos\chi)^n\,[A,\ \sin\varphi x\ \text{sinyy}\ \sin\chi z+$$

$$+\ A,\ \sin\varphi x\ \text{sinyy}\ \cos\chi z+\ldots,+A,\ \cos\varphi x\ \text{cosyy}\ \cos\chi z].$$

To satisfy the first conditions (3) $f(0,y,z,n)=0, f(x,0,z,n)=0$ and $f(x,y,0,n)=0$ for every value, we must put $A_2=A_2=\ldots=$ $=A_8=0.$

To satisfy the second conditions (3), $f(2a,y,z,n)=0$, $f(x,2b,z,n)=0$ and $f(x,y,2c,n)=0$ we put

$$\varphi=\frac{\nu\pi}{2a}\qquad \psi=\frac{\mu\pi}{2b}\qquad \chi=\frac{\lambda\pi}{2c}\,.$$

ν,μ,λ being arbitrary positive integers.

Writing moreover $C_{\nu\mu\lambda}$ instead of *A*,, and and summing from

$\nu=0$ to $\nu=2a$, from $\mu=0$ to $\mu=2b$ and $\lambda=0$ to $\lambda=2c$ we obtain a sufficiently general solution for our problem:

$$(7) \qquad f(x,y,z,n) = \sum_{\nu=0}^{2a} \sum_{\mu=0}^{2b} \sum_{\lambda=0}^{2c} \frac{1}{3^n} C_{\nu\mu\lambda} \cdot$$

$$\cdot \left(\cos\frac{\nu\pi}{2a} + \cos\frac{\mu\pi}{2b} + \cos\frac{\lambda\pi}{2c} \right)^n \sin\frac{\nu\pi x}{2a} \sin\frac{\mu\pi y}{26} \sin\frac{\lambda\pi z}{2c}.$$

Putting into this equation $n=0$ we find

$$(8) \quad f(x,y,z,0) = \sum_{\nu=0}^{2a} \sum_{\mu=0}^{2b} \sum_{\lambda=0}^{2c} C_{\nu\mu\lambda} \sin\frac{\nu\pi x}{2a} \sin\frac{\mu\pi y}{2b} \sin\frac{\lambda\pi z}{2c}.$$

To begin with, let us consider these equations corresponding to $x=1, 2, \ldots, 2a-1$, and multiply the first by sin $\frac{\nu_1\pi}{2a}$, the second by sin $\frac{2\nu_1\pi}{2a}$, and so on, the x th by sin $\frac{\nu_1\pi x}{2a}$.; adding up the results we have

$$\sum_{x=1}^{2a} f(x,y,z,0) \sin\frac{\nu_1\pi x}{2a} = \sum_{\nu=0}^{2a} \sum_{\mu=0}^{2b} \sum_{\lambda=0}^{2c} C_{\nu\mu\lambda} \cdot$$

$$\cdot \sin\frac{\mu\pi y}{2b} \sin\frac{\lambda\pi z}{2c} \sum_{x=1}^{2a} \sin\frac{\nu\pi x}{2a} \sin\frac{\nu_1\pi x}{2a}.$$

The last sum is equal to zero if $\nu \neq \nu_1$ and equal to a if $\nu=\nu_1$. Therefore we have

$$\sum_{x=1}^{2a} f(x,y,z,0) \sin\frac{\nu_1\pi x}{2a} = a \sum_{\mu=0}^{2b} \sum_{\lambda=0}^{2c} C_{\nu_1\mu\lambda} \sin\frac{\mu\pi y}{2b} \sin\frac{\lambda\pi z}{2c}.$$

We will now consider these equations for $y=1, 2, \ldots, 2b-1$, and multiply the first equation by sin $\frac{\mu_1\pi}{26}$, the second by sin $\frac{2\mu_1\pi}{2b}$, and so on the y th by sin $\frac{\mu_1\pi y}{2b}$, finally adding up the results we get

$$\sum_{x=1}^{2a} \sum_{y=1}^{2b} f(x,y,z,0) \sin\frac{\nu_1\pi x}{2a} \sin\frac{\mu_1\pi y}{2b} =$$

$$= a \sum_{\mu=0}^{2b} \sum_{\lambda=0}^{2c} c_{\nu_1\mu\lambda} \sin\frac{\lambda\pi z}{2c} \sum_{y=1}^{2b} \sin\frac{\mu\pi y}{2b} \sin\frac{\mu_1\pi y}{2b}.$$

