# HARMONIC ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

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# FOREWORD

These lecture notes are based on a course I gave first at University of Texas, Austin during the academic year 1983 - 1984 and at University of Göteborg in the fall of 1984. My purpose in those lectures was to present some of the required background in order to present the recent results on the solvability of boundary value problems in domains with “bad” boundaries. These notes concentrate on the boundary value problems for the Laplace operator; for a complete survey of results, we refer to the survey article by Carlos Kenig; I am very grateful for this kind permission to include it here. It is also my pleasure to acknowledge my gratitude to Peter Kumlin for excellent work in preparing these notes for publication.

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# Contents

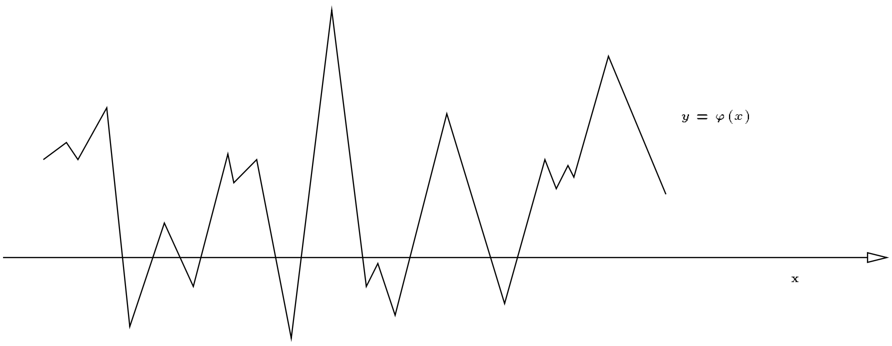
# Chapter 0

# Introduction

In this course we will study boundary value problems (BVP:s) for linear elliptic PDE:s with constant coefficients in Lipschitz-domains $\Omega$, i.e., domains where the boundary $\partial\Omega$ locally is given by the graph of Lipschitz function. We recall that a function $\varphi$ is Lipschitz if there exists a constant $M < \infty$ such that

$$|\varphi(x) - \varphi(z)| \leq M|x - z|$$

for all $x$ and $z$.




To solve the BVP:s we will reformulate the problems in terms of integral equations. It therefore becomes necessary to study singular integral operators of Calderón-Zygmund type, which we prove to be $L^p$-bounded for $1 < p < \infty$ and invertible. The $L^p$-boundedness is a consequence of the $L^p$-boundedness of the *Cauchy integral* (Coifman, McIntosh and Meyer)

$$Tf(z) = \int_\Gamma \frac{f(w)}{w - z} dw$$

where $\Gamma$ is a Lipschitz-curve (method of rotation). The invertability will be proved by a new set of ideas recently developed by Dahlberg, Kenig and Verchota. Among the BVP:s which can be solved by this technique are the

Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases}$$

Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = f & \text{on} \quad \partial\Omega \end{cases}$$

the clamped plate problem

$$\begin{cases} \Delta\Delta u = 0 & \text{in} \quad \Omega \\ u = f & \\ \frac{\partial u}{\partial n} = g & \text{on} \quad \partial\Omega \end{cases}$$

and BVP's for systems e.g. the elasticity problem

$$\begin{cases} \Delta u + \nabla \text{div } u = 0 & \text{in} \quad \Omega \\ (\nabla u + \nabla u^T) n = g & \text{on} \quad \partial\Omega \end{cases}$$

and Stoke's equation

$$\begin{cases} \Delta u = \nabla p & \text{in} \quad \Omega \\ \text{div } u = 0 & \\ u = f & \text{on} \quad \partial\Omega \end{cases}$$

where $u = (u_1, u_2, u_3)$ in $\mathbb{R}^3$.

## Fredholm theory for Dirichlet problem for domain $\Omega$ with $C^2$ boundary

We start with an example.

*Example (Dirichlet problem for a halfspace).* If the function $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, it is well known that

$$u(x, y) = p_y * f(x), \quad (x, y) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+,$$

where

$$P_y(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{y}{\left(|x|^2 + y^2\right)^{\frac{n+1}{2}}}$$

denotes the Poisson kernel, is a solution of

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^{n+1}_+ \\ u = f & \text{on} \quad \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \end{cases}$$

and that

$$(*) \qquad \sup_{y>0} \|u(\cdot, y)\|_p \le \|f\|_p.$$

Thus with $X = L^p(\mathbb{R}^n)$ and $Y = \{u : u \text{ harmonic in } \mathbb{R}^{n+1}_+ \text{ and } u \text{ satisfies } (*)\}$ we have the implication

$$f \in X \Rightarrow u \in Y.$$

However, we can also reverse the implication since a harmonic function $u$ which satisfies $(*)$ has non-tangential limits a.e. on $\partial\mathbb{R}^{n+1}_+$, the limit-function $u_0 = u(\cdot, 0) \in L^p(\mathbb{R}^n)$ and $u(x, y) = p_y * u_0(x)$.

*Sketch of a proof.* Assume $u$ harmonic function in $\mathbb{R}^{n+1}_+$ that satisfies $(*)$. The semigroup properties of $\{p_y\}_{y\ge0}$ implies

$$u(x, y + \rho) = p_y * u_\rho(x), \quad \rho > 0,\ y > 0$$

where $u_\rho(x) = u(x, \rho)$.

$$\begin{aligned} (*) &\Rightarrow u_{\rho_n} \rightharpoonup v \text{ in } L^p(\mathbb{R}^n) && \text{as } \rho_n \downarrow 0 \\ &\Rightarrow p_y * u_{\rho_n}(x) \to p_y * v(x) && \text{as } \rho_n \downarrow 0,\ y > 0. \end{aligned}$$

But $p_y * u_{\rho_n}(x) = u(x, y + \rho_n)$ and thus

$$u(x, y) = p_y * v(x) \quad \text{where} \quad v \in L^p(\mathbb{R}^n).$$

For the proof of the existence of non-tangential limits of $p_y * u_0$ we refer to e.g. Stein/Weiss [2]. $\square$

The notion of "solution of the Dirichlet problem" and any other problem, is sound only if we have such a matching between the boundary value $f$ of $u$ and the solution $u$ itself, i.e., we should not accept concepts of solution which are so weak such that the reversed implication is "impossible".

Now assume that $\Omega$ is a bounded (connected) domain in $\mathbb{R}^n$, $n \ge 3$ with $C^2$ boundary. (To avoid technicalities, we have assumed $n \ne 2$). Consider the Dirichlet problem

$$(\mathrm{D}) \qquad \begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = f \in C(\partial\Omega) \end{cases}$$

Let $r$ denote $(-1) \cdot$ (the fundamental solution) of the Laplace operator in $\mathbb{R}^n$, that is,

$$r(x) = c_n \frac{1}{|x|^{n-2}}, \quad c_n = -\frac{1}{(2-n)\omega_n} \equiv -\frac{1}{2-n} \cdot \frac{\Gamma(n/2)}{2\pi^{n/2}}$$

and set

$$R(x,y) = r(x-y).$$

For $f \in C(\partial\Omega)$ we define

$$\mathcal{D}f(P) = \int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P,Q) f(Q) d\sigma(Q) \qquad\qquad P \notin \partial\Omega$$

$$Sf(P) = \int_{\partial\Omega} R(P,Q) f(Q) d\sigma(Q) \qquad\qquad P \notin \partial\Omega$$

Thus $\mathcal{D}f$ and $Sf$ denote the double layer potential and single layer potential resp. Here $d\sigma$ is the surface measure on $\partial\Omega$ and $\frac{\partial}{\partial n_Q}$ is the directional derivative along the unit outward normal for $\partial\Omega$ at $Q$. It is immediate that

$$\Delta \mathcal{D}f(P) = 0, \quad P \in \mathbb{R}^n \setminus \partial\Omega$$

and $\mathcal{D}f$ will be our candidate for solution of (D). It remains to study the behaviour of $\mathcal{D}f$ at $\partial\Omega$.
Part of that story is

**Lemma 1.** *If $f \in C(\partial\Omega)$, then*

1) $\mathcal{D}f \in C(\bar{\Omega})$
2) $\mathcal{D}f \in C(\overline{\complement\Omega})$.

More precisely: $\mathcal{D}f$ can be extended as a continuous function from inside $\Omega$ to $\bar{\Omega}$ and from outside $\Omega$ to $\overline{\complement\Omega}$. Let $\mathcal{D}_+f$ and $\mathcal{D}_-f$ denote the restrictions of these functions to $\partial\Omega$ resp.

Set $K(P,Q) = \frac{\partial}{\partial n_Q} R(P,Q)$ for $P \neq Q$, $P,Q \in \partial\Omega$. We note that

i) $K \in C(\partial\Omega \times \partial\Omega \setminus \{(P,P) : P \in \partial\Omega\})$
ii) $|K(P,Q)| \leq \frac{C}{|P-Q|^{n-2}}$ for $P,Q \in \partial\Omega$ and some $C < \infty$.

ii) is a consequence of the regularity of the boundary and can be seen as follows:

Assume $\partial\Omega$ given by the graph of the $C^2$-function $\varphi$. Set $P = (x, \varphi(x)$ and $Q = (y, \varphi(y))$.

Then $K(P,Q) = \dfrac{1}{\omega_n} \dfrac{\langle P-Q, n_Q\rangle}{|P-Q|^n}$ where $\langle , \rangle$ is the inner product in $\ell^2(\mathbb{R}^n)$ and

$$n_Q = \frac{(\nabla\varphi(y), -1)}{\sqrt{|\nabla\varphi(y)|^2 + 1}}.$$

Since $\varphi$ is a $C^2$ function, we have that

$$\varphi(x) = \varphi(y) + \langle x - y, \nabla\varphi(y)\rangle + e(x,y) \quad \text{where} \quad |e(x,y)| = \mathcal{O}(|x-y|^2).$$

Hence

$$|K(P,Q)| \leq C\frac{|\langle P-Q, (\nabla\varphi(y), -1)\rangle|}{|P-Q|^n} \leq C\frac{|e(x,y)|}{|P-Q|^n} \leq \frac{C}{|P-Q|^{n-2}}.$$

This estimate is uniform in $P$ and $Q$ since $\partial\Omega$ compact.

For $f \in C(\partial\Omega)$ define

$$Tf(P) = \int_{\partial\Omega} K(P,Q)f(Q)d\sigma(Q), \quad P \in \partial\Omega.$$

We can now formulate

**Lemma 2 (jump relation for $\mathcal{D}$).**

1) $\mathcal{D}_+ = \frac{1}{2}I + T$
2) $\mathcal{D}_- = -\frac{1}{2}I + T$

and

**Lemma 3.** *$T : C(\partial\Omega) \to C(\partial\Omega)$ is compact.*

*Sketch of proof of Lemma 3.* Define the operators $T_n$ by

$$T_n f(P) = \int_{\partial\Omega} K_n(P,Q)f(Q)d\sigma(Q), \quad P \in \partial\Omega$$

for $f \in C(\partial\Omega)$, where

$$K_n(P,Q) = \text{sign}\ (K(P,Q)) \cdot \min(n, |K(P,Q)|), \quad n \in Z_+.$$

Thus $K_n$ is continuous on $\partial\Omega \times \partial\Omega$ and Arzela-Ascoli's theorem implies that $T_n$ is a compact operator on $C(\partial\Omega)$. Furthermore since $\|T_n\| \leq \sup_{Q\in\partial\Omega} \|K_n(\cdot, Q)\|_1 \leq C < \infty$, where $C$ is independent of $n$ we see that

$$T_n \to T$$

in the space $B = \{\text{bounded linear operators on } C(\partial\Omega)\}$. But the compact operators in $B$ form a closed subspace in $B$, and hence $T$ is compact. $\square$

*Proof of Lemma 1 and 2.* Some basic facts:

1) $\displaystyle\int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P,Q)d\sigma(Q) = 1, \quad \text{if } P \in \Omega$

   Proof: Apply Green's formula to the harmonic function $R(\Gamma, Q)$ in $\Omega \setminus B_\delta(P)$ for $\delta > 0$ small, where $B_\delta(P) = \{x \in \mathbb{R}^n : |P - x| \leq \delta\}$.

2) $\displaystyle\int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P,Q)d\sigma(Q) = 0, \quad \text{if } P \notin \bar{\Omega}.$

   Proof: Exercise.

3) $\displaystyle\int_{\partial\Omega} K(P,Q)d\sigma(Q) = \frac{1}{2}, \quad \text{if } P \in \partial\Omega$

   Proof: Exercise.

   Let $P \in \partial\Omega$. We want to show that

$$\mathcal{D}f(Q) \to \frac{1}{2}f(P) + Tf(P) \quad \text{as} \quad \Omega \ni Q \to P$$

   A: Assume $P \notin \operatorname{supp} f$: Easy.

   B: Assume $f(P) = 0$: We need.

4) $\displaystyle\exists C > 0 : \int_{\partial\Omega} \Big|\frac{\partial}{\partial n_Q} R(P,Q)\Big| d\sigma(Q) < C$ for all $P \notin \partial\Omega$

   Proof: Exercise.

   4) implies the estimate

$$\|\mathcal{D}f\|_{L^\infty(\mathbb{R}^n \setminus \partial\Omega)} \leq C\|f\|_{L^\infty(\partial\Omega)}.$$

   Choose $\{f_k\} \subset C(\partial\Omega)$ with $P \notin \operatorname{supp} f_k$ such that

$$\|f - f_k\|_{L^\infty(\partial\Omega)} \to 0 \quad \text{as} \quad k \to \infty.$$

   $T$ bounded operator implies $Tf_k(P) \to Tf(P)$ as $k \to \infty$. Hence

$$\begin{aligned}|\mathcal{D}f(Q) - Tf(P)| &\leq C\|(f - f_k)\|_{L^\infty(\mathbb{R}^n \setminus \partial\Omega)} + |\mathcal{D}f_k(Q) - Tf_k(P)| + \\ &+ |Tf_k(P) - Tf(P)| \to 0 \text{ as } k \to \infty \text{ and } \Omega \ni Q \to P.\end{aligned}$$

   C: Enough to check $f \equiv 1$.

The result follows from basic facts 1) and 3). Hence we have proved Lemma 1 and 2 part 1). Part 2) follows analogously. □

We now return to the single layer potential and observe that $Sf$ is harmonic in $\mathbb{R}^n \setminus \partial\Omega$ and continuous in $\mathbb{R}^n$ if $f \in C(\partial\Omega)$. Next we want to compare the normal derivative of $Sf$ with $\mathcal{D}f$ at $\partial\Omega$. Since $\partial\Omega$ is $C^2$ we have following result:

For $\varepsilon > 0$ small enough

$$]-\varepsilon, \varepsilon[\times \partial\Omega \ni (t, P) \to P + tn_p \in V$$

is a diffeomorphism, where $n_p$ is the outward unit normal of $\partial\Omega$ at $P$, and $V$ is a neighborhood of $\partial\Omega$. For $P \in \partial\Omega$ and $t \in ]-\varepsilon, \varepsilon[$ set

$$DSf(P + tn_p) = \int_{\partial\Omega} \frac{\partial}{\partial n_p} R(P + tn_p, Q) f(Q) d\sigma(Q).$$

The close relations between $\mathcal{D}f$ and $DSf$ is formulated in

**Lemma 4.** *If $f \in C(\partial\Omega)$ then*

1) $DSf \in C(\overline{V \cap \Omega})$

2) $DSf \in C(\overline{V \cap \complement\Omega})$

*(Compare Lemma 1).*

Let $D_+Sf$ be the restriction to $\partial\Omega$ of the function $DSf$ extended to $\overline{V \cap \Omega}$ from inside and $D_-Sf$ the restriction to $\partial\Omega$ of the function $DSf$ extended to $\overline{V \cap \complement\Omega}$ from outside. But $R(P, Q) = R(Q, P)$ so with $R^n(P, Q) = K(Q, P)$, which is the real-valued kernel in

$$T^* f(P) = \int_{\partial\Omega} K^*(P, Q) f(Q) d\sigma(Q) \quad P \in \partial\Omega,$$

we have that $T^*$ is the adjoint operator of $T$.

**Lemma 5 (jump relations for $DS$).** 1) $D_+S = -\frac{1}{2}I + T^*$

2) $D_-S = \frac{1}{2}I + T^*$.

*Proof of Lemma 4 and 5.* Let $f \in C(\partial\Omega)$ and define

$$w_f(P) = \begin{cases} \mathcal{D}f(P) + DSf(P) & P \in V \setminus \partial\Omega \\ Tf(P) + T^*f(P) & P \in \partial\Omega. \end{cases}$$

Claim: $w_f \in C(V)$.

Proof: $w_f$ continuous on $V \setminus \partial\Omega$ and on $\partial\Omega$. Hence it is enough to show taht

$$w_f(P + tn_p) \to w_f(P) \text{ uniformly for } P \in \partial\Omega \text{ as } t \to 0.$$

Assume $\chi_0 \in C(\partial\Omega)$ such that $0 \le \chi_0 \le 1$, $\chi_0 = 1$ in a neighborhood of $P$ and

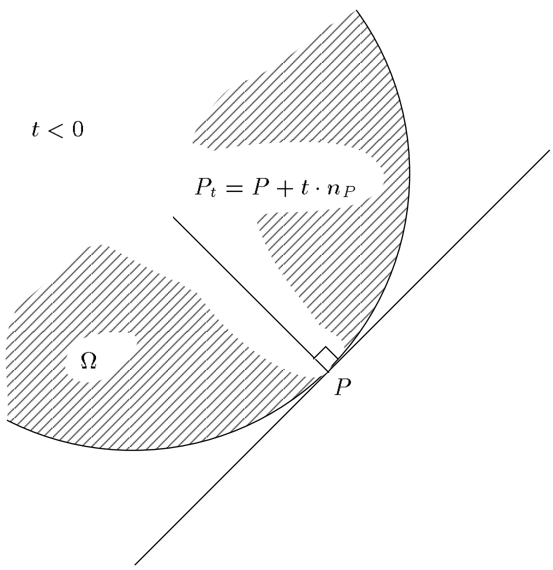
$$\text{supp } \chi_0 \subset B_\delta(P).$$

Decompose $f$ as

$$f = f_1 + f_2 \equiv \chi_0 f + (1-\chi_0) f.$$

A: $w_{f_2}(P + tn_p) \to w_{f_2}(P)$ as $t \to 0$. Easy

B: Assume $t \neq 0$




$$w_{f_1}(P_t) = C \int_{\partial\Omega} \frac{\langle Q - P_t, n_Q\rangle - \langle Q - P_t, n_p\rangle}{|Q - P_t|^n} f_1(Q) d\sigma(Q).$$

Hence

$$\begin{aligned} |w_{f_1}(P_t)| &\le C\|f\|_\infty \int_{\partial\Omega\cap B_\delta(P)} \frac{|\langle Q - P_t, n_Q - n_p\rangle|}{|Q - P_t|^n} d\sigma(Q) \le \\ &\le C\|f\|_\infty \int_{\partial\Omega\cap B_\delta(P)} \frac{|Q - P_t||Q - P|}{|Q - P_t|^n} d\sigma(Q) \le \\ &\le C\|f\|_\infty \int_{\partial\Omega\cap B_\delta(P)} \frac{d\sigma(Q)}{|Q - P_t|^{n-2}} = \mathcal{O}(\delta) \end{aligned}$$

independently of $t$, since $|n_Q - n_p| = \mathcal{O}(|Q - P|)$.

But $w_f = w_{f_1} + w_{f_2}$ and thus $w_f$ continuous on $V$. This proves the claim.

Therefore

$$Tf(P) + T^*f(P) = \mathcal{D}_+ f(P) + D_+ Sf(P) = \mathcal{D}_- f(P) + D_- Sf(P), \quad P \in \partial\Omega.$$

The jumprelations for $DS$ follow. $\square$

We now give the final argument for the existence of a solution of the Dirichlet problem in $\Omega$ and that is

$$\mathcal{D}_+ : C(\partial\Omega) \to C(\partial\Omega)$$

is onto.

Since $\mathcal{D}_+ = \frac{1}{2}I + T$, where $T$ is compact, Fredholm's Alternative theorem can be applied. Hence,

$$\frac{1}{2}I + T = \mathcal{D}_+ \quad \text{onto}$$

iff

$$\frac{1}{2}I + T^* = D_- S \quad 1-1.$$

To prove $D_- S$ is $1-1$ is easy:

Assume $D_- Sf = 0$ for some $f \in C(\partial\Omega)$. Set $v = Sf$. Then

i) $v$ harmonic in $\complement\bar{\Omega}$

ii) $v(P) = \mathcal{O}(|P|^{2-n})$ as $|P| \to \infty$

iii) $\frac{\partial v}{\partial n}|_{\partial\Omega} = 0$.

Green's formula implies

$$\int_{\complement\bar{\Omega}} |\nabla v|^2 = \int_{\complement\bar{\Omega}} v\Delta v + \int_{\partial\Omega} v\frac{\partial v}{\partial n} d\sigma = 0.$$

Thus $v = 0$ in $\complement\bar{\Omega}$. But $v \in C(\mathbb{R}^n)$ and $\Delta v = 0$ in $\Omega$.

Maximumprinciple $\Rightarrow v = 0$ in $\mathbb{R}^n \Rightarrow f = 0$.

*Remark*: The proof above is valid for domains $\Omega$ with $C^{1+\alpha}$ boundaries where $\alpha > 0$, but not for domains with boundaries with less regularity.

*Remark*: We observe that the method is *non-constructive* as a consequence of the soft arguments (i.e., compactness arguments) we have used. Hence it is not possible to solve the Dirichlet problem for, say Lipschitz-domains $\Omega$ by approximating $\Omega$ with $C^2$ domains $\Omega_k$, solve some Dirichlet problems for these and obtain an approximation of a solution for $\Omega$, since we do not have any estimates of the inverses of the $D_+$:s.

## References

[1] Folland, G: Introduction to partial differential equations. Math. Notes 17, Princeton U.P.

[2] Stein, E.M./ Weiss, G: Introduction to Fourier Analysis on Euclidean Spaces. Princeton U.P.

# Chapter 1

# Dirichlet Problem for Lipschitz Domain. The Setup

A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that

$$|\varphi(x) - \varphi(y)| \leq M|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^n$$

is called *Lipschitz function.* A bounded domain $\Omega \subset \mathbb{R}^{n+1}$ is called *Lipschitz domain* if $\partial\Omega$ can be covered by finitely many right circular cylinders $L$ whose bases are at a positive distance from $\partial\Omega$ such that to each cylider $L$ there is a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a coordinate system $(x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ such that the $y$-axis is parallel to the axis of symmetry of $L$ and $L \cap \Omega = L \cap \{(x, y) : y > \varphi(x)\}$ and $L \cap \partial\Omega = L \cap \{(x, y) : y = \varphi(x)\}$. A domain $D \subset \mathbb{R}^{n+1}$ is called *special Lipschitz domain* if there is a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $D = \{(x, y) : y > \varphi(x)\}$ and $\partial D = \{(x, y) : y = \varphi(x)\}$. In this and all proceeding chapters we reserve the notation $\Omega$ for bounded Lipschitz domains and $D$ for special Lipschitz domains respectively. With a *cone* $\Gamma$ we mean a circular cone which is open. A cone $\Gamma$ with vertex at a point $P \in \partial C$, where $C \subset \mathbb{R}^{n+1}$ is a domain, is called a *nontangential cone* if there is a cone $\Gamma'$ and a $\delta > 0$ such that

$$\emptyset \neq (\bar{\Gamma} \cap B_\delta(P)) \setminus \{P\} \subset \Gamma' \cap B_\delta(P) \subset C.$$

$B_r(Q)$ is our standard notation for the ball $\{x \in \mathbb{R}^n : |x - Q| \leq r\}$. We say that a function $u$ defined in a domain $C$ has *nontangential limit* $L$ at a point $P \in \partial C$ if

$$u(Q) \to L \quad \text{as} \quad Q \to P, \quad Q \in \Gamma$$

for all nontangential cones $\Gamma$ with vertices at $P$. Finally we define the *nontangential maximal function* $M_\beta u$ for $\beta > 1$ and function $u$ defined in Lipschitz domain $\Omega$ by

$$M_\beta u(P) = \sup\{|u(Q)| : |P - Q| < \beta \text{dist}\,(Q, \partial\Omega), Q \in \Omega\}, \quad P \in \partial\Omega.$$

One of the main results in this course will be the existence of a *solution to the Dirichlet problem*

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^{n+1} \\ u|_{\partial\Omega} = f \in L^2(\partial\Omega) \end{cases}$$

where $\Omega$ is a bounded Lipschitz domain. By this we mean that there exists a harmonic function $u$ in $\Omega$ which converges nontangentially to $f$ almost everywhere with respect to the surface measure $d\sigma(\partial\Omega)$ and that the maximal function $M_\beta u \in L^2(\partial\Omega)$ for $\beta > 1$. The starting point for our enterprise of proving the existence of a solution to Dirichlet problem for the Lipschitz domain $\Omega$ is the double layer potential

$$\mathcal{D}g(P) = \int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P,Q) g(Q) d\sigma(Q) \quad P \in \Omega,$$

where $R(P,Q)$ is the fundamental solution for Laplace equation in $\mathbb{R}^{n+1}$ (multiplied with $-1$) and $g \in L^2(\partial\Omega)$. Since $\mathcal{D}g$ is harmonic in $\Omega$, we are done if we can show that for some choice of $g$ we have the right behaviour of $\mathcal{D}g$ at $\partial\Omega$. However, this is not easy since for $K(P,Q) = \frac{\partial}{\partial n_Q} R(P,Q) \quad P,Q \in \partial\Omega \quad P \neq Q$, we only have the estimate $|K(P,Q)| \leq \frac{C}{|P-Q|^{n-1}}$ which cannot be improved in general. Thus we have to rely on the cancellation properties of $K(P,Q)$, and the operator $T$ which appeared in Chapter 0 can only be defined as a principal value operator. Before we study the case with a general bounded Lipschitz domain $\Omega$ we treat the case with a special Lipschitz domain $D$. From this we obtain the result for $\Omega$ using standard patching techniques (see Appendix 2).

Consider

$$\mathcal{D}g(P) = \int_{\partial D} \frac{\partial}{\partial n_Q} R(P,Q) g(Q) d\sigma(Q) \quad P \in D$$

where $D = \{(x,y) : y > \varphi(x)\}$ for a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$. We remark that $\varphi$ Lipschitz function implies that $\varphi'$ exists a.e. so the definition of $\mathcal{D}g$ makes sense and $\dfrac{\partial}{\partial n_Q} R(P,Q) = C_n \dfrac{\langle n_Q, P-Q \rangle}{|P-Q|^{n+1}}$, with $n_Q = \dfrac{(\nabla\varphi(x), -1)}{\sqrt{|\nabla\varphi(x)|^2 + 1}}$ for $Q = (x, \varphi(x))$, exists a.e. $d\sigma(\partial D)$. To state the first proposition, we need some more notation: For every measure $\mu$ and each $\mu$-measurable function $g$ and measurable set $A$ with $\mu(A) \neq 0$ we let $⨍_A g d\mu$, denote the mean value $\dfrac{1}{\mu(A)} \displaystyle\int_A g d\mu$. Furthermore for $g \in L^1_{\text{loc}}(\partial D)$ we define the maximal function $Mg$ by

$$Mg(P) = \sup_{r>0} ⨍_{\partial D \cap B_r(P)} |g(Q)| d\sigma(Q), \quad P \in \partial D.$$

The following result is crucial.

**Proposition 1.1.** *Let $D = \{(x, y) : y > \varphi(x)\}$ where $\varphi : \mathbb{R}^n \to R$ is a Lipschitz function with $\|\varphi'\|_\infty = A$. Let $P = (x, y) \in D$ and $P^* \in (x, \varphi(x)) \in \partial D$ and set $\rho = y - \varphi(x)$. Assume $g \in L^p(\partial D)$ for some $p$ where $1 < p < \infty$. Then*

$$|\mathcal{D}g(P) - T_\rho g(P^*)| \leq CMg(P^*)$$

*where*

$$T_\rho g(P^*) = \int_{\partial D \setminus B_\rho(P^*)} K(P^*, Q)g(Q)d\sigma(Q).$$

*The constant $C$ depends only on the dimension $a$. Before we prove this proposition, we make some remarks on the maximal function $M$.*

1) Another maximal function, which is quite similar to the *Hardy-Littlewood maximal function $M$* is $M^*$ defined by

$$M^*g(P) = \sup_{B_r(Q) \ni P} \fint_{\partial D \cap B_r(Q)} |g(Q')|d\sigma(Q'),\ P \in \partial D$$

for $g \in L^1_{\text{loc}}(\partial D)$. We immediately observe that $Mg \leq M^*g \leq CMg$ for some dimensional constant $C$, i.e., $M$ and $M^*$ are equivalent.

2) Let $\pi$ denote the projection

$$\pi : \partial D \to \mathbb{R}^n \quad \text{where} \quad (x, \varphi(x)) \mapsto x$$

and define the maximal function $\tilde{M}$ by

$$\tilde{M}g(x) = \sup_{r>0} \fint_{B_r(x)} |g \circ \pi^{-1}(y)|dy, \quad x \in \mathbb{R}^n,$$

for $g \in L^1_{\text{loc}}(\partial D)$. Since $\varphi$ is a Lipschitz function, we see that $M$ and $\tilde{M}$ are equivalent.

3) $M$ is bounded in $L^\infty$ with norm 1, i.e.,

$$\|Mg\|_\infty \leq \|g\|_\infty \quad \text{for all} \quad g \in L^\infty$$

4) $M$ is a weak $(1, 1)$ operator, i.e., there exists a $C > 0$ such that

$$|\{x : Mg(x) > \lambda\}| \leq C\frac{\|g\|_1}{\lambda} \quad \text{for all} \quad g \in L^1.$$

5) $M$ is bounded in $L^p, 1 < p < \infty$, i.e., there exists a $C_p > 0$ such that

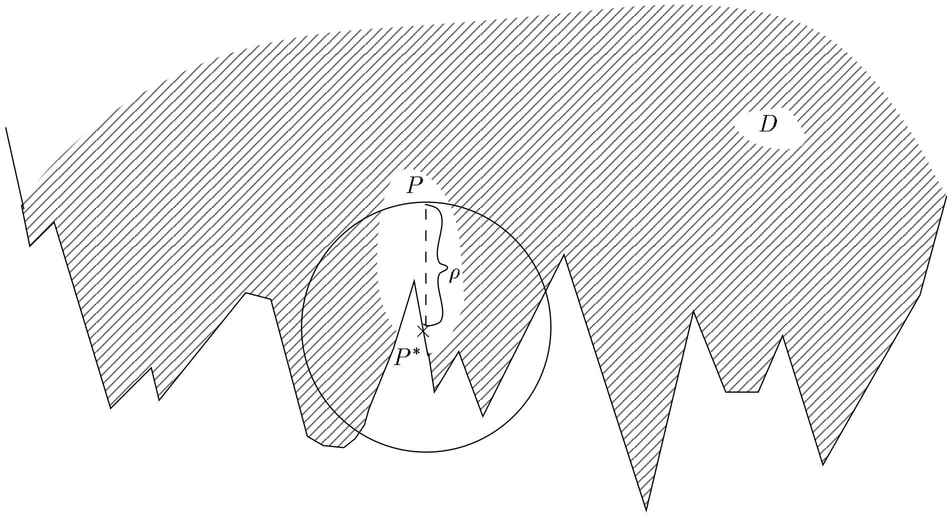
$$\|Mg\|_p \leq C_p\|g\|_p \quad \text{for all} \quad g \in L^p.$$

Here 3) is trivial, 4) can be proven by a covering lemma argument and 5) follows from 3), 4) and Marcinkiewicz' interpolation theorem (see Stein [1]). For later reference we state

**Marcinkiewicz' interpolation theorem.** Let $1 \leq p < q \leq \infty$ and let $T$ be a subadditive operator defined on $L^p + L^q$. Assume $T$ is a weak $(p,p)$ operator and a weak $(q,q)$ operator. Then $T$ is bunded on $L^r$ where $p < r < q$. An operator $T$ is a *weak* $(p,p)$ *operator* if there exists a constant $C > 0$ such that

$$|\{x : Tg(x)| > \lambda\}| \leq C\left(\frac{\|g\|_p}{\lambda}\right)^p \quad \text{for all} \quad g \in L^p \text{ and } \lambda > 0.$$

Hence, if $T$ is bounded on $L^p$, then $T$ is a weak $(p,p)$ operator, but the converse is not true in general.




*Proof of Proposition 1.1.*

$$\begin{aligned}
|\mathcal{D}g(P) - T_\rho g(P^*)| &\leq C \int_{\partial D \cap \{|P^*-Q|>\rho\}} \left| \frac{\langle n_Q, P-Q\rangle}{|P-Q|^{n+1}} - \frac{\langle n_Q, P^*-Q\rangle}{|P^*-Q|^{n+1}} \right| \cdot |g(Q)| d\sigma \\
&+ C \int_{\partial D \setminus \{|P^*-Q|>\rho\}} \left| \frac{\langle n_Q, P-Q\rangle}{|P-Q|^{n+1}} \right| |g(Q)| d\sigma(Q) \leq \\
&\leq C \int_{\partial D \cap \{|P^*-Q|>\rho\}} \frac{\rho}{(\rho + |Q-P^*|)^{n+1}} |g(Q)| d\sigma(Q) + \\
&+ C \int_{\partial D \setminus \{|P^*-Q|>\rho\}} \frac{1}{\rho^n} |g(Q)| d\sigma(Q)
\end{aligned}$$

where we have applied the mean value theorem to the first integral. The second integral is $\leq CMg(P^*)$ and the first integral can also be estimated from above with the same bound according to

**Lemma 1.1.** *Let $\psi \geq 0$ be a radial decreasing function defined in $\mathbb{R}^n$. Assume $f \in L^1 + L^\infty$ and set $mf(x) = \sup_{r>0} \fint_{B_r(x)} |f(x)| dx$ for $x \in \mathbb{R}^n$. Then $\psi * f(x) \leq Bmf(x)$ for all $x \in \mathbb{R}^n$ where $B = \int \psi(x) dx$.* $\square$

If we take this lemma for granted for a moment and set

$$\psi(x) = \frac{\rho}{(|x| + \rho)^{n+1}},$$

the first integral above is bounded from above by $CMg(P^*)$ and we are done. $\square$

*Proof of Lemma 1.1.* It is enough to prove the lemma for $0 \le f \in C_0^\infty, \psi \in C_0^\infty$ and $x = 0$. Set $S^n = \partial B_1(0)$ and $A(r) = \int_{B_r(0)} f(x)dx$.

We obtain

$$\begin{aligned}\psi * f(0) = \int_{\mathbb{R}^n} \psi(|x|)f(x)dx = \int_0^\infty \psi(r)r^{n-1} \int_{S^n} f(rw)d\sigma(w)dr = \\ = \int_0^\infty \psi(r)A'(r)dr = -\int_0^\infty \psi'(r)A(r)dr \le -\int_0^\infty \psi'(r)|B_r(0)|dr\, mf(0).\end{aligned}$$

Set $f \equiv 1$ in the calculations above and we get $-\int_0^\infty \psi'(r)B_r(0)dr = B$. The lemma is proven. $\square$

If we define the operator $T_*$ by

$$T_*g(P^*) = \sup_{\rho>0} |T_\rho g(P^*)| \quad P^* \in \partial D$$

for $g \in L^p(\partial D)$, then

$$|\mathcal{D}g(P)| \le C(T_*g(P^*) + Mg(P^*))$$

for all $P = (x, y) \in D$ and $P^* = (x, \varphi(x)) \in \partial D$. Thus *if* we can prove that $T_*$ *is bounded on* $L^p(\partial D)$, then $|\mathcal{D}g(P^*)| < \infty$ for a.e. $P^* \in \partial D$. We remark that with some additional considerations one can prove that

$$|\mathcal{D}g(Q)| \le C(T_*g(P^*) + Mg(P^*))$$

for all $Q$ in a nontangential cone $\Gamma$ with vertex at $P^* \in \partial D$, and thus $\sup_{Q\in\Gamma} |g(Q)|$ in non-tangential cones $\Gamma$ with vertices at $P^* \in \partial D$ for almost every $P^* \in \partial D$. It then follows that $\mathcal{D}g$ has finite nontangential limit a.e. $d\sigma(\partial D)$ (see Dahlberg [2]).

The limitfunction belongs to $L^p(\partial D)$. If we also can prove that the limitfunction is equal to $f$ for some choice of $g$, we are done. To be successful in our approach, we have to study the operators $T_\rho$ for $\rho > 0$ and $T_*$. This calls for some definitions. Let $\mathcal{S}(\mathbb{R}^n)$ denote the *Schwartz class* (i.e., the space of all $C^\infty$-functions in $\mathbb{R}^n$ which together with all their derivatives die out faster than any power of $x$ at infinity) with the usual topology.

$T$ is called a *singular integral operator* (SIO) if $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)^*$ is linear and continuous and there exists a kernel $K$ such that for all $\varphi$, $\psi \in C_0^\infty(\mathbb{R}^n)$ with supp $\varphi \cap$ supp $\psi = \phi$

$$\langle T\varphi, \psi\rangle = \int\int K(x, y)\varphi(y)\psi(x)dydx,$$

where $\langle , \rangle$ is the usual $\mathcal{S} - \mathcal{S}^*$ paring. We observe that $K$ does not determine $T$ uniquely. Consider for instance $Tf = f'$ for which $K = 0$ is a kernel.

We say that a kernel $K$ is of *Calderón-Zygmund type* (CZ-type) if

1) $|K(x,y)| \leq \dfrac{C}{|x-y|^n}$

2) $|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \dfrac{C}{|x-y|^{n+1}}$

3) $K(x,y) = -K(y,x)$.

The operator-kernels, we will study, will be of the form

$$K_i(x,y) = \frac{((x,\varphi(x)) - (y,\varphi(y)))_i}{|(x,\varphi(x)) - (y,\varphi(y))|^{n+1}} \quad i = 1,2,\dots,n+1$$

where $(a)_i$ denotes the $i$-th component of $a \in \mathbb{R}^{n+1}$.

We observe that these kernels are of $CZ$-type and adopt the convention that whenever we discuss kernels $K$, they are assumed to be of $CZ$-type unless we explicitly state the converse. Starting with a kernel $K$, we can form a well-defined SIO with $K$ as the kernel namely the *principal value operator* (PVO) $T$. Note that for $\varphi$, $\psi \in \mathcal{S}(\mathbb{R}^n))$

$$\int\int_{|x-y|>\varepsilon} K(x,y)\varphi(y)\psi(x)dydx = \frac{1}{2}\int\int_{|x-y|>\varepsilon} K(x,y)(\varphi(y)\psi(x) - \varphi(x)\psi(y))dydx$$

since $K(x,y) = -K(y,x)$ and thus

$$\lim_{\varepsilon\to 0}\int\int_{|x-y|>\varepsilon} K(x,y)\varphi(y)\psi(x)dydx$$

exists since $|\varphi(y)\psi(x) - \varphi(x)\psi(y)| = \mathcal{O}(|x-y|)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ decay fast enough at infinity.

Hence $T : \mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto T\varphi \in \mathcal{S}(\mathbb{R}^n)^*$ where

$$\langle T\varphi, \psi\rangle = \lim_{\varepsilon\to 0}\int\int_{|x-y|>\varepsilon} K(x,y)\varphi(y)\psi(x)dydx$$

is a SIO.

We leave the proof of continuity of $T$ as an exercise. From now on we assume that all operators $T$ are PVO with kernels $K$ of $CZ$-type.

To achieve our goal to establish the existence of a solution to the Dirichlet problem for Lipschitz domains, we will prove the following sequence of theorems

**Theorem 1.1.** *If $T$ bounded on $L^2$, then $T$ is a weak $(1,1)$ operator.*

This implies

**Theorem 1.1'.** *If $T$ bounded on $L^2$, then $T$ bounded on $L^p$ for $1 < p < \infty$.*

and

**Theorem 1.2.** *If $T$ bounded on $L^2$, then $T^*$ bounded on $L^p$ for $1 < p < \infty$ where $T^*g(x) = \sup_{\varepsilon>0} |\int_{|x-y|>\varepsilon} K(x,y)g(y)dy|$.*

Thus it is crucial for us to be able to prove $L^2$-boundedness of $T$. This is done in two steps.

**Theorem 1.3.** *If $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ Lipschitz function and $K(x,y) = \frac{1}{x-y+i(\varphi(x)-\varphi(y))}$, then the corresponding operator $T$ is $L^2$ bounded.*

**Theorem 1.4.** *If $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ Lipschitz function and*

$$K_i(x,y) = \frac{((x,\varphi(x)) - (y,\varphi(y)))_i}{|(x,\varphi(x)) - (y,\varphi(y))|^{n+1}} \quad i = 1, 2, \ldots, n+1,$$

*then the corresponding operators $T_i$ are $L^2$ bounded.*

Finally we prove

**Theorem 1.5.** *$\mathcal{D}|_{\partial D}$ is invertible.*

To prove Theorem 1.3, we will characterize those kernels $K$ of $CZ$-type which correspond to $L^2$ bounded operators $T$. This is done by a theorem of Daivd and Journé [3].

**Theorem 1.6.** *$T$ bounded on $L^2$ iff $T1 \in BMO$.*

The definition of BMO and the theorem and its proof will be discussed in Chapter 3.

# References


[1] E. M. Stein: Singular integral and differentiability properties of functions, Princeton University Press 1970.

[2] B.E.J. Dahlberg: Harmonic functions in Lipschitz domains, Proceedings of Symposia in Pure Mathematics Vol XXXV, Part 1 (1979) pp. 313-322.

[3] G. David and J.-L. Journé: A boundedness criterion for generalized Calderón-Zygmund operators, Preprint.

# Chapter 2

# Proofs of Theorem 1.1 and Theorem 1.2

We recall that $T$ is a PVO with kernel $K$ of CZ-type. In this chapter we give a proof of the following result of Calderón-Zygmund [1].

**Theorem 1.1:** If $T$ bounded on $L^2$, then $T$ is a weak $(1,1)$ operator.

The following bound on $T_*$ is due to Cotlar [3].

**Theorem 1.2:** If $T$ bounded on $L^2$, then $T_*$ is bounded on $L^p$ for $1 < p < \infty$ where $T_* f(x) = \sup_{\varepsilon>0} |\int_{|x-y|>\varepsilon} K(x,y)f(y)dy|$,

*Proof of Theorem 1.1.* The idea of the proof is the same as when $T$ is a translation-invariant $L^2$-bounded operator (See Stein [2]). Thus we show that there exists $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C\frac{\|f\|_1}{\lambda} \quad \text{for all } f \in L^1 \text{ and } \lambda > 0$$

by splitting $f$ in a good part $g$, which is a $L^2$-function and a bad part $b$. This is done with the following lemma

**Lemma (Calderón-Zygmund decomposition).** Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then there exist cubes $Q_j, \quad j = 1, 2, \ldots$ such that

1) $|Q_j \cap Q_k| = 0$ for $j \neq k$

2) $|f(x)| \leq \lambda$ a.e. for $x \in \mathbb{R}^n \setminus \cup_{j=1}^{\infty} Q_j$

3) $\lambda \leq \fint_{Q_j} |f(x)|dx < 2^n\lambda$.

