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# Lectures on Matrices

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## PREFACE

This book contains lectures on matrices given at Princeton University at various times since 1920. It was my intention to include full notes on the history of the subject, but this has proved impossible owing to circumstances beyond my control, and I have had to content myself with very brief notes (see Appendix I). A bibliography is given in Appendix II. In compiling it, especially for the period of the last twenty-five years, there was considerable difficulty in deciding whether to include certain papers which, if they had occurred earlier, would probably have found a place there. In the main, I have not included articles which do not use matrices as an algebraic calculus, or whose interest lies in some other part of mathematics rather than in the theory of matrices; but consistency in this has probably not been attained.

Since these lectures have been prepared over a somewhat lengthy period of time, they owe much to the criticism of many friends. In particular, Professor A. A. Albert and Dr. J. L. Dorroh read most of the MS making many suggestions, and the former gave material help in the preparation of the later sections of Chapter X.

J. H. M. WEDDERBURN.

*Princeton, N. J.,  
July 20, 1934.*

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## CORRIGENDA

- page 4, line 9 from top: on second  $\Sigma$  read  $j$  for  $p$
- page 6, Theorem 1, add: *and conversely, if a matrix is commutative with every other matrix, it is a scalar matrix.*
- page 7, line 12 from foot; for first and third  $A$  read  $|A|$
- page 11, lines 10, 11 from foot: for  $(Q'Q)$  read  $(Q'Q)^{-1}$
- page 13, line 9 from foot: for  $Sg_{i;x}$  read  $S\bar{g}_{i;x}$
- page 13, line 6 from foot: for  $\bar{g}_{s+1}$  read  $\bar{g}'_{s+1}$
- page 14, line 3 from top: before  $a_{ji}e_j$  read  $\Sigma$ .
- page 18, line 8 from foot: for  $j$  read  $\gamma_j$
- page 20, line 8 from foot: for  $r + 1$  read  $r - 1$
- page 30, line 13 from top: for  $=$  read  $-$
- page 31, line 4 from foot: for second  $x_i$  read  $xe_i$ ; add  $e_1 + e_2 = 1$
- page 42, equation (16): for  $1$  read  $-1$
- page 54, line 14 from foot; for (12) read (13)
- page 54, line 6 from foot: for (14) read (15)
- page 54, line 3 from foot: for (15) read (16)
- page 54, line 2 from foot: for (13) read (14)
- page 56, line 12 from top: correct term after  $\Sigma$  to read  $k_{i, \beta+j}z_i$
- page 67, lines 4, 5, 6, 7: the exponent  $n$  on the second last  $C$  should read  $\binom{n}{r}$ .
- page 68, line 11 from foot: before  $\Sigma$  read  $(-1)^r$
- page 68, line 8 from foot: before  $|\mathfrak{A}|$  read  $(-1)^r$
- page 74, line 11 from foot: for  $r = 1$  read  $r = 3$
- page 81, line 4 from foot: for  $1/\beta_1!$  read  $\beta_1!$  with similar change in last line
- page 84, line 13 from foot: interchange  $i$  and  $j$ .
- page 85, line 8 from foot: for  $\mathfrak{F}_1$  read  $\bar{\mathfrak{F}}_1$
- page 86, line 7 from top: for first  $e_1$  read  $e_i$

page 92, line 11 from foot: delete from “and if” to end of paragraph

page 101, line 6 from foot: after hermitian insert  $A = A'$

page 103, line 4 from foot: delete first 0; for  $q = t + 1$  read  $q = s + 1$

page 112, equation (23): for  $\{ \}$  read  $[ \ ]$

page 116, line 7 from top: add Every power series converges when  $x$  is nilpotent.

page 119, line 9 from top: for “at least . . . first” read “the H.C.F. of the  $t$ 's is relatively prime to  $m$ ”

page 122, line 4 from foot: multiply bracket by  $\epsilon^{\mu_i}$  and delete same inside

page 122, equation (30): for  $g_{ij}$  read  $p_{ij} = \epsilon^{-\mu_i} g_{ij}$

page 123, lines 2 and 3 from top: for  $g_{ij}$  read  $p_{ij}$

page 123, equations (32) and (33): for  $\pi$  read  $2\pi$

page 125, line 4 from top: read  $\alpha_1^{r_1}(\lambda), \alpha_2^{r_2}(\lambda), \dots, \alpha_k^{r_k}(\lambda)$

page 126, line 13 from top: for  $| |$  read  $| |$

page 126, equation (45): for first  $\alpha$  read  $a$

page 129, equation (63): in first term the bars should be heavy

page 129, line 5 from foot: for  $|x|$  read  $|x|$

page 134, line 6 from top: multiply right side of equation by 2

page 136, line 10 from top: for  $\xi_r$  read  $\xi_{r-1}$

page 137, equation (103): read  $\dot{p} = -\partial_q \zeta$

page 144, equation (24): read  $x'axa^{-1}$

page 156, line 6 from top: for second  $=$  read  $\leq$  and add “ $\leq A$ , whence  $A = \Sigma A_{ij}$ ”

page 164, line 8 from top: for primitive read minimal

page 164, line 7 from foot: for invariant read semi-invariant

page 164, last line: before “complete” insert “suitably chosen”

page 166, line 10 from foot: for equivalent read invariant

page 166, line 5 from foot: for first  $B_2$  read  $B_1$

page 167, Theorem 9: for  $j \neq k$  read  $i \neq t$

page 171, line 5 from top: delete 80

## CHAPTER I

### MATRICES AND VECTORS

**1.01 Linear transformations and vectors.** In a set of linear equations

$$\begin{aligned} \eta'_1 &= a_{11}\eta_1 + a_{12}\eta_2 + \cdots + a_{1n}\eta_n \\ \eta'_2 &= a_{21}\eta_1 + a_{22}\eta_2 + \cdots + a_{2n}\eta_n \\ &\dots\dots\dots \\ \eta'_n &= a_{n1}\eta_1 + a_{n2}\eta_2 + \cdots + a_{nn}\eta_n \end{aligned}$$

or

$$(1) \quad \eta'_i = \sum_{j=1}^n a_{ij}\eta_j \quad (i = 1, 2, \dots, n)$$

the quantities  $\eta_1, \eta_2, \dots, \eta_n$  may be regarded as the coordinates of a point  $P$  in  $n$ -space and the point  $P'(\eta'_1, \eta'_2, \dots, \eta'_n)$  is then said to be derived from  $P$  by the *linear homogeneous transformation* (1). Or, in place of regarding the  $\eta$ 's as the coordinates of a point we may look on them as the components of a vector  $y$  and consider (1) as defining an operation which transforms  $y$  into a new vector  $y'$ . We shall be concerned here with the properties of such transformations, sometimes considered abstractly as entities in themselves, and sometimes in conjunction with vectors.

To prevent misconceptions as to their meaning we shall now define a few terms which are probably already familiar to the reader. By a *scalar* or number we mean an element of the field in which all coefficients of transformations and vectors are supposed to lie; unless otherwise stated the reader may assume that a scalar is an ordinary number real or complex.

A *vector*<sup>1</sup> of order  $n$  is defined as a set of  $n$  scalars  $(\xi_1, \xi_2, \dots, \xi_n)$  given in a definite order. This set, regarded as a single entity, is denoted by a single symbol, say  $x$ , and we write

$$x = (\xi_1, \xi_2, \dots, \xi_n).$$

The scalars  $\xi_1, \xi_2, \dots, \xi_n$  are called the *coordinates* or *components* of the vector.

If  $y = (\eta_1, \eta_2, \dots, \eta_n)$  is also a vector, we say that  $x = y$  if, and only if, corresponding coordinates are equal, that is,  $\xi_i = \eta_i$  ( $i = 1, 2, \dots, n$ ). The vector

$$z = (\zeta_1, \zeta_2, \dots, \zeta_n) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots, \xi_n + \eta_n)$$

is called the *sum* of  $x$  and  $y$  and is written  $x + y$ ; it is easily seen that the operation of addition so defined is commutative and associative, and it has a unique inverse if we agree to write 0 for the vector  $(0, 0, \dots, 0)$ .

<sup>1</sup> In chapter 5 we shall find it convenient to use the name *hypernumber* for the term vector which is then used in a more restricted sense, which, however, does not conflict with the use made of it here.

If  $\rho$  is a scalar, we shall write

$$\rho x = x\rho = (\rho\xi_1, \rho\xi_2, \dots, \rho\xi_n).$$

This is the only kind of multiplication we shall use regularly in connection with vectors.

**1.02 Linear dependence.** In this section we shall express in terms of vectors the familiar notions of linear dependence.<sup>2</sup> If  $x_1, x_2, \dots, x_r$  are vectors and  $\omega_1, \omega_2, \dots, \omega_r$  scalars, any vector of the form

$$(2) \quad x = \omega_1 x_1 + \omega_2 x_2 + \dots + \omega_r x_r$$

is said to be *linearly dependent* on  $x_1, x_2, \dots, x_r$ ; and these vectors are called linearly independent if an equation which is reducible to the form

$$0 = \omega_1 x_1 + \omega_2 x_2 + \dots + \omega_r x_r$$

can only be true when each  $\omega_i = 0$ . Geometrically the  $r$  vectors determine an  $r$ -dimensional subspace of the original  $n$ -space and, if  $x_1, x_2, \dots, x_r$  are taken as the coordinate axes,  $\omega_1, \omega_2, \dots, \omega_r$  in (2) are the coordinates of  $x$ .

We shall call the totality of vectors  $x$  of the form (2) the *linear set* or *subspace*  $(x_1, x_2, \dots, x_r)$  and, when  $x_1, x_2, \dots, x_r$  are linearly independent, they are said to form a *basis* of the set. The number of elements in a basis of a set is called the *order* of the set.

Suppose now that  $(x_1, x_2, \dots, x_r), (y_1, y_2, \dots, y_s)$  are bases of the same linear set and assume  $s \geq r$ . Since the  $x$ 's form a basis, each  $y$  can be expressed in the form

$$(3) \quad y_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ir}x_r \quad (i = 1, 2, \dots, s)$$

and, since the  $y$ 's form a basis, we may set

$$x_i = b_{i1}y_1 + b_{i2}y_2 + \dots + b_{is}y_s \quad (i = 1, 2, \dots, r)$$

and therefore from (3)

$$(4) \quad y_i = \sum_{j=1}^r a_{ij}x_j = \sum_{j=1}^r a_{ij} \sum_{k=1}^s b_{jk}y_k = \sum_{k=1}^s c_{ik}y_k,$$

where  $c_{ik} = \sum_{j=1}^r a_{ij}b_{jk}$ , which may also be written

$$(5) \quad c_{ik} = \sum_{j=1}^s a_{ij}b_{jk} \quad (i = 1, 2, \dots, s)$$

if we agree to set  $a_{ij} = 0$  when  $j > r$ . Since the  $y$ 's are linearly independent, (4) can only hold true if  $c_{ii} = 1, c_{ik} = 0$  ( $i \neq k$ ) so that the determinant

<sup>2</sup> See for instance Bôcher, *Introduction to Higher Algebra*, p. 34.

$|c_{ik}| = 1$ . But from the rule for forming the product of two determinants it follows from (5) that  $|c_{ik}| = |a_{ik}| |b_{ik}|$  which implies (i) that  $|a_{ik}| \neq 0$  and (ii) that  $r = s$ , since otherwise  $|a_{ik}|$  contains the column  $a_{i, r+1}$  each element of which is 0. The order of a set is therefore independent of the basis chosen to represent it.

It follows readily from the theory of linear equations (or from §1.11 below) that, if  $|a_{ij}| \neq 0$  in (3), then these equations can be solved for the  $x$ 's in terms of the  $y$ 's, so that the conditions established above are sufficient as well as necessary in order that the  $y$ 's shall form a basis.

If  $e_i$  denotes the vector whose  $i$ th coordinate is 1 and whose other coordinates are 0, we see immediately that we may write

$$x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$$

in place of  $x = (\xi_1, \xi_2, \cdots, \xi_n)$ . Hence  $e_1, e_2, \cdots, e_n$  form a basis of our  $n$ -space. We shall call this the *fundamental basis* and the individual vectors  $e_i$  the *fundamental unit vectors*.

If  $x_1, x_2, \cdots, x_r$  ( $r < n$ ) is a basis of a subspace of order  $r$ , we can always find  $n-r$  vectors  $x_{r+1}, \cdots, x_n$  such that  $x_1, x_2, \cdots, x_n$  is a basis of the fundamental space. For, if  $x_{r+1}$  is any vector not lying in  $(x_1, x_2, \cdots, x_r)$ , there cannot be any relation

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_r x_r + \omega_{r+1} x_{r+1} = 0$$

in which  $\omega_{r+1} \neq 0$  (in fact every  $\omega$  must be 0) and hence the order of  $(x_1, x_2, \cdots, x_r, x_{r+1})$  is  $r+1$ . Since the order of  $(e_1, e_2, \cdots, e_n)$  is  $n$ , a repetition of this process leads to a basis  $x_1, x_2, \cdots, x_r, \cdots, x_n$  of order  $n$  after a finite number of steps; a suitably chosen  $e_i$  may be taken for  $x_{r+1}$ . The  $(n-r)$ -space  $(x_{r+1}, \cdots, x_n)$  is said to be *complementary* to  $(x_1, x_2, \cdots, x_r)$ ; it is of course not unique.

**1.03 Linear vector functions and matrices.** The set of linear equations given in §1.01, namely,

$$(6) \quad \eta'_i = \sum_{j=1}^n a_{ij} \eta_j \quad (i = 1, 2, \cdots, n)$$

define the vector  $y' = (\eta'_1, \eta'_2, \cdots, \eta'_n)$  as a linear homogeneous function of the coordinates of  $y = (\eta_1, \eta_2, \cdots, \eta_n)$  and in accordance with the usual functional notation it is natural to write  $y' = A(y)$ ; it is usual to omit the brackets and we therefore set in place of (6)

$$y' = Ay.$$

The function or operator  $A$  when regarded as a single entity is called a **matrix**; it is completely determined, relatively to the fundamental basis, when

the  $n^2$  numbers  $a_{ij}$  are known, in much the same way as the vector  $y$  is determined by its coordinates. We call the  $a_{ij}$  the *coordinates* of  $A$  and write

$$(7) \quad A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

or, when convenient,  $A = || a_{ij} ||$ . It should be noted that in  $a_{ij}$  the first suffix denotes the row in which the coordinate occurs while the second gives the column.

If  $B = || b_{ij} ||$  is a second matrix,  $y'' = A(By)$  is a vector which is a linear vector homogeneous function of  $y$ , and from (6) we have

$$\eta'_i = \sum_{p=1}^n a_{ip} \sum_{j=1}^n b_{pj} \eta_j = \sum_{j=1}^n d_{ij} \eta_j,$$

where

$$(8) \quad d_{ij} = \sum_{p=1}^n a_{ip} b_{pj}.$$

The matrix  $D = || d_{ij} ||$  is called the *product* of  $A$  into  $B$  and is written  $AB$ . The form of (8) should be carefully noted; in it each element of the  $i$ th row of  $A$  is multiplied into the corresponding element of the  $j$ th column of  $B$  and the terms so formed are added. Since the rows and columns are not interchangeable,  $AB$  is in general different from  $BA$ ; for instance

$$\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 2a+c & 2b+d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} a+2b & b \\ c+2d & d \end{vmatrix}.$$

The product defined by (8) is associative; for if  $C = || c_{ij} ||$ , the element in the  $i$ th row and  $j$ th column of  $(AB)C$  is

$$\sum_{q=1}^n \left( \sum_{p=1}^n a_{ip} b_{pq} \right) c_{qj} = \sum_{p=1}^n a_{ip} \left( \sum_{q=1}^n b_{pq} c_{qj} \right)$$

and the term on the right is the  $(i, j)$  coordinate of  $A(BC)$ .

If we add the vectors  $Ay$  and  $By$ , we get a vector whose  $i$ th coordinate is (cf. (6))

$$\eta'_i = \sum_{j=1}^n a_{ij} \eta_j + \sum_{j=1}^n b_{ij} \eta_j = \sum_{j=1}^n c_{ij} \eta_j$$

where  $c_{ij} = a_{ij} + b_{ij}$ . Hence  $Ay + By$  may be written  $Cy$  where  $C = ||c_{ij}||$ . We define  $C$  to be the *sum* of  $A$  and  $B$  and write  $C = A + B$ ; two matrices are then added by adding corresponding coordinates just as in the case of vectors. It follows immediately from the definition of sum and product that

$$\begin{aligned} A + B &= B + A, & (A + B) + C &= A + (B + C), \\ A(B + C) &= AB + AC, & (B + C)A &= BA + CA, \\ A(x + y) &= Ax + Ay, \end{aligned}$$

$A, B, C$  being any matrices and  $x, y$  vectors. Also, if  $k$  is a scalar and we set  $y' = Ay, y'' = ky'$ , then

$$y'' = ky' = kA(y) = A(ky)$$

or in terms of the coordinates

$$\eta_i'' = \sum_j k a_{ij} \eta_j.$$

Hence  $kA$  may be interpreted as the matrix derived from  $A$  by multiplying each coordinate of  $A$  by  $k$ .

On the analogy of the unit vectors  $e_i$  we now define the *fundamental unit matrices*  $e_{ij}$  ( $i, j = 1, 2, \dots, n$ ). Here  $e_{ij}$  is the matrix whose coordinates are all 0 except the one in the  $i$ th row and  $j$ th column whose value is 1. Corresponding to the form  $\sum \xi_i e_i$  for a vector we then have

$$(9) \quad A = \sum_{i,j=1}^n a_{ij} e_{ij}.$$

Also from the definition of multiplication in (8)

$$(10) \quad e_{ij} e_{jk} = e_{ik}, \quad e_{ij} e_{pq} = 0, \quad (j \neq p)$$

a set of relations which might have been made the basis of the definition of the product of two matrices. It should be noted that it follows from the definition of  $e_{ij}$  that

$$(11) \quad e_{ij} e_j = e_i, \quad e_{ij} e_k = 0 \quad (j \neq k),$$

$$(12) \quad A e_k = \sum_{i,j} a_{ij} e_{ij} e_k = \sum_i a_{ik} e_i.$$

Hence the coordinates of  $A e_k$  are the coordinates of  $A$  that lie in the  $k$ th column.

**1.04 Scalar matrices.** If  $k$  is a scalar, the matrix  $K$  defined by  $Ky = ky$  is called a *scalar matrix*; from (1) it follows that, if  $K = ||k_{ij}||$ , then  $k_{ii} = k$  ( $i = 1, 2, \dots, n$ ),  $k_{ij} = 0$  ( $i \neq j$ ). The scalar matrix for which  $k = 1$  is called the identity matrix of order  $n$ ; it is commonly denoted by  $I$  but, for reasons

explained below, we shall here usually denote it by 1, or by  $1_n$  if it is desired to indicate the order. When written at length we have

$$1_n = \left\| \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array} \right\|, \quad K = \left\| \begin{array}{cccc} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{array} \right\|$$

A convenient notation for the coordinates of the identity matrix was introduced by Kronecker: if  $\delta_{ij}$  is the numerical function of the integers  $i, j$  defined by

$$(13) \quad \delta_{ii} = 1, \quad \delta_{ij} = 0 \quad (i \neq j),$$

then  $1_n = \|\delta_{ij}\|$ . We shall use this Kronecker delta function in future without further comment.

**THEOREM 1.** *Every matrix is commutative with a scalar matrix.*

Let  $k$  be the scalar and  $K = \|k_{ij}\| = \|k\delta_{ij}\|$  the corresponding matrix. If  $A = \|a_{ij}\|$  is any matrix, then from the definition of multiplication

$$KA = \left\| \sum_p k_{ip} a_{pj} \right\| = \left\| \sum_p k \delta_{ip} a_{pj} \right\| = \|ka_{ij}\|$$

$$AK = \left\| \sum_p a_{ip} k_{pj} \right\| = \left\| \sum_p ka_{ip} \delta_{pj} \right\| = \|ka_{ij}\|$$

so that  $AK = KA$ .

If  $k$  and  $h$  are two scalars and  $K, H$  the corresponding scalar matrices, then  $K + H$  and  $KH$  are the scalar matrices corresponding to  $k + h$  and  $kh$ . Hence the *one-to-one* correspondence between scalars and scalar matrices is maintained under the operations of addition and multiplication, that is, the two sets are simply isomorphic with respect to these operations. So long therefore as we are concerned only with matrices of given order, there is no confusion introduced if we replace each scalar by its corresponding scalar matrix, just as in the theory of ordinary complex numbers,  $(a, b) = a + bi$ , the set of numbers of the form  $(a, 0)$  is identified with the real continuum. We shall therefore as a rule denote  $\|\delta_{ij}\|$  by 1 and  $\|k\delta_{ij}\|$  by  $k$ .

**1.05 Powers of a matrix; adjoint matrices.** Positive integral powers of  $A = \|a_{ij}\|$  are readily defined by induction; thus

$$A^2 = A \cdot A, \quad A^3 = A \cdot A^2, \dots, \quad A^m = A \cdot A^{m-1}.$$

With this definition it is clear that  $A^r A^s = A^{r+s}$  for any positive integers  $r, s$ . Negative powers, however, require more careful consideration.

Let the determinant formed from the array of coefficients of a matrix be denoted by

$$| A | = \det. A$$

and let  $\alpha_{pq}$  be the cofactor of  $a_{pq}$  in  $A$ , so that from the properties of determinants

$$(14) \quad \sum_p a_{ip} \alpha_{pj} = | A | \delta_{ij} = \sum_p \alpha_{ip} a_{pj} \quad (i, j = 1, 2, \dots, n).$$

The matrix  $\| \alpha_{ij} \|$  is called the *adjoint* of  $A$  and is denoted by  $\text{adj } A$ . In this notation (14) may be written

$$(15) \quad A(\text{adj } A) = | A | = (\text{adj } A)A,$$

so that a matrix and its adjoint are commutative.

If  $| A | \neq 0$ , we define  $A^{-1}$  by

$$(16) \quad A^{-1} = | A |^{-1} \text{adj } A.$$

Negative integral powers are then defined by  $A^{-r} = (A^{-1})^r$ ; evidently  $A^{-r} = (A^r)^{-1}$ . We also set  $A^0 = 1$ , but it will appear later that a different interpretation must be given when  $| A | = 0$ . Since  $AB \cdot B^{-1}A^{-1} = A \cdot BB^{-1} \cdot A^{-1} = AA^{-1} = 1$ , the reciprocal of the product  $AB$  is

$$(AB)^{-1} = B^{-1}A^{-1}.$$

If  $A$  and  $B$  are matrices, the rule for multiplying determinants, when stated in our notation, becomes

$$| AB | = | A | | B |.$$

In particular, if  $AB = 1$ , then  $| A | | B | = 1$ ; hence, if  $| A | = 0$ , there is no matrix  $B$  such that  $AB = 1$  or  $BA = 1$ . The reader should notice that, if  $k$  is a scalar matrix of order  $n$ , then  $| k | = k^n$ .

If  $A = 0$ ,  $A$  is said to be *singular*; if  $A \neq 0$ ,  $A$  is *regular* or non-singular. When  $A$  is regular,  $A^{-1}$  is the only solution of  $AX = 1$  or of  $XA = 1$ . For, if  $AX = 1$ , then

$$A^{-1} = A^{-1} \cdot 1 = A^{-1}AX = X.$$

If  $AX = 0$ , then either  $X = 0$  or  $A$  is singular; for, if  $A^{-1}$  exists,

$$0 = A^{-1}AX = X.$$

If  $A^2 = A \neq 0$ , then  $A$  is said to be *idempotent*; for example  $e_{11}$  and  $\begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix}$  are idempotent. A matrix a power of which is 0 is called *nilpotent*. If the lowest power of  $A$  which is 0 is  $A^r$ ,  $r$  is called the *index* of  $A$ ; for example, if  $A = e_{12} + e_{23} + e_{34}$ , then

$$A^2 = e_{13} + e_{24}, \quad A^3 = e_{14}, \quad A^4 = 0,$$

so that the index of  $A$  in this case is 4.

1.06 **The transverse of a matrix.** If  $A = || a_{ij} ||$ , the matrix  $|| a'_{ij} ||$  in which  $a'_{ij} = a_{ji}$  is called the *transverse*<sup>3</sup> of  $A$  and is denoted by  $A'$ . For instance the transverse of

$$\left\| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right\| \text{ is } \left\| \begin{array}{ccc} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{array} \right\|.$$

The transverse, then, is obtained by the interchange of corresponding rows and columns. It must be carefully noted that this definition is relative to a particular set of fundamental units and, if these are altered, the transverse must also be changed.

**THEOREM 2.** *The transverse of a sum is the sum of the transverses of the separate terms, and the transverse of a product is the product of the transverses of the separate factors in the reverse order.*

The proof of the first part of the theorem is immediate and is left to the reader. To prove the second it is sufficient to consider two factors. Let  $A = || a_{ij} ||$ ,  $B = || b_{ij} ||$ ,  $C = AB = || c_{ij} ||$  and, as above, set  $a'_{ij} = a_{ji}$ ,  $b'_{ij} = b_{ji}$ ,  $c'_{ij} = c_{ji}$ ; then

$$c'_{ij} = c_{ji} = \sum_p a_{jp} b_{pi} = \sum_p b'_{ip} a'_{pj}$$

whence

$$(AB)' = C' = B'A'.$$

The proof for any number of factors follows by induction.

If  $A = A'$ ,  $A$  is said to be *symmetric* and, if  $A = -A'$ , it is called *skew-symmetric* or *skew*. A scalar matrix  $k$  is symmetric and the transverse of  $kA$  is  $kA'$ .

**THEOREM 3.** *Every matrix can be expressed uniquely as the sum of a symmetric and a skew matrix.*

For if  $A = B + C$ ,  $B' = B$ ,  $C' = -C$ , then  $A' = B' + C' = B - C$  and therefore

$$B = (A + A')/2, \quad C = (A - A')/2.$$

Conversely  $2A = (A + A') + (A - A')$  and  $A + A'$  is symmetric,  $A - A'$  skew.

<sup>3</sup> It is also called the transposed or conjugate of  $A$ . It is sometimes written  $\check{A}$ .

1.07 **Bilinear forms.** A scalar bilinear form in two variable vectors,  $x = \sum \xi_i e_i$ ,  $y = \sum \eta_i e_i$ , is a function of the form

$$(17) \quad A(x, y) = \sum_{i,j=1}^n a_{ij} \xi_i \eta_j.$$

There is therefore a one-to-one correspondence between such forms and matrices,  $A = || a_{ij} ||$  corresponding to  $A(x, y)$ . The special form for which  $A = || \delta_{ij} || = 1$  is of very frequent occurrence and we shall denote it by  $S$ ; it is convenient to omit the brackets and write simply

$$(18) \quad Sxy = \xi_1 \eta_1 + \xi_2 \eta_2 + \cdots + \xi_n \eta_n$$

and, because of the manner in which it appears in vector analysis, we shall call it the *scalar* of  $xy$ . Since  $S$  is symmetric,  $Sxy = Syx$ .

The function (17) can be conveniently expressed in terms of  $A$  and  $S$ ; for we may write  $A(x, y)$  in the form

$$A(x, y) = \sum_{i=1}^n \xi_i \left( \sum_{j=1}^n a_{ij} \eta_j \right) = SxAy.$$

It may also be written

$$\sum_{j=1}^n \left( \sum_{i=1}^n a_{ij} \xi_i \right) \eta_j = SA'xy = SyA'x;$$

hence

$$(19) \quad SxAy = SyA'x,$$

so that the form (17) is unaltered when  $x$  and  $y$  are interchanged if at the same time  $A$  is changed into  $A'$ . This gives another proof of Theorem 2. For

$$Sx(AB)'y = SyABx = SBxA'y = SxB'A'y,$$

which gives  $(AB)' = B'A'$  since  $x$  and  $y$  are independent variables.

1.08 **Change of basis.** We shall now investigate more closely the effect of a change in the fundamental basis on the coordinates of a vector or matrix. If  $f_1, f_2, \dots, f_n$  is a basis of our  $n$ -space, we have seen (§1.02) that the  $f$ 's are linearly independent. Let

$$(20) \quad f_i = \sum_{j=1}^n p_{ji} e_j = P e_i \quad (i = 1, 2, \dots, n)$$

$$P = || p_{ij} ||.$$

Since the  $f$ 's form a basis, the  $e$ 's are linearly expressible in terms of them, say

$$(21) \quad e_i = \sum_{j=1}^n q_{ji} f_j,$$

and, if  $Q = || q_{ij} ||$ , this may be written

$$(22) \quad e_i = \sum_j q_{ij} \sum_k p_{kj} e_k = PQe_i \quad (i = 1, 2, \dots, n).$$

Hence  $PQ = 1$ , which is only possible if  $|P| \neq 0$ ,  $Q = P^{-1}$ .

Conversely, if  $|P| \neq 0$ ,  $Q = P^{-1}$ , and  $f_i = Pe_i$  as in (20), then (22) holds and therefore also (21), that is, the  $e$ 's, and therefore also any vector  $x$ , are linearly expressible in terms of the  $f$ 's. We have therefore the following theorem.

**THEOREM 4.** *If  $f_i = Pe_i$  ( $i = 1, 2, \dots, n$ ), the vectors  $f_i$  form a basis if, and only if,  $|P| \neq 0$ .*

If we have fewer than  $n$  vectors, say  $f_1, f_2, \dots, f_r$ , we have seen in 1.02 that we can choose  $f_{r+1}, \dots, f_n$  so that  $f_1, f_2, \dots, f_n$  form a basis. Hence

**THEOREM 5.** *If  $f_1, f_2, \dots, f_r$  are linearly independent, there exists at least one non-singular matrix  $P$  such that  $Pe_i = f_i$  ( $i = 1, 2, \dots, r$ ).*

We shall now determine how the form  $Sxy$ , which was defined relatively to the fundamental basis, is altered by a change of basis. As above let

$$(23) \quad f_i = Pe_i, \quad e_i = P^{-1}f_i = Qf_i, \quad |P| \neq 0, \quad (i = 1, 2, \dots, n)$$

be a basis and

$$x = \sum \xi_i e_i = \sum \xi'_i f_i, \quad y = \sum \eta_i e_i = \sum \eta'_i f_i$$

variable vectors; then from (23)

$$x = Q \sum \xi_i f_i = P \sum \xi'_i e_i, \quad y = Q \sum \eta_i f_i = P \sum \eta'_i e_i$$

and

$$\sum \xi'_i e_i = P^{-1}x = Qx, \quad \sum \eta'_i e_i = Qy.$$

Let us set temporarily  $S_e xy$  for  $Sxy$  and also put  $S_f xy = \sum \xi'_i \eta'_i$ , the corresponding form with reference to the new basis; then

$$(24) \quad \begin{aligned} S_f xy &= S_e QxQy = S_e xQ'Qy \\ S_e xy &= S_f PxPy. \end{aligned}$$

Consider now a matrix  $A = || a_{ij} ||$  defined relatively to the fundamental basis and let  $A_1$  be the matrix which has the same coordinates when expressed in terms of the new basis as  $A$  has in the old. From the definition of  $A$  and from  $\xi_j = S_e e_j x$  we have

$$Ax = \sum_{i,j} a_{ij} \xi_j e_i = \sum_{i,j} a_{ij} e_i S_e e_j x$$

and hence

$$(25) \quad \begin{aligned} A_1x &= \sum a_{ij}\xi'_j f_i = \sum a_{ij}f_i S_j f_j x = \sum a_{ij}Q^{-1}e_i S_e Q f_j Qx \\ &= Q^{-1}\sum a_{ij}e_i S_e Qx = Q^{-1}AQx. \end{aligned}$$

We have therefore, remembering that  $Q = P^{-1}$ ,

**THEOREM 6.** *If  $f_i = Pe_i$  ( $i = 1, 2, \dots, n$ ) is a basis and  $A$  any matrix, the matrix  $PAP^{-1}$  has the same coordinates when expressed in terms of this basis as  $A$  has in terms of the fundamental basis.*

The matrix  $Q^{-1}AQ$  is said to be *similar* to  $A$  and to be the *transform* of  $A$  by  $Q$ . Obviously the transform of a product (sum) is the product (sum) of the transforms of the individual factors (terms) with the order unaltered. For instance  $Q^{-1}ABQ = Q^{-1}AQ \cdot Q^{-1}BQ$ .

Theorem 6 gives the transformation of the matrix units  $e_{ij}$  defined in §1.03 which corresponds to the vector transformation (23); the result is that, if  $f_{ij}$  is the unit in the new system corresponding to  $e_{ij}$ , then

$$f_{ij} = Pe_{ij}P^{-1}$$

which is readily verified by setting

$$A = e_{ij} = e_i S_e e_j, \quad A_1 = f_{ij} = f_i S_j f_j$$

in (25). The effect of the change of basis on the form of the transverse is found as follows. Let  $A^*$  be defined by

$$S_j x A y = S_j y A^* x;$$

then

$$\begin{aligned} S_j y A^* x &= S_j x A y = S_e Q x Q A y = S_e x Q' Q A y = S_e Q y (Q') A' Q' Q x \\ &= S_j y (Q' Q) A' Q' Q x. \end{aligned}$$

Hence

$$(26) \quad A^* = (Q' Q) A' Q' Q.$$

**1.09 Reciprocal and orthogonal bases.** With the same notation as in the previous section we have  $S_j f_i f_j = 0$  ( $i \neq j$ ),  $S_j f_i f_j = 1$ . Hence

$$\delta_{ij} = S_j f_i f_j = S_e Q f_i Q f_j = S_e f_i Q' Q f_j.$$

If, therefore, we set

$$(27) \quad f'_j = Q' Q f_j \quad (j = 1, 2, \dots, n),$$

we have, on omitting the subscript  $e$  in  $S_e$ ,

$$(28) \quad S_j f'_i = \delta_{ij} \quad (i, j = 1, 2, \dots, n).$$

Since  $|Q'Q| \neq 0$ , the vectors  $f'_1, f'_2, \dots, f'_n$  form a basis which we say is *reciprocal* to  $f_1, f_2, \dots, f_n$ . This definition is of course relative to the fundamental

basis since it depends on the function  $S$  but, apart from this the basis  $(f'_i)$  is uniquely defined when the basis  $(f_i)$  is given since the vectors  $f_i$  determine  $P$  and  $Q = P^{-1}$ .

The relation between  $(f'_i)$  and  $(f_i)$  is a reciprocal one; for

$$f'_i = Q'Qf_i = Q'QPe_i = Q'e_i$$

and, if  $R = (Q')^{-1}$ , we have  $f_i = R'Rf'_i$ .

If only the set  $(f_1, f_2, \dots, f_r)$  is supposed given originally, and this set of linearly independent vectors is extended by  $f_{r+1}, \dots, f_n$  to form a basis of the  $n$ -space, then  $f'_{r+1}, \dots, f'_n$  individually depend on the choice of  $f_{r+1}, \dots, f_n$ . But (28) shows that, if  $Sf_i x = 0$  ( $i = 1, 2, \dots, r$ ), then  $x$  belongs to the linear set  $(f'_{r+1}, \dots, f'_n)$ ; hence this linear set is uniquely determined although the individual members of its basis are not. We may therefore without ambiguity call  $\mathfrak{F}' = (f'_{r+1}, \dots, f'_n)$  reciprocal to  $\mathfrak{F} = (f_1, f_2, \dots, f_r)$ ;  $\mathfrak{F}'$  is then the set of all vectors  $x$  for which  $Sxy = 0$  whenever  $y$  belongs to  $\mathfrak{F}$ .

In a later chapter we shall require the following lemma.

LEMMA 1. *If  $(f_1, f_2, \dots, f_r)$  and  $(f'_{r+1}, \dots, f'_n)$  are reciprocal, so also are  $(B^{-1}f_1, B^{-1}f_2, \dots, B^{-1}f_r)$  and  $(B'f'_{r+1}, B'f'_{r+2}, \dots, B'f'_n)$  where  $B$  is any non-singular matrix.*

For  $SB'f'_i B^{-1}f_j = Sf'_i BB^{-1}f_j = Sf'_i f_j = \delta_{ij}$ .

Reciprocal bases have a close connection with reciprocal or inverse matrices in terms of which they might have been defined. If  $P$  is non-singular and  $Pe_i = f_i$  as above, then  $P = \sum f_i Se_i(\ )$  and, if  $Q = \sum e_i Sf'_i(\ )$ , then

$$PQ = \sum e_i Sf'_i f_j Se_j(\ ) = \sum \delta_{ij} e_j Se_j(\ ) = 1$$

so that  $Q = P^{-1}$ .

If  $QQ' = 1$ , the bases  $(f_i)$  and  $(f'_i)$  are identical and  $Sf_i f_j = \delta_{ij}$  for all  $i$  and  $j$ ; the basis is then said to be *orthogonal* as is also the matrix  $Q$ . The inverse of an orthogonal matrix and the product of two or more orthogonal matrices are orthogonal; for, if  $RR' = 1$ ,

$$(RQ)(RQ)' = RQQ'R' = RR' = 1.$$

Suppose that  $h_1, h_2, \dots, h_r$  are real vectors which are linearly independent and for which  $Sh_i h_j = \delta_{ij}$  ( $i \neq j$ ); since  $h_i$  is real, we have  $Sh_i h_i \neq 0$ . If  $r < n$ , we can always find a real vector  $x$  which is not in the linear set  $(h_1, \dots, h_r)$  and, if we put

$$h_{r+1} = x - \sum_1^r h_i Sh_i x / Sh_i h_i,$$

then  $h_{r+1} \neq 0$  and  $Sh_i h_{r+1} = 0$  ( $i = 1, 2, \dots, r$ ). Hence we can extend the original set to form a basis of the fundamental  $n$ -space. If we set  $f_i = h_i / (Sh_i h_i)^{1/2}$ , then  $Sf_i f_j = \delta_{ij}$  even when  $i = j$ ; this modified basis is called an *orthogonal basis* of the set.

If the vectors  $h_i$  are not necessarily real, it is not evident that  $x$  can be chosen so that  $Sh_{r+1}h_{r+1} \neq 0$  when  $Sh_ih_i \neq 0$  ( $i = 1, 2, \dots, r$ ). This may be shown as follows. In the first place we cannot have  $Syh_{r+1} = 0$  for every  $y$ , and hence  $Sh_{r+1}h_{r+1} \neq 0$  when  $r = n - 1$ . Suppose now that for every choice of  $x$  we have  $Sh_{r+1}h_{r+1} = 0$ ; we can then choose a basis  $h_{r+1}, \dots, h_n$  supplementary to  $h_1, \dots, h_r$  such that  $Sh_ih_i = 0$  ( $i = r + 1, \dots, n$ ) and  $Sh_ih_j = 0$  ( $i = r + 1, \dots, n; j = 1, 2, \dots, r$ ). Since we cannot have  $Sh_{r+1}h_i = 0$  for every  $h_i$  of the basis of the  $n$ -space, this scalar must be different from 0 for some value of  $i > r$ , say  $r + k$ . If we then put  $h'_{r+1} = h_{r+1} + h_{r+k}$  in place of  $h_{r+1}$ , we have  $Sh_ih'_{r+1} = 0$  ( $i = 1, 2, \dots, r$ ) as before and also

$$\begin{aligned} Sh'_{r+1}h'_{r+1} &= Sh_{r+1}h_{r+1} + Sh_{r+k}h_{r+k} + 2Sh_{r+1}h_{r+k} \\ &= 2Sh_{r+1}h_{r+k} \neq 0. \end{aligned}$$

We can therefore extend the basis in the manner indicated for real vectors even when the vectors are complex.

When complex coordinates are in question the following lemma is useful; it contains the case discussed above when the vectors used are real.

**LEMMA 2.** *When a linear set of order  $r$  is given, it is always possible to choose a basis  $g_1, g_2, \dots, g_n$  of the fundamental space such that  $g_1, \dots, g_r$  is a basis of the given set and such that  $Sg_i\bar{g}_j = \delta_{ij}$  where  $\bar{g}_i$  is the vector whose coordinates are the conjugates of the coordinates of  $g_i$  when expressed in terms of the fundamental basis.*

The proof is a slight modification of the one already given for the real case. Suppose that  $g_1, \dots, g_s$  are chosen so that  $Sg_i\bar{g}_j = \delta_{ij}$  ( $i, j = 1, 2, \dots, s$ ) and such that  $(g_1, \dots, g_s)$  lies in the given set when  $s < r$  and when  $s > r$ , then  $g_1, \dots, g_r$  is a basis of this set. We now put

$$g'_{s+1} = x - \sum_1^s g_i Sg_i x / S\bar{g}_i g_i$$

which is not 0 provided  $x$  is not in  $(g_1, \dots, g_s)$  and, if  $s < r$ , will lie in the given set provided  $x$  does. We may then put

$$g_{s+1} = g'_{s+1} / (Sg'_{s+1}\bar{g}_{s+1})^{\frac{1}{2}}$$

and the lemma follows readily by induction.