The last sum is equal to zero if $\mu \neq \mu_1$ and equal to b if $\mu = \mu_1$. We have

$$\sum_{x=1}^{2a} \sum_{y=1}^{2b} f(x,y,z,0) \sin \frac{\nu_1 \pi x}{2a} \sin \frac{\mu_1 \pi y}{2b} = ab \sum_{\lambda=0}^{2c} C_{\nu_1 \mu_1 \lambda} \sin \frac{\lambda \pi z}{2c}.$$

Considering these equations for $z=1, 2, \ldots, 2c-1$ and multiplying the first by $\sin \frac{\lambda_1 \pi}{2c}$, the second by $\sin \frac{2\lambda_1 \pi}{2c}$, the z th by $\sin \frac{\lambda_1 \pi z}{2c}$, moreover adding up the results, we find

$$\sum_{x=1}^{2a} \sum_{y=1}^{2b} \sum_{z=1}^{2c} f(x,y,z,0) \sin \frac{\nu_1 \pi x}{2a} \sin \frac{\mu_1 \pi y}{2b} \sin \frac{\lambda_1 \pi z}{2c} =$$

$$= ab \sum_{\lambda=0}^{2c} C_{\nu_1 \mu_1 \lambda} \sin \frac{\lambda \pi z}{2c} \sin \frac{\lambda_1 \pi z}{2c}.$$

The last sum is equal to zero if $\lambda \neq \lambda_1$ and equal to c if $\lambda = \lambda_1$. Therefore the second member is equal to $abc\, C_{\nu_1 \mu_1 \lambda_1}$. *The* first member is according to (2) equal to

$$\sin \frac{\nu_1 \pi x_1}{2a} \sin \frac{\mu_1 \pi y_1}{2b} \sin \frac{\lambda_1 \pi z_1}{2c}.$$

Therefore writing ν, μ, λ instead of ν_1, μ_1, λ_1 we get

$$C_{\nu \mu \lambda} = \frac{1}{abc} \sin \frac{\nu \pi x_1}{2a} \sin \frac{\mu \pi y_1}{2b} \sin \frac{\lambda \pi z_1}{2c}. \tag{9}$$

Finally the required. probability will be:

$$f(x,y,z,n) = \sum_{\nu=0}^{2a} \sum_{\mu=0}^{2b} \sum_{\lambda=0}^{2c} \frac{1}{3^n abc} \left(\cos \frac{\nu \pi}{2a} + \cos \frac{\mu \pi}{2b} + \cos \frac{\lambda \pi}{2c} \right)^n.$$

(10)

$$. \sin \frac{\nu \pi x}{2a} \sin \frac{\mu \pi y}{2b} \sin \frac{\lambda \pi z}{2c} \sin \frac{\nu \pi x_1}{2a} \sin \frac{\mu \pi y_1}{2b} \sin \frac{\lambda \pi z_1}{2c}.$$

Particular case of $x=a$, $y=b$, and $z=c$. Since in this case $\sin \frac{1}{2}\nu\pi$, $\sin \frac{1}{2}\mu\pi$ and $\sin \frac{1}{2}\lambda\pi$ are equal to zero if ν, μ or λ are even, therefore we shall consider only the odd values, and put $\nu = 2i+1$, $\mu = 2j+1$ and $\lambda = 2k+1$. Moreover, introducing the new variables $\xi = x-a$, $\eta = y-b$ and $\zeta = z-c$ we find

$$P = \sum_{i=0} \sum_{j=0} \sum_{k=0} \frac{1}{3^n\, abc} \left(\cos \frac{2i+1}{2a}\pi + \cos \frac{2j+1}{2b}\pi + \cos \frac{2k+1}{2c}\pi \right)^n \cdot \tag{11}$$

$$\cdot \cos \frac{2i+1}{2a} \pi\xi_1 \cos \frac{2j+1}{2b} \pi\eta_1 \cos \frac{2k+1}{2c} \pi\zeta_1$$

where $\xi_1 = x_1 - a$, $\eta_1 = y_1 - b$ and $\zeta_1 = z_1 - c$.