The proof is based on a recursive stop-time argument and can be found e.g. Stein [2]. $\square$

Now set

$$g(x) = \begin{cases} f(x) & x \in \mathbb{R}^n \setminus \cup_{j=i}^{\infty} Q_j \\ \fint_{Q_j} |f(x)|dx & x \in Q_j \end{cases} \qquad j = 1, 2, \dots$$

and $b(x) = f(x) - g(x)$. We immediately observe that

1) $\mathrm{supp}\ b \subset \cup_{j=1}^{\infty} \bar{Q}_j$

2) $\int_{Q_j} b(x)dx = 0 \quad j = 1, 2, \dots$

and

3) $\|g\|_2 \leq 2^n\lambda\|f\|_1$.

Furthermore,

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| &\leq |\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\}| + |\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\}| \\ &\leq |\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\}| + |\cup_{j=1}^{\infty} 2Q_j| + |\{x \in \mathbb{R}^n \setminus \cup_{j=1}^{\infty} 2Q_j : |Tb(x)| > \frac{\lambda}{2}\}| \end{aligned}$$

where $2Q_j$ is the cube with the same center as $Q_j$, which we denote $y_j$, with sides parallel with $Q_j$ and with doubled sidelengths compared with $Q_j$. Here

$$|\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2}\|Tg\|_2 \leq \frac{C}{\lambda^2}\|g\|_2 \leq C\frac{1}{\lambda}\|f\|_1$$

and

$$|\cup_{j=1}^{\infty} 2Q_j| \leq 2^n \sum_{j=1}^{\infty} |Q_j| \leq \frac{2^n}{\lambda}\|f\|_1$$

so it remains to estimate $|\{x \in \mathbb{R}^n \setminus \cup_{j=1}^{\infty} 2Q_j : |Tb(x)| > \frac{\lambda}{2}\}|$ and this is the point where we use the properties of the kernel $K$. Set

$$b_j(x) = \begin{cases} b(x) & x \in Q_j \\ 0 & \text{otherwise} \end{cases}$$

For, $x \notin 2Q_j$ we have

$$Tb_j(x) = \int_{Q_j} (K(x, y) - K(x, y_j))b_j(y)dy$$

and thus

$$|Tb_j(x)| \leq C \int_{Q_j} \frac{|y - y_j|}{|x - y|^{n+1}} |b_j(y)| dy$$

where we used $\int_{Q_j} b_j(y) dy = 0$. Integrating these inequalities gives

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \cup_{j=1}^{\infty} 2Q_j} |Tb(x)| dx &\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n \setminus 2Qj} |Tb_j(x)| dx \leq \\ &\leq C \sum_{j=1}^{\infty} \int_{Q_j} |b_j(y)| dy = C\|b\|_1 \leq C\|f\|_1 \end{aligned}$$

and consequently

$$|\{x \in \mathbb{R}^n \setminus \cup_{j=1}^{\infty} 2Qj : |Tb(x)| > \frac{\lambda}{2}\}| \leq \frac{2}{\lambda} \|Tb\|_{L^1(\mathbb{R}^n \setminus \cup_{j=1}^{\infty} 2Qj)} \leq C \frac{\|f\|_1}{\lambda}.$$

The proof is done. □

**Corollary:** If $T$ is bounded on $L^2$, then $T$ is bounded on $L^p$ for $1 < p < \infty$.

*Proof.* Marcinkiewicz' interpolation theorem and Theorem 1.1 implies that $T$ is bounded on $L^p$ for $1 < p \leq 2$. But the adjoint operator $T^*$ of $T$ is a PVO with CZ-kernel $K^*(x, y) = K(y, x)$ and $T^*$ is bounded on $L^2$. An application of Marcinkiewicz' interpolation theorem and Theorem 1.1 to $T^*$ gives $T^*$ bounded on $L^p$ for $1 < p \leq 2$. Hence $T = (T^*)^*$ is bounded on $L^p$ for $2 \leq p < \infty$ by duality and we are done. □

*Proof of Theorem 1.2.* The proof is an easy consequence of *Cotlar's inequality*, i.e., if $T$ bounded on $L^2$ then

$$T_* f \leq C(T)(Mf + MTF) \tag{2.1}$$

where $M$ is the Harcy-Littlewood maximal function; since $M$ is bounded on $L^p$ and $T$ is bounded on $L^p$ according to the corollary above.

Remains to show Cotlar's inequality: It is enough to prove (2.1) for $x = 0$. Fix an $\varepsilon > 0$. We will show that

$$T_\varepsilon f(0) \leq C(Mf(0) + MTF(0)) \tag{2.2}$$

where $C$ is independent of $\varepsilon$. Set

$$f_1(x) = \begin{cases} f(x) & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon \end{cases}$$

and $f_2(x) = f(x) - f_1(x)$. Thus $T_\varepsilon f(0) = Tf_2(0)$. The strategy is to prove that for $|x| < \frac{\varepsilon}{2}$ we have

$$|Tf_2(x) - Tf_2(0)| \leq CMf(0) \tag{2.3}$$

where the constant $C$ is independent of $x$ and $\varepsilon$. Assume (2.3) to be true for a moment and argue as follows.

$$|T_\varepsilon f(0)| = |Tf_2(0)| \leq |Tf_2(x)| + \tilde{C}Mf(0) \leq |Tf(x)| + |Tf_1(x)| + \tilde{C}Mf(0).$$

Consider the two cases

1) $\frac{1}{3}|T_\varepsilon f(0)| \leq \tilde{C}Mf(0)$.

2) $\frac{1}{3}|T_\varepsilon f(0)| > \tilde{C}Mf(0)$.

For case 1) inequality (2.2) is trivial. For case 2) set $\lambda = |T_\varepsilon f(0)|$, $B = B_{\frac{\varepsilon}{2}}(0)$ and define

$$E_1 = \{x \in B : |Tf(x)| > \frac{\lambda}{3}\}$$
$$E_2 = \{x \in B : |Tf_1(x)| > \frac{\lambda}{3}\}.$$

Here $B = E_1 \cup E_2$ and thus

$$1 \leq \frac{|E_1|}{|B|} + \frac{|E_2|}{|B|}.$$

We see that

$$|E_1| \leq \frac{3}{\lambda}\int_B |Tf(x)|dx \leq \frac{3|B|}{\lambda}MTf(0)$$

and

$$|E_2| \leq \frac{C}{\lambda}\|f_1\|_1 \leq \frac{C|B|}{\lambda}Mf(0)$$

where we have used that $T$ is a weak $(1,1)$ operator. Thus $\lambda \leq C(Mf(0) + MTf(0))$ and it only remains to show (2.3), which is a straightforward calculation:

$$\begin{aligned}|Tf_2(x) - Tf_2(0)| &\leq \int_{|y|>\varepsilon} |K(x,y) - K(0,y)||f(y)|dy \leq \\ &\leq C\int_{|y|>\varepsilon} \frac{\varepsilon}{|y|^{n+1}}|f(y)|dy \leq C\int \min\big(\frac{\varepsilon}{|y|^{n+1}}, \varepsilon^{-n}\big)|f(y)|dy \leq CMf(0)\end{aligned}$$

for $|x| < \frac{\varepsilon}{2}$ where we have applied Lemma 1.2. This completes the proof. □

## References


[1] A. P. Calderon/A. Zygmund: On the existence of certain singular integrals, Acta Math. 88 (1952) pp. 85-139.

[2] E. M. Stein: Singular integrals and differentiability properties of functions. Princeton University Press 1970.

[3] M. Cotlar: Some generalizations of the Hardy-Littlewood maximal theorem Rev. Mat. Cuyana 1(1955) pp. 85-104.

# Chapter 3

# Proof of Theorem 1.6

We begin this section by introducing some of the tools we need to prove the $L^2$-boundedness of $T$.

BMO$_p(\mathbb{R}^n)$: For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we set

$$\|f\|_{p,*} = \sup_{Q \text{ cube}} \Big( \fint_Q |f(x) - f_Q|^p dx \Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

where $f_Q = \fint_Q f(x)dx$ and define the space BMO$_p$ to consist of those functions $f$ such that $\|f\|_{p,*} < \infty$. Thus $(\text{BMO}_p, \| \, \|_{p,*})$ becomes a semi-normed vectorspace with semi-norm vanishing on the constant functions. The letters BMO stand for bounded mean oscillation.

Examples of BMO-functions: 1) $\log |x| \in \text{BMO}_p(\mathbb{R}^n)$ 2) $L^\infty(\mathbb{R}^n) \subset \text{BMO}_p(\mathbb{R}^n)$
3) $\displaystyle\int \log |x-y| d\mu(y) \in \text{BMO}_p(\mathbb{R}^n)$ for finite measures $\mu$.

To be able to work with this space, we only need to know three basic facts.

**Fact 1:** Let $f \in L^1_{\text{loc}}$. If for all cubes $Q$, there exist constants $C_Q$ such that $(\fint_Q |f(x) - C_Q|^p dx)^{1/p} \leq \ell$, then $f \in \text{BMO}_p$ and $\|f\|_{p,*} \leq 2\ell$.

*Proof.* Exercise. □

This fact can be used to prove following proposition.

**Proposition 3.1.** *If $T$ bounded on $L^2$, then $T : L^\infty \to BMO$.*

*Proof.* The first step in the proof is to give a definition of the function $Tf$ where $f \in L^\infty$. We therefore introduce $\{Q_j\}$ = the set of all cubes $Q$ with centers with rational coordinates and with rational sidelengths. Set $E = \cup_j \partial Q_j$. For each pair $(x_1, x_2) \in (\mathbb{R}^n \setminus E) \times (\mathbb{R}^n \setminus E)$ choose a cube $Q \in \{Q_j\}$ such that $x_1, x_2 \in Q$. Set $f_1 = f \cdot \chi_{2Q}$ and $f_2 = f - f_1$. Define

$$F(x_1, x_2) = Tf_1(x_1) - Tf_1(x_2) + \int_{\mathbb{R}^n} (K(x_1, y) - K(x_2, y)) f_2(y) dy.$$

We note that $F$ is defined a.e. and that $F$ is independent of $Q$ (as long as $x_1, x_2 \in Q$). Check it! Furthermore, for a.e. $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$, $F(x, x_1) - F(x, x_2)$ is a constant (regarded as a function of $x$). We now define $Tf$ as the class $x \to F(x, x_1)$ for a.e. $x_1 \in \mathbb{R}^n$.

It remains to show that $T : L^\infty \to$ BMO is bounded. It is enough to show that

$$⨍_Q |F(x, x_Q) - Tf_1(x_Q)| dx \le C \|f\|_{L^\infty}, \quad f \in L^\infty(\mathbb{R}^n)$$

for all cubes $Q \in \{Q_j\}$.

But

$$⨍_Q |Tf_1(x)| dx \le (⨍_Q |Tf_1(x)|^2 dx)^{1/2} \le C(⨍_Q |f_1(x)|^2 dx)^{1/2} \le C\|f\|_\infty$$

since

$$\begin{aligned} &|\int_{\mathbb{R}^n} (K(x, y) - K(x_Q, y)) f_2(y) dy| \le |\int_{\mathbb{R}^n \setminus 2Q} |K(x, y) - K(x_Q, y)||f(y)| dy \le \\ &\le C \int_{\mathbb{R}^n \setminus 2Q} \frac{|x - x_Q|}{|y - x_Q|^{n+1}} dy \|f\|_\infty \le C\|f\|_\infty \text{ for } x \in Q. \end{aligned}$$

The proposition follows. ☐

### Fact 2: John-Nirenberg inequality

**Theorem:** Let $\varphi \in$ BMO $(\mathbb{R}^n)$. Then there exists constants, $C > 0, \alpha > 0$, depending only on $n$, such that

$$|\{x \in Q : |\varphi(x) - \varphi_Q| > \lambda\}| \le C|Q| \exp\big(-\frac{\alpha\lambda}{\|\varphi\|_*}\big)$$

for all $\lambda > 0$ and cubes $Q$.

*Sketch of a proof.* It is enough to show that

$$\sup_{Q \text{ cube}} ⨍_Q \exp\big(\frac{\alpha}{\|\varphi\|_*}|\varphi(x) - \varphi_Q|\big) dx \le C < \infty.$$

Assume $\|\varphi\|_* = 1$ and $\varphi \in L^\infty$. Since the constants $C$ and $\alpha$ will be independent of $\|\varphi\|_\infty$, the result follows for a general $\varphi$. Fix a cube $Q$. Consider all cubes $Q_j$ in the dyadic mesh of $Q$ and choose a $t > 1$. Let $\tilde{Q}_j$ denote those dyadic cubes which are maximal with respect to inclusion satisfying

$$⨍_{\tilde{Q}_j} |\varphi(x) - \varphi_Q| dx > t$$

and

$$|\varphi(x) - \varphi_Q| \leq t \text{ a.e. for } x \in Q \setminus \cup_{j=1}^{\infty} \tilde{Q}_j.$$

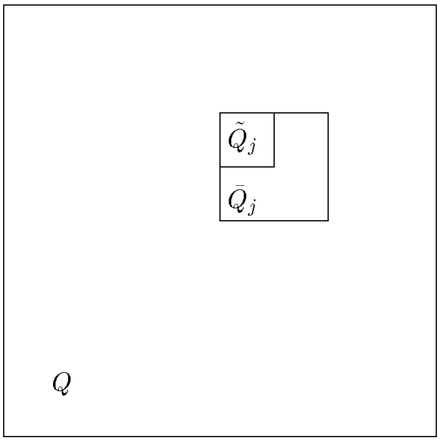
Clearly $\tilde{Q}_j \subset Q$ and

$$| \cup_{j=1}^{\infty} \tilde{Q}_j | \leq \frac{1}{t} \|\varphi - \varphi_Q\|_{L^1(Q)} \leq \frac{1}{t}|Q|.$$

The maximality of $\tilde{Q}_j$ implies that

$$⨍_{\bar{Q}_j} |\varphi(x) - \varphi_Q| dx \leq t$$

where $\bar{Q}_j$ is the minimal cube in the dyadic mesh of $Q$ with respect to inclusion for which $\tilde{Q}_j \not\subseteq \bar{Q}_j$. Furthermore




$$\begin{aligned}
|\varphi_{\tilde{Q}_j} - \varphi_Q| &\leq |\varphi_{\tilde{Q}_j} - \varphi_{\bar{Q}_j}| + |\varphi_{\bar{Q}_j} - \varphi_Q| \leq \\
&\leq ⨍_{\tilde{Q}_j} |\varphi(x) - \varphi_{\bar{Q}_j}| dx + t \leq \\
&\leq 2^n ⨍_{\bar{Q}_j} |\varphi(x) - \varphi_{\bar{Q}_j}| dx + t \leq \\
&\leq (2^n + 1)t.
\end{aligned}$$

Set $X(\alpha, Q) = \sup_{Q_j \in \text{dyadic mesh of} Q} ⨍_{Q_j} \exp(\alpha|\varphi(x) - \varphi_{Q_j}|)dx$ which is $< \infty$ since $\varphi \in L^\infty$. From the properties of $\tilde{Q}_j$ it follows that

$$\begin{aligned}
⨍_Q \exp(\alpha|\varphi(x) - \varphi_Q|)dx &\leq \frac{1}{|Q|} \int_{Q \setminus \cup_{j=1}^{\infty} \tilde{Q}_j} e^{\alpha t} dx + \\
&+ \frac{1}{|Q|} \sum_{j=1}^{\infty} |\tilde{Q}_j| ⨍_{\tilde{Q}_j} \exp(\alpha|\varphi(x) - \varphi_{\tilde{Q}_j}|) dx \exp(\alpha t(2^n + 1)) \leq \\
&\leq e^{\alpha t} + \frac{1}{t} \exp(\alpha t(2^n + 1)) X(\alpha, Q)
\end{aligned}$$

Take supremum over all cubes $Q$. Thus

$$\sup_{Q\,\text{cube}} X(\alpha, Q)[1 - \frac{1}{t}\exp(\alpha t(2^n + 1))] \leq e^{\alpha t}$$

which implies $\sup_{Q\,\text{cube}} X(\alpha, Q) \leq C$ if $\alpha > 0$ small enough. The proof is done. $\square$

**Remark:** It is an easy consequence of John-Nirenberg's inequality that the norms $\| \; \|_{p,*}$ and $\| \; \|_* \equiv \| \; \|_{1,*}$ are equivalent for $1 < p < \infty$.

*Proof.* For every $1 < p < \infty$ and $\beta > 0$ there exists a $C > 0$ such that $x^p \leq C \exp(\beta x)$ for $x > 0$. Choose $\beta = \frac{\alpha}{2}$ and apply the inequality above. Hence

$$\begin{aligned}
&\| \frac{\varphi}{\|\varphi\|_*} \|_{p,*} \leq C \sup_{Q\,\text{cube}} ⨍_Q \exp\big(\frac{\alpha}{2} \frac{|\varphi(x) - \varphi_Q|}{\|\varphi\|_*}\big) dx = \\
&= C \sup_{Q\,\text{cube}} \frac{1}{|Q|} \int_0^\infty \exp\big(\frac{\alpha}{2} t\big) d\big(|\{x \in Q : \frac{|\varphi(x) - \varphi_Q|}{\|\varphi\|_*} > t\}|\big) \leq \\
&\leq C \sup_{Q\,\text{cube}} \frac{1}{|Q|} \int_0^\infty \exp\big(\frac{\alpha}{2} t\big) C|Q| \exp(-\alpha t) \cdot (-\alpha) dt = C
\end{aligned}$$

and $\|\varphi\|_{p,*} \leq C\|\varphi\|_*$. The inequality $\|\varphi\|_* \leq \|\varphi\|_{p,*}$ follows from Hölder's inequality. $\square$

**Fact 3:** Connection between BMO and Carleson measures.

Carleson measures originally appeared as answers to the following question.

**Question:** Which positive measures $\mu$ on $\mathbb{R}^{n+1}_+$ have the property

$$\int\int_{\mathbb{R}^{n+1}_+} |P_y f(x)|^2 d\mu(x, y) \leq C(\mu) \|f\|_2^2 \quad \text{for all} \quad f \in L^2(\mathbb{R}^n)$$

where $P_y f(x) = p_y * f(x)$ with the Poisson kernel $p_y(x) = c_n \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}$?

To obtain a necessary condition on $\mu$ consider $f = \chi_Q$, i.e., $f$ is the characteristic function for a cube $Q \subset \mathbb{R}^n$. We immediately observe that $P_y f(x) \geq C > 0$ for $\{(x, y) : x \in \frac{1}{2}Q,\ 0 < y < \ell(Q)\}$ where $\ell(Q) =$ side length of $Q$. Set $\tilde{Q} = \{(\xi, \eta) \in \mathbb{R}^{n+1}_+ : \xi \in Q,\ 0 < \eta < \ell(Q)\}$. Hence

$$(C) \qquad \mu(\tilde{Q}) \leq C|Q| \quad \text{for all cubes} \quad Q \subset \mathbb{R}^n$$

is a necessary condition on $\mu$. We call a positive measure $\mu$ a Carleson measure if $\mu$ satisfies (C) and $\inf\{C : \mu(\tilde{Q}) \leq C|Q| \text{ for all cubes } Q\}$ is called the Carleson norm for $\mu$.

**Lemma 3.1.** *Let $\mu$ be a continuous function in $\mathbb{R}^{n+1}_+$ and set*

$$u^*(x) = \sup\{|u(\xi,\eta)| : |x-\xi| < \eta\}, \quad x \in \mathbb{R}^n.$$

*Let $\mu$ be a Carleson measure. Then*

$$\mu(\{(x,y) \in \mathbb{R}^{n+1}_+ : |u(x,y)| > \lambda\}) \leq C|\{x \in \mathbb{R} : u^*(x) > \lambda\}|$$

*for all $\lambda > 0$, where $C$ only depends on $n$ and the Carleson norm of $\mu$.*

*Proof.* The lemma is a consequence of following geometric fact: For every covergin $\{Q_j\}$ of countably many cubes there is a subcovering $\{Q'_j\}$ such that $\cup Q_j = \cup Q'_j$ and each $x \in \cup Q_j$ belongs to at most $2^n$ of the $Q'_j$'s. We leave the proof of this fact as an exercise. For $\lambda > 0$ set

$$E_\lambda = \{(x,y) \in \mathbb{R}^{n+1}_+ : |u(x,y)| > \lambda\}$$

and for each $(x,y) \in E_\lambda$ define

$$\begin{aligned}\tilde{Q}(x,y) &= \{(\xi,\eta) \in \mathbb{R}^{n+1}_+ : \|\xi - x\| < y,\, 0 < \eta < y\}\\ Q(x,y) &= \{\xi \in \mathbb{R}^n : \|\xi - x\| < y\}\end{aligned}$$

where $\|x\| = \max_{i=1,\dots,n} |x_i|$. Select a covering of $E_\lambda$ consisting of countably many cubes $\tilde{Q}(x,y)$ by a compactness argument. It is obvious that

$$u^*(\xi) > \lambda \text{ for all } \xi \in Q(x,y)$$

and hence

$$\begin{aligned}\mu(E_\lambda) &\leq \mu(\cup\tilde{Q}(x,y)) = \mu(\cup\tilde{Q}(x,y)') \leq\\ &\leq \sum \mu(\tilde{Q}(x,y)') \leq C \sum |Q(x,y)'| \leq\\ &\leq 2^n C|\{\xi \in \mathbb{R}^n : u^*(\xi) > \lambda\}|.\end{aligned}$$

□

We can now answer the question posed above by

**Theorem 3.1.** *If $\mu$ is a Carleson measure on $\mathbb{R}^{n+1}_+$, then*

$$\int\int_{\mathbb{R}^{n+1}_+} |P_y f(x)|^p d\mu(x,y) \leq C(p,\mu,n)\|f\|_p^p, \quad 1 < p \leq \infty.$$

*Proof.* Let $p_y(x)$ denote the Poisson kernel. If $|\bar{x} - x| < y$, then

$$p_y(\bar{x} - t) \leq Cp_y(x-t) \text{ for all } t \in \mathbb{R}^n$$

where $C$ is independent of $x, \bar{x}$ and $y$. Furthermore

$$P_y f(x) \equiv p_y * f(x) \leq CMf(x)$$

and thus

$$\sup\{|P_y f(\bar{x})| : |\bar{x} - x| < y\} \leq CMf(x).$$

Lemma 3.1 implies Theorem 3.1 since $M$ is bounded on $L^p$ for $1 < p \leq \infty$. □

**Remark:** Theorem 3.1 is also valid for all operators of the form

$$P_t f(x) = \varphi_t * f(x)$$

where $\varphi$ is a smooth function which decays at infinity and such that $|\varphi(x)| \leq \psi(x)$ for some radial function $\psi \in L^1(\mathbb{R}^n)$. $\varphi_t(x)$ denotes $\frac{1}{t^n}\varphi(\frac{x}{t})$. We leave the proof of this remark as an exercise.

We now introduce two families of operators denoted $P_t$ and $Q_t$ of which the first is an approximation of the identity and the second is an "approximation of the zero operator". Let $\varphi, \psi$ be smooth functions that decay at infinity such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$ and $\int_{\mathbb{R}^n} \psi(x)dx = 0$. Define $P_t$ and $Q_t$ by

$$\widehat{P_t f}(\xi) = \hat{\varphi}(t\xi)\hat{f}(\xi)$$
$$\widehat{Q_t f}(\xi) = \hat{\psi}(t\xi)\hat{f}(\xi)$$

for nice functions $f$ in $\mathbb{R}^n$. We immediately observe

**Lemma 3.2.** *If $f \in L^2(\mathbb{R}^n)$, then*

$$\int_0^\infty \|Q_t f\|_2^2 \frac{dt}{t} \leq C(\psi)\|f\|_2^2.$$

*Proof.* Apply Plancherel's formula. □

**Theorem 3.2.** *If $f \in BMO\ (\mathbb{R}^n)$, then*

$$d\mu(x,t) = |Q_t f(x)|^2 \frac{dxdt}{t}$$

*is a Carleson measure with Carleson norm $\leq C(\psi)\|f\|_*^2$.*

To carry through the argument in the proof of this theorem, we need a lemma.

**Lemma 3.3.** *If $f \in BMO\ (\mathbb{R}^n)$ and $Q_0$ is the unit cube (centered at 0), then*

$$\int_{\mathbb{R}^n} \frac{|f(x) - f_{Q_0}|}{1 + |x|^{n+1}} dx \leq C\|f\|_*$$

*where $C$ only depends on $n$.*

*Proof.* For $a > 0$ let $aQ$ denote the cube with sides parallel with the sides of $Q$ and of lengths $a$ times the sidelengths of $Q$ and with the same center as $Q$. We observe that for every cube $Q \subset \mathbb{R}^n$

$$|f_Q - f_{2Q}| \leq \frac{1}{|Q|}\int_Q |f(x) - f_{2Q}|dx \leq \frac{2^n}{|2Q|}\int_{2Q} |f(x) - f_{2Q}|dx \leq 2^n\|f\|_*.$$

Set $Q_j = 2^j Q_0$ for $j \in \mathbb{N}$ and assume $\|f\|_* = 1$. Here

$$|f_{Q_{j+1}} - f_{Q_j}| \leq 2^n \quad \text{for} \quad j \in \mathbb{N}$$

which implies

$$|f_{Q_{j+1}} - f_{Q_0}| \leq (j+1)2^n \quad \text{for} \quad j \in \mathbb{N}.$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x) - f_{Q_0}|}{1 + |x|^{n+1}} dx &= \sum_{j=0}^{\infty} \int_{Q_{j+1} \setminus Q_j} \frac{|f(x) - f_{Q_0}|}{1 + |x|^{n+1}} dx + \\
+ \int_{Q_0} \frac{|f(x) - f_{Q_0}|}{1 + |x|^{n+1}} dx &\leq \sum_{j=0}^{\infty} \Big( \int_{Q_{j+1}} \frac{|f(x) - f_{Q_{j+1}}|}{2^{j(n+1)}} dx + \\
+ \int_{Q_{j+1}} \frac{|f_{Q_{j+1}} - f_{Q_0}|}{2^{j(n+1)}} dx \Big) + 1 &\leq \sum_{j=0}^{\infty} (2^{n-j} + (j+1)2^{2n-j}) + 1 = C
\end{aligned}$$

which completes the proof. $\square$

*Proof of Theorem 3.2.* $Q_t f(x)$ is a well-defined function in $\mathbb{R}^{n+1}_+$ since $Q_t 1 = 0$. We want to prove that for each cube $Q \subset \mathbb{R}^n$

$$(*) \qquad \int\int_{\tilde{Q}} |Q_t f(x)|^2 \frac{dxdt}{t} \leq C(\psi) \|f\|_*^2 |Q|.$$

It is enough to consider $Q =$ unit cube $Q_0$ since BMO is scale- and translation invariant, i.e., $\|f\|_* = \|f_t^s\|_*$ where $f_t^s(x) = f(t(x-s))$ and $\frac{dt}{t}$ is scale invariant. Furthermore we may assume $f_{2Q_0} = 0$ since $Q_t 1 = 0$. Thus we have to prove $(*)$ for $Q = Q_0$ and *all* $f \in$ BMO with $f_{2Q_0} = 0$. Set $f_1 = f\chi_{2Q_0}$ and $f_2 = f - f_1$.

Then $Q_t f = Q_t f_1 + Q_t f_2$ and we obtain

$$\begin{aligned}
&\int\int_{Q_0} |Q_t f_1|^2 \frac{dxdt}{t} \leq \int\int_{\mathbb{R}^{n+1}_+} |Q_t f_1|^2 \frac{dxdt}{t} \leq \\
&\leq C(\psi) \|f_1\|_2^2 \leq C(\psi) \|f\|_*^2
\end{aligned}$$

from lemma 3.2 and for $(x,t) \in \tilde{Q}_0$ we obtain

$$\begin{aligned}
|Q_t f_2(x)| &\leq \int_{\mathbb{R}^n \setminus 2Q_0} \frac{1}{t^n} |\psi(\frac{x-z}{t})| |f_2(z)| dz \leq \\
&\leq C(\psi) \int_{\mathbb{R}^n \setminus 2Q_0} \frac{t}{t^{n+1} + |x-z|^{n+1}} |f_2(z)| dz \leq \\
&\leq C(\psi) t \int_{\mathbb{R}^n} \frac{|f(z)|}{1 + |z|^{n+1}} dz \leq C(\psi) t \|f\|_*
\end{aligned}$$

according to lemma 3.3. This completes the proof. $\square$

We have now prepared the tools we need to prove the theorem of David and Journé.

**Theorem 1.6:** If $T$ is a PVO with $CZ$ type kernel $K$, then

$$T \text{ is bounded on } L^2 \text{ iff } T1 \in \mathrm{BMO}.$$

Here the "only if"-part follows from Proposition 3.1. The "if"-part is the hard part. The proof we present is due to Coifman/Meyer [1]. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi$ radial with support in the unit ball $B_1(0)$ and such that

$$\hat{\varphi}(|\xi|) = 1 + \mathcal{O}(|\xi|^4) \quad \text{as} \quad |\xi| \to 0.$$

Define $P_t$ as above by $\widehat{P_t f}(\xi) = \hat{\varphi}(t|\xi|)\hat{f}(\xi)$. Analogously define $Q_t$ by $\widehat{Q_t f}(\xi) = (t|\xi|)^2\varphi(t|\xi|)f(\xi)$ and $R_t$ by $\widehat{R_t f}(\xi) = (|\xi|)^{-1}\hat{\varphi}'(t|\xi|)\hat{f}(\xi)$. Hence $P_t, Q_t$ and $R_t$ commutes and

$$\frac{d}{dt}P_t^2 = \frac{2}{t}R_t Q_t.$$

This implies

$$\frac{1}{2}\frac{d}{dt}P_t^2 T P_t^2 = \frac{1}{t}(R_t Q_t T P_t^2 + P_t^2 T R_t Q_t).$$

The idea is as follows: We want to show

$$|\langle \eta_1, T\eta_2\rangle| \le C\|\eta_1\|_2\|\eta_2\|_2 \text{ for all } \eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^n).$$

We note that $\langle \eta_1, P_t^2 T P_t^2 \eta_2\rangle \to 0$ as $t \to \infty$ and hence it is enough to prove

$$|\int_0^\infty \frac{d}{dt}\langle \eta_1, P_t^2 T P_t^2 \eta_2\rangle dt| \le C\|\eta_1\|_2\|\eta_2\|_2 \text{ for all } \eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^n)$$

since $P_0$ is the identity operator.

Furthermore, it is enough to prove

$$|\int_0^\infty \langle \eta_1, R_t Q_t T P_t^2 \eta_2\rangle \frac{dt}{t}| \le C\|\eta_1\|_2\|\eta_2\|_2 \text{ for all } \eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^n)$$

since $P_t, Q_t, R_t$ are selfadjoint operators and $T^* = -T$. We need the following estimate.

**Lemma 3.4.** *Let $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball $B_1(0)$ and assume*

$$\int_{\mathbb{R}^n} \psi(x)dx = 0.$$

*Then*

$$|\langle \psi_t^x, T\varphi_t^y\rangle| \le C p_t(x-y)$$

*where $\psi_t^x(z) = \frac{1}{t^n}\psi\big(\frac{z-x}{t}\big), \varphi_t^y(z) = \frac{1}{t^n}\varphi\big(\frac{z-y}{t}\big)$ and $p_t$ is the Poisson kernel.*

*Proof.* The argument consists of a straightforward calculation where we use the $CZ$ type properties of the kernel $K$.

$$2\langle \psi_t^x, T\varphi_t^y\rangle = \lim_{\varepsilon\downarrow 0}\int\int_{|\xi-\eta|>\varepsilon} K(\xi,\eta)(\psi_t^x(\xi)\varphi_t^y(\eta) - \psi_t^x(\eta)\varphi_t^y(\xi))d\eta d\xi$$
$$= \lim_{\varepsilon\downarrow 0}\int\int_{|\xi-\eta|>\varepsilon} K(t\xi+y, t\eta+y)(\psi_1^{\frac{x-y}{t}}(\xi)\varphi(\eta) - \psi_1^{\frac{x-y}{t}}(\eta)\varphi(\xi))d\eta d\xi.$$

Hence, it is enough to prove the lemma for $y=0$.

Assume $|x| < 10t$: Then we obtain

$$|\langle \psi_t^x, T\varphi\rangle| \le \frac{1}{t^n}\frac{C}{(1+(\frac{|x|}{t})^2)^{\frac{n+1}{2}}} = Cp_t(x).$$

Assume $|x| \ge 10t$: Then we obtain

$$|\langle \psi_t^x, T\varphi\rangle| = |\int\int (K(\xi,\eta) - K(x,\eta))\psi_t^x(\xi)\varphi_t(\eta)d\eta d\xi \le$$
$$\le \int\int |K(t\xi, t\eta) - K(x, t\eta)|\psi_1^{\frac{x}{t}}(\xi)\varphi(\eta)d\eta d\xi \le$$
$$\le C\frac{t}{|x|^{n+1}} \le C\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} = Cp_t(x)$$

which concludes the proof of the lemma. □

Now set $L_t = Q_tTP_t$ where

$$L_tf(x) = \int_{\mathbb{R}^n} 1_t(x,y)f(y)dy$$

with $|1_t(x,y)| \le Cp_t(x-y)$ according to Lemma 3.4. We recall that it is enough to show that

$$|\int_0^\infty \langle \eta_1, R_tQ_tTP_t^2\eta_2\rangle\frac{dt}{t}| \le C\|\eta_1\|_2\|\eta_2\|_2 \text{ for all } \eta_1,\eta_2\in\mathcal{S}(\mathbb{R}^n).$$

Hence it is enough to show

$$|\int_0^\infty \langle \eta_1, R_tQ_tTP_t^2\eta_2\rangle\frac{dt}{t}| \le C(\|\eta_1\|_2^2 + \|\eta_2\|_2^2) \text{ for all } \eta_1,\eta_2\in\mathcal{S}(\mathbb{R}^n).$$

But

$$|\int_0^\infty \langle \eta_1, R_tQ_tTP_t^2\eta_2\rangle\frac{dt}{t}| \le \int_0^\infty |\langle R_t\eta_1, Q_tTP_t^2\eta_2\rangle|\frac{dt}{t} \le$$
$$\le \int_0^\infty \|r_t\eta_1\|_2^2\frac{dt}{t} + \int_0^\infty \|Q_tTP_t^2\eta_2\|_2^2\frac{dt}{t} \equiv I + II.$$

Here

$$I=\int_0^\infty \|\frac{\hat{\varphi}'(t|\xi|)}{t|\xi|}\hat{\eta}_1(\xi)\|_2^2\frac{dt}{t}\le C\|\hat{\eta}_1\|_2^2=C\|\eta_1\|_2^2.$$

To cope with II, we rewirte $Q_tTP_t^2\eta_2=L_tP_t\eta_2$ as

$$\begin{aligned}&((L_tP_t)\eta_2)(x)=L_t[P_t\eta_2-P_t\eta_2(x)](x)+P_t\eta_2(x)L_t1(x)=\\&=L_t[P_t\eta_2-P_t\eta_2(x)](x)+P_t\eta_2(x)Q_tT1(x)\end{aligned}$$

<u>But</u> $T1\in\mathrm{BMO}$ implies $|Q_tT1|^2\frac{dxdt}{t}$ is a Carleson measure according to Theorem 3.2 and hence

$$\int_0^\infty \|P_t\eta_2(x)Q_tT1\|_2^2\frac{dt}{t}=\int\int_{\mathbb{R}_+^{n+1}}|P_t\eta_2(x)|^2|Q_tT1|^2\frac{dxdt}{t}\le C\|\eta_2\|_2^2$$

where we have used the remark to Theorem 3.1. Furthermore using Jensen's inequality

$$\begin{aligned}&A(x,t)\equiv|L_t(P_t\eta_2-P_t\eta_2(x))(x)|^2=\\&=|\int_{\mathbb{R}^n}1_t(x,y)(P_t\eta_2(y)-P_2\eta_2(x))dy|^2\le C\int_{\mathbb{R}^n}p_t(x-y)|P_t\eta_2(y)-P_t\eta_2(x)|^2dy\end{aligned}$$

and thus

$$\begin{aligned}&\int_0^\infty\|L_t[P_t\eta_2-P_t\eta_2(x)](x)\|_2^2\frac{dt}{t}=\int_0^\infty\int_{\mathbb{R}^n}A(x,t)\frac{dxdt}{t}\le\\&\le C\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}p_t(x-y)|P_t\eta_2(x)-P_t\eta_2(y)|^2dydx\frac{dt}{t}=\\&=C\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}p_t(x)|P_t\eta_2(x+y)-P_t\eta_2(y)|^2dxdy\frac{dt}{t}\\&=C\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}p_t(x)|(P_t\eta_2(\widehat{\cdot+x)-}P_t\eta_2(\cdot))(\xi)|^2d\xi dx\frac{dt}{t}=\\&=C\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}p_t(x)|\hat{\varphi}(t|\xi|)|^2|1-e^{i\langle x,\xi\rangle}|^2|\hat{\eta}_2(\xi)|^2d\xi dx\frac{dt}{t}.\end{aligned}$$

But

$$\int_{\mathbb{R}^n}p_t(x)|1-e^{i\langle x,\xi\rangle}|^2dx=2-2e^{-|\xi|t}.$$

Thus

$$\begin{aligned}&\int_0^\infty\|L_t[P_t\eta_2-P_t\eta_2(x)](x)\|_2^2\frac{dt}{t}\le\\&\le C\int_{\mathbb{R}^n}\int_0^\infty 2(1-e^{-|\xi|t})|\hat{\varphi}(t|\xi|)|^2\frac{dt}{t}|\hat{\eta}_2(\xi)|^2d\xi\\&\le C\int_{\mathbb{R}^n}(\int_0^{\frac{1}{|\xi|}}(1-e^{-|\xi|t})\frac{dt}{t}+\int_{\frac{1}{|\xi|}}^\infty|\hat{\varphi}(t|\xi|)|^2\frac{dt}{t})|\hat{\eta}_2(\xi)|^2d\xi\le\\&\le C\|\eta_2\|_2^2.\end{aligned}$$

This completes the proof. $\square$

## References

[1] R. R. Coifman/ Y. Meyer: personal communication.

# Chapter 4

# Proof of Theorem 1.3

In this section we prove

**Theorem 1.3:** If $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz function and $K_\varphi(x,y) = \dfrac{1}{x-y+i(\varphi(x)-\varphi(y))}$, then the corresponding operator $T_\varphi$ is bounded on $L^2$ and $\|T_\varphi\| \le C(\|\varphi'\|_\infty)$.

We immediately observe that the kernel $K_\varphi$ is of $CZ$ type and thus the $L^2$ boundedness of the PVO $T_\varphi$ can be proved using Theorem 1.6, i.e., it is enough to prove $T_\varphi 1 \in \text{BMO}$. However, this is not easy.

We give the proof in two steps.

A: There exists an $\varepsilon_0 > 0$ such that if $\|\varphi'\|_\infty \le \varepsilon_0$ then $\|T_\varphi\| \le C(\|\varphi'\|_\infty)$.

B: Removal of the constraint $\|\varphi'\|_\infty \le \varepsilon_0$.

Part A was proved by Calderón [1]. Part B was proved by Coifman/McIntosh/Meyer [2]. The proof we present is due to David [3].

Proof of A: Assume $\varphi \in C_0^\infty(\mathbb{R}^n)$ and define

$$K_N(x,y) = \frac{(\varphi(x)-\varphi(y))^N}{(x-y)^{N+1}} \quad N = 0,1,2,\dots .$$

$T_N$ corresponding PVO with kernel $K_N$.

We remark that the $T_N$'s are called *commutators* and arise naturally when one tries to construct a calculus of singular integral operators to handle differential equations with non-smooth coefficients. We refer to Calderón [4] for an extensive discussion of commutators and PDE's. Since $T = \sum_{N=0}^\infty (-i)^N T_N$, it is enough to prove that

$$\|T_N\| \le C^{N+1} \|\varphi'\|_\infty^N \quad N = 0,1,2,\dots .$$

We also note that it is enough to prove that there exists an $\varepsilon_1 > 0$ such that if $\|\varphi'\|_\infty \leq \varepsilon_1$, then

$$\|T_N\| \leq C^N \quad N = 1, 2, \dots .$$

since $T_0$ is the Hilbert transform and this operator is $L^2$ bounded. There are many proofs of this fact and one proof is supplied by Theorem 1.6. To prove that $T_N \quad N = 1, 2, \dots$ are $L^2$ bounded we make the following observation.

**Lemma 4.1.** *If $\varphi \in C_0^\infty$, then*

$$T_{N+1}(1) = T_N(\varphi'), \quad N = 0, 1, \dots .$$

*Proof.* The lemma is a consequence of the identity

$$\frac{d}{dy}\left(\frac{\varphi(x) - \varphi(y)}{x - y}\right)^{N+1} = (N+1)\frac{(\varphi(x) - \varphi(y))^{N+1}}{(x-y)^{N+2}} - (N+1)\frac{(\varphi(x) - \varphi(y))^N}{(x-y)^{N+1}}\varphi'(y)$$

and

$$\begin{aligned} T_{N+1}1(x) &= \lim_{\varepsilon \downarrow 0} \int_{|x-y| > \varepsilon} \frac{(\varphi(x) - \varphi(y))^{N+1}}{(x-y)^{N+2}} dy = \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{N+1}\left(\frac{\varphi(x) - \varphi(x - \varepsilon)}{\varepsilon} - \frac{\varphi(x) - \varphi(x+\varepsilon)}{-\varepsilon}\right) + T_N\varphi'(x) = T_N\varphi'(x). \end{aligned}$$

□

A recursion argument using Proposition 3.1, Lemma 4.1 and the fact that $\varphi$ Lipschitz function implies $\varphi' \in L^\infty$ shows that $T_N, N = 0, 1, 2, \dots$ are $L^2$ bounded. What remains to be done is to show that

$$\|T_N\| \leq C^N \quad N = 1, 2, \dots ,$$

for some choice of $C > 0$. Here, of course, $\| \ \|$ denotes the norm $\| \ \|_{L^2 \to L^2}$. To conclude the proof of A we note that

1: If $K$ is a kernel of $CZ$ type such that

$$|K(x,y)| + (|\nabla_x K(x,y)| + |\nabla_y K(x,y)|)|x - y| \leq \frac{C_1}{|x-y|^n}$$

and if $\|T1\|_* \leq C_2$, then

$$\|T\| \leq D_n(C_1 + C_2)$$

for some constant $D_n$ which only depends on dimension $n$.

2: Under the same assumptions as in 1

$$\|T\|_{L^\infty \to BMO} \le D_n(\|T\| + C_1)$$

This follows from the proofs of Proposition 3.1 and Theorem 1.6. Hence

$$\|T_{N+1}\| \le C_1 + C_2\|T_N\| \quad N = 0, 1, 2, \ldots$$

for some constants $C_1$, $C_2$ independent of $N$ and

$$\|T_N\| \le C^N \quad N = 1, 2, \ldots$$

for some constant $C > 0$ follows. The proof of A is completed.

Proof of B: We start with three lemmas.