If  $U$  is the matrix  $\Sigma e_i Sg_i$ , then  $\bar{U} = \Sigma e_i S\bar{g}_i$  and

$$(29) \quad U\bar{U}' = 1.$$

Such a matrix is called a *unitary* matrix and the basis  $g_1, g_2, \dots, g_n$  is called a unitary basis. A real unitary matrix is of course orthogonal.

1.10 **The rank of a matrix.** Let  $A = || a_{ij} ||$  be a matrix and set (cf. (12) §1.03)

$$h_i = Ae_i = a_{ij}e_j;$$

then, if

$$x = \sum \xi_i e_i = \sum e_i S e_i x$$

is any vector, we have

$$Ax = A \sum e_i S e_i x = \sum A e_i S e_i x$$

or

$$(30) \quad Ax = \sum_1^n h_i S e_i x.$$

Any expression of the form  $Ax = \sum_1^m a_i S b_i x$ , where  $a_i, b_i$  are constant vectors, is a linear homogeneous vector function of  $x$ . Here (30) shows that it is never necessary to take  $m > n$ , but it is sometimes convenient to do so. When we are interested mainly in the matrix and not in  $x$ , we may write  $A = \sum a_i S b_i$  ( ) or, omitting the brackets, merely

$$(31) \quad A = \sum a_i S b_i.$$

It follows readily from the definition of the transverse that

$$(32) \quad A' = \sum b_i S a_i.$$

No matter what vector  $x$  is,  $Ax$ , being equal to  $\sum a_i S b_i x$ , is linearly dependent on  $a_1, a_2, \dots, a_m$  or, if the form (30) is used, on  $h_1, h_2, \dots, h_n$ . When  $|A| \neq 0$ , we have seen in Theorem 4 that the  $h$ 's are linearly independent but, if  $A$  is singular, there are linear relations connecting them, and the order of the linear set  $(a_1, a_2, \dots, a_m)$  is less than  $n$ .

Suppose in (31) that the  $a$ 's are not linearly independent, say

$$a_s = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_{s-1} a_{s-1},$$

then on substituting this value of  $a_s$  in (31) we have

$$A = a_1 S (b_1 + \alpha_1 b_s) + \dots + a_{s-1} S (b_{s-1} + \alpha_{s-1} b_s) + \sum_{s+1}^m a_i S b_i,$$

an expression similar to (31) but having at least one term less. A similar reduction can be carried out if the  $b$ 's are not linearly independent. After a finite number of repetitions of this process we shall finally reach a form

$$(33) \quad A = \sum_1^r c_i S d_i$$

in which  $c_1, c_2, \dots, c_r$  are linearly independent and also  $d_1, d_2, \dots, d_r$ . The integer  $r$  is called the *rank* of  $A$ .

It is clear that the value of  $r$  is independent of the manner in which the reduction to the form (33) is carried out since it is the order of the linear set  $(Ae_1, Ae_2, \dots, Ae_n)$ . We shall, however, give a proof of this which incidentally yields some important information regarding the nature of  $A$ .

Suppose that by any method we have arrived at two forms of  $A$

$$A = \sum_1^r c_i Sd_i = \sum_1^s p_i Sq_i,$$

where  $(c_1, c_2, \dots, c_r)$  and  $(d_1, d_2, \dots, d_r)$  are spaces of order  $r$  and  $(p_1, p_2, \dots, p_s), (q_1, q_2, \dots, q_s)$  spaces of order  $s$ , and let  $(c'_{r+1}, c'_{r+2}, \dots, c'_n), \dots, (q'_{s+1}, q'_{s+2}, \dots, q'_n)$  be the corresponding reciprocal spaces. Then

$$Aq'_j = \sum_1^s p_i Sq_i q'_j = p_j \quad (j = 1, 2, \dots, s)$$

and also  $Aq'_j = \sum c_i Sd_i q'_j$ . Hence each  $p_j$  lies in  $(c_1, c_2, \dots, c_r)$ . Similarly each  $c_i$  lies in  $(p_1, p_2, \dots, p_s)$  so that these two subspaces are the same and, in particular, their orders are equal, that is,  $r = s$ . The same discussion with  $A'$  in place of  $A$  shows that  $(d_1, d_2, \dots, d_r)$  and  $(q_1, q_2, \dots, q_s)$  are the same. We shall call the spaces  $\mathfrak{G}_l = (c_1, c_2, \dots, c_r)$ ,  $\mathfrak{G}_r = (d_1, d_2, \dots, d_r)$  the left and right *grounds* of  $A$ , and the total space  $\mathfrak{G} = (c_1, \dots, c_r, d_1, \dots, d_r)$  will be called the (total) ground of  $A$ .

If  $x$  is any vector in the subspace  $\mathfrak{N}_r = (d'_{r+1}, d'_{r+2}, \dots, d'_n)$  reciprocal to  $\mathfrak{G}_r$ , then  $Ax = 0$  since  $Sd_i d'_j = 0$  ( $i \neq j$ ). Conversely, if

$$0 = Ax = \sum c_i Sd_i x,$$

each multiplier  $Sd_i x$  must be 0 since the  $c$ 's are linearly independent; hence every solution of  $Ax = 0$  lies in  $\mathfrak{N}_r$ . Similarly every solution of  $A'x = 0$  lies in  $\mathfrak{N}_l = (c'_{r+1}, c'_{r+2}, \dots, c'_n)$ . We call  $\mathfrak{N}_r$  and  $\mathfrak{N}_l$  the right and left *nullspaces* of  $A$ ; their order,  $n - r$ , is called the *nullity* of  $A$ .

We may summarize these results as follows.

**THEOREM 7.** *If a matrix  $A$  is expressed in the form  $\sum_1^r a_i S b_i$ , where  $\mathfrak{G}_l = (a_1, a_2, \dots, a_r)$  and  $\mathfrak{G}_r = (b_1, b_2, \dots, b_r)$  define spaces of order  $r$ , then, no matter how the reduction to this form is carried out, the spaces  $\mathfrak{G}_l$  and  $\mathfrak{G}_r$  are always the same. Further, if  $\mathfrak{N}_l$  and  $\mathfrak{N}_r$  are the spaces of order  $n - r$  reciprocal to  $\mathfrak{G}_l$  and  $\mathfrak{G}_r$ , respectively, every solution of  $Ax = 0$  lies in  $\mathfrak{N}_r$  and every solution of  $A'x = 0$  in  $\mathfrak{N}_l$ .*

The following theorem is readily deduced from Theorem 7 and its proof is left to the reader.

**THEOREM 8.** *If  $A, B$  are matrices of rank  $r, s$ , the rank of  $A + B$  is not greater than  $r + s$  and the rank of  $AB$  is not greater than the smaller of  $r$  and  $s$ .*

**1.11 Linear dependence.** The definition of the rank of a matrix in the preceding section was made in terms of the linear dependence of vectors associated with the matrix. In this section we consider briefly the theory of linear dependence introducing incidentally a notation which we shall require later.

Let  $x_i = \sum_{j=1}^n \xi_{ij}e_j$  ( $i = 1, 2, \dots, r; r \leq n$ ) be a set of  $r$  vectors. From the rectangular array of their coordinates

$$(34) \quad \begin{array}{cccc} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \xi_{r1} & \xi_{r2} & \cdots & \xi_{rn} \end{array}$$

there can be formed  $n!/r!(n-r)!$  different determinants of order  $r$  by choosing  $r$  columns out of (34), these columns being taken in their natural order. If these determinants are arranged in some definite order, we may regard them as the coordinates of a vector in space of order  $n!/r!(n-r)!$  and, when this is done, we shall denote this vector by<sup>4</sup>

$$(35) \quad |x_1 x_2 \cdots x_r|$$

and call it a *pure vector of grade  $r$* . It follows from this definition that  $|x_1 x_2 \cdots x_r|$  has many of the properties of a determinant; its sign is changed if two  $x$ 's are interchanged, it vanishes when two  $x$ 's are equal and, if  $\lambda$  and  $\mu$  are scalars,

$$(36) \quad |(\lambda x_1 + \mu x_1') x_2 \cdots x_r| = \lambda |x_1 x_2 \cdots x_r| + \mu |x_1' x_2 \cdots x_r|.$$

If we replace the  $x$ 's in (35) by  $r$  different units  $e_{i_1}, e_{i_2}, \dots, e_{i_r}$ , the result is clearly not 0: we thus obtain  $\binom{n}{r}$  vectors which we shall call the *fundamental unit vectors of grade  $r$* ; and any linear combination of these units, say

$$\sum \xi_{i_1 i_2 \dots i_r} |e_{i_1} e_{i_2} \cdots e_{i_r}|,$$

is called a *vector of grade  $r$* . It should be noticed that not every vector is a pure vector except when  $r$  equals 1 or  $n$ .

If we replace  $x_i$  by  $\sum \xi_{ij}e_j$  in (35), we get

$$|x_1 x_2 \cdots x_r| = \sum \xi_{1j_1} \xi_{2j_2} \cdots \xi_{rj_r} |e_{j_1} e_{j_2} \cdots e_{j_r}|,$$

where the summation extends over all permutations  $j_1, j_2, \dots, j_r$  of  $1, 2, \dots, n$  taken  $r$  at a time. This summation may be effected by grouping together the

<sup>4</sup> If it had been advisable to use here the indeterminate product of Grassmann, (35) would appear as a determinant in much the ordinary sense (cf. §5.09).

sets  $j_1, j_2, \dots, j_r$  which are permutations of the same combination  $i_1, i_2, \dots, i_r$ , whose members may be taken to be arranged in natural order, and then summing these partial sums over all possible combinations  $i_1, i_2, \dots, i_r$ . Taking the first step only we have

$$\Sigma \xi_{1j_1} \xi_{2j_2} \cdots \xi_{rj_r} | e_{j_1} e_{j_2} \cdots e_{j_r} | = \Sigma \delta_{j_1 \cdots j_r}^{i_1 \cdots i_r} \xi_{1i_1} \cdots \xi_{ri_r} | e_{i_1} e_{i_2} \cdots e_{i_r} |$$

where  $\delta_{j_1 \cdots j_r}^{i_1 \cdots i_r}$  is the sign corresponding to the permutations  $(\begin{smallmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{smallmatrix})$  and this equals  $|\xi_{1i_1} \cdots \xi_{ri_r}| |e_{i_1} \cdots e_{i_r}|$ . We have therefore

$$(37) \quad |x_1 x_2 \cdots x_r| = \sum_{(i)}^* |\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}| |e_{i_1} e_{i_2} \cdots e_{i_r}|,$$

where the asterisk on  $\Sigma$  indicates that the sum is taken over all  $r$ -combinations of  $1, 2, \dots, n$  each combination being arranged in natural order.

**THEOREM 9**  $|x_1 x_2 \cdots x_r| = 0$  if, and only if,  $x_1, x_2, \dots, x_r$  are linearly dependent.

The first part of this theorem is an immediate consequence of (36). To prove the converse it is sufficient to show that, if  $|x_1 x_2 \cdots x_{r-1}| \neq 0$ , then there exist scalars  $\alpha_1, \alpha_2, \dots, \alpha_{r-1}$  such that

$$x_r = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{r-1} x_{r-1}.$$

Let  $x_i = \sum_j \xi_{ij} e_j$ . Since  $|x_1 x_2 \cdots x_{r-1}| \neq 0$ , at least one of its coordinates is not 0, and for convenience we may suppose without loss of generality that

$$(38) \quad |\xi_{11} \xi_{22} \cdots \xi_{r-1, r-1}| \neq 0.$$

Since  $|x_1 x_2 \cdots x_r| = 0$ , all its coordinates equal 0 and in particular

$$|\xi_{11} \xi_{22} \cdots \xi_{r-1, r-1} \xi_{ri}| = 0 \quad (i = 1, 2, \dots, n).$$

If we expand this determinant according to the elements of its last column, we get a relation of the form

$$\beta_1 \xi_{ri} + \beta_2 \xi_{2i} + \cdots + \beta_r \xi_{r-1, i} = 0$$

where the  $\beta$ 's are independent of  $i$  and  $\beta_1 \neq 0$  by (38). Hence we may write

$$(39) \quad \xi_{ri} = \alpha_1 \xi_{1i} + \cdots + \alpha_{r-1} \xi_{r-1, i} \quad (i = 1, 2, \dots, n)$$

the  $\alpha$ 's being independent of  $i$ . Multiplying (39) by  $e_i$  and summing with regard to  $i$ , we have

$$x_r = \alpha_1 x_1 + \cdots + \alpha_{r-1} x_{r-1},$$

which proves the theorem.

If  $(a_1, a_2, \dots, a_m)$  is a linear set of order  $r$ , then some set of  $r$   $a$ 's form a basis, that is, are linearly independent while each of the other  $a$ 's is linearly

dependent on them. By a change of notation, if necessary, we may take  $a_1, a_2, \dots, a_r$  as this basis and write

$$(40) \quad a_{r+i} = \sum_{j=1}^r \beta_{ij} a_j, \quad (i = 1, 2, \dots, m-r).$$

We shall now discuss the general form of all linear relations among the  $a$ 's in terms of the special relations (40); and in doing so we may assume the order of the space to be equal to or greater than  $m$  since we may consider any given space as a subspace of one of arbitrarily higher dimensionality.

Let

$$(41) \quad \sum_1^m \gamma_j a_j = 0$$

be a relation connecting the  $a$ 's and set

$$c = \sum_1^m \gamma_j e_j.$$

Then (40), considered as a special case of (41), corresponds to setting for  $c$

$$(42) \quad c_i = - \sum_{j=1}^r \beta_{ij} e_j + e_{r+i}, \quad (i = 1, 2, \dots, m-r);$$

and there is clearly no linear relation connecting these vectors so that they define a linear set of order  $m-r$ . Using (40) in (41) we have

$$\sum_{j=1}^r \left( \gamma_j + \sum_{i=1}^{m-r} \gamma_{r+i} \beta_{ij} \right) a_j = 0$$

and, since  $a_1, a_2, \dots, a_r$  are linearly independent, we have

$$j = - \sum_{i=1}^{m-r} \beta_{ij} \gamma_{r+i} \quad (j = 1, 2, \dots, r)$$

whence

$$(43) \quad c = \sum_1^m \gamma_j e_j = - \sum_{i=1}^{m-r} \gamma_{r+i} \sum_{j=1}^r \beta_{ij} e_j + \sum_{i=1}^{m-r} \gamma_{r+i} e_{r+i} = \sum_{i=1}^{m-r} \gamma_{r+i} c_i,$$

so that  $c$  is linearly dependent on  $c_1, c_2, \dots, c_{m-r}$ . Conversely, on retracing these steps in the reverse order we see that, if  $c$  is linearly dependent on these vectors, so that  $\gamma_{r+i}$  ( $i = 1, 2, \dots, m-r$ ) are known, then from (43) the  $\gamma_j$  ( $j = 1, 2, \dots, r$ ) are defined in such a way that  $c = \sum_1^m \gamma_j e_j$  and  $\sum_1^m \gamma_j a_j =$

0. We have therefore the following theorem.

**THEOREM 10.** *If  $a_1, a_2, \dots, a_m$  is a linear set of order  $r$ , there exist  $m - r$  linear relations  $\sum_{j=1}^m \gamma_j a_j = 0$  ( $i = 1, 2, \dots, m - r$ ) such that (i) the vectors  $c_i = \sum_{j=1}^m \gamma_j e_j$  are linearly independent and (ii) if  $\sum \gamma_j a_j = 0$  is any linear relation connecting the  $a$ 's, and if  $c = \sum \gamma_j e_j$ , then  $c$  belongs to the linear set  $(c_1, c_2, \dots, c_{m-r})$ .*

This result can be translated immediately in terms concerning the solution of a system of ordinary linear equations or in terms of matrices. If  $a_j = \sum_i a_{ij} e_i$ , then (41) may be written

$$\begin{aligned}
 & a_{11}\gamma_1 + a_{21}\gamma_2 + \dots + a_{m1}\gamma_m = 0 \\
 & \dots\dots\dots \\
 & a_{1n}\gamma_1 + a_{2n}\gamma_2 + \dots + a_{mn}\gamma_m = 0
 \end{aligned}
 \tag{44}$$

a system of linear homogeneous equations in the unknowns  $\gamma_1, \gamma_2, \dots, \gamma_m$ . Hence (44) has solutions for which some  $\gamma_i \neq 0$  if, and only if, the rank  $r$  of the array

$$\begin{array}{cccc}
 a_{11} & a_{21} & \dots & a_{m1} \\
 a_{12} & a_{22} & \dots & a_{m2} \\
 \cdot & \cdot & \dots & \cdot \\
 a_{1n} & a_{2n} & \dots & a_{mn}
 \end{array}
 \tag{45}$$

is less than  $m$  and, when this condition is satisfied, every solution is linearly dependent on the set of  $m - r$  solutions given by (42) which are found by the method given in the discussion of Theorem 9.

Again, if we make (45) a square array by the introduction of columns or rows of zeros and set  $A = || a_{ij} ||$ ,  $c = \sum \gamma_j e_j$ , then (41) becomes  $A'c = 0$  and Theorem 10 may therefore be interpreted as giving the properties of the null-space of  $A'$  which were derived in §1.10.

## CHAPTER II

### ALGEBRAIC OPERATIONS WITH MATRICES. THE CHARACTERISTIC EQUATION

**2.01 Identities.** The following elementary considerations enable us to carry over a number of results of ordinary scalar algebra into the algebra of matrices. Suppose  $f(\lambda_1, \lambda_2, \dots, \lambda_r)$ ,  $g(\lambda_1, \lambda_2, \dots, \lambda_r)$  are integral algebraic functions of the scalar variables  $\lambda_i$  with scalar coefficients, and suppose that

$$f(\lambda_1, \lambda_2, \dots, \lambda_r) = g(\lambda_1, \lambda_2, \dots, \lambda_r)$$

is an algebraic identity; then, when  $f(\lambda_1, \dots, \lambda_r) - g(\lambda_1, \dots, \lambda_r)$  is reduced to the standard form of a polynomial, the coefficients of the various powers of the  $\lambda$ 's are zero. In carrying out this reduction no properties of the  $\lambda$ 's are used other than those which state that they obey the laws of scalar multiplication and addition: if then we replace  $\lambda_1, \lambda_2, \dots, \lambda_r$  by *commutative* matrices  $x_1, x_2, \dots, x_r$ , the reduction to the form 0 is still valid step by step and hence

$$f(x_1, x_2, \dots, x_r) = g(x_1, x_2, \dots, x_r).$$

An elementary example of this is

$$(1 - x^2) = (1 - x)(1 + x)$$

or, when  $xy = yx$ ,

$$x^2 - y^2 = (x - y)(x + y).$$

Here, if  $xy \neq yx$ , the reader should notice that the analogue of the algebraic identity becomes

$$x^2 - y^2 = x(x + y) - (x + y)y,$$

which may also be written  $x^2 - y^2 = (x - y)(x + y) + (yx - xy)$ .

**2.02 Matric polynomials in a scalar variable.** By a matric polynomial in a scalar variable  $\lambda$  is meant a matrix that can be expressed in the form

$$(1) \quad P(\lambda) = p_0\lambda^r + p_1\lambda^{r+1} + \dots + p_r \quad (p_0 \neq 0),$$

where  $p_0, p_1, \dots, p_r$  are constant matrices. The coordinates of  $P(\lambda)$  are scalar polynomials in  $\lambda$  and hence, if

$$(2) \quad Q(\lambda) = q_0\lambda^s + q_1\lambda^{s-1} + \dots + q_s \quad (q_0 \neq 0)$$

is also a matric polynomial,  $P(\lambda) \equiv Q(\lambda)$  if, and only if,  $r = s$  and the coefficients of corresponding powers of  $\lambda$  are equal, that is,  $p_i = q_i$  ( $i = 1, 2, \dots, r$ ). If  $|q_0| \neq 0$ , the degree of the product  $P(\lambda)Q(\lambda)$  (or  $Q(\lambda)P(\lambda)$ ) is exactly  $r + s$  since the coefficient of the highest power  $\lambda^{r+s}$  which occurs in the product is  $p_0q_0$

(or  $q_0 p_0$ ) which cannot be 0 if  $p_0 \neq 0$  and  $|q_0| \neq 0$ . If, however, both  $|p_0|$  and  $|q_0|$  are 0, the degree of the product may well be less than  $r + s$ , as is seen from the examples

$$(e_{11}\lambda + 1)(e_{22}\lambda + 1) = e_{11}e_{22}\lambda^2 + (e_{11} + e_{22})\lambda + 1 = (e_{11} + e_{22})\lambda + 1,$$

$$\left\| \begin{array}{cc} \lambda & 1 \\ \lambda & 1 \end{array} \right\| \left\| \begin{array}{cc} 1 & -1 \\ -\lambda & \lambda \end{array} \right\| = 0.$$

Another noteworthy difference between matrix and scalar polynomials is that, when the determinant of a matrix polynomial is a constant different from 0, its inverse is also a matrix polynomial: for instance

$$(e_{12}\lambda + 1)^{-1} = -e_{12}\lambda + 1,$$

$$[(e_{12} + e_{23})\lambda + 1]^{-1} = e_{13}\lambda^2 - (e_{12} + e_{23})\lambda + 1.$$

We shall call such polynomials *elementary polynomials*.

**2.03 The division transformation.** The greater part of the theory of the division transformation can be extended from ordinary algebra to the algebra of matrices; the main precaution that must be taken is that it must not be assumed that every element of the algebra has an inverse and that due allowance must be made for the peculiarities introduced by the lack of commutativity in multiplication.

**THEOREM 1.** *If  $P(\lambda)$  and  $Q(\lambda)$  are the polynomials defined by (1) and (2), and if  $|q_0| \neq 0$ , there exist unique polynomials  $S(\lambda)$ ,  $R(\lambda)$ ,  $S_1(\lambda)$ ,  $R_1(\lambda)$ , of which  $S$  and  $S_1$  if not zero, are of degree  $r - s$  and the degrees of  $R$  and  $R_1$  are  $s - 1$  at most, such that*

$$P(\lambda) \equiv S(\lambda)Q(\lambda) + R(\lambda) \equiv Q(\lambda)S_1(\lambda) + R_1(\lambda).$$

If  $r < s$ , we may take  $S_1 = S = 0$  and  $R_1 = R = P$ ; in so far as the existence of these polynomials is concerned the theorem is therefore true in this case. We shall now assume as a basis for a proof by induction that the theorem is true for polynomials of degree less than  $r$  and that  $r \leq s$ . Since  $|q_0| \neq 0$ ,  $q_0^{-1}$  exists and, as in ordinary scalar division, we have

$$P(\lambda) - p_0 q_0^{-1} \lambda^{r-s} Q(\lambda) = (p_1 - p_0 q_0^{-1} q_1) \lambda^{r-1} + \dots = P_1(\lambda).$$

Since the degree of  $P_1$  is less than  $r$ , we have by hypothesis  $P_1(\lambda) = P_2(\lambda)Q(\lambda) + R(\lambda)$ , the degrees of  $P_2$  and  $R$  being less, respectively, than  $r - s$  and  $s$ ; hence

$$P(\lambda) = (p_0 q_0^{-1} \lambda^{r-s} + P_2(\lambda))Q(\lambda) + R(\lambda) = S(\lambda)Q(\lambda) + R(\lambda)$$

as required by the theorem. The existence of the right hand quotient and remainder follows in the same way.

It remains to prove the uniqueness of  $S$  and  $R$ . Suppose, if possible, that  $P = SQ + R = TQ + U$  where  $R$  and  $S$  are as above and  $T, U$  are poly-

nomials the degree of  $U$  being less than  $s$ ; then  $(S - T)Q = U - R$ . If  $S - T \neq 0$ , then, since  $|q_0| \neq 0$ , the degree of the polynomial  $(S - T)Q$  is at least as great as that of  $Q$  and is therefore greater than the degree of  $U - R$ . It follows immediately that  $S - T = 0$ , and hence also  $U - R = 0$ ; which completes the proof of the theorem.

If  $Q$  is a scalar polynomial, that is, if its coefficients  $q$  are scalars, then  $S = S_1$ ,  $R = R_1$ ; and, if the division is exact, then  $Q(\lambda)$  is a factor of each of the coordinates of  $P(\lambda)$ .

**THEOREM 2.** *If the matrix polynomial (1) is divided on the right by  $\lambda - a$ , the remainder is*

$$p_0 a^r + p_1 a^{r-1} + \cdots + p_r$$

and, if it is divided on the left, the remainder is

$$a^r p_0 + a^{r-1} p_1 + \cdots + p_r.$$

As in ordinary algebra the proof follows immediately from the identity

$$\lambda^s - a^s = (\lambda - a)(\lambda^{s-1} + \lambda^{s-2}a + \cdots + a^{s-1})$$

in which the order of the factors is immaterial since  $\lambda$  is a scalar.

If  $P(\lambda)$  is a scalar polynomial, the right and left remainders are the same and are conveniently denoted by  $P(a)$ .

2.04 Theorem 1 of the preceding section holds true as regards the existence of  $S$ ,  $S_1$ ,  $R$ ,  $R_1$ , and the degree of  $R$ ,  $R_1$  even when  $|q_0| = 0$  provided  $|Q(\lambda)| \neq 0$ . Suppose the rank of  $q_0$  is  $t < n$ ; then by §1.10 it has the form  $\sum_1^t \alpha_i S \beta_i$

or, say,  $h \left( \sum_1^t e_{ii} \right) k$  where  $h$  and  $k$  are non-singular matrices for which  $h e_i = \alpha_i$ ,

$k' e_i = \beta_i$  ( $i = 1, 2, \dots, t$ ). If  $c_1 = \sum_{i=1}^n e_{ii}$ , then

$$(3) \quad Q_1 = (c_1 \lambda + 1) h^{-1} Q$$

is a polynomial whose degree is not higher than the degree  $s$  of  $Q$  since  $c_1 h^{-1} q_0 = 0$  so that the term in  $\lambda^{s+1}$  is absent. Now, if  $\eta = |h^{-1}|$ , then

$$|Q_1| = |c_1 \lambda + 1| |h^{-1}| |Q| = (1 + \lambda)^n - t \eta |Q|,$$

so that the degree of  $|Q_1|$  is greater than that of  $|Q|$  by  $n - t$ . If the leading coefficient of  $Q_1$  is singular, this process may be repeated, and so on, giving  $Q_1, Q_2, \dots$ , where the degree of  $|Q_i|$  is greater than that of  $|Q_{i-1}|$ . But the degree of each  $Q_i$  is less than or equal to  $s$  and the degree of the determinant of a polynomial of the  $s$ th degree cannot exceed  $ns$ . Hence at some stage the leading coefficient of, say,  $Q_j$  is not singular and, from the law of formation (3) of the successive  $Q$ 's, we have  $Q_j(\lambda) = H(\lambda)Q(\lambda)$ , where  $H(\lambda)$  is a matrix polynomial.

By Theorem 1,  $Q_i$  taking the place of  $Q$ , we can find  $S^*$  and  $R$ , the latter of degree  $s - 1$  at most, such that

$$P(\lambda) = S^*(\lambda)H(\lambda)Q(\lambda) + R(\lambda) = S(\lambda)Q(\lambda) + R(\lambda).$$

The theorem is therefore true even if  $|q_0| = 0$  except that the quotient and remainder are not necessarily unique and the degree of  $S$  may be greater than  $r - s$ , as is shown by taking  $P = \lambda^2 - 1$ ,  $Q = e_{11}\lambda + 1$ , when we have

$$P = (e_{22}\lambda^2 + e_{11}\lambda - 1)Q = (e_{22}\lambda^2 + e_{11}\lambda - 1 + e_{12})Q - e_{12}.$$

**2.05 The characteristic equation.** If  $x$  is a matrix, the scalar polynomial

$$(4) \quad f(\lambda) = |\lambda - x| = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$$

is called the *characteristic function* corresponding to  $x$ . We have already seen (§1.05 (15)) that the product of a matrix and its adjoint equals its determinant; hence

$$(\lambda - x) \operatorname{adj}(\lambda - x) = |\lambda - x| = f(\lambda).$$

It follows that the polynomial  $f(\lambda)$  is exactly divisible by  $\lambda - x$  so that by the remainder theorem (§2.03, Theorem 2)

$$(5) \quad f(x) = 0.$$

As a simple example of this we may take  $x = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$ . Here

$$f(\lambda) = (\lambda - \alpha)(\lambda - \delta) - \beta\gamma = \lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma,$$

and

$$f(x) = \begin{vmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \gamma\alpha + \delta\gamma & \gamma\beta + \delta^2 \end{vmatrix} - (\alpha + \delta) \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} + (\alpha\delta - \beta\gamma) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0.$$

The following theorem is an important extension of this result.

**THEOREM 3.** If  $f(\lambda) = |\lambda - x|$  and  $\theta(\lambda)$  is the highest common factor of the first minors of  $|\lambda - x|$ , and if

$$(6) \quad \varphi(\lambda) = f(\lambda)/\theta(\lambda),$$

the leading coefficient of  $\theta(\lambda)$  being 1 (and therefore also that of  $\varphi(\lambda)$ ), then

- (i)  $\varphi(x) = 0$ ;
- (ii) if  $\psi(\lambda)$  is any scalar polynomial such that  $\psi(x) = 0$ , then  $\varphi(\lambda)$  is a factor of  $\psi(\lambda)$ , that is,  $\varphi(\lambda)$  is the scalar polynomial of lowest degree and with leading coefficient 1 such that  $\varphi(x) = 0$ ;
- (iii) every root of  $f(\lambda)$  is a root of  $\varphi(\lambda)$ .

The coordinates of  $\operatorname{adj}(\lambda - x)$  are the first minors of  $|\lambda - x|$  and therefore by hypothesis  $[\operatorname{adj}(\lambda - x)]/\theta(\lambda)$  is integral; also

$$\frac{\operatorname{adj}(\lambda - x)}{\theta(\lambda)} (\lambda - x) = \frac{f(\lambda)}{\theta(\lambda)} = \varphi(\lambda);$$

hence  $\varphi(x) = 0$  by the remainder theorem.

If  $\psi(\lambda)$  is any scalar polynomial for which  $\psi(x) = 0$ , we can find scalar polynomials  $M(\lambda)$ ,  $N(\lambda)$  such that  $M(\lambda)\varphi(\lambda) + N(\lambda)\psi(\lambda) \equiv \zeta(\lambda)$ , where  $\zeta(\lambda)$  is the highest common factor of  $\varphi$  and  $\psi$ . Substituting  $x$  for  $\lambda$  in this scalar identity and using  $\varphi(x) = 0 = \psi(x)$  we have  $\zeta(x) = 0$ ; if, therefore,  $\psi(x) = 0$  is a scalar equation of lowest degree satisfied by  $x$ , we must have  $\psi(\lambda) = \zeta(\lambda)$ , apart from a constant factor, so that  $\psi(\lambda)$  is a factor of  $\varphi(\lambda)$ , say

$$(7) \quad \varphi(\lambda) = h(\lambda)\psi(\lambda).$$

Since  $\psi(x) = 0$ ,  $\lambda - x$  is a factor of  $\psi(\lambda)$ , say  $\psi(\lambda) = (\lambda - x)g(\lambda)$ , where  $g$  is a matric polynomial; hence

$$\psi(\lambda) = \frac{\varphi(\lambda)}{h(\lambda)} = \frac{f(\lambda)}{h(\lambda)\theta(\lambda)} = (\lambda - x)g(\lambda).$$

Hence

$$g(\lambda) = \frac{f(\lambda)}{\theta(\lambda)h(\lambda)(\lambda - x)} = \frac{\text{adj}(\lambda - x)}{\theta(\lambda)h(\lambda)}$$

and this cannot be integral unless  $h(\lambda)$  is a constant in view of the fact that  $\theta(\lambda)$  is the highest common factor of the coordinates of  $\text{adj}(\lambda - x)$ ; it follows that  $\psi(\lambda)$  differs from  $\varphi(\lambda)$  by at most a constant factor.

A repetition of the first part of this argument shows that, if  $\psi(x) = 0$  is any scalar equation satisfied by  $x$ , then  $\varphi(\lambda)$  is a factor of  $\psi(\lambda)$ .

It remains to show that every root of  $f(\lambda)$  is a root of  $\varphi(\lambda)$ . If  $\lambda_1$  is any root of  $f(\lambda) = |\lambda - x|$ , then from  $\varphi(\lambda) = g(\lambda)(\lambda - x)$  we have

$$\varphi(\lambda_1) = g(\lambda_1)(\lambda_1 - x)$$

so that the determinant,  $[\varphi(\lambda_1)]^n$ , of the scalar matrix  $\varphi(\lambda_1)$  equals  $|g(\lambda_1)| |\lambda_1 - x|$ , which vanishes since  $|\lambda_1 - x| = f(\lambda_1)$ . This is only possible if  $\varphi(\lambda_1) = 0$ , that is, if every root of  $f(\lambda)$  is also a root of  $\varphi(\lambda)$ .

The roots of  $f(\lambda)$  are also called the *roots*<sup>1</sup> of  $x$ ,  $\varphi(\lambda)$  is called the *reduced characteristic function* of  $x$ , and  $\varphi(x) = 0$  the *reduced equation* of  $x$ .

2.06 A few simple results are conveniently given at this point although they are for the most part merely particular cases of later theorems. If  $g(\lambda)$  is a scalar polynomial, then on dividing by  $\varphi(\lambda)$ , whose degree we shall denote by  $\nu$ , we may set  $g(\lambda) = q(\lambda)\varphi(\lambda) + r(\lambda)$ , where  $q$  and  $r$  are polynomials the degree of  $r$  being less than  $\nu$ . Replacing  $\lambda$  by  $x$  in this identity and remembering that  $\varphi(x) = 0$ , we have<sup>2</sup>  $g(x) = r(x)$ , that is, any polynomial can be replaced by an equivalent polynomial of degree less than  $\nu$ .

<sup>1</sup> They are also called the *latent* roots of  $x$ .

<sup>2</sup> If  $g(\lambda)$  is a matric polynomial whose coefficients are not all commutative with  $x$ , the meaning of  $g(x)$  is ambiguous; for instance,  $x$  may be placed on the right of the coefficients, or it may be put on the left. For such a polynomial we can say in general that it can be replaced by an equal polynomial in which no power of  $x$  higher than the  $(\nu - 1)$ th occurs.

If  $g(\lambda)$  is a scalar polynomial which is a factor of  $\varphi(\lambda)$ , say  $\varphi(\lambda) = h(\lambda)g(\lambda)$ , then  $0 = \varphi(x) = h(x)g(x)$ . It follows that  $|g(x)| = 0$ ; for if this were not so, we should have  $h(x) = [g(x)]^{-1}\varphi(x) = 0$ , whereas  $x$  can satisfy no scalar equation of lower degree than  $\varphi$ . Hence, if  $g(\lambda)$  is a scalar polynomial which has a factor in common with  $\varphi(x)$ , then  $g(x)$  is singular.

If a scalar polynomial  $g(\lambda)$  has no factor in common with  $\varphi(\lambda)$ , there exist scalar polynomials  $M(\lambda), N(\lambda)$  such that  $M(\lambda)g(\lambda) + N(\lambda)\varphi(\lambda) \equiv 1$ . Hence  $M(x)g(x) = 1$ , or  $[g(x)]^{-1} = M(x)$ . It follows immediately that any finite rational function of  $x$  with scalar coefficients can be expressed as a scalar polynomial in  $x$  of degree  $\nu - 1$  at most. It should be noticed carefully however that, if  $x$  is a variable matrix, the coefficients of the reduced polynomial will in general contain the variable coordinates of  $x$  and will not be integral in these unless the original function is integral. It follows also that  $g(x)$  is singular only when  $g(\lambda)$  has a factor in common with  $\varphi(\lambda)$ .

Finally we may notice here that similar matrices have the same reduced equation; for, if  $g$  is a scalar polynomial,  $g(y^{-1}xy) = y^{-1}g(x)y$ . As a particular case of this we have that  $xy$  and  $yx$  have the same reduced equation if, say,  $y$  is non-singular; for  $xy = y^{-1} \cdot yx \cdot y$ . If both  $x$  and  $y$  are singular, it can be shown<sup>3</sup> that  $xy$  and  $yx$  have the same characteristic equation, but not necessarily the same reduced equation as is seen from the example  $x = e_{12}, y = e_{22}$ .

**2.07 Matrices with distinct roots.** Because of its importance and comparative simplicity we shall investigate the form of a matrix all of whose roots are different before considering the general case. Let

$$(8) \quad f(\lambda) = |\lambda - x| = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where no two roots are equal and set

$$(9) \quad f_i(\lambda) = \frac{(\lambda - \lambda_1) \cdots (\lambda - \lambda_{i-1})(\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_n)}{(\lambda_i - \lambda_1) \cdots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \cdots (\lambda_i - \lambda_n)} = \frac{f(\lambda)/f'(\lambda_i)}{\lambda - \lambda_i}.$$

By the Lagrange interpolation formula  $\sum_i f_i(\lambda) \equiv 1$ ; hence

$$(10) \quad f_1(x) + f_2(x) + \cdots + f_n(x) = 1.$$

Further,  $f(\lambda)$  is a factor of  $f_i(\lambda)f_j(\lambda)$  ( $i \neq j$ ) so that

$$(11) \quad f_i(x)f_j(x) = 0 \quad (i \neq j);$$

hence multiplying (10) by  $f_i(x)$  and using (11) we have

$$(12) \quad [f_i(x)]^2 = f_i(x).$$

Again,  $(\lambda - \lambda_i)f_i(\lambda) = f(\lambda)/f'(\lambda_i)$ ; hence  $(x - \lambda_i)f_i(x) = 0$ , that is,

$$(13) \quad xf_i(x) = \lambda_i f_i(x),$$

<sup>3</sup> For example, by replacing  $y$  by  $y + \delta$ ,  $\delta$  being a scalar, and considering the limiting case when  $\delta$  approaches 0.

whence, summing with regard to  $i$  and using (10), we have

$$(14) \quad x = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots + \lambda_n f_n(x).$$

If we form  $x^r$  from (14),  $r$  being a positive integer, it is immediately seen from (11) and (12), or from the Lagrange interpolation formula, that

$$(15) \quad x^r = \lambda_1^r f_1 + \lambda_2^r f_2 + \cdots + \lambda_n^r f_n,$$

where  $f_i$  stands for  $f_i(x)$ , and it is easily verified by actual multiplication that, if no root is 0,

$$x^{-1} = \lambda_1^{-1} f_1 + \lambda_2^{-1} f_2 + \cdots + \lambda_n^{-1} f_n$$

so that (15) holds for negative powers also. The matrices  $f_i$  are linearly independent. For if  $\Sigma \gamma_i f_i = 0$ , then

$$0 = f_i \Sigma \gamma_i f_i = \gamma_i f_i^2 = \gamma_i f_i$$

whence every  $\gamma_i = 0$  seeing that in the case we are considering  $f(\lambda)$  is itself the reduced characteristic function so that  $f_i(x) \neq 0$ .

From these results we have that, if  $g(\lambda)$  is any scalar rational function whose denominator has no factor in common with  $\varphi(\lambda)$ , then

$$(16) \quad g(x) = g(\lambda_1) f_1 + g(\lambda_2) f_2 + \cdots + g(\lambda_n) f_n.$$

It follows from this that the roots of  $g(x)$  are  $g(\lambda_i)$  ( $i = 1, 2, \dots, n$ ). For setting  $y = g(x)$ ,  $\mu_i = g(\lambda_i)$ , we have as above

$$\psi(y) = \Sigma \psi(\mu_i) f_i,$$

$\psi(\lambda)$  being a scalar polynomial. Now  $\psi(y) f_i = \psi(\mu_i) f_i$ ; hence, if  $\psi(y) = 0$ , then also  $\psi(\mu_i) = 0$  ( $i = 1, 2, \dots, n$ ); and conversely. Hence if the notation is so chosen that  $\mu_1, \mu_2, \dots, \mu_r$  are the distinct values of  $\mu_i$ , the reduced characteristic function of  $y = g(x)$  is

$$\prod_1^r (\lambda - \mu_i).$$

**2.08** If the determinant  $|\lambda - x| = f(\lambda)$  is expanded in powers of  $\lambda$ , it is easily seen<sup>4</sup> that the coefficient  $a_r$  of  $\lambda^{n-r}$  is  $(-1)^r$  times the sum of the principal minors of  $x$  of order  $r$ ; this coefficient is therefore a homogeneous polynomial of degree  $r$  in the coordinates of  $x$ . In particular,  $-a_1$  is the sum of the coordinates in the main diagonal: this sum is called the *trace* of  $x$  and is denoted by  $\text{tr } x$ .