If a, b, and c increase indefinitely, this becomes a triple integral. But on the other hand we may then determine this probability by combinatorial analysis.

The total number of ways possible in n steps is 6^n. To determine the number of ways starting from $\xi=0$, $\eta=0$ and $\zeta=0$ and ending in ξ_1, η_1, ζ_1 let us remark that the number of steps taken in the positive direction of the ξ axis must be equal to $i+\xi_1$ and in the negative direction to i, where i may vary from zero to every possible value, Moreover the number of steps in the positive direction of the η axis must be equal to $s+\eta_1$ and in the negative direction to s, where s varies also from zero to every compatible value. Finally the number of steps in the positive direction of the ζ axis is $\omega - i - s + \zeta_1$ and in the negative direction $\omega - i - s$. Since the sum of the steps is equal to n we have

$$2\omega = n - \xi_1 - \eta_1 - \zeta_1 .$$

Hence the required number of ways is given by the number of permutations of n elements, among which there are $i+\xi_1$ elements equal to α, and i elements equal to $-\alpha$ then $s+\eta_1$ elements equal to β and s elements equal to $-\beta$ moreover $\omega - i - s + \xi_1$ elements equal to γ and $\omega - i - s$ elements equal to $-\gamma$. But this number of permutations must still be summed for every possible value of i and s. Therefore we have

$$\sum_{i=0}^{\omega+1} \sum_{s=0}^{\omega-i+1} \frac{n!}{i!\,(i+\xi_1)!\; s!\,(s+\eta_1)!\,(\omega-i-s+\zeta_1)!\,(\omega-i-s)!}$$

This may be written in the form of a sum of a product of binomial coefficients

$$\binom{n}{\omega} \sum_{i=0}^{\omega+1} \binom{\omega}{i} \binom{n-\omega}{i+\xi_1} \sum_{s=0}^{\omega-i+1} \binom{\omega-i}{s} \binom{n-\omega-\xi_1-i}{n-\omega-\xi_1-\eta_1-i-s} .$$

The second sum is, in consequence of *Couchy's* formula (14) § 22, equal to

$$\binom{n-\xi_1-2i}{n-\omega-\xi_1-\eta_1-i} = \binom{n-\xi_1-2i}{\omega+\eta_1-i}.$$

Therefore the required probability will be

$$(12) \qquad P = \frac{1}{6^n}\binom{n}{\omega}\sum_{i=0}^{\omega+1}\binom{\omega}{i}\binom{n-\omega}{i+\xi_1}\binom{n-\xi_1-2i}{\omega+\eta_1-i}.^{*}$$

Remark. Starting from formula (11) we may easily derive the formula corresponding to the problem in two dimensions and also that of the problem in one dimension,

First we have to put into (11) $\zeta_1=0$ and then $c=1$; thus we obtain the probability that a mobile starting from $\xi=0$, $\eta=0$, $\zeta=0$ reaches the point ξ_1, η_1, 0 without touching the planes $\xi=$ +a, $\eta= \pm b$ and $\zeta= \pm 1$. But then the corresponding ways will all be in the plane $\zeta=0$. This is therefore the solution of the problem in two dimensions. Only the total number of ways will now be 4” instead of 6^n, hence we have still to multiply the probability (11) by $\left(\frac{6}{4}\right)^n$. Then it will become identical with formula (9) of § 186.

In the same manner, starting from this formula we may solve the problem in one dimension, putting first $\eta_1=0$ and then $b=1$. The corresponding ways will all be on the ξ axis. The total number of ways will now be 2” instead of 4^n, hence we have still to multiply the probability (9) § 186 by $\left(\frac{4}{2}\right)^n$. The formula obtained will be identical with formula (25) § 184.

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* *Courant*'s formula (loc. cit. 60) is erroneous. The formula given by *Army.* without demonstration, is another form of formula (12).

GENERAL INDEX

The numbers refer to pages.

QUOTATIONS