**Lemma 4.2.** *There exists an $\varepsilon_0 > 0$ such that if*

$$\varphi(x) = Ax + \psi(x), \quad x \in \mathbb{R}$$

*where $A \in \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ Lipschitz function with $\|\psi'\|_\infty \le \varepsilon_0$ then*

$$\|T_\varphi\| \le C_0$$

*for some constant $C_0 > 0$ which is independent of $A$.*

*Proof.* Repeating the argument above, we see that if $h : \mathbb{R} \to C$ Lipschitz function with $\|h'\| \le \delta < 1$ then $\|T_h\| \le C_\delta$ where the constant $C_\delta$ is independent of $h$. Furthermore consider $T_\varphi$ with kernel

$$K_\varphi(x, y) = \frac{1}{x - y + i(Ax + \psi(x) - Ay - \psi(y))} = \frac{1}{1 + iA} \cdot \frac{1}{x - y + i(h(x) - h(y))}$$

where $h(x) = \frac{\psi(x)}{1+iA}$. Then the first observation gives the desired result. $\square$

**Lemma 4.3 (David [3]).** *Assume $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz function and $L \in \mathbb{R}$ such that*

$$|\varphi(x) + Lx - (\varphi(y) + Ly)| \le M|x - y| \quad x, y \in \mathbb{R}.$$

*Let $I \subset \mathbb{R}$ be an interval.*

*Then there exists a Lipschtz function $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ and a $\tilde{L} \in \mathbb{R}$ such that*

*(i)* $|\{x \in I : \varphi(x) = \tilde{\varphi}(x)\}| \ge \frac{3}{8}|I|$

*(ii)* $|\tilde{\varphi}(x) + \tilde{L}x - (\tilde{\varphi}(y) + \tilde{L}y)| \le \frac{9}{10}M|x - y| \quad x, y \in \mathbb{R}.$

Remark: $\tilde{L} \in [L-M, L+M]$.

*Proof.* Without loss of generality we can assume that $I = [0,1], M = 1, L = -\frac{4}{5}, \quad U \equiv \{x \in I : \varphi'(x) + L \geq 0\}$ has measure $\geq \frac{1}{2}$. Check this!

Hence $-\dfrac{1}{5} \leq \varphi' \leq \dfrac{9}{5}$.

Define $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(0) & x < 0 \\ \sup_{0 \leq y \leq x} \varphi(y) & 0 \leq x \leq 1 \\ \sup_{0 \leq y \leq 1} \varphi(y) & 1 < x. \end{cases}$$

Then $\tilde{\varphi}$ is an increasing function and

$$\tilde{\varphi}(x+h) = \sup_{0 \leq y \leq x+h} \varphi(y) \leq \sup_{0 \leq y \leq x} \varphi(y) + \frac{9}{5}h = \tilde{\varphi}(x) + \frac{9}{5}h$$

for each $h > 0$ such that $x, x+h \in [0,1]$. Thus $0 \leq \tilde{\varphi}' \leq \frac{9}{5}$ and $\tilde{\varphi}$ satisfies (ii) with $\tilde{L} = \frac{-9}{10}$. Remains to check that (i) is satisfied. Set

$$E = \{x \in I : \varphi(x) = \tilde{\varphi}(x)\}.$$

Then $0 \in E$ and $E$ closed implies that

$$I \setminus E \equiv \Omega = \cup_k I_k$$

where the components $I_k$ are of the form $]a_k, b_k[$ or $]a_k, 1]$ where $0 < a_k < b_k \leq 1$. But $\tilde{\varphi}$ is constant on each interval $I_k$ and thus

$$\begin{array}{ll} \varphi(a_k) = \tilde{\varphi}(a_k) = \tilde{\varphi}(b_k) = \varphi(b_k) & \text{if} \quad I_k = ]a_k, b_k[ \\ \varphi(a_k) = \tilde{\varphi}(a_k) = \tilde{\varphi}(1) \geq \varphi(1) & \text{if} \quad I_k = ]a_k, 1]. \end{array}$$

Hence $\displaystyle\int_{I_k} \varphi'(x)dx \leq 0$ for each $k$ and we obtain

$$\int_\Omega \varphi'(x)dx \leq 0.$$

Finally

$$0 \geq \int_\Omega \varphi'(x)dx = \int_{\Omega \cap U} \varphi'(x)dx + \int_{\Omega \cap (I \setminus U)} \varphi'(x)dx \geq \frac{4}{5}|\Omega \cap U| - \frac{1}{5}|\Omega \cap (I \setminus U)|$$

implies

$$|\Omega \cap U| \leq \frac{1}{4}|\Omega \cap (I \setminus U)|$$

and this gives us

$$|\Omega| = |\Omega \cap U| + |\Omega \cap (I \setminus U)| \leq \frac{5}{4}|\Omega \cap (I \setminus U)| \leq \frac{5}{8}.$$

Hence

$$|E| \geq \frac{3}{8}$$

and the proof is done. □

**Lemma 4.4 (John [6]).** *Assume $f : \mathbb{R} \to \mathbb{R}$ measurable. Assume there exists an $\alpha > 0$ and a continuous function $C : \Sigma \to \mathbb{R}$, where $\Sigma = \{(a,b) \in \mathbb{R}^2 : a < b\}$, such that $|\{x \in I : |f(x) - C(I)| < \alpha\}| > \frac{1}{3}|I|$ for* each *interval $I = (a,b)$. Then $f \in BMO\,(\mathbb{R})$ and $\|f\|_* \leq C\alpha$ where $C$ is independent of $\alpha, f$ and the function $C$.*

*Proof.* It is enough to prove the lemma for $\alpha = 1$. The arguments are similar to those one uses to prove John-Nirenberg's inequality once we have proved the following

Claim: If $I \subset J \subset \mathbb{R}$ are intervals such that $|J| = 2|I|$, then $|C(I) - C(J)| \leq 15$.

Proof of claim: Let $\Lambda$ denote the interval with endpoints $C(I)$ and $C(J)$ and assume to obtain a contradiction that $|\Lambda| > 15$. Then there exists points $z_k \in \Lambda$, $k = 1, 2, \ldots, 6$ such that $z_k \in \Lambda$ and $\min_{k \neq 1} |z_k - z_1| > 2$. Set $M_k = \{x \in J : |f(x) - z_k| < 1\}$. The sets $M_k$ are mutually disjoint. Furthermore the points $z_k$ can be chosen such that $|J| > \sum_{k=1}^{6} |M_k|$. Since $C : \Sigma \to \mathbb{R}$ is continuous, there exist intervals $I_k, k = 1, 2, \ldots, 6$ such that $I \subset I_k \subset J$ and $C(I_k) = z_k$. Hence

$$|J| > \sum_{k=1}^{6} |M_k| \geq \sum_{k=1}^{6} |M_k \cap I_k| \geq 6 \cdot \frac{1}{3}|I| = 2|I|.$$

This contradicts $|J| = 2|I|$ and claim is proven. □

We now show that under the hypothesis in the lemma with $\alpha = 1$

$$\int_I |f(x) - C(I)| dx \leq C|I|$$

for each interval $I \subset \mathbb{R}$. Observe that we do not know whether $f \in L^1_{\text{loc}}$ or not. Since the assumptions on $f$ are scale- and translationinvariant we can assume $I = [0,1]$. We can also assume $C(I) = 0$.

Now, set $\lambda_k = 100k \quad k = 1, 2, \ldots$ and let $\Omega_k = \cup_j I_j^k$ be the union of those intervals in the dyadic mesh of $I$ which are maximal with respect to inclusion with the property $|C(I_j^k)| > \lambda_k$. We see that $I \notin \Omega_1$ and $\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \ldots$ . For each $I_j^k$ in $\Omega_k$ there exists an interval $\tilde{I}_j^k$ in the dyadic mesh of $I$ which is minimal with respect to inclusion and with the property $I_j^k \not\subseteq \tilde{I}_j^k$. Hence $|C(\tilde{I}_j^k)| \leq \lambda_k$ and the claim above implies

$$\lambda_k < |C(I_j^k)| < \lambda_k + 15.$$

We claim that if we can prove that

$$|\Omega_{k+6}| \leq \frac{1}{2}|\Omega_k|$$

we are done.

*Proof.* Consider $A_k = \{x \in I : |f(x)| > \lambda_k + 2\}$ which is a measurable set and take a point of density $x_0 \in A_k$. Then there exists a sufficiently small interval $J$ in the dyadic mesh of $I$ such that $x_0 \in J$ and

$$|\{x \in J : |f(x)| > \lambda_k + 2\}| \geq \frac{99}{100}|J|.$$

This implies $C(J) > \lambda_k$ and thus $x_0 \in J \subset \Omega_k$. Hence $A_k \subset \Omega_k$ except for a set of measure 0 and

$$|A_k| \leq |\Omega_k|.$$

But $|\Omega_{k+6}| \leq \frac{1}{2}|\Omega_k|$ implies that there exist constants $B$, $C > 0$ such that

$$|A_k| \leq Be^{-C(\lambda_k+2)} \quad k = 1, 2, \dots .$$

Choosing a slightly larger constant $C$ we get

$$|\{x \in I : |f(x)| > \lambda\}| \leq Be^{-C\lambda}.$$

The remaining argument is the same as in the proof of the remark on page 29. $\square$

Finally we give the argument for $|\Omega_{k+6}| \leq \frac{1}{2}|\Omega_k|$.

*Proof.* Set $E_k = \{x \in I : |f(x) - \lambda_k| < 16\} \quad k = 1, 2, \dots$. Hence $E_k$ are mutually disjoint and

$$|E_k \cap I_j^k| \geq |\{x \in I_j^k : |f(x) - C(I_j^k)| < 1\}| \geq \frac{1}{3}|I_j^k|.$$

Furthermore, for $k \geq k_0$ we obtain

$$|I_{j_0}^{k_0} \cap E_k| \geq \frac{1}{3}|I_{j_0}^{k_0} \cap \Omega_k|$$

by taking the union overall $I_j^k \subset I_{j_0}^{k_0}$ and using $|E_k \cap I_j^k| \geq \frac{1}{3}|I_j^k|$. Hence

$$\begin{aligned}|I_{j_0}^{k_0}| &\geq \sum_{k_0<k\leq k_0+6} |I_{j_0}^{k_0} \cap E_k| \geq \frac{1}{3} \sum_{k_0<k\leq k_0+6} |I_{j_0}^{k_0} \cap \Omega_k| \geq \\ &\geq 2|I_{j_0}^{k_0} \cap \Omega_{k_0+6}|\end{aligned}$$

which implies $|\Omega_{k+6}| \leq \frac{1}{2}|\Omega_k|$. $\square$

We have now prepared all the machinery we need to prove part B. The idea is to make an *induction* argument where part $A$ is the base and the inductionstep is the following: Let $M > 0$. If there exists a constant

$$C = C(\frac{9}{10}M)$$

such that $\|T_{\tilde{\varphi}}\| \leq C$ for all Lipschitz functions $\tilde{\varphi}$ with $osc\,\tilde{\varphi} \leq \frac{9}{10}M$ then there exists a constant $C = C(M)$ such that $\|T_\varphi\| \leq C$ for all Lipschitz functions $\varphi$ with $osc\,\varphi \leq M$. Here we let $osc\,\varphi$ denote $\inf\{M : |\varphi(x) + Lx - (\varphi(y) + Ly)| \leq M|x-y|$ for all $x \neq y$ and some $L \in \mathbb{R}\}$.

We remark that Lemma 4.2 implies that $\|T_\varphi\|$ only depends on $osc\,\varphi$ and not on $\|\varphi'\|_\infty$. We observe that it is enough to prove the inductionstep for Lipschitz-funtions $\varphi$ with $\varphi \in C^\infty$ and it is enough to prove that there exists a constant $C = C(M)$ such that

$$\|T_\varphi\|_{L^\infty \to \mathrm{BMO}} \leq C$$

for all Lipschitz-functions $\varphi \in C^\infty$ with $osc\,\varphi \leq M$. Take $f \in L^\infty$ with $\|f\|_\infty = 1$. Let $I$ be an interval and let $x_I$ denote the center of $I$. Set $z(x) = x + i\varphi(x)$. Decompose $f$ in $f_1$ and $f_2$ where $f_1 = f\chi_I$ and $f_2 = f - f_1$. Finally set $C_f(I) = T_\varphi f_2(x_I)$. We see that $C$ is a continuous function with respect to the endpoints of $I$. Without loss of generality we assume $I = [0,1]$. From Lemma 4.3 we obtain a Lipschitz function $\psi$ such that $osc\,\psi \leq \frac{9}{10}M$ and $E \equiv \{x \in I : \varphi(x) = \psi(x)\}$ has measure $\geq \frac{3}{8}$. Set $z^*(x) = x + i\psi(x)$. Our purpose is to use the characterization of BMO functions which is given in Lemma 4.4 by showing that

$$|\{x \in I : |T_\varphi f(x) - C_f(I)| < C(M)\}| > \frac{1}{3}$$

for constant $C = C(M)$ chosen large enough and which is independent of $f$. If this is done, Theorem 1.3 follows. We start with

$$\begin{aligned}&|T_\varphi f(x) - C_f(I)| = |T_\varphi f(x) - T_\varphi f_2(x_I)| \leq \\ &\leq |T_\varphi f_2(x) - T_\varphi f_2(x_I) + |T_\varphi f_1(x) - T_\psi f_1(x)| + |T_\psi f_1(x)|\end{aligned}$$

where $x$ belongs to a subset of $I$ which we will define later.

For $x \in I$,

$$|T_\varphi f_2(x) - T_\varphi f_2(x_I)| \le \int_{\mathbb{R}\setminus I} \left|\frac{z(x)-z(\frac{1}{2})}{(z(x)-z(y))(z(\frac{1}{2})-z(y))}\right| |f(y)| dy \le$$
$$\le C'(M) \int_{R\setminus I} \frac{dy}{|x-y||\frac{1}{2}-y|}$$

which implies

$$\int_I |T_\varphi f_2(x) - T_\varphi f_2(x_I)| dx \le C'(M).$$

For $x \in E \subset I$,

$$|T_\varphi f_1(x) - T_\psi f_1(x)| \le \int_{I\setminus E} \left|\frac{z(y)-z^*(y)}{(z(x)-z(y))(z^*(x)-z^*(y))}\right| |f(y)| dy.$$

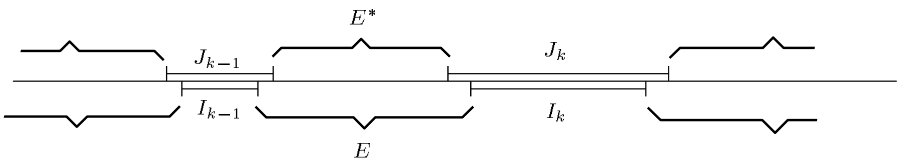
But $I \setminus E = \cup I_k$ where the components $I_k$ are intervals and

$$|z(y) - z^*(y)| \le C'(M)|I_k| \quad \text{for} \quad y \in I_k.$$

Hence

$$|T_\varphi f_1(x) - T_\psi f_1(x)| \le \sum_k \int_{I_k} \frac{C'(M)|I_k|}{|x-y|^2} dy \quad x \in E.$$

Set $J_k = \frac{1001}{1000} I_k$ and $E^* = E \cap \complement(\cup_k J_k)$




Then $|E \setminus E^*| \le \frac{1}{1000}$ and

$$\int_{E^*} |T_\varphi f_1(x) - T_\psi f_1(x)| dx \le C'(M) \sum_k \int_{I_k} \int_{J_k} \frac{|I_k|}{|x-y|^2} dx dy \le C'(M) \sum_k \int_{I_k} dy \le C'(M)$$

Finally, from the hypothesis in the induction step

$$\int_I |T_\psi f_1(x)| dx \le \Big( \int_I |T_\psi f_1(x)|^2 dx \Big)^{1/2} \le C\Big(\frac{9}{10} M\Big).$$

But

$$|\{x \in I : |T_\varphi f_2(x) - T_\varphi f_2(x_I)| \ge \frac{C'(M)}{3}\}| \le \frac{1}{1000}$$
$$|\{x \in E^* : |T_\varphi f_1(x) - T_\psi f_1(x)| \ge \frac{C'(M)}{3}\}| \le \frac{1}{1000}$$
$$|\{x \in I : |T_\psi f_1(x)| \ge \frac{C'(M)}{3}\}| \le \frac{1}{1000}$$

if $C'(M)$ large enough. Hence $|\{x \in E^* : |T_\varphi f(x) - Tf_2(x_I)| < C'(M)\}| \geq |E| - \frac{4}{1000} > \frac{1}{3}$ if $C'(M)$ large enough and where $C'(M)$ is independent of $f$. (Note that we have assumed $\|f\|_\infty = 1$.) Lemma 4.4 concludes that

$$\|T_\varphi\| \leq CC'(M) = C(M)$$

and the induction step is proved. $\square$

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# Chapter 5

# Proof of Theorem 1.4

In this chapter our aim is to prove that the double layer potential $\mathcal{D}$ defined in a special Lipschitz domain $D$ and restricted to $\partial D$ is bounded on $L^2(\partial D)$ which corresponds to proving

**Theorem 1.4:** If $\varphi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz function and

$$K_i(x,y) = \frac{((x,\varphi(x)) - (y,\varphi(y)))_i}{|(x,\varphi(x)) - (y,\varphi(y))|^{n+1}} \quad i = 1,2,\dots,n+1$$

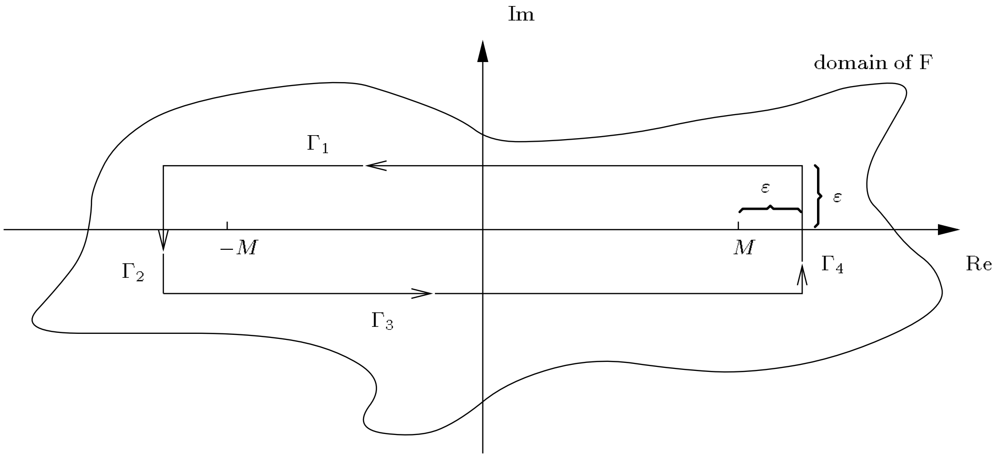
then the corresponding operators $T_i$ are bounded on $L^2$.

The proof of this runs in two steps.

**A:** Assume $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz function with $\|\varphi'\|_\infty \leq M$ and $F$ holomorphic function in a neighborhood of $[-M,M] \times \{0\} \subset \mathbb{C}$. Set $K(x,y) = \dfrac{1}{x-y} \cdot F\big(\dfrac{\varphi(x)-\varphi(y)}{x-y}\big)$. Then $K$ is a CZ type kernel and the corresponding PVO $T$ is bounded on $L^2$ where the bound only depend on $\|\varphi'\|_\infty$ and $F$.

**B:** Theorem 1.4

*Proof of Step A.* Choose $\varepsilon > 0$ so small that the curve $\Gamma$ defined in the figure below belongs to the domain of $F$.




Set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We obtain

$$\begin{aligned} Tf(x) &= \int_{\mathbb{R}} \frac{1}{x-y} F\big(\frac{\varphi(x)-\varphi(y)}{x-y}\big) f(y)dy = \\ &= \frac{1}{2\pi i} \int_{\Gamma} F(w) \int_{\mathbb{R}} \frac{f(y)}{(x-y)(w - \frac{\varphi(x)-\varphi(y)}{x-y})} dydw = \\ &= \frac{1}{2\pi i} \sum_{k=1}^{4} \int_{\Gamma_k} F(w) \int_{\mathbb{R}} \frac{f(y)}{w(x-y) - (\varphi(x)-\varphi(y))} dydw. \end{aligned}$$

It remains to estimate

$$\int_{\mathbb{R}} \frac{f(y)}{w(x-y) - (\varphi(x)-\varphi(y))} dy$$

uniformly in $w \in \Gamma_k$ for $k = 1, 2, 3$ and $4$.

**Case 1:** $w \in \Gamma_1$. Thus $w = \xi + i\varepsilon$ where $\xi \in [-M-\varepsilon, M+\varepsilon]$ and

$$w(x-y) - (\varphi(x)-\varphi(y)) = i\varepsilon\big[x - y + i\big(\frac{\varphi(x)+\xi x}{\varepsilon} - \frac{\varphi(y)+\xi y}{\varepsilon}\big)\big].$$

Hence, if we set $\varphi$ in Theorem 1.3 equal to $\dfrac{\varphi(x)+\xi x}{\varepsilon}$ we get that

$$\int_{\mathbb{R}} \frac{f(y)}{w(x-y) - (\varphi(x)-\varphi(y))} dy$$

is bounded on $L^2$ uniformly in $w \in \Gamma$.

**Case 2:** $w \in \Gamma_2$. Thus $w = -M - \varepsilon + i\eta$ where $\eta \in [-\varepsilon, \varepsilon]$ and

$$w(x-y) - (\varphi(x)-\varphi(y)) = (-M-\varepsilon)x - \varphi(x) - ((-M-\varepsilon)y - \varphi(y)) + i\eta(x-y).$$

Set $\psi(x) = (-M-\varepsilon)x - \varphi(x)$. Then $-2M-\varepsilon \le \psi'(x) \le -\varepsilon$. Thus $\psi^{-1}$ exists and $\psi^{-1}$ Lipschitz function. After a change of variables it follows that

$$\int_{\mathbb{R}} \frac{f(y)}{w(x-y)-(\varphi(x)-\varphi(y))}dy = \int_{\mathbb{R}} \frac{f\circ\psi^{-1}(t)\cdot(\psi^{-1})'(t)}{\psi(x)-t+i\eta(\psi^{-1}(\psi(x))-\psi^{-1}(t))}dt =$$

$$= (T_{\eta\psi^{-1}}(f\circ\psi^{-1}\cdot(\psi^{-1})'))\circ\psi(x)$$

using obvious notation and hence

$$\|\int_{\mathbb{R}} \frac{f(y)}{w(x-y)-(\varphi(x)-\varphi(y))}dy\|_2 \le C\|T_{\eta\psi^{-1}}(f\circ\psi^{-1}(\psi^{-1})')\|_2 \le$$

$$\le C\|f\circ\psi^{-1}\cdot(\psi^{-1})'\|_2 \le C\|f\|_2.$$

The curves $\Gamma_3$ and $\Gamma_4$ are treated similarily and part A is proved. □

*Proof of step B.* To lift the result form $\mathbb{R}$ to $\mathbb{R}^n$ we apply the *method of rotation.* To simplify the notation, we only consider the kernel

$$K(x,z) = \frac{\varphi(x)-\varphi(z)}{(|x-z|^2+(\varphi(x)-\varphi(z))^2)^{\frac{n+1}{2}}}$$

$x, z \in \mathbb{R}^n$. Assume $f \in C_0^\infty$.

$$Tf(x) = \int_{\mathbb{R}^n} K(x,z)f(z)dz =$$
$$= \frac{1}{2}\int_{\mathbb{R}^n} (K(x,x+z)f(x+z)+K(x,x-z)f(x-z))dz.$$

Introduce polar coordinates. Then

$$Tf(x) = \int_{S^{n-1}} T_\omega f(x)d\omega \text{ where } T_\omega f(x) = \int_{-\infty}^{\infty} K(x,x+r\omega)f(x+r\omega)|r|^{n-1}dr$$

and it is enough to prove that $\|T_\omega f\|_2 < C\|f\|_2$ where $C$ independent of $\omega$. Let $E_\omega$ denote the ortogonal complement of the 1 dimension space $\{t\omega : t \in \mathbb{R}\} \subset \mathbb{R}^n$. Any $x \in \mathbb{R}^n$ can uniquely be written as $t\omega + y$, $y \in E_\omega$ and

$$T_\omega f(t\omega+y) = \int_{-\infty}^{\infty} K(t\omega+y,(t+r)\omega+y)f((t+r)\omega+y)|r|^{n-1}dr =$$
$$= \int_{-\infty}^{\infty} K(t\omega+y,s\omega+y)f(s\omega+y)|s-t|^{n-1}ds$$
$$= \int_{-\infty}^{\infty} \frac{\frac{\varphi(t\omega+y)-\varphi(s\omega+y)}{t-s}}{(t-s)(1+\frac{\varphi(t\omega+y)-\varphi(s\omega+y)}{t-s})^2)^{\frac{n+1}{2}}}f(s\omega+y)ds$$

Finally, we apply Part A with $F(z) = \dfrac{z}{(1+z^2)^{\frac{n+1}{2}}}$ which is holomorphic in a neighborhood of the real axis. Hence

$$\|T_\omega f(x)\|_2^2 \leq C(\|\varphi'\|_\infty, F) \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |f(t\omega + y)|^2 dt dy = C(\|\varphi'\|_\infty, F)\|f\|_2^2.$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ the result in Part B follows. $\square$

# Chapter 6

# Dirichlet Problem for Lipschitz domains. The final arguments for the $L^2$-theory

We are now able to complete the proof of the following theorems where, as usual, $\varphi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz function and $D = \{(x, y) \in R^{n+1} : \varphi(x) < y\}$.

**Theorem 6.1.** *If $f \in L^2(\partial D)$, then there exists a $u$ such that*

$$\begin{cases} \Delta u = 0 & in \quad D \\ u|_{\partial D} = f & on \quad \partial D \end{cases}$$

*where the boundary values are taken non-tangentially a.e. $\partial D$ and $M_\beta u \in L^2(\partial D)$ with $\|M_\beta u\|_2 \le C\|f\|_2$ where $C$ only depends on $\beta > 1$ and $\|\varphi'\|_\infty$.*

**Theorem 6.2.** *If $f \in L^2(\partial D)$, then there exists a $u$ such that*

$$\begin{cases} \Delta u = 0 & in \quad D \\ \frac{\partial u}{\partial n}|_{\partial D} = f & on \quad \partial D \end{cases}$$

*where the boundary values are taken in the sense $n_p \cdot \nabla u(Q) \to f(P)$ as $Q \to P$ non-tangentialy a.e. $\partial D$ and $M_\beta(\nabla u) \in L^2(\partial D)$ with $\|M_\beta(\nabla u)\|_2 \le C\|f\|_2$ where $C$ only depends on $\beta > 1$ and $\|\varphi'\|_\infty$.*

We recall that

$$M_\beta u(P) = \sup\{|u(Q)| : |P - Q| < \beta \text{ dist } (Q, \partial D)\}, \quad P \in \partial D$$

and that $n_p$ denotes the outward unit normal at $P \in \partial D$ which is defined a.e. on $\partial D$.

**Corollary 6.1.** Theorem 6.1 is valid if $L^2$ replaced by $L^p$ for $2 \le p \le \infty$.

This is a straightforward consequence of the maximum principle and interpolation between $L^2$ and $L^\infty$, but apart from this result we discuss the $L^p$-theory for the Dirichlet problem and Neumann problem in Chapter 7. The proofs of Theorems 6.1 and 6.2 involves the layer potentials

$$\mathcal{D}f(P) = \int_{\partial D} \frac{\partial}{\partial n_Q} r(P,Q) f(Q) d\sigma(Q), \quad P \in D$$
$$\mathcal{S}f(P) = \int_{D} r(P,Q) f(Q) d\sigma(Q), \quad P \in D$$

where $r(P,Q) = c_n|P-Q|^{1-n}$. With $Q = (x, \varphi(x)) \in \partial D$ and $P = (z,y) \in D$

$$\mathcal{D}f(z,y) = c_n \int_{\mathbb{R}^n} \frac{y - \varphi(x) - (z-x)\cdot\nabla\varphi(x)}{(|x-z|^2 + (\varphi(x)-y)^2)^{\frac{n+1}{2}}} f(x) dx$$
$$\mathcal{S}f(z,y) = c_n \int_{\mathbb{R}^n} \frac{\sqrt{1+|\nabla\varphi(x)|^2}}{(|x-z|^2 + (\varphi(x)-y)^2)^{\frac{n-1}{2}}} f(x) dx.$$

From Proposition 1.1, the remark on page 17 and Theorems 1.2 and 1.4 it follows that

$$\|M_\beta(\mathcal{D}f)\|_p \leq C(\beta, \|\varphi'\|_\infty)\|f\|_p, \qquad f \in L^p$$

for $1 < p < \infty$. Thus $\mathcal{D}|_{\partial D} f(P) = \lim_{\substack{D \ni Q \to P \\ \text{non-tangentially}}} \mathcal{D}f(Q)$ exists and it remains to show that $\mathcal{D}|_{\partial D}$ is invertible on $L^p$. We now observe that $D_- = \{(x,y) \in R^{n+1} : \varphi(x) > y\}$ is also a special Lipschitz domain and we have the following jump relations at the interface between $D$ and $D_-$.

**Lemma 6.1.** *Let $f \in L^2(\partial D)$ and let*

$$Tf(P) = \lim_{\varepsilon \downarrow 0} \int_{|P-Q|>\varepsilon} \frac{\partial}{\partial n_Q} r(P,Q) f(Q) d\sigma(Q), \quad P \in \partial D$$

*Then*

$$\mathcal{D}|_{\partial D} f(p) = \frac{1}{2} f(P) + Tf(P) \qquad a.e.\ \partial D$$
$$\mathcal{D}|_{\partial D_-} f(P) = -\frac{1}{2} f(P) + Tf(P) \qquad a.e.\ \partial D$$
$$\frac{\partial}{\partial n_p} \mathcal{S}|_{\partial D} f(P) = -\frac{1}{2} f(P) + T^* f(P) \qquad a.e.\ \partial D$$
$$\frac{\partial}{\partial n_P} \mathcal{S}|_{\partial D_-} f(P) = \frac{1}{2} f(P) + T^* f(P) \qquad a.e.\ \partial D$$

*where $T^*$ is the adjoint operator of $T$.*

We observe that it is equivalent to solve the Dirichlet problem in $D$ and the Neumann problem in $D_-$. The jumprelations are similar to those in the case of regular boundary in chapter 0 with the difference that here the operator $T$ has to be interpreted as a PVO. The main ingredient in the proof is Proposition 1.1. The idea is to approximate $\partial D$ by a $C^2$-boundary and use the result from Chapter 0. We leave the proof as an exercise.

The final step in the proof of Theorem 6.1 and 6.2 is to prove that $\pm\frac{1}{2}I + T$ and $\pm\frac{1}{2}I + T^*$ are invertible on $L^2$. The proof of this is due to Verchota [1], who used an indentity due to Rellich [2]. See also Jerison/Kenig [3].

**Lemma 6.2.** *Let $f \in L^2(\partial D)$ and $u = \mathcal{S}f|_{\bar{D}}$. Then there exists $C_1, C_2 > 0$ such that*

$$C_1 \int_{\partial D} (\frac{\partial u}{\partial n})^2 d\sigma \leq \int_{\partial D} |\nabla_t u|^2 d\sigma \leq C_2 \int_{\partial D} (\frac{\partial u}{\partial n})^2 d\sigma$$

*where $\nabla_t$ denotes the "tangential derivative".*

Remark: Let $\pi : \partial D \to \mathbb{R}^n$ denote the projection mapping $(x, \varphi(x)) \mapsto x$. Then $\nabla_t u$ is $\nabla(u \circ \pi^{-1})$ lifted with $\pi^{-1}$ to $\partial D$.

*Proof.* Assume $f$ has compact support. Set $e = (\underline{0}, 1)$. Since $\Delta u = 0$ in $D$ we get

$$\text{div } (|\nabla u|^2 e - 2\frac{\partial u}{\partial y}\nabla u) = 0$$

called *Rellich identity*. Apply the divergence theorem.

Hence

$$(*) \qquad \int_{\partial D} |\nabla u|^2 \langle e, n\rangle d\sigma = 2\int_{\partial D} \frac{\partial u}{\partial y}\frac{\partial u}{\partial n} d\sigma.$$

Since $0 < c_0 \leq \langle e, n\rangle \leq 1$ for Lipschitz domain, we obtain

$$c_0 \int_{\partial D} |\nabla u|^2 d\sigma \leq 2\int_{\partial D} |\nabla u| |\frac{\partial u}{\partial n}| d\sigma$$

and Schwartz inequality implies

$$\int_{\partial D} |\nabla u|^2 d\sigma \leq C \int_{\partial D} (\frac{\partial u}{\partial n})^2 d\sigma$$

Remains to prove the reversed inequality.

$$\frac{\partial u}{\partial y} = \langle \nabla u, e\rangle$$

where $e = \langle e, n\rangle n + e_t$ on $\partial D$. Hence

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial n}\langle e, n\rangle + \langle \nabla u, e_t\rangle$$

which we introduce in formula $(*)$ above together with $|\nabla u|^2 = (\frac{\partial u}{\partial n})^2 + |\nabla_t u|^2$ and $|\langle \nabla u, e_t\rangle| \leq |\nabla_t u|$.

Then we obtain

$$\int_{\partial D} |\nabla_t u|^2 \langle e, n\rangle d\sigma = \int_{\partial D} (\frac{\partial u}{\partial n})^2 \langle e, n\rangle d\sigma + 2\int_{\partial D} \frac{\partial u}{\partial n}\langle \nabla u, e_t\rangle d\sigma.$$

Thus

$$\int_{\partial D} (\frac{\partial u}{\partial n})^2 d\sigma \leq C\Big(\int_{\partial D} |\nabla_t u|^2 d\sigma + \Big(\int_{\partial D} (\frac{\partial u}{\partial n})^2 d\sigma\Big)^{\frac{1}{2}}\Big(\int_{\partial D} |\nabla_t u|^2 d\sigma\Big)^{\frac{1}{2}}\Big)$$

and

$$\int_{\partial D} (\frac{\partial u}{\partial n})^2 d\sigma \leq C\int_{\partial D} |\nabla_t u|^2 d\sigma$$

$\square$

To prove that $\pm\frac{1}{2}I + T$ and $\pm\frac{1}{2}I + T^*$ are invertible on $L^2$, it is enough to prove that $\pm\frac{1}{2}I + T^*$ are invertible. We first claim that there exists a $C > 0$ such that

$$\|(\pm\frac{1}{2}I + T^*)f\|_2 \geq C\|f\|_2 \quad f \in L^2(\partial D)$$

*Proof.* Assume $\|(\frac{1}{2}I + T^*)f\|_2 = \varepsilon\|f\|_2$ for some $f \in L^2(\partial D)$. But

$$\begin{aligned}\|\frac{1}{2}f + T^*f\|_2 =& \|\frac{\partial}{\partial n}\mathcal{S}|_{\partial D_-}f\|_2 \approx \|\nabla_t \mathcal{S}|_{\partial D_-}f\|_2 = \\ &= \|\nabla_t \mathcal{S}|_{\partial D}f\|_2 \approx \|\frac{\partial}{\partial n}\mathcal{S}|_{\partial D}f\|_2 = \|-\frac{1}{2}f + T^*f\|_2\end{aligned}$$

since $\nabla_t \mathcal{S}f$ is continuous across the boundary.

Now

$$f = (\frac{1}{2}f + T^*f) - (-\frac{1}{2}f + T^*f)$$

and

$$\|\frac{1}{2}f + T^*f\|_2 \approx \|-\frac{1}{2}f + T^*f\|_2$$

implies that $\varepsilon$ above cannot be too small. The proof is completed. $\square$

The final step for proving the invertability of $\pm\frac{1}{2}I+T^*$ is done with a method of continuity argument. Let $U^s$ donote the operator $\pm\frac{1}{2}I+T^*$ where $\varphi$ is replaced by $s\varphi$. We note that $U^s : L^2 \to L^2$ bounded such that for $0 \leq s \leq 1$

$$\|U^s f\|_2 \geq C\|f\|_2, \quad C \text{ independent of } s \tag{6.1}$$

$$\|U^s f - U^t f\|_2 \leq C|t-s|\|f\|_2, \quad C \text{ independent of } s \text{ and } t \tag{6.2}$$

$$U^0 = \frac{1}{2}I \quad \text{invertible} \tag{6.3}$$

The invertability follows easily.

*Proof.* Set $S = \{s \in [0,1] : U^s \text{ invertible}\}$. Then that $S \neq \phi$ follows from (6.3), that $S$ is open which follows from (6.2) and that $S$ is closed is a consequence of (6.1) and (6.2). We only indicate the closedness of $S$. Assume $s_j \to s$ and $U(s_j)$ invertible. Take $g \in L^2$ and $f_j \in L^2$ such that $U(s_j)f_j = g$. (6.1) implies $f_j \rightharpoonup f$ in $L^2$ for a subsequence. We can also assume $U(s)f_j \rightharpoonup U(s)f$.

Claim: $U(s)f = g$.

Take an $h \in L^2$. Then

$$|\langle U(s)f - g, h\rangle| \leq |\langle U(s)f - U(s)f_j, h\rangle| + |\langle (U(s) - U(s_j))f_j, h\rangle| \to 0$$

as $j \to \infty$. Hence $U(s)f = g$. $\square$

Remark on uniqueness of solutions in Theorem 6.1 and 6.2: The $u$ appearing in Theorem 6.1 is unique while the $u$ appearing in Theorem 6.2 is uniquely defined up to an additive constant.

## References


[1] G. C. Verchota: Layer potentials and boundary value problems for Laplace's equation in Lispchitz domain. (to appear in J. Funct. Anal.)

[2] F. Rellich: Darstellung der eigenwerte von $\nabla u + \lambda u$ durch ein Randintegral Math. Z. 46 (1940) pp. 635 - 646.

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# Chapter 7

# Existence of solutions to Dirichlet and Neumann problems for Lipschitz domains. The optimal $L^p$-results

In this chapter we give parts of the proofs of

**Theorem 7.1.** *For every Lipschitz domain $D = \{(x,y) \in \mathbb{R}^{n+1} : \varphi(x) < y\}$ for $\varphi : \mathbb{R}^n \to \mathbb{R}$ Lipschitz function there exists an $\varepsilon = \varepsilon(D) > 0$ such that for all $2 - \varepsilon < p \leq \infty$ and all $f \in L^p(\partial D)$ there exists an $u$ such that*

$$\begin{aligned} \Delta u &= 0 \quad in \quad D \\ u|_{\partial D} &= f \quad on \quad \partial D \end{aligned}$$

*where the boundary values are taken non-tangentially a.e. $\partial D$ and such that $M_\beta u \in L^p(\partial D)$ with $\|M_\beta u\|_p \leq C\|f\|_p$ where $C$ only depends on $\beta > 1$ and $\|\varphi'\|_\infty$.*

**Theorem 7.2.** *For every Lipschiatz domain $D$ as above there exists an $\varepsilon = \varepsilon(D) > 0$ such that for all $1 < p < 2 + \varepsilon$ and all $f \in L^p(\partial D)$ there exists an $u$ such that*

$$\begin{aligned} \Delta u &= 0 \quad in \quad D \\ \frac{\partial u}{\partial n}|_{\partial D} &= f \quad on \quad \partial D \end{aligned}$$

*where the boundary values are taken in the sense $n_p \cdot \nabla u(Q) \to f(P)$ as $Q \to P$ non-tangentialy a.e. $\partial D$ and such that $M_\beta \nabla u \in L^p(\partial D)$ with $\|M_\beta \nabla u\|_p \leq C\|f\|_p$ where $C$ only depends on $\beta > 1$ and $\|\varphi'\|_\infty$.*

Remark: The ranges $2 - \varepsilon < p \leq \infty$ for the Dirichlet problem and $1 < p < 2 + \varepsilon$ for the Neumann problem are optimal. The estimate $\|M_\beta \nabla u\|_1 \leq C\|f\|_1$ fails even for smooth regions and for each $p > 2$ it is possible to construct a Lipschitz domain such that $\|M_\beta \nabla u\|_p \le C\|f\|_p$ fails. The situation is analogous for the Dirichlet problem. See Dahlberg [1]. The extension $2 - \varepsilon < p < 2$ for the Dirichlet problem and $2 < p < 2 + \varepsilon$ for the Neumann problem is done by a real variable argument using the result for $p = 2$ an a "good $\lambda$ inequality". The extension to $1 < p < 2$ in the Neumann problem is shown by establishing that for $f \in H^1_{at}$, the atomic $H^1$ on $\partial D$, the solution of the Neumann problem with data $f$ satisfies $\|M_\beta \nabla u\|_1 \le C\|f\|_{H^1_{at}}$. This is done by estimating the maximal function of gradient of the $L^2$ solutions of atoms using the regularity theory for uniformly elliptic equations in self-adjoint form. The full result then follows by interpolation. The extension to $2 < p \le \infty$ in the Dirichlet problem is a consequence of the maximum principle and interpolation.

We begin to discuss the regularity theory for uniformly elliptic equations in self-adjoint form. Let $A(x) = (a_{ij}(x))$ be a $n \times n$ dimensional symmetric matrix valued function in $D$ where the entries $a_{ij}(x)$ are bounded real-valued measurable functions. We assume that $A(x)$ is uniformly elliptic on $D$, i.e., there exists a $\lambda \ge 1$ such that

$$\frac{1}{\lambda}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \text{ for all } \xi \in R^n$$

Let $L$ denote the operator $\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij}(x) \frac{\partial}{\partial x_j}$. We call $u$ a (weak) *solution* of $Lu = 0$ in $D$ if $u \in L^2_{1,\mathrm{loc}}(D)$ and $\displaystyle\int_D \langle A\nabla u, \nabla\varphi\rangle dx = 0$ for all $\varphi \in C_0^\infty(D)$. Here $L^2_{1,\mathrm{loc}}$ denotes the space of functions in $L^2_{\mathrm{loc}}(D)$ with distributional derivatives of first order in $L^2_{\mathrm{loc}}(D)$. We say that $u$ is a *subsolution (supersolution)* of $Lu = 0$ in $D$ if $u \in L^2_{1,\mathrm{loc}}(D)$ and $\displaystyle\int_D \langle A\nabla u, \nabla\varphi\rangle dx \le 0$ ($\displaystyle\int_D \langle A\nabla u, \nabla\varphi\rangle dx \ge 0$) for all $0 \le \varphi \in C_0^\infty(D)$. The main result is

**Theorem 7.3 (DeGiorgi [2], Nash [3]).** *If $u$ is a solution of $Lu = 0$ in $D$, then $u$ is Hölder continuous.*

This follows from

**Theorem 7.4 (Harnack's inequality).** *If $u \ge 0$ and $Lu = 0$ in $D$ and if $K \subset D$ is a compact set, then*

$$ess\ sup_K u \le C\ ess\ inf_K u$$

*where $C = C(n, \lambda, K, D)$.*

Remark: Harnack's inequality is a quantitative version of the maximum principle.

Remark: For notational convenience we let min and max denote ess inf and ess sup, resp.