If  $y$  is an arbitrary matrix,  $\mu$  a scalar variable, and  $z = x + \mu y$ , the coefficients of the characteristic equation of  $z$ , say

$$(17) \quad z^n + b_1 z^{n-1} + \cdots + b_n = 0,$$

<sup>4</sup> For instance, by differentiating  $|\lambda - x|$   $n - r$  times with respect to  $\lambda$  and then setting  $\lambda = 0$ .

are polynomials in  $\mu$  of the form

$$(18) \quad b_s = a_{s0} + \mu a_{s1} + \cdots + \mu^s a_{ss}, \quad (a_{s0} = a_s, a_{00} = 1)$$

and the powers of  $z$  are also polynomials in  $\mu$ , say

$$(19) \quad z^r = x^r + \mu \begin{Bmatrix} x & y \\ r-1 & 1 \end{Bmatrix} + \mu^2 \begin{Bmatrix} x & y \\ r-2 & 2 \end{Bmatrix} + \cdots + \mu^r y^r$$

where  $\begin{Bmatrix} x & y \\ s & t \end{Bmatrix}$  is obtained by multiplying  $s$   $x$ 's and  $t$   $y$ 's together in every possible way and adding the terms so obtained, e.g.,

$$\begin{Bmatrix} x & y \\ 2 & 1 \end{Bmatrix} = x^2y + xyx + yx^2.$$

If we substitute (18) and (19) in (17) and arrange according to powers of  $\mu$ , then, since  $\mu$  is an independent variable, the coefficients of its several powers must be zero. This leads to a series of relations connecting  $x$  and  $y$  of the form

$$(20) \quad \sum_{i,j} a_{ij} \begin{Bmatrix} x & y \\ n-s-i+j & s-j \end{Bmatrix} = 0 \quad (s = 0, 1, 2, \dots)$$

where  $a_{ij}$  are the coefficients defined in (18) and  $\begin{Bmatrix} x & y \\ n-s-i+j & s-j \end{Bmatrix}$  is

replaced by 0 when  $j > s$ . In particular, if  $s = 1$ ,

$$\begin{Bmatrix} x & y \\ n-1 & 1 \end{Bmatrix} + a_1 \begin{Bmatrix} x & y \\ n-2 & 1 \end{Bmatrix} + \cdots + a_{n-1}y + a_{11}x^{n-1} + \cdots + a_{n1} = 0$$

which, when  $xy = yx$ , becomes

$$f'(x)y = -(a_{11}x^{n-1} + \cdots + a_{n1}) = g(x).$$

When  $x$  has no repeated roots,  $f'(\lambda)$  has no root in common with  $f(\lambda)$  and  $f'(x)$  has an inverse (cf. §2.06) so that  $y = g(x)/f'(x)$  which can be expressed as a scalar polynomial in  $x$ ; and conversely every such polynomial is commutative with  $x$ . We therefore have the following theorem:

**THEOREM 4.** *If  $x$  has no multiple roots, the only matrices commutative with it are scalar polynomials in  $x$ .*

**2.09 Matrices with multiple roots.** We shall now extend the main results of §2.07 to matrices whose roots are not necessarily simple. Suppose in the first place that  $x$  has only one distinct root and that its reduced characteristic function is  $\varphi(\lambda) = (\lambda - \lambda_1)^r$ , and set

$$\eta_1^i = \eta_i = (x - \lambda_1)^i = (x - \lambda_1)\eta_{i-1} \quad (i = 1, 2, \dots, r-1);$$

then

$$\eta_1^r = 0, \quad x\eta_{r-1} = \lambda_1\eta_{r-1}, \quad x\eta_i = \lambda_1\eta_i + \eta_{i+1} \quad (i = 1, 2, \dots, r-2)$$

and

$$x^s = (\lambda_1 + \eta_1)^s = \lambda_1^s + s\lambda_1^{s-1}\eta_1 + \binom{s}{2}\lambda_1^{s-2}\eta_1^2 + \dots$$

where the binomial expansion is cut short with the term  $\eta_1^{\nu-1}$  since  $\eta_1^\nu = 0$ . Again, if  $g(\lambda)$  is any scalar polynomial, then

$$g(x) = g(\lambda_1 + \eta_1) = g(\lambda_1) + g'(\lambda_1)\eta_1 + \dots + \frac{g^{(\nu-1)}(\lambda_1)}{(\nu-1)!} \eta_1^{\nu-1}.$$

It follows immediately that, if  $g^{(\kappa)}(\lambda)$  is the first derivative of  $g(\lambda)$  which is not 0 when  $\lambda = \lambda_1$  and  $(\kappa-1)s < \nu \leq \kappa s$ , then the reduced equation of  $g(x)$  is

$$[g(x) - g(\lambda_1)]^\kappa = 0.$$

It should be noted that the first  $\nu-1$  powers of  $\eta_1$  are linearly independent since  $\varphi(\lambda)$  is the reduced characteristic function of  $x$ .

2.10 We shall now suppose that  $x$  has more than one root. Let the reduced characteristic function be

$$(21) \quad \varphi(\lambda) = \prod_{i=1}^r (\lambda - \lambda_i)^{\nu_i} \quad (\sum \nu_i = \nu, r > 1)$$

and set

$$(22) \quad h_i(\lambda) = \varphi(\lambda)/(\lambda - \lambda_i)^{\nu_i}.$$

We can determine two scalar polynomials,  $M_i(\lambda)$  and  $N_i(\lambda)$ , of degrees not exceeding  $\nu_i - 1$  and  $\nu - \nu_i - 1$ , respectively, such that

$$M_i(\lambda)h_i(\lambda) + (\lambda - \lambda_i)^{\nu_i}N_i(\lambda) \equiv 1, \quad M_i(\lambda_i) \neq 0.$$

If we set

$$(23) \quad \varphi_i(\lambda) = M_i(\lambda)h_i(\lambda),$$

then  $1 - \sum \varphi_i(\lambda)$  is exactly divisible by  $\varphi(\lambda)$  and, being of degree  $\nu - 1$  at most, must be identically 0; hence

$$(24) \quad \sum_1^r \varphi_i(\lambda) = 1.$$

Again, from (22) and (23),  $\varphi(\lambda)$  is a factor of  $\varphi_i(\lambda)\varphi_j(\lambda)$  ( $i \neq j$ ) and hence on multiplying (24) by  $\varphi_i(\lambda)$  we have

$$(25) \quad [\varphi_i(\lambda)]^2 \equiv \varphi_i(\lambda), \quad \varphi_i(\lambda)\varphi_j(\lambda) \equiv 0, \text{ mod } \varphi(\lambda) \quad (i \neq j).$$

Further, if  $g(\lambda)$  is a scalar polynomial, then

$$(26) \quad \begin{aligned} g(x) &= \sum_1^r g(\lambda)\varphi_i(\lambda) \\ &= \sum_1^r [g(\lambda_i) + g'(\lambda_i)(\lambda - \lambda_i) + \dots + \frac{g^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!} (\lambda - \lambda_i)^{\nu_i-1}] \varphi_i(\lambda) + R \end{aligned}$$

where  $R$  has the form  $\Sigma C_i(\lambda)(\lambda - \lambda_i)^{\nu_i}\varphi_i(\lambda)$ ,  $C_i$  being a polynomial, so that  $R$  vanishes when  $x$  is substituted for  $\lambda$ .

2.11 If we put  $x$  for  $\lambda$  in (23) and set  $\varphi_i$  for  $\varphi_i(x)$ , then (24) and (25) show that

$$(27) \quad \varphi_i^2 = \varphi_i, \quad \varphi_i\varphi_j = 0 \quad (i \neq j), \quad \sum_1^r \varphi_i = 1.$$

It follows as in §2.07 that the matrices  $\varphi_i$  are linearly independent and none is zero, since  $\varphi_i(\lambda_i) \neq 0$  so that  $\varphi(\lambda)$  is not a factor of  $\varphi_i(\lambda)$ , which would be the case were  $\varphi_i(x) = 0$ . We now put  $x$  for  $\lambda$  in (26) and set

$$(28) \quad \eta_i = (x - \lambda_i)\varphi_i \quad (i = 1, 2, \dots, r).$$

Since the  $\nu_i$ th power of  $(\lambda - \lambda_i)\varphi_i(\lambda)$  is the first which has  $\varphi(\lambda)$  as a factor,  $\eta_i$  is a nilpotent matrix of index  $\nu_i$  (cf. §1.05) and, remembering that  $\varphi_i^2 = \varphi_i$ , we have

$$(29) \quad \eta_i^j = (x - \lambda_i)^j\varphi_i \neq 0 \quad (j < \nu_i), \quad \eta_i\varphi_i = \eta_i = \varphi_i\eta_i,$$

$$(30) \quad x\varphi_i = \lambda_i\varphi_i + \eta_i, \quad x\eta_i^j = \lambda_i\eta_i^j + \eta_i^{j+1},$$

equation (26) therefore becomes

$$(31) \quad g(x) = \sum_1^r \left[ g(\lambda_i)\varphi_i + g'(\lambda_i)\eta_i + \dots + \frac{g^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!} \eta_i^{\nu_i-1} \right]$$

and in particular

$$(32) \quad x = \sum_1^r (\lambda_i\varphi_i + \eta_i) = \Sigma x_i.$$

The matrices  $\varphi_i$  and  $\eta_i$  are called the *principal idempotent* and *nilpotent elements* of  $x$  corresponding to the root  $\lambda_i$ . The matrices  $\varphi_i$  are uniquely determined by the following conditions: if  $\psi_i$  ( $i = 1, 2, \dots, r$ ) are any matrices such that

$$(33) \quad \begin{aligned} & \text{(i) } x\psi_i = \psi_i x, \\ & \text{(ii) } (x - \lambda_i)\psi_i \text{ is nilpotent,} \\ & \text{(iii) } \sum_i \psi_i = 1, \quad \psi_i^2 = \psi_i \neq 0, \end{aligned}$$

then  $\psi_i = \varphi_i$  ( $i = 1, 2, \dots, r$ ). For let  $\theta_{ij} = \varphi_i\psi_j$ ; from (i)  $\theta_{ij}$  also equals  $\psi_j\varphi_i$ . From (ii) and (28)

$$\eta_i = x\varphi_i - \lambda_i\varphi_i, \quad \xi_j = x\psi_j - \lambda_j\psi_j$$

are both nilpotent and, since  $\eta_i$  and  $\varphi_i$  are polynomials in  $x$ , they are commutative with  $\psi_j$  and therefore with  $\xi_j$ ; also

$$\begin{aligned} x\theta_{ij} &= \lambda_i\theta_{ij} + (x - \lambda_i)\varphi_i\psi_j = \lambda_i\theta_{ij} + \eta_i\psi_j \\ &= \lambda_j\theta_{ij} + (x - \lambda_j)\varphi_i\psi_j = \lambda_j\theta_{ij} + \xi_j\varphi_i. \end{aligned}$$

Hence  $(\lambda_i - \lambda_j)\theta_{ij} = \xi_j\varphi_i - \eta_i\psi_j$ . But if  $\mu$  is the greater of the indices of  $\xi_j$  and  $\eta_i$ , then, since all the matrices concerned are commutative, each term of  $(\xi_j\varphi_i - \eta_i\psi_j)^{2\mu}$  contains  $\xi_j^\mu$  or  $\eta_i^\mu$  as a factor and is therefore 0. If  $\theta_{ij} \neq 0$ , this is impossible when  $i \neq j$  since  $\theta_{ij}$  is idempotent and  $\lambda_i - \lambda_j \neq 0$ . Hence  $\varphi_i\psi_j = 0$  when  $i \neq j$  and from (iii)

$$\psi_j = \psi_j\Sigma\varphi_i = \psi_j\varphi_j = \varphi_j\Sigma\psi_i = \varphi_j$$

which proves the uniqueness of the  $\varphi$ 's.

2.12 We shall now determine the reduced equation of  $g(x)$ . If we set  $g_i$  for  $g(x)\varphi_i$ , then

$$(34) \quad \begin{aligned} g_i &= g(\lambda_i)\varphi_i + g'(\lambda_i)\eta_i + \cdots + \frac{g^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!} \eta_i^{\nu_i-1} \\ &= g(\lambda_i)\varphi_i + \zeta_i, \end{aligned}$$

say, and if  $s_i$  is the order of the first derivative in (34) which is not 0, then  $\zeta_i$  is a nilpotent matrix whose index  $k_i$  is given by  $k_i = 1 < \nu_i/s_i \leq k_i$ .

If  $\Phi(\lambda)$  is a scalar polynomial, and  $\gamma_i = g(\lambda_i)$ ,

$$\Phi(g(x)) = \sum \Phi(g_i)\varphi_i = \sum \left[ \Phi(\gamma_i)\varphi_i + \Phi'(\gamma_i)\zeta_i + \cdots + \frac{\Phi^{(k_i-1)}(\gamma_i)}{(k_i-1)!} \zeta_i^{k_i-1} \right]$$

so that  $\Phi(g(x)) = 0$  if, and only if,  $g(\lambda_i)$  is a root of  $\Phi(\lambda)$  of multiplicity  $k_i$ . Hence, if

$$\Psi(\lambda) = \Pi[\lambda - g(\lambda_i)]^{k_i}$$

where when two or more values of  $i$  give the same value of  $g(\lambda_i)$ , only that one is to be taken for which  $k_i$  is greatest, then  $\Psi(\lambda)$  is the reduced characteristic function of  $g(x)$ . As a part of this result we have the following theorem.

**THEOREM 5.** *If  $g(\lambda)$  is a scalar polynomial and  $x$  a matrix whose distinct roots are  $\lambda_1, \lambda_2, \dots, \lambda_r$ , the roots of the matrix  $g(x)$  are<sup>5</sup>*

$$g(\lambda_1), g(\lambda_2), \dots, g(\lambda_r).$$

If the roots  $g(\lambda_i)$  are all distinct, the principal idempotent elements of  $g(x)$  are the same as those of  $x$ ; for condition (33) of §2.11 as applied to  $g(x)$  are satisfied by  $\varphi_i$  ( $i = 1, 2, \dots, r$ ), and these conditions were shown to characterize the principal idempotent elements completely.

2.13 **The square root of a matrix.** Although the general question of functions of a matrix will not be taken up till a later chapter, it is convenient to give here one determination of the square root of a matrix  $x$ .

<sup>5</sup> That these are roots of  $g(x)$  follows immediately from the fact that  $\lambda - x$  is a factor of  $g(\lambda) - g(x)$ ; but it does not follow so readily from this that the only roots are those given except, of course, when  $r = n$  and all the quantities  $g(\lambda_i)$  are distinct.

If  $\alpha$  and  $\beta$  are scalars,  $\alpha \neq 0$ , and  $(\alpha + \beta)^{\frac{1}{2}}$  is expanded formally in a Taylor series,

$$(\alpha + \beta)^{\frac{1}{2}} = \alpha^{\frac{1}{2}} \sum_0^{\infty} \delta_r \left(\frac{\beta}{\alpha}\right)^r$$

then, if  $S_\nu = \alpha^{\frac{1}{2}} \sum_0^{\nu-1} \delta_r (\beta/\alpha)^r$ , it follows that

$$(35) \quad S_\nu^2 = \alpha + \beta + \alpha T_\nu$$

where  $T_\nu$  is a polynomial in  $\beta/\alpha$  which contains no power of  $\beta/\alpha$  lower than the  $\nu$ th. If  $a$  and  $b$  are commutative matrices and  $a$  is the square of a known non-singular matrix  $a^{\frac{1}{2}}$ , then (35) being an algebraic identity in  $\alpha$  and  $\beta$  remains true when  $a$  and  $b$  are put in their place.

If  $x_i = \lambda_i \varphi_i + \eta_i$  is the matrix defined in §2.11 (32), then so long as  $\lambda_i \neq 0$ , we may set  $\alpha = \lambda_i \varphi_i$ ,  $\beta = \eta_i$  since  $\lambda_i \varphi_i = (\lambda_i^{\frac{1}{2}} \varphi_i)^2$ ; and in this case the Taylor series terminates since  $\eta_i^{\nu_i} = 0$ , that is,  $T_{\nu_i} = 0$  and the square of the terminating series for  $(\lambda_i \varphi_i + \eta_i)^{\frac{1}{2}}$  in powers of  $\eta_i$  equals  $\lambda_i \varphi_i + \eta_i$ . It follows immediately from (32) and (27) that, if  $x$  is a matrix no one of whose roots is 0, the square of the matrix

$$(36) \quad x^{\frac{1}{2}} = \sum_1^r \lambda_i^{\frac{1}{2}} \left[ \varphi_i + \frac{1}{2} \lambda_i^{-1} \eta_i - \dots + (-1)^{\nu_i-2} \frac{(2\nu_i - 4)!}{2^{2\nu_i-3} (\nu_i - 2)! (\nu_i - 1)!} \left(\frac{\eta_i}{\lambda_i}\right)^{\nu_i-1} \right]$$

is  $x$ .

If the reduced equation of  $x$  has no multiple roots, (36) becomes

$$(37) \quad x^{\frac{1}{2}} = \Sigma \lambda_i^{\frac{1}{2}} \varphi_i$$

and this is valid even if one of the roots is 0. If, however, 0 is a multiple root of the reduced equation,  $x$  may have no square root as, for example, the matrix  $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$ .

Formula (36) gives  $2^r$  determinations of  $x^{\frac{1}{2}}$  but we shall see later that an infinity of determinations is possible in certain cases.

**2.14 Reducible matrices.** If  $x = x_1 + x_2$  is the direct sum of  $x_1$  and  $x_2$  and  $e_1, e_2$  are the corresponding idempotent elements, that is,

$$e_i x = x_i = x_i, \quad e_i e_j = 0 \quad (i \neq j; i, j = 1, 2),$$

then  $x^r = x_1^r + x_2^r$  ( $r \geq 2$ ) and we may set as before  $1 = x^0 = x_1^0 + x_2^0 = e_1 + e_2$ . Hence, if  $f(\lambda) = \lambda^m + b_1 \lambda^{m-1} + \dots + b_m$  is any scalar polynomial, we have

$$f(x) = e_1 f(x_1) + e_2 f(x_2) = f(x_1) + f(x_2) - b_m,$$

and if  $g(\lambda)$  is a second scalar polynomial

$$f(x)g(x) = e_1f(x_1)g(x_1) + e_2f(x_2)g(x_2).$$

Now if  $f_i(\lambda)$  is the reduced characteristic function of  $x_i$  regarded as a matrix in the space determined by  $e_i$ , then the reduced characteristic function of  $x_i$  as a matrix in the original fundamental space is clearly  $\lambda f_i(\lambda)$  unless  $\lambda$  is a factor of  $f_i(\lambda)$  in which case it is simply  $f_i(\lambda)$ . Further the reduced characteristic function of  $x = x_1 + x_2$  is clearly the least common multiple of  $f_1(\lambda)$  and  $f_2(\lambda)$ ; for if

$$\psi(\lambda) = f_1(\lambda)g_1(\lambda) = f_2(\lambda)g_2(\lambda)$$

then

$$\begin{aligned} \psi(x_1 + x_2) &= e_1\psi(x_1) + e_2\psi(x_2) \\ &= e_1f_1(x_1)g_1(x_1) + e_2f_2(x_2)g_2(x_2) = 0. \end{aligned}$$

## CHAPTER III

### INVARIANT FACTORS AND ELEMENTARY DIVISORS

**3.01 Elementary transformations.** By an elementary transformation of a matrix polynomial  $a(\lambda) = || a_{ij} ||$  is meant one of the following operations on the rows or columns.

**Type I.** The operation of adding to a row (column) a different row (column) multiplied by a scalar polynomial  $\theta(\lambda)$ .

**Type II.** The operation of interchanging two rows (columns).

**Type III.** The operation of multiplying a row (column) by a constant  $k \neq 0$ .

These transformations can be performed algebraically as follows.

*Type I.* Let

$$P_{ij} = 1 + \theta(\lambda)e_{ij} \quad (i \neq j),$$

$\theta(\lambda)$  being a scalar polynomial; then  $|P_{ij}| = 1$  and

$$P_{ij}a = \sum_{p,q} a_{pq}e_{pq} + \theta \sum_q a_{jq}e_{iq}$$

which is the matrix derived from  $a(\lambda)$  by adding  $\theta$  times the  $j$ th row to the  $i$ th. The corresponding operation on the columns is equivalent to forming the product  $aP_{ji}$ .

*Type II.* Let  $Q_{ij}$  be the matrix

$$Q_{ij} = 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji} \quad (i \neq j)$$

that is,  $Q_{ij}$  is the matrix derived from the identity matrix by inserting 1 in place of 0 in the coefficients of  $e_{ij}$  and  $e_{ji}$  and 0 in place of 1 in the coefficients of  $e_{ii}$  and  $e_{jj}$ ; then  $|Q_{ij}| = -1$  and

$$Q_{ij}a = \sum_{p,q} a_{pq}e_{pq} - \sum_q a_{iq}e_{iq} - \sum_q a_{jq}e_{jq} + \sum_q a_{jq}e_{iq} + \sum_q a_{iq}e_{jq}$$

that is,  $Q_{ij}a$  is derived from  $a$  by interchanging the  $i$ th and  $j$ th rows. Similarly  $aQ_{ij}$  is obtained by interchanging the  $i$ th and  $j$ th columns.

Since any permutation can be effected by a succession of transpositions, the corresponding transformation in the rows (columns) of a matrix can be produced by a succession<sup>1</sup> of transformations of Type II.

*Type III.* This transformation is effected on the  $r$ th row (column) by multiplying on the left (right) by  $R = 1 + (k - 1)e_{rr}$ ; it is used only when it is convenient to make the leading coefficient in some term equal to 1.

<sup>1</sup> The transformation corresponding to the substitution  $\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$  is  $Q = \sum_i e_{ip_i}$ .

The inverses of the matrices used in these transformations are

$$P_{ij}^{-1} = 1 - \theta e_{ij}, \quad Q_{ij}^{-1} = Q_{ij}, \quad R^{-1} = 1 + (k^{-1} - 1)e_{rr};$$

these inverses are elementary transformations. The transverses are also elementary since  $P'_{ij} = P_{ji}$ , and  $Q_{ij}$  and  $R$  are symmetric.<sup>2</sup>

A matrix polynomial  $b(\lambda)$  which is derived from  $a(\lambda)$  by a sequence of elementary transformations is said to be *equivalent* to  $a(\lambda)$ ; every such polynomial has the form  $p(\lambda)a(\lambda)q(\lambda)$  where  $p$  and  $q$  are products of elementary transformations. Since the inverse of an elementary transformation is elementary,  $a(\lambda)$  is also equivalent to  $b(\lambda)$ . Further, the inverses of  $p$  and  $q$  are polynomials so that these are what we have already called elementary polynomials; we shall see later that every elementary polynomial can be derived from 1 by a sequence of elementary transformations.

In the following sections we require two lemmas whose proofs are almost immediate.

LEMMA 1. *The rank<sup>3</sup> of a matrix is not altered by an elementary transformation.*

For if  $|P| \neq 0$ ,  $AP$  and  $PA$  have the same rank as  $A$  (§1.10).

LEMMA 2. *The highest common factor of the coordinates of a matrix polynomial is not altered by an elementary transformation.*

This follows immediately from the definition of elementary transformations.

3.02 The normal form of a matrix. The theorem we shall prove in this section is as follows.

THEOREM 1. *If  $a(\lambda)$  is a matrix polynomial of rank  $r$ , it can be reduced by elementary transformations to a diagonal matrix*

$$(1) \quad \sum_1^r \alpha_i(\lambda)e_{ii} = \left\| \begin{array}{ccccccc} \alpha_1(\lambda) & & & & & & \\ & \alpha_2(\lambda) & & & & & \\ & & \ddots & & & & \\ & & & \alpha_r(\lambda) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array} \right\| = P(\lambda)a(\lambda)Q(\lambda),$$

<sup>2</sup> The definition of an elementary transformation given above is the most convenient but not the only possible one. All three transformations have the form  $T = 1 + xSy$  with the condition that  $1 + Sxy$  is not 0 and is independent of  $\lambda$ .

<sup>3</sup> By the rank of a matrix polynomial is meant the order of the highest minor which does not vanish identically. For particular values of  $\lambda$  the rank may be smaller than  $r$ ; there are always values of  $\lambda$  for which it equals  $r$  and it cannot be greater.

where the coefficient of the highest power of  $\lambda$  in each polynomial  $\alpha_i(\lambda)$  is 1,  $\alpha_i$  is a factor of  $\alpha_{i+1}, \dots, \alpha_r$  ( $i = 1, 2, \dots, r-1$ ), and  $P(\lambda), Q(\lambda)$  are elementary polynomials.

We shall first show that, if the coordinate of  $a(\lambda)$  of minimum degree  $m$ , say  $a_{pq}$ , is not a factor of every other coordinate, then  $a(\lambda)$  is equivalent to a matrix in which the degree of the coordinate of minimum degree is less than  $m$ .

Suppose that  $a_{pq}$  is not a factor of  $a_{pi}$  for some  $i$ ; then we may set  $a_{pi} = \beta a_{pq} + a'_{pi}$  where  $\beta$  is integral and  $a'_{pi}$  is not 0 and is of lower degree than  $m$ . Subtracting  $\beta$  times the  $q$ th column from the  $i$ th we have an equivalent matrix in which the coordinate<sup>4</sup>  $(p, i)$  is  $a'_{pi}$  whose degree is less than  $m$ . The same reasoning applies if  $a_{pq}$  is not a factor of every coordinate  $a_{iq}$  in the  $q$ th column.

After a finite number of such steps we arrive at a matrix in which a coordinate of minimum degree, say  $k_{pq}$ , is a factor of all the coordinates which lie in the same row or column, but is possibly not a factor of some other coordinate  $k_{ij}$ . When this is so, let  $k_{pj} = \beta k_{pq}, k_{iq} = \gamma k_{pq}$  where  $\beta$  and  $\gamma$  are integral. If we now add  $(1 - \beta)$  times the  $q$ th column to the  $j$ th,  $(p, j)$  and  $(i, j)$  become respectively

$$k'_{pj} = k_{pj} + (1 - \beta)k_{pq} = k_{pq}, \quad k'_{ij} = k_{ij} + (1 - \beta)k_{iq} = k_{ij} + (1 - \beta)\gamma k_{pq}.$$

Here either the degree of  $k'_{ij}$  is less than that of  $k_{pq}$ , or  $k'_{pj}$  has the minimum degree and is not a factor of  $k'_{ij}$ , which lies in the same column, and hence the minimum degree can be lowered as above.

The process just described can be repeated so long as the coordinate of lowest degree is not a factor of every other coordinate and, since each step lowers the minimum degree, we derive in a finite number of steps a matrix  $\|b'_{ij}\|$  which is equivalent to  $a(\lambda)$  and in which the coordinate of minimum degree is in fact a divisor of every other coordinate; and further we may suppose that  $b'_{11} = \alpha_1(\lambda)$  is a coordinate of minimum degree and set  $b'_{1i} = \gamma_i b'_{11}, b'_{j1} = \delta_j b'_{11}$ . Subtracting  $\gamma_i$  times the first column from the  $i$ th and then  $\delta_j$  times the first row from the  $j$ th ( $i, j = 2, 3, \dots, n$ ) all the coordinates in the first row and column except  $b'_{11}$  become 0, and we have an equivalent matrix in the form

$$(2) \quad \begin{array}{cccc} \alpha_1(\lambda) & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & b_{32} & b_{33} & \cdots & b_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & b_{n2} & b_{n3} & \cdots & b_{nn} \end{array}$$

in which  $\alpha_1$  is a factor of every  $b_{ij}$ . The coefficient of the highest power of  $\lambda$  in  $\alpha_1$  may be made 1 by a transformation of type III.

The theorem now follows readily by induction. For, assuming it is true for

<sup>4</sup> That is, the coordinate in the  $p$ th row and  $i$ th column.



The definitions of this and the preceding sections have all been made relative to the fundamental basis. But we have seen in §1.08 that, if  $a_1$  is the matrix with the same array of coordinates as  $a$  but relative to another basis, then there exists a non-singular constant matrix  $b$  such that  $a = b^{-1}a_1b$  so that  $a$  and  $a_1$  are equivalent matrices. In terms of the new basis  $a_1$  has the same invariant factors as  $a$  does in terms of the old and  $a$ , being equivalent to  $a_1$ , has therefore the same invariant factors in terms of the new basis as it has in the old. Hence the invariant and determinantal factors of a matrix polynomial are independent of the (constant) basis in terms of which its coordinates are expressed.

The results of this section may be summarized as follows.

**THEOREM 2.** *Two matrix polynomials are equivalent if, and only if, they have the same invariant factors.*

**3.04 Non-singular linear polynomials.** In the case of linear polynomials Theorem 2 can be made more precise as follows.

**THEOREM 3.** *If  $a\lambda + b$  and  $c\lambda + d$  are non-singular linear polynomials which have the same invariant factors, and if  $|c| \neq 0$ , there exist non-singular constant matrices  $p$  and  $q$  such that*

$$p(a\lambda + b)q = c\lambda + d.$$

We have seen in Theorem 2 that there exist elementary polynomials  $P(\lambda)$ ,  $Q(\lambda)$  such that

$$(3) \quad c\lambda + d = P(\lambda)(a\lambda + b)Q(\lambda).$$

Since  $|c| \neq 0$ , we can employ the division transformation to find matrix polynomials  $p_1, q_1$  and constant matrices  $p, q$  such that

$$P(\lambda) = (c\lambda + d)p_1 + p, \quad Q(\lambda) = q_1(c\lambda + d) + q.$$

Using this in (3) we have

$$(4) \quad c\lambda + d = p(a\lambda + b)q + (c\lambda + d)p_1(a\lambda + b)Q + P(a\lambda + b)q_1(c\lambda + d) - (c\lambda + d)p_1(a\lambda + b)q_1(c\lambda + d)$$

and, since from (3)

$$(a\lambda + b)Q = P^{-1}(c\lambda + d), \quad P(a\lambda + b) = (c\lambda + d)Q^{-1},$$

we may write in place of (4)

$$(5) \quad p(a\lambda + b)q = [1 - (c\lambda + d)(p_1P^{-1} + Q^{-1}q_1 - p_1(a\lambda + b)q_1)](c\lambda + d) = [1 - (c\lambda + d)R](c\lambda + d)$$

where  $R = pP_1^{-1} + Q^{-1}q_1 - p_1(a\lambda + b)q_1$ , which is integral in  $\lambda$  since  $P$  and  $Q$  are elementary. If  $R \neq 0$ , then, since  $|c| \neq 0$ , the degree of the right side

of (5) is at least 2, whereas the degree of the left side is only 1; hence  $R = 0$  so that (5) gives  $p(a\lambda + b)q = c\lambda + d$ . Since  $c\lambda + d$  is not singular, neither  $p$  nor  $q$  can be singular, and hence the theorem is proved.

When  $|c| = 0$  (and therefore also  $|a| = 0$ ) the remaining conditions of Theorem 3 are not sufficient to ensure that we can find constant matrices in place of  $P$  and  $Q$ , but these conditions are readily modified so as to apply to this case also. If we replace  $\lambda$  by  $\lambda/\mu$  and then multiply by  $\mu$ ,  $a\lambda + b$  is replaced by the homogeneous polynomial  $a\lambda + b\mu$ ; and the definition of invariant factors applies immediately to such polynomials. In fact, if  $|a| \neq 0$ , the invariant factors of  $a\lambda + b\mu$  are simply the homogeneous polynomials which are equivalent to the corresponding invariant factors of  $a\lambda + b$ . If, however,  $|a| = 0$ , then  $|a\lambda + b\mu|$  is divisible by a power of  $\mu$  which leads to factors of the form  $\mu^i$  in the invariant factors of  $a\lambda + b\mu$  which have no counterpart in those of  $a\lambda + b$ .

If  $|c| = 0$  but  $|c\lambda + d| \neq 0$ , there exist values,  $\lambda_1 \neq 0$ ,  $\mu_1$ , such that  $|c\lambda_1 + d\mu_1| \neq 0$  and, if we make the transformation

$$(6) \quad \lambda = \lambda_1\alpha, \quad \mu = \mu_1\alpha + \beta,$$

$a\lambda + b\mu$ ,  $c\lambda + d\mu$  become  $a_1\alpha + b_1\beta$ ,  $c_1\alpha + d_1\beta$  where  $a_1 = a\lambda_1 + b\mu_1$ ,  $c_1 = c\lambda_1 + d\mu_1$ , and therefore  $|c_1| \neq 0$ . Further, when  $a\lambda + b\mu$  and  $c\lambda + d\mu$  have the same invariant factors, this is also true of  $a_1\alpha + b_1\beta$  and  $c_1\alpha + d_1\beta$ . Since  $|c_1| \neq 0$ , the proof of Theorem 3 is applicable, so that there are constant non-singular matrices  $p, q$  for which  $p(a_1\alpha + b_1\beta)q = c_1\alpha + d_1\beta$ , and on reversing the substitution (6) we have

$$p(a\lambda + b\mu)q = c\lambda + d\mu.$$

Theorem 3 can therefore be extended as follows.

**THEOREM 4.** *If the non-singular polynomials  $a\lambda + b\mu$ ,  $c\lambda + d\mu$  have the same invariant factors, there exist non-singular constant matrices  $p, q$  such that  $p(a\lambda + b\mu)q = c\lambda + d\mu$*

An important particular case of Theorem 3 arises when the polynomials have the form  $\lambda - b$ ,  $\lambda - d$ . For if  $p(\lambda - b)q = \lambda - d$ , on equating coefficients we have  $pq = 1$ ,  $pbq = d$ ; hence  $b = p^{-1}dp$ , that is,  $b$  and  $d$  are similar. Conversely, if  $b$  and  $d$  are similar, then  $\lambda - b$  and  $\lambda - d$  are equivalent, and hence we have the following theorem.

**THEOREM 5.** *Two constant matrices  $b, d$  are similar if, and only if,  $\lambda - b$  and  $\lambda - d$  have the same invariant factors.*

**3.05 Elementary divisors.** If  $D = |a\lambda + b|$  is not identically zero and if  $\lambda_1, \lambda_2, \dots, \lambda_s$  are its distinct roots, say

$$D = (\lambda - \lambda_1)^{v_1}(\lambda - \lambda_2)^{v_2} \dots (\lambda - \lambda_s)^{v_s},$$







where  $a_i^k$  is the block of terms derived by forming the  $k$ th power of  $a_i$  regarded as a matrix of order  $\nu_i$ .

Since  $D_n$  equals  $|\lambda - a|$ , it is the characteristic function of  $a$  (or  $A$ ) and, since  $D_{n-1}$  is the highest common factor of the first minors, it follows from Theorem 3 of chapter 2 that  $\alpha_n$  is the reduced characteristic function.

If we add the  $f$ 's together in groups each group consisting of all the  $f$ 's that correspond to the same value of  $\lambda_i$ , we get a set of idempotent matrices, say  $\varphi_1, \varphi_2, \dots, \varphi_r$ , corresponding to the distinct roots of  $a$ , say  $\alpha_1, \alpha_2, \dots, \alpha_r$ . These are the principal idempotent elements of  $a$ ; for (i)  $a\varphi_i = \varphi_i a$ , (ii)  $(a - \alpha_i)\varphi_i$  is nilpotent, (iii)  $\Sigma\varphi_i = \Sigma f_i = 1$  and  $\varphi_i\varphi_j = 0$  ( $i \neq j$ ) so that the conditions of §2.11 are satisfied.

When the same root  $\alpha_i$  occurs in several elementary divisors, the corresponding  $f$ 's are called *partial idempotent elements* of  $a$ ; they are not unique as is seen immediately by taking  $a = 1$ .

If  $\alpha$  is one of the roots of  $A$ , the form of  $A - \alpha$  is sometimes important. Suppose that  $\lambda_1 = \lambda_2 = \dots = \lambda_p = \alpha$ ,  $\lambda_i \neq \alpha$  ( $i > p$ ) and set

$$b_i = a_i - \alpha f_i,$$

the corresponding array in the  $i$ th block of  $a - \alpha$  (cf. (10), (11)) being

$$(16) \quad \begin{array}{ccccccc} \lambda_i - \alpha & & & & & & 1 \\ & \lambda_i - \alpha & & & & & 1 \\ & & \ddots & & & & \vdots \\ & & & \ddots & & & \vdots \\ & & & & \ddots & & 1 \\ & & & & & \ddots & \vdots \\ & & & & & & \lambda_i - \alpha. \end{array}$$

In the case of the first  $p$  blocks  $\lambda_i - \alpha = 0$  and the corresponding  $b_1, b_2, \dots, b_p$  are nilpotent, the index of  $b_i$  being  $\nu_i$  and, assuming  $\nu_1 \geq \nu_2 > \dots \geq \nu_p$  as before,  $(A - \alpha)^k$  has the form

$$P^{-1}(A - \alpha)^k P = \left\| \begin{array}{cccc} b_1^k & & & \\ & b_2^k & & \\ & & \ddots & \\ & & & b_s^k \end{array} \right\|$$

or, when  $k \geq \nu_1$

$$(17) \quad P^{-1}(A - \alpha)^k P = \left\| \begin{array}{cccc} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \ddots \\ & & & & b_{p+1}^k \\ & & & & & \ddots \\ & & & & & & b_s^k \end{array} \right\|$$

Since none of the diagonal coordinates of  $b_{p+1}, \dots, b_s$  are 0, the rank of  $(A - \alpha)^k$ , when  $k \geq \nu_1$ , is exactly  $n - \sum_{i=1}^p \nu_i = \sum_{i=p+1}^s \nu_i$  and the nullspace of  $(A - \alpha)^k$  is then the same as that of  $(A - \alpha)^{\nu_1}$ . Hence, if there exists a vector  $z$  such that  $(A - \alpha)^k z = 0$  but  $(A - \alpha)^{k-1} z \neq 0$ , then (i)  $k \leq \nu_1$ , (ii)  $z$  lies in the nullspace of  $(A - \alpha)^{\nu_1}$ .

**3.08 Invariant vectors.** If  $A$  is a matrix with the elementary divisors given in the statement of Theorem 6, then  $\lambda - A$  is equivalent to  $\lambda - a$  and by Theorem 5 there is a non-singular matrix  $P$  such that  $A = PaP^{-1}$ . If we denote the unit vectors corresponding to the rows and columns of  $a_i$  in (10) by  $e_1^i, e_2^i, \dots, e_{\nu_i}^i$  and set

$$(18) \quad x_j^i = \begin{cases} Pe_j^i & (j = 1, 2, \dots, \nu_i; i = 1, 2, \dots, s) \\ 0 & (j < 1 \text{ or } > \nu_i, \text{ or } i < 1 \text{ or } > s) \end{cases}$$

then

$$ae_1^i = \lambda_i e_1^i, ae_2^i = \lambda_i e_2^i + e_1^i, \dots, ae_{\nu_i}^i = \lambda_i e_{\nu_i}^i + e_{\nu_i-1}^i$$

and hence

$$(19) \quad Ax_j^i = \lambda_i x_j^i + x_{j-1}^i \quad (j = 1, 2, \dots, \nu_i; i = 1, 2, \dots, s).$$

The vectors  $x_j^i$  are called a set of *invariant vectors*<sup>6</sup> of  $A$ .

The matrix  $A$  can be expressed in terms of its invariant vectors as follows. We have from (10)

$$a_i = \sum_j (\lambda_i e_j^i + e_{j-1}^i) S e_j^i = \sum_j e_j^i S (\lambda_i e_j^i + e_{j+1}^i)$$

and hence, if

$$(20) \quad y_j^i = (P')^{-1} e_j^i = (PP')^{-1} x_j^i,$$

then

$$(21) \quad A = \sum_{i,j} (\lambda_i x_j^i + x_{j-1}^i) S y_j^i = \sum_{i,j} x_j^i S (\lambda_i y_j^i + y_{j+1}^i)$$

where it should be noted that the  $y$ 's form a system reciprocal to the  $x$ 's and that each of these systems forms a basis of the vector space since  $|P| \neq 0$ .

If we form the transverse of  $A$ , we have from (21)

$$(22) \quad A' = \sum_{i,j} (\lambda_i y_j^i + y_{j+1}^i) S x_j^i$$

<sup>6</sup> If homogeneous coordinates are used so that vectors represent points, an invariant vector is usually called a pole.

so that the invariant vectors of  $A'$  are obtained by forming the system reciprocal to the  $x$ 's and inverting the order in each group of vectors corresponding to a given elementary divisor; thus

$$A'y_{\nu_i}^i = \lambda_i y_{\nu_i}^i, A'y_{\nu_i-1}^i = \lambda_i y_{\nu_i-1}^i + y_{\nu_i}^i, \dots, A'y_1^i = \lambda_i y_1^i + y_2^i.$$

A matrix  $A$  and its transverse clearly have the same elementary divisors and are therefore similar. The matrix which transforms  $A$  into  $A'$  can be given explicitly as follows. Let  $q_i$  be the symmetric array

$$\begin{matrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{matrix} \quad (\nu_i \text{ rows and columns}).$$

It is easily seen that  $q_i a_i = a'_i q_i$  and hence, if  $Q$  is the matrix

$$Q = \begin{vmatrix} q_1 & & & & \\ & q_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & q_s \end{vmatrix}$$

we have  $Qa = a'Q$ , and a short calculation gives  $A' = R^{-1}AR$  where  $R$  is the symmetric matrix

$$(23) \quad R = PQ^{-1}P' = PQP'.$$

If the elementary divisors of  $A$  are simple, then  $Q = 1$  and  $R = PP'$ .