*Proof of Theorem 7.3.* This is done using Harnack's inequality. Assume $Lu = 0$ in $D = \{x \in \mathbb{R}^n : |x| < 2\}$. Set

$$\begin{matrix} M(r) = \max_{|x|\le r} u \\ m(r) = \min_{|x|\le r} u \end{matrix} \qquad r < 1.$$

Then $M(r) - u$ and $u - m(r)$ are solutions and $\geq 0$ in $\{x \in R^n : |x| \leq r\}$. Hence

$$M(r) - m(\frac{r}{2}) \leq C(M(r) - M(\frac{r}{2}))$$
$$M(\frac{r}{2}) - m(r) \leq C\big(m(\frac{r}{2}) - m(r)\big)$$

Add these inequalities and set $\eta(r) = M(r) - m(r)$. We obtain

$$\eta(\frac{r}{2}) \leq \frac{C-1}{C+1}\eta(r)$$

and hence

$$\eta(r) \leq r^\alpha \eta(1) \quad \text{for some} \quad \alpha > 0.$$

Note that we have used the same constant $C$ in the repeated uses of Harnack's inequality. This is justified by the scale invariance properties of $C$. We leave it as an exercise to check this. $\square$

*Proof of Theorem 7.4.* (The proof is due to Moser [4]). Set $Q(h) = \{x \in \mathbb{R}^n : |x_j| < \frac{h}{2}\}$. By a covering argument it is enough to prove the theorem for $K = \overline{Q(1)}$ and $D = Q(4)$. Thus we assume $u \geq 0$ and $Lu = 0$ in $Q(4)$. Set

$$\varphi(p, h) = \big(\frac{1}{|Q(h)|}\int_{Q(h)} u^p dx\big)^{1/p} \equiv \big(⨍_{Q(h)} u^p dx\big)^{1/p}$$

for $0 < h < 4$ and $-\infty < p < \infty$. For $-\infty < p < 0$ we study $u + \varepsilon$ for $\varepsilon > 0$ small and then let $\varepsilon$ tend to zero in the estimates. Since

$$\max_{Q(h)} u = \lim_{p\to\infty} \varphi(p, h)$$
$$\min_{Q(h)} u = \lim_{p\to-\infty} \varphi(p, h),$$

the theorem is equivalent to show that for some $C = C(\lambda, n)$ we have

$$\varphi(\infty, 1) \leq C\varphi(-\infty, 1).$$

The proof is based on three general inequalties relating integrals of functions $v = v(x)$ to integrals of the gradient of $v$. We assume $n \geq 3$.

**Inequality A** (Poincare's inequality)

$$⨍_{Q(h)} |v - v_{Q(h)}|^2 dx \le Ch^2 ⨍_{Q(h)} |\nabla v|^2 dx$$

where $v_{Q(h)} = ⨍_{Q(h)} v dx$.

**Inequality B** (Sobolev's inequality). Set $\mathcal{K} = \dfrac{n}{n-2}$.

$$(⨍_{Q(h)} |v|^{2\mathcal{K}} dx)^{\frac{1}{\mathcal{K}}} \le C(h^2 ⨍_{Q(h)} |\nabla v|^2 dx + ⨍_{Q(h)} |v|^2 dx)$$

**Inequality C** (John-Nirenberg's inequality). If $\|v\|_* \le 1$, then there exists $\alpha > 0$ and $C$ only depeding on $n$ such that

$$\int_{Q(2)} e^{\alpha v} dx \int_{Q(2)} e^{-\alpha v} dx \le C.$$

The proof will be given in two parts viz. $\square$

**Proposition 7.1.** *If $u \ge 0$, subsolution in $Q(4)$, then*

$$\max_{Q(1)} u \le C\big(\frac{p}{p-1}\big)^2 \big(⨍_{Q(2)} u^p dx\big)^{1/p} \quad for \ p > 1$$

**Proposition 7.2.** *If $u > 0$, supersolution in $Q(4)$, then*

$$(⨍_{Q(3)} u^p dx)^{1/p} \le C\big(\frac{1}{\mathcal{K} - p}\big)^2 \min_{Q(1)} u \quad for \quad 0 < p < \mathcal{K}.$$

Since $\mathcal{K} > 1$, the theorem follows from the propositions. The proof of the propositions will be done by estimating $\varphi(p,h)/\varphi(p',h')$ for $p > p'$ and derive the desired estimates by iteration. We need the following lemma.

**Lemma 7.1.** *If $u > 0$ is a subsolution in $D$ and $v = u^\beta$, then for any function $\eta \in C_0^\infty(D)$ one has*

$$\int_D \eta^2 |\nabla v|^2 dx \le C\big(\frac{\beta}{2\beta - 1}\big)^2 \int_D |\nabla \eta|^2 v^2 dx \quad if \quad \beta > \frac{1}{2}.$$

*The same assertion is true for supersolutions if $\beta < \frac{1}{2}$.*

*Proof.* $u$ subsolution in $D$ implies that $\int_D \langle A\nabla u, \nabla\varphi\rangle dx \leq 0$ for all $0 \leq \varphi \in C_0^\infty(D)$. Choose $\eta \in C_0^\infty(D)$ and $\alpha > 0$ and set $\varphi = u^\alpha \eta^2$. (This $\varphi$ does not belong to $C_0^\infty(D)$, but we have the approximation business as an exercise to the reader). This implies

$$\int_D \langle A\nabla u, \nabla u\rangle \alpha u^{\alpha-1}\eta^2 dx + \int_D \langle A\nabla u, \nabla\eta\rangle 2\eta u^\alpha dx \leq 0.$$

The uniform ellipticity of $A$ implies

$$\frac{1}{\lambda}\int_D |\nabla u|^2 \alpha u^{\alpha-1}\eta^2 dx + \int_D \langle A\nabla u, \nabla\eta\rangle 2\eta u^\alpha dx \leq 0.$$

Introduce $v = u^\beta$ where $\beta = \dfrac{\alpha+1}{2}$. Then we obtain

$$\frac{\alpha}{\beta^2}\int_D |\nabla v|^2\eta^2 dx \leq C\int_D |\langle\nabla u, \nabla\eta\rangle| \cdot |\eta| \cdot |u^{\frac{\alpha+1}{2}}||u^{\frac{\alpha-1}{2}}|dx \leq$$

$$\leq C\frac{1}{\beta}\int_D (|\nabla\eta|v)(|\nabla v|\cdot|\eta|)dx$$

Thus

$$\int_D |\nabla v|^2\eta^2 dx \leq c(\frac{\beta}{2\beta-1})^2\int_D |\nabla\eta|^2 v^2 dx, \quad \beta > \frac{1}{2}.$$

The argument for $u$ supersolution and $\beta < \frac{1}{2}$ is similar. $\square$

Now let $0 < h' < h < 2h' < 4$ and choose $\eta \in C_0^\infty(Q(h))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $Q(h')$ and $|\nabla\eta| \leq C\dfrac{1}{h-h'}$. Since $u$ solution to $Lu = 0$ in $Q(h)$, $u$ is both a subsolution and a supersolution and Lemma 7.1 implies that for $\beta \neq \frac{1}{2}$

$$⨍_{Q(h')} |\nabla u^\beta|^2 dx \leq C(\frac{2\beta}{(2\beta-1)})\frac{1}{(h-h')^2}\int_{Q(h)} u^{2\beta}dx.$$

Set $p = 2\beta$. Inequality $B$ gives us

$$(⨍_{Q(h')} u^{p\mathcal{K}}dx)^{\frac{1}{\mathcal{K}}} \leq C(\frac{p}{p-1})^2\frac{1}{\left(\frac{h}{h'}-1\right)^2}⨍_{Q(h)} u^p dx,$$

i.e,

$$\varphi(\mathcal{K}p, h') \leq C(\frac{p}{p-1})^{\frac{2}{p}}(\frac{h}{h'}-1)^{-\frac{2}{p}}\varphi(p,h) \quad \text{if } p > 1$$

and

$$\varphi(\mathcal{K}p, h') \geq C(\frac{h}{h'}-1)^{-\frac{2}{p}}\varphi(p,h) \quad \text{if } p < 0.$$

Now for $p > 1$ let

$$\begin{aligned} p_\nu &= \mathcal{K}^\nu p \\ h_\nu &= 1 + 2^{-\nu} \qquad\qquad \nu = 0, 1, 2, \ldots \\ h'_\nu &= h_{\nu+1} \end{aligned}$$

We find since $\prod_{i=1}^{\nu} (\frac{\mathcal{K}^i p}{\mathcal{K}^i p - 1}) \leq C$ that

$$\varphi(p_{\nu+1}, h_{\nu+1}) \leq C^{\nu/\mathcal{K}\nu} \varphi(p_\nu, h_\nu) \quad \text{for} \quad \nu = 1, 2, \ldots$$

and iteration yields

$$\varphi(p_{\nu+1}, h_{\nu+1}) \leq C(\frac{p}{p-1})^2 \varphi(p, h) = C(\frac{p}{p-1})^2 \varphi(p, 2).$$

But $\limsup_{\nu\to\infty} \varphi(p_\nu, h_\nu) \geq \varphi(+\infty, 1)$ and the proof of Proposition 7.1 is completed.

To prove Proposition 7.2, we first note that

$$\varphi(-\infty, 1) \geq C\,\varphi(-q, 2) \quad q > 0$$

follows if we apply the same iteration technique as above to $p < 0$ and especially to $-q < 0$ close to 0. What remains to be shown is that

$$\varphi(-q, 2) \geq C\varphi(q, 2)$$

and

$$\varphi(q, 2) \geq C\varphi(p, 3)$$

where $0 < p < \mathcal{K}$ is the parameter that appears in the proposition. The first inequality follows from Inequality $C$ if $q \leq \alpha$.

*Proof.* It is enough to show that $v = \log u \in$ BMO $(Q(2))$. Take any cube $Q \subset Q(2)$ and choose $\eta \in C_0(Q(3))$ such that $\eta = 1$ on $Q$. Since $u$ is a supersolution

$$\int_D \langle A\nabla u, \nabla\varphi \rangle dx \geq 0 \quad \text{for all } 0 \leq \varphi \in C_0^\infty(D).$$

Choose $\varphi = \eta^2 \frac{1}{u}$. We get

$$\int_{Q(3)} \langle A\nabla u, \nabla u \rangle \eta^2 \frac{1}{u^2} dx \leq C \int_{Q(3)} \langle A\nabla u, \nabla \eta \rangle \frac{1}{u} \eta dx$$

and with Schwarz' inequality and the uniform ellipticity

$$\int_{Q(3)} |\nabla u|^2 \eta^2 \frac{1}{u^2} dx \leq C \int_{Q(3)} |\nabla \eta| \, |\nabla u| \frac{1}{u} \eta dx.$$

Hence

$$\int_{Q} |\nabla v|^2 dx \leq C \int_{Q(3)} |\nabla \eta| dx$$

and Inequality $A$ implies

$$\int_{Q} |v - v_Q|^2 dx \leq C$$

where $C$ independent of $Q \subset Q(2)$. The inequality

$$\varphi(-q, 2) \geq \varphi(q, 2)$$

follows from Inequality $C$. □

Finally, for $0 < p < \mathcal{K}$ choose a $q > 0$ such that $q\mathcal{K}^{\nu} = p$ for some $\nu \in N$ and $q \leq \alpha$ in Ineauality $C$. Finitely many applicaitons of Lemma 7.1 and Inequality $B$ regarding $u$ as a positive supersolution gives

$$\varphi(q, 2) \geq C\varphi(p, 3).$$

This concludes the proof of Proposition 7.2. □

The rest of the proof the Theorem 7.1 and Theorem 7.2 can be found in Dahlberg/Kenig [5]. See Appendix. There can also be found the corresponding results for bounded Lipschitz domains using a patching technique. We finally remark that the solution of the Dirichlet problem is unique and the solution of the Neumann problem is unique to an additive constant. □

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# Appendix 1

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# Recent Progress on Boundary Value Problems on Lipschitz Domains

by

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# Introduction

In this note we will describe, and sketch the proofs of some recent developments on boundary value problems on Lipschitz domains.

In 1977, B. E. J. Dahlberg was able to show the solvability of the Dirichlet problem for Laplace's equation on a Lipschitz domain $D$, and with $L^2(\partial D, d\sigma)$ data and optimal estimates. In fact, he proved that given a Lipschitz domain $D$, there exists $\varepsilon = \varepsilon(D)$ such that this can be done for data in $L^p(\partial D, d\sigma)$, $2 - \varepsilon \leq p \leq \infty$. (See [6], [7] and [8]). Also, simple examples show that given $p < 2$, there exists a Lipschitz domain $D$ where this fails in $L^p(\partial D, d\sigma)$. Dahlberg's method consisted of a careful analysis of the harmonic measure. His techniques relied on positivity, Harnack's inequality and the maximum principle, and thus, they were not applicable to the Neumann problem, to systems of equations, or to higher order equations. In 1978, E. Fabes, M. Jodeit, Jr. and N.Riviere ([15]) were able to utilize A. P. Calderon's theorem ([1]) on the boundedness of the Cauchy integral on $C^1$ curves, to extend the classical method of layer potentials to $C^1$ domains. They were thus able to resolve the Dirichlet and Neumann problem for Laplace's equation, with $L^p(\partial D, d\sigma)$ data, and optimal estimates, for $C^1$ domains. They relied on the Fredholm theory, exploiting the compactness of the layer potentials in the $C^1$ case. In 1979, D. Jerison and C. Kenig [20], [21] were able to give a simplified proof of Dahlberg's results, using an integral identity that goes back to Rellich ([33]). However, the method still relied on positivity. Shortly afterwards, D. Jerison and C. Kenig, ([22]) were also able to treat the Neumann problem on Lipschitz domains, with $L^2(\partial D, d\sigma)$ data and optimal estimates. To do so, they combined the Rellich type formulas with Dahlberg's results on the Dirichlet problem. This still relied on positivity, and dealt only with the $L^2$ case, leaving the corresponding $L^p$ theory open.

In 1981, R. Coifman, A. McIntosh and Y. Meyer [3] established the boundedness of the Cauchy integral on any Lipschitz curve, opening the door to the applicability of the method of layer potentials to Lipschitz domains. This method is very flexible, does not rely on positivity, and does not in principle differentiate between a single equation or a system of equations. The difficulty then becomes the solvability of the integral equations, since unlike in the $C^1$ case, the Fredholm theory is not applicable, because on a Lipschitz domain operators like the double layer potential are not compact.

For the case of the Laplace equation, with $L^2(\partial D, \partial\sigma)$ data, this difficulty was overcome by G. C. Verchota ([36]) in 1982, in his doctoral dissertation. He made the key observation that the Rellich identities mentioned before are the appropriate substitutes to compactness, in the case of Lipschitz domains. Thus, Verchota was able to recover the $L^2$ results of Dahlberg [7] and of Jerison and Kenig [22], for Laplace's equation on a Lipschitz domain, but using the method of layer potentials.

This paper is divided into two sections. The first section which consists of two parts, deals with Laplace's equation on Lipschitz domains. The first part explains the $L^2$ results of

Verchota mentioned above. The second part deals with a sketch of recent joint work of B. Dahlberg and C. Kenig (1984) ([9]). We were able to show that given a Lipschitz domain $D \subset \mathbb{R}^n$, there exists $\varepsilon = \varepsilon(D)$ such that one can solve the Neumann problem for Laplace's equation with data in $L^p(\partial D, \partial\sigma)$, $1 < p \leq 2 + \varepsilon$. Easy examples show that this range of $p$'s is optimal. Moreover, we showed that the solution can be obtained by the method of layer potentials, and that Dahlbergs solution of the $L^p$ Dirichlet problem can also be obtained by the method of layer potentials. We also obtained endpoint estimates for the Hardy space $H^1(\partial D, d\sigma)$, which generalize the results for $n = 2$ in [25] and [26], and for $C^1$ domains in [16]. The key idea in this work is that one can estimate the regularity of the so-called Neumann function for $D$, by using the De Giorgi-Nash regularity theory for elliptic equations with bounded measurable coefficients. This, combined with the use of the so-called 'atoms' yields the desired results.

The second section, which consists of three parts, deals with higher order problems. In parts 1 and 2, we treat $L^2$ boundary value problems for systems of equations. Part 1 deals with the systems of elastostatics, whicle part 2 deals with the Stokes system of hydrostatics. The results in part 1 are joint work of B. Dahlberg, C. Kenig and G. Verchota (see [12]), while the results in part 2 are joint work of E. Fabes, C. Kenig and G. Verchota (see [17]). The results obtained had not been previously available for general Lipschitz domains, although a lot of work has been devoted to the case of piecewise linear domains. (See [27], [28] and their bibliographies). For the case of $C^1$ domains, our results for the systems of elastostatics had been previously obtained by A. Gutierrez ([19]), using compactness and the Fredholm theory. This is, of course, not available for the case of Lipschitz domains. We are able to use once more the method of layer potentials. Invertability is shown again by means of Rellich type formulas. This works very well in the Dirichlet problem for the Stokes system (see part 2), but serious difficulties occur for the systems of elastostatics (see part 1). These difficulties are overcome by proving a Korn type inequality at the boundary. The proof of this inequality proceeds in three steps. One first establishes it for the case of small Lipschitz constant. One then proves an analogous inequality for non-tangential maximal functions on any Lipschitz domain, by using the ideas of G. David ([13]), on icreasing the Lipschitz constant. Finally, one can remove the non-tangential maximal function, using the results on the Dirichlet problem for the Stokes system, which are established in part 2. See parts 1 and 2 for the details. Some partial results in this direction were previously announced in [26]. The third part of Section 2 deals with the Dirichlet problem for the biharmonic equation $\Delta^2$ ( a fourth order elliptic equation), on an arbitrary Lipschitz domain in $\mathbb{R}^n$. This sketches joint work of B. Dahlberg, C. Kenig and G. Verchota ([11]). The case of $C^1$ domains in the plane was previsouly treated by J. Cohen and J. Gosselin [2], using layer potentials and compactness. We are able to reduce the problem, for an arbitrary Lipschitz domain in $\mathbb{R}^n$, to a bilinear estimate for harmonic functions. This is a Lipschitz domain version of the paraproduct of J. M. Bony. See part 3 of Section 2 for further details.

Compete proofs of the results explained in Section 1, part 2, and Section 2, will appear in future publicaitons.

**Acknowledgements:** As was mentioned before, the results in Section 1, part 2 are joint work with B. Dahlberg, the results in Section 2, part 1, joint work with B. Dahlberg and G. Verchota, the results in Section 2, part 2, joint work with E. Fabes and G. Verchota, and the results in Seciton 2, part 3, joint work with B. Dahlberg and G. Verchota. It is a great pleasure to express my gratitutde to B. Dahlberg, E. Fabes and G. Verchota, for their contributions to our joint work. I would also like to thank A. McIntosh for pointing out the applicability of the continuity method in Seciton 1, part 1, and for pointing out to us the work of Nečas ([30]). I also would like to thank G. David for making his unpublished result (Lemma 2.1.10 in Section 2, part 1) available to us.

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# Section 1: Laplace's equation

## Part 1: The theory on a Lipschitz domain, for Laplace's equation, by the method of layer potentials

A bounded Lipschitz domain $D \subset \mathbb{R}^n$ is one which is locally given by the domain above the graph of a Lipschitz function. For such a domain $D$, the non-tangential region of opening $\alpha$ at a point $Q \in \partial D$ is $\Gamma_\alpha(Q) = \{X \in D : |X - Q| < (1+\alpha)\text{dist}\,(X, \partial D)\}$. All the results in this paer are valid, when suitably interpreted for all bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, with the non-tangential approach regions defined above. For simplicity, in this exposition we will restrict ourselves to the case $n \geq 3$ (and sometimes even to the case $n = 3$), and to domains $D \subset \mathbb{R}^n$, $D = \{(x,y) : y > \varphi(x)\}$, where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, with Lipschitz constant $M$, i.e., $|\varphi(x) - \varphi(x')| \leq M|x - x'|$. $D^- = \{(x,y) : y < \varphi(x)\}$. For fixed $M' < M, \Gamma_e(x) = \{(z,y) : (y - \varphi(x)) > -M'|z - x|\} \subset D^-$, and $\Gamma_i(x) = \{(z,y) : (y - \varphi(x)) > M'|z - x|\} \subset D$. Points in $D$ will usually be denoted by $X$, while points on $\partial D$ by $Q = (x, \varphi(x))$ or simply by $x$. $N_x$ or $N_Q$ will denote the unit normal to $\partial D = \Lambda$ at $Q = (x, \varphi(x))$. If $u$ is a function defined on $\mathbb{R}^n \setminus \Lambda$, and $Q \in \partial D, u^{\pm}(Q)$ will denote $\lim_{\substack{X \to Q \\ X \in \Gamma_i(Q)}} u(X)$ or $\lim_{\substack{X \to Q \\ X \in \Gamma_e(Q)}} u(X)$, respectively. If $u$ is a function defined on $D, N(u)(Q) = \sup_{X \in \Gamma_i(Q)} |u(X)|$.

We wish to solve the problems

$$(\mathrm{D}) \begin{cases} \Delta u = 0 & \text{in} \quad D \\ u|_{\partial D} = f \in L^2(\partial D, d\sigma) \end{cases}, \qquad (\mathrm{N}) \begin{cases} \Delta u = 0 & \text{in} \quad D \\ \frac{\partial u}{\partial N}|_{\partial D} = f \in L^2(\partial D, d\sigma) \end{cases}$$

The results here are

**Theorem 1.1.1:** There exists a unique $u$ such that $N(u) \in L^2(\partial D, d\sigma)$, solving (D), where the boundary values are taken non-tangentially a.e. Moreover, the solution $u$ has the form

$$u(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X - Q, N_Q \rangle}{|Q - X|^n} g(Q) d\sigma(Q),$$

for some $g \in L^2(\partial D, d\sigma)$.

**Theorem 1.1.2.** There exists a unique $u$ tending to 0 at $\infty$, such that $N(\nabla u) \in L^2(\partial D, d\sigma)$, solving (N) in the sense that $N_Q \cdot \nabla u(X) \to f(Q)$ as $X \to Q$ non-tantentially a.e. Moreover, the solution $u$ has the form

$$u(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X - Q|^{n-2}} g(Q) d\sigma(Q),$$

for some $g \in L^2(\partial D, d\sigma)$.

In order to prove the above theorems, we introduce

$$Kg(x) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X - Q, N_Q\rangle}{|X-Q|^n} g(Q) d\sigma(Q)$$

and

$$Sg(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X-Q|^{n-2}} g(Q) d\sigma(Q).$$

If $Q = (X, \varphi(x))$, $X = (z, y)$, then

$$Kg(z,y) = \frac{1}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{y - \varphi(x) - (z-x)\cdot\nabla\varphi(x)}{[|x-z|^2 + [\varphi(x) - \varphi(z)]^2]^{\frac{n}{2}}} g(x) dx$$

$$Sg(z,y) = \frac{-1}{\omega_n(n-2)} \int_{\mathbb{R}^{n-1}} \frac{\sqrt{1 + |\nabla\varphi(x)|^2}}{[|x-z|^2 + [\varphi(x) - y]^2]^{\frac{n-2}{2}}} g(x) dx.$$

**Theorem 1.1.3.** a) If $g \in L^p(\partial D, d\sigma), 1 < p < \infty$, then $N(\nabla Sg), N(Kg)$ also belong to $L^p(\partial D, d\sigma)$ and their norms are bounded by $C\|g\|_{L^p(\partial D, d\sigma)}$.

(b) $$\lim_{\varepsilon\to 0} \frac{1}{\omega_n} \int_{|x-z|>\varepsilon} \frac{\varphi(z) - \varphi(x) - (z-x)\cdot\nabla\varphi(x)}{[|x-z|^2 + [\varphi(x)-\varphi(z)]^2]^{n/2}} g(x) dx = Kg(z)$$

exists a.e. and

$$\|Kg\|_{L^p(\partial D, d\sigma)} \le C\|g\|_{L^p(\partial D, d\sigma)},\ 1 < p < \infty,$$

$$\lim_{\varepsilon\to 0} \frac{-1}{\omega_n} \int_{|z-x|>\varepsilon} \frac{(z-x, \varphi(z) - \varphi(x))\sqrt{1 + |\nabla\varphi(x)|^2}}{[|2-x|^2 + [\varphi(z) - \varphi(x)]^2]^{n/2}} g(x) dx$$

exists a.e. and in $L^p(\partial D, d\sigma)$, and its $L^p$ norm is bounded by $C\|g\|_{L^p(\partial D, d\sigma)}$, $1 < p < \infty$.

(c) $$(Kg)^{\pm}(Q) = \pm\frac{1}{2} g(Q) + Kg(Q)$$

$$(\nabla Sg)^{\pm}(z) = \pm\frac{1}{2} g(z) N_z + \frac{1}{\omega_n} \lim_{\varepsilon\to 0} \int_{|z-x|>\varepsilon} \frac{(z-x, \varphi(z)-\varphi(x))\sqrt{(1 + |\nabla\varphi(x)|^2)}}{[|z-x|^2 + [\varepsilon(z) - \varepsilon(x)]^2]^{n/2}} g$$

**Corollary 1.1.4.** $(N_z \nabla Sg)^{\pm}(z)^{\pm} = \frac{1}{2} g(z) - K^* g(z)$, where $K^*$ is the $L^2(\partial D, d\sigma)$ adjoint of $K$.

The proof of Theorem 1.1.3 is an easy consequence of the deep results of Coifman-McIntosh-Meyer ([3]).

It is easy to see that (at least the existence part) of Theorems 1.1. and 1.1.2 will follow immediately if we can show that $(\frac{1}{2}I + K)$ and $\frac{1}{2}I + K^*)$ are invertible on $L^2(\partial D, d\sigma)$. This is the result of G. Verchota ([36]).

**Theorem 1.1.5.** $(\pm\frac{1}{2}I + K)$, $(\pm\frac{1}{2}I + K^*)$ are invertible on $L^2(\partial D, d\sigma)$.

In order to do so, we show that if $f \in L^2(\partial D, d\sigma), \|(\frac{1}{2}I + K^*)f\|_{L^2(\partial D, d\sigma)} \approx \|(\frac{1}{2}I - K^*)f\|_{L^2(\partial D, d\sigma)}$, where the constants of equivalence depend only on the Lipschitz constant $M$. Let us take this for granted, and show, for example, that $\frac{1}{2}I + K^*$ is invertible. To do this, note first that if $T = \frac{1}{2}I + K^*$, $\|Tf\|_{L^2} \geq C\|f\|_{L^2}$, where $C$ depends only on the Lipschitz constant $M$. For $0 \leq t \leq 1$, consider the operator $T_t = \frac{1}{2}I + K_t^*$, where $K_t^*$ is the operator corresponding to the domain defined by $t\varphi$. Then, $T_0 = \frac{1}{2}I, T_1 = T$, and $\frac{\partial}{\partial t}T_t : L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$, $1 < p < \infty$ with bound independent of $t$, by the theorem of Coifman-McIntosh-Meyer. Moreover, for each $t$, $\|T_tf\|_{L^2} \geq C\|f\|_{L^2}$, $C$ independent of $t$. The invertibility of $T$ now follows from the continuity mehtod:

**Lemma 1.1.6.** Suppose that $T_t : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ satisfy

(a) $\|T_tf\|_{L^2} \geq C_1\|f\|_{L^2}$

(b) $\|T_tf - T_sf\|_{L^2} \leq C_2|t-s|\|f\|_{L^2}$, $0 \leq t, s \leq 1$

(c) $T_0 : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ is invertible.

Then, $T_1$ is invertible.

The proof of 1.1.6 is very simple. We are thus reduced to proving

$$\text{(1.1.7)} \quad \|(\frac{1}{2}I + K^*)f\|_{L^2(\partial D, d\sigma)} \approx \|(\frac{1}{2}I - K^*)f\|_{L^2(\partial D, d\sigma)}.$$

In order to prove (1.1.7), we will use the following formula, which goes back to Rellich [33] (see also [31], [30], [22]).

**Lemma 1.1.8.** Assume that $u \in$ Lip $(\bar{D}), \Delta u = 0$ in $D$, and $u$ and its derivatives are suitablly small at $\infty$. Then if $e_n$ is the unit vector in the direction of the $y$-axis,

$$\int_{\partial D} \langle N_Q, e_n \rangle |\nabla u|^2 d\sigma = 2\int_{\partial D} \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial N} d\sigma.$$

**Proof.** Observe that div $(e_n|\nabla u|^2) = \frac{\partial}{\partial y}|\nabla u|^2 = 2\frac{\partial}{\partial y}\nabla u\cdot\nabla u$, while div $(\frac{\partial u}{\partial y}\nabla u) = \frac{\partial}{\partial y}\nabla u\cdot\nabla u + \frac{\partial u}{\partial y}\cdot$ div $\nabla u = \frac{\partial}{\partial y}\nabla u \nabla u$. Stokes' theorem now gives the lemma.

We will now deduce a few consequences of the Rellich identity. Recall that $N_x = (-\nabla\varphi(x), 1)/\sqrt{1+|\nabla\varphi(x)|^2}$, so that $\frac{1}{(1+M^2)^{1/2}} \le \langle N_x, e_n\rangle \le 1$.

**Corollary 1.1.9.** Let $u$ be as in 1.1.8, and let $T_1(x), T_2(x), \dots, T_{n-1}(x)$ be an orthogonal basis for the tangent plane to $\partial D$ at $(x,\varphi(x))$. Let $|\nabla_t u(x)|^2 = \sum_{j=1}^{n-1} |\langle\nabla u(x), T_j(x)\rangle|^2$. Then,

$$\int_{\partial D} (\frac{\partial u}{\partial N})^2 d\sigma \le C\int_{\partial D} |\nabla_t u|^2 d\sigma.$$

**Proof.** Let $\alpha = e_n - \langle N_x, e_n\rangle N_x$, so that $\alpha$ is a linear combination of $T_1(x), T_2(x), \dots, T_{n-1}(x)$. Then,

$$\frac{\partial u}{\partial y} = \langle N_x, e_n\rangle \frac{\partial u}{\partial N} + \langle \alpha, \nabla u\rangle.$$

Also,

$$|\nabla u|^2 = (\frac{\partial u}{\partial N})^2 + |\nabla_t u|^2,$$

and so,

$$\begin{aligned}&\int_{\partial D}\langle N_x, e_n\rangle(\frac{\partial u}{\partial N})^2 d\sigma + \int_{\partial D}\langle N_x, e_n\rangle|\nabla_t u|^2 d\sigma = \\ &= 2\int_{\partial D}\langle N_x, e_n\rangle(\frac{\partial u}{\partial N})^2 + 2\int_{\partial D}\langle\alpha,\nabla u\rangle(\frac{\partial u}{\partial N})d\sigma.\end{aligned}$$

Hence,

$$\int_{\partial D}\langle N_x, e_n\rangle(\frac{\partial u}{\partial N})^2 d\sigma = \int_{\partial D}\langle N_x, e_n\rangle|\nabla_t u|^2 d\sigma - 2\int_{\partial D}\langle\alpha,\nabla u\rangle\frac{\partial u}{\partial N}d\sigma.$$

So,

$$\int_{\partial D}(\frac{\partial u}{\partial N})^2 d\sigma \le C\int_{\partial D}|\nabla_t u|^2 d\sigma + C(\int_{\partial D}|\nabla_t u|^2 d\sigma)^{1/2}(\int_{\partial D}(\frac{\partial u}{\partial N})^2 d\sigma)^{1/2},$$

and the corollary follows.

**Corollary 1.1.10.** Let $u$ be as in 1.1.8. Then,

$$\int_{\partial D} |\nabla_t u|^2 d\sigma \leq c \int_{\partial D} (\frac{\partial u}{\partial N})^2 d\sigma.$$

**Proof.** $\displaystyle\int_{\partial D} |\nabla u|^2 d\sigma \leq 2\big(\int_{\partial D} |\nabla u|^2 d\sigma\big)^{1/2}\big(\int_{\partial D} |\frac{\partial u}{\partial N}|^2 d\sigma\big)^{1/2}$, and the corollary follows.

**Corollary 1.1.11.** Let $u$ be as in 1.1.8. Then

$$\int_{\partial D} |\nabla_t u|^2 d\sigma \approx \int_{\partial D} |\frac{\partial u}{\partial N}|^2 d\sigma.$$

In order to prove 1.1.7, let $u = Sg$. Because of 1.1.3c, $\nabla_t u$ is continuous across the boundary, while by 1.1.4,

$$(\frac{\partial u}{\partial N})^{\pm} = (\pm\frac{1}{2}I - K^*)g.$$

We now apply 1.1.11 in $D$ and $D^-$, to obtain 1.1.7. This finishes the proof of 1.1.1 and 1.1.2.

We now turn our attention to $L^2$ regularity in the Dirichlet problem.

**Definition 1.1.12.** $f \in L_1^p(\Lambda), 1 < p < \infty$, if $f(x, \varphi(x))$ has a distributional gradient in $L^p(\mathbb{R}^{n-1})$. It is easy to check that if $F$ is any extension to $\mathbb{R}^n$ of $f$, then $\nabla_x F(x, \varphi(x))$ is well defined, and belongs to $L^p(\Lambda)$. We call this $\nabla_t f$. The norm in $L_1^p(\Lambda)$ will be $\|\nabla_t f\|_{L^p(\Lambda)}$.

**Theorem 1.1.13.** The single layer potential $S$ maps $L^2(\Lambda)$ into $L_1^2(\Lambda)$ boundedly, and has a bounded inverse.

**Proof.** The boundedness follows from 1.1.3 a). Because of the $L^2$-Neumann theory, and 1.1.11, $\|\nabla_t S(f)\|_{L^2(\Lambda)} \geq C\|\frac{\partial S}{\partial N}(f)\|_{L^2(\Lambda)} \geq C\|f\|_{L^2(\Lambda)}$. The argument used in the proof of 1.1.5 now proves 1.1.13.

**Theorem 1.1.14.** Given $f \in L_1^2(\Lambda)$, there exists a harmonic function $u$, with $\|N(\nabla u)\|_{L^2(\Lambda)} \leq C\|\nabla_t f\|_{L^2(\Lambda)}$, and such that $\nabla_t u = \nabla_t f$ (a.e.) non-tangentially on $\Lambda$. $u$ is unique (modulo constants), and we can choose $u = S(g)$, where $g \in L^2(\Lambda)$.

The existence part of 1.1.14 follows directly from 1.1.13.

**Part 2: The $L^p$ theory for Laplace's equation on a Lipschitz domain**

The main results in this section are:

**Theorem 1.2.1.** There exists $\varepsilon = \varepsilon(M) > 0$ such that, given $f \in L^p(\partial D, d\sigma)$, $2-\varepsilon \le p < \infty$, there exists a unique $u$ harmonic in $D$, with $N(u) \in L^p(\partial D, d\sigma)$, such that $u$ converges non-tangentially almost everywhere to $f$. Moreover, the solution $u$ has the form

$$u(x) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X-Q, N_Q\rangle}{|X-Q|^n} g(Q) d\sigma(Q),$$

for some $g \in L^p(\partial D, d\sigma)$.

**Theorem 1.2.2.** The exists $\varepsilon = \varepsilon(M) > 0$, such that, given $f \in L^p(\partial D, d\sigma)$, $1 < p \le 2+\varepsilon$, there exists a unique $u$ harmonic in $D$, tending to 0 at $\infty$, with $N(\nabla u) \in L^p(\partial D, d\sigma)$, such that $N_Q \cdot \nabla u(X)$ covnergens non-tangentially a.e. to $f(Q)$. Moreover, $u$ has the form

$$u(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X-Q|^{n-2}} g(Q) d\sigma(Q),$$

for some $g \in L^p(\partial D, d\sigma)$.

**Theorem 1.2.3.** There exists $\varepsilon = \varepsilon(M) > 0$ such that given $f \in L^p_1(\Lambda)$, $1 < p \le 2+\varepsilon$, there exists a harmonic funciton $u$, with

$$\|N(\nabla u)\|_{L^p(\Lambda)} \le C \|\nabla_t f\|_{L^p(\Lambda)},$$

and such that $\nabla_t u = \nabla_t f$ a.e. non-tangentially on $\Lambda$, $u$ is unique (modulo constants). Moreover, $u$ has the form

$$u(x) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X-Q|^{n-2}} g(Q) d\sigma(Q),$$

for some $g \in L^p(\partial D, d\sigma)$.

The case $p = 2$ of the above theorems was discussed in Part 1. The first part of 1.2.1 (i.e., without the representation formula), is due to B. Dahlberg (1977) ([7]). Theorem 1.2.3 was first proved by G. Verchota (1982) ([36]). The representation formula in 1.2.1, Theorem 1.2.2, and the proof that we are going to present of 1.2.3 are due to B. Dahlberg and C. Kenig (1984) ([9]). Just like in Seciton 1, 1.2.1, 1.2.2, and 1.2.3 follow from:

**Theorem 1.2.4.** There exists $\varepsilon = \varepsilon(M) > 0$ such taht $(\pm\frac{1}{2}I - K^*)$ is invertible in $L^p(\partial D, d\sigma), 1 < p \leq 2 + \varepsilon, (\pm\frac{1}{2}I + K)$ is invertible in $L^p(\partial D, d\sigma), 2 - \varepsilon \leq p < \infty$, and $S : L^p(\partial D, \partial\sigma) \to L_1^p(\partial D, d\sigma)$ is invertible, $1 < p \leq 2 + \varepsilon$.

In order to prove Theorem 1.2.4, just as in Part 1, it is enough to show that if $u = Sf$, $f$ since, then, for $1 < p \leq 2 + \varepsilon$,

$$\|\nabla_t u\|_{L^p(\partial D, d\sigma)} \approx \left\|\frac{\partial u}{\partial N}\right\|_{L^p(\partial D, d\sigma)}.$$

This will be done by proving the following two theorems:

**Theorem 1.2.5.** Let $\Delta u = 0$ in $D$. Then $\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leq C\left\|\frac{\partial u}{\partial N}\right\|_{L^p(\partial D, d\sigma)}$, $1 < p \leq 2 + \varepsilon$.

**Theorem 1.2.6.** Let $\Delta u = 0$ in $D$. Then

$$\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leq C\|\nabla_t u\|_{L^p(\partial D, d\sigma)},\ 1 < p \leq 2 + \varepsilon.$$

We first turn our attention to the case $1 < p < 2$ of Theorem 1.2.5. In order to do so, we introduce some definitions. A surface ball $B$ in $\Lambda$ is a set of the form $(x, \varphi(x))$, where $x$ belongs to a ball in $\mathbb{R}^{n-1}$.

**Definition 1.2.7.** An atom $a$ on $\Lambda$ is a function supported in a surface ball $B$, with $\|a\|_{L^\infty} \leq 1/\sigma(B)$, and with $\int_\Lambda a\,d\sigma = 0$.

Notice that atoms are in particular $L^2$ functions. The following interpolation theorem will be of importance to us.

**Theorem 1.2.8.** Let $T$ be a linear operator such that $\|Tf\|_{L^2(\Lambda)} \leq C\|f\|_{L^2(\Lambda)}$, and such that for all atoms $a$, $\|Ta\|_{L^1(\Lambda)} \leq C$. Then, for $1 < p < 2$, $\|Tf\|_{L^p(\Lambda)} \leq C\|f\|_{L^p(\Lambda)}$.

For a proof of this theorem, see [5]. Thus, in order to establish the case $1 < p < 1$ of 1.2.5, it suffices to show that if $a = \dfrac{\partial u}{\partial N}$ is an atom, then $\|N(\nabla u)\|_{L^1(\Lambda)} \leq C$. By dilation and translation invariance we can assume that $\varphi(0) = 0$, supp $a \subset B_1 = \{(x, \varphi(x)) : |x| < 1\}$. Let $B^*$ be a large ball centred at $(0, 0)$ in $\mathbb{R}^n$, which contains $(x, \varphi(x)), |x| < 2$. The diameter of $B^*$ depends only on $M$. Since $\|a\|_{L^2(\Lambda)} \leq \dfrac{1}{\sigma(B_1)^{1/2}} = C$, by the $L^2$-Neuman theory,

$$\int_{\partial D \cap B^*} N(\nabla u) \leq C \int_{\partial D \cap B^*} N(\nabla u)^2 d\sigma \leq c.$$

Thus, we only have to estimate $\int_{CB^*\cap\partial D}(\nabla u)d\sigma$. We will do so by appealing to the regularity theory for divergence form elliptic equations. Consider the bi-Lipschitzian mapping $\Phi : D \to D^-$ given by $\Phi(x,y) = (x, \varphi(x) - [y - \varphi(x)])$. Define $u^*$ on $D^-$ by the formula $u^* = u \circ \Phi^{-1}$. A simple calculation shows that, in $D^-$, $u^*$ verifies (in the weak sense) the equation div $(A(x,y)\nabla u^*) = 0$, where $A(x,y) = \dfrac{1}{J\phi(X)} \cdot (\Phi')^t(X) \cdot (\Phi')(X)$, where $X = \Phi^{-1}(x,y)$. It is easy to see that $A \in L^\infty(D^-)$, and $\langle A(x,y)\xi, \xi\rangle \geq C|\xi|^2$. Notice also that supp $\frac{\partial u}{\partial N} \subset B_1 \subset B^* \cap \partial D$. Define now

$$B(x,y) = \begin{cases} I & \text{for} \quad (x,y) \in D \\ A(x,y) & \text{for} \quad (x,y) \in D^-, \end{cases}$$

and

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{for} \quad (x,y) \in D \\ u^*(x,y) & \text{for} \quad (x,y) \in D^-. \end{cases}$$

Because $\dfrac{\partial u}{\partial N} = 0$ in $\partial D \setminus B^*$, it is very easy to see that $\tilde{u}$ is a (weak) solution in $\mathbb{R}^n \setminus B^*$ of the divergence form elliptic equation with bounded measurable coefficients, $L\tilde{u} =$ div $B(x,y)\nabla\tilde{u} = 0$. In order to estimate $u$, (and hence $\nabla u$) at $\infty$, we use the following theorem of J. Serrin and H. Weinberger ([34]).

**Theorem 1.2.9.** Let $\tilde{u}$ solve $L\tilde{u} = 0$ in $\mathbb{R}^n \setminus B^*$, and suppose that $\|\tilde{u}\|_{L^\infty(\mathbb{R}^n\setminus B^*)} < \infty$. Let $g(X)$ solve $Lg = 0$ in $|X| > 1$, with $g(X) \approx |X|^{2-n}$. Then, $\tilde{u}(X) = \tilde{u}_\infty + \alpha g(X) + v(X)$, where $Lv = 0$ in $\mathbb{R}^n \setminus B^*$, and $|v(X)| \leq C\|\tilde{u}\|_{L^\infty(\mathbb{R}^n\setminus B^*)} \cdot |X|^{2-n-\nu}$, where $\nu > 0, C > 0$ depend only on the ellipticity constants of $L$., Moreover, $\alpha = c\int B(X)\nabla\tilde{u}(X) \cdot \nabla\psi(X)$, where $\psi \in C^\infty(\mathbb{R}^n), \psi = 0$ for $X$ in $2B^*$, and $\psi \equiv 1$ for large $X$.