If the roots  $\lambda_i$  of the elementary divisors (12) are all different, the nullity of  $(A - \lambda_i)$  is 1, and hence  $x_1^i$  is unique to a scalar multiplier. But the remaining  $x_j^i$  are not unique. In fact, if the  $x$ 's denote one choice of the invariant vectors, we may take in place of  $x_j^i$

$$z_j^i = k_1^i x_j^i + k_2^i x_{j-1}^i + \dots + k_j^i x_1^i \quad (j = 1, 2, \dots, \nu_i)$$

where the  $k$ 's are any constant scalars subject to the condition  $k_1^i \neq 0$ . Suppose now that  $\lambda_1 = \lambda_2 = \dots = \lambda_p = \alpha$ ,  $\lambda_i \neq \alpha$  ( $i > p$ ) and  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_p$  as in §3.07. We shall say that  $z_1, z_2, \dots, z_k$  is a *chain*<sup>7</sup> of invariant vectors belonging to the exponent  $k$  if

$$(24) \quad \begin{aligned} z_i &= (A - \alpha)^k - i z_k \neq 0 & (i = 1, 2, \dots, k) \\ (A - \alpha)^k z_k &= 0. \end{aligned}$$

It is also convenient to set  $z_i = 0$  for  $i < 0$  or  $> k$ . We have already seen that  $k \leq \nu_1$  and that  $z_k$  lies in the nullspace of  $(A - \alpha)^{\nu_1}$ ; and from (17) it is

<sup>7</sup> We shall say that the chain is *generated* by  $z_k$ .

seen that the nullspace of  $(A - \alpha)^{v_i}$  has the basis  $(x_j^i; j = 1, 2, \dots, \nu_i, i = 1, 2, \dots, p)$ .

Since  $z_k$  belongs to the nullspace of  $(A - \alpha)^{v_i}$ , we may set

$$(25) \quad z_k = \sum_{i=1}^p \sum_{j=1}^{\nu_i} \zeta_{ij} x_j^i$$

and therefore by repeated application of (15) with  $\lambda_i = \alpha$

$$(26) \quad (A - \alpha)^r z_k = \sum_{i,j} \zeta_{ij} x_j^i - 2.$$

From this it follows that, in order that  $(A - \alpha)^k z_k = 0$ , only values of  $j$  which are less than or equal to  $k$  can actually occur in (25) and in order that  $(A - \alpha)^{k-1} z_k \neq 0$  at least one  $\zeta_{ik}$  must be different from 0; hence

$$(27) \quad \begin{aligned} z_k &= \sum_i (\zeta_{ik} x_k^i + \zeta_{i,k-1} x_{k-1}^i + \dots) \\ z_{k-1} &= \sum_i (\zeta_{ik} x_{k-1}^i + \zeta_{i,k-1} x_{k-2}^i + \dots) \\ &\dots\dots\dots \\ z_1 &= \sum_i \zeta_{i1} x_1^i. \end{aligned}$$

Finally, if we impose the restriction that  $z_k$  does not belong to any chain pertaining to an exponent greater than  $k$ , it is necessary and sufficient that  $k$  be one of the numbers  $\nu_1, \nu_2, \dots, \nu_p$  and that no value of  $i$  corresponding to an exponent greater than  $k$  occur in (27).

3.09 The actual determination of the vectors  $x_j^i$  can be carried out by the processes of §3.02 and §3.04 or alternatively as follows. Suppose that the first  $s_1$  of the exponents  $\nu_i$  equal  $n_1$ , the next  $s_2$  equal  $n_2$ , and so on, and finally the last  $s_q$  equal  $n_q$ . Let  $\mathfrak{N}_1$  be the nullspace of  $(A - \alpha)^{n_1}$  and  $\mathfrak{N}'_1$  the nullspace of  $(A - \alpha)^{n_1-1}$ ; then  $\mathfrak{N}_1$  contains  $\mathfrak{N}'_1$ . If  $\mathfrak{M}_1$  is a space complementary to  $\mathfrak{N}'_1$  in  $\mathfrak{N}_1$ , then for any vector  $x$  in  $\mathfrak{M}_1$  we have  $(A - \alpha)^r x = 0$  only when  $r \geq n_1$ . Also, if  $x_1, x_2, \dots, x_{m_1}$  is a basis of  $\mathfrak{M}_1$ , the vectors

$$(28) \quad (A - \alpha)^r x_i, (r = 0, 1, \dots, n_1 - 1)$$

are linearly independent; for, if

$$\sum_{r=s}^{n_1-1} \sum_i \xi_{ir} (A - \alpha)^r x_i = 0,$$

some  $\xi_{ir}$  being different from 0, then multiplying by  $(A - \alpha)^{n_1-s-1}$  we have

$$(A - \alpha)^{n_1-1} \sum_i \xi_{is} x_i = 0,$$

which is only possible if every  $\xi_i = 0$  since  $x_1, x_2, \dots, x_{m_1}$  form a basis of  $\mathfrak{M}_1$  and for no other vector of  $\mathfrak{M}_1$  is  $(A - \alpha)^{n_1 - 1} x = 0$ . The space defined by (28) clearly lies in  $\mathfrak{N}_1$ ; we shall denote it by  $\mathfrak{R}_1$ . If we set  $\mathfrak{N}_1 = \mathfrak{N}_2 + \mathfrak{R}_1$  where  $\mathfrak{N}_2$  is complementary to  $\mathfrak{R}_1$  in  $\mathfrak{N}_1$ , then  $\mathfrak{N}_2$  contains all vectors which are members of sets belonging to the exponents  $n_2, n_3, \dots$  but not lying in sets with the exponent  $n_1$ .

We now set  $\mathfrak{N}_2 = \mathfrak{N}'_2 + \mathfrak{M}_2$  where  $\mathfrak{N}'_2$  is the subspace of vectors  $x$  in  $\mathfrak{N}_2$  such that  $(A - \alpha)^{n_2 - 1} x = 0$ . As before the elements of  $\mathfrak{M}_2$  generate sets with exponent  $n_2$  but are not members of sets with higher exponents; and by a repetition of this process we can determine step by step the sets of invariant vectors corresponding to each exponent  $n_i$ .

## CHAPTER IV

### VECTOR POLYNOMIALS. SINGULAR MATRIC POLYNOMIALS

**4.01 Vector polynomials.** If a matric polynomial in  $\lambda$  is singular, the elements of its nullspace may depend on  $\lambda$ . We are therefore led to consider vectors whose coordinates are polynomials in a scalar variable  $\lambda$ ; such a vector is called a *vector polynomial*. Any vector polynomial can be put in the form

$$z(\lambda) = z_0\lambda^m + z_1\lambda^{m-1} + \cdots + z_m$$

where  $z_0, z_1, \dots, z_m$  are vectors whose coordinates are independent of  $\lambda$  and, if  $z_0 \neq 0$ ,  $m$  is called the degree of  $z(\lambda)$ . In a linear set with a basis composed of vector polynomials we are usually only concerned with those vectors that have integral coordinates when expressed in terms of the basis and, when this is so, we shall call the set an *integral set*. In a basis of an integral set the degree of an element of maximum degree will be called the degree of the basis.

In practice an integral set is often given in terms of a sequence of vectors which are not linearly independent and so do not form a basis. For the present therefore we shall say that the sequence of vector polynomials

$$(1) \quad z_1(\lambda), z_2(\lambda), \dots, z_k(\lambda)$$

defines the integral set of all vectors of the form  $\sum \zeta_i(\lambda)z_i(\lambda)$  where  $\zeta$ 's are scalar polynomials, and show later that this is really an integral set by finding for it an integral basis. The sequence (1) is said to have rank  $r$  if  $|z_i z_{i_2} \cdots z_{i_s}|$  vanishes identically in  $\lambda$  for all choices of  $s$   $z$ 's when  $s > r$  and is not identically 0 for some choice of the  $z$ 's when  $s = r$ .

The theory of integral sets can be expressed entirely in terms of matric polynomials, but it will make matters somewhat clearer not to do so at first. By analogy with matrices we define an elementary transformation of a sequence of vector polynomials as follows. An elementary transformation of the sequence (1) is the operation of replacing it by a sequence  $z'_1, z'_2, \dots, z'_k$  where:

$$\begin{aligned} \text{Type I: } z'_i &= z_i + \sum_{p \neq i} \zeta_p z_p, z'_q = z_q, (q \neq i), \\ \text{Type II: } z'_i &= z_j, z'_j = z_i, z'_q = z_q (q \neq i, j), \\ \text{Type III: } z'_p &= \rho_p z_p, (p = 1, 2, \dots, k) \end{aligned}$$

where the  $\zeta$ 's are scalar polynomials and the  $\rho$ 's constants none of which is 0.

The rank of a sequence is not altered by an elementary transformation, and two sequences connected by an elementary transformation are equivalent in the sense that every vector polynomial belonging to the integral set defined by the one also belongs to the integral set defined by the other.

Two sequences which can be derived the one from the other by elementary transformations are said to be equivalent. The corresponding integral sets

may also be said to be equivalent; and if only transformations with constant coefficients are used, the equivalence is said to be strict. Equivalence may also be defined as follows. If  $P$  is an elementary matrix which turns any vector of the integral linear set  $(z_1, z_2, \dots, z_k)$  into a vector of the same set, then it is easily shown that this set is equivalent to  $(Pz_1, Pz_2, \dots, Pz_k)$  and conversely; we also say that the linear set  $(z_1, z_2, \dots, z_k)$  is *invariant under  $P$*  although the individual elements of the basis are not necessarily unchanged. If the restriction that  $P$  leaves  $(z_1, z_2, \dots, z_k)$  invariant is not imposed, the two sets are said to be similar.

**4.02 The degree invariants.** We have seen in the previous section that the sequence in terms of which an integral set is defined may be transformed by elementary transformations without altering the integral set itself. We shall now show how we may choose a normalized basis and determine certain invariants connected with the set. Let the vectors (1), when written in full, be

$$(2) \quad z_i(\lambda) = \lambda^{m_i} z_{i0} + \lambda^{m_i - 1} z_{i1} + \dots + z_{im_i},$$

and suppose the notation so arranged that  $m_1 \leq m_2 \leq \dots \leq m_k$ . Suppose further that the leading coefficients  $z_{10}, z_{20}, \dots, z_{s-1,0}$  are linearly independent but that

$$z_{s0} = \sum_1^{s-1} \eta_i z_{i0},$$

the  $\eta$ 's being constants not all 0; then  $m_s \geq m_i$  ( $i = 1, 2, \dots, s-1$ ) and

$$z'_s = z_s - \sum_1^{s-1} \eta_i \lambda^{m_s - m_i} z_i$$

is either 0 or has a lower degree than  $z_s$ , and it may replace  $z_s$  in the sequence.

After a finite number of elementary transformations of this kind we arrive at a sequence equivalent to (1) which consists of a number  $p$  of vector polynomials  $x_1, x_2, \dots, x_p$  in which the leading coefficients are linearly independent followed by  $k - p$  zero-vectors. Now if we form  $|x_1 x_2 \dots x_p|$  using the notation of (2) with  $x$ 's in place of  $z$ 's, the term of highest degree is  $\lambda^{m_1 + \dots + m_p} |x_{10} x_{20} \dots x_{p0}|$ , which is not 0 since the leading vectors  $x_{10}, x_{20}, \dots, x_{p0}$  are linearly independent. But the rank of a sequence is not changed by elementary transformations; hence  $p = r$  and we have the following theorem.

**THEOREM 1.** *If  $z_1, z_2, \dots, z_k$  is a sequence of vector polynomials of rank  $r$ , the set of vectors of the form  $\sum_i \zeta_i(\lambda) z_i(\lambda)$ , the  $\zeta$ 's being scalar polynomials, form an integral set with a basis of order  $r$  which may be so chosen that the leading coefficients of its constituent vectors are linearly independent.*

When a basis of an integral set satisfies the conditions of this theorem and its elements are arranged in order of ascending degree, we shall call it a *normal basis*.

*Corollary.* If  $x_1, x_2, \dots, x_r$  is a normal basis with the degrees  $m_1 \leq m_2 \leq \dots \leq m_r$ , and if  $\xi_1, \xi_2, \dots, \xi_r$  are scalar polynomials, then the degree of the vector polynomial  $x = \sum_1^s \xi_i x_i$  ( $\xi_s \neq 0$ ) is not less than  $m_s$ .

**THEOREM 2.** If  $x_1, x_2, \dots, x_r$  is a normal basis of an integral set and  $m_1 \leq m_2 \leq \dots \leq m_r$  the corresponding degrees, and if  $y_1, y_2, \dots, y_r$  is any other basis with the degrees  $n_1 \leq n_2 \leq \dots \leq n_r$ , then

$$m_1 \leq n_1, m_2 \leq n_2, \dots, m_r \leq n_r.$$

Further, the exponents  $m_1, m_2, \dots, m_r$  are the same for all normal bases.

Let  $s$  be the first integer for which  $n_s < m_s$  so that  $n_i \leq n_s < m_s$  for  $i \leq s$ . Since  $(x_1, x_2, \dots, x_r)$  is a basis, we may set

$$y_i = \sum_p \xi_{ip}(\lambda) x_p(\lambda) \quad (i = 1, 2, \dots, s).$$

Here no value of  $p$  greater than  $s - 1$  is admissible since the degree  $n_i$  of  $y_i$  is less than  $m_s$ . This would mean that the rank of  $y_1, y_2, \dots, y_s$  was less than  $s$ , which is impossible since they form part of a basis. Hence  $m_s \leq n_s$  for all values of  $s$ .

If both bases are normal, it follows immediately that  $m_i \leq n_i$  and also  $n_i \leq m_i$ , whence  $m_i = n_i$ , that is, the set of exponents  $m_1, m_2, \dots, m_r$  is the same for all normal bases. We shall call these exponents the *degree invariants* of the integral set.

**4.03 Elementary sets.** If  $z_1(\lambda), z_2(\lambda), \dots, z_r(\lambda)$  is a basis of an integral set, but not necessarily a normal basis, the  $r$ -vector  $|z_1 z_2 \dots z_r|$ , which we call the determinant of the basis, is not identically 0 but may vanish for certain values of  $\lambda$ . If it vanishes for  $\lambda = \lambda_1$ , then  $z_1(\lambda_1), z_2(\lambda_1), \dots, z_r(\lambda_1)$  are linearly dependent, that is, there is a relation  $\Sigma \zeta_i z_i(\lambda_1) = 0$ ; we may assume  $\zeta_1 \neq 0$  without loss of generality. It follows that  $\Sigma \zeta_i z_i(\lambda)$  has a factor of the form  $(\lambda - \lambda_1)^\alpha$ ,  $\alpha \geq 1$ , and hence

$$z'_1(\lambda) = \frac{\Sigma \zeta_i z_i(\lambda)}{(\lambda - \lambda_1)^\alpha}$$

is integral; and, since  $\zeta_1 \neq 0$ , every element of  $(z_1(\lambda), z_2(\lambda), \dots, z_r(\lambda))$  is integrably expressible in terms of  $(z'_1(\lambda), z_2(\lambda), \dots, z_r(\lambda))$ . Moreover, since

$$|z'_1 z_2 \dots z_r| = \frac{\zeta_1}{(\lambda - \lambda_1)^\alpha} |z_1 z_2 \dots z_r|,$$

the determinant of the new basis is of lower degree than that of the old and so, if we continue this process, we shall arrive after a finite number of steps at a basis  $(x_1(\lambda), x_2(\lambda), \dots, x_r(\lambda))$  whose elements are linearly independent for all values of  $\lambda$ . A set which has a basis of this kind will be called an *elementary integral set*; and it is readily shown that every basis of an elementary integral set has the given property, namely, that its elements are linearly independent for every value of  $\lambda$ . These results are summarized as follows.

**THEOREM 3.** *Every integral set of order  $r$  is contained in an elementary set of the same order.*

We also have

**THEOREM 4.** *Let  $x_1, x_2, \dots, x_r$  be a basis of an elementary set. If  $r < n$ , there exists a complementary elementary basis  $x_{r+1}, \dots, x_n$  such that  $|x_1 x_2 \dots x_n| \neq 0$  for any value of  $\lambda$  and this basis can be so chosen that its degree does not exceed that of  $x_1, x_2, \dots, x_r$ .*

**THEOREM 5.** *If  $x_1, x_2, \dots, x_r$  is a basis of an elementary set, there exists an elementary matrix polynomial  $X$  such that  $x_i = X e_i$  ( $i = 1, 2, \dots, r$ ).*

For let  $y$  be a constant vector which for some value<sup>1</sup> of  $\lambda$  is not linearly dependent on  $x_1, x_2, \dots, x_r$  so that we do not have identically  $y = \sum \eta_i x_i$  for any  $\eta$ 's which are scalar polynomials. If for some value of  $\lambda$ , say  $\lambda_1$ , we have  $y = \sum \xi_i x_i(\lambda_1)$ , the  $\xi$ 's being constants, then  $y - \sum \xi_i x_i(\lambda)$  has the factor  $\lambda - \lambda_1$  and, as in the proof of Theorem 3, we can modify  $y$  step by step till we arrive at a vector polynomial  $x_{r+1}$  such that  $x_1, x_2, \dots, x_r, x_{r+1}$  form an elementary basis. The degree at each step of this process does not exceed that of the original basis since only constant multipliers are used. This procedure may be continued till a basis of order  $n$  is reached, which proves Theorem 4.

The proof of Theorem 5 is immediate; in fact, using the basis derived in the proof of Theorem 4,  $X = \sum_1^n x_i S e_i$  satisfies the required conditions and  $|X| = |x_1 x_2 \dots x_n|$ , which does not vanish for any value of  $\lambda$ .

As a converse to Theorem 5 we have that, if  $X$  is an elementary matrix, then  $x_i = X e_i$  ( $i = 1, 2, \dots, r$ ) is a basis of the elementary set  $(x_1, x_2, \dots, x_r)$ .

**4.04** If  $z_1, z_2, \dots, z_k$  is a sequence of vector polynomials of rank  $r$ , we may always assume  $k \leq n$  by merely increasing the order of the fundamental space, if necessary. Setting  $z_i = \sum \zeta_{ij} e_j$ , let us consider the matrix polynomial

<sup>1</sup> If the question of degree is not important, any vector polynomial satisfying this condition may take the place of a constant vector.

$$Z = \left\| \begin{array}{cccccc} \zeta_{11} & \cdots & \zeta_{1k} & 0 & \cdots & 0 \\ \zeta_{21} & \cdots & \zeta_{2k} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \zeta_{n1} & \cdots & \zeta_{nk} & 0 & \cdots & 0 \end{array} \right\| = \sum_1^k z_i S e_i.$$

The elementary transformations used in §4.02 in finding a basis of the integral set correspond when applied to  $Z$  to a combination of elementary transformations, as defined as §3.01, and because these transformations involve columns only, they correspond to multiplying  $Z$  on the right by an elementary polynomial  $Q_1$ . Similarly, if

$$Z = \sum e_i S y_i, \quad y_i = \sum \zeta_{ij} e_j,$$

the process of finding a basis for  $y_1, y_2, \dots, y_n$ , whose rank is  $r$ , corresponds to multiplying  $Z$  on the left by an elementary polynomial  $P_1$ .

We shall now suppose that  $k = r$  so that  $Q_1 = 1$ ; then  $P_1 Z$  has the form

$$Z_1 = P_1 Z = \left\| \begin{array}{cccccc} \omega_{11} & \cdots & \omega_{1r} & 0 & \cdots & 0 \\ \omega_{21} & \cdots & \omega_{2r} & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \omega_{r1} & \cdots & \omega_{rr} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right\|$$

We now bring  $Z_1$  to the normal form of §3.02, say

$$PZ_1Q = PP_1ZQ = \left\| \begin{array}{cccccc} \zeta_1 & & & & & \\ & \zeta_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \zeta_r & \\ & & & & & 0 \\ & & & & & \cdot \\ & & & & & 0 \end{array} \right\|$$

where  $\zeta_1, \zeta_2, \dots, \zeta_r$  are the invariant factors of  $Z$  (or  $Z_1$ ) and in doing so only the first  $r$  rows and columns are involved so that

$$(3) \quad Qe_i = e_i, \quad (i = r + 1, r + 2, \dots, n).$$

Therefore, if  $x_i = (PP_1)^{-1}e_i$ , we have successively

$$PP_1ZQe_i = \zeta_i e_i, ZQe_i = \zeta_i (PP_1)^{-1}e_i = \zeta_i x_i, \quad (i = 1, 2, \dots, r)$$

and, if  $Q = \| q_{ij} \|$ ,

$$Qe_i = q_{1i}e_1 + q_{2i}e_2 + \dots + q_{ri}e_r, \quad (i = 1, 2, \dots, r)$$

and hence

$$(4) \quad \zeta_i x_i = ZQe_i = q_{1i}z_1 + q_{2i}z_2 + \dots + q_{ri}z_r, \quad (i = 1, 2, \dots, r).$$

But from (3) and the fact that  $|Q|$  is a constant different from 0, it follows that the determinant of the coefficients in (4) is also a constant different from 0, and hence these equations can be solved for the  $z$ 's in terms of the  $x$ 's giving, say

$$z_i = \sum b_{ji} \zeta_j x_j \quad (i = 1, 2, \dots, r)$$

where the  $b$ 's are scalar polynomials.

Returning now to the case  $k \geq r$ , we see that, since we can pass to the case  $k = r$  by elementary transformations, the  $\zeta$ 's are still the invariant factors of  $Z = \sum_1^k z_i S e_i$ . They are therefore also invariants of the integral set independently of the basis chosen to represent it, and so we shall call them the *invariant factors* of the set.

We can now state the following theorem.

**THEOREM 6.** *If  $\zeta_1, \zeta_2, \dots, \zeta_r$  are the invariant factors of an integral set of vector polynomials, we can find a basis of the form*

$$\zeta_1 x_1, \zeta_2 x_2, \dots, \zeta_r x_r$$

where  $x_1, x_2, \dots, x_r$  define an elementary set.

**4.05 Linear elementary bases.** We shall derive in this section a canonical form for a basis of an elementary linear set. If

$$(5) \quad z_1, z_2, \dots, z_r, z_i = x_i \lambda + y_i,$$

is a basis of an elementary linear set, it is convenient, though not necessary, to associate with it the matrix

$$(6) \quad A\lambda - B = \sum_1^r g_i S z_i = \sum_1^r g_i S (x_i \lambda + y_i)$$

where  $g_1, g_2, \dots, g_r$  is a sequence of linearly independent constant vectors. When this is done, it should be noted that multiplying  $A\lambda - B$  on the right by an elementary matrix  $P$  corresponds to replacing (5) by the similar sequence  $P'z_1, \dots, P'z_r$ . Multiplying on the left by  $P$  has no immediate interpretation in terms of the sequence except when

$$Pg_i = \sum_{j=1}^r p_{ji} g_j \quad (i = 1, 2, \dots, r)$$

in which case we can write

$$P(A\lambda - B) = \sum_{i=1}^r g_i S \sum_{j=1}^r p_{ij} z_j = \sum_{i=1}^r g_i S z'_i$$

and the set  $(z'_1, z'_2, \dots, z'_r)$  is equivalent to (5); when  $P$  is constant, the equivalence is strict.

Instead of restricting ourselves to the matrix (6), we shall only assume to begin with that  $A\lambda - B$  is a linear matrix polynomial of rank  $r < n$ . The nullspace  $\mathfrak{N}$  of  $A\lambda - B$  is then an elementary integral set, a normalized basis of which we shall take to be

$$(7) \quad a_i(\lambda) = a_{i0}\lambda^{m_i} + a_{i1}\lambda^{m_i-1} + \dots + a_{im_i} \quad (i = 1, 2, \dots, n - r).$$

From  $(A\lambda - B)a_i = 0$  we have

$$Aa_{i0} = 0, Aa_{i1} = Ba_{i0}, \dots, 0 = Ba_{im_i},$$

or, if we set  $a_{it} = 0$  for  $t < 0$  or  $t > m_i$ ,

$$(8) \quad Aa_{it} = Ba_{i,t-1} \quad (t = 0, 1, 2, \dots, m_i + 1).$$

We shall now show that the vectors  $a_{ij}$  ( $i = 1, 2, \dots, n - r; j = 0, 1, \dots, m_i$ ) are linearly independent. Assume that  $a_{ij}$  are linearly independent for  $(i = 1, 2, \dots, p - 1; j = 0, 1, \dots, m_i)$  and  $(i = p; j = 0, 1, \dots, q - 1)$  but that

$$(9) \quad a_{pq} + \sum_{j=0}^{q-1} \alpha_{pj} a_{pj} + \sum_{i=1}^{p-1} \sum_{j=0}^{m_i} \alpha_{ij} a_{ij} = 0.$$

Let  $s'$  be the greatest value of  $j$  for which some  $\alpha_{ij} \neq 0$  and let  $s$  be the greater of  $s'$  and  $q$ . If we set

$$(10) \quad c_t = a_{p,q-s+t} + \sum_{j=0}^{q-1} \alpha_{pj} a_{p,j-s+t} + \sum_{i=1}^{p-1} \sum_{j=0}^{m_i} \alpha_{ij} a_{i,j-s+t},$$

then  $c_{-1} = 0, c_s = 0$  and

$$c_{s-1} = a_{p,q-1} + \sum_{j=1}^{q-1} \alpha_{pj} a_{p,j-1} + \sum_{i=1}^{p-1} \sum_{j=1}^{m_i} \alpha_{ij} a_{i,j-1},$$

which is not 0 by hypothesis, except perhaps when  $q = 0$  and every  $\alpha_{ij}$  ( $j \neq 0$ ) in (9) is 0, which, however, is not possible since by Theorem 1 the leading coefficients  $a_{i0}$  in (7) are linearly independent. Also from (8) it follows that  $Ac_t = Bc_{t-1}$ , and hence

$$(11) \quad c(\lambda) = c_0\lambda^s + c_1\lambda^{s-1} + \dots + c_{s-1}$$

is a null-vector of  $A\lambda - B$  of degree less than  $m_p$ . But every such integral null-vector is linearly dependent on  $a_1, a_2, \dots, a_{p-1}$  with integral coefficients, say

$$(12) \quad c(\lambda) = \sum_1^{p-1} \gamma_i(\lambda) a_i(\lambda);$$

and this gives

$$c_{s-1} = c(0) = \sum_1^{p-1} \gamma_i(0) a_i(0) = \sum_1^{p-1} \gamma_i(0) a_{im_i}$$

which is impossible since  $c_{s-1}$  is obtained from (9) by lowering the second subscript in each term and no such subscript greater than  $m_i$  can occur in any  $a_{ij}$ . Hence the  $a_{ij}$  are linearly independent.

In order to simplify the notation we shall now set

$$(13) \quad a_{ij} = Qe_j^i \quad (i = 1, 2, \dots, n-r; j = 0, 1, \dots, m_i)$$

where  $Q$  is a constant non-singular matrix and  $e_j^i$  are fundamental units rearranged by setting, say,  $e_k = e_j^i$  when  $k = \sum_{q=1}^{i-1} (m_q + 1) + j + 1$ ; as before  $e_j^i = 0$  for  $j < 0$  and  $j > m_i$ . We shall denote the space defined by the  $e_j^i$  by  $\mathfrak{M}_1$  and the complementary space by  $\mathfrak{M}_2$ ; since the bases of  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  can be chosen as sequences of fundamental units, they are reciprocal as well as complementary.

We return to the particular case in which  $A\lambda - B$  is given by (6). Corresponding to (12) we define a new set of vectors  $w$  by

$$(14) \quad w_i = Q'z_i \quad (i = 1, 2, \dots, r)$$

and when this is done a normal basis of the nullspace  $\mathfrak{N}_1$  of  $(A\lambda - B)Q = \sum g_i S w_i$  is given by

$$(15) \quad b_i = e_0^i \lambda^{m_i} + e_1^i \lambda^{m_i-1} + \dots + e_{m_i}^i \quad (i = 1, 2, \dots, n-r).$$

We have seen in §1.10 that  $b_1, b_2, \dots, b_{n-r}$  is the space reciprocal to  $w_1, w_2, \dots, w_r$ . Now in  $\mathfrak{M}_1$  the  $\sum_1^{n-r} m_i$  vectors

$$(16) \quad f_j^i = e_{j-1}^i - \lambda e_j^i \quad (i = 1, 2, \dots, n-r; j = 1, 2, \dots, m_i)$$

are linearly independent; and they form the set reciprocal to (14) in  $\mathfrak{M}_1$  since  $Sf_j^i b_p = 0$  for all  $i, j, p$  and the sum of the orders of the two sets is  $\sum m_i + (n-r)$  which is the order of  $\mathfrak{M}_1$ . Hence the total set  $(w_1, w_2, \dots, w_r)$  reciprocal to  $(b_1, b_2, \dots, b_{n-r})$  is composed of (15) together with  $\mathfrak{M}_2$ . We shall call this form of basis a *canonical* basis of the set (13). We can now state the following theorems.

**THEOREM 7.** *A linear elementary set of order  $r$  has a basis of the form*

$$(17) \quad g_j^0 \quad (j = 1, 2, \dots, m), \quad g_j^i = a_{i, j-1} + \lambda a_{ij} \quad (i = 1, 2, \dots, \nu; j = 1, 2, \dots, m_i)$$

where the constant vectors  $g_j^0$ ,  $a_{ij}$  are linearly independent for all  $j$  and  $i$  and the integers  $m_i$  are those degree invariants of the reciprocal set that are not 0.

We shall call each set  $g_j^i$  ( $j = 1, 2, \dots, m_i$ ) a *chain of index  $m_i$* , and define the integers  $m_1, m_2, \dots, m_r$  as the *Kronecker invariants* of the set. A basis of the form (17) will be called *canonical*.

**THEOREM 8.** *Two linear elementary sets are similar if, and only if, they have the same Kronecker invariants and the same order.*

It should be noted that, if  $r$  is the order of the set, then

$$(18) \quad m + \sum_1^r m_i = r, \quad m + \sum_1^r (m_i + 1) \leq n, \quad \nu \leq n - r.$$

If  $r = n$ , all the Kronecker invariants are 0 and there are no chains in the basis.

If  $z_1, z_2, \dots, z_r$  is a normal basis of an elementary linear set, the first  $m$  being constant and the rest linear in  $\lambda$ , and  $g_j^0, g_j^i$  is a canonical basis, the notation being that of Theorem 7, then clearly the set  $g_j^0$  ( $j = 1, 2, \dots, m$ ) is strictly equivalent to  $(z_1, z_2, \dots, z_m)$  and the remaining vectors have the form

$$(19) \quad g_j^i = w_j^i + u_j^i + \lambda v_j^i$$

where  $u_j^i$  and  $v_j^i$  belong to  $(z_1, z_2, \dots, z_m)$  and the  $w_j^i$  are constant linear combinations of  $z_{m+1}, \dots, z_r$ . Since a canonical basis is also normal,

$$(20) \quad g_j^0 \quad (j = 1, 2, \dots, m), \quad w_j^i \quad (i = 1, 2, \dots, \nu; j = 1, 2, \dots, m_i)$$

is a normal basis strictly equivalent to  $(z_1, z_2, \dots, z_r)$ . Now (19) may be written

$$(21) \quad w_j^i = g_j^i - u_j^i - \lambda v_j^i = a_{i, j-1} - u_j^i + \lambda(a_{ij} - v_j^i) = z_j^i + b_j^i$$

where  $b_j^i = v_j^i - u_j^i$  is a constant vector of the linear set  $(g_1^0, g_2^0, \dots, g_m^0)$  and

$$(22) \quad z_j^i = a_{i, j-1} - v_j^i + \lambda(a_{ij} - v_j^i).$$

Here (22) together with the  $g_j^0$  form a canonical basis which from (21) is strictly equivalent to (20) and therefore to  $(z_1, z_2, \dots, z_r)$ . We therefore have the following theorem.

**THEOREM 9.** *Every normal basis of a linear elementary set is strictly equivalent to some canonical basis.*

**4.06 Singular linear polynomials.** Let  $A\lambda + B$  be a matrix polynomial of rank  $r < n$ . Its left and right grounds are linear integral sets of rank  $r$ , and by Theorems 3 and 7 we can find canonical bases in terms of which the vectors of the two grounds can be integrally expressed, say

$$(23) \quad z_1, z_2, \dots, z_r \quad \text{and} \quad w_1, w_2, \dots, w_r,$$

respectively, where the first  $\alpha$   $z$ 's and  $\beta$   $w$ 's are constant and the rest linear in  $\lambda$ . When  $A\lambda + B$  is expressed in terms of these bases, then, remembering that no second degree term can appear, we see that it has the form

$$(24) \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (h_{ij}\lambda + k_{ij})z_iSw_j + \sum_{i=1}^{r-\alpha} \sum_{j=1}^{\beta} k_{\alpha+i,j}z_{\alpha+i}Sw_j + \sum_{i=1}^{\alpha} \sum_{j=1}^{r-\beta} k_{i,\beta+j}z_iSw_{\beta+j}.$$

The row vectors

$$(25') \quad p_i = \sum_{j=1}^{\beta} (h_{ij}\lambda + k_{ij})w_j + \sum_{j=1}^{r-\beta} k_{i,\beta+j}w_{\beta+j} \quad (i = 1, 2, \dots, \alpha)$$

$$(25'') \quad p_{\alpha+i} = \sum_{j=1}^{\beta} k_{\alpha+i,j}w_j \quad (i = 1, 2, \dots, r - \alpha)$$

form a set of  $r$  linearly independent vectors and, since the set (25'') depends only on  $\beta$   $w$ 's, we must have  $r - \alpha \leq \beta$ . Setting  $\gamma = \alpha + \beta - r$  we may replace  $w_{\gamma+1}, \dots, w_{\beta}$  by  $p_{\alpha+1}, \dots, p_r$  in (23) without destroying the canonical form of the basis. A similar change can be made independently in the  $z$  basis by replacing  $z_{\gamma+j}$  by  $\sum_1^{\alpha} k_{i,\beta+\gamma}z_{\beta+j}$  ( $j = 1, 2, \dots, r - \beta = \alpha - \gamma$ ).

When we assume that these changes have been made to begin with, we may take in place of (24)

$$(26) \quad A\lambda + B = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} (h_{ij}\lambda + k_{ij})z_iSw_j + \sum_{i=1}^{r-\alpha} z_{\alpha+i}Sw_{\gamma+i} + \sum_{j=1}^{r-\beta} z_{\gamma+j}Sw_{\beta+j}.$$

Figure 1 shows schematically the effect of this change of basis. To begin with the coefficients in (24) may be arranged in a square array  $AR$  of side  $r$ ; the

		$\gamma$	$\beta$	$r$	$\alpha + \beta$
$A$	$B$	$C$	$D$		
$\gamma$	$E$	$F$	$G$	$H$	
$\alpha$	$J$	$K$	$L$	$M$	
$r$	$N$	$P$	$Q$	$R$	
$\alpha + \rho$					

FIG. 1

first double sum corresponds to the rectangle  $AL$ , the second to  $JQ$  and the third to  $CM$ , and the rectangle  $LR$  contains only zeros. After the transformation which leads to (26), the only change in the scheme is that in  $JQ$  the part  $JP$  is now zero and the square  $KQ$  has 1 in the main diagonal and zeros elsewhere, and  $CM$  also takes a similar form.

If we set

$$z'_{\alpha+i} = z_{\alpha+i} + \sum_{j=1}^{\alpha} (h_{j,\gamma+i}\lambda + k_{j,\gamma+i})z_j \quad (i = 1, 2, \dots, r - \alpha)$$

$$w'_{\beta+i} = w_{\beta+i} + \sum_{j=1}^{\gamma} (h_{\gamma+i,j}\lambda + k_{\gamma+i,j})w_j \quad (i = 1, 2, \dots, r - \beta)$$

then  $z_1, \dots, z, z'_{\alpha+1}, \dots, z'_r$  and  $w_1, \dots, w_\beta, w'_{\beta+1}, \dots, w'_r$  are still elementary bases of the right and left grounds, and in terms of them (26) becomes

$$\sum_{i,j=1}^{\gamma} (h_{ij}\lambda + k_{ij})z_i S w_j + \sum_{i=1}^{r-\alpha} z'_{\alpha+i} S w_{\gamma+i} + \sum_{i=1}^{r-\beta} z_{\gamma+i} S w'_{\beta+i}$$

The number of terms in these summations after summing for  $j$  is  $\gamma + (r - \alpha) + (r - \beta) = r$ . Hence the rank of the square array  $h_{ij}\lambda + k_{ij}$  ( $i, j = 1, 2, \dots, \gamma$ ) is  $\gamma$  and by a change of variable of the form  $\lambda - \lambda_1 = \lambda'$ , if necessary, we can secure that the array  $k_{ij}$  is also of rank  $\gamma$ .

The transformation just employed disturbs the canonical form of the basis and we have now to devise a different transformation which will avoid this. Let us set in place of  $w_1, w_2, \dots, w_\gamma$

$$w'_j = w_j - \sum_{i=\gamma+1}^{\beta} p_{i,j-\gamma} w_i \quad (j = 1, 2, \dots, \gamma)$$

where the  $p$ 's are constants to be determined later, and for brevity set also

$$h_j = \sum_{i=1}^{\gamma} h_{ij} z_i, \quad k_j = \sum_{i=1}^{\gamma} k_{ij} z_i \quad (j = 1, 2, \dots, \beta);$$

since the rank of  $k_{ij}$  ( $i = 1, 2, \dots, \gamma$ ) is  $\gamma$ , the vectors  $k_1, k_2, \dots, k_\gamma$  form a basis of  $(z_1, z_2, \dots, z_\gamma)$ . After this change of basis the part of the first double sum (cf. (26)) which corresponds to  $i = 1, 2, \dots, \gamma; j = \gamma + 1, \dots, \beta$  is

$$(27) \quad \sum_{j=\gamma+1}^{\beta} \left[ h_j \lambda + k_j + \sum_{i=1}^{\gamma} p_{i,j-\gamma} (h_i \lambda + k_i) \right] S w_j.$$

Consider now a single chain of  $z$ 's of index  $s$  which by a suitable change of notation we may suppose to be  $z_{\alpha+1}, z_{\alpha+2}, \dots, z_{\alpha+s}$ ; we shall seek to determine the  $p$ 's so that the corresponding part of (27) shall become

$$(28) \quad \sum_{j=\gamma+1}^{\gamma+s} (g_{j-1-\gamma} + \lambda g_{j-\gamma}) S w_j,$$



Here the first summation can be reduced to a canonical form without affecting the rest of the expression; we therefore neglect it in the meantime and deal only with the remaining terms. This is equivalent to putting  $\gamma = 0$ , and in making this change it is convenient to alter the notation so as to indicate the chains in the bases.

As in §4.05 let the chains of the  $z$  and  $w$  bases be

$$f_{j-1}^i + \lambda f_j^i \quad (j = 1, 2, \dots, s_i; i = 1, 2, \dots, \nu_1)$$

and

$$g_{q-1}^p + \lambda g_q^p \quad (q = 1, 2, \dots, t_p; p = 1, 2, \dots, \nu_2)$$

respectively, and denote the constant vectors of the respective bases by  $z_q^p$  and  $w_j^i$  where  $i, j, p, q$  take the values indicated above since, when  $\gamma = 0$ , we have  $\sum_1^{\nu_1} s_i = r - \alpha = \beta$ ,  $\sum_1^{\nu_2} t_q = r - \beta = \alpha$ . We have then to determine a canonical form for the matrix

$$(30) \quad \sum_{i,j,p,q} (h_q^p{}^i \lambda + k_q^p{}^i) z_q^p S w_j^i + \sum_{i,j} (f_{j-1}^i + \lambda f_j^i) S w_j^i + \sum_{p,q} z_q^p S (g_{q-1}^p + \lambda g_q^p),$$

and in doing so we shall show that the first summation can be eliminated by a proper choice of the bases of the chains.