Let us assume for the time being that $u$ is bounded, and let us show that if $\alpha$ is as in 1.2.9, then $\alpha = 0$. Pick a $\psi$ as in 1.2.9. In $D, B(X) = I$, and so

$$\int_D B\nabla u\nabla\psi = \int_D \nabla u \cdot \nabla\psi = \lim_{\varepsilon\to 0}\int_{D_\delta^\varepsilon} \nabla u \cdot \nabla\psi,$$

where

$$D_\delta^\varepsilon = \{(x,y) : |(x,y)| < \rho,\ y > \varphi(x) + \varepsilon\},$$

and $\rho$ is large. The right-hand side equals

$$\lim_{\varepsilon\to 0}\int_{\partial D_\rho^\varepsilon} \psi \cdot \frac{\partial u}{\partial N} = \lim_{\varepsilon\to 0}\int_{\partial D_\rho^\varepsilon} [\psi - 1]\frac{\partial u}{\partial N},$$

since, by the harmonicity of $u$,

$$\int_{\partial D_\rho^\varepsilon} \frac{\partial u}{\partial N} = 0.$$

Let

$$\partial D^{\varepsilon}_{\rho,1} = \{(x,y) \in \partial D^{\varepsilon}_{\rho} : y = \varphi(x) + \varepsilon\},$$

and $\partial D^{\varepsilon}_{\rho,2} = \partial D^{\varepsilon}_{\rho} \setminus \partial D^{\varepsilon}_{\rho,1}$. Then,

$$\lim_{\varepsilon\to 0} \int_{\partial D^{\varepsilon}_{\rho}} [\psi - 1] \frac{\partial u}{\partial N} = \lim_{\varepsilon\to 0} \int_{\partial D^{\varepsilon}_{\rho,1}} [\psi - 1] \frac{\partial u}{\partial N} + \lim_{\varepsilon\to 0} \int_{\partial D^{\varepsilon}_{\rho,2}} [\psi - 1] \frac{\partial u}{\partial N} = \int_{\partial D} [\psi - 1] a =$$
$$= \int_{\partial D} \psi a - \int_{\partial D} a = \int_{\partial D} \psi a = 0,$$

since $\psi \equiv 0$ on supp $a$. Moreover $\int_{D^-} B\nabla\tilde{u}\nabla\psi = \int_D \nabla u \cdot \nabla \psi_*$, where $\psi_* = \psi \circ \Phi$, by our construction of $B$. The last term is also 0 by the same argument, and so $\alpha = 0$. We now show that $u$ (and hence $\tilde{u}$) is bounded. We will assume that $n \geq 4$ for simplicity. Since $\|a\|_{L^2(\Lambda)} \leq C$, we know that $u(X) = C_n \displaystyle\int_{\partial D} \frac{f(Q)}{|X-Q|^{n-2}} d\sigma(Q)$, with $\|f\|_{L^2(\Lambda)} \leq C$. Now, for $X \in D_1 = \{(x,y) : y > \varphi(x) + 1\}$, $\dfrac{1}{|X-Q|^{n-2}} \leq \dfrac{C}{1+|Q|^{n-2}} \in L^2(\Lambda)$ and so $u \in L^\infty(D_1)$. Let now $B$ be any ball in $\mathbb{R}^n$ so that $2B \subset \mathbb{R}^n \setminus B^*$, $B$ is of unit size, and such that a fixed fraction of $B$ is contained in $D_1$. Since $N(\nabla u) \in L^2(\Lambda)$, with norm less than $C$, $\int_{2B\cap D} |\nabla u|^2 \leq C$, and moreover on $B \cap D_1, |u(x)| \leq C$. Therefore, by the Poincare inequality $\int_{2B} \tilde{u}^2 \leq C$. But, since $\tilde{u}$ solves $L\tilde{u} = 0$, $\max_B \tilde{u} \leq C\big(\int_{2B} |\tilde{u}|^2\big)^{1/2} \leq C$, ([29]). Therefore, $\tilde{u} \in L^\infty(\mathbb{R}^n \setminus B^*)$, $\|\tilde{u}\|_{L^\infty(\mathbb{R}^n\setminus B^*)} \leq C$. Hence, since $\alpha = 0, \nabla u = \nabla v$, and $|v(x,y)| \leq C/(|x|+|y|)^{n-2+\nu}$, $\nu > 0$. For $R \geq R_0 = \operatorname{diam} B^*$, set $b(R) = \int_{A_R} N(\nabla u)^2$, where $A_R = \{(x,\varphi(x)) : R < |x| < 2R\}$.

For each fixed $R$, let

$$N_1(\nabla u)(x) = \sup\{|\nabla u(z,y)| : (z,y) \in \Gamma_i(x), \text{dis } ((z,y), \partial D) \leq \delta R\},$$
$$N_2(\nabla u)(x) = \sup\{|\nabla u(z,y)| : (z,y) \in \Gamma_i(x), \text{dist } ((z,y), \partial D) \geq \delta R\}.$$

In the set where the sup in $N_2$ is taken, $u$ is harmonic, and the distance of any point $X$ to the boundary is comparable to $|X|$. Thus, using our bound on $v$, we see that $N_2(\nabla u)(x) \leq C/|X|^{n-1+\nu} \approx C/R^{n-1+\nu}$, and so $\int_{A_R} N_2(\nabla u)^2 \leq CR^{1-n-2\nu}$. Let now

$$\Omega_\tau = \{(x,y) : \varphi(x) < y < \varphi(x) + CR, \tau R < |X| < r^{-1}R\},\ \tau \in \big(\frac{1}{4}, \frac{1}{2}\big).$$

By the $L^2$-Neumann theory in $\Omega_\tau$, $\int_{A_R} N_1(\nabla u)^2 d\sigma \leq C \int_{\partial\Omega_\tau} |\nabla u|^2 d\sigma$. Integrating in $\tau$ from 1/4 to 1/2 gives

$$\int_{A_R} N_1(\nabla u)^2 d\sigma \leq \frac{C}{R} \int_{\Omega_{1/4}\setminus\Omega_{1/2}} |\nabla u|^2 dX \leq \frac{C}{R^3} \int_{C_1 R < |X| < C_2 R} u^2,$$

since $L\tilde{u} = 0$ (see [29] for example). The right-hand side is bounded by $\dfrac{C}{R^3}\dfrac{1}{R^{2(n-2)-2\nu}}$. Then,

$$\int_{A_R} N(\nabla u) \leq C\Big(\int_{A_R} N(\nabla u)^2\Big)^{1/2} R^{\frac{n-1}{2}} \leq C R^{-\nu}.$$

Choosing now $R = 2^j$, and adding in $j$, we obtain the desired estimate.

We now turn to the case $1 < p < 2$ of 1.2.6. We need a further definition.

**Definition 1.2.10.** A function $a$ is an $H_1^1$ atom if $A = \nabla_t a$ satisfies (a) supp $A \subset B$, a surface ball, (b) $\|A\|_{L^\infty} \leq 1/\sigma(B)$, (c) $\int A d\sigma = 0$.

We will use the following interpolation result:

**Theorem 1.2.11.** Let $T$ be a linear operator such that

$$\|Tf\|_{L^2(\Lambda)} \leq C\|f\|_{L_1^2(\Lambda)}$$

and

$$\|Ta\|_{L^1(\Lambda)} < C$$

for all $H_1^1$ atoms $a$. Then, for $1 < p < 2$,

$$\|Tf\|_{L^p(\Lambda)} \leq C\|f\|_{L_1^p(\Lambda)}.$$

Hence, all we need to show is that if $\Delta u = 0$, $\nabla_t u = \nabla_t a$, and $a$ is a unit size $H_1^1$ atom, $N(\nabla u) \in L^1(\Lambda)$. But note that if we let

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & (x,y) \in D \\ -u^*(x,y) & (x,y) \in D^- \end{cases},$$

then $\tilde{u}$ is a weak solution of $L\tilde{u} = 0$ in $\mathbb{R}^n \setminus B^*$, since $u|_{\partial D \setminus B^*} = 0$. Then, $\tilde{u} = \tilde{u}_\infty + \alpha g + v$, but $\alpha = 0$ since $\tilde{u} - \tilde{u}_\infty$ must change sign at $\infty$. The argument is then identical to the one given before.

Before we pass to the case $2 < p < 2 + \varepsilon$, we would like to point out that using the techniques described above, one can develop the Stein-Weiss Hardy space theory on an arbitrary Lipschitz domain in $\mathbb{R}^n$. This generalizes the results for $n = 2$ obtained in [24] and [25], and the results for $C^1$ domains in [16].

Some of the results one can obtain are the following: Let

$$H_{at}^1(\partial D) = \{\Sigma \lambda_i a_i : \Sigma|\lambda_i| < \infty, a_i \text{ is an atom}\},$$
$$H_{1,at}^1(\partial D) = \{\Sigma \lambda_i a_i : \Sigma|\lambda_i| < +\infty, a_i \text{ is an } H_1^1 \text{ atom}\}.$$

**Theorem 1.2.12.** a) Given $f \in H^1_{at}(\partial D)$, there exists a unique harmonic function $u$, which tends to 0 at $\infty$, such that $N(\nabla u) \in L^1(\partial D)$, and such that $N_Q \cdot \nabla u(X) \to f(Q)$ non-tangentially a.e. Moreover, $u(X) = S(g)(X), g \in H^1_{at}$. Also, $u|_{\partial D} \in H^1_{1,at}(\partial D)$. b) Given $f \in H^1_{1,at}$, there exists a unique (modulo constants) harmonic function $u$, such that $N(\nabla u) \in L^1(\partial D)$, and such that $\nabla_t u|_{\partial D} = \nabla_t f$ a.e. Moreover, $u = S(g), g \in H^1_{at}$, and $\dfrac{\partial u}{\partial N} \in H^1_{at}(\partial D)$. c) If $u$ is harmonic, and $N(\nabla u) \in L^1(\partial D)$, then $\dfrac{\partial u}{\partial N} \in H^1_{at}(\partial D), u|_{\partial D} \in H^1_{1,at}(\partial D)$. d) $f \in H^1_{at}(\partial D)$ if and only if $N(\nabla Sf) \in L^1(\partial D)$, if and only if $(\dfrac{1}{2}I - K^*)f \in H^1_{at}(\partial D)$.

We turn now to the $L^p$ theory, $2 < p < 2 + \varepsilon$. In this case, the results are obtained as automatic real variable consequences of the fact that the $L^2$ results hold for all Lipschitz domains. We will now show that $\|N(\nabla u)\|_{L^p(\Lambda)} \le C \|\dfrac{\partial u}{\partial N}\|_{L^p(\Lambda)}$ for $2 < p < 2 + \varepsilon$.

The geometry will be clearer if we do it in $\mathbb{R}^n_+$, and then we transfer it to $D$ by the bi-Lipschitzian mapping

$$\Phi : \mathbb{R}^n_+ \to D,\ \Phi(x, y) = (x, y + \varphi(x)).$$

We will systematically ignore the distinction between sets in $\mathbb{R}^n_+$ and their images under $\Phi$. Let

$$\gamma = \{(x, y) \in \mathbb{R}^n_+ : |x| < y\},\ \gamma^* = \{(x, y) \in \mathbb{R}^n_+ : \alpha|x| < y\},$$

where $\alpha$ is a small constant to be chosen. Let

$$m(x) = \sup_{(z,y)\in x+\gamma} |\nabla u(z, y)|,$$

and

$$m^*(x) = \sup_{(z,y)\in x+\gamma^*} |\nabla u(z, y)|.$$

Our aim is to show that there is a small $\varepsilon_0 > 0$ such that

$$\int m^{2+\varepsilon} dx \le c \int |f|^{2+\varepsilon} dx$$

for all $0 < \varepsilon \le \varepsilon_0$, where $f = \dfrac{\partial u}{\partial N}$. Let $h = M(f^2)^{1/2}$, where $M$ denotes the Hardy-Littlewood maximal operator. Let

$$E_\lambda = \{x \in \mathbb{R}^{n-1} : m^*(x) > \lambda\}.$$

We claim that

$$\int_{\{m^*>\lambda; h\le\lambda]} \le C\lambda^2 |E_\lambda| + C\alpha \int_{\{m^*>\lambda\}} m^2.$$

Let us assume the claim, and prove the desired estimate. First, note that

$$\int_{E_\lambda} m^2 \leq \int_{\{m^*>\lambda; h\leq\lambda\}} m^2 + \int_{\{h>\lambda\}} m^2 \leq C\lambda^2|E_\lambda| + C\alpha \int_{\{m^*>\lambda\}} m^2 + \int_{\{h>\lambda\}} m^2,$$

by the claim. Choose now and fix $\alpha$ so that $C\cdot\alpha < 1/2$. Then,

$$\int_{E_\lambda} m^2 \leq C\lambda^2|E_\lambda| + C\int_{\{h\geq\lambda\}} m^2.$$

For $\varepsilon > 0$,

$$\begin{aligned}\int m^{2+\varepsilon} &= \varepsilon\int_0^\infty \lambda^{\varepsilon-1}\int_{\{m>\lambda\}} m^2 d\lambda \leq \varepsilon\int_0^\infty \lambda^{\varepsilon-1}\int_{E_\lambda} m^2 d\lambda \leq \\ &\leq C\varepsilon\int_0^\infty \lambda^{1+\varepsilon}|\{m^*>\lambda\}|d\lambda + C\varepsilon\int_0^\infty \lambda^{\varepsilon-1}\Big(\int_{h>\lambda} m^2\Big)d\lambda.\end{aligned}$$

By a well-knwon inequality (see [18] for example), $|E_\lambda| \leq C_\alpha|\{m>\lambda\}|$. Thus,

$$\begin{aligned}\int m^{2+\varepsilon} &\leq C\varepsilon\int_0^\infty \lambda^{1+\varepsilon}|\{m>\lambda\}|d\lambda + C\varepsilon\int_0^\infty \lambda^{\varepsilon-1}\Big(\int_{h>\lambda} m^2\Big)d\lambda \leq \\ &\leq C\varepsilon\int m^{2+\varepsilon} + C\int m^2 n^\varepsilon.\end{aligned}$$

If we now choose $\varepsilon_0$ so that

$$C\varepsilon_0 < 1/2, \text{ for } \varepsilon < \varepsilon_0, \int m^{2+\varepsilon} \leq C\int m^2 n^\varepsilon.$$

If we now use Holder's inequality with exponents $\frac{2+\varepsilon}{2}$ and $\frac{2+\varepsilon}{\varepsilon}$, we see that

$$\int m^{2+\varepsilon} \leq C\Big(\int m^{2+\varepsilon}\Big)^{\frac{2}{2+\varepsilon}}\Big(\int M(f^2)^{\frac{2+\varepsilon}{2}}\Big)^{\frac{\varepsilon}{2+\varepsilon}},$$

and the desired inequality follows from the Hardy-Littlewood maximal theorem.

It remains to establish the claim. Let $\{Q_k\}$ be a Whitney decomposition of the set $E_\lambda = \{m^* > \lambda\}$, such that $3Q_k \subset E_\lambda$, and $\{3Q_k\}$ has bounded overlap. Fix $k$, we can assume that there exists $x \in Q_k$ such that $h(x) \leq \lambda$, and hence, $\int_{2Q_k} f^2 \leq C\lambda^2|Q_k|$. For $1 \leq \tau \leq 2$, let $Q_{k,\tau} = \tau Q_k$, and

$$\tilde{Q}_{k,\tau} = \{(x,y) : x \in \tau Q_k,\ 0 < y < \tau \text{ length } (Q_k)\}.$$

$\tilde{Q}_{k,\tau}$ (and $\Phi(\tilde{Q}_{k,\tau})$) is a Lipschitz domain, uniformly in $k,\tau$. Also, by construction of $Q_k$, there exists $x_k$ with dist $(x_k, Q_k) \approx$ length $(Q_k)$ and such that $m^*(x_k) \leq \lambda$. Let

$$\begin{aligned}A_{k,\tau} &= \partial Q_{k,\tau} \cap x_k + \gamma^* \\ B_{k,\tau} &= \partial Q_{k,\tau} \cap \mathbb{R}^n_+ \setminus A_{k,\tau},\end{aligned}$$

so that

$$\partial Q_{k,\tau} = Q_{k,\tau} \cup A_{k,\tau} \cup B_{k,\tau}.$$

Note that the height of $B_{k,\tau}$ is dominated by $C\alpha$ length $(Q_k)$, and that $|\nabla u| \leq \lambda$ on $A_{k,\tau}$. Let $m_1$ be the maximal function of $\nabla u$, corresponding to the domain $\tilde{Q}_{k,\tau}$ (i.e., where the cones are truncated at height $\approx \ell(Q_k)$). Then, for $x \in Q_k, m(x) \leq m_1(x) + \lambda$. Also,

$$\int_{Q_k} m_1^2 \leq \int_{\partial \tilde{Q}_{k,\tau}} m_1^2 \leq (\text{using the } L^2\text{-theory on } \tilde{Q}_{k,\tau}) \leq$$

$$C \int_{B_{k,\tau}} |\nabla u|^2 d\sigma + c \int_{A_{k,\tau}} |\nabla u|^2 d\sigma + c \int_{2Q_k} f^2 \leq C \int_{B_{k,\tau}} |\nabla u|^2 d\sigma + C\lambda^2 |Q_k|.$$

Integrating in $\tau$ between 1 and 2, we see that

$$\int_{Q_k} m_1^2 \leq \frac{C}{\ell(Q_k)} \int_0^{\alpha \ell(Q_k)} \int_{2Q_k} |\nabla u|^2 + C\lambda^2 |Q_k| \leq C\alpha \int_{2Q_k} m^2 + C\lambda^2 |Q_k|.$$

Thus,

$$\int_{Q_k} m^2 \leq C\alpha \int_{2Q_k} m^2 + C\lambda^2 |Q_k|.$$

Adding in $k$, we see that

$$\int_{\{m^* > \lambda, h \leq \lambda\}} m^2 \leq C\lambda^2 |E_\lambda| + C\alpha \int_{\{m^* > \lambda\}} m^2,$$

which is the claim. Note also that the same argument gives the estimate $\|N(\nabla u)\|_p \leq C\|\nabla_t u\|_p, 2 < p < 2 + \varepsilon$, and the $L^p$ theory is thus completed.

# Section 2. Higher order boundary value problems

### Part 1: The systems of elastostatics

In this part we will sketch the extension of the $L^2$ results for the Laplace equation to the systems of linear elastostatics on Lipschitz domains. These results are joint work of B. Dahlberg, C. Kenig and G. Verchota, and will be discussed in detail in a forthcoming paper ([12]). Here we will describe some of the main ideas in that work. For simplicity, here we restrict our attention to domains $D$ above the graph of a Lipschitz function $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

Let $\lambda > 0, \mu \geq 0$ be constants (Lame moduli). We will seek to solve the following boundary value problems, where $\vec{u} = (u^1, u^2, u^3)$

$$\text{(2.1.1)} \qquad \begin{aligned} &\mu\Delta\vec{u} + (\lambda+\mu)\nabla \text{div } \vec{u} = 0 \quad \text{in} \quad D \\ &\vec{u}|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma) \end{aligned}$$

$$\text{(2.1.2)} \qquad \begin{aligned} &\mu\Delta\vec{u} + (\lambda+\mu)\nabla \text{div } \vec{u} = 0 \quad \text{in} \quad D \\ &\lambda(\text{div } \vec{u})N + \mu\{\nabla\vec{u} + (\nabla\vec{u})^t\}N|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma). \end{aligned}$$

(2.1.1) corresponds to knowing the displacement vector $\vec{u}$ on the boundary of $D$, while (2.1.2) corresponds to knowing the surface stresses on the boundary of $D$. We seek to solve (2.1.1) and (2.1.2) by the method of layer potentials. In order to do so, we introduce the Kelvin matrix of fundamental solutions (see [27] for example),

$$\Gamma(X) = (\Gamma_{ij}(X)),$$

where

$$\Gamma_{ij}(X) = \frac{A}{4\pi}\frac{\delta_{ij}}{|X|} + \frac{C}{4\pi}\frac{X_i X_j}{|X|^3},$$

and

$$A = \frac{1}{2}[\frac{1}{\mu} + \frac{1}{2\mu+\lambda}],\ C = \frac{1}{2}[\frac{1}{\mu} - \frac{1}{2\mu+\lambda}].$$

We will also introduce the stress operator $T$, where

$$T\vec{u} = \lambda(\text{div } \vec{u})N + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\}N.$$

The double layer potential of a density $\vec{g}(Q)$ is then given by

$$\vec{u}(X) = \mathcal{K}\vec{g}(X) = \int_{\partial D}\{T(Q)\Gamma(X-Q)\}^t\vec{g}(Q)d\sigma(Q),$$

where the operator $T$ is applied to each column of the matrix $\Gamma$.

The single layer potential of a denisty $\vec{g}(Q)$ is

$$\vec{u}(X) = S\vec{g}(X) = \int_{\partial D}\Gamma(X-Q)\cdot\vec{g}(Q)d\sigma(Q).$$

Our main results here parallel those of Section 1, Part 1. They are

**Theorem 2.1.3.** (a) There exists a unique solution of problem 2.1.1 in $D$, with $N(\vec{u}) \in L^2(\partial D, d\sigma)$. Moreover, the solution $u$ has the form $\vec{u}(x) = \mathcal{K}\vec{g}(x)$, $\vec{g} \in L^2(\partial D, d\sigma)$.

(b) There exists a unique solution of (2.1.2) in $D$, which is 0 at infinity, with $N(\nabla\vec{u}) \in L^2(\partial D, d\sigma)$. Moreover, the solution $\vec{u}$ has the form $\vec{u}(X) = S\vec{g}(X)$, $\vec{g} \in L^2(\partial D, d\sigma)$.

(c) If the data $\vec{f}$ in 2.1.1 belongs to $L^2_1(\partial D, d\sigma)$, then we can solve (2.1.1), with $N(\nabla\vec{u}) \in L^2(\partial D, d\sigma)$.

The proof of Theorem 2.1.3 starts out following the pattern we used to prove 1.1.1, 1.1.2 and 1.1.14. We first show, as in Theorem 1.1.3, that the following lemma holds:

**Lemma 2.1.4.** Let $\mathcal{K}\vec{g}, S\vec{g}$ be defined as above, so that they both solve $\mu\Delta\vec{u} + (\lambda + \mu)\nabla \text{ div } \vec{u} = 0$ in $\mathbb{R}^3 \setminus \partial D$. Then,

(a)
$$\|N(\mathcal{K}\vec{g})\|_{L^p(\partial D, d\sigma)} \leq C\|\vec{g}\|_{L^p(\partial D, d\sigma)},$$
$$\|N(\nabla S\vec{g})\|_{L^p(\partial D, d\sigma)} \leq C\|\vec{g}\|_{L^p(\partial D, d\sigma)}, \text{ for } 1 < p < \infty.$$

(b)
$$(\mathcal{K}\vec{g})^{\pm}(P) = \pm\frac{1}{2}\vec{g}(P) + K\vec{g}(P)$$
$$\left(\frac{\partial}{\partial X_i}(S\vec{g})_j\right)^{\pm}(P) = \pm\left\{\frac{A+C}{2}n_i(P)g_j(P) - n_i(P)\cdot n_j(P)\langle N_p, g(P)\rangle\right\} + $$
$$+ \left(p.v. \int_{\partial D} \frac{\partial}{\partial P_i}\Gamma(P-Q)\vec{g}(Q)d\sigma(Q)\right)_j,$$
where $K\vec{g}(P) = p.v. \int_{\partial D}\{T(Q)\Gamma(P-Q)\}^t\vec{g}(Q)d\sigma(Q)$, and $A, C$ are the constants in the definition of the fundamental solution.

Thus, just as in Section 1, part 1 reduces to proving the invertibility on $L^2(\partial D, d\sigma)$ of $\pm\frac{1}{2}I + K$, $\pm\frac{1}{2}I + K^*$, and the invertibility from $L^2(\partial D, d\sigma)$ onto $L^2_1(\partial D, d\sigma)$ of $S$. Just as before, using the jump relations, it suffices to show that if $\vec{u}(X) = S\vec{g}(X)$, then
$$\|T\vec{u}\|_{L^2(\partial D, d\sigma)} \approx \|\nabla_t\vec{u}\|_{L^2(\partial D, d\sigma)}.$$
Before explaining the difficulties in doing so, it is very useful to explain the stress operator $T$ (and thus the boundary value problem 2.1.2), from the point of view of the theory of constant coefficient second order elliptic systems. We go back to working on $\mathbb{R}^n$, and use the summation convention.

Let $a^{rs}_{ij}$, $1 \leq r, s \leq m$, $1 \leq i, j \leq n$ be constants satisfying the ellipticity condition
$$a^{rs}_{ij}\xi_i\xi_j\eta^r\eta^s \geq C|\xi|^2|\eta|^2$$
and the symmetry condition $a^{rs}_{ij} = a^{sr}_{ji}$. Consider vector valued functions $\vec{u} = (u^1, \dots, u^m)$ on $\mathbb{R}^n$ satisfying the divergence form system
$$\frac{\partial}{\partial X_i}a^{rs}_{ij}\frac{\partial}{\partial X_j}u^s = 0 \text{ in } D.$$

From variational considerations, the most natural boundary conditions are Dirichlet conditions ($\vec{u}|_{\partial D}= \vec{f}$) or Neumann type conditions, $\dfrac{\partial \vec{u}}{\partial \nu} = n_i a_{ij}^{rs}\dfrac{\partial u^s}{\partial X_j} = f_r$. The interpretation of problem 2.1.2 in this context is that we can find constants $a_{ij}^{rs}, 1 \le i,j \le 3, 1 \le r,s \le 3$, which satisfy the ellipticity condition and the symmetry condition, and such that $\mu\Delta\vec{u}+(\lambda+\mu)\nabla\mathrm{div}\ \vec{u} = 0$ in $D$ if and only if $\dfrac{\partial}{\partial X_i} a_{ij}^{rs}\dfrac{\partial u^s}{\partial X_j} = 0$ in $D$, and with $T\vec{u} = \dfrac{\partial}{\partial \nu}\vec{u}$. In order to obtain the equivalence between the tangential derivatives and the stress operator, we need an identity of the Rellich type. Such identities are available for general constant coefficient systems (see [32], [30]).

**Lemma 2.1.5 (The Rellich, Payne-Weinberger, Nečas identities).** Suppose that $\dfrac{\partial}{\partial X_i} a_{ij}^{rs}\dfrac{\partial}{\partial X_j} u^s = 0$ in $D$, $a_{ij}^{rs} = a_{ji}^{sr}$, $\vec{h}$ is a constant vector in $\mathbb{R}^n$, and $\vec{u}$ and its derivatives are suitably small at $\infty$. Then,

$$\int_{\partial D} h_\ell n_\ell a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} d\sigma = 2\int_{\partial D} h_i \frac{\partial u^r}{\partial X_i} n_\ell a_{\ell j}^{rs}\frac{\partial u^s}{\partial X_j} d\sigma.$$

**Proof.** Apply the divergence theorem to the formula

$$\frac{\partial}{\partial X_\ell}\big[(h_\ell a_{ij}^{rs} - h_i a_{\ell j}^{rs} - h_j a_{i\ell}^{rs})\frac{\partial u^r}{\partial X_i}\cdot\frac{\partial u^s}{\partial X_j}\big] = 0$$

**Remark 1:** Note that if we are dealing with the case $m = 1, a_{ij} = I$, and we choose $\vec{h} = e_n$, we recover the identity we used before for Laplace's equation.

**Remark 2:** Note that if we had the stronger ellipticity assumption that $a_{ij}^{rs}\xi_i^r\xi_j^s \ge C\sum_{\ell,t}|\xi_\ell^t|^2$, we would have, if $\partial D = \{(x,\varphi(x)) : \varphi : \mathbb{R}^{n-1}\to\mathbb{R}, \|\nabla\varphi\|_\infty \le M\}$, that $\|\nabla_t u\|_{L^2(\partial D, d\sigma)} \approx \|\frac{\partial u}{\partial \nu}\|_{L^2(\partial D, d\sigma)}$. In fact, if we take $\vec{h} = e_n$, then we would have

$$\begin{aligned}\sum_r \int_{\partial D} |\nabla u^r|^2 d\sigma &\le c\int_{\partial D} h_\ell n_\ell a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} d\sigma = \\ &= 2C\int_{\partial D} h_i\frac{\partial u^r}{\partial X_i}\cdot n_\ell a_{\ell j}^{rs}\frac{\partial u^s}{\partial X_j} d\sigma \le 2C\big(\sum_r\int_{\partial D}|\nabla u^r|^2 d\sigma\big)^{1/2}\big(\int_{\partial D}|\frac{\partial u}{\partial \nu}|^2 d\sigma\big)^{1/2}.\end{aligned}$$

Thus, $\sum_r \int_{\partial D}|\nabla u^r|^2 d\sigma \le C\int_{\partial D}|\frac{\partial u}{\partial \nu}|^2 d\sigma$.

For the opposite inequality, observe that for each $r, s, j$ fixed, the vector $h_i n_\ell a_{\ell j}^{rs} - h_\ell n_\ell a_{ij}^{rs}$ is perpendicular to $N$. Because of Lemma 2.1.5

$$\int_{\partial D} h_\ell n_\ell a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} d\sigma = 2\int_{\partial D}(h_\ell n_\ell a_{ij}^{rs} - h_i n_\ell a_{\ell j}^{rs})\frac{\partial u^\ell}{\partial X_i}\cdot\frac{\partial u^s}{\partial X_j} d\sigma.$$

Hence,

$$\int_{\partial D} |\nabla u|^2 d\sigma \le C\big(\int_{\partial D} |\nabla_t u|^2 d\sigma\big)^{1/2}\big(\int_{\partial D} |\nabla u|^2 d\sigma\big)^{1/2},$$

and so

$$\int_{\partial D} |\frac{\partial u}{\partial \nu}|^2 d\sigma \le c\int_{\partial D} |\nabla u|^2 d\sigma \le c\int_{\partial D} |\nabla_t u|^2.$$

**Remark 3:** In the case in which we are interested, i.e., the case of the systems of elastostatics,

$$a_{ij}^{rs}\frac{\partial u^s}{\partial X_i}\cdot\frac{\partial u^r}{\partial X_j} = \lambda(\text{div }\vec{u})^2 + \frac{\mu}{2}\sum_{i,j}\big(\frac{\partial u^j}{\partial X_i} + \frac{\partial u^i}{\partial X_j}\big)^2,$$

which clearly does not satisfy

$$a_{ij}^{rs}\xi_i^r\xi_j^s \ge C\sum_{\ell,t}|\xi_\ell^t|^2,$$

since the quadratic form involves only the symmetric part of the matrix $(\xi_i^r)$. In this case, of course $\dfrac{\partial\vec{u}}{\partial\nu} = T\vec{u} = \lambda(\text{div }\vec{u})N + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\}N$.

**Remark 4:** The inequality

$$\|\nabla\vec{u}\|_{L^2(\partial D, d\sigma)} \le C\|\nabla_t\vec{u}\|_{L^2(\partial D, d\sigma)}$$

holds in the general case, directly from Lemma 2.1.5, by a more complicated algebraic argument. In fact, as in Remark 2,

$$\int_{\partial D} h_\ell n_\ell a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j}d\sigma = 2\int_{\partial D}(h_\ell n_\ell a_{ij}^{rs} - h_i n_\ell a_{\ell j}^{rs})\frac{\partial u^\ell}{\partial X_i}\cdot\frac{\partial u^s}{\partial X_j}d\sigma,$$

and for fixed $r, s, j, (h_\ell n_\ell a_{ij}^{rs} - h_i n_\ell a_{\ell j}^{rs})$ is a tangential vector. Thus,

$$\int_{\partial D} h_\ell n_\ell a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j}d\sigma \le C\big(\int_{\partial D}|\nabla_t\vec{u}|^2 d\sigma\big)^{1/2}\big(\int_{\partial D}|\nabla\vec{u}|^2 d\sigma\big)^{1/2}.$$

Consider now the matrix $d_{rs} = (a_{ij}^{rs}n_i n_j)^{-1}$. This is a strictly positive matrix, since $a_{ij}^{rs}\xi_i\xi_j\eta^r\eta^s \ge C|\xi|^2|\eta|^2$. Moreover,

$$\begin{aligned}
& d_{rs}\big(\frac{\partial u}{\partial\nu}\big)_r\big(\frac{\partial u}{\partial\nu}\big)_s - a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} = \\
&= d_{rt}n_i a_{ij}^{rt}\frac{\partial u^t}{\partial X_j}\cdot n_\ell a_{\ell k}^{sm}\frac{\partial u^m}{\partial X_k} - a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} = \\
&= d_{rs}n_k a_{k\ell}^{rt}\frac{\partial u^t}{\partial X_\ell}\cdot n_m a_{mv}^{s\tau}\frac{\partial u^\tau}{\partial X_v} - a_{v\ell}^{t\tau}\frac{\partial u^t}{\partial X_v}\frac{\partial u^\tau}{\partial X_\ell} = \\
&= d_{rs}n_k a_{kv}^{rt}\frac{\partial u^t}{\partial X_v}\cdot n_m a_{m\ell}^{s\tau}\frac{\partial u^\tau}{\partial X_\ell} - a_{v\ell}^{t\tau}\frac{\partial u^t}{\partial X_v}\frac{\partial u^\tau}{\partial X_\ell} = \\
&= \{d_{rs}n_k a_{kv}^{rt}n_m a_{m\ell}^{s\tau} - a_{v\ell}^{t\tau}\}\frac{\partial u^t}{\partial X_v}\frac{\partial u^\tau}{\partial X_\ell}.
\end{aligned}$$

Now, note that for $t, \tau, \ell$ fixed $\{d_{rs} n_k a^{rt}_{kv} n_m a^{s\tau}_{m\ell} - a^{t\tau}_{v\ell}\}$ is perpendicular to $N$, by our definition of $d_{rs}$, and the symmetry of $a^{rs}_{ij}$:

$$\begin{aligned} &d_{rs} n_k a^{rt}_{kv} n_m a^{st}_{m\ell} n_v - a^{t\tau}_{v\ell} n_v = a^{rt}_{kv} n_k n_v d_{rs} a^{st}_{m\ell} n_m - a^{t\tau}_{m\ell} n_m = \\ &= a^{tr}_{vk} n_v n_k d_{rs} a^{st}_{m\ell} n_m - a^{t\tau}_{m\ell} n_m = \delta_{ts} a^{st}_{m\ell} n_m - a^{t\tau}_{m\ell} n_m = a^{t\tau}_{m\ell} n_m - a^{t\tau}_{m\ell} n_m = 0. \end{aligned}$$

Therefore,

$$\int_{\partial D} h_\ell n_\ell d_{rs} (\frac{\partial \vec{u}}{\partial \nu})_r (\frac{\partial \vec{u}}{\partial \nu})_s d\sigma \leq c( \int_{\partial D} |\nabla_t \vec{u}|^2 d\sigma)^{1/2} ( \int_{\partial D} |\nabla \vec{u}|^2 d\sigma)^{1/2}$$

Now,

$$\begin{aligned} &(\frac{\partial \vec{u}}{\partial \nu})_r - a^{rs}_{kj} n_k n_j \frac{\partial u^s}{\partial N} = n_i a^{rs}_{ij} \frac{\partial u^s}{\partial X_j} - a^{rs}_{kj} n_k n_j n_i \frac{\partial u^s}{\partial X_i} = \\ &= n_i a^{rs}_{ij} \frac{\partial u^s}{\partial X_j} - a^{rs}_{ki} n_k n_j n_i \frac{\partial u^s}{\partial X_j} = \{ n_i a^{rs}_{ij} - a^{rs}_{ki} n_k n_i n_j \} \frac{\partial u^s}{\partial X_j} = \\ &= \{ n_i a^{rs}_{ij} - a^{rs}_{ik} n_k n_i n_j \} \frac{\partial u^s}{\partial X_j}. \end{aligned}$$

But, for $i, r, s$ fixed, $a^{rs}_{ij} - a^{rs}_{ik} n_k n_j$ is perpendicular to $N$, and so

$$\begin{aligned} &\int_{\partial D} h_\ell n_\ell d_{rs} \{ a^{rt}_{kj} n_k n_j \frac{\partial u^t}{\partial N} \} \{ a^{s\tau}_{i\ell} n_i n_\ell \frac{\partial u^\tau}{\partial N} \} d\sigma \leq \\ &\leq C \{ ( \int_{\partial D} |\nabla_t \vec{u}|^2 \partial \sigma)^{1/2} ( \int_{\partial D} |\nabla \vec{u}|^2 d\sigma)^{1/2} + \int_{\partial D} |\nabla_t \vec{u}|^2 d\sigma \}. \end{aligned}$$

We now choose $\vec{h} = e_n$, so that $h_\ell n_\ell \geq C$, and recall that $(d_{rs})$ and $(a^{rt}_{kj} n_k n_j)$ are strictly positive definite matrices. We then see that

$$\int_{\partial D} |\frac{\partial \vec{u}}{\partial N}|^2 d\sigma \leq C \{ ( \int_{\partial D} |\nabla_t \vec{u}|^2 d\sigma)^{1/2} ( \int_{\partial D} |\nabla \vec{u}|^2 d\sigma)^{1/2} + \int_{\partial D} |\nabla_t \vec{u}|^2 d\sigma \}.$$

Now, as $|\nabla \vec{u}|^2 = |\nabla_t \vec{u}|^2 + |\frac{\partial \vec{u}}{\partial N}|^2$, the remark follows.

**Remark 5:** In order to show that $\int_{\partial D} |\nabla_t \vec{u}|^2 d\sigma \leq C \int_{\partial D} |T \vec{u}|^2 d\sigma$, it suffices to show that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq c \int_{\partial D} |\lambda (\text{div } \vec{u}) I + \mu \{ \nabla \vec{u} + \nabla \vec{u}^t \}|^2 d\sigma.$$

In fact, if this inequality holds, we would clearly have that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq C \int_{\partial D} |\nabla \vec{u} + \nabla \vec{u}^t|^2 d\sigma$$

(Korn type inequality at the boundary). The Rellich-Payne/Weinberger-Nečas identity is, in this case (with $\vec{h} = e_n$),

$$\int_{\partial D} n_n \{\frac{\mu}{2}|\nabla \vec{u} + \nabla \vec{u}^t|^2 + \lambda(\text{div } \vec{u})^2\} d\sigma =$$
$$= 2\int_{\partial D} \frac{\partial \vec{u}}{\partial y} \cdot \{\lambda(\text{div } \vec{u})N + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\}N\} d\sigma.$$

But then,

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq C\Big(\int_{\partial D} |\nabla \vec{u}|^2 d\sigma\Big)^{1/2}\Big(\int_{\partial D} |\lambda(\text{div } \vec{u})N + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\}N|^2 d\sigma\Big)^{1/2}.$$

The rest of part 1 is devoted to sketching the proof of the above inequality.

**Theorem 2.1.6.** Let $\vec{u}$ solve $\mu\Delta\vec{u} + (\lambda+\mu)\nabla$ div $\vec{u} = 0$ in $D, \vec{u} = S(\vec{g})$, where $\vec{g}$ is nice. Then, there exists a constant $C$, which depends only on the Lipschitz constant of $\varphi$ so that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq C\int_{\partial D} |\lambda(\text{div } \vec{u})I + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\}|^2 d\sigma.$$

The proof of the above theorem proceeds in two steps. They are:

**Lemma 2.1.7.** Let $\vec{u}$ be as in Theorem 2.1.6. Then,

$$\int_{\partial D} N(\nabla \vec{u})^2 d\sigma \leq c\int_{\partial D} N(\lambda(\text{div } \vec{u})I + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\})^2 d\sigma.$$

**Lemma 2.1.8.** Let $\vec{u}$ be as in Theorem 2.1.6. Then,

$$\int_{\partial D} N(\lambda(\text{div } \vec{u})I + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\})^2 d\sigma \leq C\int_{\partial D} |\lambda(\text{div } \vec{u})I + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\}|^2 d\sigma.$$

Lemma 2.1.7 is proved by first doing so in the case when the Lipschitz constant is small, and then passing to the general case by using the ideas of G. David ([13]). Lemma 2.1.8 is proved by observing that if $\vec{v}$ is any row of the matrix $\lambda(\text{div } \vec{u})I + \mu\{\nabla \vec{u} + \nabla \vec{u}^t\}$, then $\vec{v}$ is a solution of the Stokes system

$$\text{(S)} \qquad \begin{cases} \Delta \vec{v} = \nabla p \text{ in } D \\ \text{div } \vec{v} = 0 \text{ in } D \\ \vec{v}|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma) \end{cases}$$

This is checked directly by using the system of equations $\mu\Delta\vec{u} + (\lambda+\mu)\nabla\text{div } \vec{u} = 0$. One then invokes the following Theorem of E. Fabes, C. Kenig and G. Verchota, whose proof will be presented in the next section.

**Theorem 2.1.9.** Given $\vec{f} \in L^2(\partial D, d\sigma)$, there exists a unique solution $(\vec{v}, p)$ to system (S) with $p$ tending to 0 at $\infty$, and $N(\vec{v}) \in L^2(\partial D, d\sigma)$. Moreover,

$$\|N(\vec{v})\|_{L^2(\partial D, d\sigma)} \leq C\|\vec{f}\|_{L^2(\partial D, d\sigma)}.$$

We now turn to a sketch of the proof of Lemma 2.1.7. We will need the following unpublished real variable lemma of G. David ([14]).

**Lemma 2.1.10.** Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of two variables $t \in \mathbb{R}, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Assume that for each $x$, the function $t \to F(t, x)$ is Lipschitz, with Lipschitz constant less than or equal to $M$, and for each $i$, $1 \leq i \leq n$, the function $x_i \to F(t, x)$ is Lipschitz, with Lipschitz constant less than or equal to $M_i$, for any choice of the other variables. Given an interval $I \times J = I \times J_1 \times \ldots \times J_n$, where the $J_i$'s and $I$ are 1-dimensional compact intervals, there exists a function $G(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ with the following properties:

(a) $G(t, x) \geq F(t, x)$ on $I \times J$

(b) If $E = \{(t, x) \in I \times J : F(t, x) = G(t, x)\}$, then $|E| \geq \frac{3}{8}|I||J|$.

(c) For each $i$, the function $G(t, x_1, x_2, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n)$ is Lipschitz, with Lipschitz constant less than or equal to $M_i$, and one of the following statements is true: Either for each $x$, $-M \leq \dfrac{\partial G}{\partial t}(t, x) \leq \dfrac{4M}{5}$, or for each $x$, $\dfrac{-4M}{5} \leq \dfrac{\partial G}{\partial t}(t, x) \leq M$.

The proof of this lemma is the same as in the 1-dimensional case, treating $x$ as a parameter (see [13]).

Before we procedd with the proof of Lemma 2.1.7, we would like to point out that in the analogue of Lemma 2.1.7 for bounded domains, a normalization is necessary since if $\vec{u}(x)$ solves the systems of elastostatics so does $\vec{u}(x) + \vec{a} + BX$, where $\vec{a}$ is a constant vector, while $B$ is any antisymmetric $3 \times 3$ matrix. The right-hand side of the inequality in the Lemma of course remains unchanged, while the left-hand side increases if $B$ 'increases'. The most convenient normalization is that for some fixed point $X^*$ in the domain $\nabla\vec{u}(X^*) - \nabla\vec{u}(X^*)^t = 0$. This also gives uniqueness modulo constants to problem 2.1.2 in bounded domains.

We now need to introduce some definitions. Let $D_0 \subset \mathbb{R}^n_+$ be a fixed, $C^\infty$ domain with $\{(x, 0) : \|x\| = \max|x_i| \leq 1\} \subset \partial D_0$,

$$\{(x, y) : 0 < y < 1, \|x\| \leq 1\} \subset D_0 \subset \{(x, y) : 0 < y < 2, \|x\| < 2\}.$$

If $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz, with $\|\nabla\varphi\| \leq M$, we construct the mapping $T_\varphi : \mathbb{R}^n_+ \to \mathbb{R}^n$ by $T_\varphi(x, y) = (x, cy + \eta_y * \varphi(x))$ where $\eta \in C_0^\infty(\mathbb{R}^{n-1})$ is radial, $\int \eta = 1$, and $c = c(M)$ is chosen so that $T_\varphi(\mathbb{R}^n_+) \subset \{(x,y) : y > \varphi(x)\}$, and so that $T_\varphi$ is a bi-Lipschitzian mapping. Also, it is clear that $T_\varphi$ is smooth for $(x,y)$ with $y > 0$, and $T_\varphi(x,0) = (x, \varphi(x))$. We will denote by $A_\varphi$ the point $T_\varphi(0,1)$. Lemma 2.1.7 is an easy consequence of the following result.