It will simplify the notation if we consider first only two chains, one of index  $s$  in the  $z$ -basis and the other of index  $t$  in the  $w$ -basis and, omitting the superscripts, choose the notation so that these chains are  $f_0 + \lambda f_1, \dots, f_{s-1} + \lambda f_s$  and  $g_0 + \lambda g_1, \dots, g_{t-1} + \lambda g_t$ . We now modify these by adding  $a_{j-1} + \lambda a_j$  to  $f_{j-1} + \lambda f_j$  and  $b_{i-1} + \lambda b_i$  to  $g_{i-1} + \lambda g_i$  choosing

$$a_j = \sum_{i=1}^t \alpha_{ij} z_i \quad (j = 0, 1, \dots, s), \quad b_i = \sum_{j=1}^s \beta_{ij} w_j \quad (i = 0, 1, \dots, t)$$

in such a way as to eliminate the corresponding terms in the first summation of (30). To do this we must choose the  $\alpha$ 's and  $\beta$ 's so that

$$(31) \quad h_{ij} = \alpha_{ij} + \beta_{ij}, \quad k_{ij} = \alpha_{i,j-1} + \beta_{i-1,j} \quad (i = 1, 2, \dots, t; j = 1, 2, \dots, s).$$

For  $j > 1$  this gives  $\alpha_{i,j-1} = h_{i,j-1} - \beta_{i,j-1}$  and hence if  $l_{ij} = k_{ij} - h_{i,j-1}$  we may write

$$(32') \quad k_{i1} = l_{i1} = \alpha_{i0} + \beta_{i-1,1} \quad (i = 1, 2, \dots, t; j = 2, 3, \dots, s).$$

$$(32'') \quad l_{ij} = \beta_{i-1,j} - \beta_{i,j-1}$$

If we give  $\alpha_{i0}$  ( $i = 1, 2, \dots, t$ ) arbitrary values, (32') defines  $\beta_{i1}$  for  $i = 0, 1, \dots, t-1$  and leaves  $\beta_{11}$  arbitrary; then  $j = 2$  in (32'') gives  $\beta_{i2}$  for  $i = 0, 1, \dots, t-1$  and leaves  $\beta_{12}$  arbitrary, and so on: and when the  $\beta$ 's are found in this way, certain of them being arbitrary, the first equation of (31) gives the remaining  $\alpha$ 's.

Combining every  $z$  chain in this way with each  $w$  chain in turn, we finally eliminate all the terms in the quadruplé sum in (30), and (29) may therefore, by a proper choice of the two bases, be replaced by

(33)

$$A\lambda + B = \sum_{i,j=1}^{\gamma} (h_{ij}\lambda + k_{ij})z_i S w_j + \sum_{i,j} (f_{j-1}^i + \lambda f_j^i) S w_j^i + \sum_{p,q} z_q^p S (g_{q-1}^p + \lambda g_q^p)$$

where no two of the linear sets

$$(34') \quad (z_1, z_2, \dots, z_\gamma), \quad (f_0^i, f_1^i, \dots, f_{s_i}^i, i = 1, 2, \dots, \nu_1), \\ (z_1^p, z_2^p, \dots, z_{t_p}^p, p = 1, 2, \dots, \nu_2)$$

have any vector in common, and also no two of

$$(34'') \quad (w_1, w_2, \dots, w_\gamma), \quad (w_1^i, w_2^i, \dots, w_{s_i}^i, i = 1, 2, \dots, \nu_1), \\ (g_0^p, g_1^p, \dots, g_{t_p}^p, p = 1, 2, \dots, \nu_2)$$

have any vector in common.

We shall now for the moment suppose that the order  $n$  of the fundamental space is taken so large that we can introduce vectors  $z_0^p$  ( $p = 1, 2, \dots, \nu_2$ ) into the third set in (34') without causing the three spaces to overlap, and also  $w_0^i$  ( $i = 1, 2, \dots, \nu_1$ ) into the second set of (34''). As a matter of convenience we can then find two constant non-singular matrices  $P, Q$  such that

$$Pz_i = e_i = Q'w_i, \quad Pf_j^i = e_j^i = Q'w_j^i, \quad Pz_q^p = e_q^{p_1+p} = Q'g_q^p$$

where the range of the affices is as in (34) and where

$$e_j^i = e_k, \quad k = \gamma + \sum_{\alpha=1}^{i-1} (s_\alpha + 1) + j + 1, \quad e_q^{p_1+p} = e_k, \\ k = \gamma + \sum_{i=1}^{\nu_1} (s_i + 1) + \sum_{\alpha=1}^{p-1} (t_\alpha + 1) + q + 1$$

and, when this is done,

$$(35) \quad P(A\lambda + B)Q = \sum_{i,j=1}^{\gamma} (h_{ij}\lambda + k_{ij})e_i S e_j + \sum_{i,i} (e_{j-1}^i + \lambda e_j^i) S e_j^i \\ + \sum_{p,q} e_q^{p_1+p} S (e_{q-1}^{p_1+p} + \lambda e_q^{p_1+p}).$$

This matrix is composed of a number of blocks of terms arranged along the main diagonal, the remaining coordinates being 0. It must be carefully observed however that, owing to the introduction of the vectors  $z_0^p, w_0^i$  and to the fact that a chain of index  $s$  depends on  $s + 1$  constant vectors, the total number

of rows and columns employed is greater than the rank  $r$  by the total number of chains in the left and right grounds.

The first summation in (35) gives a block  $\| h_{ij}\lambda + k_{ij} \|$  of  $\gamma$  rows and columns. Each chain in the second and third summation gives a block of the respective forms

$$\begin{array}{cc}
 \begin{array}{cccccc}
 0 & 1 & 0 & \cdots & 0 & 0 \\
 0 & \lambda & 1 & \cdots & 0 & 0 \\
 0 & 0 & \lambda & \cdots & 0 & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & 0 & \cdots & \lambda & 1 \\
 0 & 0 & 0 & \cdots & 0 & \lambda
 \end{array} &
 \begin{array}{cccccc}
 0 & 0 & 0 & \cdots & 0 & 0 \\
 1 & \lambda & 0 & \cdots & 0 & 0 \\
 0 & 1 & \lambda & \cdots & 0 & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & 0 & \cdots & \lambda & 0 \\
 0 & 0 & 0 & \cdots & 1 & \lambda
 \end{array}
 \end{array}
 \tag{36}$$

If we take  $A\lambda + B\mu$  in place of  $A\lambda + B$  and calculate the invariant factors these forms show that we obtain the invariant factors of  $\sum_i (h_{ij}\lambda + k_{ij}\mu)e_iSe_j$  together with a number of 1's from the blocks of type (36), the number contributed by each being one less than the number of rows it contains, that is, the index of the corresponding chain. This gives the following theorem.

**THEOREM 10.** *Two matrix polynomials  $A\lambda + B\mu$  and  $C\lambda + D\mu$  are strictly equivalent if, and only if, they have the same invariant factors and their respective right and left grounds have the same Kronecker invariants.*

That these conditions are necessary is obvious; that they are sufficient follows readily from the form (33) derived above. In the first place, since the Kronecker invariants are the same for both, the second and third summations in (33) have the same form for both and are therefore strictly equivalent by means of transformations which do not change the terms in the first summation. Secondly, the first summation in both yields the same invariant factors since the number of 1's due to the remaining terms depends only on the number of chains, which is the same for both; hence these summations are strictly equivalent and, because of the linear independence of the constant vectors involved, the equivalence is obtainable by transformations which do not affect the remaining terms.

When the first summation in (35) is in canonical form, we shall say that  $A\lambda + B$  is in its canonical form. This is however not altogether satisfactory since the space necessary for this form may be of greater order than  $n$ . If  $\nu$  is the greater of  $\nu_1$  and  $\nu_2$ , (33) shows that the minimum order of the enveloping space is  $\gamma + \sum s_i + \sum t_p + \nu$ . A canonical form for this number of dimensions can be obtained as follows. Pair the blocks of the first and second types of (36) till all of one type are used up, taking the order of the constituents in, say, the order of (36); then in the composite block formed from such a pair discard

the first column and also the row which contains the first row of the second block. This gives a canonical form for such a pair, namely,

$$\begin{array}{cccccccc}
 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & \lambda & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & \lambda & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
 (37) & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & \lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & 0 & 1 & \lambda & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & 0 & 0 & 1 & \lambda & \cdots & 0 & 0 \\
 & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda & 0 \\
 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & \lambda.
 \end{array}$$

If the number of chains in the left and right grounds is not the same, there will of course be blocks of one of the types (36) left unpaired.

## CHAPTER IX

### THE AUTOMORPHIC TRANSFORMATION OF A BILINEAR FORM

9.01 If the variables of a bilinear form whose matrix is  $a$  are transformed cogrediently by a matrix  $x$ , the matrix of the new bilinear form is  $x'ax$ ; when this new form is identical with the old, the transformation is said to be *automorphic*. The problem of finding all automorphic transformations of  $a$  is therefore equivalent to solving the equation

$$(1) \quad x'ax = a.$$

We shall assume for the present that  $|a| \neq 0$  in which case also  $|x| \neq 0$ .

It follows from (1) that  $x'a = ax^{-1}$ . Hence, if  $f(\lambda)$  is a scalar polynomial

$$(1') \quad [f(x)]'a = f(x')a = af(x^{-1}).$$

In particular, if  $f(\lambda) = (1 - \lambda)/(1 + \lambda)$  and  $y = f(x^{-1})$ , then

$$(2) \quad y = \frac{1 - x^{-1}}{1 + x^{-1}} = \frac{x - 1}{x + 1} = -f(x)$$

provided  $|x + 1| \neq 0$ . Hence from (1')  $y'a = -ay$  so that

$$(3) \quad y' = -aya^{-1}.$$

Conversely, if  $y$  satisfies (3) and  $|1 - y| \neq 0$ , then  $x = (1 + y)/(1 - y)$  is a solution of (1) such that  $|x + 1| \neq 0$ . For from (3)  $f(y')a = af(-y)$  so that

$$x'a = \frac{1 + y'}{1 - y'}a = a \frac{1 - y}{1 + y} = ax^{-1}.$$

Similarly, if  $|x - 1| \neq 0$ , we may set  $x = (1 - y)/(1 + y)$  and then  $y$  is a solution of (3) such that  $|1 + y| \neq 0$ , and conversely. The effect of the transformation (2) is therefore to reduce the solution of (1), which is quadratic in  $x$ , to that of (3), which is linear in  $y$ , except when both 1 and  $-1$  are roots of  $x$ . It is because (3) is linear that it is more convenient than (1); in particular if we regard

$$(3') \quad y'a + ay = 0, \quad y = \|\eta_{pa}\|, \quad a = \|\alpha_{pa}\|,$$

as a system of  $n^2$  linear homogeneous equations in the  $\eta$ 's, then the rank of the system gives the number of parameters which enter into the solution when those values of  $y$  (or of  $x$ ) are excluded for which both 1 and  $-1$  are roots.

Since the main problem is thus reduced to the solution of linear equations, it may be regarded as solved; the solution, however, can be given a somewhat more definite form as we shall now show.

**9.02 The equation  $y' = \pm aya^{-1}$ .** We shall consider in place of (3) the more general equation

$$(4) \quad y' = \delta aya^{-1}, \quad \delta = \pm 1.$$

Forming the transverse we get  $y = \delta a'^{-1}y'a'$  or  $y' = \delta a'y a'^{-1}$ , whence

$$(5) \quad ya^{-1}a' = a^{-1}a'y, \quad a'y a'^{-1} = aya^{-1}$$

so that  $y$  is commutative with  $a^{-1}a'$ . Now from (4) we have  $y = \delta a^{-1}y'a$  and hence

$$2y = y + \delta a^{-1}y'a.$$

But if  $b$  is any matrix commutative with  $a^{-1}a'$ , then

$$(6) \quad y = b + \delta a^{-1}b'a$$

is a solution of (4); for on substituting this value of  $y$  we get

$$y' - \delta aya^{-1} = b' + \delta a'ba'^{-1} - \delta aba^{-1} - b' = 0$$

since, as in (5),  $a'ba'^{-1} = aba^{-1}$ . It was noted above that  $y$  has this form and it therefore follows that the general solution of (3) is obtained by setting

$$(7) \quad y = b - a^{-1}b'a, \quad ba^{-1}a' = a^{-1}a'b.$$

It should be noted, however, that two different values of  $b$  may give rise to the same value of  $y$ .

**9.03** We are now able to give a solution of (1) under the restriction that either  $|x + 1| \neq 0$  or  $|x - 1| \neq 0$ . Since the first condition is transformed into the second if  $x$  is changed into  $-x$ , it is sufficient for the present to assume that  $|x + 1| \neq 0$ , and in this case the value of  $y$  given by (2) is finite. In terms of  $y$  we have

$$x = \frac{1 + y}{1 - y} = \frac{1 + b - a^{-1}b'a}{1 - b + a^{-1}b'a} = (a - ab + b'a)^{-1}(a + ab - b'a)$$

or, if

$$(8) \quad c = ab - b'a,$$

then

$$(9) \quad \begin{aligned} x &= (a - c)^{-1}(a + c), & |x + 1| &\neq 0, \\ x &= (c - a)^{-1}(a + c), & |x - 1| &\neq 0. \end{aligned}$$

It follows as in §9.01 that, if  $x$  has this form, it is a solution of (1)

In place of (8) we may define  $c$  by

$$(10) \quad c' = -a'a^{-1}c = -ca^{-1}a'.$$

For from (7) and (8)

$$c' = b'a' - a'b = -a'a^{-1}c = -ca^{-1}a'$$

and, if  $c$  is given by (10) and  $x$  by (9), then

$$\begin{aligned} x' &= (a' + c') (a' - c')^{-1} = (a' - ca^{-1}a') (a' + ca^{-1}a')^{-1} \\ &= (1 - ca^{-1}) (1 + ca^{-1})^{-1} = (1 + ca^{-1})^{-1} (1 - ca^{-1}) = a(a + c)^{-1} (a - c)a^{-1} \\ &= ax^{-1}a^{-1}. \end{aligned}$$

If  $a$  is symmetric, (8) or (10) gives  $c' = -c$ , and  $c$  is otherwise arbitrary except that  $|a - c| \neq 0$ ; in particular if  $a = 1$ , (9) reduces to the form of an orthogonal matrix already given in 6.03. Similarly if  $a$  is skew, (10) shows that  $c$  is an arbitrary symmetric matrix subject to the condition that  $|a - c| \neq 0$ .

The case in which  $a$  is symmetric can also be handled as follows. We can set  $a = b^2$  where  $b$  is symmetric and, if

$$y = b^{-1}xb,$$

equation (1) gives  $yy' = 1$  so that  $y$  is orthogonal. Conversely, if  $y$  is any orthogonal matrix  $x = byb^{-1}$  is a solution of (1).

**9.04 Principal idempotent and nilpotent elements.** Since  $x$  is similar to  $(x')^{-1}$ , the elementary divisors which correspond to roots other than  $\pm 1$  occur in pairs with reciprocal roots. If we arrange these roots in pairs  $g_r, g_r^{-1}$  and denote the corresponding principal idempotent elements by  $e_r$  and  $e_{-r}$ , respectively, we may set

$$(11) \quad x = \Sigma [g_r(e_r + \xi_r) + g_r^{-1}e_{-r}(1 + \xi_{-r})^{-1}] + \theta_1(e_1 + \xi_1) - \theta_2(e_{-1} + \xi_{-1})$$

where the  $\xi$ 's are nilpotent,  $e_1, e_{-1}$  are the principal idempotent elements corresponding to 1 and  $-1$ , if present as roots, and  $\theta_1, \theta_2$  are either 0 or 1. The form of  $x^{-1}$  is then

$$(12) \quad x^{-1} = \Sigma [g_r(e_{-r} + \xi_{-r}) + g_r^{-1}e_r(1 + \xi_r)^{-1}] + \theta_1e_1(1 + \xi_1)^{-1} - \theta_2e_{-1}(1 + \xi_{-1})^{-1}$$

and (11) gives

$$(13) \quad \begin{aligned} e'_r &= ae_{-r}a^{-1}, & e'_{-r} &= ae_ra^{-1} \\ \xi'_r &= a\xi_{-r}a^{-1}, & \xi'_{-r} &= a\xi_ra^{-1} \end{aligned}$$

$$(13') \quad \begin{aligned} e'_1 &= ae_1a^{-1}, & e'_{-1} &= ae_{-1}a^{-1} \\ -\xi'_1 &= -a \frac{\xi_1}{1 + \xi_1} a^{-1}, & \xi'_{-1} &= -a \frac{\xi_{-1}}{1 + \xi_{-1}} a^{-1}. \end{aligned}$$

We require also the form of  $x + x^{-1}$  and  $x - x^{-1}$ ; if

$$(14) \quad \alpha_r = \xi_r + \xi_{-r}, \quad \beta_r = \xi_r - \xi_{-r}$$

then

$$\begin{aligned} x + x^{-1} &= \Sigma [g_r(e_r + e_{-r} + \alpha_r) + g_r^{-1}(e_r + e_{-r})(1 + \alpha_r)^{-1}] \\ &\quad + \theta_1[(e_1 + \xi_1) + e_1(1 + \xi_1)^{-1}] - \theta_2[(e_{-1} + \xi_{-1}) + e_{-1}(1 + \xi_{-1})^{-1}] \\ x - x^{-1} &= \Sigma [g_r(e_r - e_{-r} + \beta_r) - g_r(e_r - e_{-r})(1 + \beta_r)^{-1}] \\ &\quad + \theta_1[(e_1 + \xi_1) - e_1(1 + \xi_1)^{-1}] - \theta_2[(e_{-1} + \xi_{-1}) - e_{-1}(1 + \xi_{-1})^{-1}] \end{aligned}$$

or if

$$\begin{aligned}
 \gamma_r &= g_r \alpha_r - g_r^{-1} \frac{\alpha_r}{1 + \alpha_r}, & \gamma_1 &= \frac{\xi_1^2}{1 + \xi_1}, & \gamma_{-1} &= \frac{\xi_{-1}^2}{1 + \xi_{-1}} \\
 (15) \quad \delta_r &= g_r \xi_r + g_r^{-1} \frac{\xi_r}{1 + \xi_r}, & \delta_{-r} &= g_r \xi_{-r} + g_r^{-1} \frac{\xi_{-r}}{1 + \xi_{-r}}, \\
 \delta_1 &= 2\xi_1 - \gamma_1, & \delta_{-1} &= 2\xi_{-1} - \gamma_{-1}
 \end{aligned}$$

we have

$$(16) \quad x + x^{-1} = \Sigma [(g_r + g_r^{-1})(e_r + e_{-r}) + \gamma_r] + \theta_1(2e_1 + \gamma_1) - \theta_2(2e_{-1} + \gamma_{-1})$$

$$(17) \quad x - x^{-1} = \Sigma [\{(g_r - g_r^{-1})e_r + \delta_r\} - \{(g_r - g_r^{-1})e_{-r} + \delta_{-r}\}] + \theta_1\delta_1 - \theta_2\delta_{-1}$$

where the elements grouped together are principal elements.

The principal idempotent elements of  $x - x^{-1}$  are also principal idempotent elements of  $x$  except that roots 1 and  $-1$  of  $x$  both give the same root 0 of  $x - x^{-1}$ ; no root of  $x$  other than  $\pm 1$  leads to the coalescing of roots in  $x - x^{-1}$ . If we put

$$(18) \quad 2u = x + x^{-1}, \quad 2v = x - x^{-1},$$

then  $u$  is a solution of (4) with  $\delta = 1$  and  $v$  is a solution with  $\delta = -1$ ; also

$$(19) \quad x^2 - 2vx - 1 = 0$$

which has the formal solution

$$(20) \quad x = v + (v^2 + 1)^{\frac{1}{2}}.$$

Here  $v^2 + 1 = u^2$  so that  $(v^2 + 1)^{\frac{1}{2}}$  exists whenever  $x$  is a solution of (1). Conversely, if  $v_1$  is any solution of (4) with  $\delta = -1$  and if  $u_1$  is a determination of  $(v_1^2 + 1)^{\frac{1}{2}}$  such that

$$(21) \quad u_1' a = a u_1,$$

then  $x$  is a solution of (1); for

$$x'a = v_1'a + u_1'a = -av_1 + au_1 = ax^{-1}$$

since

$$(u_1 + v_1)(u_1 - v_1) = u_1^2 - v_1^2 = 1.$$

If  $v^2 + 1$  has no zero root, determinations of  $(v^2 + 1)^{\frac{1}{2}}$  always exist which are polynomials in  $v^2$  and therefore satisfy (21); but even in this case this does not give all solutions. The situation is as follows. The general form of  $v$  is given by (17) if we replace  $g_r - g_r^{-1}$  by, say,  $2k_r$  and  $\theta_1\delta_1 - \theta_2\delta_{-1}$  by  $\delta_0$ . When  $k_r$  is given, then  $g_r$  is determined; from (13) and (14) we have  $(\delta_r' + \delta_{-r}')a = a(\delta_r + \delta_{-r})$  and therefore, if  $k_r^2 + 1 \neq 0$ , the part of  $(v^2 + 1)^{\frac{1}{2}}$  corresponding to  $e_r + e_{-r}$  exists and satisfies (21); we therefore get all valid expressions for this part of  $(v^2 + 1)^{\frac{1}{2}}$  by using the form of the square root given in §8.05 with the restriction

that the only sets of partial units that may be used are those that satisfy (21). However, since  $(v^2 + 1)^{\frac{1}{2}} = u$ , (16) and (17) show that we need only use the idempotent elements  $e_r + e_{-r}$ , which are determined by  $v$ , in those parts of the square root which do not depend on the zero root of  $v$ ;  $e_1$  and  $e_{-1}$ , however, are not defined by  $v$  so that it is necessary in any particular case to consider how  $e_0$  and  $\delta_0$  can be broken up into parts which have the required property.

If  $v$  has a zero root with the principal idempotent and nilpotent parts  $e_0, \delta_0$ , then  $e^2 = e$  shows that, although  $\delta'_0 a = -a\delta_0$ , we have

$$(21') \quad e'_0 a = ae_0.$$

We therefore seek to divide  $e_0$  into two idempotent parts,  $e_1$  and  $e_{-1}$ , which are commutative with  $v$  and therefore with  $\delta_0$ . In forming the square root we then attach the value  $+1$  to  $e_1$  and  $-1$  to  $e_{-1}$ .

If  $k_r^2 + 1 = 0$ , then  $g_r = i$  and the corresponding part of  $(v^2 + 1)^{\frac{1}{2}}$  is  $2i(\delta_r + \delta_{-r}) + \delta_r^2 + \delta_{-r}^2$ , and it is readily shown from (15) that this has a square root. The details are left to the reader.

If  $b$  is a solution of  $b'a = -ab$ , then so are also  $t = \tan b$  and  $v = \tan 2b$ . A short calculation then gives  $x = (1 + t)/(1 - t)$  subject to the restrictions already given; this shows the relations between the rational and irrational solutions.

**9.05 The exponential solution.** Some of the difficulties of the solution in §9.04 can be avoided by setting

$$(22) \quad x = \exp(z) \equiv \epsilon^z, \quad z = \text{Log } x$$

where a principal determination of  $\log x$  is to be used. Since this determination of  $\log x$  is a polynomial in  $x$  and  $x'ax = a$ , we have

$$(23) \quad z' = \text{Log } x' = \text{Log } ax^{-1}a^{-1} = a(\text{Log } x^{-1})a^{-1}$$

and therefore

$$(24) \quad 1 = x'axa = \epsilon^{z'}\epsilon^{aza^{-1}} = \epsilon^{z'+aza^{-1}}.$$

From (11)

$$(25) \quad z = \Sigma[(\log g_r) e_r + \eta_r - (\log g_r) e_{-r} + \eta_{-r}] + \theta_1 \eta_1 + \theta_2 (\pi i e_{-1} + \eta_{-1})$$

$$\eta_s = \xi_s - \frac{1}{2} \xi_s^2 + \dots \quad (s = r, -r, 1, -1)$$

and from (13)

$$(26) \quad \begin{aligned} e'_s &= ae_{-s}a^{-1}, & \eta'_s &= -a\eta_{-s}a^{-1} & (s = r, -r), \\ e'_s &= ae_s a^{-1}, & \eta'_s &= -a\eta_s a^{-1} & (s = 1, -1). \end{aligned}$$

Hence

$$\begin{aligned} z' + aza^{-1} &= \Sigma[(\log g_r) e'_r + \eta'_r - (\log g_r) e'_{-r} + \eta'_{-r}] + \theta_1 \eta'_1 + \theta_2 (\pi i e'_{-1} + \eta'_{-1}) \\ &+ \Sigma[(\log g_r) e'_{-r} - \eta'_{-r} - (\log g_r) e'_r - \eta'_r] - \theta_1 \eta'_1 + \theta_2 (\pi i e'_{-1} - \eta'_{-1}) \\ &= 2\theta_2 \pi i e'_{-1}, \end{aligned}$$

and therefore, if we set

$$(27) \quad \zeta = \theta e_{-1} \ (\theta \equiv \theta_2 = 0, 1), \quad w = z - \pi i \zeta,$$

we have

$$(28) \quad w' + awa^{-1} = 0, \quad \zeta' - a\zeta a^{-1} = 0,$$

and

$$(29) \quad w = \Sigma [(\log g_r)e_r + \eta_r - (\log g_{-r})e_{-r} + \eta_{-r}] + \theta_1\eta_1 + \theta_2\eta_{-1}.$$

The general value of  $x$  can therefore be expressed in terms of the solution of the equation discussed in §9.02.

If we now start with  $w$  as a solution of (28) and define  $x$  by  $x = \epsilon^w$ , then

$$x' = \epsilon^{w'} = \epsilon^{-awa^{-1}} = a\epsilon^{-w}a^{-1} = ax^{-1}a^{-1}$$

and therefore  $x$  is a solution of (1); to obtain every solution, however, we must add the terms  $\pi i \zeta$  to  $w$ .

If  $e_0$  is the principal idempotent element corresponding to the root 0, then (29) shows that the presence of the  $\zeta$ -term depends on the division of  $e_0$  into two parts  $e_1$  and  $e_{-1}$  which satisfy the second set of equations in (26); and corresponding to these we have nilpotent parts  $\eta_1$  and  $\eta_{-1}$  which give rise to 1 and  $-1$ , respectively, as roots of  $x$ , or 0,  $\pi i$  as roots of  $z$ .

A form which gives rational parameters is obtained from the exponential solution as follows. Let

$$(30) \quad t = \tanh (z/2) = \frac{\epsilon^z - 1}{\epsilon^z + 1} = \frac{x - 1}{x + 1}$$

then

$$(31) \quad x = \frac{1 + t}{1 - t},$$

and

$$ata^{-1} = \frac{axa^{-1} - 1}{axa^{-1} + 1} = \frac{1 - ax^{-1}a^{-1}}{1 + ax^{-1}a^{-1}} = \frac{1 - x'}{1 + x'} = -t'$$

so that (31) gives a solution of (1). If, however,  $|x + 1| = 0$ , then  $t$  becomes infinite so that (31) cannot give directly any  $x$  which has  $-1$  as a root. This difficulty arises from the fact that  $\tanh(\theta/2) \rightarrow \infty$  when  $\theta \rightarrow \pi i$ ; but, since  $(t + 1)(t - 1) = \epsilon^z$  for all values of  $t$  which do not have an infinite root, that is, one corresponding to a root  $(2k + 1)\pi i$  of  $z$ , hence  $x$  will be a solution of (1) so long as the coordinates of  $z$  are continuous functions of the parameters involved and the limiting value of  $x$  is finite and determinate.

**9.06 Matrices which admit a given transformation.** In (1) we may regard  $x$  as given and  $a$  as unknown; the problem then is to find all matrices  $a$  such that

$$(32) \quad x'ax = a.$$

If we associate with  $a = || a_{ij} ||$  the corresponding tensor of grade 2,

$$u = \sum a_{ij} e_i e_j,$$

we see immediately that (23) corresponds to setting (cf. §5.10)

$$(33) \quad \Pi_2(x)u = u.$$

Hence there is a solution if, and only if,  $x$  has at least one pair of reciprocal roots; in this case  $\Pi_2(x)$  has one or more roots equal to 1 and the various invariant elements corresponding to this root give a linearly independent set of determinations of  $a$ .

When it is required that  $|a| \neq 0$ , another form of solution is preferable. In this case  $x' = ax^{-1}a^{-1}$ ; but, since  $x$  and  $x'$  are similar, we also have  $x' = pxp^{-1}$ , where, if  $p_1$  is one determination of  $p$ , the general form is

$$(34) \quad p = p_1 b, \quad bx = xb, \quad |b| \neq 0.$$

Hence it is necessary that  $x^{-1}$  be similar to  $x$ , say

$$(35) \quad x^{-1} = q_1^{-1} x q_1,$$

which gives immediately

$$(36) \quad a = p_1 b q_1.$$

Conversely, if  $p_1$ ,  $b$ , and  $q_1$  satisfy the given conditions, it follows immediately that (36) gives a solution of (32).

## CHAPTER V

### COMPOUND MATRICES

5.01 In chapter I it was found necessary to consider the adjoint of  $A$  which is a matrix whose coordinates are the first minors of  $|A|$ . We shall now consider a more general class of matrices, called compound matrices, whose coordinates are minors of  $|A|$  of the  $r$ th order; before doing so, however, it is convenient to extend the definition of  $Sxy$  to apply to vectors of higher grade.

5.02 **The scalar product** Let  $x_i = \sum \xi_{ij} e_j$ ,  $y_i = \sum \eta_{ij} e_j$  ( $i = 1, 2, \dots$ ) be arbitrary vectors, then, by equation (37) §1.11 we have

$$(1) \quad |x_1 x_2 \cdots x_r| = \sum_{(i)}^* |\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}| |e_{i_1} e_{i_2} \cdots e_{i_r}|,$$

and hence it is natural to extend the notion of the scalar product by setting

$$(2) \quad S|x_1 x_2 \cdots x_r| |y_1 y_2 \cdots y_r| = \sum_{(i)}^* |\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}| |\eta_{1i_1} \eta_{2i_2} \cdots \eta_{ri_r}|.$$

We then have the following lemma which becomes the ordinary rule for multiplying together two determinants when  $r = n$ .

**LEMMA 1.**

$$(3) \quad S|x_1 x_2 \cdots x_r| |y_1 y_2 \cdots y_r| = |Sx_i y_j|.$$

For  $S|x_1 x_2 \cdots x_r| |e_{i_1} e_{i_2} \cdots e_{i_r}| = |\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}|$ , hence

$$\begin{aligned} S|x_1 x_2 \cdots x_r| |y_1 e_{i_2} \cdots e_{i_r}| &= \sum_{i_1} \eta_{1i_1} |\xi_{1i_1} \cdots \xi_{ri_r}| \\ &= \left( \sum_{i_1} \eta_{1i_1} \xi_{1i_1} \right) |\xi_{2i_2} \cdots \xi_{ri_r}| = |Sx_1 y_1, \xi_{2i_2} \cdots \xi_{ri_r}|; \end{aligned}$$

again

$$\begin{aligned} S|x_1 x_2 \cdots x_r| |y_1 y_2 e_{i_3} \cdots e_{i_r}| &= \sum_{i_2} \eta_{2i_2} S|x_1 \cdots x_r| |y_1 e_{i_2} \cdots e_{i_r}| \\ &= \sum_{i_2} \eta_{2i_2} |Sx_1 y_1, \xi_{2i_2} \cdots \xi_{ri_r}| \\ &= |Sx_1 y_1, Sx_2 y_2, \xi_{3i_3} \cdots \xi_{ri_r}|. \end{aligned}$$

The lemma follows easily by a repetition of this process.

The Laplace expansion of a determinant can clearly be expressed as a scalar product. This is most easily done by introducing the notion of the comple-

ment of a vector relative to the fundamental basis. If  $i_1, i_2, \dots, i_r$  is a sequence of distinct integers in natural order each less than or equal to  $n$  and  $i_{r+1}, \dots, i_n$  the remaining integers up to and including  $n$ , also arranged in natural order, the complement of  $|e_{i_1}e_{i_2} \dots e_{i_r}|$  relatively to the fundamental basis is defined as<sup>1</sup>

$$(4) \quad |e_{i_1}e_{i_2} \dots e_{i_r}|_c = (-1)^{\sum i_\alpha + r(r+1)/2} |e_{i_{r+1}}e_{i_{r+2}} \dots e_{i_n}|$$

and the complement of  $|x_1x_2 \dots x_r|$  by

$$(5) \quad |x_1x_2 \dots x_r|_c = \sum_{(i)}^* |\xi_{1i_1}\xi_{2i_2} \dots \xi_{ri_r}| |e_{i_1}e_{i_2} \dots e_{i_r}|_c,$$

which is a vector of grade  $n - r$ .

Laplace's expansion of a determinant in terms of minors of order  $r$  can now be expressed in the following form.

LEMMA 2.

$$(6) \quad S |x_1x_2 \dots x_r|_c |x_{r+1}x_{r+2} \dots x_n| = |\xi_{11}\xi_{22} \dots \xi_{nn}| = |Sx_ie_j| \\ = S |x_1 \dots x_n| |e_1 \dots e_n| = (-1)^{r(n-r)} S |x_1x_2 \dots x_r| |x_{r+1} \dots x_n|_c.$$

Further as an immediate consequence of (5) we have

LEMMA 3.

$$(7) \quad S |x_1x_2 \dots x_r|_c |y_1y_2 \dots y_r|_c = S |x_1x_2 \dots x_r| |y_1y_2 \dots y_r|.$$

5.03 Compound matrices. If  $A = \sum a_{ij}e_{ij}$ , then, as in (1),

$$|Ax_1Ax_2 \dots Ax_r| = \sum_{(i)}^* |\xi_{1i_1} \dots \xi_{ri_r}| |Ae_{i_1} \dots Ae_{i_r}|.$$

But  $Ae_j = \sum_i a_{ij}e_i$ ; so a second application of (1) gives

$$|Ax_1Ax_2 \dots Ax_r| = \sum_{(i)}^* \sum_{(j)}^* |\xi_{1i_1} \dots \xi_{ri_r}| |a_{i_1j_1} \dots a_{i_rj_r}| |e_{j_1} \dots e_{j_r}|.$$

But the determinants  $|\xi_{1i_1} \dots \xi_{ri_r}|$  are the coordinates of the  $r$ -vector  $|x_1x_2 \dots x_r|$ ; hence  $|Ax_1 \dots Ax_r|$  is a linear vector form in  $|x_1x_2 \dots x_r|$  in the corresponding space of  $\binom{n}{r}$  dimensions. We denote this vector function or matrix by  $C_r(A)$  and write

$$(8) \quad |Ax_1Ax_2 \dots Ax_r| = C_r(A) |x_1x_2 \dots x_r|.$$

We shall call  $C_r(A)$  the  $r$ th compound of  $A$ . Important particular cases are

$$(8') \quad C_1(A) = A, \quad C_n(A) = |A|,$$

<sup>1</sup> The Grassmann notation cannot be conveniently used here since it conflicts with the notation for a determinant. It is sometimes convenient to define the complement of  $|e_1e_2 \dots e_n|$  as 1.

and, if  $k$  is a scalar,

$$(8'') \quad C_r(k) = k^r.$$

THEOREM 1.

$$(9) \quad C_r(AB) = C_r(A)C_r(B).$$

For

$$\begin{aligned} |ABx_1ABx_2 \cdots ABx_r| &= C_r(A) |Bx_1Bx_2 \cdots Bx_r| \\ &= C_r(A)C_r(B) |x_1x_2 \cdots x_r|. \end{aligned}$$

Corollary. If  $|A| \neq 0$ , then

$$(10) \quad [C_r(A)]^{-1} = C_r(A^{-1}).$$

THEOREM 2.

$$(11) \quad [C_r(A)]' = C_r(A').$$

$$\begin{aligned} \text{For } S |x_1x_2 \cdots x_r| C_r(A) |y_1y_2 \cdots y_r| &= |Sx_iAy_j| = |SA'x_iy_j| \\ &= S |A'x_1 \cdots A'x_r| |y_1 \cdots y_r| = S |y_1 \cdots y_r| C_r(A') |x_1 \cdots x_r|. \end{aligned}$$

THEOREM 3. If  $A = \sum_1^m a_i S b_i$ , then

$$(12) \quad C_r(A) = \sum_{(i)}^* |a_i a_{i_2} \cdots a_{i_r}| S |b_{i_1} b_{i_2} \cdots b_{i_r}|.$$

This theorem follows by direct substitution for  $A$  in the left-hand side of (8). It gives a second proof for Theorem 2.

If  $r = m$ , (12) consists of one term only, and this term is 0 unless  $m$  is the rank of  $A$ , a property which might have been made the basis of the definition of rank. In particular, if  $X = \sum_1^r e_i S x_i$ ,  $Y = \sum_1^r y_i S e_i$ , then  $C_r(X) = |e_1 e_2 \cdots e_r| S |x_1 x_2 \cdots x_r|$ ,  $C_r(Y) = |y_1 y_2 \cdots y_r| S |e_1 e_2 \cdots e_r|$  so that  $C_r(XY) = |e_1 e_2 \cdots e_r| S |x_1 x_2 \cdots x_r| |y_1 y_2 \cdots y_r| S |e_1 e_2 \cdots e_r|$ . But  $XY = \sum_{i,j} e_i S x_i y_j S e_j$  so that  $C_r(XY) = |S x_i y_j| |e_1 e_2 \cdots e_r| S |e_1 e_2 \cdots e_r|$ .

Comparing these two forms of  $C_r(XY)$  therefore gives another proof of the first lemma of §5.02.

If we consider the complement of  $|Ax_1Ax_2 \cdots Ax_r|$  we arrive at a new matrix  $C^r(A)$  of order  $\binom{r}{c}$  which is called the *rth supplementary compound* of  $A$ . From (7) and (12) we have

$$\begin{aligned} (13) \quad |Ax_1Ax_2 \cdots Ax_r|_c &= \sum_i^* |a_{i_1} \cdots a_{i_r}|_c S |b_{i_1} \cdots b_{i_r}|_c |x_1 \cdots x_r|_c \\ &= C^r(A) |x_1x_2 \cdots x_r|_c \end{aligned}$$

and derive immediately the following which are analogous to Theorems 1 and 2.

THEOREM 4.

$$(14) \quad C^r(AB) = C^r(A)C^r(B).$$

THEOREM 5.

$$(15) \quad [C^r(A)]' = C^r(A').$$

The following theorems give the connection between compounds and supplementary compounds and also compounds of compounds.

THEOREM 6.

$$(16) \quad C^r(A')C_{n-r}(A) = |A| = C^{n-r}(A)C_r(A').$$

This is the Laplace expansion of the determinant  $|A|$ . Using equation (6) and setting  $|e|$  for  $|e_1e_2 \cdots e_n|$  we have

$$\begin{aligned} |A|S|x_1x_2 \cdots x_r|_c|x_{r+1} \cdots x_n| &= |A|S|x_1 \cdots x_n||e| \\ &= S|Ax_1 \cdots Ax_n||e| \\ &= S|Ax_1 \cdots Ax_r|_c|Ax_{r+1} \cdots Ax_n| \\ &= SC^r(A)|x_1 \cdots x_r|_cC_{n-r}(A)|x_{r+1} \cdots x_n| \\ &= S|x_1 \cdots x_r|_cC^r(A')C_{n-r}(A)|x_{r+1} \cdots x_n| \end{aligned}$$

and, since the  $x$ 's are arbitrary, the first part of the theorem follows. The second part is proved in a similar fashion.

Putting  $r = n - 1$  in (16) gives the following corollary.

$$\text{Corollary.} \quad \text{adj } A = C^{n-1}(A').$$

THEOREM 7.

$$(17) \quad |C_r(A)| = |A|^{\binom{n-1}{r-1}} = |C^r(A)|.$$

For from (16) with  $A'$  in place of  $A$ , and from the fact that the order of  $C_r(A)$  is  $\binom{n}{r}$ , we have

$$|A|^{\binom{n}{r}} = |C^r(A)C_{n-r}(A')| = |C^r(A)||C_{n-r}(A')|$$

and, since  $|A|$  is irreducible when the coordinates of  $A$  are arbitrary variables, it follows that  $|C^r(A)|$  is a power of  $|A|$ . Considerations of degree then show that the theorem is true when the coordinates are variables and, since the identity is integral, it follows that it is also true for any particular values of these variables.

THEOREM 8.

$$(18) \quad |A|^{\binom{n-1}{r-1}} C_s(C_r(A)) = |A|^s C^{\binom{n}{r}-s}(C^{n-r}(A))$$

$$(19) \quad |A|^{\binom{n-1}{r-1}} C_s(C^r(A)) = |A|^s C^{\binom{n}{r}-s}(C_{n-r}(A)).$$

Using (15), (16) and (17) we get

$$C_s(C^{n-r}(A'))C_s^{(n-r)-s}(C^{n-r}(A)) = |C^{n-r}(A)| = |A|^{(n-r)}$$

therefore

$$\begin{aligned} |A|^{(n-r)} C_s(C_r(A)) &= C_s(C_r(A))C_s(C^{n-r}(A'))C_s^{n-s}(C^{n-r}(A)) \\ &= C_s(C_r(A)C^{n-r}(A'))C_s^{n-s}(C^{n-r}(A)) \\ &= C_s(|A|)C_s^{n-s}(C^{n-r}(A)) \\ &= |A|^s C_s^{n-s}(C^{n-r}(A)). \end{aligned}$$

Similarly

$$C_s(C^{n-r}(A'))C_s(C_r(A)) = C_s(|A|) = |A|^s$$

and therefore

$$\begin{aligned} |A|^s C_s^{(n-r)-s}(C^{n-r}(A)) &= C_s^{(n-r)-s}(C^{n-r}(A))C_s(C^{n-r}(A'))C_s(C_r(A)) \\ &= |C^{n-r}(A)| C_s(C_r(A)) \\ &= |A|^{(n-r)} C_s(C_r(A)). \end{aligned}$$

An important particular case is  $C_s(C^{n-1}(A)) = |A|^{s-1}C^{n-s}(A)$  whence  
(20)  $C_s(\text{adj } A) = C_s(C^{n-1}(A')) = |A|^{s-1}C^{n-s}(A')$ .

**5.04 Roots of compound matrices.** If  $A$  has simple elementary divisors and its roots are  $\lambda_1, \lambda_2, \dots, \lambda_n$ , the corresponding invariant vectors being  $a_1, a_2, \dots, a_n$ , then the roots of  $C_r(A)$  are the products  $\lambda_{i_1}\lambda_{i_2} \dots \lambda_{i_r}$  in which no two subscripts are the same and the subscripts are arranged in, say, numerical order; and the invariant vector corresponding to  $\lambda_{i_1}\lambda_{i_2} \dots \lambda_{i_r}$  is  $|a_{i_1}a_{i_2} \dots a_{i_r}|$ . For there are  $\binom{n}{r}$  distinct vectors of this type and

$$C_r(A) |a_{i_1}a_{i_2} \dots a_{i_r}| = |Aa_{i_1} \dots Aa_{i_r}| = \lambda_{i_1}\lambda_{i_2} \dots \lambda_{i_r} |a_{i_1}a_{i_2} \dots a_{i_r}|.$$

Similarly for  $C^r(A)$  the roots and invariants are  $\lambda_{i_1}\lambda_{i_2} \dots \lambda_{i_r}$  and  $|a_{i_1}a_{i_2} \dots a_{i_r}|$ .

It follows from considerations of continuity that the roots are as given above even when the elementary divisors are not simple.

**5.05 Bordered determinants.** Let  $A = ||a_{ij}|| = \sum_{j=1}^n a_j s e_j$ ,  $a_j = \sum_i a_{ij} e_i$ , be any matrix and associate with it two sets of vectors

$$\begin{aligned} X: x_i &= \sum_{j=1}^n \xi_{ij} e_j, \\ & \quad (i = 1, 2, \dots, r) \\ Y: y_i &= \sum_{j=1}^n \eta_{ij} e_j. \end{aligned}$$

Consider the bordered determinant

$$(21) \quad \Delta_r = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \xi_{11} & \cdots & \xi_{r1} \\ a_{21} & a_{22} & \cdots & a_{2n} & \xi_{12} & \cdots & \xi_{r2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \xi_{1n} & \cdots & \xi_{rn} \\ \eta_{11} & \eta_{12} & \cdots & \eta_{1n} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \eta_{r1} & \eta_{r2} & \cdots & \eta_{rn} & 0 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} A & X \\ Y' & 0_r \end{vmatrix}$$

where  $r < n$ , and  $0_r$  is a square block of 0's with  $r$  rows and columns.

If we introduce  $r$  additional fundamental units  $e_{n+1}, \dots, e_{n+r}$ ,  $\Delta_r$  can be regarded as the determinant of a matrix  $\mathfrak{A}$  of order  $n+r$ , namely,

$$\mathfrak{A} = \sum_1^n a_i S e_i + \sum_1^r x_i S e_{n+i} + \sum_1^r e_{n+i} S y_i = \sum_1^{n+r} c_i S d_i.$$

If now we form  $|\mathfrak{A}| = S |e| C_{n+r}(\mathfrak{A}) |e|$  as in §5.03, we have

$$C_{n+r}(\mathfrak{A}) = \sum_{(i)}^* |c_{i_1} \cdots c_{i_{n+r}}| S |d_{i_1} \cdots d_{i_{n+r}}| \quad (i = 1, 2, \dots, n+r).$$

In this form any  $|c_{i_1} \cdots c_{i_{n+r}}|$  which contains more than  $n$  out of  $a_1, \dots, a_n, x_1, \dots, x_r$  is necessarily 0; also, if it does not contain all the  $x$ 's, the corresponding  $|d_{i_1}, \dots, d_{i_{n+r}}|$  will contain more than  $n$  out of  $e_1, \dots, e_n, y_1, \dots, y_r$  and is consequently 0. We therefore have

$$C_{n+r}(\mathfrak{A}) = \sum_{(i)}^* |a_{i_1} a_{i_2} \cdots a_{i_{n-r}} x_1 x_2 \cdots x_r e_{n+1} \cdots e_{n+r}| \\ \times S |e_{i_1} e_{i_2} \cdots e_{i_{n-r}} y_1 y_2 \cdots y_r e_{n+1} \cdots e_{n+r}| \quad (i = 1, 2, \dots, n)$$

and hence, passing back to space of  $n$  dimensions,

$$|\mathfrak{A}| = \sum_i^* S |e| |a_{i_1} \cdots a_{i_{n-r}} x_1 \cdots x_r| S |e_{i_1} \cdots e_{i_{n-r}} y_1 \cdots y_r| |e| \\ = \Sigma^* S |x_1 \cdots x_r| |a_{i_1} \cdots a_{i_{n-r}}| S |e_{i_1} \cdots e_{i_{n-r}}| |y_1 \cdots y_r| \\ = S |x_1 \cdots x_r| C^{n-r}(A) |y_1 \cdots y_r|.$$

This relation shows why the bordered determinant is frequently used in place of the corresponding compound in dealing with the theory of forms.

**5.06 The reduction of bilinear forms.** The Lagrange method of reducing quadratic and bilinear forms to a normal form is, as we shall now see, closely connected with compounds.

If  $A$  is any matrix, not identically 0, there exist vectors  $x_1, y_1$  such that  $Sx_1A y_1 \neq 0$ ; then, setting  $A = A_1$  for convenience, the matrix

$$A_2 = A_1 - A_1 y_1 \frac{SA'_1 x_1}{Sx_1 A_1 y_1}$$

has its rank exactly 1 less than that of  $A$ . For, if  $A_1 z = 0$ , then

$$A_2 z = A_1 z - A_1 y_1 \frac{SA'_1 x_1 \cdot z}{Sx_1 A_1 y_1} = A_1 z - A_1 y_1 \frac{Sx_1 A_1 z}{Sx_1 A_1 y_1} = 0$$

and, conversely if  $A_2 z = 0$ , then

$$A_1 z = A_1 y_1 \frac{Sx_1 A_1 z}{Sx_1 A_1 y_1} = k A_1 y_1,$$

say, or  $A_1(z - ky_1) = 0$ . The null-space of  $A_2$  is therefore obtained from that of  $A_1$  by adding  $y_1$  to its basis, which increases the order of this space by 1 since  $A_1 y_1 \neq 0$ .

If  $A_2 \neq 0$ , this process may be repeated, that is, there exist  $x_2, y_2$  such that  $Sx_2 A_2 y_2 \neq 0$  and the rank of

$$A_3 = A_2 - A_2 y_2 \frac{SA'_2 x_2}{Sx_2 A_2 y_2}$$

is 1 less than that of  $A_2$ . If the rank of  $A$  is  $r$ , we may continue this process by setting

$$(22) \quad A_{s+1} = A_s - A_s y_s \frac{SA'_s x_s}{Sx_s A_s y_s} \quad (s = 1, 2, \dots, r)$$

where  $Sx_s A_s y_s \neq 0$  and  $A_1 = A, A_{r+1} = 0$ ; we then have

$$(23) \quad A = \sum_{s=1}^r A_s y_s \frac{SA'_s x_s}{Sx_s A_s y_s} = \sum_1^r \mathfrak{A}_s$$

where  $\mathfrak{A}_s = A_s y_s \frac{SA'_s x_s}{Sx_s A_s y_s}$  is a matrix of rank 1. Generally speaking, one may take  $x_s = y_s$  and it is of some interest to determine when this is not possible. If  $Sx B x = 0$  for every  $x$ , we readily see that  $B$  is skew. For then  $Se_i B e_i = Se_j B e_j = S(e_i + e_j) B (e_i + e_j) = 0$  and therefore

$$0 = S(e_i + e_j) B (e_i + e_j) = Se_i B e_i + Se_j B e_j + Se_i B e_j + Se_j B e_i,$$

that is,  $Se_i B e_j = -Se_j B e_i$  and hence  $B' = -B$ . Hence we may take  $x_s = y_s$  so long as  $A_s \neq -A'_s$ .

5.07 We shall now derive more explicit forms for the terms in (23) and show how they lead to the Sylvester-Francke theorems on compound determinants.

Let  $x^1, x^2, \dots, x^r, y^1, y^2, \dots, y^r$  be variable vectors and set

$$(24) \quad \begin{aligned} J &= S | x^1 x^1 x^2 \cdots x^r | C_{r+1}(A_s) | y_s y^1 y^2 \cdots y^r | \\ &= S | x^1 x^1 x^2 \cdots x^r | | A_s y_s A_s y^1 \cdots A_s y^r |; \end{aligned}$$

then from (22)

$$\begin{aligned} J &= S | x_s x^1 \cdots x^r | | A_s y_s A_{s+1} y^1 \cdots A_{s+1} y^r | \\ &= | S x_s A_s y_s S x^1 A_{s+1} y^1 \cdots S x^r A_{s+1} y^r |. \end{aligned}$$

If the  $x$ 's denote rows in this determinant, the first row is

$$S x_s A_s y_s, S x_s A_{s+1} y^1, \cdots, S x_s A_{s+1} y^r$$

each term of which is 0 except the first, since  $x_s$  lies in the null-space of  $A'_{s+1}$ , and  $S x_s A_s y_s \neq 0$ . Hence

$$(25) \quad J = S x_s A_s y_s | S x^1 A_{s+1} y^1 \cdots S x^r A_{s+1} y^r |$$

and therefore from (24)

$$\begin{aligned} (26) \quad S | x_s x^1 \cdots x^r | C_{r+1}(A_s) | y_s y^1 \cdots y^r | \\ = S x_s A_s y_s S | x^1 \cdots x^r | C_r(A_{s+1}) | y^1 \cdots y^r |. \end{aligned}$$

Repeated application of this relation gives

$$\begin{aligned} (27) \quad S | x_s x_{s+1} \cdots x_{s+t-1} x^1 x^2 \cdots x^r | C_{r+t}(A_s) | y_s \cdots y_{s+t-1} y^1 \cdots y^r | \\ = S x_s A_s y_s S x_{s+1} A_{s+1} y_{s+1} \cdots S x_{s+t-1} A_{s+t-1} y_{s+t-1} S | x^1 \cdots x^r | \\ \cdot C_r(A_{s+t}) | y^1 \cdots y^r |, \end{aligned}$$

a particular case of which is

$$\begin{aligned} (27') \quad S | x_1 x_2 \cdots x_{s-1} x | C_s(A) | y_1 \cdots y_{s-1} y | \\ = S x_1 A_1 y_1 \cdots S x_{s-1} A_{s-1} y_{s-1} S x_s A_s y_s. \end{aligned}$$

To simplify these and similar formulae we shall now use a single letter to indicate a sequence of vectors; thus we shall set  $X_{s, s+t-1}$  for  $x_s x_{s+1} \cdots x_{s+t-1}$  and  $Y^r$  for  $y^1 y^2 \cdots y^r$ ; also  $C_{r, s}$  for  $C_r(A_s)$ . Equations (26) and (27) may then be written

$$(26a) \quad S | x_s X^r | C_{r+1, s} | y_s Y^r | = S x_s A_s y_s S | X^r | C_{r, s+1} | Y^r |,$$

$$(27a) \quad S | X_{s, s+t-1} X^r | C_{r+t, s} | Y_{s, s+t-1} Y^r | = \prod_{i=s}^{s+t-1} S x_i A_i y_i S | X^r | C_{r, s+i} | Y^r |.$$

We get a more convenient form for (26a), namely

$$\begin{aligned} (28) \quad S | X_{s, t} X^r | C_{r+t-s+1, s} | Y_{s, t} Y^r | \\ = S x_s A_s y_s S | X_{s+1, t} X^r | C_{r+t-s, s+1} | Y_{s+1, t} Y^r | \end{aligned}$$

by replacing  $r$  by  $r+t-s$  and then changing  $x^1 x^2 \cdots x^{r+t-s}$  into  $x_{s+1} \cdots x_t x^1 \cdots x^r$  along with a similar change in the  $y$ 's. Putting  $s = 1, 2, \cdots, t$  in succession and forming the product of corresponding sides of

the equations so obtained from (28) we get after canceling the common factors, which are not identically 0 provided that  $r + t$  is not greater than the rank of  $A$ ,

$$(29) \quad S | X_t X^r | C_{r+t, t} | Y_t Y^r | = \prod_1^t S x_i A_i y_i \cdot S | X^r | C_{r, t+1} | Y^r |,$$

or from (27')

$$(30) \quad S | X_t X^r | C_{r+t} | Y_t Y^r | = S | X_t | C_t | Y_t | S | X^r | C_{r, t+1} | Y^r |$$

which may also be written in the form

$$(30') \quad K \equiv \frac{S | X_t X^r | C_{r+t} | Y_t Y^r |}{S | X_t | C_t | Y_t |} = | S x^i A_{t+1} y^j |;$$

in particular

$$(31) \quad \frac{S | X_t x | C_{t+1}(A) | Y_t y |}{S | X_t | C_t(A) | Y_t |} = S x A_{t+1} y.$$

This gives a definition of  $A_{t+1}$  which may be used in place of (22); it shows that this matrix depends on  $2t$  vector parameters. It is more convenient for some purposes to use the matrix  $A^{(t)}$  defined by

$$(32) \quad S x A^{(t)} y = S | X_t x | C_{t+1}(A) | Y_t y |.$$

From (31) we then have  $S x^i A_{t+1} y^j = S x^i A^{(t)} y^j / S | X_t | C_t | Y_t |$  and therefore from (30')

$$(33) \quad K = \frac{| S x^i A^{(t)} y^j |}{[S | X_t | C_t | Y_t |]^r} = \frac{S | X^r | C_r(A^{(t)}) | Y^r |}{[S | X_t | C_t(A) | Y_t |]^r}$$

Hence

$$(34) \quad S | X_t X^r | C_{r+t}(A) | Y_t Y^r | = \frac{S | X^r | C_r(A^{(t)}) | Y^r |}{[S | X_t | C_t(A) | Y_t |]^{r-1}}$$

which is readily recognized as Sylvester's theorem if the  $x$ 's are replaced by fundamental units and the integral form of (33) is used.

**5.08 Invariant factors.** We shall now apply the above results in deriving the normal form of §3.02. We require first, however, the following lemma.

**LEMMA 4.** *If  $A(\lambda)$  is a matrix polynomial, there exists a constant vector  $y$  and a vector polynomial  $x$  such that  $Sx Ay$  is the highest common factor of the coordinates of  $A$ .*

Let  $y = \sum \eta_i e_i$  be a vector whose coordinates are variables independent of  $\lambda$ . Let  $\alpha_1$  be the H. C. F. of the coordinates of  $A = || a_i ||$  and set

$$A = \alpha_1 B, \quad B y = \sum \eta_i b_i e_i = \sum \beta_i e_i.$$

There is no value  $\lambda_1$  of  $\lambda$  independent of the  $\eta$ 's for which every  $\beta_i = 0$ ; for if this were so,  $\lambda - \lambda_1$  would be a factor of each  $b_i$ , and  $\alpha_1$  could not then be the H. C. F. of the  $a_{ij}$ . Hence the resultant of  $\beta_1, \beta_2, \dots, \beta_n$  as polynomials in  $\lambda$  is not identically 0 as a polynomial in the  $\eta$ 's; there are therefore values of the  $\eta$ 's for which this resultant is not 0, and for these values the  $\beta$ 's have no factor common to all. There then exist scalar polynomials  $\xi_1, \xi_2, \dots, \xi_n$  such that  $\Sigma \xi_i \beta_i = 1$  and therefore, if  $x = \Sigma \xi_i e_i$ , we have  $SxBy = 1$  or  $SxAy = \alpha_1$ .

Returning now to the form of  $A$  given in §5.06, namely

$$A = \sum_1^r \frac{A_s y_s S A'_s x_s}{S x_s A_s y_s},$$

we can as above choose  $x_s, y_s$  in such a manner that  $S x_s A_s y_s = \alpha_s$  is the highest common factor of the coordinates of  $A_s$  and, when this is done,  $v_s = A_s y_s / \alpha_s$ ,  $u_s = A'_s x_s / \alpha_s$  are integral in  $\lambda$ . We then have

$$(35) \quad A = \sum_1^r \frac{A_s y_s S A'_s x_s}{\alpha_s} = \Sigma \alpha_s v_s S u_s.$$

Moreover  $A_s y_i = 0 = A'_s x_i$  when  $i < s$  and therefore in

$$S | x_1 \cdots x_r | | A_1 y_1 A_2 y_2 \cdots A_r y_r | = | S x_s A_s y_i | = | S A'_s x_i y_i |$$

all terms on one side of the main diagonal are 0 so that it reduces to  $S x_1 A_1 y_1 \cdots S x_r A_r y_r = \alpha_1 \alpha_2 \cdots \alpha_r$ . Hence, dividing by  $\alpha_1 \cdots \alpha_r$  and replacing  $A_s y_i / \alpha_s$  by  $v_i$  as above, we see that  $| x_1 \cdots x_r |$  and  $| v_1 \cdots v_r |$  are not 0 for any value of  $\lambda$ , and therefore the constituent vectors in each set remain linearly independent for all values of  $\lambda$ . It follows in the same way that the sets  $u_1, \dots, u_r$  and  $y_1, \dots, y_r$ , respectively, are also linearly independent for all values of  $\lambda$ , that is, these four sets are elementary sets. It follows from Theorem 5 §4.03, that we can find elementary polynomials  $P$  and  $Q$  such that

$$P v_i = e_i = Q' u_i \quad (i = 1, 2, \dots, r),$$

and hence

$$(36) \quad PAQ = P \left( \sum_1^r \alpha_s v_s S u_s \right) Q = \sum_1^r \alpha_s e_s S e_s,$$

which is the normal form of §3.02.

**5.09 Vector products.** Let  $x_i = \Sigma \xi_{ij} e_j$ , ( $i = 1, 2, \dots, r$ ) be a set of arbitrary vectors and consider the set of all products of the form  $\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}$ , arranged in some definite order. These products may then be regarded as the coordinates of a hypernumber<sup>2</sup> of order  $n^r$  which we shall call the *tensor product*<sup>3</sup>

<sup>2</sup> The term 'hypernumber' is used in place of vector, as defined in §1.01 since we now wish to use the term 'vector' in a more restricted sense.

<sup>3</sup> This product was called by Grassmann the general or indeterminate product.

of  $x_1, x_2, \dots, x_r$  and we shall denote it by  $x_1 x_2 \dots x_r$ . In particular if we take all the products  $e_{i_1} e_{i_2} \dots e_{i_r}$  ( $i_1, i_2, \dots, i_r = 1, 2, \dots, n$ ) each has all its coordinates zero except one, which has the value 1, and no two are equal. Further

$$x_1 x_2 \dots x_r = \sum \xi_{1 i_1} \xi_{2 i_2} \dots \xi_{r i_r} e_{i_1} e_{i_2} \dots e_{i_r}.$$

If we regard the products  $e_{i_1} e_{i_2} \dots e_{i_r}$  as the basis of the set of hypernumbers, we are naturally led to consider sums of the type

$$w = \sum \omega_{i_1 i_2 \dots i_r} e_{i_1} e_{i_2} \dots e_{i_r}$$

where the  $\omega$ 's are scalars; and we shall call such a hypernumber a *tensor* of grade  $r$ . It is readily seen that the product  $x_1 x_2 \dots x_r$  is distributive and homogeneous with regard to each of its factors, that is,

$$x_1 (\lambda x_2 + \mu y_2) x_3 \dots x_r = \lambda x_1 x_2 \dots x_r + \mu x_1 y_2 x_3 \dots x_r.$$

The product of two tensors of grade  $r$  and  $s$  is then defined by assuming the distributive law and setting

$$(e_{i_1} e_{i_2} \dots e_{i_r})(e_{j_1} e_{j_2} \dots e_{j_s}) = e_{i_1} \dots e_{i_r} e_{j_1} \dots e_{j_s}.$$

It is easily shown that the product so defined is associative; it is however not commutative as is seen from the example

$$\begin{aligned} x_1 x_2 - x_2 x_1 &= \sum \sum (\xi_{1 i_1} \xi_{2 i_2} - \xi_{1 i_2} \xi_{2 i_1}) e_{i_1} e_{i_2} \\ &= \sum_{(i)}^* \begin{vmatrix} \xi_{1 i_1} & \xi_{1 i_2} \\ \xi_{2 i_1} & \xi_{2 i_2} \end{vmatrix} (e_{i_1} e_{i_2} - e_{i_2} e_{i_1}). \end{aligned}$$

Here the coefficients of  $e_{i_1} e_{i_2} - e_{i_2} e_{i_1}$  ( $i_1 < i_2$ ) are the coordinates of  $|x_1 x_2|$  so that this tensor might have been defined in terms of the tensor product by setting

$$|x_1 x_2| = x_1 x_2 - x_2 x_1.$$

In the same way, if we form the expression<sup>4</sup>

$$f(x_1, x_2, \dots, x_r) = \sum \text{sgn}(i_1, i_2, \dots, i_r) x_{i_1} x_{i_2} \dots x_{i_r},$$

and expand it in terms of the coordinates of the  $x$ 's and the fundamental units, it is readily shown that the result is

$$\sum_{(i)}^* |\xi_{1 i_1} \xi_{2 i_2} \dots \xi_{r i_r}| f(e_{i_1}, e_{i_2}, \dots, e_{i_r}).$$

<sup>4</sup> The determinant of a square array of vectors  $x_{ij}$  ( $i, j = 1, 2, \dots, r$ ) may be defined as

$$|x_{ij}| = \sum \text{sgn}(i_1, i_2, \dots, i_r) x_{1 i_1} x_{2 i_2} \dots x_{r i_r}.$$

In this definition the row marks are kept in normal order and the column marks permuted; a different expression is obtained if the rôles of the row and column marks are interchanged but, as these determinants seem to have little intrinsic interest, it is not worth while to develop a notation for the numerous variants of the definition given above.

Here the scalar multipliers are the same as the coordinates of  $|x_1 x_2 \cdots x_r|$  and hence the definition of §1.11 may now be replaced by

$$|x_1 x_2 \cdots x_r| = \sum \text{sgn}(i_1, i_2, \dots, i_r) x_{i_1} x_{i_2} \cdots x_{i_r},$$

which justifies the notation used. We then have

$$(37) \quad |x_1 x_2 \cdots x_r| = \sum_{(i)}^* |\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}| |e_{i_1} e_{i_2} \cdots e_{i_r}|.$$

It is easily seen that the tensors  $|e_{i_1} e_{i_2} \cdots e_{i_r}|$  are linearly independent and (37) therefore shows that they form a basis for the set of vectors of grade  $r$ . Any expression of the form

$$\sum \xi_{i_1 i_2 \dots i_r} |e_{i_1} e_{i_2} \cdots e_{i_r}|$$

is called a vector of grade  $r$  and a vector of the form (37) is called a pure vector of grade  $r$ .

**5.10 The direct product.** If  $A_i = ||a_p^{(i)}||$  ( $i = 1, 2, \dots, r$ ) is a sequence of matrices of order  $n_i$ , then

$$(38) \quad A_1 x_1 A_2 x_2 \cdots A_r x_r = \sum_{i,j} a_{i_1 j_1}^{(1)} a_{i_2 j_2}^{(2)} \cdots a_{i_r j_r}^{(r)} \xi_{1j_1} \xi_{2j_2} \cdots \xi_{rj_r} e_{i_1} e_{i_2} \cdots e_{i_r} \\ = \mathfrak{A}(x_1 x_2 \cdots x_r)$$

where  $\mathfrak{A}$  is a linear homogeneous tensor function of  $x_1 x_2 \cdots x_r$ , that is, a matrix in space of  $n^r$  dimensions. This matrix is called the direct product<sup>5</sup> of  $A_1, A_2, \dots, A_r$  and is denoted by  $A_1 \times A_2 \times \cdots \times A_r$ . Obviously

$$(39) \quad A_1 B_1 \times A_2 B_2 \times \cdots = (A_1 \times A_2 \times \cdots)(B_1 \times B_2 \times \cdots),$$

and the form of (38) shows that

$$(40) \quad (A_1 \times A_2 \times \cdots)' = A_1' \times A_2' \times \cdots.$$

From (39) we have, on putting  $r = 1$  for convenience,

$$A_1 \times A_2 \times A_3 = (A_1 \times 1 \times 1)(1 \times A_2 \times 1)(1 \times 1 \times A_3).$$

Making  $A_i = 1$  ( $i = 2, 3, \dots, r$ ) in (38) we have

$$A_1 x_1 x_2 \cdots x_r = \sum a_{i_1 j_1}^{(1)} \xi_{1j_1} \xi_{2i_2} \cdots \xi_{ri_r} e_{i_1} e_{i_2} \cdots e_{i_r}$$

and hence the determinant of the corresponding matrix equals  $|A_1|^{n^{r-1}}$ . Treating the other factors in the same way we then see that

$$(41) \quad |A_1 \times A_2 \times \cdots \times A_r| = |A_1 A_2 \cdots A_r|^{n^{r-1}}.$$

Again if as in §5.04 we take  $x_1$  as an invariant vector of  $A_1$ ,  $x_2$  as an invariant vector of  $A_2$ , and so on, and denote the roots of  $A_i$  by  $\lambda_{ij}$ , we see that the roots

<sup>5</sup> This definition may be generalized by taking  $x_1, x_2, \dots$  as vectors in different spaces of possibly different orders. See also §7.03.

of  $A_1 \times A_2 \times \cdots \times A_r$  are the various products  $\lambda_{1i_1}\lambda_{2i_2} \cdots \lambda_{ri_r}$ . When the roots of each matrix are distinct, this gives equation (41) and, since this is an integral relation among the coefficients of the  $A$ 's, it follows that it is true in general.

An important particular case arises when each of the matrices in (38) equals the same matrix  $A$ ; the resultant matrix is denoted by  $\Pi_r(A)$ , that is

$$(42) \quad \Pi_r(A) = A \times A \times \cdots \quad (r \text{ factors}).$$

It is sometimes called the *product transformation*. Relations (39), (40), and (41) then become

$$(43) \quad \Pi_r(AB) = \Pi_r(A)\Pi_r(B), \quad \Pi_r(A)' = \Pi_r(A'), \quad |\Pi_r(A)| = |A|^{r^{n-1}}.$$

**5.11 Induced or power matrices.** If  $x_1, x_2, \cdots, x_r$  are arbitrary vectors, the symmetric expression obtained by forming their products in every possible order and adding is called a *permanent*. It is usually denoted by  $\left| \begin{smallmatrix} x_1 & x_2 & \cdots & x_r \\ x_1 & x_2 & \cdots & x_r \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_r \end{smallmatrix} \right|$  but it will be more convenient here to denote it by  $\{x_1 x_2 \cdots x_r\}$ ; and similarly, if  $\alpha_{ij}$  is a square array of scalars, we shall denote by  $\{\alpha_{1i_1} \alpha_{2i_2} \cdots \alpha_{ri_r}\}$  the function  $\Sigma \alpha_{1i_1} \alpha_{2i_2} \cdots \alpha_{ri_r}$  in which the summation stretches over every permutation of 1, 2,  $\cdots$ ,  $r$ .

If some of the  $x$ 's are equal, the terms of  $\{x_1 x_2 \cdots x_r\}$  become equal in sets each of which has the same number of terms. If the  $x$ 's fall into  $s$  groups of  $i_1, i_2, \cdots, i_s$  members, respectively, the members in each group being equal to one another, then

$$\frac{\{x_1 x_2 \cdots x_r\}}{i_1! i_2! \cdots i_s!} \quad (\Sigma i_j = r)$$

has integral coefficients. For the present we shall denote this expression by  $\{x_1 x_2 \cdots x_r\}^*$ , but sometimes it will be more convenient to use the more explicit notation

$$\left\{ \begin{array}{cccc} x_1 & x_2 & \cdots & x_s \\ i_1 & i_2 & \cdots & i_s \end{array} \right\}$$

in which  $i_1$  of the  $x$ 's equal  $x_1$ ,  $i_2$  equal  $x_2$ , etc.; this notation is, in fact, that already used in §2.08, for instance,

$$\begin{aligned} \{x \ x \ y\} &= 2x^2y + 2xyx + 2yx^2 \\ \left\{ \begin{array}{cc} x & y \\ 2 & 1 \end{array} \right\} &= x^2y + xyx + yx^2 = \frac{1}{2}\{xxy\}. \end{aligned}$$

The same convention applies immediately to  $\{\alpha_{1i_1} \alpha_{2i_2} \cdots \alpha_{ri_r}\}$ .

In the notation just explained we have

$$(44) \quad \{x_1 x_2 \cdots x_r\} = \Sigma'' \{\xi_{1i_1} \xi_{2i_2} \cdots \xi_{ri_r}\}^* \{e_{i_1} e_{i_2} \cdots e_{i_r}\}$$

where the summation  $\Sigma''$  extends over all combinations  $i_1 i_2 \cdots i_r$  of the first  $n$  integers repetition being allowed. This shows that the set of all permanents

of grade  $r$  has the basis  $\{e_{i_1}e_{i_2} \cdots e_{i_r}\}$  of order  $(n + r - 1)!/r!(n - 1)!$ . From (44) we readily derive

$$(45) \quad \{Ax_1Ax_2 \cdots Ax_r\} = \sum_{i,j}'' \{a_{i_1j_1}a_{i_2j_2} \cdots a_{i_rj_r}\}^* \{\xi_{1i_1} \cdots \xi_{ri_r}\}^* \{e_{i_1} \cdots e_{i_r}\}$$

which is a linear tensor form in  $\{x_1x_2 \cdots x_r\}$ . We may therefore set

$$(46) \quad \{Ax_1Ax_2 \cdots Ax_r\} = P_r(A)\{x_1x_2 \cdots x_r\},$$

where  $P_r(A)$  is a matrix of order  $(n + r - 1)!/r!(n - 1)!$  whose coordinates are the polynomials in the coordinates of  $A$  which are given in (45); this matrix is called the  $r$ th *induced* or *power* matrix of  $A$ . As with  $C_r(A)$  and  $\Pi_r(A)$  it follows that

$$(47) \quad \begin{aligned} P_r(AB) &= P_r(A)P_r(B), \quad P_r(A)' = P_r(A'), \\ |P_r(A)| &= |A| \binom{n+r-1}{r-1}; \end{aligned}$$

also the roots of  $P_r(A)$  are the various products of the form  $\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} \cdots \lambda_r^{\alpha_r}$  for which  $\Sigma\alpha_i = r$ .

**5.12 Associated matrices.** The matrices considered in the preceding sections have certain common properties; the coordinates of each are functions of the variable matrix  $A$  and, if  $T(A)$  stands for any one of them, then

$$(48) \quad T(AB) = T(A)T(B).$$

Following Schur, who first treated the general problem of determining all such matrices, we shall call any matrix with these properties an *associated* matrix. If  $S$  is any constant matrix in the same space as  $T(A)$ , then  $T_1(A) = ST(A)S^{-1}$  is clearly also an associated matrix; associated matrices related in this manner are said to be *equivalent*.

Let the orders of  $A$  and  $T(A)$  be  $n$  and  $m$  respectively and denote the corresponding identity matrices by  $1_n$  and  $1_m$ ; then from (48)

$$(49) \quad T^2(1_n) = T(1_n), \quad T(1_n)T(A) = T(A) = T(A)T(1_n).$$

If  $s$  is the rank of  $T(1_n)$ , we can find a matrix  $S$  which transforms  $T(1_n)$  into a diagonal matrix with  $s$  1's in the main diagonal and zeros elsewhere; and we may without real loss of generality assume that  $T(1_n)$  has this form to start with, and write

$$T(1_n) = \left\| \begin{array}{cc} 1_s & 0 \\ 0 & 0 \end{array} \right\|$$

The second equation of (49) then shows that  $T(A)$  has the form

$$T(A) = \left\| \begin{array}{cc} T_s(A) & 0 \\ 0 & 0 \end{array} \right\|$$

and we shall therefore assume that  $s = m$  so that  $T(1_n) = 1_m$ . It follows from this that  $|T(A)| \neq 0$  so that  $T(A)$  is not singular for every  $A$ ; we shall then say that  $T$  is non-singular.

A non-singular associated matrix  $T(A)$  is reducible (cf. §3.10) if it can be expressed in the form  $T(A) = T_1(A) + T_2(A)$  where, if  $E_1 = T_1(1_n)$ ,  $E_2 = T_2(1_n)$ , so that  $E_1 + E_2 = 1_m$ , then

$$\begin{aligned} T_1(A) &= E_1 T(A) E_1, & T_2(A) &= E_2 T(A) E_2 \\ E_1 T(A) E_2 &= 0 & E_2 T(A) E_1 &= 0 \end{aligned}$$

so that

$$\begin{aligned} E_1^2 &= E_1, & E_2^2 &= E_2 \\ E_1 E_2 &= 0 & E_2 E_1 &= 0 \end{aligned}$$

and there is therefore an equivalent associated matrix  $t(A)$  which has the form

$$t(A) = \begin{vmatrix} t_1(A) & 0 \\ 0 & t_2(A) \end{vmatrix}$$

When  $T(A)$  is reducible in this manner we have

$$\begin{aligned} T_1(AB) &= E_1 T(AB) E_1 = E_1 T(A) T(B) E_1 \\ &= E_1 T(A) (E_1 + E_2) T(B) E_1 \\ &= E_1 T(A) E_1 T(B) E_1 = T_1(A) T_1(B) \end{aligned}$$

so that  $T_1(A)$  and  $T_2(A)$  are separately associated matrices. We may therefore assume  $T(A)$  irreducible without loss of generality since reducible associated matrices may be built up out of irreducible ones by reversing the process used above.

5.13 We shall now show that, if  $\lambda$  is a scalar variable, then  $T(\lambda)$  is a power of  $\lambda$ . To begin with we shall assume that the coordinates of  $T(\lambda)$  are rational functions in  $\lambda$  and that  $T(1)$  is finite; we can then set  $T(\lambda) = T_1(\lambda)/f(\lambda)$  where  $f(\lambda)$  is a scalar polynomial whose leading coefficient is 1 and the coordinates of  $T_1(\lambda)$  are polynomials whose highest common factor has no factor in common with  $f(\lambda)$ . If  $\mu$  is a second scalar variable, (48) then gives

$$\frac{T_1(\lambda)T_1(\mu)}{f(\lambda)f(\mu)} = \frac{T_1(\lambda\mu)}{f(\lambda\mu)},$$

hence  $f(\lambda\mu)$  is a factor of  $f(\lambda)f(\mu)$ , from which it follows readily that  $f(\lambda\mu) = f(\lambda)f(\mu)$ ; so that  $f(\lambda)$  is a power of  $\lambda$  and also

$$(50) \quad T_1(\lambda\mu) = T_1(\lambda)T_1(\mu).$$

We also have  $f(1) = 1$  and hence  $T_1(1_n) = T(1_n) = 1_m$ .

Let  $T_1(\lambda) = F_0 + \lambda F_1 + \cdots + \lambda^s F_s (F_s \neq 0)$ ; then from (50)

$$F_0 + \lambda\mu F_1 + \cdots + \lambda^s \mu^s F_s = (F_0 + \lambda F_1 + \cdots)(F_0 + \mu F_1 + \cdots)$$

which gives

$$F_i^2 = F_i, F_i F_j = 0 \quad (i \neq j), \quad (i, j = 0, 1, \dots, s).$$

Now

$$T_1(\lambda)T(A) = f(\lambda)T(\lambda)T(A) = f(\lambda)T(\lambda A) = T(A)T_1(\lambda);$$

therefore

$$\Sigma F_i T(A) \lambda^i = \Sigma T(A) F_i \lambda^i$$

and hence on comparing powers of  $\lambda$  we have

$$F_i T(A) = T(A) F_i$$

and, since  $\Sigma F_i = T_1(1) = 1_m$  and we have assumed that  $T(A)$  is irreducible, it follows that every  $F_i = 0$  except  $F_s$ , which therefore equals  $1_m$ . Hence  $T_1(\lambda) = \lambda^s$  and, since  $f(\lambda)$  is a power of  $\lambda$ , we may set

$$(51) \quad T(\lambda) = \lambda^r.$$

Since  $T(\lambda A) = T(\lambda)T(A) = \lambda^r T(A)$ , we have the following theorem.

**THEOREM 9.** *If  $T(A)$  is irreducible, and if  $T(\lambda)$  is a rational function of the scalar variable  $\lambda$ , then  $T(\lambda) = \lambda^r$  and the coordinates of  $T(A)$  are homogeneous functions of order  $r$  in the coordinates of  $A$ .*

The restriction that  $T(\lambda)$  is rational in  $\lambda$  is not wholly necessary. For instance, if  $q$  is any whole number and  $\epsilon$  a corresponding primitive root of 1, then  $T^q(\epsilon) = 1_m$  and from this it follows without much difficulty that  $T(\epsilon) = \epsilon^s$  where  $s$  is an integer which may be taken to be the same for any finite number of values of  $q$ . It follows then that, if  $T(\lambda) = ||t_{ij}(\lambda)||$ , the functions  $t_{ij}(\lambda)$  satisfy the equation

$$t_{ij}(\epsilon\lambda) = \epsilon^s t_{ij}(\lambda)$$

and under very wide assumptions as to the nature of the functions  $t_{ij}$  it follows from this that  $T(\lambda)$  has the form  $\lambda^r$ . Again, if we assume that  $T(\lambda) = \lambda^\alpha \sum_{-\alpha}^{\alpha} T_r \lambda^r$ , then  $T(\lambda)T(\mu) = T(\lambda\mu)$  gives immediately

$$T_r \mu^{r+\alpha} = T(\mu)$$

so that only one value of  $r$  is admissible and for this value  $T_r = 1$  as before.

5.14 If the coordinates of  $T(A)$  are rational functions of the coordinates  $a_{ij}$  of  $A$ , so that  $r$  is an integer, we can set  $T(A) = T_1(A)/f(A)$  where the coordinates of  $T_1(A)$  are integral in the  $a_{ij}$  and  $f(A)$  is a scalar polynomial in these variables which has no factor common to all the coordinates of  $T_1(A)$ . As in (50) we then have

$$T_1(AB) = T_1(A)T_1(B), f(AB) = f(A)f(B).$$

It follows from the theory of scalar invariants that  $f(A)$  can be taken as a positive integral power of  $|A|$ ; we shall therefore from this point on assume that the coordinates of  $T(A)$  are homogeneous polynomials in the coordinates of  $A$  unless the contrary is stated explicitly. We shall call  $r$  the *index* of  $T(A)$ .

**THEOREM 10.** *If  $T(A)$  is an associated matrix of order  $m$  and index  $r$ , and if the roots of  $A$  are  $\alpha_1, \alpha_2, \dots, \alpha_n$ , then the roots of  $T(A)$  have the form  $\alpha_1^{r_1} \alpha_2^{r_2} \dots \alpha_n^{r_n}$  where  $\sum r_i = r$ . The actual choice of the exponents  $r$  depends on the particular associated matrix in question but, if  $\alpha_1^{r_1} \alpha_2^{r_2} \dots \alpha_n^{r_n}$  is one root, all the distinct quantities obtained from it by permuting the  $\alpha$ 's are also roots.*

If the roots of  $A$  are arbitrary variables, then  $A$  is similar to a diagonal matrix  $A_1 = \sum \alpha_i e_{ii}$ . We can express  $T(A_1)$  as a polynomial<sup>6</sup> in the  $\alpha$ 's, say

$$(52) \quad T(A_1) = \sum \alpha_1^{r_1} \alpha_2^{r_2} \dots \alpha_n^{r_n} F_{r_1 r_2 \dots r_n}$$

where the  $F$ 's are constant matrices. If now  $B = \sum \beta_i e_{ii}$  is a second variable diagonal matrix, the relation  $T(A_1 B) = T(A_1) T(B)$  gives as in (50)

$$(53) \quad \begin{aligned} F_{r_1 r_2 \dots r_n}^2 &= F_{r_1 r_2 \dots r_n} \\ F_{r_1 r_2 \dots r_n} F_{s_1 s_2 \dots s_n} &= 0 \quad ((r_1, r_2, \dots) \neq (s_1, s_2, \dots)) \end{aligned}$$

and hence  $T(A_1)$  can be expressed as a diagonal matrix with roots of the required form; these roots may of course be multiple since the rank of  $F_{r_1 \dots r_n}$  is not necessarily 1, the elementary divisors are, however, simple.