**Lemma 2.1.11.** Given $M > 0$ and $\varphi$ with $\|\nabla\varphi\| \leq M$, there exists a constant $C = C(M)$ such that for all functions $\vec{u}$ in $D_\varphi$, which are Lipschitz in $\bar{D}_\varphi$, which satisfy $\mu\Delta\vec{u} + (\lambda + \mu)\nabla\text{div } \vec{u} = 0$ in $D_\varphi$ and $\nabla\vec{u}(A_\varphi) = \nabla\vec{u}(A_\varphi)^t$, we have

$$\|N_\varphi(\nabla\vec{u})\|_{L^2(\partial D, d\sigma)} \leq C\|N_\varphi(\lambda(\text{div } \vec{u})I + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\})\|_{L^2(\partial D, d\sigma)}.$$

Here $N_\varphi$ is the non-tangential maximal operator corresponding to the domain $D_\varphi$.

This lemma will be proved by a series of propositions. Before we proceed, we need to introduce one more definition. We say that proposition $(M, \varepsilon)$ holds if whenever $\varphi$ is such that $\|\nabla\varphi\| \leq M$, and there exists a constant vector $\vec{a}$ with $\|\vec{a}\| \leq M$ so that $\|\nabla\varphi - \vec{a}\| \leq \varepsilon$, then for all Lipschitz functions $\vec{u}$ on $\bar{D}_\varphi$, with $\mu\Delta\vec{u} + (\lambda + \mu)\nabla\text{div } \vec{u} = 0$ in $D_\varphi$, with $\nabla\vec{u}(A_\varphi) = \nabla\vec{u}^t(A_\varphi)$ we have

$$\|N_\varphi(\nabla\vec{u})\|_{L^2(\partial D, d\sigma)} \leq C\|N_\varphi(\lambda(\text{div } \vec{u})I + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\}\|_{L^2(\partial D, d\sigma)},$$

where $C = C(M, \varepsilon)$.

Note that if proposition $(M, \varepsilon)$ holds, then the corresponding estimates automatically hold for all translates, rotates or dilates of the domains $D_\varphi$ when $\varphi$ satisfies the conditions in proposition $(M, \varepsilon)$. In the rest of this section, a coordinate chart will be a translate, rotate or dilate of a domain $D_\varphi$. The bottom $B_\varphi$ of $\partial D_\varphi$ will be $T_\varphi(\partial D_0 \cap (x, 0) : x \in \mathbb{R}^{n-1})$.

**Proposition 2.1.12.** Given $M > 0$, there exists $\varepsilon = \varepsilon(M)$ so that propostion $(M, \varepsilon)$ holds.

We will not give the proof of Proposition 2.1.12 here. We will just make a few remarks about its proof. First, in this case the stronger estimate $\|N_\varphi(\nabla\vec{u})\|_{L^2(\partial D, d\sigma)} \leq C\|\lambda(\text{div } \vec{u})N + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\}N\|_{L^2(\partial D, d\sigma)}$ holds. This is because in this case, the domain $D_\varphi$ is a small perturbation of the smooth domain $D_{ax}$. For the smooth domain $D_{ax}$, we can solve problem 2.1.2 by the method of layer potentials (see [27], for example). If $\varepsilon$ is small, a perturbation analysis based on the theorem of Coifman-McIntosh-Meyer ([3]) shows that this is still the case. This easily gives the estimate claimed above.

**Proposition 2.1.13.** For all $M > 0, \varepsilon > 0, \alpha \in (0, 0.1)$, if proposition $(M, \varepsilon)$ holds, then propostion $(1 - \alpha M, 1.1\varepsilon)$ holds.

We postpone the proof of Proposition 2.1.13, and show first how Proposition 2.1.12 and Proposition 2.1.13 yield Lemma 2.1.11.

**Proof of Lemma 2.1.11.** We will show that proposition $(M,\varepsilon)$ holds for any $M,\varepsilon$. Fix $M,\varepsilon$, and choose $N$ so large that if $\varepsilon(10M)$ is as in Proposition 2.1.12, then $(1.1)^N\varepsilon(10M)\geq \varepsilon$. Pick now $\alpha_j>0$ so that $\prod_{j=1}^{N}(1-\alpha_j)=1/10$. Then, since proposition $(10M,\varepsilon(10M))$ holds, by Proposition 2.1.12, applying Proposition 2.1.13 $N$ times we see that proposition $(M,\varepsilon)$ holds.

We will now sketch the proof of Proposition 2.1.13. We first note that it suffices to show that

$$\|N_\varphi(\nabla\vec{u})\|_{L^2(\partial D,d\sigma)}\leq C\|\tilde{N}_\varphi(\lambda(\text{div }\vec{u})I+\mu\{\nabla\vec{u}+\nabla\vec{v}^t\}\|_{L^2(\partial D,d\sigma)}$$

where $\tilde{N}_\varphi$ is the non-tangential maximal operator with a wider opening of the non-tangential region. This follows because of classical arguments relating non-tangential maximal functions with different openings (see [18]) for example). Pick now $\varphi$ with $\|\nabla\varphi\|\leq(1-\alpha)M$, and such that there exists $\vec{a}$ with $\|\nabla\varphi-\vec{a}\|\leq 1.1\varepsilon$, $\|\vec{a}\|\leq(1-\alpha)M$. We will choose $\tilde{N}_\varphi$ as follows: Since $\partial D_\varphi\setminus B_\varphi$ is smooth, it is easy to see that we can find a finite number of coordinate charts (i.e., rotates, translates and dilates of $D_\psi$), which are entirely contained in $D_\varphi$, such that their bottoms $B_\psi$ are contained in $\partial D_\varphi$, such that $T_\psi((x,0):\|x\|<1/2)$ cover $\partial D_\varphi$, and such that the $\psi$'s involved satisfy $\|\nabla\psi\|\leq(1-\dfrac{\alpha}{2})M$ and there exist $\vec{a}_\psi$ such that $\|\vec{a}_\psi\|\leq(1-\dfrac{\alpha}{2})M$, and $\|\nabla\psi-\vec{a}_\psi\|\leq 1.11\varepsilon$. The non-tantential region defining $\tilde{N}_\varphi$, on $T_\psi((x,0):\|x\|<\dfrac{1}{2})$ is defined as follows: let $F\subset\{(x,0):\|x\|<1/2\}$ be a closed set. Consider the cone on $\mathbb{R}^n_+,\gamma=\{(x,y)\in\mathbb{R}^n_+:b|x|<y\}$, where $b$ is a small constant. Consider now the domain $D_F$ on $\mathbb{R}^n_+$, given by $D_F=\cup_{x\in F}((x,0)+\gamma)$. Then $D_F$ is the domain above the graph of a Lipschitz function $\theta$, for which $\|\nabla\theta\|\leq cb$, for some absolute constant $c$ (independent of $F$). It is also easy to see that we can take now $b$ so small, depending only on $M$ and $\varepsilon$ such that $T_\psi(D_F)$ is the domain above the graph of a Lipschitz function $\tilde{\psi}$, with $\tilde{\psi}\geq\psi$, and which statisfies

$$\|\nabla\tilde{\psi}\|\leq(1-\frac{\alpha}{10})M,\quad\|\nabla\tilde{\psi}-\vec{a}_\psi\|\leq 1.111\varepsilon.$$

The non-tangential region defining $\tilde{N}_\varphi$, for $Q\in T_\psi((x,0):\|x\|<1/2)$ is then the image under $T_\psi$ of $(x,0)+\gamma$, with $b$ chosen as above, suitably truncated, and where $Q=T_\psi((x,0)$. Let now, to lighten notation, $m=N_\varphi(\nabla\vec{u}),\bar{m}=\tilde{N}_\varphi(\lambda(\text{div }\vec{u})I+\mu\{\nabla\vec{u}+\nabla\vec{u}^t\})$.

For $t>0$, consider the open-set $E_t=\{m>t\}$. We now produce a Whitney type decomposition of $E_t$ into a family of disjoint sets $\{U_j\}$ with the property that each $U_j$ is contained in $T_\psi((x,0):\|x\|<1.2)$ for a coordinate chart $D_\psi$, each $U_j$ containes $T_\psi(I_j)$, where $I_j$ is a cube in $\|x\|<1/2$, and is contained in $T_\psi(\bar{I}_j)$, where $\bar{I}_j$ is a fixed multiple of $I_j$. Finally, we can also assume that there exists a constant $\eta_0$ such that if diam $(U_j)\leq\eta_0$, there exists a point $Q_j$ in $\partial D_\varphi$, with dist $(Q_j,U_j)\approx$ diam $U_j$, such that $m(Q_j)\leq t$. Let now

$\beta > 1$ be given. We claim that there exists $\delta > 0$ so small that if $E_j = U_j \cap \{m > \beta t, \bar{m} \leq \delta t\}$ then $\sigma(E_j) \leq (1-\eta_M)\sigma(U_j)$, where $\eta_M > 0$. Assume the claim for the time being. Then,

$$\begin{aligned}\int_{\partial D_\varphi} m^2 d\sigma &= 2\int_0^\infty t\sigma(E_t)dt = 2\beta^2 \int_0^\infty t\sigma(E_{\beta t})dt = \sum_j 2\beta^2 \int_0^\infty t\sigma(Q_j \cap E_{\beta t})dt \leq \\ &\leq \sum_j 2\beta^2 \int_0^\infty t\sigma(E_j)dt + 2\beta^2 \int_0^\infty t\sigma(\bar{m} > \delta t)dt \leq \\ &\leq \sum_j 2\beta^2(1-\eta_M) \int_0^\infty t\sigma(Q_j)dt + 2\frac{\beta^2}{\delta^2}\int_0^\infty t\sigma\{\bar{m} > t\}dt = \\ &= \beta^2 \cdot (1-\eta_M)\int_{\partial D_\varphi} m^2 d\sigma + \frac{\beta^2}{\delta^2}\int_{\partial D_\varphi} \bar{m}^2 d\sigma.\end{aligned}$$

Thus, if we choose $\beta > 1$, but so that $\beta^2 \cdot (1-\eta_M) < 1$, the desired result follows. It remains to establish the claim. We argue by contradiction. Suppose not, then $\sigma(E_j) > (1-\eta_M)\sigma(U_j)$. Let $\tilde{E}_j = T_\psi^{-1}(E_j)$. If $\eta_M$ is chosen sufficiently small, we can gurarantee that $|\tilde{E}_j \cap I_j| \geq .99|I_j|$. Let now $F_j = \tilde{E}_j \cap I_j$, and construct now the Lipschitz function $\tilde{\psi}$ corresponding to it, as in the definition of $\tilde{N}_\varphi$. Thus, $\tilde{\psi} \geq \psi$, $\|\nabla\tilde{\psi}\| \leq (1-\frac{\alpha}{10})M$, $\|\nabla\tilde{\psi} - \vec{a}_\psi\| \leq 1.111\varepsilon$. We now apply Lemma 2.1.10 to $\tilde{\psi}$, one variable at a time, to find a Lipschitz function $f$, with $f \geq \tilde{\psi}$ on $I_j$, such that if $\bar{F}_j = \{x \in I_j : f = \tilde{\psi}\}$, then $|\bar{F}_j \cap F_j| \geq c\sigma(U_j)$, with $\|\nabla f\| \leq (1-\frac{\alpha}{10})M$, and such that there exists $\vec{a}_f$, with $\|\vec{a}_f\| \leq (1-\frac{\alpha}{10})M$ so that $\|\nabla f - \vec{a}_f\| \leq \frac{4}{5}1.111\varepsilon < \varepsilon$. We can also arrange the truncation of our non-tangential regions in such a way that on the appropriate rotate, translate and dilate of $D_f$ (which of course is contained in the corresponding coordinate chart associated to $D_\psi$, which is contained in $D_\varphi$),

$$|\lambda(\text{div } \vec{u})I + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\}| \leq \delta t.$$

To lighten the exposition, we will still denote by $D_f$ the translate, rotate and dilate of $D_f$. Note that proposition $(M,\varepsilon)$ applies to it it. We divide the sets $U_j$ into two types. Type I are those with diam $U_j \geq \eta_0$, and type II those for which diam $U_j \leq \eta_0$. We first deal with the $U_j$ of type I. In this case, $D_f$ has diameter of the order of 1. Because of the solvability of problem 2.1.2 for balls, and our normalization, we see that on a ball $B \subset D_\varphi$, diam $B \approx 1, A_\varphi \in B$, we have

$$\int_B |\nabla\vec{u}|^2 \leq c\int_B |\lambda\text{div } \vec{u}I + \mu\{\nabla\vec{u} + \nabla\vec{u}^t\}|^2.$$

Joining $A_f$ to $A_\varphi$ by a finite number of balls, and using interior regularity results for the system $\mu\Delta\vec{u} + (\lambda+\mu)\nabla$ div $\vec{u} = 0$, we see that $|\nabla\vec{u}(A_f)| \leq C\delta t$, for some absolute constant

$C$. Then

$$\begin{aligned}
C\sigma(U_j)\beta^2 t^2 &\leq \int_{T_\psi(\bar{F}_j\cap F_j)} m^2 d\sigma \leq c\int_{\partial D_f} N_f^2(\nabla u)d\sigma \leq \\
&\leq C\sigma(U_j)\delta^2 t^2 + C\int_{\partial D_f} N_f^2(\nabla\vec{u} - \frac{[\nabla\vec{u}(A_f)-\nabla\vec{u}^t(A_f)]}{2})d\sigma \leq \\
&\leq C\sigma(U_j)\delta^2 t^2 + C\int_{\partial D_f} N_f^2(\lambda(\text{div }\vec{u})I + \mu\{\nabla\vec{u}+\nabla\vec{u}^t\})^2 d\sigma,
\end{aligned}$$

by $(M,\varepsilon)$. The last quantity is also bounded by $C\sigma(U_j)\delta^2t^2$, which is a contradiction for small $\delta$. Now, assume that $U_j$ is of type II. Note that in this case there exists $Q_j \in \partial D_\varphi$, with dist $(Q_j, U_j) \approx$ diam $U_j$, and such that $|\nabla\vec{u}(x)| \leq t$ for all $x$ in the non-tantential region associated to $Q_j$. Because of this, it is easy to see, using the arguments we used to bound $|\nabla\vec{u}(A_f)|$ in case I, that for all $X$ in a neighborhood of $A_f$ and also on the top part of $D_f$, we have that $|\nabla\vec{u}(X)| \leq t + C\delta t$. Since for $Q \in T_\psi(\bar{F}_j \cap F_j), m(Q) \geq \beta t$, and $\beta > 1$, if $\delta$ is small enough, we see that we must have $N_f(\nabla\vec{u})(Q) \geq m(Q)$. Hence,

$$N_f(\nabla\vec{u} - [\frac{\nabla\vec{u}(A_f)-\nabla\vec{u}^t(A_f)}{2}])(Q) \geq (\beta - 1 - C\delta)t \geq \frac{(\beta-1)}{2}t$$

if $\delta$ is small and $Q \in T_\psi(\bar{F}_j \cap F_j)$. Thus, applying $(M,\varepsilon)$ to $D_f$, we see that

$$\begin{aligned}
C(\beta-1)^2t^2\sigma(U_j) &\leq \int_{T_\psi(\bar{F}_j\cap F_j)} N_f(\nabla\vec{u} - [\frac{\nabla\vec{u}(A_f)-\nabla\vec{u}^t(A_f)}{2}])^2 d\sigma \leq \\
&\leq \int_{\partial D_f} N_f(\nabla\vec{u} - [\frac{\nabla\vec{u}(A_f)-\nabla\vec{u}^t(A_f)}{2}])^2 d\sigma \leq C\sigma(U_j)\delta^2t^2,
\end{aligned}$$

a contradiction if $\delta$ is small. This finishes the proof of Proposition 2.1.13, and hence of Lemma 2.1.11.

**Part 2: The Stokes system of linear hydrostatics**

In this part I will sketch the proof of the $L^2$ results for the Stokes system of hydrostatics. These results are joint work of E. Fabes, C. Kenig and G. Verchota ([17]). We will keep using the notation introduced in Part 1.

We seek a vector valued function $\vec{u} = (u^1, u^2, u^3)$ and a scalar valued function $p$ satisfying

$$\text{(2.2.1)} \qquad \begin{cases} \Delta\vec{u} = \nabla p \text{ in } D \\ \text{div } \vec{u} = 0 \text{ in } D \\ \vec{u}|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma) \text{ in the non-tangential sense} \end{cases}$$

**Theorem 2.2.2 (Also Theorem 2.1.9).** Given $\vec{f} \in L^2(\partial D, d\sigma)$, there exists a unique solution $(\vec{u}, p)$ to (2.2.1), with $p$ tending to 0 at $\infty$, and $N(\vec{u}) \in L^2(\partial D, d\sigma)$. Moreover, $\vec{u}(X) = \mathcal{K}\vec{g}(X)$, with $\vec{g} \in L^2(\partial D, d\sigma)$. ($\mathcal{K}$ will be defined below).

In order to sketch the proof of 2.2.2, we introduce the matrix $\Gamma(X)$ of fundamental solutions (see the book of Ladyzhenskaya [28]), $\Gamma(X) = (\Gamma_{ij}(X))$, where $\Gamma_{ij}(X) = \frac{1}{8\pi}\frac{\delta_{ij}}{|X|} + \frac{1}{8\pi}\frac{X_i X_j}{|X|^3}$, and its corresponding pressure vector

$$q(X) = (q^i(X)), \text{ where } q^i(X) = \frac{X_i}{4\pi|X|^3}.$$

Our solution of (2.2.2) will be given in the form of a double layer potential,

$$\vec{u}(X) = \mathcal{K}\vec{g}(X) = -\int_{\partial D} \{H'(Q)\Gamma(X-Q)\}\vec{g}(Q)d\sigma(Q),$$

where

$$(H'(Q)\Gamma(X-Q))_{i\ell} = \delta_{ij}q^\ell(X-Q)n_j(Q) + \frac{\partial \Gamma_{i\ell}}{\partial Q_j}(X-Q)n_j(Q).$$

We will also use the single layer potential

$$\vec{u}(X) = S\vec{g}(X) = \int_{\partial D} \Gamma(X,Q)\vec{g}(Q)d\sigma(Q).$$

In the same way as one establishes 2.1.4,

**Lemma 2.2.3.** Let $\mathcal{K}\vec{g}, S\vec{g}$ be defined as above, with $\vec{g} \in L^2(\partial D, d\sigma)$. Then, they both solve $\Delta\vec{u} = \nabla p$ in $D$, and $D_-$ div $\vec{u} = 0$ in $D$ and $D_-$. Also

(a) $\|N(\mathcal{K}\vec{g})\|_{L^2(\partial D, d\sigma)} \leq C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$,

(b) $(\mathcal{K}\vec{g})^{\pm}(P) = \pm\frac{1}{2}\vec{g}(P) - p.v.\int_{\partial D}\{H'(Q)\Gamma(P-Q)\}\vec{g}(Q)d\sigma(Q)$

(c) $\|N(\nabla S\vec{g})\|_{L^2(\partial D, d\sigma)} \leq C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$

(d) $(\frac{\partial}{\partial X_i}(S\vec{g})_j)^{\pm}(P) = \pm\{\frac{n_i(P)g_j(P)}{2} - \frac{n_i(P)n_j(P)}{2}\langle N_p, \vec{g}(P)\rangle\}$
$+ p.v.\int_{\partial D}\frac{\partial}{\partial P_i}\Gamma(P,Q)\vec{g}(Q)d\sigma(Q)$

(e) $(HS\vec{g})^{\pm}(P) = \pm\frac{1}{2}\vec{g}(P) + p.v.\int_{\partial D}\{H(P)\Gamma(P-Q)\}\vec{g}(Q)d\sigma(Q)$, where

$$(H(X)\Gamma(X-Q))_{i\ell} = n_j(x)\frac{\partial \Gamma_{i\ell}}{\partial X_j}(X-Q) - \delta_{ij}q^\ell(X-Q)n_j(X).$$

For the proof of this lemma in the case of smooth domains, see [28].

The proof of Theorem 2.2.2. (at least the existence part of it), reduces to the invertibility in $L^2(\partial D, d\sigma)$ of the operator $\frac{1}{2}I+K$, where $K\vec{g}(P) = -p.v. \int_{\partial D}\{H'(Q)\Gamma(P-Q)\}\vec{g}(Q)d\sigma(Q)$. As in previous cases, it is enough to show

$$\|(\frac{1}{2}I - K^*)\vec{g}\|_{L^2(\partial D, d\sigma)} \approx \|(\frac{1}{2}I + K^*)\vec{g}\|_{L^2(\partial D, d\sigma)}. \tag{2.2.4}$$

This is shown by using the following two integral identities.

**Lemma 2.2.5.** Let $\vec{h}$ be a constant vector in $\mathbb{R}^n$, and suppose that $\Delta\vec{u} = \Delta p$, div $\vec{u} = 0$, in $D$, and that $\vec{u}, p$ and their derivatives are suitablly small at $\infty$. Then,

$$\int_{\partial D} h_\ell n_\ell \frac{\partial u^s}{\partial X_j}\cdot\frac{\partial u^s}{\partial X_j}d\sigma = 2\int_{\partial D}\frac{\partial u^s}{\partial N}\cdot h_\ell\frac{\partial u^s}{\partial X_\ell}d\sigma - 2\int_{\partial D} p n_s h_\ell \frac{\partial u^s}{\partial X_\ell}d\sigma.$$

**Lemma 2.2.6.** Let $\vec{h}, p$ and $\vec{u}$ be as in 2.2.5. Then,

$$\int_{\partial D} h_\ell n_\ell p^2 d\sigma = 2\int_{\partial D} h_r\frac{\partial u^r}{\partial N}p d\sigma - 2\int_{\partial D} h_r\frac{\partial u^r}{\partial X_i}\frac{\partial u^i}{\partial N}d\sigma + 2\int_{\partial D} h_r n_s\frac{\partial u^s}{\partial X_j}\frac{\partial u^r}{\partial X_i}d\sigma.$$

The proofs of 2.2.5 and 2.2.6 are simple applicaitons of the properties of $\vec{u}, p$, and the divergence theorem.

Choosing $\vec{h} = e_3$, we see that, from 2.2.6 we obtain

**Corollary 2.2.7.** Let $\vec{u}, p$ be as in 2.2.6. Then , $\int_{\partial D} p^2 d\sigma \le c\int_{\partial D}|\nabla\vec{u}|^2 d\sigma$, where $C$ depends only on $M$.

A consequence of Corollary 2.2.7 and Lemma 2.2.5, is that if $\frac{\partial\vec{u}}{\partial\nu} = \frac{\partial\vec{u}}{\partial N} - p\cdot N$, then we have

**Corollary 2.2.8.** Let $\vec{u}, p$ be as in 2.2.5. Then,

$$\int_{\partial D}|\frac{\partial\vec{u}}{\partial\nu}|^2 d\sigma \approx \int_{\partial D}|\nabla_t\vec{u}|^2 d\sigma + \sum_j\int_{\partial D}|n_s\frac{\partial u^s}{\partial X_j}|^2 d\sigma,$$

where the constants of equivalence depend only on $M$.

**Proof.** 2.2.5 clearly implies, by Schwartz's inequality, that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq C \int_{\partial D} |\frac{\partial \vec{u}}{\partial \nu}|^2 d\sigma.$$

Moreover, arguing as in the second part of the Remark 2 after 2.1.5, we see that 2.2.5 shows that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq C \int_{\partial D} |\nabla_t \vec{u}|^2 d\sigma + |\int_{\partial D} p n_s h_\ell \frac{\partial u^s}{\partial X_\ell} d\sigma|.$$

By Corollary 2.2.7, the right-hand side is bounded by

$$C\Big(\int_{\partial D} |\nabla \vec{u}|^2 d\sigma\Big)^{1/2}\Big(\sum_j \int_{\partial D} |n_s \frac{\partial u^s}{\partial X_j}|^2 d\sigma\Big)^{1/2} + c\int_{\partial D} |\nabla_t u|^2 d\sigma$$

2.2.8 follows now, using 2.2.7 once more.

To prove 2.2.4, let $\vec{u} = S(\vec{g})$. By d) in 2.2.3, $\nabla_t \vec{u}$ and $n_s \frac{\partial u^s}{\partial X_j}$ are continuous across $\partial D$. Using this fact, 2.2.3 e) and Corollary 2.2.8, 2.2.4 follows.

In closing this part, we would like to point out another boundary value problem for the Stokes system, which is of physical siginificance, the so-called slip boundary condition

$$(2.2.9) \qquad \begin{cases} \Delta \vec{u} = \nabla p \;\text{ in }\; D \\ \operatorname{div} \vec{u} = 0 \;\text{ in }\; D \\ ((\nabla \vec{u} + \nabla \vec{u}^t)N - p \cdot N)|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma). \end{cases}$$

This problem is very similar to (2.1.2). Using the techniques introduced in Part 1, together with the observation that if $\Delta \vec{u} = \nabla p$, div $\vec{u} = 0$ in $D$, the same is true for each row $\vec{v}$ of the matrix $[\nabla \vec{u} + \nabla \vec{u}^t - pI]$, we have obtained

**Theorem 2.2.10.** Given $\vec{f} \in L^2(\partial D, d\sigma)$, there exists a unique solution $(\vec{u}, p)$ to (2.2.9), which tends to 0 at $\infty$, and with $N(\nabla \vec{u}) \in L^2(\partial D, d\sigma)$. Moreover, $\vec{u}(X) = S(\vec{g})(X)$, with $\vec{g} \in L^2(\partial D, d\sigma)$.

## Part 3: The Dirichlet problem for the biharmonic equation on Lipschitz domains

This part deals with the Dirichlet problem for $\Delta^2$ on an arbitrary Lipschitz domain in $\mathbb{R}^n$. The results are joint work of B. Dahlberg, C. Kenig and G. Verchota ([11]). We continue using the notation introduced before.

We seek a function $u$ defined in $D$, such that

$$(2.3.1) \qquad \begin{cases} \Delta^2 u = 0 \text{ in } D \\ u|_{\partial D} = f \in L^2_1(\partial D, d\sigma), \\ \frac{\partial u}{\partial N}|_{\partial D} = g \in L^2(\partial D, d\sigma) \end{cases}$$

where the boundary values are taken non-tangentially a.e.

**Theorem 2.3.2.** There exists a unique $u$ solving (2.3.1), with

$$N(\nabla u) \in L^2(\partial D, d\sigma), \quad \|N(\nabla u)\|_{L^2(\partial D, d\sigma)} \leq C\{\|g\|_{L^2(\partial D, d\sigma)} + \|f\|_{L^2_1(\partial D, d\sigma)}\},$$

where $C$ depends only on $M$.

We will only discuss existence. By 1.1.14, we may assume $f = 0$ on $\partial D$. Let $G(X, Y)$ be the Green function for $\Delta$ on $D$. Then, since $u|_{\partial D} = 0$, we have $u(X) = \int_D G(X, Y)\Delta u(Y)dy$. Notice that $w(y) = \Delta u(y)$ is harmonic in $D$. We claim that $w(Y) = \dfrac{\partial}{\partial y} v(Y)$, where $v$ is a harmonic function in $D$, with $L^2(\partial D, d\sigma)$ Dirichlet data, and that the operator $T : v|_{\partial D} \rightarrow \dfrac{\partial u}{\partial N}|_{\partial D}$ is an invertible map from $L^2(\partial D, d\sigma)$ onto $L^2(\partial D, d\sigma)$. This would establish 2.3.2. In fact, by using the Green's potential representation, Fubini's theorem, and the fact that $\dfrac{\partial}{\partial N} G(-, Y)$ is the density of harmonic measure at $Y \in D$,

$$\int_{\partial D} vTv d\sigma = \int_D v(Y)\frac{\partial}{\partial y} v(Y)dY = \frac{1}{2}\int_{\mathbb{R}^{n-1}} v(x, \varphi(x))^2 dx \geq C\int_{\partial D} v^2 d\sigma.$$

This shows that if $T : L^2(\partial D, d\sigma) \rightarrow L^2(\partial D, d\sigma)$ is bounded, it will have a bounded inverse. To establish the boundedness of $T$, note that if $h$ is harmonic in $D$, then the argument given above shows that

$$\int_{\partial D} hTv d\sigma = \int_D \frac{\partial v}{\partial y}(Y)h(Y)dY.$$

All we need therefore, is the following bilinear estimate.

**Theorem 2.3.3.** If $v, h$ are harmonic in $D$, tend to 0 at $\infty$, then

$$\left|\int_D \frac{\partial v}{\partial y}(Y) \cdot h(Y)dY\right| \leq C\|v\|_{L^2(\partial D, d\sigma)} \cdot \|h\|_{L^2(\partial D, d\sigma)}.$$

**Proof.** This theorem is a generalization to Lipschitz domains of the fact that the paraproduct of two $L^2$ functions is in $L^1$ (see [4]).

In order to establish the inequality, because of the invertibility of the double layer potential (the representation formula in 1.1.1), we can assume that

$$h(Y) = \frac{1}{\omega_n}\int_{\partial D}\frac{\langle Y-Q, N_Q\rangle}{|Y-Q|^{n-2}}g(Q)d\sigma(Q),$$

with

$$\|g\|_{L^2(\partial D, d\sigma)} \le C\|h\|_{L^2(\partial D, d\sigma)}.$$

Thus, since

$$\frac{\langle Y-Q, N_Q\rangle}{|Y-Q|^{n-2}} = C_n\frac{\partial}{\partial N_Q}(\frac{1}{|Y-Q|^{n-2}}),$$

it suffices to show that

$$\|\frac{\partial}{\partial N_Q}\int_D\frac{1}{|Y-Q|^{n-2}}\frac{\partial v}{\partial y}(Y)dY\|_{L^2(\partial D, d\sigma)} \le C\|v\|_{L^2(\partial D, d\sigma)}.$$

In order to do so, we will obtain a respresentation formula for

$$\frac{\partial}{\partial N_Q}\int_D\frac{1}{|Y-Q|^{n-2}}\frac{\partial v}{\partial y}(Y)dY.$$

Fix $Q \in \partial D$, and let $B$ satisfy $\Delta_Y B(Y-Q) = \dfrac{1}{|Y-Q|^{n-2}}$, i.e., $B$ is the fundamental solution for $\Delta^2$ (for example, if $n \ge 5, B(Y) = C_n|Y|^{4-n}$). We recall the definition of the Riesz transforms $v_j = R_j v, j = 1, \ldots, n-1$. They are harmonic functions, which, together with $v$ satisfy the generalized Cauchy-Riemann equations (see [35]), i.e., $\dfrac{\partial v}{\partial X_j} = \dfrac{\partial}{\partial y}R_j v$, and $\dfrac{\partial v}{\partial y} = -\sum_{j=1}^{n-1}\dfrac{\partial}{\partial x_j}R_j v$. If $Y = (x, y)$, then $\dfrac{1}{|Y-Q|^{n-2}}\dfrac{\partial}{\partial y}v(Y) = \Delta_y B$. $\dfrac{\partial v}{\partial y} =$ (using the summation convention)

$$\begin{aligned}(\frac{\partial^2}{\partial x_j^2}B + \frac{\partial^2}{\partial y^2}B)\frac{\partial}{\partial y}v &= \frac{\partial^2}{\partial x_j^2}B\frac{\partial}{\partial y}v - \frac{\partial^2 B}{\partial x_j\partial y}\frac{\partial v}{\partial y} + \\ &+ \frac{\partial^2}{\partial x_j\partial y}B\frac{\partial R_j v}{\partial y} - \frac{\partial^2 B}{\partial y^2}\cdot\frac{\partial}{\partial x_j}R_j v.\end{aligned}$$

Let now $e_1, e_2, \ldots, e_{n-1}, e_n$ be the standard basis of $\mathbb{R}^n$, with $e_n$ pointing in the direction of the $y$ axis. Then, we can rewrite the right-hand side as

$$\begin{aligned}&\langle(-\frac{\partial^2 B}{\partial x_1\partial y}, \frac{-\partial^2 B}{\partial x_2\partial y}, \ldots, \frac{-\partial^2 B}{\partial x_{n-1}\partial y}, \sum_{j=1}^{n-1}\frac{\partial^2 B}{\partial x_j^2}), \nabla v\rangle + \\ &+ \sum_{j=1}^{n-1}\langle\frac{\partial^2 B}{\partial x_j\partial y}e_n, \nabla R_j v\rangle - \sum_{j=1}^{n-1}\langle\frac{\partial^2 B}{\partial y^2}e_j, \nabla R_j v\rangle.\end{aligned}$$

Let $\vec{\alpha} = \left(-\frac{\partial^2 B}{\partial x_1 \partial y}, \frac{-\partial^2 B}{\partial x_2 \partial y}, \ldots, \frac{-\partial^2 B}{\partial x_{n-1}\partial y}, \sum_{j=1}^{n-1} \frac{\partial^2 B}{\partial x_j^2}\right)$, $\vec{\beta}_j = \dfrac{\partial^2 B}{\partial x_j \partial y} e_n - \dfrac{\partial^2 B}{\partial y^2} e_j$. Note that div $\vec{\beta}_j = 0$, and div $\vec{\alpha} = 0$, and that

$$\int_D \frac{1}{|Y-Q|^{n-2}} \frac{\partial}{\partial y} v(Y) dY = \int_D \langle \vec{\alpha}, \nabla v\rangle + \sum_{j=1}^{n-1} \int_D \langle \vec{\beta}_j, \nabla R_j v\rangle =$$
$$= \int_{\partial D} v(P) \cdot \langle \vec{\alpha}(P), N_p\rangle d\sigma(P) + \sum_{j=1}^{n-1} \int_{\partial D} R_j v(P) \cdot \langle \vec{\beta}_j(P), N_p\rangle d\sigma(p),$$

by the divergence theorem. This can be rewritten as

$$\int_{\partial D} [(-n_j(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_n} B(P-Q) + n_n(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_j} B(P-Q)] v(P) d\sigma(P) +$$
$$+ \sum_{j=1}^{n-1} \int_{\partial D} [n_n(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_n} B(P-Q) - n_j(P) \frac{\partial^2}{\partial P_n^2} B(P-Q)] R_j v(P) d\sigma(P).$$

Hence

$$\frac{\partial}{\partial N_Q} \int_D \frac{1}{|Y-Q|^{n-2}} \frac{\partial}{\partial y} v(Y) dY =$$
$$= \int_{\partial D} [-n_j(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_n} \langle \nabla B(P-Q), N_Q\rangle + n_n(P) \frac{\partial^2}{\partial P_n^2} \langle \nabla B(P-Q, N_Q\rangle] v(P) d\sigma(P) +$$
$$+ \sum_{j=1}^{n-1} \int_{\partial D} [n_n(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_n} \langle \nabla B(P-Q), N_Q\rangle - n_j(P) \frac{\partial^2}{\partial P_n^2} \langle \nabla B(P-Q), N_Q\rangle R_j v(P) d\sigma(P).$$

But, by the theorem of Coifman-McIntosh-Meyer, [3], $\dfrac{\partial}{\partial P_j} \dfrac{\partial}{\partial P_i} \dfrac{\partial}{\partial Q_k} B(P-Q)$ is the kernel of a bounded operator in $L^2(\partial D, d\sigma)$. Thus,

$$\| \frac{\partial}{\partial N_Q} \int_D \frac{1}{|Y-Q|^{n-2}} \frac{\partial v}{\partial y}(Y) dY | \leq C\{\|v\|_{L^2(\partial D, d\sigma)} + \sum_{j=1}^{n-1} \|R_j v\|_{L^2(\partial D, d\sigma)}\}.$$

Finally, we invoke a result of Dahlberg ([8]), who showed that

$$\|R_j v\|_{L^2(\partial D, d\sigma)} \leq C \|v\|_{L^2(\partial D, d\sigma)}.$$

This concludes the proof of 2.3.3.

As a final comment, we would like to point out that in this exposition we have emphasized non-tangential maximal function estimates, but that optimal Sobolev space estimates also hold. For example, the solution $\vec{u}$ of (2.1.1) is in the Sobolev space $H^{1/2}(D)$, the one of (2.1.2) in the Sobolev space $H^{3/2}(D)$, and the same is true for $\vec{u}$ in 2.1.3 c). The solution of (2.2.1) is in $H^{1/2}(D)$, while the one of (2.2.9) is in $H^{3/2}(D)$. Finally, the solution $u$ of 2.3.1 is in $H^{3/2}(D)$. All of these results can be proved in a unified fashion using a variant of the proof of Lemma 2.1.11. The details will appear in a forthcoming paper of B. Dahlberg and C. Kenig [10].

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# Appendix 2

# Hardy spaces and the Neumann Problem in $L^p$ for Laplace's equation in Lipschitz domains

By Björn E. J. Dahlberg* and Carlos E. Kenig*

## 1. Introduction

The purpose of this paper is to give optimal results for the solvability of the Neumann problem in Lipschitz domains with data in $L^p$. We also obtain corresponding end point results for Hardy spaces. Our main theorem asserts that if $D \subset \mathbf{R}^n, n \geq 3$, is a bounded Lipschitz domain with connected boundary, then there exists $\varepsilon = \varepsilon(D) > 0$ such that, for all $f \in L^p(\partial D, d\sigma)$, with $1 < p < 2 + \varepsilon$, and $\int_{\partial D} f d\sigma = 0$, there is a unique (modulo constants) harmonic function $u$ in $D$ with

$$\|M(\nabla u)\|_{L^p(d\sigma)} \leq C_p(D)\|f\|_{L^p(d\sigma)}, \tag{1.1}$$

and

$$\frac{\partial u}{\partial n} = f \text{ on } \partial D. \tag{1.2}$$

Here $\sigma$ is the surface measure on $\partial D$, and $M(\nabla u)$, the non-tangential maximal function of $\nabla u$, the gradient of $u$, is defined, for $Q \in \partial D$ by (for example)

$$M(F)(Q) = \sup\{|F(X)| : X \in D, |X - Q| < 2 \text{ dist } (X, \partial D)\}. \tag{1.3}$$

It is known (see [4]) that if $v$ is harmonic in a Lipschitz domain and $M(v) < \infty$ a.e. on $\partial D$, then $v$ has non-tangential limits a.e. on $\partial D$. Here "almost everywhere" is taken with respect to the surface measure on $\partial D$, and the existence of the non-tangential limit means that $\lim\limits_{\substack{X \to Q \\ X \in \Gamma_\alpha(Q)}} v(X)$ exists and is finite for all $\alpha > 0$, where

$$\Gamma_\alpha(Q) = \{X \in D : |X - Q| < (1 + \alpha) \text{ dist } (X, \partial D)\}.$$

Consequently, if (1.1) holds, then $\nabla u$ has non-tangential limits a.e. on $\partial D$, and the meaning of the generlized normal derivative $\partial u / \partial n$ in (1.2) is the limit of $\langle \nabla u(X), n(Q) \rangle$ as $X \to Q \in \partial D$ non-tangentially. Here $\langle A, B \rangle$ denotes the inner product in $\mathbf{R}^n$, and $n(Q)$ the unit normal to $\partial D$ at $Q$. As is well known, $n(Q)$ exists for a.e. $Q$, since $D$ is a Lipschitz domain.

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The result was first established for the case $p = 2$ by Jerison and Kenig ([13]) and for the case of $C^1$ domains by Fabes, Jodeit and Riviere ([7]). Our extension of the Lipschitz domain case to the range $1 < p < 2+\varepsilon$ is done by two different methods. The extension to $2 < p < 2+\varepsilon$ is done by a real variable argument, using the result for $p = 2$, and a variant of the 'good $\lambda$' inequalities. The extension to $1 < p < 2$ is accomplished by establishing that for $f \in H^1_{at}(\partial D)$, the atomic $H^1$ space on $\partial D$, there exists a unique (modulo constants) solution $u$ of the Neumann problem with data $f$ which satisfies

$$\|M(\nabla u)\|_{L^1(d\sigma)} \leq C(D)\|f\|_{H^1_{at}(\partial D)}.$$

This is proved by estimating the non-tangential maximal functions of gradients of the $L^2$-solutions with data atoms. We do this in turn by using the regularity theory for uniformly elliptic operators in selfadjoint form. The full result then follows by interpolation. Combining our estimates for atoms with the techniques in [8], we are able to obtain a generalization of the Stein-Weiss theory of Hardy spaces, valid for Lipschitz domains in $\mathbf{R}^n$. This generalizes the results for $C^1$ domains in [8], and some of the two dimensional results in [14].

The range $1 < p < 2 + \varepsilon$ is optimal for the Neumann problem. The estimate (1.1) fails for $p = 1$ even for smooth regions (Hardy space results are the appropriate analogue). Moreover, for each $p_0 > 2$ it is possible to construct a Lipschitz domain $D$, for which (1.1) fails for $p = p_0$. (See for example [16] for the relevant examples.) The situation is similar to the case of the Dirichlet problem (see Dahlberg, [5]), where one has for the solution $u$ of the problem $\Delta u = 0$ in $D$, $u = f$ on $\partial D$,

$$\|M(u)\|_{L^p(d\sigma)} \leq c_p(D)\|f\|_{L^p(d\sigma)}, \tag{1.4}$$

whenever $2 - \varepsilon < p \leq \infty$ for an $\varepsilon = \varepsilon(D) > 0$. The relationship between the results (1.1) and (1.3) can be best understood by the use of the method of integral equations. For $f$ on $\partial D$, let

$$\mathcal{D}(f)(X) = \frac{1}{\omega_n(2-n)}\int_{\partial D} f(Q)\frac{\partial}{\partial n_Q}(|X-Q|^{2-n})d\sigma(Q)$$

be the double layer potential of $f$, and let

$$S(f)(X) = \frac{1}{\omega_n(n-2)}\int_{\partial D} f(Q)|X-Q|^{2-n}d\sigma(Q)$$

be the single layer potential of $f$.

A consequence of the boundedness of the Cauchy integral on Lipschitz curves (see Coifman, McIntosh and Meyer [2]), is that $M(\mathcal{D}(f))$ and $M(\nabla S(f))$ take $L^p(d\sigma)$ into $L^p(d\sigma)$, $1 < p < \infty$. In the classical case when the domain $D$ is smooth, one can use the Fredholm theory to see that the operators $\mathcal{D}(f)|_{\partial D}$ and $(\partial/\partial n)S(f)|_{\partial D}$ are invertible on $L^p(d\sigma)$, $1 < p < \infty$.

This is the result that was extended to $C^1$ domains by Fabes, Jodeit and Riviere [7], using the work of A. P. Caldeón [1.A]. However, this compactness argument does not extend to the case of Lipschitz domains (see for example [16] for simple couterexamples). The invertibility of the layer potentials for general Lipschitz domains was established in $L^2(d\sigma)$ by Verchota. As a consequence of (1.1) and the analogous estimate

$$\|M(\nabla u)\|_{L^p(d\sigma)} \le C_p(D)\|\nabla_T f\|_{L^p(d\sigma)}, \quad 1 < p < 2+\varepsilon, \tag{1.5}$$

$\varepsilon = \varepsilon(D)$, where $u$ is the solution of the Dirichlet problem, and $\nabla_T$ denotes the tangential gradient, we are able to establish that the operator $f \to \mathcal{D}(f)|_{\partial D}$ is one-to-one and onto on $L^p(d\sigma), 2-\varepsilon < p < \infty$, and that the operator $f \to (\partial/\partial n)S(f)|_{\partial D}$ is one-to-one and onto $L^p_0(d\sigma), 1 < p < 2+\varepsilon$, where

$$L^p_0(d\sigma) = \{f \in L^p(d\sigma) : \int_{\partial D} f d\sigma = 0\}.$$

This is again the optimal range of $p$'s in both cases.