Since the associated matrices of similar matrices are similar, it follows that the roots of the characteristic equation of  $T(A)$  are given by those terms in (52) for which  $F_{r_1 r_2 \dots r_n} \neq 0$ ; and, since this equation has coefficients which are polynomials in the coordinates of  $A$ , the roots of  $T(A)$  remain in this form even when the roots of  $A$  are not necessarily all different.

The rest of the theorem follows from the fact that the trace of  $T(A_1)$  equals that of  $T(A)$  which is rational in the coordinates of  $A$  and is therefore symmetric in the  $\alpha$ 's.

**THEOREM 11.** *The value of the determinant of  $T(A)$  is  $|A|^{rm/n}$  and  $rm/n$  is an integer.*

For  $T(A)T(\text{adj}A) = T(|A|) = |A|^r$  and therefore  $|T(A)|$  is a power of  $|A|$ , say  $|A|^s$ . But  $T(A)$  is a matrix of order  $m$  whose coordinates are polynomials in the coordinates of  $A$ . Hence  $sn = mr$  and  $rm/n$  is an integer.

**5.15 Transformable systems.** From a scalar point of view each of the associated matrices discussed in §§5.03–5.11 can be characterized by a set of scalar functions  $f_k$  ( $k = 1, 2, \dots, m$ ) of one or more sets of variables ( $\xi_i$ ,

<sup>6</sup> If we merely assume that  $T(A_1)$  is a convergent series of the form (52), equation (53) still holds. It follows that there are only a finite number of terms in (52) since (53) shows that there is no linear relation among those  $F_{r_1 \dots r_n}$  which are not zero. Let  $F_i$  be the sum of those  $F_{r_1 \dots r_n}$  for which  $\sum r_i$  has a fixed value  $\rho_i$ ; then  $T(\lambda) = \sum \lambda^{\rho_i} F_i$ , and as before only one value of  $\rho_i$  is admissible when  $T(A)$  is irreducible.

$j = 1, 2, \dots, n), i = 1, 2, \dots, r$ , which have the following property: if the  $\xi$ 's are subjected to a linear transformation

$$\xi'_{ij} = \sum_{s=1}^n a_{js} \xi_{is} \quad (j = 1, 2, \dots, n; i = 1, 2, \dots, r)$$

and if  $f'_k$  is the result of replacing  $\xi_{ij}$  by  $\xi'_{ij}$  in  $f_k$ , then

$$f'_k = \sum_{s=1}^m \alpha_{ks} f_s.$$

where the  $\alpha$ 's are functions of the  $a_{ij}$  and are independent of the  $\xi$ 's. For instance, corresponding to  $C_2(A)$  we have

$$f_i \equiv f_{pq} = \begin{vmatrix} \xi_{1p} & \xi_{1q} \\ \xi_{2p} & \xi_{2q} \end{vmatrix} \quad (p, q = 1, 2, \dots, n; p < q)$$

for which

$$\alpha_{ij} \equiv \alpha_{pq, rs} = \begin{vmatrix} a_{pr} & a_{qr} \\ a_{ps} & a_{qs} \end{vmatrix}.$$

We may, and will, always assume that there are no constant multipliers such that  $\sum \lambda_i f_i = 0$ . Such systems of functions were first considered by Sylvester; they are now generally called *transformable systems*.

If we put  $T(A) = \|\alpha_{ij}\|$ , we have immediately  $T(AB) = T(A)T(B)$ , and consequently there is an associated matrix corresponding to every transformable system. Conversely, there is a transformable system corresponding to an associated matrix. For if  $X = \|\xi_{ij}\|$  is a variable matrix and  $c$  an arbitrary constant vector in the space of  $T(A)$ , then the coordinates of  $T(X)c$  form a transformable system since  $T(A)T(X)c = T(AX)c$  and  $c$  can be so determined that there is no constant vector  $b$  such that  $SbT(X)c \equiv 0$ .

The basis  $f_k$  ( $k = 1, 2, \dots, m$ ) may of course be replaced by any basis which is equivalent in the sense of linear dependence, the result of such a change being to replace  $T(A)$  by an equivalent associated matrix. If in particular there exists a basis

$$g_1, g_2, \dots, g_{k_1}, h_1, h_2, \dots, h_{k_2} \quad (k_1 + k_2 = k)$$

such that the  $g$ 's and the  $h$ 's form separate transformable systems, then  $T(A)$  is reducible; and conversely, if  $T(A)$  is reducible, there always exists a basis of this kind.

**5.16 Transformable linear sets.** If we adopt the tensor point of view rather than the scalar one, an associated matrix is found to be connected with a linear set  $\mathfrak{F}$  of constant tensors, derived from the fundamental units  $e_i$ , such that, when  $e_i$  is replaced by  $Ae_i$  ( $i = 1, 2, \dots, n$ ) in the members of the basis of  $\mathfrak{F}$ , then the new tensors are linearly dependent on the old; in other words

the set  $\mathfrak{F}$  is invariant as a whole under any linear transformation  $A$  of the fundamental units. For instance, in the case of  $C_2(A)$  cited above,  $\mathfrak{F}$  is the linear set defined by

$$|e_i e_j| \quad (i, j = 1, 2, \dots, n; i < j).$$

We shall call a set which has this property a *transformable linear set*.

Let  $u_1, u_2, \dots, u_m$  be a transformable linear set of tensors of grade  $r$  and let  $u'_i$  be the tensor that results when  $e_j$  is replaced by  $Ae_j$  ( $j = 1, 2, \dots, n$ ) in  $u_i$ . Since the set is transformable, we have

$$u'_i = \sum_j \alpha_{ji} u_j = T(A)u_i \quad (i = 1, 2, \dots, m)$$

where the  $\alpha_{ij}$  are homogeneous polynomials in the coordinates of  $A$  of degree  $r$ . If we employ a second transformation  $B$ , we then have

$$u''_i = T(A)T(B)u_i, \quad u''_i = T(AB)u_i \quad (i = 1, 2, \dots, m)$$

and therefore  $T(A)$  is an associated matrix.

We have now to show that there is a transformable linear set corresponding to every associated matrix. In doing this it is convenient to extend the notation  $Suv$  to the case where  $u$  and  $v$  are tensors of grade  $r$ . Let  $E_i$  ( $i = 1, 2, \dots, n^r$ ) be the unit tensors of grade  $r$  and

$$u = \sum \psi_i E_i, \quad v = \sum \varphi_i E_i$$

any tensors of grade  $r$ ; we then define  $Suv$  by

$$Suv = \left( \sum_1^{n^r} \psi_i \varphi_i \right) / r!$$

where the numerical divisor is introduced solely in order not to disagree with the definition of §5.02.

Let  $x_i = \sum \xi_{ij} e_j$  ( $i = 1, 2, \dots, r$ ) be a set of variable vectors and  $X_i$  ( $i = 1, 2, \dots, s$ ) the set of tensors of the form  $x_1^{j_1} x_2^{j_2} \dots x_r^{j_r}$  ( $\sum j_i = r$ ); we can then put any product  $\xi_{11}^{\beta_{11}} \xi_{12}^{\beta_{12}} \dots \xi_{rn}^{\beta_{rn}}$  for which  $\sum \beta_{ij} = r$  in the form  $k S E_i X_j$ ,  $k$  being a numerical factor. This can be done in more than one way as a rule; in fact, if  $\sum_j \beta_{ij} = \beta_i$ , then

$$\xi_{11}^{\beta_{11}} \dots \xi_{1n}^{\beta_{1n}} = \frac{1}{\beta_1!} S e_1^{\beta_{11}} \dots e_n^{\beta_{1n}} x_1^{\beta_1}$$

and from the definition of  $Suv$  it is clear that the factors in  $e_1^{\beta_{11}} \dots e_n^{\beta_{1n}}$  can be permuted in any way without altering the value of the scalar. It follows that

$$\xi_{11}^{\beta_{11}} \dots \xi_{1n}^{\beta_{1n}} = \frac{1}{\beta_{11}! \beta_{12}! \dots \beta_{1n}!} S \left\{ \begin{matrix} e_1 & \dots & e_n \\ \beta_{11} & \dots & \beta_{1n} \end{matrix} \right\} x_1^{\beta_1}$$

and repeating this process we get

$$k_1 \xi_{11}^{\beta_{11}} \xi_{12}^{\beta_{12}} \cdots \xi_{rn}^{\beta_{rn}} = S \left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{11} & \cdots & \beta_{1n} \end{matrix} \right\} \cdots \left\{ \begin{matrix} \cdots & e_n \\ \cdots & \beta_{rn} \end{matrix} \right\} x_1^{\beta_1} x_2^{\beta_2} \cdots x_r^{\beta_r}$$

where  $k_1$  is a numerical factor whose value is immaterial for our present purposes.

If  $f$  is any homogeneous polynomial in the variables  $\xi_{ij}$  of degree  $\rho$ , it can be expressed uniquely in the form

$$f = \sum \varphi_{\beta_{11} \cdots \beta_{rn}} S \left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{11} & \cdots & \beta_{1n} \end{matrix} \right\} \cdots \left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{r1} & \cdots & \beta_{rn} \end{matrix} \right\} x_1^{\beta_1} \cdots x_r^{\beta_r}$$

where the inner summation extends over the partitions of  $\beta_i$  into  $\beta_{i1}, \beta_{i2}, \cdots, \beta_{in}$  ( $i = 1, 2, \cdots, r$ ) and the outer over all values of  $\beta_1, \beta_2, \cdots, \beta_r$  for which  $\sum \beta_i = \rho$ . We may therefore write

$$f = \sum_1^s SF_j X_j$$

where, as above,  $X_j = x_1^{\beta_1} x_2^{\beta_2} \cdots x_r^{\beta_r}$  and

$$F_j \equiv F_{\beta_1 \beta_2 \cdots \beta_r} = \sum \varphi_{\beta_{11} \cdots \beta_{rn}} \left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{11} & \cdots & \beta_{1n} \end{matrix} \right\} \cdots \left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{r1} & \cdots & \beta_{rn} \end{matrix} \right\}.$$

The expression of  $f$  in this form is unique. In the first place,  $F_j \neq 0$  unless each  $\varphi_{\beta_{11} \cdots \beta_{rn}}$  is zero, since the set of tensors of the form

$$\left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{11} & \cdots & \beta_{1n} \end{matrix} \right\} \cdots \left\{ \begin{matrix} e_1 & \cdots & e_n \\ \beta_{r1} & \cdots & \beta_{rn} \end{matrix} \right\} \quad (\sum \beta_{ij} = \rho)$$

are clearly linearly independent. Further, if  $\sum SF_j X_j \equiv 0$ , then each  $SF_j X_j$  is zero since each gives rise to terms of different type in the  $\xi_{ij}$ ; and finally the form of  $F_j$  shows that  $SF_j X_j = 0$  only if  $F_j = 0$  since in

$$SF_j X_j = k_1 \sum \varphi_{\beta_{11} \cdots \beta_{rn}} \xi_{11}^{\beta_{11}} \cdots \xi_{rn}^{\beta_{rn}}$$

each term of the summation is of different type in the  $\xi_{ij}$ .

Let  $(f_k)$  be a transformable system; we can now write uniquely

$$(54) \quad f_k = \sum_j SF_{kj} X_j \quad (k = 1, 2, \cdots, m)$$

and we may set

$$F = \sum_1^{n^r} f_i E_i = \sum_{i,j} E_i SF_{ij} X_j$$

where  $f_i \equiv 0$  when  $i > m$ . If we transform the  $x$ 's by  $A = || a_{ij} ||$  and denote  $\Pi_r(A)$  temporarily by  $\Pi$ , then  $X_j$  becomes  $\Pi X_j$  and  $F$  is transformed into  $F^*$  where

$$(55) \quad F^* = \sum_{i,j} E_i SF_{ij} \Pi X_j = \sum_{i,j} E_i S \Pi' F_{ij} X_j.$$

But the  $f$ 's form a transformable system and hence by this transformation  $f_i$  becomes

$$f'_i = \sum_k \alpha_{ik} f_k$$

so that

$$(56) \quad F^{*i} = \sum_{k,i} \alpha_{ik} f_k E_i = \sum_i E_i S \sum_k \alpha_{ik} \sum_j F_{kj} X_j.$$

Comparing (55) and (56) we have

$$(57) \quad \sum_j S \left[ \sum_k \alpha_{ik} F_{kj} - \Pi' F_{ij} \right] X_j = 0$$

and therefore, as was proved above, each of the terms of the summation is zero, that is,

$$(58) \quad \Pi' F_{ij} = \sum_k \alpha_{ik} F_{kj}$$

and therefore, if  $j$  is kept fixed, the linear set

$$(59) \quad (F_{1j}, F_{2j}, \dots)$$

is transformable provided  $F_{1j}, F_{2j}, \dots$  are linearly independent.

If there is no  $j$  for which the set (59) is linearly independent we proceed as follows. Let  $f_{ij} = SF_{ij}X_j$  so that

$$(60) \quad \begin{aligned} f_1 &= f_{11} + f_{12} + \dots + f_{1s} \\ f_2 &= f_{21} + f_{22} + \dots + f_{2s} \\ &\dots\dots\dots \\ &\dots\dots\dots \\ f_m &= f_{m1} + f_{m2} + \dots + f_{ms}. \end{aligned}$$

If the removal of any column of this array leaves the new  $f_i$  so defined linearly independent, they form a transformable system which defines the same associated matrix as the original system; we shall therefore suppose that the removal of any column leads to linear relations among the rows, the coefficients of these relations being constants. Remove now the first column; then by non-singular constant combinations of the rows we can make certain of them, say the first  $m_1$ , equal 0, the remainder being linearly independent. On applying the same transformation to the rows of (60), which leaves it still a transformable system, we see that we may replace (60) by an array of the form

$$(61) \quad \begin{aligned} f_1 &= f_{11} \\ &\dots\dots\dots \\ f_{m_1} &= f_{m_1,1} \\ f_{m_1+1} &= f_{m_1+1,1} + f_{m_1+1,2} + \dots + f_{m_1+1,s} \\ &\dots\dots\dots \\ f_m &= f_{m1} + f_{m2} + \dots + f_{ms} \end{aligned}$$

where  $f_{m_1+i} - f_{m_1+i,1}$  ( $i = 1, 2, \dots, m - m_1$ ) are linearly independent. It follows that  $f_1, \dots, f_{m_1}$  are transformed among themselves and so form a transformable system. For these functions are transformed in the same way as  $f_{11}, f_{21}, \dots, f_{m_1,1}$ , and if the last  $m - m_1$  rows of (61) were involved in the transformation, this would mean that  $f_{11}, \dots, f_{m_1,1}$ , when transformed, would depend on  $f_{m_1+1,1}$  etc., which is impossible owing to the linear independence of  $f_{m_1+i} - f_{m_1+i,1}$  ( $i = 1, 2, \dots, m - m_1$ ).

Corresponding to the first column of (61) we have tensors  $F_{11}, F_{21}, \dots, F_{m_1}$  and we may suppose this basis so chosen that  $F_{i1}$  ( $i = 1, 2, \dots, p$ ) are linearly independent and  $F_{j1} = 0$  for  $j > p$ ; and this can be done without disturbing the general form of (61). If  $p = m$ , we have a transformable system of the type we wish to obtain and we shall therefore assume that  $p < m$ . We may also suppose the basis so chosen that  $S\bar{F}_{i1}F_{j1} = \delta_{ij}$  ( $i, j = 1, 2, \dots, p$ ) as in Lemma 2, §1.09. It follows from what we have proved above that  $F_{11}, F_{21}, \dots, F_{m_1,1}$  is a transformable set.

Let  $A$  be a real matrix, the corresponding transformation of the  $F$ 's, being, as in (58),

$$(62) \quad F_{i1}^* = \sum_j \alpha_{ij} F_{j1} = \Pi' F_{i1}, \quad (i = 1, 2, \dots, p);$$

we then have

$$(63) \quad \bar{F}_{i1}^* = \sum_j \bar{\alpha}_{ij} \bar{F}_{j1} = \Pi'(A) \bar{F}_{i1}$$

so that the  $\bar{F}_{i1}$  also forms a transformable set. Since  $F_{11}, \dots, F_{m_1,1}$  form a transformable set,  $\alpha_{ij}$  and  $\bar{\alpha}_{ij}$  are 0 when  $i > m_1$  and  $j \leq m_1$  no matter what matrix  $A$  is. Now

$$\alpha_{ij} = S\bar{F}_{j1}F_{i1}^* = S\bar{F}_{j1}\Pi'(A)F_{i1} = S\Pi(A)\bar{F}_{j1}F_{i1} = S\Pi'(A')\bar{F}_{j1}F_{i1}$$

which equals 0 for  $i \leq m_1, j > m_1$  since by (63)  $\Pi'(A')\bar{F}_{j1}$  is derived from  $\bar{F}_{j1}$  by the transformation  $A'$  on the  $x$ 's and for  $j \leq m_1$  is therefore linearly dependent on  $\bar{F}_{j1}$  ( $j = 1, 2, \dots, m_1$ ). Hence the last  $m - m_1$  rows in (61) also form a transformable system, which is only possible if the system  $f_1, f_2, \dots, f_m$  is reducible. If  $T(A)$  is irreducible, the corresponding transformable system is irreducible and it follows now that there also corresponds to it an irreducible transformable set of tensors.

5.17 We have now shown that to every associated matrix  $T(A)$  of index  $r$  and order  $m$  there corresponds a transformable linear set of constant tensors  $F_1, F_2, \dots, F_m$  of grade  $r$  whose law of transformation is given by (62). Also since  $\Pi'(A) = \Pi(A')$ , we have

$$(64) \quad \Pi F_i = \Sigma \alpha'_{ik} F_k, \quad \Pi \bar{F}_i = \Sigma \bar{\alpha}'_{ik} \bar{F}_k$$

where  $T(A') = || \alpha'_{ij} ||$ .

Since  $F_1, F_2, \dots, F_m$  are linearly independent, we can find a supplement to this set in the set of all tensors of grade  $r$ , say

$$G_1, G_2, \dots, G_\mu \quad (\mu = n^r - m)$$

such that

$$(65) \quad S\bar{F}_i G_j = 0.$$

It is convenient also to choose bases for both sets such that

$$(65') \quad S\bar{F}_i F_j = \delta_{ij} = S\bar{G}_i G_j.$$

Since the two sets together form a basis for the space of  $\Pi$ , we can set

$$\Pi' G_j = \Sigma \beta_{ki} F_k + \Sigma \gamma_{ki} G_k$$

and this gives

$$\beta_{ij} = S\bar{F}_i \Pi' G_j = S G_j \Pi \bar{F}_i,$$

which is 0 from (64) and (65), hence the  $G$ 's are transformed among themselves by  $\Pi'$ . This means, however, that  $\Pi'$  is reducible, and when it is expressed in terms of the basis  $(F_1, \dots, F_m, G_1, \dots, G_\mu)$ , the part corresponding to  $(F_1, \dots, F_m)$  has the form  $\|\alpha_{ij}\|$  and is therefore similar to  $T(A)$ . Hence:

**THEOREM 12.** *Every irreducible associated matrix  $T(A)$  of index  $r$  is equivalent to an irreducible part of  $\Pi_r(A)$ , and conversely.*

**5.18 Irreducible transformable sets.** If  $F$  is a member of a transformable linear set  $\mathfrak{F} = (F_1, F_2, \dots, F_m)$ , the total set of tensors derived from  $F$  by all linear transformations of the fundamental units clearly form a transformable linear set which is contained in  $\mathfrak{F}$ , say  $\mathfrak{F}_1$ ; and we may suppose the basis of  $\mathfrak{F}$  so chosen that  $\mathfrak{F}_1 = (F_1, F_2, \dots, F_k)$  and  $S\bar{F}_i F_j = \delta_{ij}$  ( $i, j = 1, 2, \dots, m$ ). Let  $G$  be an element of  $(F_{k+1}, \dots, F_m)$  and  $G'$  a transform of  $G$  so that

$$G' = \sum_{i=1}^m \gamma_i F_i.$$

Then  $S\bar{F}_i G' = \gamma_i$ . But  $S\bar{F}_i G' = S\bar{F}'_i G$ , where  $F'_i$  is the transform of  $F_i$  obtained by the transverse of the transformation which produced  $G'$  from  $G$  so that  $\bar{F}'_i$  is in  $\mathfrak{F}_1$  for  $i \leq k$ . Hence  $\gamma_i = 0$  for  $i = 1, 2, \dots, k$ , that is,  $(F_{k+1}, \dots, F_m)$  is also a transformable set; and so, when the original set is irreducible, we must have  $\mathfrak{F}_1 = \mathfrak{F}$ . If we say that  $F$  generates  $\mathfrak{F}$ , this result may be stated as follows.

**LEMMA 5.** *An irreducible transformable linear set is generated by any one of its members.*

We may choose  $F$  so that it is homogeneous in each  $e_i$ ; for if we replace, say,  $e_1$  by  $\lambda e_1$ , then  $F$  has the form  $\Sigma \lambda^k H_k$  and by the same argument as in §5.13, any  $H_k$  which is not 0 is homogeneous in  $e_1$  and belongs to  $\mathfrak{F}$ . A repetition of

this argument shows that we may choose  $F$  to be homogeneous in each of the fundamental units which occur in it. If  $r$  is the grade of  $F$ , we may assume that  $F$  depends on  $e_1, e_2, \dots, e_s$ , and, if  $k_1, k_2, \dots, k_s$  are the corresponding degrees of homogeneity, then  $\Sigma k_i = r$  and, when convenient, we may arrange the notation so that  $k_1 \geq k_2 \geq \dots \geq k_s$ .

If we now replace  $e_1$  in  $F$  by  $e_1 + \lambda e_i$  ( $i > s$ ), the coefficient  $H$  of  $\lambda$  is not 0, since  $i > s$ , and  $H$  becomes  $k_1 F$  when  $e_1$  is replaced by  $e_1$ ; it therefore forms a generator of  $\mathfrak{F}$  in which the degree of  $e_1$  is one less than before. It follows that, when  $r \leq n$ , we may choose a generator which is linear and homogeneous in  $r$  units  $e_1, e_2, \dots, e_r$ . It is also readily shown that such a tensor defines an irreducible transformable linear set if, and only if, it forms an irreducible set when the transformations of the units are restricted to permuting the first  $r$   $e$ 's among themselves. Further, since the choice of fundamental units is arbitrary, we may replace them by variable vectors  $x_1, x_2, \dots, x_r$ . For instance, the transformable sets associated with  $\Pi_r, P_r$  and  $C_r$  are  $x_1 x_2 \dots x_r$ ,  $\{x_1 x_2 \dots x_r\}$  and  $|x_1 x_2 \dots x_r|$ , respectively, and of these the first is reducible and the other two irreducible.

5.19 It is not difficult to calculate directly the irreducible transformable sets for small values of  $r$  by the aid of the results of the preceding paragraph. If we denote  $x_1, x_2, \dots$  by  $1, 2, \dots$ , the following are generators for  $r = 2, 3$ .

	generator	$r = 2$	order
2.1	{12}		$n(n+1)/2$
2.2	12		$n(n-1)/2$
		$r = 3$	
3.1	{123}		$n(n+1)(n+2)/6$
3.2	1 23		$n(n^2-1)/3$
3.3	1{23}		$n(n^2-1)/3$
3.4	123		$n(n-1)(n-2)/6$ .

This method of determining the generators directly is tedious and the following method is preferable.<sup>7</sup> Any generator has the form

$$w_1 = \Sigma \omega_{i_1 i_2 \dots i_r} x_{i_1} x_{i_2} \dots x_{i_r}$$

and if  $q_{i_1 \dots i_r}$  denotes the substitution  $\begin{pmatrix} 1, 2, \dots, r \\ i_1, i_2, \dots, i_r \end{pmatrix}$ , we may write

$$\begin{aligned} w_1 &= \Sigma \omega_{i_1 i_2 \dots i_r} q_{i_1 i_2 \dots i_r} x_{i_1} x_{i_2} \dots x_{i_r} \\ &= q_1(x_1 x_2 \dots x_r) \end{aligned}$$

where  $q_1$  may be regarded (see chap. 10) as an element of the algebra  $S$  whose units are the operators  $q$  of the symmetric group on  $r$  letters. Now  $w_1$  generates a transformable set and hence, if  $w_i = q_i(x_1 \dots x_r)$  ( $i = 1, 2, \dots$ ) is a

<sup>7</sup> Fuller details of the actual determination of the generators will be found in Weyl: *Gruppentheorie und Quantentheorie*, 2 ed. chap. 5.

basis of the set, and  $Q$  is the set of elements  $q_1, q_2, \dots$  in  $S$ , then the set of elements  $Qq = (q_1q, q_2q, \dots)$  must be the same as the set  $Q$ , that is, in the terminology of chapter 10,  $Q$  is a semi-invariant subalgebra of  $S$ ; conversely any such semi-invariant subalgebra gives rise to a transformable set and this set is irreducible if the semi-invariant subalgebra is minimal, that is, is contained in no other such subalgebra.

It follows now from the form derived for a group algebra such as  $S$  that we get all independent generators as follows. In the first place the operators of  $S$  can be divided into sets<sup>8</sup>  $S_k$  ( $k = 1, 2, \dots, t$ ) such that (i) the product of an element of  $S_k$  into an element of  $S_j$  ( $k \neq j$ ) is zero; (ii) in the field of complex numbers a basis for each  $S_k$  can be chosen which gives the algebra of matrices of order  $n_k^2$ ; and in an arbitrary field  $S$  is the direct product of a matrix algebra and a division algebra; (iii) there exists a set of elements  $u_{k1}, u_{k2}, \dots, u_{k\nu_k}$  in  $S_k$  such that  $\sum_i u_{ki}$  is the identity of  $S_k$  and  $u_{ki}^2 = u_{ki} \neq 0, u_{ki}u_{kj} = 0$  ( $i \neq j$ )

and such that the set of elements  $u_{ki}S_ku_{ki}$  is a division algebra, which in the case of the complex field contains only one independent element; (iv) the elements of  $S_k$  can be divided into  $\nu_k$  sets  $u_{ki}S_k$  ( $i = 1, 2, \dots$ ) each of which is a minimal semi-invariant subalgebra of  $S$  and therefore corresponds to an irreducible transformable set.

<sup>8</sup> It is shown in the theory of groups that  $t$  equals the number of partitions of  $r$ .

## CHAPTER VI

### SYMMETRIC, SKEW, AND HERMITIAN MATRICES

**6.01 Hermitian matrices.** If we denote by  $\bar{x}$  the matrix which is derived from  $x$  by replacing each coordinate by its conjugate imaginary, then  $x$  is called a *hermitian* matrix if

$$(1) \quad \bar{x} = x'.$$

We may always set  $x = x_1 + ix_2$  where  $x_1$  and  $x_2$  are real and (1) shows that, when  $x$  is hermitian,

$$(2) \quad x'_1 = x_1, x'_2 = -x_2,$$

so that the theory of real symmetric and real skew matrices is contained in that of the hermitian matrix. The following are a few properties which follow immediately from the definition; their proof is left to the reader.

If  $x$  and  $y$  are hermitian and  $a$  is arbitrary, then

$$x + y, \bar{x}, x', ax\bar{a}', xy + yx, i(xy - yx),$$

are all hermitian.

Any matrix  $x$  can be expressed uniquely in the form  $a + ib$  where  $2a = x + \bar{x}'$ ,  $2b = -i(x - \bar{x}')$  are hermitian.

The product of two commutative hermitian matrices is hermitian. In particular, any integral power of a hermitian matrix  $x$  is hermitian; and, if  $g(\lambda)$  is a scalar polynomial with real coefficients,  $g(x)$  is hermitian.

**THEOREM 1.** *If  $a, b, c, \dots$  are hermitian matrices such that  $a^2 + b^2 + c^2 + \dots = 0$ , then  $a, b, c, \dots$  are all 0.*

If  $\Sigma a^2 = 0$ , its trace is 0; but  $\Sigma a^2 = \Sigma a\bar{a}'$  and the trace of the latter is the sum of the squares of the absolute values of the coordinates of  $a, b, \dots$ ; hence each of these coordinates is 0.

**THEOREM 2.** *The roots of a hermitian matrix are real and its elementary divisors are simple.*

Let  $x$  be a hermitian matrix and  $g(\lambda)$  its reduced characteristic function. Since  $g(x) = 0$ , we have  $0 = \bar{g}(\bar{x}) = \bar{g}(x')$  and, since  $x$  and  $x'$  have the same reduced characteristic function, it follows that  $g(\lambda) \equiv \bar{g}(\lambda)$ , that is, the coefficients of  $g$  are real. Suppose that  $\xi_1 = \alpha + i\beta$  ( $\beta \neq 0$ ) is a root of  $g(\lambda)$ ; then  $\xi_2 = \alpha - i\beta \neq \xi_1$  is also a root, and we may set

$$(3) \quad g(\lambda) = (\lambda - \xi_1)(g_1(\lambda) + ig_2(\lambda)) = (\lambda - \xi_2)(g_1(\lambda) - ig_2(\lambda))$$



In view of the opening paragraph of this section Theorems 1-4 apply also to real symmetric matrices; they apply also to real skew matrices except that Theorem 2 must be modified to state that the roots are pure imaginaries.

**6.02 The invariant vectors of a hermitian matrix.** Let  $H$  be a hermitian matrix,  $\alpha_1, \alpha_2$  two different roots, and  $a_1, a_2$  the corresponding invariant vectors so chosen that  $Sa_i\bar{a}_i = 1$ ; then, since  $Ha_1 = \alpha_1 a_1$ ,  $\bar{H}\bar{a}_1 = \alpha_1 \bar{a}_1$ , we have

$$\alpha_1 Sa_2 \bar{a}_1 = Sa_2 \bar{H} \bar{a}_1 = S\bar{H}' a_2 \bar{a}_1 = \alpha_2 Sa_2 \bar{a}_1$$

and, since  $\alpha_1 \neq \alpha_2$ , we must have  $Sa_2 \bar{a}_1 = 0$ . Again, if  $\alpha$  is a repeated root of order  $s$  and  $a_1, a_2, \dots, a_s$  a corresponding set of invariant vectors we may choose these vectors (cf. §1.09) so that  $Sa_i \bar{a}_j = \delta_{ij}$ . The invariant vectors may therefore be so chosen that they form a unitary set and

$$(4) \quad H = \sum \alpha_i a_i S \bar{a}_i.$$

If  $U$  is the matrix defined by

$$(5) \quad Ue_i = a_i \quad (i = 1, 2, \dots, n),$$

then

$$(6) \quad U\bar{U}' = 1,$$

so that  $U$  is unitary, and if  $A$  is the diagonal matrix  $\sum_1^r \alpha_i e_i S e_i$ , then

$$(7) \quad H = UA\bar{U}' = UAU^{-1}.$$

We may therefore say:

**THEOREM 5.** *A hermitian matrix can be transformed to its canonical form by a unitary matrix.*

If  $H$  is a real symmetric matrix, the roots and invariant vectors are real, and hence  $U$  is a real orthogonal matrix. Hence

**THEOREM 6.** *A real symmetric matrix can be transformed to its canonical form by a real orthogonal matrix.*

If  $T$  is a real skew matrix,  $h = iT$  is hermitian. The non-zero roots of  $T$  are therefore pure imaginaries and occur in pairs of opposite sign. The invariant vectors corresponding to the zero roots are real and hence by the proof just given they may be taken orthogonal to each other and to each of the other invariant vectors. Hence, if the rank of  $T$  is  $r$ , we can find a real orthogonal matrix which transforms it into a form in which the last  $n - r$  rows and columns are zero.

Let  $ia$  be a root of  $T$  which is not 0 and  $a = b + ic$  a corresponding invariant vector; then  $Ta = ia a$  so that

$$Tb = -ac, \quad Tc = ab.$$

Hence

$$-\alpha Sc^2 = ScTb = -SbTc = -\alpha Sb^2, \quad -\alpha Sbc = SbTb = 0,$$

which gives

$$Sb^2 = Sc^2, \quad Sbc = 0.$$

We can then choose  $a$  so that  $Sb^2 = Sc^2 = 1$  and can therefore find a real orthogonal matrix which transforms  $T$  into

$$(8) \quad \begin{matrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \\ & 0 & \alpha_2 \\ & -\alpha_2 & 0 \\ & & & \ddots \\ & & & & \ddots \end{matrix} = \Sigma \alpha_j (e_{2j-1} S e_{2j} - e_{2j} S e_{2j-1}).$$

We have therefore the following theorem.

**THEOREM 7.** *If  $T$  is a real skew matrix, its non-zero roots are pure imaginaries and occur in pairs of opposite sign; its rank is even; and it can be transformed into the form (8) by a real orthogonal matrix.*

**6.03 Unitary and orthogonal matrices.** The following properties of a unitary matrix follow immediately from its definition by equation (6).

The product of two unitary matrices is unitary.

The transform of a hermitian matrix by a unitary matrix is hermitian.

The transform of a unitary matrix by a unitary matrix is unitary.

If  $H_1$  and  $H_2$  are hermitian, a short calculation shows that

$$(9) \quad U_1 = \frac{1 - iH_1}{1 + iH_1}, \quad U_2 = \frac{iH_2 - 1}{iH_2 + 1}$$

are unitary (the inverses used exist since a hermitian matrix has only real roots). Solving (9) for  $H_1$  and  $H_2$  on the assumption that the requisite inverses exist we get

$$H_1 = \frac{i(U_1 - 1)}{U_1 + 1}, \quad H_2 = \frac{i(U_2 + 1)}{U_2 - 1}.$$

These are hermitian when  $U_1$  and  $U_2$  are unitary, and therefore any unitary matrix which has no root equal to  $-1$  can be put in the first form while the second can be used when  $U$  has no root equal to  $1$ .

**THEOREM 8.** *The absolute value of each root of a unitary matrix equals 1.*

Let  $\alpha + i\beta$  be a root and  $a + ib$  a corresponding invariant vector; then

$$U(a + ib) = (\alpha + i\beta)(a + ib), \quad \bar{U}(a - ib) = (\alpha - i\beta)(a - ib).$$

Hence

$$\begin{aligned} Sa^2 + Sb^2 &= S(a + ib)(a - ib) = S(a + ib)U'\bar{U}(a - ib) = SU(a + ib)\bar{U}(a - ib) \\ &= (\alpha^2 + \beta^2)S(a + ib)(a - ib) = (\alpha^2 + \beta^2)S(a^2 + b^2), \end{aligned}$$

so that  $\alpha^2 + \beta^2 = 1$ .

*Corollary.*

$$(10) \quad \begin{aligned} U^{-1}(a + ib) &= (\alpha - i\beta)(a + ib), \\ U'(a - ib) &= \bar{U}^{-1}(a - ib) = (\alpha + i\beta)(a - ib). \end{aligned}$$

**THEOREM 9.** *The elementary divisors of a unitary matrix are simple.*

For, if we have

$$U(a_1 + ib_1) = (\alpha + i\beta)(a_1 + ib_1), \quad U(a_2 + ib_2) = (\alpha + i\beta)(a_2 + ib_2) + (a_1 + ib_1),$$

then from (10)

$$\begin{aligned} (\alpha + i\beta)S(a_1 - ib_1)(a_2 + ib_2) &= SU'(a_1 - ib_1)(a_2 + ib_2) = S(a_1 - ib_1)U(a_2 + ib_2) \\ &= (\alpha + i\beta)S(a_1 - ib_1)(a_2 + ib_2) + S(a_1 - ib_1)(a_1 + ib_1) \end{aligned}$$

which is impossible since  $S(a_1 - ib_1)(a_1 + ib_1) = Sa_1^2 + Sb_1^2 \neq 0$ .

The results of this section apply immediately to real<sup>1</sup> orthogonal matrices; it is however convenient to repeat (9).

**THEOREM 10.** *If  $U$  is a real orthogonal matrix, it can be expressed in the form  $(1 + T)/(1 - T)$  if it has no root equal to 1 and in the form  $(T - 1)/(T + 1)$  if it has no root equal to  $-1$ , the matrix  $T$  being a real skew matrix in both cases; and any real matrix of this form which is not infinite, is a real orthogonal matrix.*

**6.04 Hermitian and quasi-hermitian forms.** Let  $H$  be a hermitian matrix and  $x = u + iv$  a vector of which  $u$  and  $iv$  are the real and imaginary parts; then the bilinear form  $f = S\bar{x}Hx$  is called a *hermitian form*. Such a form is real since

$$\bar{f} = Sx\bar{H}\bar{x} = SxH'\bar{x} = S\bar{x}Hx = f.$$

In particular, if  $x$  and  $H$  are real,  $f$  is a real quadratic form and, if  $H = iT$  is a pure imaginary,  $T$  is skew and  $f = 0$ .

If we express  $H$  in terms of its invariant vectors, say  $H = \sum \alpha_i a_i S \bar{a}_i$  and then put  $x = \sum \xi_i a_i$ , the form  $f$  becomes  $f = \sum \alpha_i \xi_i \bar{\xi}_i$ . This shows that, if all the roots of  $H$  are positive, the value of  $f$  is positive for all values of  $x$ ;  $H$  and  $f$  are then said to be *positive definite*. Similarly if all the roots are negative,  $H$  and  $f$  are *negative definite*. If some roots are 0 so that  $f$  vanishes for some value of  $x \neq 0$ , we say that  $H$  and  $f$  are *semi-definite*, positive or negative as the case may be. It follows immediately that, when  $H$  is semi-definite,  $S\bar{x}Hx$  can only vanish if  $Hx = 0$ .

<sup>1</sup> The first part of the theorem applies also to complex orthogonal matrices.

THEOREM 11. *If  $H$  and  $K$  are hermitian and  $H$  is definite, the elementary divisors of  $H\lambda - K$  are real and simple.*

Since  $H\lambda - K$  and  $\bar{\nu}(H\lambda - K)$  are equivalent, we may suppose that  $H$  is positive definite. Its roots are then positive so that

$$H^{\frac{1}{2}} = \sum \alpha_i^{\frac{1}{2}} a_i S \bar{a}_i$$

has real roots and hence is also hermitian so that  $H^{-\frac{1}{2}}KH^{-\frac{1}{2}}$  is hermitian. But

$$H\lambda - K = H^{\frac{1}{2}}(\lambda - H^{-\frac{1}{2}}KH^{-\frac{1}{2}})H^{\frac{1}{2}}$$

so that  $H\lambda - K$  is equivalent to  $\lambda - H^{-\frac{1}{2}}KH^{-\frac{1}{2}}$  which has real and simple elementary divisors by Theorem 2.

In order to include the theory of complex symmetric matrices we shall now define a type of matrix somewhat more general than the hermitian matrix and closely connected with it. If  $A = A(\lambda)$  is a matrix whose coefficients are analytic functions of a scalar variable  $\lambda$ , we shall call it *quasi-hermitian* if

$$(11) \quad A'(\lambda) = A(-\lambda).$$

For convenience we shall set  $A''(\lambda)$  for  $A(-\lambda)$  with a similar convention for vector functions.

If  $A = B + \lambda C$ ,  $B$  and  $C$  being functions of  $\lambda^2$ , then  $A'' = B - \lambda C$  so that, if  $A$  is quasi-hermitian,  $B$  is symmetric and  $C$  skew just as in the case of a hermitian matrix except that now  $B$  and  $C$  are not necessarily real. If  $A$  is any matrix,

$$2P' \equiv A' + A'' = 2P'', \quad 2Q' \equiv (A' - A'')/\lambda = 2Q''$$

so that any matrix can be expressed in the form  $P + \lambda Q$  where  $P$  and  $Q$  are quasi-hermitian.

If  $x = u + \lambda v$ , where  $u$  and  $v$  are vectors which are functions of  $\lambda^2$  and if  $A$  is quasi-hermitian, then

$$(12) \quad f(\lambda) = Sx''Ax = f(-\lambda)$$

is called a *quasi-hermitian form*. Again, if  $|1 + \lambda A| \neq 0$ , and we set  $Q = (1 - \lambda A)/(1 + \lambda A)$ , then

$$Q' = \frac{(1 - \lambda A')}{1 + \lambda A'} = \frac{1 - \lambda A''}{1 + \lambda A''} = (Q'')^{-1}$$

so that

$$(13) \quad Q'Q'' = 1.$$

We shall call such a matrix *quasi-orthogonal*.