At this point we would like to point out that the case $p = 2$ of (1.5) is due to Jerison and Kenig [13], while the general case is due to Verchota [25]. Here we will also give a new proof of (1.5), analogous to our proof of (1.1). We will also present endpoint results on the invertibility of $\mathcal{D}$ and $(\partial/\partial n)S$ on BMO and $H^1_{at}$ respectively. For a more complete description of these results, we refer to the body of the paper.

Capitial letters $X, Y, Z$ will denote points of a fixed domain $D \subset \mathbf{R}^n$, while $P, Q$ will be reserved for points in $\partial D$. Lower case letters $x, y, z$ are reserved for points in $\mathbf{R}^{n-1}$, while the letters $s, t$ will be reserved for real numbers. As was mentioned before, in the sequel we assume that $n \ge 3$. The results remain valid when $n = 2$ with the obvious modifications.

# 2. The Neumann problem on graphs

We begin by treating the case when

$$D = \{(x, y) \in \mathbf{R}^n : y > \varphi(x), x \in \mathbf{R}^{n-1}\},$$

where $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ is Lipschitz continous; i.e., $|\varphi(x) - \varphi(x')| \le m|x - x'|$.

We start out by reviewing the Neumann problem with data $f \in L^2(\Lambda)$, $\Lambda = \partial D$. Let us a priori assume that $f$ is bounded and has compact support.

Let

$$u(X) = S(f)(X) = \frac{1}{\omega_n(n-2)} \int_\Lambda f(Q)|X - Q|^{2-n} d\sigma(Q),$$

be the single layer potential of $f$. Since $u$ is harmonic in $D$, we have the identity

$$\operatorname{div}\big(|\nabla u|^2 e - 2\frac{\partial u}{\partial y}\nabla u\big) = 0, \tag{2.1}$$

where $e = (0,1)$.

The results in [2] show that

$$\|M(\nabla u)\|_{L^2(\Lambda)} \le C\|f\|_{L^2(\Lambda)}.$$

Since $\nabla u(X) = O(|X|^{1-n})$ as $X \to \infty$, it follows from (2.1), the estimate above, and the divergence theorem applied in the domains $D_{\rho,\varepsilon} = \{(x,y) : y > \varphi(x) + \varepsilon, |x|^2 + y^2 < \rho^2\}$, that

$$\int_\Lambda \big(|\nabla u|^2\langle e, n\rangle - 2\frac{\partial u}{\partial n}\cdot\frac{\partial u}{\partial y}\big)d\sigma = 0, \tag{2.2}$$

where the derivatives on $\Lambda$ are taken as the non-tangential limits from $D$, of the corresponding expressions in $D$. This is the Rellich ([21]) identity on $\Lambda$.

To exploit (2.2), let $T_1(x), T_2(x), \dots, T_{n-1}(x)$ be an orthonormal basis for the tangent plane to $\Lambda$ at $(x, \varphi(x))$. The $T_i(x)$ exist for a.e. $x$. Let

$$|\nabla_T u(x,\varphi(x))|^2 = \sum_{j=1}^{n-1} |\langle \nabla u(x,\varphi(x)), T_i(x)\rangle|^2.$$

A well-known argument (see [15], Corollary 2.1.11 for example) shows that there are constants $c_1, c_2$, that depend only on the Lipschitz constant $m$ of $\varphi$ such that

$$C_1\int_\Lambda \big|\frac{\partial u}{\partial n}\big|^2 d\sigma \le \int_\Lambda |\nabla_T u|^2 d\sigma \le C_2\int_\Lambda \big|\frac{\partial u}{\partial n}\big|^2 d\sigma. \tag{2.3}$$

Note that in the above estimate the values of the derivatives of $u$ are taken as the limits from above the graph, but the same estimate can be obtained by taking limits from below the graph. Let now $Tf$ and $T_- f$ denote the normal derivatives of $Sf$ as a function in $D$ and $\mathbf{R}^n \setminus \bar{D}$ respectivly.

We then have (see [15])

$$\|Tf\|_{L^2(\Lambda)} \ge c\|f\|_{L^2(\Lambda)}, \tag{2.4}$$

where $c = c(m)$. To establish (2.4), we recall the classical jump relation $T + T_- = I$, where $I$ is the identity operator (again, see [15] for a proof of this in our case). To prove (2.4) we only need to remark that $|\nabla_T u|^2$ is continuous a.e. across $\Lambda$ (see [15]). Thus, (2.4) follows by application of (2.3) in $D$ and $\mathbf{R}^n \setminus \bar{D}$, together with the jump relation. The boundedness of $T$ ([2]) together with (2.4) immediately show that $T$ is one-to-one, with closed range.

To see that the range of $T$ is all of $L^2(\Lambda)$, we let $U : L^2(\mathbf{R}^{n-1}) \to L^2(\mathbf{R}^{n-1})$ be the pullback of $T$, i.e., $Ug = T(g \circ F^{-1}) \circ F$, where $F : \mathbf{R}^{n-1} \to \Lambda$ is given by $F(x) = (x, \varphi(x))$. Letting $U_s$ denote the operator corresponding to the graph of $x \to s\varphi(x), 0 \leq s \leq 1$, we see easily, using the results in [2], that $\|U_s - U_{s_1}\| \leq C|s - s_1|, 0 \leq s, s_1 \leq 1$. (See [15] for the details.) Since $U_0 = \frac{1}{2}I$, the continuity method shows that $U_1$, and hence $T$, is onto (see [15], Lemma 2.1.6).

**Theorem 2.5.** *Given any $f \in L^2(\Lambda)$ there is a harmnonic function $u$ in $D$ such that $\|M(\nabla u)\|_{L^2(\Lambda)} \leq C(m)\|f\|_{L^2(\Lambda)}$, and $\partial u/\partial n = f$ a.e. on $\Lambda$, in the sense of non-tangential convergence. Any two such harmonic functions differ by a constant. Let $g = T^{-1}f$. If $n \geq 4$ one such function is given by $S(g)$. If $n = 3$ one such solution is given by*

$$u(X) = \int_{\partial D} g(Q)\{|X - Q|^{2-n} - |X_0 - Q|^{2-n}\}d\sigma(Q),$$

*where $X_0$ is any fixed point in $\mathbf{R}^n \setminus \bar{D}$.*

*Proof.* The existence part and the representation formulas follow from the invertibility of $T$, and the results of Coifman-McIntosh-Meyer [2] mentioned before. It remains to establish uniqueness. We present here an argument which will be very useful for us later on in treating the $L^p$ case $1 < p < 2$. Let $\omega$ be harmonic in $D$, with $M(\nabla\omega) \in L^2(\Lambda)$, and $\partial\omega/\partial n = 0$ a.e. on $\Lambda$. Note first that $\int_{D_R} |\nabla\omega|^2 dX \leq CR$, where $D_R = D \cap \{(x, y) : |x|^2 + y^2 < R^2\}$. Consider now the bi-Lipschitzian mapping $\phi : \bar{D} \to \mathbf{R}^n \setminus D$ given by

$$\phi(x, y) = (x, \varphi(x) - [y - \varphi(x)]) = (x, 2\varphi(x) - y).$$

Define $\omega^*$ on $\mathbf{R}^n \setminus \bar{D}$ by the formula $\omega^* = \omega \circ \phi^{-1}$. A simple calculation shows that in $\mathbf{R}^n \setminus \bar{D}$, $\omega^*$ verifies (in the weak sense) the equation div $(A(x, y)\nabla\omega^*) = 0$, where $A(x, y) = 1/[J\phi(X)]\phi'(X)\phi'^t(X)$, where $X = \phi^{-1}(x, y)$, $\phi'$ is the Jacobian matrix of $\phi$, and $J\phi$ the Jacobian determinant of $\phi$. It is easy to see that $A \in L^\infty(\mathbf{R}^n \setminus \bar{D})$, and $\langle A(x, y)\xi, \xi\rangle \geq C|\xi|^2$, where $C = C(m)$. Let now

$$B(x, y) = \begin{cases} I & \text{for } (x, y) \in D \\ A(x, y) & \text{for } (x, y) \in \mathbf{R}^n \setminus \bar{D} \end{cases}$$

and extend $\omega(x, y)$ to all of $\mathbf{R}^n$ by setting it equal to $\omega^*(x, y)$ in $\mathbf{R}^n \setminus \bar{D}$. Since $M(\nabla\omega) \in L^2(\Lambda)$, and $\partial\omega/\partial n = 0$ a.e. on $\Lambda$, it is easy to see that the extended $\omega$ is a weak solution in all of $\mathbf{R}^n$ of the divergence form elliptic equation with bounded measurable coefficients div $B(x, y)\nabla\omega = 0$. The extended $\omega$ also satisfies the estimate $\int_{|x|<R} |\nabla\omega|^2 dX \leq CR$. By the Poincaré inequality we see that

$$\int_{|X|<R} |\omega - \omega_R|^2 dX \leq CR^3,$$

where $\omega_R$ is the average of $\omega$ over the ball $|X| < R$. By the theorem of De Giorgi-Nash ([6], [20]) $\omega$ is locally Hölder continuous in $\mathbf{R}^n$. By the $L^\infty$ estimate of Moser ([19]), it follows that

$$\sup_{|x|<R/2} |\omega(X) - \omega_R| \le C\big(R^{-n}\int_{|x|<R} |\omega - \omega_R|^2 dX\big)^{1/2} \le C.$$

Therefore, the oscillation of $\omega$ over the ball of radius $R$ remains bounded. By the Liouville theorem of Moser ([19]) $\omega$ is a constant. This finishes the proof of Theorem 2.5.

In order to pass to the $L^p$ theory, we need to recall some definitions. An atom $a$ is a bounded function on $\Lambda$ with support in a surface ball $B = B(Q,r) = \{P \in \partial D : |P - Q| \le r\}$, such that $\|a\|_\infty \le 1/\sigma(B)$ and $\int a d\sigma = 0$. The atomic Hardy space

$$H^1_{at}(\Lambda) = \{f \in L^1(\Lambda) : f = \sum \lambda_j a_j, \text{ where } a_j \text{ is an atom and } \sum |\lambda_j| < \infty\}.$$

has norm, for $f \in H^1_{at}(\Lambda)$,

$$\|f\|_{H^1_{at}(\Lambda)} = \inf\{\sum |\lambda_j| : f = \sum \lambda_j a_j, a_j \text{ atoms}\}.$$

For general facts concerning atomic Hardy spaces, see the survey article by Coifman and Weiss ([3]). In the next lemma we study the action of the gradient of the single layer potential on atoms. The lemma can be proved if one combines the arguments in [3] with the results in [2].

**Lemma 2.6.** *Let $a$ be an atom on $\Lambda$, and $f = M(\nabla S(a))$. Then,*

(a)
$$\int_\Lambda f d\sigma \le C,$$

(b)
$$\Big(\int_\Lambda f^2 d\sigma\Big)\Big(\int_\Lambda f^2(Q)|Q - Q_a|^{(n-1)(1+\varepsilon)} d\sigma\Big)^{1/\varepsilon} \le C,$$

(c)
$$\int \frac{\partial S}{\partial n}(a) d\sigma = \int \frac{\partial S}{\partial \tilde{T}_j}(a) d\sigma = 0, \quad 1 \le j \le n-1,$$

*where $C$ and $\varepsilon > 0$ depend only on $m$, $Q_a$ is the center of the support of $a$, and the tangential vector fields $\tilde{T}_j$, are given by $\tilde{T}_j = (0, \dots, 1, 0, \dots, \partial\varphi/\partial x_j)(1 + |\nabla\varphi|^2)^{-1/2}$, where the 1 is on the $j$th slot.*

We will now establish the analogue of Lemma 2.6 for the solution of the Neumann problem with data $a$. This is the central point of our paper.

**Lemma 2.7.** *Let $a$ be an atom on $\Lambda$ and let $u$ be a solution in $D$ of the Neumann problem with data $a$, given by Theorem 2.5. Let $f = M(\nabla u)$. Then,*

(a)
$$\int_\Lambda f d\sigma \le C,$$

(b)
$$\Big(\int_\Lambda f^2 d\sigma\Big)\Big(\int_\Lambda f^2(Q)|Q-Q_a|^{(n-1)(1+\varepsilon)}d\sigma\Big)^{1/\varepsilon}\Big) \le C,$$

(c)
$$\int_\Lambda f^2(Q)|Q-Q_a|^{n-1}d\sigma \le C,$$

*where $C$ and $\varepsilon$ are positive constants which depend only on $m$.*

*Proof.* Because of the translation and dilation invariance of the estimates, we can assume supp $a \subset \{(x,\varphi(x)) : |x| \le 1\}, Q_a = (0,0)$ and $\|a\|_\infty \le 1$. Pick $g \in L^2(\Lambda)$ such that $\partial S/\partial n(g) = a$ on $\Lambda$. We clearly have $\|g\|_{L^2(\Lambda)} \le C$. Let

$$D^* = \{(x,y) : |x| < 2, \varphi(x) < y < \varphi(x)+2\}.$$

We first claim that there is a constant $C = C(m)$ such that for some choice of $u$ we have

$$|u(X)| \le C \text{ in } D \setminus D^*. \tag{2.8}$$

In order to establish (2.8), we argue as in the uniqueness part of the proof of Theorem (2.5), and extend $u$ by reflection to $\mathbf{R}^n \setminus \bar{D}$ so that $Lu = 0$ in $\mathbf{R}^n \setminus \{(x,\varphi(x)) : |x| \le 1\}$, where $L$ is a uniformly elliptic operator in divergence from, with bounded measurable coefficients. (Here we use the support property of $a$.)

Assume first that $n \ge 4$, and let $u(X) = S(g)(X)$ in $D$. By the reflection, and Schwarz's inequality, we have that $|u(X)| \le C$ in $\{X \in \mathbf{R}^n : \text{dist }(X,\Lambda) \ge 1\}$. Let $\omega = \max\{0, |u| - c\}$. $\omega$ is a non-negative subsolution of $L$ in $\mathbf{R}^n \setminus \{(x,\varphi(x)) : |x| \le 1\}$, and there is a constant $d > 0$ such that for all $n$-dimensional balls $B$ centered in $\mathbf{R}^n \setminus \{(x,y) : |x| < 2, |y-\varphi(x)| < 2\}$, of radius $r_0 = r_0(m)$, we have that $|\{X \in B : \omega(X) = 0\}| \ge d$. Furthermore, on such balls $B$, $\int_B |\nabla\omega|^2 \le C$, by the $L^2$ estiamte for the non-tangential maximal function of $\nabla u$. Therefore, by a standard variant of the Poincaré inequality, we have $\int_B \omega^2 \le C$. The sub-mean value inequality for $L$-subsolutions ([19]) now establishes (2.8) in the case $n \ge 4$. It remains to show (2.8) when $n = 3$. In this case, we let, for $X \in D, u(X) = \int_\Lambda g(Q)\{|X-Q|^{2-n} - |X_0 - Q|^{2-n}\}d\sigma(Q)$. Schwarz's inequality now shows that $|u(X)| \le C\log(2+\text{dist }(X,\Lambda))$ whenever dist $(X,\Lambda) \ge 1$. Let now $X = (x,y)$ and set $\omega(X) = \max(|u(X)| - c, 0)$ for $y > \varphi(x)+1$, and 0 otherwise. If $c$ is chosen large enough, then $\omega$ is zero in a neighborhood of $\{(x,y) : y = \varphi(x)+1\}$, and so $\omega$ is a subharmonic function in all of $\mathbf{R}^n$. If $\Gamma_0 = \{(x,y) : y + y_0 > -2m|x|\}$, then $\omega$ is 0 on $\partial\Gamma_0$ if $y_0$ is large enough, and $\omega$ has logarithmic growth at $\infty$. By the Phragmen-Lindelöf Theorem (see [17]), $\omega$ is identically 0, which yields (2.8) as in the case $n \geq 4$.

Let $g$ be the fundamental solution of $L$ in $\mathbf{R}^n$, with pole at 0. As is well-known, there are constants $C_1$, and $C_2$ which depend only on the ellipticity constants of $L$ (and hence only on $m$), such that

$$C_1|X|^{2-n} \leq g(X) \leq C_2|X|^{2-n}$$

(see [18]). By (2.8) and the asymptotic expansion of Serrin and Weinberger (Theorem 7 of [22]), there are constants $\alpha, \beta, \gamma, \nu$ and $R_0, 0 < \nu, R_0$, such that

$$u(X) = \alpha g(X) + \beta + v(X),$$

for $X \in \mathbf{R}^n \setminus \{(x, \varphi(x)) : |x| \leq 1\}$ where $|v(X)| \leq \gamma |X|^{2-n-\nu}$ for $|X| \geq R_0$, where $\nu^{-1}$ can be bounded in terms of $m$, while $\alpha, \beta, \gamma$ and $R_0$ can be bounded in terms of $\nu$ and $\|u\|_{L^\infty(D \setminus D^*)}$. We next claim that $\alpha = 0$. To show this, we recall that if $L = \mathrm{div}\, B\nabla$, Theorem 7 of [22] shows that $\alpha = b\int \langle B\nabla u, \nabla\psi\rangle$, where $b$ is a constant that depends on $g$, and $\psi$ is any $C^\infty$ function on $\mathbf{R}^n$, which is 0 for $|x| \leq R_0$ and equals 1 for $|x| \geq R_1 > R_0$. Let $R$ be large, and set, for $0 < \tau$,

$$D(\tau) = \{(x, y) : \varphi(x) + \tau < y < \varphi(x) + R, |x| \leq R\}.$$

Clearly, for $R$ large enough, $\psi \equiv 1$ on $A(\tau) = \partial D(\tau) - B(\tau)$ where $B(\tau) = \{(x, \varphi(x) + \tau) : |x| \leq R\}$. Hence,

$$\begin{aligned}
\int_D \langle B\nabla u, \nabla\psi\rangle &= \int_D \langle \nabla u, \nabla\psi\rangle = \lim_{\tau\to 0}\int_{D(\tau)} \langle \nabla u, \nabla\psi\rangle \\
&= \lim_{\tau\to 0}\int_{\partial D(\tau)} \psi\frac{\partial u}{\partial n} = \lim_{\tau\to 0}\int_{\partial D(\tau)} (\psi - 1)\frac{\partial u}{\partial n} \\
&= \lim_{\tau\to 0}\int_{B(\tau)} (\psi - 1)\frac{\partial u}{\partial n} = \int_\Lambda (\psi - 1) a\, d\sigma = 0.
\end{aligned}$$

By our construction of $B$ (see the proof of Theorem 2.5), $\int_{\mathbf{R}^n \setminus \bar{D}} \langle B\nabla u, \nabla\psi\rangle = \int_D \langle \nabla u, \nabla\tilde{\psi}\rangle$, where $\tilde{\psi} = \psi \circ \phi$. This quanity is also zero. Let

$$a(R) = \int_{\Lambda(R)} M(\nabla u)^2 d\sigma,$$

where

$$\Lambda(R) = \{(x, \varphi(x)) : R \leq |x| \leq 2R\},$$

and let

$$a(1) = \int_{\{(x,\varphi(x)) : |x| \leq 2\}} M(\nabla u)^2 d\sigma.$$

By Theorem 2.5, $a(1) \leq C\|a\|^2_{L^2(\Lambda)} \leq C$.

For $Q \in \Lambda$, let $\gamma(Q) = \{X \in D : |X - Q| < 2 \text{ dist } (X, \Lambda)\}$ and for $Q \in \Lambda(R)$ set $\gamma_1(Q) = \{X \in \gamma(Q) : |X - Q| < R\}$, $\gamma_2(Q) = \gamma(Q) \setminus \gamma_1(Q)$, and $M_i(Q) = \sup\{|\nabla u(X)| : X \in \gamma_i(Q)\}$, $i = 1, 2$. Observe that if $X \in \gamma_2(Q)$, then $u$ is harmonic in $B = \{Y : |Y - X| \leq \delta R\}$, where $\delta = \delta(m)$ is small enough, and $\sup_{Y \in B} |u(Y) - \beta| \leq CR^{2-n-\nu}$. Since $|\nabla u(X)| \leq CR^{-1-n} \int_B |u(y) - \beta| dY \leq CR^{1-n-\nu}$, it follows that

$$\int_{\Lambda(R)} (M_2(\nabla u))^2 d\sigma \leq CR^{1-n-2\nu}. \tag{2.9}$$

For $\tau \in I = [1/4, 1/2]$, set

$$\Omega_\tau = \{(x, y) : \varphi(x) < y < \varphi(x) + \theta R, \tau R < |x| < \tau^{-1} R\},$$

where $\theta$ is chosen so that for $Q \in \Lambda(R), \overline{\gamma_1(Q)} \subset\subset \Omega_\tau$.

From the $L^2$ Neumann theory for bounded Lipschitz domains ([13]), it follows that

$$\int_{\Lambda(R)} M_1(|\nabla u|)^2 d\sigma \leq C \int_{\partial\Omega_\tau} (\frac{\partial u}{\partial n})^2 d\sigma \leq C \int_{\partial\Omega_\tau \cap D} |\nabla u|^2 d\sigma,$$

with $C$ depending only on $m$, since $\partial u / \partial n$ on $\partial\Omega_\tau \cap \partial D$. Integrating in $\tau$ on $I$ yields

$$\begin{aligned} \int_{\Lambda(R)} M_1(|\nabla u|^2) d\sigma &\leq CR^{-1} \int_{\Omega_{1/4} \setminus \Omega_{1/2}} |\nabla u|^2 dX \\ &\leq CR^{-3} \int_{C_1 R \leq |x| \leq C_2 R} u^2 dX \leq CR^{1-n-2\nu}, \end{aligned}$$

where the next to the last inequality follows from the inequality of Caciopolli for solutions of $Lu = 0$ (see [12], for example). Putting this together with (2.9), we see that

$$\int_{\Lambda(R)} M(\nabla u)^2 d\sigma \leq CR^{1-n-2\nu},$$

which easily yields the lemma.

We shall next study the boundary value properties of harmonic functions in $D$, with $M(\nabla u) \in L^1(\Lambda)$.

**Lemma 2.10.** *Suppose that $u$ is harmonic in $D$, and $M(\nabla u) \in L^1(\Lambda)$. Then, $\nabla u$ has non-tangential limits a.e. on $\Lambda$, and if $\partial u / \partial n = \langle \nabla u(Q), n(Q) \rangle$, then $\partial u / \partial n \in H^1_{at}(\Lambda)$, with*

$$\left\| \frac{\partial u}{\partial n} \right\|_{H^1_{at}(\Lambda)} \leq C \|M(\nabla u)\|_{L^1(\Lambda)}. \tag{2.11}$$

*Proof.* We first remark that by the extension theorem of Varoupoulos ([24]), given $g$ continuous and with compact support in $\Lambda$, there is a continuous function $G$ in $\bar{D}$, which agrees with $g$ on $\Lambda$, which has bounded support, and such that $|\nabla G|dX$ is a Carleson measure, with Carleson norm bounded by $C(m)\|g\|_{\mathrm{BMO}(\Lambda)}$, i.e., for all $Q \in \Lambda$, and all $r \geq 0$, we have

$$\int_{\{X \in D : |X-Q|<r\}} |\nabla G| dX \leq C(m) r^{n-1} \|g\|_{\mathrm{BMO}(\Lambda)},$$

where $\|g\|_{\mathrm{BMO}(\Lambda)}$ is the smallest constant $\gamma$, such that for all surface balls $B$ on $\Lambda$ there is a constant $\beta(B)$ such that

$$\int_B |g - \beta(B)| \leq \gamma \sigma(B).$$

The existence of the non-tangential boundary values of $\nabla u$ follows from [4]. For $\tau > 0$ let $u_\tau(X) = u(X + (0, \tau))$. Then,

$$\begin{aligned} \left| \int_\Lambda g \frac{\partial u}{\partial n} d\sigma \right| &= \lim_{\tau \to 0} \left| \int_\Lambda g \frac{\partial u_\tau}{\partial n} d\sigma \right| \\ &= \lim_{\tau \to 0} \left| \int_D \langle \nabla G, \nabla u_\tau \rangle dX \right| \leq C \|g\|_{\mathrm{BMO}(\Lambda)} \int_\Lambda M(\nabla u) d\sigma, \end{aligned}$$

by the basic property of Carleson measures (see [11]). Recall now that VMO($\Lambda$) is the closure in BMO ($\Lambda$) of the space of continuous functions with compact support, and that the dual space of VMO ($\Lambda$) is $H^1_{at}(\Lambda)$ (see [3]). This concludes the proof of the lemma.

We are now in position to solve the Neumann problem on a Lipschitz graph, with data in $H^1_{at}(\Lambda)$.

**Theorem 2.12.** *Let $f \in H^1_{at}(\Lambda)$. Then there exists a harmonic function $u$ in $D$ with $M(\nabla u) \in L^1(\Lambda)$, and $\partial u / \partial n = f$ non-tangentially a.e. on $\Lambda$. The function $u$ is unique modulo constants. Furthermore, there are constants $C_1 = C_1(m), C_2 = C_2(m), C_3 = C_3(m)$ such that*

$$C_1 \|f\|_{H^1_{at}(\Lambda)} \leq \|M(\nabla u)\|_{L^1(\Lambda)} \leq C_2 \|f\|_{H^1_{at}(\Lambda)},$$

*and*

$$\left\| \frac{\partial u}{\partial \tilde{T}_j} \right\|_{H^1_{at}(\Lambda)} \leq C_3 \|f\|_{H^1_{at}(\Lambda)}.$$

*Proof.* The exisatence of $u$ follows directly from Lemma 2.7 (a). In order to show uniqueness, let us assume that $\omega$ is harmonic in $D, M(\nabla \omega) \in L^1(\Lambda)$, and $\partial \omega / \partial n = 0$ non-tangentialy a.e. on $\Lambda$. We want to conclude that $\omega$ is a constant. From the sub-mean value property of $|\nabla \omega|$ it follows that

$$|\nabla \omega(X)| \leq C \{\text{dist } (X, \Lambda)\}^{1-n}.$$

Therefore, adding a suitable constant to $\omega$, we have

$$|\omega(X)| \leq C\{\text{dist }(X,\Lambda)\}^{2-n}.$$

For $\tau > 0$, let $\omega_\tau(X) = \omega(X + (0,\tau))$. By the estimates above, Sobolev's inequality and the assumption on $M(\nabla\omega)$, we have that

$$\int_\Lambda |\omega_\tau|^{(n-1)/(n-2)} d\sigma \leq C,$$

with $C$ independent of $\tau$. Let now $L$ be the divergence form operator used in the proof of Lemma 2.7, and let $G(X,Y)$ be its fundamental solution in $\mathbf{R}^n$. Fix $X \in D$, $|X| < R$, and let $\psi \in C_0^\infty(\mathbf{R}^n)$, be identically 1 for $|X| < R$, and 0 for $|X| > 2R$. We can also assume that $R|\nabla\psi| + R^2|\Delta\psi| \leq C$, where $C$ is independent of $R$. Let $G(Y) = G(X,Y) + G(X^*,Y)$, where $X^*$ is the reflection of $X$. Then,

$$\begin{aligned}
\omega_\tau(X) &= \int_\Lambda G \cdot \psi \frac{\partial \omega_\tau}{\partial n} d\sigma + \int_\Lambda G\omega_\tau \frac{\partial\psi}{\partial n} d\sigma \\
&\quad + \int_D G \cdot \{2\langle \nabla\psi, \nabla\omega_\tau\rangle + \omega_\tau \Delta\psi\} dy \\
&= I + II + III.
\end{aligned}$$

Set $F(R) = \{X : R < |X| < 2R\}$. Then

$$|II| \leq CR^{1-n} \int_{\Lambda\cap F(R)} |\omega_\tau| d\sigma \leq C\big(R^{1-n} \int_{\Lambda\cap F(R)} |\omega_\tau|^{(n-1)/(n-2)} d\sigma\big)^{(n-2)/(n-1)} \to 0$$

as $R \to \infty$, and

$$\begin{aligned}
|III| &\leq C \int_{D\cap F(R)} R^{1-n} |\nabla\omega_\tau| dY + C \int_{D\cap F(R)} R^{-n}|\omega_\tau| \\
&\leq CR^{2-n} \int_\Lambda M(\nabla\omega) d\sigma + C\big(R^{-n} \int_{D\cap F(R)} |\omega_\tau|^{(n-1)/(n-2)} dY\big)^{(n-2)/(n-1)} \\
&\to 0 \text{ as } R \to \infty.
\end{aligned}$$

Hence

$$\omega_\tau(X) = \int_\Lambda G \cdot \frac{\partial\omega_\tau}{\partial n} d\sigma.$$

Since $G \in L^\infty(\Lambda)$, the dominated convergence theorem shows that $\omega(X) = \lim_{\tau\to 0} \omega_\tau(X) = 0$. The estimate $\|M(\nabla u)\|_{L^1(\Lambda)} \leq C_2\|f\|_{H^1_{at}(\Lambda)}$ follows by construction, while the estimate $C_1\|f\|_{H^1_{at}(\Lambda)} \leq \|M(\nabla u)\|_{L^1(\Lambda)}$ follows by Lemma 2.10. Finally, it is enough to establish the estimate $\|\partial u/\partial \tilde{T}_j\|_{H^1_{at}(\Lambda)} \leq C_3\|f\|_{H^1_{at}(\Lambda)}$ when $f$ is an atom. One first shows that $\int(\partial u/\partial \tilde{T}_j) d\sigma = 0$. We see that this follows by considering the functions $u_\tau, \tau > 0$, and then passing to the limit. This fact, togther with (b) of Lemma 2.7 shows that $\partial u/\partial \tilde{T}_j$ is a molecule and the estimate follows by the general theory of [3].

We shall now treat the Neumann problem with $L^p$ data.

**Theorem 2.13.** *There exists a positive number $\varepsilon = \varepsilon(n,m)$ such that for all $f \in L^p(\Lambda), 1 < p < 2+\varepsilon$ there is a harmonic function $u$ in $D$ with $\|M(\nabla u)\|_{L^p(\Lambda)} \le C\|f\|_{L^p(\Lambda)}$ and $\partial u/\partial n = f$ non-tangentially a.e. on $\Lambda$. Furthermore, $u$ is unique modulo constants.*

*Proof.* We first remark that uniqueness for $1 < p < (n-1)$ follows by the same argument as in the uniqueness part of Theorem 2.11. The case $n = 3, p = 2$ of uniqueness is proved in Theorem 2.5. The case $n = 3, 2 < p < 2+\varepsilon$ of uniqueness will be treated later on. Next we note that existence, in the range $1 < p < 2$ follows by interpolation. We shall now treat existence in the case $p > 2$. We remark that this in fact follows from an abstract argument of A. P. Calderón ([1.B]). We present here an alternative proof, which also yields uniqueness. Let $f \in L^\infty(\Lambda)$ have compact support and let $u$ be the $L^2$ Neumann solution with data $f$, given by Theorem 2.5. Let $H = \{(x,y) \in \mathbf{R}^n, y > 0\}$ and $\phi : H \to D$ be given by $\phi(x,y) = (x, y+\varphi(x))$ where $\phi$ is a bi-Lipschitzian mapping between $H$ and $D$. Put $v = \nabla u \circ \phi$, and for $x \in \mathbf{R}^{n-1}$, set $m(x) = \sup_{\gamma(x)} |v|$, where $\gamma(x) = \{(x',y) \in H : y > |x'-x|\}$. Similarly, set $\gamma^*(x) = \{(x',y) \in H : \alpha y > |x'-x|\}$, and $m^*(x) = \sup_{\gamma^*(x)} |v|$, where $\alpha \in (0,1)$ is a number to be chosen later. Our aim is to show that there exists $\varepsilon = \varepsilon(n,m) > 0$ such that, if $0 < \delta \le \varepsilon$, then

$$\int_{\mathbf{R}^{n-1}} m^{2+\delta} dx \le C \int_\Lambda f^{2+\delta} d\sigma. \tag{2.14}$$

This clearly yields the desired existence results, in the range $2 < p \le 2+\varepsilon$. In order to establish (2.14), we need to introduce a bit more notation. Let $g(x) = f \circ \phi(x,0)$, $x \in \mathbf{R}^{n-1}$, and $h(x) = \sup_I (1/|I| \int_I g^2 dx)^{1/2}$, where the sup is taken over all cubes $I$ in $\mathbf{R}^{n-1}$, that contain $x$. Finally, for $\lambda > 0$ let $E_\lambda = \{x \in \mathbf{R}^{n-1} : m^*(x) > \lambda\}$. We will show that, for $\alpha$ sufficiently small,

$$\int_{\{m^* > \lambda;\, h \le \lambda\}} \le C\lambda^2 |E_\lambda| + C\alpha \int_{\{m^* > \lambda\}} m^2 dx. \tag{2.15}$$

Let us assume (2.15) for the time being, and use it to establish (2.14). From (2.15) it immediately follows that, if $\alpha$ is chosen sufficeintly small,

$$\int_{E_\lambda} m^2 dx \le C\lambda^2 |E_\lambda| + C \int_{\{h > \lambda\} \cap E_\lambda} m^2 dx.$$

Let now $\delta$ and $N$ be positive numbers. Then,

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} [\min\{m,N\}]^{2+\delta} dx &= \delta \int_0^N \lambda^{\delta-1} \Big( \int_{\{m > \lambda\}} m^2 dx \Big) d\lambda \\ &\le C\delta \int_0^N \lambda^{\delta+1} |E_\lambda| d\lambda + C\delta \int_0^N \lambda^{\delta-1} \Big( \int_{\{h > \lambda\}} m^2 dx \Big) d\lambda.\end{aligned}$$

By a classical argument (see [11]), $|E_\lambda| \le C_\alpha |\{m > \lambda\}|$, and so

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} [\min\{m,N\}]^{2+\delta} dx &\le C\delta/(2+\delta) \int_{\mathbf{R}^{n-1}} [\min(m,N)]^{2+\delta} dx \\ &\quad + C \int_{\mathbf{R}^{n-1}} \min(h,N)^\delta m^2 dx.\end{aligned}$$

Thus, if we choose $\delta$ small enough, we have

$$\int_{\mathbf{R}^{n-1}} [\min\{m, N\}]^{2+\delta} dx \leq C \int_{\mathbf{R}^{n-1}} h^{\delta} m^2 dx.$$

The boundedness of $f$ implies that $h$ is also bounded, and hence the right-hand side is finite. By monotone convergence, we see that

$$\int_{\mathbf{R}^{n-1}} m^{2+\delta} dx \leq C \int_{\mathbf{R}^{n-1}} h^{\delta} m^2 dx < +\infty;$$

(2.14) now follows from Hölder's inequality and the fact that $\int_{\mathbf{R}^{n-1}} h^{2+\delta} \leq C \int_{\Lambda} f^{2+\delta} d\sigma$. It remains to establish (2.15). Let $\{I_k\}$ be a Whitney decomposition on $E_\lambda$, such that the cubes $3I_k \subset E_\lambda$, and $\{3I_k\}$ has bounded overlap (see [25]). Here $3I_k$ denotes the cube with the same center as $I_k$, and sides 3 times those of $I_k$. We are only interested in Whitney cubes $I_k$ such that $I_k \cap \{h \leq \lambda\} \neq \emptyset$. For $2 < \tau < 3$, let $I_{k,\tau} = \{(x,y) : x \in \tau I_k, 0 < y < \tau l(I_k)\}$, where $l(I_k)$ denotes the side length of $I_k$. For a set $F \subset H$, we let $\tilde{F} = \phi(F)$. Clearly $\tilde{I}_{k,\tau}$ is a Lipschitz domain, with Lipschitz constants bounded independently of $k, \tau$. Since $I_k$ is a Whitney cube, there exists a point $x_k \in \mathbf{R}^{n-1} \setminus E_\lambda$ with dist $(x_k, I_k) \leq C_n l(I_k)$. Put $A_{k,\tau} = \partial I_{k,\tau} \cap \gamma^*(x_k)$, $B_{k,\tau} = (\partial I_{k,\tau} \cap H) \setminus A_{k,\tau}$. Since $\tau \in (2,3)$, the height of $B_{k,\tau}$ is bounded by $C\alpha l(I_k)$, i.e., $\sup\{y : (x,y) \in B_{k,\tau}\} \leq C\alpha l(I_k)$. Also, $|v| \leq \lambda$ on $A_{k,\tau}$. Since $I_k \cap \{h \leq \lambda\} \neq \emptyset$, we have

$$\int_{\widetilde{3I_k}} f^2 d\sigma \leq c \int_{3I_k} g^2 dx \leq C\lambda^2 |I_k|.$$

From the $L^2$ Neumann theory for bounded Lipschitz domains, applied to $\tilde{I}_{k,\tau}$ ([13]), we find that

$$\begin{aligned} \int_{I_k} m_1^2 dx &\leq C \int_{\widetilde{\partial I_{k,\tau}}} (\frac{\partial u}{\partial n})^2 d\sigma \\ &\leq C \int_{\widetilde{B_{k,\tau}}} |\nabla u|^2 d\sigma + C\lambda^2 |I_k|, \end{aligned}$$

where $m_1(x) = \sup\{|v(x', y)| : |x - x'| < y < \theta l(I_k)\}$, and $\theta > 0$ is chosen so small that $\{X \in H : |X - (x,0)| < \theta l(I_k)\}$ is contained in $I_{k,\tau}$, for all $x \in I_k, \tau \in (2,3)$.

If $\alpha$ is chosen small enough, then, for all $x \in I_k$, we have that

$$\gamma(X) \setminus \{(x', y) : |x - x'| < y < \theta l(I_k)\} \subset \gamma^*(x_k),$$

and so, for all $\tau \in (2,3)$, we have

$$\int_{I_k} m^2 dx \leq c \int_{\widetilde{B_{k,\tau}}} |\nabla u|^2 d\sigma + C\lambda^2 |I_k|.$$

Integration in $\tau$ from 2 to 3 gives

$$\int_{I_k} m^2 dx \le C\alpha \int_{3I_k} m^2 dx + C\lambda^2 |I_k|,$$

which gives (2.14) by additon of $k$.

For the uniqueness, in the case $n = 3, 2 < p < 2 + \varepsilon$, note that, keeping the notation we used above, if $M(\nabla u) \in L^p, 2 < p < 2 + \varepsilon$, and $\partial u/\partial n = 0$ non-tangentially a.e. on $\Lambda$, the argument used before shows that

$$\int_{E_\lambda} m^2 dx \le C\lambda^2 |E_\lambda|.$$

But then,

$$\begin{aligned}\int m^{2+\delta} dx &\le \int_0^\infty \delta\lambda^{\delta-1} \Big(\int_{E_y} m^2 dx\Big) d\lambda \\ &\le C \int_0^\infty \delta\lambda^{\delta-1} |\{m^* > \lambda\}| d\lambda \le C\delta(2+\delta)^{-1} \int m^{2+\delta} dx,\end{aligned}$$

which shows that $m$ is $\equiv 0$ if $\delta$ is small enough.

# 3. Regularity properties for the Dirichlet problem on graphs

We continue treating domains of the form $D = \{(x, y) \in \mathbf{R}^n : y > \varphi(x),\ x \in \mathbf{R}^{n-1}\}$, where $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ is Lipschitz continuous, i.e., $|\varphi(x) - \varphi(x')| \le m|x - x'|$.

We will say that $f \in L_1^p(\Lambda)$, $1 < p < \infty$, if $g(x) = f(x, \varphi(x))$ has a gradient in $L^p(\mathbf{R}^{n-1})$. It is easy to check that this is equivalent to the fact that if $F$ is any extension of $f$ to $\mathbf{R}^n$, $|\nabla_T F|$ (defined as in the remarks following (2.2)) belongs to $L^p(\lambda)$. It is also easy to see that this is equivalent to the fact that, for any extension $F$ of $f$, $\partial F/\partial \tilde{T}_j$, $j = 1, \dots, n-1$, belong to $L^p(\Lambda)$, where $\tilde{T}_1, \dots, \tilde{T}_{n-1}$ are the vector fields introduced in Lemma 2.6. Moreover, $\partial F/\partial \tilde{T}_j$ is independent of the particular choice of the extension $F$, and depends only on $f$. We put $\|f\|_{L_1^p(\Lambda)} = \|\nabla_x g\|_{L^p(\mathbf{R}^{n-1})}$. Clearly, $L_1^p(\Lambda)$ is a Banach space modulo constants.

We start out by studying the properties of the single layer potential on the $L_1^2(\Lambda)$ spaces. The following lemma is the graph version of results of $G$. Verchota ([25]).

**Lemma 3.1.** *The single layer potential $S$ maps $L^2(\Lambda)$ onto $L_1^2(\Lambda)$ boundedly, and has a bounded inverse.*

*Proof.* The boundedness follows by the definition of $L^2_1(\Lambda)$, and the theorem of Coifman, McIntosh and Meyer ([2]). From (2.3) and (2.4), it follows that

$$\|\nabla_T S(f)\|_{L^2(\Lambda)} \geq C_1 \left\| \frac{\partial}{\partial n} S(f) \right\|_{L^2(\Lambda)} \geq C\|f\|_{L^2(\Lambda)}. \tag{3.2}$$

The lemma now follows as in the $L^2$-Neumann case.

**Theorem 3.3.** *For every $f \in L^2_1(\Lambda)$ there is a harmonic function $u$ in $D$ with $M(\nabla u) \in L^2(\Lambda)$, and such that $\partial u/\partial \tilde{T}_j = \partial f/\partial \tilde{T}_j$ non-tangentially a.e. on $\Lambda, 1 \leq j \leq n-1$. Furthermore, $u$ is unique modulo constants, and*

$$\|M(\nabla u)\|_{L^2(\Lambda)} \leq c\|f\|_{L^2_1(\Lambda)},$$

*where $C = C(n,m)$.*

*Proof.* The existence follows from Lemma 3.1. To show uniqueness, it is enough to show that if $u$ is harmonic in $D, M(\nabla u) \in L^2(\Lambda)$, and $\partial u/\partial \tilde{T}_j = 0$, $j = 1, \ldots, n-1$ non-tantentialy a.e. on $\Lambda$, then $u$ is a constant. By our assumption on $u$, $|\nabla_T u| = 0$ a.e. on $\Lambda$. By the uniqueness in the Neumann problem (Theorem 2.5),

$$u(X) = C + \int_\Lambda g(Q)\{|X-Q|^{2-n} - |X_0 - Q|^{2-n}\} d\sigma(Q),$$

where $X_0 \in \mathbf{R}^n \setminus \bar{D}$, and $g \in L^2(\Lambda)$. By (3.2), $g \equiv 0$, and so $u$ is a constant.

A vector-valued function $A : \Lambda \to \mathbf{R}^N$ is a vector-valued atom if $A$ is supported on a surface ball $B = \{P \in \Lambda : |P-Q| < r\}$ for some $Q \in \Lambda$, and $r > 0, \|A\|_{L^2(\Lambda)} \leq \{\sigma(B)\}^{-1/2}$ and $\int_\Lambda A d\sigma = 0$. We say that $f \in H^1_{1,at}(\Lambda)$ if there are functions $f_j \in L^2_1(\Lambda)$ with

$$\left( \frac{\partial}{\partial \tilde{T}_1} f_j, \frac{\partial}{\partial \tilde{T}_2} f_j, \ldots, \frac{\partial}{\partial \tilde{T}_{n-1}} f_j \right)$$

being vector-valued atoms and

$$\frac{\partial}{\partial \tilde{T}_k} f = \sum_{j=1}^{\infty} \lambda_j \frac{\partial}{\partial \tilde{T}_k} f_j, \quad k = 1, \ldots, n-1, \ \sum |\lambda_j| < \infty. \tag{3.3}$$

We also set $\|f\|_{H^1_{1,at}(\Lambda)} = \inf \sum |\lambda_j|$, where the $\lambda_j$'s are as in (3.4). Note that $H^1_{1,at}(\Lambda)$ is a Banach space modulo constants.

**Lemma 3.5.** *The single layer potential $S$ maps $H^1_{at}(\Lambda)$ into $H^1_{1,at}(\Lambda)$ boundedly.*

*Proof.* The proof is standard and will be omitted.

**Theorem 3.6.** *Given $f \in H^1_{1,at}(\Lambda)$ there is a harmonic function $u$ in $D$ with $M(\nabla u)$ in $L^1(\Lambda)$, and $\partial u/\partial \tilde{T}_j = \partial f/\partial \tilde{T}_j$, $j = 1, \ldots, n-1$ non-tangentially a.e. on $\Lambda$. Moreover $u$ is unique modulo constants, and*

$$\|M(\nabla u)\|_{L^1(\Lambda)} \leq C\|f\|_{H^1_{1,at}(\Lambda)}.$$

*Proof.* For the existence part of the theorem, it is enough to assume $\varphi(0) = 0$, and to show that if $((\partial f/\partial \tilde{T}_1), \ldots, (\partial f/\partial \tilde{T}_{n-1}))$ is a vector-valued atom supported in $\{Q \in \Lambda : |Q| < 1\}$ and $u$ is the $L^2_1$ solution of the Dirichlet problem with data $f$, given in Theorem 3.3, then

$$\|M(\nabla u)\|_{L^1(\Lambda)} \leq C$$

where $C$ depends only on the Lipschitz constant of $\Lambda$. By adding a suitable constant to $f$, we may assume that $f$ has support in $B_1 = \{Q \in \Lambda, |Q| < R_0\}$ where $R_0$ depends only on the Lipschitz constant of $\Lambda$. Furthermore by Sobolev's inequality, $\|f\|_{L^2(\Lambda)} \leq C$. By the $L^2$ theory for the Dirichlet problem (see [5]) $|u(X)| \leq C = C(m)$ for $X \in D, |X| > 2R_0$, and $u(X)$ takes the boundary value zero continuously on $\Lambda \setminus B_1$. Let $\omega(x) = 0$ for $\mathbf{R}^n \setminus D$, and $\omega(X) = |u(X)|$ for $X \in D$, so that $\omega$ is subharmonic in $\mathbf{R}^n \setminus \bar{B}_1$. By the Phragmen-Lindelöf theory (see [17]) we have $|\omega(X)| \leq C|X|^{2-n-\alpha}$, where $C$ and $\alpha$ only depend on $m$, and $|x| > 2R_0$. Arguing as in the corresponding Neumann problem, we obtain the existence and the estiamte in Theorem 3.7. We remark that instead of the Phragmen-Lindelöf theory we could have used an odd reflection of $u$ to extend $u$ as a solution of $Lu = 0$ in $\mathbf{R}^n \setminus B_1$, and use the Serrin-Weinberger asymptotic expansion just as in the case of the Neumann problem. To show uniqueness, we assume that $u$ is harmonic in $D, M(\nabla u) \in L^1(\Lambda)$, and $\partial u/\partial \tilde{T}_j = 0, j = 1, \ldots, n-1$, non-tngentially a.e. on $\Lambda$. We must conclude that $u$ is a constant in $D$. As in the corresponding uniqueness theorem for the Neumann problem, we have $|\nabla u(X)| \leq C\{\text{dist }(X, \Lambda)\}^{1-n}$, and after we add a suitable constant $|u(X)| \leq C\{\text{dist }(x, \Lambda)\}^{2-n}$. Thus, by Sobolev's inequality, $\int_\Lambda |u_\tau|^{(n-1)/(n-2)} d\sigma \leq c$, and so $u = 0$ a.e. on $\Lambda$. By the uniqueness in the Neumann problem, it is enough to show that $f = 0$ a.e., where $f = \partial u/\partial n$. Let $b$ be a Lipschitz function on $\Lambda$, with compact support. Let $\omega$ be the harmonic extension of $b$ to $D$. By the Phragmen-Lindelöf principle, $|\omega(X)| \leq C|X|^{2-n-\delta}$ for $X \in D$, $X$ large, where $C > 0, \delta > 0$. We will now show that for $s, t > 0$ we have

$$\int_\Lambda \omega_s \frac{\partial u_t}{\partial n} d\sigma = \int_\lambda u_t \frac{\partial \omega_s}{\partial n} d\sigma.$$

In fact, let $R > 0$ be large, and let

$$\Omega_R = \{(x, y) : |x| < R, \varphi(x) < y < \varphi(x) + R\}.$$

We then have $\partial\Omega_R = \Lambda_R \cup S_R \cup T_R$, where

$$\begin{aligned} \Lambda_R &= \{(x, \varphi(x)) : |x| < R\}, \\ S_R &= \{(x, y) : |x| = R, \varphi(x) < y < \varphi(x) + R\}, \end{aligned}$$

and

$$T_R = \{(x, y) : |x| < R, \quad y = \varphi(x) + R\}.$$

Applying Green's theorem in $\Omega_R$, we see that

$$\int_{\partial\Omega_R} \omega_s \frac{\partial u_t}{\partial n} d\sigma = \int_{\partial\Omega_R} u_t \frac{\partial \omega_s}{\partial n} d\sigma.$$

Since $\omega_s \in L^\infty(\bar{D})$, and $N(\nabla u) \in L^1(\Lambda)$,

$$\int_{\Lambda_R} \omega_s (\partial u_t / \partial n) d\sigma \to \int_\Lambda \omega_s (\partial u_t / \partial n) d\sigma.$$

Also,

$$\left| \int_{T_R} \omega_s \frac{\partial u_t}{\partial n} d\sigma \right| \le C R^{2-n-\delta} R^{1-n} R^{n-1} \xrightarrow[R\to\infty]{} 0,$$

while

$$\left| \int_{S_R} \omega_s \frac{\partial u_t}{\partial n} d\sigma \right| \le R^{2-n-\delta} \int_{\Lambda \setminus \Lambda_R} M(\nabla u) d\sigma \xrightarrow[R\to\infty]{} 0.$$

Similarly, we know that, for $s$ fixed $|\nabla \omega_s(X)| \le C|X|^{2-n-\delta}$, and it is locally bounded in $\bar{D}$. Thus, since $u_t \in L^{(n-1)/(n-2)}(\Lambda)$, $u_t(\partial\omega_s/\partial n) \in L^1(\Lambda)$, and so $\int_{\Lambda_R} u_t(\partial\omega_s/\partial n) d\sigma \to \int_\Lambda u_t(\partial\omega_s/\partial n) d\sigma$. Also,

$$\left| \int_{T_R} u_t \frac{\partial \omega_s}{\partial n} d\sigma \right| \le C R^{2-n} \cdot R^{1-n-\delta} \cdot R^{n-1} \to 0,$$

while

$$\begin{aligned} \frac{1}{R} \int_R^{2R} \left| \int_{S_R} u_t \frac{\partial \omega_s}{\partial n} d\sigma \right| &\le \frac{C}{R} \int_{\Omega_{2R} \setminus \Omega_R} |u_t| |\nabla \omega_s| dX \\ &\le C \frac{R^{2-n-\delta}}{R} \int_{\Omega_{2R} \setminus \Omega_R} |u_t| dX \\ &\le C \frac{R^{2-n-\delta}}{R} \left( \int_{\Omega_{2R}} |u_t|^{(n-1)/(n-2)} dx \right)^{(n-2)/(n-1)} \cdot R^{n/n-1} \\ &\le C \frac{R^{2-n-\delta}}{R} \cdot R^{(n-2)/(n-1)} R^{n/n-1} = C R^{3-n-\delta} \xrightarrow[R\to\infty]{} 0. \end{aligned}$$

Thus, by a choice of an appropriate sequence of $R_j$'s tending to $\infty$, the claim follows. Letting $t \downarrow 0$ we see that $\int_\Lambda \omega_s f d\sigma = 0$, and then letting $s \downarrow 0$ we have $\int_\Lambda b f d\sigma = 0$ as desired.

**Corollary 3.7.** *The single layer potential $S$ is a bounded operator from $H^1_{at}(\Lambda)$ onto $H^1_{1,at}(\Lambda)$. It has a bounded inverse, whose norm depends only on the Lipschitz constant of $\Lambda$.*

We now turn to regularity results for the solution of the Dirichlet problem when the data are in $L^p_1(\Lambda)$. This is a new proof of results of Verchota ([25]).

**Theorem 3.8.** *There exists a positive number $\varepsilon = \varepsilon(n,m)$ such that for all $f \in L^p_1(\Lambda)$, $1 < p < 2+\varepsilon$, there is a harmonic function $u$ in $D$ with $M(\nabla u)$ in $L^p(\Lambda)$, and $\partial u/\partial \tilde{T}_j = \partial f/\partial \tilde{T}_j$, $j = 1, \dots, n-1$ non-tangentially a.e. on $\Lambda$. Moreover, $u$ is unique modulo constants and*

$$\|M(\nabla u)\|_{L^p(\Lambda)} \leq C\|f\|_{L^p_1(\Lambda)},$$

*were $C$ depends only on $p, n$ and $m$.*

*Proof.* The case $2 < p < 2+\varepsilon$ follows in the same way as in the Neumann case. Since $S$ is invertible from $H^1_{at}(\Lambda)$ onto $H^1_{1,at}(\Lambda)$ and from $L^2(\Lambda)$ onto $L^2_1(\Lambda)$, it follows by interpolation that $S$ is invertible from $L^p(\Lambda)$ onto $L^p_1(\Lambda)$, $1 < p < 2$, which gives existence for $1 < p < 2$. Uniqueness follows in the same way as in the $H^1_{1,at}(\Lambda)$ case.

We conclude this section by giving the invertibility properties of layer potentials.

**Theorem 3.9.** *There exists a number $\varepsilon = \varepsilon(n,m) > 0$ such that $S$ maps $L^p(\Lambda)$ boundedly onto $L^p_1(\Lambda)$, with a bounded inverse, for $1 < p < 2+\varepsilon$. Furthermore $S$ is a bounded invertible mapping from $H^1_{at}(\Lambda)$ onto $H^1_{1,at}(\Lambda)$. The operators $\partial S/\partial n$ and $\mathcal{D}$ are bounded and invertible on $L^p(\Lambda)$ for $1 < p < 2+\varepsilon$ and $2-\varepsilon < p < \infty$ respectively. Furthermore $\partial S/\partial n$ is a bounded invertible mapping on $H^1_{at}(\Lambda)$, and $\mathcal{D}$ is a bounded invertible mapping on BMO $(\Lambda)$.*

# 4. Bounded Lipschitz domains

In this seciton we will sketch the localization arguments which are necessary to extend the results in the last two sections to the case of general bounded Lipschitz domains in $\mathbf{R}^n$. The $L^2$ theory in the Neumann problem and the $L^2$-regularity in the Dirichlet problem have been treated in [13] and [25]. The $L^p$ regularity in the Dirichlet problem has been treated in [25].

From now on we will assume that $D \subset \mathbf{R}^n$, $n \geq 3$, is a bounded Lipschitz domain such that $D^* = \mathbf{R}^n \setminus \bar{D}$ is connected. Atoms are defined as in the graph case, and the atomic

Hardy space $H^1_{at}(\partial D)$ is also defined as in the graph case. We say that $f \in L^p_1(\partial D)$ if $f \in L^p(\partial D, d\sigma)$ and for each coordinate chart $(Z, \varphi)$, there are $L^p(Z \cap \partial D)$ functions $g_1, \ldots, g_{n-1}$ so that

$$\int_{\mathbf{R}^{n-1}} h(x) g_j(x, \varphi(x)) dx = \int_{\mathbf{R}^{n-1}} \frac{\partial}{\partial x_j} h(x) f(x, \varphi(x)) dx$$

for all $h \in C_0^\infty(Z \cap \mathbf{R}^{n-1})$. It is easy to see that given $f \in L^p_1(\partial D)$, it is possible to define a unique vector $\nabla_T f \in \mathbf{R}^n$, at almost every $Q \in \partial D$ so that $\|\nabla_T f\|_{L^p(\partial D, d\sigma)}$ is equivalent to the sum over alla the coordinate cylinders in a given covering of $\partial D$ of the $L^p$ norms of the locally defined functions $g_j$ for $f$, occurring in the definition of $L^p_1(\partial D)$. The resulting vector field, $\nabla_T f$, will be called the tangential gradient of $f$. If $F$ is a function defined on $\mathbf{R}^n$, $\nabla_T F$ is orthogonal to the normal vector $n$, and $\nabla F = \nabla_T F + (\partial F/\partial n) \cdot n$. In local coordinates, $\nabla_T f$ may be realized as

$$\begin{aligned}&(g_1(x, \varphi(x)), g_2(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0)\\ &- \langle (g_1(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0), n_{(x,\varphi(x))} \rangle n_{(x,\varphi(x))};\end{aligned}$$

$L^p_1(\partial D)$ may be normed by $\|f\|_{L^p_1(\partial D)} = \|f\|_{L^p(\partial D)} + \|\nabla_T f\|_{L^p(\partial D)}$.

Before we proceed to define the space $H^1_{1,at}(\partial D)$, we will make a few remarks about it in the graph case. We say that $f$ is an $H^1_{1,at}(\Lambda) - L^2$ atom if $f$ is in $L^2_1(\Lambda)$, it is supported in a surface ball $B$, and $A = ((\partial/\partial \tilde{T}_1) f, \ldots, (\partial/\partial \tilde{T}_{n-1}) f)$ (which automatically verifies $\int_\Lambda A d\sigma = 0$) verifies $\|A\|_{L^2(\Lambda)} \leq \sigma(B)^{-1/2}$. We say that $f \in \tilde{H}^1_{1,at}(\lambda)$ if $f \in L^{(n-1)/(n-2)}(\Lambda)$, and there exist $H^1_{1,at}(\Lambda) - L^2$ atoms $f_j$ and numbers $\lambda_j$ with $\sum |\lambda_j| < +\infty$, such that $f = \sum_{j=1}^\infty \lambda_j f_j$, where the sum is taken in the sense of $L^{(n-1)/(n-2)}(\Lambda)$. Moreover, if $f \in H^1_{1,at}(\Lambda)$, there exists a constant $c$ such that $f - c \in \tilde{H}^1_{1,at}(\Lambda)$. Let $\phi : \mathbf{R}^{n-1} \to \Lambda$ be given by $\phi(x) = (x, \varphi(x))$. Then $f \in \tilde{H}^1_{1,at}(\mathbf{R}^{n-1})$ if and only if $g(x) = C_n \int_{\mathbf{R}^{n-1}} (h(y)/|x-y|^{n-2}) dy$, where $h \in H^1_{1,at}(\mathbf{R}^{n-1})$. In fact, such $g(x)$ clearly belong to $L^{(n-1)/(n-2)}(\mathbf{R}^{n-1})$, and Lemma 3.5 shows that they are in fact in $\tilde{H}^1_{1,at}(\mathbf{R}^{n-1})$. Conversely, if $g \in \tilde{H}^1_{1,at}(\mathbf{R}^{n-1})$, then $g(x) = C_n \int_{\mathbf{R}^{n-1}} (h(y)/|x-y|^{n-2}) dy$, where $h(y) = \sum_{j=1}^n R_j (\partial/\partial_{y_j}) g$, where $R_j$ are the classical Riesz transforms. Note that if we define $\tilde{H}^1_{1,at}(\mathbf{R}^{n-1})$ by using $H^1_{1,at}(\mathbf{R}^{n-1}) - L^p$ atoms, $1 < p \leq \infty$, we obtain the same characterization of $\tilde{H}^1_{1,at}(\mathbf{R}^{n-1})$, which shows that all these spaces coincide, and have comparable norms. The same fact of course remains true for $\tilde{H}^1_{1,at}(\Lambda)$ This allows one to show in a very simple fashion that if $\theta$ is a Lipschitz function with compact support in $\Lambda$, and $f \in \tilde{H}^1_{1,at}(\Lambda)$, then $\theta f$ also belongs to $\tilde{H}^1_{1,at}(\Lambda)$. Our final remark is that if $f \in \tilde{H}^1_{1,at}(\Lambda)$, and $u$ is the solution to the Dirichlet problem constructed in Theorem 3.7, then $u|_\Lambda = f$, in the sense of non-tangential convergence, $\int_\Lambda (|u_\tau|)^{(n-1)/(n-2)} d\sigma \leq C$, and $|u(X)| \leq C\{\text{dist } (X, \Lambda)\}^{2-n}$. Moreover, the uniqueness then follows without the addendum 'modulo constants'.

We are now ready to define $H^1_{1,at}(\partial D)$. We say that $f$ is an $H^1_{1,at}(\partial D) - L^2$ atom if $f$ is supported in a coordinate cylinder $(Z, \varphi)$, and if $\Lambda$ is the graph of $\varphi$, $f$ is an $H^1_{1,at}(\lambda) - L^2$ atom. The space $H^1_{1,at}(\partial D)$ is then defined as the absolutely convergent sums of $H^1_{1,at}(\partial D) -$

$L^2$ atoms, where the convergence of the sum takes place in the $L^{(n-1)/(n-2)}(\partial D)$ norms. It is a Banach space, and if we replace $L^2$ atoms by $L^p$ atoms, $1 < p \leq \infty$, we obtain the same space, with an equivalent norm. Also, if $\theta \in \text{Lip }(\partial D)$, and $f \in H^1_{1,at}(\partial D), \theta f \in H^1_{1,at}(\partial D)$, and, if $f \in H^1_{1,at}(\partial D)$, then $f \in L^{(n-1)/(n-2)}(\partial D)$. Also, $L^p_1(\partial D) \subset H^1_{1,at}(\partial D)$, for any $1 < p \leq \infty$.

The non-tangential regions $\Gamma_\alpha(Q), Q \in \partial D$, are defined as $\Gamma_\alpha(Q) = \{X \in D : |X - Q| < (1+\alpha)\text{dist }(X, \partial D)\}$, while the non-tangential maximal function $M(\omega)(Q) = \sup_{X \in \Gamma_1(Q)} |\omega(X)|$. Finally, we recall that a bounded Lipschitz domain $\Omega$ is called a starlike Lipschitz domain (with respect to the origin) if there exists $\varphi : S^{n-1} \to \mathbf{R}$, where $\varphi$ is strictly positive, and $|\varphi(\theta) - \varphi(\theta')| \leq m|\theta - \theta'|, \theta, \theta' \in S^{n-1}$ such that, in polar coordinates $(r, \theta), \Omega = \{(r, \theta) : 0 \leq r < \varphi(\theta)\}$.

Note that if $D$ is an arbitrary bounded Lipschitz domain, and $(Z, \varphi)$ is a coordinate chart, with $\|\nabla\varphi\|_\infty \leq m$, then, for appropriate $\delta > 0, a > 0, b > 0$ which depend only on $m$, the domain $D \cap U$ is a starlike Lipschitz domain with respect to $X_0 = (0, b\delta)$, where $U = \{(x, y) : |x| < \delta, |t| < a\delta\}$.

**Lemma 4.1.** *Let $\Omega$ be a starlike Lipschitz domain, and let $u$ be the $L^2$-solution of the Neumann problem with data an atom $a$, centred at $Q_0 \in \partial\Omega$. Then, there exists a constant $C$, which depends only on the Lipschitz constants of $D$ such that*

(a) $$\|M(\nabla u)\|_{L^1(\partial\Omega)} \leq C,$$

(b) $$\int_{\partial\Omega} M(\nabla u)^2 |Q - Q_0|^{n-1} d\sigma \leq C,$$

(c) $$\|u\|_{H^1_{1,at}(\partial\Omega)} \leq C,$$

*if we subtract from $u$ an appropriate constant.*

*Proof.* We may assume that the size of the support of $a$ is small. We may also assume that $\Omega \subset \{y < \varphi(x)\} = D$, where $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ is Lipschitz with norm depending only on the Lipschitz character of $\Omega$, that $\partial\Omega \cap \partial D \supset \{|X - Q_0| < r_0\} \cap \partial\Omega$, where $r_0$ depends only on the Lipschitz characater of $\partial\Omega$, that $Q_0$ is the origin and that supp $a \subset \{|X - Q_0| < r_0\}$.

Let $v$ be the solution of the Neuman problem in $D$, with data $a$, given by Lemma 2.7, and let $\omega$ be the $L^2$-solution of the Neumann problem in $\Omega$, with data $\partial\omega/\partial n = 0$ on $\partial\Omega \cap \partial D$, and $\partial\omega/\partial n = -\partial v/\partial n|_{\partial\Omega}$ on $\partial\Omega \setminus (\partial\Omega \cap \partial D)$. We clearly have $u = v + \omega$, and from this the lemma follows.

**Lemma 4.2.** *Let $\Omega$ be a bounded, starlike Lipschitz domain, and let $u$ be harmonic in $\Omega$, with $M(\nabla u) \in L^1(\partial\Omega)$ and either $\nabla_T u = 0$ or $\partial u/\partial n = 0$ non-tangentially a.e. on $\partial\Omega$. Then, $u$ is a constant.*

*Proof.* Assume first that $\partial u/\partial n = 0$ non-tangentially a.e. on $\partial\Omega$. We can show that $u$ is a constant using a variant of the uniqueness proof in Theorem 2.11, using a radial reflection across our starlike surface.

If $\nabla_T u = 0$, is constant on $\partial\Omega$, we have, if $b \in \mathrm{Lip}\,(\partial\Omega)$, and $\omega$ is its harmonic extension, that (with $u_r(x) = u(rx)$)

$$\int_{\partial\Omega} \omega_s \frac{\partial u_r}{\partial n} d\sigma = \int_{\partial\Omega} u_r \frac{\partial \omega_s}{\partial n} d\sigma.$$

If we let $r \to 1$, the right-hand side tends to 0, while the left-hand side tends to $\int_{\partial\Omega} b(\partial u/\partial n)$ $d\sigma = 0$, and so $\partial u/\partial n = 0$ a.e. on $\partial\Omega$. Therefore $u$ is constant by the previous result.

We are now in a position to give the solution of the Neumann problem with $H^1_{at}(\partial D)$ data, for a general bounded Lipschitz domain $D$.

**Theorem 4.3.** *Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain. If $u$ is harmonic in $D$, with $M(\nabla u) \in L^1(\partial D)$, then $\partial u/\partial n \in H^1_{at}(\partial D)$ and*

$$\left\| \frac{\partial u}{\partial n} \right\|_{H^1_{at}(\partial D)} \leq C \| M(\nabla u)\|_{L^1(\partial D)}. \tag{4.4}$$

*If $f \in H^1_{at}(\partial D)$, then there is a harmonic function $u$ with $M(\nabla u) \in L^1(\partial D)$ and $\partial u/\partial n = f$ non-tangentially a.e. on $\partial D$. Furthermore, $u$ is unique modulo constants, and*

$$\| M(\nabla u)\|_{L^1(\partial D)} \leq C \|f\|_{H^1_{at}(\partial D)}; \tag{4.5}$$

*$u$ can be chosen so that*

$$\|u\|_{H^1_{1,at}(\partial D)} \leq C \|f\|_{H^1_{at}(\partial D)}. \tag{4.6}$$

*Proof.* As in the proof of Lemma 2.10, the estimate (4.4) follows from Green's formula, the extension theorem of Varopoulos ([24]) and the fact that the dual of VMO $(\partial D)$ is $H^1_{at}(\partial D)$. (See [8] for the exact form of the Varopoulos extension theorem that is needed here.)

In the case when $D$ is a bounded starlike Lipschitz domain, the rest of the theorem follows from Lemma 4.1 and Lemma 4.2.

We now pass to the general case. We first establish uniqueness in the general case. Thus, $M(\nabla u) \in L^1(\partial D)$, and $\partial u/\partial n = 0$ a.e. on $\partial D$. We can cover a neighborhood of $\partial D$ in $D$, with finitely many bounded starlike Lipschitz domains $\Omega_i \subset D$, such that $M_{\Omega_i}(\nabla u)$, the non-tangential maximal function relative to the domain $\Omega_i$, is in $L^1(\partial\Omega_i)$. Thus, if $v_i = u|_{\Omega_i}$, we have $\partial v_i/\partial n \in H^1_{at}(\partial\Omega_i)$. If also $\partial\Omega_i \supset B(Q_i, 3r) \cap \partial D$, for some $r > 0, Q_i \in \partial D$, we can take the atoms in the atomic decomposition of $\partial v_i/\partial n$ to have supports that are so small that they are all contained in $\partial\Omega_i/B(Q_i, 2r)$ (since $\partial v_i/\partial n = 0$ on $B(Q_i, 3r) \cap \partial D$). It then follows from (b) in Lemma 4.1, and the uniqueness for starlike Lipschitz domains, that $M(\nabla u) \in L^2(B(Q_i, r) \cap \partial D)$. Since $\cup_i B(Q_i, r) \cap \partial D$ can be taken to be $\partial D$, it follows that $M(\nabla u) \in L^2(\partial D)$, and hence $u$ is a constant by the $L^2$-theory (see [13] or [25]).

To show (4.5), it is enough to show that if $a$ is an atom with support contained in a ball of radius $r$, with $r \leq r_0 = r_0(D)$, then $\|M(\nabla u)\|_{L^1(\partial D)} \leq C(D)$, where $u$ is the solution of the $L^2$-Neumann problem with data $a$. For $\theta \in (\frac{1}{4}, 10)$ let $D(\theta)$ be a domain of the form $\{(x, y) : \varphi(x) < y < \varphi(x) + \rho_1\theta, |x| < \rho_2\theta\}$, where $\varphi$ is a Lispchitz function. We can choose numbers $\rho_1, \rho_2$ and coordinate systems so that the domains $D(\theta)$ are starlike Lipschitz domains contained in $D$, for $\frac{1}{4} < \theta < 10$. The number $r_0$ is chosen in such a way that there are finitely many $D_\nu(\theta)$, $1 \leq \nu \leq N = N(D)$ such that $\cup\partial D_\nu(1/4) \cap \partial D = \partial D$, and such that, for any $\nu$ we have that either the support of $a$ is contained in $\partial D \cap \partial D_\nu(4)$ or supp $a \cap \partial D_\nu(3) = \emptyset$.

We first claim that for each compact set $K \subset D$, we have

$$\sup_K |\nabla u| \leq C = C(K, D). \tag{4.7}$$

To see this, pick $\eta \in L^\infty(D)$, supp $\eta \subset K$ and $\int \eta(X)dX = 0$. Letting $\omega(X) = C_n \int |X - Y|^{2-n}\eta(Y)dY$, we have that $\Delta\omega = \eta$, $\int_{\partial D}(\partial\omega/\partial n)d\sigma = 0$, and $\|\partial\omega/\partial n\|_{L^\infty(\partial D)}$
$\leq C(K, D)\|\eta\|_{L^1(K)}$.

Let $h$ solve the Neumann problem in $D$ with data $\partial\omega/\partial n$ and $\int_D h(x)dx = 0$. Then,

$$\int_D u\eta dX = \int_D u\Delta(\omega - h) = \int_{\partial D}(\omega - h)\frac{\partial u}{\partial n}.$$

If we now note that the normal derivative of $\omega - h$ is 0 on $\partial D$, and we use locally the graph reflection argument that we used in the proof of Lemma 2.7, it follows that $\|\omega - h\|_{L^\infty(\partial D)} \leq C(K, D)\|\eta\|_{L^1(K)}$ which yields (4.7).

Let $M^{\nu,\theta}$ be the non-tangential maximal operator associated to the domain $D_\nu(\theta)$. We can choose a suitable compact set $K \subset D$ so that, for all $\theta \in (1/4, 10)$ we have

$$\int_{\partial D} M(\nabla u)d\sigma \leq \sum_v \int_{\partial D_\nu(\theta)} |M^{\nu,\theta}(\nabla u)|d\sigma + C\sup_K |\nabla u|. \tag{4.8}$$

In order to apply (4.8), we shall first study the case when (supp $a) \cap D_\nu(3) = \emptyset$. From the

$L^2$-Neumann theory, it follows that for $1/4 < \theta < 3$ we have

$$\begin{aligned}\Big(\int_{\partial D_\nu(1/4)} M^{\nu,1/4}(\nabla u)d\sigma\Big)^2 &\le c\int_{\partial D_\nu(\theta)} M^{\nu,\theta}(\nabla u)^2 d\sigma\\ &\le C\int_{\partial D_\nu(\theta)\setminus\partial D}\Big(\frac{\partial u}{\partial n}\Big)^2 d\sigma \le\\ &\le C\int_{\partial D_\nu(\theta)\setminus\partial D}|\nabla u(X)|d\sigma.\end{aligned}$$

Integrating in $\theta$ from $1/2$ to $1$ now gives

$$\begin{aligned}\Big(\int_{\partial D_\nu(1/4)\cap\partial D} M^{\nu,1/4}(\nabla u)d\sigma\Big)^2 &\le C\int_{D_\nu(2)}|\nabla u(X)|^2 dX\\ &\le C\Big(\int_{D_\nu(3)}|\nabla u(X)|dX\Big)^2.\end{aligned}$$

The last inequality follows from the graph reflection and the reversed Hölder inequality for the gradient of the solution of a uniformly elliptic equation in divergence form (see [12]) together with the fact that one can lower the exponent on the right-hand side of such a reversed Hölder inequality. This last fact was proved by the present authors; see [10]. It is possible to use the graph reflection because supp $\partial u/\partial n \cap D_\nu(3) = \emptyset$. Hence, $\int_{D_\nu(1/4)} M^{\nu,1/4}(\nabla u)d\sigma \le C\int_{D_\nu(3)}|\nabla u(X)|dX$, and therefore, given $\varepsilon > 0$ there is a compact $K_\varepsilon \subset D$ such that

$$\int_{D_\nu(1/4)} M^{\nu,1/4}(\nabla u)d\sigma \le C\varepsilon\int_{\partial D} M(\nabla u)d\sigma + C(\varepsilon)\int_{K_\varepsilon}|\nabla u|. \tag{4.9}$$

If supp $a \subset D_\nu(4)$, we let $v$ solve the Neumann problem in $D_\nu(4)$, with data $a$ on $\partial D_\nu(4)\cap\partial D$, and $0$ elsewhere on $\partial D_\nu(4)$. Let $\omega = u - v$. Since $\partial\omega/\partial n = 0$ on $\partial D_\nu(4)\cap\partial D$ we have, from the argument leading to (4.9), that

$$\int_{\partial D_\nu(1/4)\cap\partial D} M^{\nu,1/4}(\nabla\omega)d\sigma \le C\int_{D_\nu(3)}|\nabla\omega|dX$$

and therefore

$$\begin{aligned}\int_{\partial D_\nu(1/4)\cap\partial D} M^{\nu,1/4}(\nabla u)d\sigma &\le C\varepsilon\int_{\partial D} M(\nabla u)d\sigma\\ &\quad + C(\varepsilon)\int_{K_\varepsilon}|\nabla u(X)|dX + C.\end{aligned} \tag{4.10}$$

Using now (4.8), (4.9) and (4.10), and the weak estimate (4.7), we see that

$$\int_{\partial D} M(\nabla u)d\sigma \le C\varepsilon\int_{\partial D} M(\nabla u)d\sigma + C(\varepsilon) + C,$$

and so, if we choose $\varepsilon$ small enough

$$\int_{\partial D} M(\nabla u) d\sigma \leq C = C(D),$$

which yields (4.5). Finally, note that because of (4.7), we can subtract a constant $C$ from $u$ so that $\sup_K |u - C| \leq C_K$ for all $K$ compact in $D$. Let $v = u - C$. We claim that $\|v\|_{H^1_{1,at}(\partial D_\nu(1))} \leq C$. In fact, we know that, because of the Poincaré inequality on $\partial D_\nu(1)$, $\int_{\partial D_\nu(1)} |v|^{(n-1)/(n-2)} d\sigma \leq C$. But, by Lemma 4.1, there exists a constant $C_\nu$ so that $\|v - C_\nu\|_{H^1_{1,at}(\partial D_\nu(1))} \leq C$. But then, $\int_{\partial D_\nu(1)} |v - C_\nu|^{(n-1)/(n-2)} d\sigma \leq C$, and thus $|C_\nu| \leq C$. Therefore $\|v\|_{H^1_{1,at}(\partial D_\nu(1))} \leq C$, and (4.6) follows for the case of atoms. The general case follows from this.

We shall next study the regularity in the Dirichlet problem with $H^1_{1,at(\partial D)}$ data.

**Lemma 4.11.** *Let $f$ be an $H^1_{1,at}(\partial D) - L^2$ atom. If $u$ solves the Dirichlet problem with boundary values $f$, then*

(a)
$$\int_{\partial D} M(\nabla u) d\sigma \leq C.$$

(b)
$$\Big(\int_{\partial D} M(\nabla u)^2 d\sigma\Big)\Big(\int_{\partial D} M(\nabla u)^2 |Q - Q_0|^{(\varepsilon+1)(n-1)} d\sigma\Big)^{1/\varepsilon} \leq C,$$

*where $Q_0$ is the center of the support of $f$.*

(c)
$$\int_{\partial D} M(\nabla u)^2 |Q - Q_0|^{n-1} d\sigma \leq C.$$

*Here $C$ and $\varepsilon > 0$ are independent of the $H^1_{1,at}(\partial D) - L^2$ atom $f$.*

*Proof.* If we perform a change of scale so that the support of $f$ is of size 1, we see that the arguments in the graph case (Theorem 3.7), yield the proof of Lemma 4.11.

**Theorem 4.12.** *Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain. If $u$ is harmonic in $D$, with $M(\nabla u) \in L^1(\partial D)$, then $u \in H^1_{1,at}(\partial D)$, and*

$$\|u\|_{H^1_{1,at}(\partial D)} \leq C \|M(\nabla u)\|_{L^1(\partial D)}. \tag{4.13}$$

*If $f \in H^1_{1,at}(\partial D)$, then there is a harmonic function $u$ with $M(\nabla u) \in L^1(\partial D)$ and $u = f$ non-tangentially a.e. on $\partial D$. Furthermore, $u$ is unique,*

$$\|M(\nabla u)\|_{L^1(\partial D)} \leq C \|f\|_{H^1_{1,at}(\partial D)}.$$

*Proof.* Uniqueness follows from the uniqueness in Theorem 4.3. Next note that existence in the range $1 < p < 2$ follows by interpolation between Theorem 4.3 and the $L^2$ results. Existence in the case $p > 2$ follows by a minor modification of the corresponding part of the proof of Theorem 2.12. In fact, the main difference is that in the bounded case there are two kinds of Whitney cubes $I_k$, the small ones and the big ones. The small ones are treated just as in 2.12, while the big ones are of diameter comparable to that of $D$, and hence $m_1$ is comparable to $m$ on them. The rest of the proof is identical, and is therefore omitted.

Our next theorem deals with regularity in the Dirichlet problem with $L_1^p(\partial D)$ data. It was first proved in [25].

**Theorem 4.14.** *Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain. There exists a positive number $\varepsilon = \varepsilon(D)$ such that for all $f \in L_1^p(\partial D)$, $1 < p < 2+\varepsilon$, there is a harmonic function $u$ in $D$, with $M(\nabla u)$ in $L^p(\partial D)$, and $u = f$ non-tangentially a.e. on $\partial D$. Moreover, $u$ is unique and*

$$\|M(\nabla u)\|_{L^p(\partial D)} \leq C\|f\|_{L_1^p(\partial D)}$$

*where $C$ depends only on $p$ and $D$.*

*Proof.* Uniqueness follows from Theorem 4.12. Existence follows just as in Theorem 4.13 in the range $2 < p < 2 + \varepsilon$, while the case $1 < p < 2$ follows by interpolation.

We will now study the Neumann problem and regularity in the Dirichlet problem for the domain $D^* = \mathbf{R}^n \setminus \bar{D}$. The $L^2$ theory for $D^*$ can be found in [25]. We will let $M^*$ be the non-tangential maximal operator associated to $D^*$, where the non-tangential regions are truncated. We let $\tilde{H}_{at}^1(\partial D)$ be defined as $H_{at}^1(\partial D)$, but add the constant 1 to the atoms.

**Theorem 4.15.** *Given $f \in \tilde{H}_{at}^1(\partial D)$, there exists a harmonic function $u$ in $D^*$ with $\partial u/\partial n = f$ non-tangentially a.e. on $\partial D, u(X) = o(1)$ at $\infty$,*

$$\|M^*(\nabla u)\|_{L^1(\partial D)} \leq C\|f\|_{\tilde{H}_{at}^1(\partial D)}, \quad \text{and} \quad \|u\|_{H_{1,at}^1(\partial D)} \leq C\|f\|_{\tilde{H}_{at}^1(\partial D)}.$$

*Moreover, $u$ is unique. There exists $\varepsilon = \varepsilon(D) > 0$ such that if $f \in L^p(\partial D)$, $1 < p < 2 + \varepsilon$, then $\|M^*(\nabla u)\|_{L^p(\partial D)} \leq C\|f\|_{L^p(\partial D)}$ where $C = C(p, D)$.*

*Proof.* The uniqueness reduces to the $L^2$-uniqueness just as in Theorem 4.3. For existence in the $\tilde{H}_{at}^1(\partial D)$ case, the atom 1 is taken care of by the $L^2$-theory. The existence and the estimate $\|M^*(\nabla u)\|_{L^1(\partial D)} \le C$ for the other atoms are the same as in the proof of Theorem 4.3, the only difference being that the estiamte

$$\left|\int_{D^*} u\eta\right| \le C(K, D^*)\|\eta\|_{L^1(K)}$$

is valid for all $\eta \in L^\infty(D^*)$, supp $\eta \subset K$, since $u(X) = o(1)$ at $\infty$. This fact also shows, by a small variation of the argument used in Theorem 4.3, that $\|u\|_{H^1_{1,at}(\partial D)} \le C$. The case $1 < p < 2 + \varepsilon$ of the theorem follows in the same way as in Theorem 4.13. Note also that if $M^*(\nabla u) \in L^1(\partial D)$, and $u(X) = o(1)$ at $\infty$, then $\partial u/\partial n \in \tilde{H}^1_{at}(\partial D)$. This is proved in a similar way to (4.4) in Theorem 4.3.

**Theorem 4.16.** *Given $f \in H^1_{1,at}(\partial D)$, there exists a harmonic function $u$ in $D^*$ with $u = f$ on $\partial D$ non-tangentially a.e., $u(X) = o(1)$ at $\infty$,*

$$\|M^*(\nabla u)\|_{L^1(\partial D)} \le C\|f\|_{H^1_{1,at}(\partial D)} \quad \textit{and} \quad \left\|\frac{\partial u}{\partial n}\right\|_{\tilde{H}^1_{at}(\partial D)} \le C\|f\|_{H^1_{1,at}(\partial D)}.$$

*Moreover, $u$ is unique. There exists $\varepsilon = \varepsilon(D) > 0$ such that if $f \in L^p_1(\partial D)$, $1 < p < 2 + \varepsilon$, then $\|M^*(\nabla u)\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)}$, where $C = C(p, D)$.*

*Proof.* Uniqueness follows as in the proof of uniqueness in Theorem 4.12. Existence for atoms follows in the same way as in Lemma 4.11. The estimate $\|\partial u/\partial n\|_{\tilde{H}^1_{at}(\partial D)} \le C\|f\|_{H^1_{1,at}(\partial D)}$ follows because of the remark before the statement of Theorem 4.16. The case $1 < p < 2 + \varepsilon$ follows in the same way as in Theorem 4.14.

We are now ready to prove the sharp invertibility properties of the layer potentials.

For $P, Q \in \partial D$, $P \neq Q$, let

$$K(P, Q) = \frac{1}{\omega_n}\langle Q - P, n(Q)\rangle,$$

where $\omega_n$ is the surface area of the unit sphere in $\mathbf{R}^n$, and put $Tf(P) = p.v. \int_{\partial D} K(P, Q)f(Q)d\sigma(Q)$. Also, let

$$Sf(P) = \frac{1}{\omega_n(n-2)}\int_{\partial D} |P - Q|^{2-n} f(Q)d\sigma(Q).$$

The boundedness properties of these operators are the same as for the corresponding operators in the graph case.

**Theorem 4.17.** *There is a number $q_0 = q_0(D)$, $q_0 \in (2,\infty)$, such that $\frac{1}{2}I - T^*$ is an invertible mapping from $L^p_0(\partial D)$ onto $L^p_0(\partial D)$ for $1 < p < q_0$, where $L^p_0(\partial D) = \{f \in L^p(\partial D) : \int_{\partial D} f d\sigma = 0\}$. Also, $\frac{1}{2}I - T^*$ is invertible from $H^1_{at}(\partial D)$ onto $H^1_{at}(\partial D)$. There is a number $p_0 = p_0(D)$, $p_0 \in (1,2)$ such that $\frac{1}{2}I + T$ is an invertible mapping of $L^p(\partial D)$ onto $L^p(\partial D)$ for $p_0 < p < \infty$. Also, $\frac{1}{2}I + T$ is invertible from $BMO\,(\partial D)$ onto $BMO\,(\partial D)$. There is a number $r_0 = r_0(D)$, $r_0 \in (2,\infty)$ such that $S$ is an invertible mapping of $L^p(\partial D)$ onto $L^p_1(\partial D)$. Also, $S$ is an invertible mapping from $H^1_{at}(\partial D)$ onto $H^1_{1,at}(\partial D)$.*

*Proof.* The proof of this theorem is the same as the corresponding $L^2$ case presented in [25], using the results of this ection. Finally, we give representation formulas for the solutions of the Dirichlet and Neumann problem, using layer potential.

**Theorem 4.18.** *Let $D \subset \mathbf{R}^n$ be a bounded Lipschitz domain, whose complement is connected. Let $q_0, p_0, r_0$ be the numbers given in Theorem 4.17. Let $f \in L^p(\partial D)$, $p_0 < p < \infty$, and let $u(X)$ be the unique solution of the Dirichlet problem given in [5]. Then $u(X) = (1/\omega_n)\int_{\partial D}(\langle X - Q, n(Q)\rangle / |X-Q|^n)(\frac{1}{2}I + T)^{-1}(f)(Q)d\sigma(Q)$. The same holds when $f \in BMO\,(\partial D)$, and $u$ is the unique solution of the Dirichlet problem given in [9]. Let $f \in L^p(\partial D)$, $1 < p < q_0$, $\int_{\partial D} f d\sigma = 0$, and let $u(X)$ be the unique (modulo constants) solution of the Neumann problem given in Theorem 4.13. Then,*

$$u(X) = \frac{1}{\omega_n(n-2)}\int_{\partial D}|X-Q|^{2-n}(\frac{1}{2}I - T^*)^{-1}(f)(Q)d\sigma(Q).$$

*The same holds when $f \in H^1_{at}(\partial D)$, and $u$ is as in Theorem 4.3. Let $f \in L^p_1(\partial D)$, $1 < p < r_0$ and let $u(X)$ be the unique solution of the Dirichlet problem given in Theorem 4.14. Then,*

$$u(X) = \frac{1}{\omega_n(n-2)}\int_{\partial D}|X-Q|^{2-n}S^{-1}(f)(Q)d\sigma(Q).$$

*The same holds when $f \in H^1_{1,at}(\partial D)$, and $u$ is as in Theorem 4.12.*

*Proof.* The proof follows from well-known properties of layer potentials (see [25], for example), the uniqueness in all the theorems mentioned and Theorem 4.17.

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