**6.05 Reduction of a quasi-hermitian form to the sum of squares.** We have seen in §5.06 that any matrix  $A$  of rank  $r$  can be expressed in the form

$$(14) \quad A = \sum_{s=1}^r A_s y_s \frac{S A'_s x_s}{S x_s A_s y_s}$$

where

$$A_{s+1} = A_s - A_s y_s \frac{SA'_s x_s}{Sx_s A_s y_s}, \quad A_1 = A, \quad Sx_s A_s y_s \neq 0.$$

and the null space of  $A_{s+1}$  is obtained by adding  $(y_1, y_2, \dots, y_s)$  to the null space of  $A$  and the null space of  $A'_{s+1}$  by adding  $(x_1, x_2, \dots, x_s)$  to that of  $A'$ .

Suppose now that  $A$  is quasi-hermitian and replace  $y_s, x_s$  by  $z_s, z''_s$  and set  $z_s = u_s + \lambda v_s, A_s = B_s + \lambda C_s$  so that

$$Sz''_s A_s z_s = Su_s B_s u_s + \lambda^2 (2Su_s C_s v_s - Sv_s B_s v_s)$$

and, so long as  $A_s$  is not 0, we can clearly choose  $z_s$  so that  $Sz''_s A_s z_s \neq 0$ . Each matrix  $A_s$  is then quasi-hermitian since  $A'_s = A''_s$ , and

$$(15) \quad A = \sum_1^r \frac{A_s z_s SA''_s z''_s}{Sz''_s A_s z_s}.$$

If  $x$  is an arbitrary vector and

$$f = f(\lambda^2) = Sx A x'', \quad \varphi_s(\lambda) = Sx A_s z_s = \psi_s(\lambda^2) + \lambda \chi_s(\lambda^2)$$

then  $\psi_s$  and  $\chi_s$  are linear functions of the coordinates of  $x$  which are linearly independent and

$$(16) \quad f = \sum_s \frac{Sx A_s z_s SA''_s z''_s \cdot x''}{Sz''_s A_s z_s} = \sum_s \frac{\varphi_s(\lambda) \varphi''_s(\lambda)}{Sz''_s A_s z_s} = \sum_s \frac{\psi_s^2(\lambda^2) - \lambda^2 \chi_s^2(\lambda^2)}{Sz''_s A_s z_s}$$

which is the required expression for  $f(\lambda^2)$  in terms of squares.

If  $A$  is hermitian, then  $\lambda = i$  and  $\psi_s, \chi_s, Sz''_s A_s z_s = S\bar{z}_s A_s z_s$  are real and, if  $S\bar{z}_s A_s z_s = \alpha_s^{-1}$ , (16) becomes

$$(17) \quad f = \sum_1^r \alpha_s \varphi_s \bar{\varphi}_s.$$

If  $\lambda = 0$ , then  $A$  is symmetric and

$$(18) \quad f = Sx A x = \sum_1^r \alpha_s \varphi_s^2$$

where the terms are all real if  $A$  is real.

In terms of the matrices themselves these results may be expressed as follows.

**THEOREM 12.** *If  $A$  is a hermitian matrix of rank  $r$ , there exist an infinity of sets of vectors  $a_s$  and real constants  $\alpha_s$  such that*

$$(19) \quad A = \sum_1^r \alpha_s a_s S \bar{a}_s;$$

and, if  $A$  is symmetric, there exists an infinity of sets of vectors  $a_s$  and constants  $\alpha_s$  such that

$$(20) \quad A = \sum_1^r \alpha_s a_s S a_s$$

$a_s$  and  $\alpha_s$  being real if  $A$  is real.

If  $\pi$  of the  $\alpha$ 's in (19) are positive and  $\nu$  are negative, the difference  $\sigma = \pi - \nu$  is called the *signature* of  $A$ . A given hermitian matrix may be brought to the form (19) in a great variety of ways but, as we shall now show, the signature is the same no matter how the reduction is carried out. Let  $K_1$  be the sum of the terms in (19) for which  $\alpha_s$  is positive and  $-K_2$  the sum of the terms for which it is negative so that  $A = K_1 - K_2$ ; the matrices  $K_1$  and  $K_2$  are positive semi-definite and, if  $k_1$  and  $k_2$  are their ranks, we have  $r = k_1 + k_2$ . Suppose that by a different method of reduction we get  $A = M_1 - M_2$  where  $M_1$  and  $M_2$  are positive semi-definite matrices of ranks  $m_1$  and  $m_2$  and  $m_1 + m_2 = r$ ; and suppose, if possible, that  $k_2 < m_2$ . The orders of the null spaces of  $K_2$  and  $M_1$  relative to the right ground of  $A$  are  $r - k_2$  and  $r - m_1 = m_2$  and, since  $r - k_2 + m_2 > r$ , there is at least one vector  $x$  in the ground of  $A$  which is common to both these null spaces, that is,

$$Ax = K_1x = -M_2x \neq 0,$$

and hence  $S\bar{x}K_1x = -S\bar{x}M_2x$ . But both  $K_1$  and  $M_2$  are positive semi-definite; hence we must have  $S\bar{x}K_1x = 0$  which by §6.04 entails  $K_1x = 0$ . We have therefore arrived at a contradiction and so must have  $k_2 = m_2$  which is only possible when the signature is the same in both cases.

In the case of a skew matrix the reduction given by (16) is not convenient and it is better to modify it as follows. Let  $A' = -A$  and set

$$(21) \quad A_{s+1} = A_s + \frac{A_s y_s S A_s x_s}{S x_s A_s y_s} - \frac{A_s x_s S A_s y_s}{S x_s A_s y_s},$$

$$A_1 = A, \quad S x_s A_s y_s \neq 0.$$

So long as  $A_s \neq 0$ , the condition  $S x_s A_s y_s \neq 0$  can always be satisfied by a suitable choice of  $x_s$  and  $y_s$  and it is easily proved as in §5.06 that the null space of  $A_{s+1}$  is obtained from that of  $A_s$  by adding  $x_s, y_s$ ; also  $A_s$  is skew so that we must necessarily have  $x_s \neq y_s$ . It follows that the rank of  $A$  is even and

$$(22) \quad A = \sum_1^{r/2} \frac{A_s x_s S A_s y_s - A_s y_s S A_s x_s}{S x_s A_s y_s} = \sum \alpha_s (a_{2s-1} S a_{2s} - a_{2s} S a_{2s-1})$$

where each term in the summation is a skew matrix of rank 2 and

$$\alpha_s^{-1} = S x_s A_s y_s, \quad a_{2s-1} = A_s x_s, \quad a_{2s} = A_s y_s.$$

This form corresponds to the one given in Theorem 12 for symmetric matrices.

If we put

$$\begin{aligned}
 T &= \sum_1^{r/2} (e_{2s-1}Se_{2s} - e_{2s}Se_{2s-1}) = -T' \\
 (23) \quad R &= \sum_1^{r/2} \alpha_s(e_{2s-1}Se_{2s} - e_{2s}Se_{2s-1}) = -R' \\
 P &= \sum_1^{r/2} (\alpha_s a_{2s} Se_{2s} + a_{2s-1} Se_{2s-1}), \quad Q = \sum_1^{r/2} (a_{2s} Se_{2s} + a_{2s-1} Se_{2s-1})
 \end{aligned}$$

then (22) may be put in the form  $A = PTP' = QRQ'$ . When  $r = n$ , the determinant of  $T$  equals 1 and therefore  $|A| = |P|^2$ . The following theorem summarizes these results.

**THEOREM 13.** *If  $A$  is a skew matrix of rank  $r$ , then (i)  $r$  is even; (ii)  $A$  can be expressed by rational processes in the form*

$$(24) \quad A = \sum_1^{r/2} \alpha_s (a_{2s-1} S a_{2s} - a_{2s} S a_{2s-1}) = PTP' = QRQ'$$

where  $P, Q, R$  and  $T$  are given by (23); (iii) if  $r = n$ , the determinant of  $A$  is a perfect square, namely  $|P|^2$ ; (iv) if  $x$  and  $y$  are any vectors and  $w = \sum \alpha_s |a_{2s-1} a_{2s}|$ , then

$$(25) \quad Sx Ay = S |xy| w.$$

The following theorem contains several known properties of hermitian matrices.

**THEOREM 14.** *If  $T(A)$  is an associated matrix for which  $T'(A) = T(A')$ , then, when  $A$  is quasi-hermitian,  $T(A)$  is also quasi-hermitian.*

For  $A' = A''$  gives  $T'(A) = T(A') = T(A'') = T''(A)$ .

Particular cases of interest are: If  $A$  is hermitian,  $T(A)$  is hermitian. If  $T(\mu A) = \mu^s T(A)$  and  $A$  is skew, then  $T(A)$  is skew if  $s$  is odd, symmetric if  $s$  is even.

**6.06 The Kronecker method of reduction.** Let  $A = \sum_1^r x_i S y_i$  be a quasi-hermitian matrix of rank  $r$ ; then

$$(26) \quad \sum y_i S x_i = A' = A'' = \sum x_i'' S y_i'',$$

from which it follows that  $y_i$  is linearly dependent on  $x_1'', x_2'', \dots, x_r''$ , say

$$y_i = \sum_{j=1}^r q_{ij} x_j'', \quad |q_{ij}| \neq 0, \quad (i = 1, 2, \dots, r).$$

Using this value of  $y_i$  we have

$$A = \sum q_{ij} x_i S x_j'', \quad A' = \sum q_{ji} x_i'' S x_j, \quad A'' = \sum q_{ij}'' x_i'' S x_j$$

and therefore

$$(27) \quad q_{ij} = q_{ji}''.$$

Further, since  $|q_{ij}| \neq 0$ , we can find  $s_{ij}$  ( $i, j = 1, 2, \dots, r$ ) so that

$$\sum_j q_{ij} s_{jk} = \delta_{ik}$$

and then (27) gives  $s_{ij} = s_{ji}''$ .

Let  $x_1, \dots, x_r, x_{r+1}, \dots, x_n$  be a basis and  $z_1, z_2, \dots, z_n$  the reciprocal basis.

Then, if  $w_i = \sum_1^r s_{ji} z_j''$ , the basis reciprocal to  $y_1'', \dots, y_r'', x_{r+1}, \dots, x_n$  is  $w_1'', \dots, w_r'', z_{r+1}, \dots, z_n$ . Hence

$$P = \sum_1^r w_i S z_i = \sum s_{ij} z_j'' S z_i$$

is quasi-hermitian. Further, if  $u = \sum_1^r \xi_i x_i$ , then  $Su''Pu = \sum \xi_i'' \xi_j s_{ij}$ ; and we can choose  $u$  so that this form is not 0. We also have

$$AP = \sum_1^r x_i S y_i \sum_1^r w_j S z_j = \sum_1^r x_i S z_i,$$

whence  $APu = u$ .

Let

$$(28) \quad A_{s+1} = A_s - \frac{u_s S u_s''}{S u_s'' P_s u_s}, \quad A_1 = A, \quad P_1 = P,$$

where  $P_s$  is formed from  $A_s$  in the same way as  $P$  is from  $A$  and  $u_s$  is a vector of the left ground of  $A_s$  such that  $Su_s'' P_s u_s \neq 0$ ; also, as above,  $A_s P_s u = u$  for any vector  $u$  in the left ground of  $A_s$  and  $A_s$  in quasi-hermitian. The vector  $u_s''$  belongs to the right ground of  $A_s$  and therefore every vector of the null space of  $A_s$  lies in the null space of  $A_{s+1}$ ; also

$$\begin{aligned} A_{s+1} P_s u_s &= A_s P_s u_s - u_s \frac{S u_s'' P_s u_s}{S u_s'' P_s u_s} \\ &= u_s - u_s = 0. \end{aligned}$$

Hence the null space of  $A_{s+1}$  is derived from that of  $A_s$  by adding  $P_s u_s$  to it. It then follows as in §6.06 that  $A$  can be expressed in the form

$$(29) \quad A = \sum_1^r \frac{u_s S u_s''}{S u_s'' P_s u_s}$$

which is analogous to (16) and may be used in its place in proving Theorem 12.

We may also note that, if  $Q$  is the matrix defined by  $Q'e_j = x_j$  ( $j = 1, 2, \dots, n$ ), then

$$A = Q' \sum_1^n q_{ij} e_i S e_j Q'' = Q' B Q''$$

where  $B$  is the quasi-hermitian matrix  $\sum_1^n q_{ij} e_i S e_j$ . It may be shown by an argument similar to that used for hermitian matrices that a basis for the  $x$ 's may be so chosen that  $Q$  is quasi-orthogonal provided  $A$  is real.

**6.07 Congredient transformation.** If  $SxAy$  and  $SxB y$  are two bilinear forms, the second is said to be derived from the first by a congruent transformation if there exists a non-singular matrix  $P$  such that  $SxAy = SPxBy$ , that is,

$$(30) \quad A = P'BP.$$

When this relation holds between  $A$  and  $B$ , we shall say they are *congruent*.

From (30) we derive immediately  $A' = P'B'P$  and therefore, if

$$R = \frac{A + A'}{2} = R', \quad S = \frac{A - A'}{2} = -S',$$

$$U = \frac{B + B'}{2} = U', \quad V = \frac{B - B'}{2} = -V',$$

then

$$R + \lambda S = P'(U + \lambda V)P$$

so that  $R + \lambda S$  and  $U + \lambda V$  are strictly equivalent.

Suppose conversely that we are given that  $R + \lambda S$  and  $U + \lambda V$ , which are quasi-hermitian, are strictly equivalent so that there exist constant non-singular matrices  $p, q$  such that

$$R + \lambda S = p(U + \lambda V)q$$

or

$$(31) \quad R = pUq, \quad S = pVq;$$

then, remembering that  $R$  and  $U$  are symmetric,  $S$  and  $V$  skew, we have

$$R = q'U p', \quad S = q'V p'$$

Equating these two values of  $R$  and  $S$ , respectively, we get

$$(q')^{-1}pU = U p' q^{-1}, \quad (q')^{-1}pV = V p' q^{-1}$$

or, if  $W$  stands for  $U$  or  $V$  indifferently, and

$$(32) \quad J = (q')^{-1}p,$$

we have

$$JW = WJ',$$

repeated application of which gives

$$J^r W = W(J')^r.$$

From this it follows that, if  $f(\lambda)$  is a scalar polynomial,

$$(33) \quad f(J)W = Wf(J') = W(f(J))'.$$

In particular, since  $|J| \neq 0$ , we may choose  $f(\lambda)$  so that  $f(J)$  is a square root of  $J$  and, denoting this square root by  $K$ , we have  $KW = WK'$  or

$$W = K^{-1}WK', \quad K^2 = J, \quad (W = U \text{ or } V).$$

Using this in (31) we have

$$R = pK^{-1}UK'q, \quad S = pK^{-1}VK'q$$

and from (32)  $p = q'J = q'K^2$  or

$$pK^{-1} = q'K = (K'q)'.$$

Hence, if we put  $P' = q'K$ , there follows

$$R = P'UP, \quad S = P'VP$$

or

$$A = R + S = P'(U + V)P = P'BP.$$

We therefore have the following theorem, which is due to Kronecker.

**THEOREM 15.** *A necessary and sufficient condition that  $A$  and  $B$  be cogredient is that  $A + \lambda A'$  and  $B + \lambda B'$  shall be strictly equivalent.*

If  $A$  and  $B$  are symmetric, these polynomials become  $A(1 + \lambda)$  and  $B(1 + \lambda)$  which are always strictly equivalent provided the ranks of  $A$  and  $B$  are the same. Hence quadratic forms of the same rank are always cogredient, as is also evident from Theorem 12 which shows in addition that  $P$  may be taken real if the signatures are the same.

The determination of  $P$  from (31) is unaltered if we suppose  $S$  symmetrical instead of skew, or  $R$  skew instead of symmetrical. Hence

**THEOREM 16.** *If  $R, S, U, V$  are all symmetric or all skew, and if  $R + \lambda S$  and  $U + \lambda V$  are strictly equivalent, we can find a constant non-singular matrix  $P$  such that*

$$R + \lambda S = P'(U + \lambda V)P,$$

that is, the corresponding pairs of forms are cogredient.

In the case of a hermitian form  $S\bar{x}Ax$  changing  $x$  into  $Px$  replaces  $A$  by  $\bar{P}'AP$  and we have in place of (30)

$$(34) \quad A = \bar{P}'BP.$$

If we put  $B = \Sigma \beta_s b_s \bar{S} b_s$ , then

$$\bar{P}'BP = \Sigma \beta_s \bar{P}'b_s \bar{S} b_s P = \Sigma \beta_s \bar{P}'b_s S P' \bar{b}_s = \Sigma \beta_s c_s \bar{S} c_s.$$

where  $c_s = \bar{P}'b_s$ . Equation (34) can therefore hold only if the signature as well as the rank is the same for  $B$  as for  $A$ . Conversely, if  $A = \Sigma \alpha_s a_s \bar{S} a_s$  and  $A$  and  $B$  have the same signature and rank the notation may be so arranged that  $\alpha_s$  and  $\beta_s$  have the same signs for all  $s$ ; then any matrix for which

$$\bar{P}'b_s = \begin{pmatrix} \alpha_s \\ \beta_s \end{pmatrix} a_s \quad (s = 1, 2, \dots, r), \quad |P| \neq 0$$

where  $r$  is the common rank of  $A$  and  $B$ , clearly satisfies (33).<sup>2</sup> Hence

**THEOREM 17.** *Two hermitian forms are cogredient if, and only if, they have the same rank and signature.*

The reader will readily prove the following extension of Theorem 16 by the aid of the artifice used in the proof of Theorem 11

**THEOREM 18.** *If  $A, B, C, D$  are hermitian matrices such that  $A + \lambda B$  and  $C + \lambda D$  are (i) equivalent (ii) both definite for some value of  $\lambda$ , there exists a constant non-singular matrix  $P$  such that*

$$A + \lambda B = \bar{P}'(C + \lambda D)P.$$

**6.08 Real representation of a hermitian matrix.** Any matrix  $H = A + iB$  in which  $A$  and  $B$  are real matrices of order  $n$  can be represented as a real matrix of order  $2n$ . For the matrix of order 2

$$i_2 = \left\| \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right\|$$

satisfies the equation  $i_2^2 = -1$  and, on forming the direct product of the original set of matrices of order  $n$  and a set of order 2 in which  $i_2$  lies, we get a set of order  $2n$  in which  $H$  is represented by

$$\mathfrak{H} = A + i_2 B = \left\| \begin{array}{cc} A & -B \\ B & A \end{array} \right\|$$

As a verification of this we may note that

$$\left\| \begin{array}{cc} A & -B \\ B & A \end{array} \right\| \left\| \begin{array}{cc} C & -D \\ D & C \end{array} \right\| = \left\| \begin{array}{cc} AC - BD & -(AD + BC) \\ AD + BC & AC - BD \end{array} \right\|$$

which corresponds to

$$(A + iB)(C + iD) = AC - BD + i(AD + BC).$$

<sup>2</sup> The proof preceding Theorem 15 generalizes readily up to equation (33); at that point, however, if  $K = f(J)$ , we require  $\bar{K}' = f(\bar{J}')$ , which is only true when the coefficients of  $f(\lambda)$  are real.

This representation has the disadvantage that a complex scalar  $\alpha + i\beta$  is represented by

$$\begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix}$$

which is not a scalar matrix although it is commutative with every matrix of the form  $\mathfrak{S}$ . Consequently, if  $H$  has a complex root, this root does not correspond to a root of  $\mathfrak{S}$ . If, however, all the roots of  $H$  are real, the relation  $HK = \alpha K$  is represented by  $\mathfrak{S}\mathfrak{K} = \alpha\mathfrak{K}$  when  $\alpha$  is real so that  $\alpha$  is a root of both  $H$  and  $\mathfrak{S}$ .

To prove the converse of this it is convenient to represent the vector  $x + iy$  in the original space by  $(x, y)$  in the extended space. Corresponding to

$$(A + iB)(x + iy) = Ax - By + i(Bx + Ay)$$

we then have

$$\begin{vmatrix} A & -B \\ B & A \end{vmatrix} (x, y) = (Ax - By, Bx + Ay).$$

If therefore  $\mathfrak{S}$  has a real root  $\alpha$  and  $(x, y)$  is a corresponding invariant vector so that

$$\mathfrak{S}(x, y) = \alpha(x, y) = (\alpha x, \alpha y),$$

we have

$$Ax - By = \alpha x, \quad Bx + Ay = \alpha y,$$

which gives

$$(A + iB)(x + iy) = \alpha(x + iy).$$

It follows that invariant vectors in the two representations correspond provided they belong to real roots. This gives

**THEOREM 19.** *To every real root of  $H = A + iB$  there corresponds a real root of*

$$\mathfrak{S} = \begin{vmatrix} A & -B \\ B & A \end{vmatrix}$$

*and vice-versa.*

In this representation  $\bar{H}$  and  $H'$  correspond to

$$\begin{vmatrix} A & B \\ -B & A \end{vmatrix} \quad \begin{vmatrix} A' & -B' \\ B' & A' \end{vmatrix},$$

respectively, and hence, if  $H$  is hermitian,  $B' = -B$  so that  $\mathfrak{S}$  is symmetric. The theory of hermitian matrices of order  $n$  can therefore be made to depend on that of real symmetric matrices of order  $2n$ . For example, if we have proved of real symmetric matrices that they have real roots and simple elementary divisors, it follows that the same is true of hermitian matrices, thus reversing the order of the argument made in §6.01.

## CHAPTER VII

### COMMUTATIVE MATRICES

7.01 We have already seen in §2.08 how to find all matrices commutative with a given matrix  $x$  which has no repeated roots. We shall now treat the somewhat more complicated case in which  $x$  is not so restricted. If

$$(1) \quad xy = yx$$

then  $x^r y = y x^r$  so that, if  $f(\lambda)$  is a scalar polynomial, then  $f(x)y = yf(x)$ . In particular, if  $f_i$  is a principal idempotent element of  $x$ , then  $f_i y = y f_i$ . Remembering that  $\Sigma f_i = 1$  we may set

$$x = \Sigma f_i x = \Sigma x_i, \quad y = \Sigma f_i y = \Sigma y_i,$$

and also, by §2.11,  $x_i = \lambda_i f_i + z_i$ , where  $z_i$  is nilpotent. Since  $y_i x_j = 0 = x_j y_i$  ( $i \neq j$ ), the determination of all matrices  $y$  which satisfy (1) is reduced to finding  $y$  so that

$$y_i x_i = x_i y_i, \quad y_i = f_i y = y f_i.$$

We can therefore simplify the notation by first assuming that  $x$  has only one principal idempotent element, 1, and one root which may be taken to be 0 without loss of generality;  $x$  is then nilpotent.

Let  $e_1, e_2, \dots, e_s$  be the partial idempotent elements of  $x$  and let their ranks be  $n_1, n_2, \dots, n_s$ ;  $x$  is then composed of blocks of the form

$$\begin{array}{cccccc} 0 & 1 & 0 & \cdots & 0 & \\ 0 & 0 & 1 & \cdots & 0 & \\ 0 & 0 & 0 & \cdots & 0 & \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \cdots & 1 & \\ 0 & 0 & 0 & \cdots & 0 & \end{array} \quad (n_i \text{ rows and columns})$$

provided the fundamental basis is suitably chosen. To simplify the notation further we divide the array of  $n^2$  units  $e_{ij}$  into smaller arrays formed by separating off the first  $n_1$  rows, then the next  $n_2$  rows, and so on, and then making a similar division of the columns (see figure 1). And when this is done, we shall denote the units in the block in which the  $i$ th set of rows meets the  $j$ th set of columns by

$$e_{pq}^{ij} \quad (i, j = 1, 2, \dots, s; p = 1, 2, \dots, n_i; q = 1, 2, \dots, n_j).$$

It is also convenient to put  $e_{p,q}^{i,j} = 0$  for  $p > n_i$  or  $q > n_j$ .

	$n_1$	$n_2$	$n_3$	$\dots$
$n_1$	11	12	13	$\dots$
$n_2$	21	22	23	$\dots$
$n_3$	31	32	33	$\dots$
	.	.	.	$\dots$

FIG. 1

The expression for  $x$  is now

$$x = \sum_{i=1}^s \sum_{p=1}^{n_i-1} e_{p,p+1}^{i,i} = \sum_{i=1}^s x_i$$

and we may set

$$y = \sum_{i,j,p,q} \eta_{p,q}^{i,j} e_{p,q}^{i,j} = \sum_{i,j} y_{ij}$$

where

$$y_{ij} = e_i y e_j = \sum_{p=1}^{n_i} \sum_{q=1}^{n_j} \eta_{p,q}^{i,j} e_{p,q}^{i,j}.$$

The equation  $xy = yx$  is then equivalent to

$$(2) \quad x_i y_{ij} = y_{ij} x_j \quad (i, j = 1, 2, \dots, s).$$

If we now suppress for the moment the superscripts  $i, j$ , which remain constant in a single equation in (2), we may replace (2) by

$$\sum_{p=1}^{n_i-1} e_{p,p+1} \sum_{l=1}^{n_i} \sum_{m=1}^{n_j} \eta_{lm} e_{lm} = \sum_{l=1}^{n_i} \sum_{m=1}^{n_j} \eta_{lm} e_{lm} \sum_{q=1}^{n_j-1} e_{q,q+1}$$

or

$$(3) \quad \sum_{m=1}^{n_j} \sum_{p=1}^{n_i-1} \eta_{p+1,m} e_{pm} = \sum_{l=1}^{n_i} \sum_{q=1}^{n_j-1} \eta_{lq} e_{l,q+1}.$$

Equating corresponding coefficients then gives

$$(4) \quad \eta_{p+1,q+1} = \eta_{pq}.$$

Since  $q \geq 1$  on the right of (3), it follows that  $\eta_{p+1,1} = 0$  ( $p = 1, 2, \dots, n_i - 1$ ) and, since  $p \leq n_i - 1$  on the left,  $\eta_{n_i,q} = 0$  ( $q = 1, 2, \dots, n_j - 1$ ) and hence from (4)

$$(5) \quad \eta_{p+t,t} = 0 = \eta_{n_i-t,q-t}$$

where  $p = 0, 1, \dots, n_i - t, q = t + 1, s + 2, \dots, n_j - 1, t = 0, 1, \dots$ .

From (4) we see that in  $y_{ij}$  all coordinates in an oblique line parallel to the main diagonal of the original array have the same value; from the first part of (5) those to the left of the oblique  $AB$  through the upper left hand corner



If we arrange the notation so that  $n_1 \leq n_2 \leq \dots \leq n_s$ , a simple enumeration shows that the number of independent parameters in  $y$  is

$$(2s - 1)n_1 + (2s - 3)n_2 + \dots + n_s.$$

We have therefore the following theorem which is due to Frobenius.

**THEOREM 1.** *If the elementary divisors of  $x$  are  $(\lambda - \lambda_i)^{n_{ij}}$ ,  $i = 1, 2, \dots, r$ ,  $j = 1, 2, \dots, s_i$ , where  $\lambda_1, \lambda_2, \dots, \lambda_r$  are all different and  $n_{i1} \leq n_{i2} \leq \dots \leq n_{is_i}$ , then the general form of a matrix commutative with  $x$  depends on*

$$\sum_{i=1}^r \sum_{j=1}^{s_i} (2s - 2j + 1)n_{ij}$$

*independent parameters.*

**7.02 Commutative sets of matrices.** The simple condition  $xy = yx$  may be replaced by the more stringent one that  $y$  is commutative with every matrix which is commutative with  $x$ . To begin with we shall merely assume that  $y$  is commutative with each of a particular set of partial idempotent elements  $e_i$ ; as in the previous section we may assume that  $x$  has only one principal idempotent element.

In order that  $e_i y = y e_i$  for every  $i$  it is necessary and sufficient that  $y_{ij} = 0$  when  $i \neq j$ ; if  $u_1, u_2, \dots, u_s$  are the partial nilpotent elements of  $x$  corresponding to  $e_1, e_2, \dots, e_s$  and we set  $m_i = n_i - 1$ , this gives for  $y$

$$(7) \quad y = \sum_i (\eta_{i0} e_i + \eta_{i1} u_i + \dots + \eta_{im_i} u_i^{m_i}).$$

If we now put  $z = \sum_i (\beta_i e_i + u_i)$ , where no  $\beta_i = 0$ , and if  $g(\lambda)$  is any scalar polynomial, then (cf. §2.11)

$$g(z) = \Sigma g(\beta_i e_i + u_i) = \Sigma (g(\beta_i) e_i + g'(\beta_i) u_i + \dots + g^{(m_i)}(\beta_i) u_i^{m_i} / m_i!)$$

and when  $y$  is given, we can always find  $g(\lambda)$  so that

$$\eta_{ik} = g^{(k)}(\beta_i) / k!$$

provided the  $\beta$ 's are all different. Hence every  $y$ , including  $x$  itself, can be expressed as a polynomial in  $z$ .

We now impose the more exacting condition that  $y$  is permutable with every matrix permutable with  $x$ . Let  $n_{ij}$  ( $i \neq j$ ) be the matrix of the same form as  $y_{ij}$  in §7.01 but with zero coordinates everywhere except in the principal oblique; for example in (6)  $u_{23}$  is obtained by putting  $f_0 = 1$  and making every other coordinate 0. We then have

$$e_i u_{ij} = u_{ij} e_j, \quad u_i u_{ij} = u_{ij} u_j.$$

Hence  $yu_{ij} = u_{ij}y$  gives  $y_{ii}u_{ij} = u_{ij}y_{ii}$  and therefore from (7)

$$\begin{aligned} (\eta_{i0}e_i + \eta_{i1}u_i + \cdots + \eta_{im_i}u_i^{m_i})u_{ij} &= u_{ij}(\eta_{j0}e_j + \eta_{j1}u_j + \cdots + \eta_{jm_j}u_j^{m_j}) \\ &= (\eta_{j0}e_i + \eta_{j1}u_i + \cdots + \eta_{jm_j}u_i^{m_j})u_{ij} \end{aligned}$$

from which we readily derive for all  $i, j$  and  $k$

$$\eta_{ik} = \eta_{jk}$$

with the understanding that  $\eta_{ik}$  does not actually occur when  $k > m_i$ . When  $r$  is the matrix used in deriving (6), these conditions give in place of (6)

$$(8) \quad \begin{array}{cccccccc} a_0 & a_1 & . & . & . & . & . & . \\ . & a_0 & . & . & . & . & . & . \\ . & . & a_0 & a_1 & a_2 & . & . & . \\ . & . & . & a_0 & a_1 & . & . & . \\ . & . & . & . & a_0 & . & . & . \\ . & . & . & . & . & a_0 & a_1 & a_2 & a_3 & a_4 \\ . & . & . & . & . & . & a_0 & a_1 & a_2 & a_3 \\ . & . & . & . & . & . & . & a_0 & a_1 & a_2 \\ . & . & . & . & . & . & . & . & a_0 & a_1 \\ . & . & . & . & . & . & . & . & . & a_0 \end{array}$$

Comparing this form with (7) we see that  $y$  is now a scalar polynomial in  $x$ , which in the particular case given above becomes  $g(x - \alpha)$  where

$$g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4.$$

The results of this section may be summarized as follows.

**THEOREM 2.** *Any matrix which is commutative, not only with  $x$ , but also with every matrix commutative with  $x$ , is a scalar polynomial in  $x$ .*

**7.03 Rational methods.** Since the solution of  $xy - yx = 0$  for  $y$  can be regarded as equivalent to solving a system of linear homogeneous equations, the solution should be expressible rationally in terms of suitably chosen parameters; the method of §7.01, though elementary and direct, cannot therefore be regarded as wholly satisfactory. The following discussion, which is due to Frobenius, avoids this difficulty but is correspondingly less explicit.

As before let  $xy = yx$  and set  $a = \lambda - x$ ; also let  $b = L^{-1}aM^{-1}$  be the normal form of  $a$ . If  $u$  is an arbitrary polynomial in  $\lambda$  and we set

$$P = L^{-1}(au + y)L, \quad Q = M(ua + y)M^{-1},$$

then

$$Pb = PL^{-1}aM^{-1} = L^{-1}(au + y)aM^{-1} = L^{-1}a(va + y)M^{-1} = bQ.$$

Conversely, if  $Pb = bQ$  and, using the division transformation, we set

$$(9) \quad LPL^{-1} = av + y, \quad M^{-1}QM = v_1a + y_1,$$

where  $y$  and  $y_1$  are constants, then

$$0 = Pb - bQ = L^{-1}(av + y)aM^{-1} - L^{-1}a(v_1a + y_1)M^{-1}$$

or  $a(v - v_1)a = ay_1 - ya$ . Here the degree on the left is at least 2 and on the right only 1 and hence by the usual argument both sides of the equation vanish. This gives

$$ava = av_1a, \quad ay_1 = ya$$

whence  $v_1 = v$  and, since  $a = \lambda - x$ , also  $y_1 = y$  so that  $xy = yx$ .

Hence we can find all matrices commutative with  $x$  by finding all solutions of

$$(10) \quad Pb = bQ.$$

Let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be the invariant factors<sup>1</sup> of  $a$  and  $n_1, n_2, \dots, n_n$  the corresponding degrees so that  $b$  is the diagonal matrix  $\Sigma \alpha_i e_{ii}$ , and let  $P = \| P_{ij} \|$ ,  $Q = \| Q_{ij} \|$ ; then

$$(11) \quad P_{ij}\alpha_j = \alpha_i Q_{ij}.$$

By the division transformation we may set

$$P_{ij} = R_{ij}\alpha_i + p_{ij}, \quad Q_{ij} = S_{ij}\alpha_i + q_{ij}$$

and then from (10) we have

$$R_{ij} = S_{ij}, \quad p_{ij}\alpha_j = \alpha_i q_{ij}$$

or, if  $p = \| p_{ij} \|$ ,  $q = \| q_{ij} \|$ ,

$$(12) \quad pb = bq.$$

Hence  $P = p$ ,  $Q = q$  is a solution of (10) for which the degree of  $p_{ij}$  is less than that of  $\alpha_i$  and the degree of  $q_{ij}$  is less than that of  $\alpha_j$ . It is then evident that, when the general solution  $p$ ,  $q$  of (12) is found, then the general solution of (10) has the form

$$P = bR + p, \quad Q = Rb + q$$

where  $R$  is an arbitrary matrix polynomial in  $\lambda$ . We are however not concerned with  $R$ ; for

$$LPL^{-1} = LbRL^{-1} + LpL^{-1} = aM^{-1}RL^{-1} + LpL^{-1}$$

so that in (9) the value of  $y$  depends on  $p$  only.

<sup>1</sup> Since we may add a scalar to  $x$  we may clearly assume that the rank of  $a$  is  $n$ .

The general solution of (12) is given by

$$(13) \quad \begin{aligned} p_{ij} &= \frac{\alpha_i}{\alpha_j} s_{ij}, & p_{ji} &= s_{ji} \\ q_{ij} &= s_{ij}, & q_{ji} &= \frac{\alpha_i}{\alpha_j} s_{ji} \end{aligned} \quad (i \geq j)$$

where  $s_{\lambda\mu}$  is an arbitrary polynomial whose degree is at most  $n_\mu - 1$  and which therefore depends on  $n_\mu$  parameters. It follows that the total number of parameters in the value of  $y$  is that already given in §7.01.

**7.04 The direct product.** We shall consider in this section some properties of the direct product which was defined in §5.10.

**THEOREM 3.** *If  $f_{ij}$  ( $i, j = 1, 2, \dots, m$ ) is a set of matrices, of order  $n$ , for which*

$$(14) \quad f_{ij}f_{pq} = \delta_{in}f_{iq}, \quad \sum_{i=1}^m f_{ii} = 1,$$

*then  $m$  is a factor of  $n$  and any matrix of order  $n$  can be expressed uniquely in the form  $\sum a_{ij}f_{ij}$  where each  $a_{ij}$  is commutative with every  $f_{pq}$ ; and, if  $n = mr$ , the rank of each  $f_{pq}$  is  $r$ .*

For, if  $x$  is an arbitrary matrix and we set

$$(15) \quad a_{ij} = \sum_{k=1}^m f_{ki}x f_{jk},$$

a short calculation shows:

- (i)  $x = \sum a_{ij}f_{ij}$ ;
- (ii)  $a_{ij}f_{pq} = f_{pq}a_{ij}$  for all  $i, j, p, q$ ;
- (iii) the set  $\mathfrak{A}$  of all matrices of the form (15) is closed under the operations of addition and multiplication;
- (iv) if  $b_{11}, b_{12}, \dots$  are members of  $\mathfrak{A}$ , then  $\sum b_{ij}f_{ij}$  is zero if, and only if, each  $b_{ij} = 0$ .

If  $(a_1, a_2, \dots, a_l)$  is a basis of  $\mathfrak{A}$ , it follows that

$$(a_p f_{ij}; p = 1, 2, \dots, l; i, j = 1, 2, \dots, m)$$

is equivalent to the basis  $(e_{ij}, i, j = 1, 2, \dots, n)$  of the set of matrices of order  $n$ . This basis contains  $lm^2$  independent elements and hence  $n^2 = lm^2$  so that  $n = mr$ ,  $l = r^2$ . Let  $r_{ij}$  be the rank of  $f_{ij}$ . Since  $f_{ii} = f_{ij}f_{ji}$ , it follows from Theorem 8 of chapter I that  $r_{ii} \leq r_{ji}$ ; also from  $f_{ji}f_{ii} = f_{ji}$  we have  $r_{ji} \leq r_{ii}$ ; hence  $r_{ii} = r_{ji}$  and therefore each  $r_{ji}$  has the same value. Finally, since  $1 = \sum f_{ii}$ , and  $f_{ii}f_{jj} = 0$  ( $i \neq j$ ) and  $r_{ii} = r_{jj}$ , we have  $mr_{ii} = n$  and hence each  $r_{ij} = r$ .

We shall now show that a basis  $g_{ij}$  can be chosen for  $\mathfrak{A}$  which satisfies the relations (14) with  $r$  in place of  $m$ . Since the rank of  $f_{ii}$  is  $r$ , we can set

$$(16) \quad f_{ii} = \sum_1^r \alpha_{ik} S \beta_{ik} \quad (i = 1, 2, \dots, m)$$

where the sets of vectors  $(\alpha_{ik})$  and  $(\beta_{ik})$  ( $i = 1, 2, \dots, m; k = 1, 2, \dots, r$ ) each form a basis of the  $n$ -space since  $\sum_{i=1}^m f_{ii} = 1$ . If  $(\alpha'_{ik}), (\beta'_{ik})$  are the corresponding reciprocal sets and

$$p_{ii} = \sum_1^r \beta'_{ik} S \alpha'_{ik} \quad (i = 1, 2, \dots, m)$$

we have, since  $S \alpha_{ik} \alpha'_{js} = \delta_{ij} \delta_{ks}$ ,

$$\Sigma p_{ii} = \Sigma f_{ii} \Sigma p_{ii} = \Sigma \Sigma \alpha_{ik} S \beta_{ik} \beta'_{ij} S \alpha'_{ij} = \Sigma \alpha_{ik} S \alpha'_{ik} = 1,$$

and similarly

$$f_{ii} p_{ii} = \sum_k \alpha_{ik} S \alpha'_{ik}, \quad f_{ii} p_{jj} = 0 \quad (i \neq j).$$

Hence

$$(17) \quad f_{ii} = f_{ii} \Sigma p_{ii} = \sum_k \alpha_{ik} S \alpha'_{ik},$$

that is  $\beta_{ik} = \alpha'_{ik}$ .

Since  $f_{ij} = f_{ii} f_{ij} f_{jj}$ , the left ground of  $f_{ij}$  is the same as that of  $f_{ii}$  and its right ground is the same as that of  $f_{jj}$ . Let

$$f_{ij} = \Sigma \alpha_{ik} S \gamma_{jk}.$$

The vectors  $\gamma_{jk}$  ( $k = 1, 2, \dots, r$ ) then form a basis for the set  $\alpha'_{jk}$  ( $k = 1, 2, \dots, r$ ) and, since the basis chosen for this set in (16) is immaterial, we may suppose  $\gamma_{jk} = \alpha'_{jk}$  ( $j = 1, 2, \dots, m; k = 1, 2, \dots, r$ ), that is,

$$f_{ij} = \sum_k \alpha_{ik} S \alpha'_{jk}.$$

Similarly we may set  $f_{j1} = \Sigma \alpha_{jk} S \theta_{1k}$  and then since

$$\sum_k \alpha_{jk} S \alpha'_{1k} = f_{j1} = f_{j1} f_{j1} = \sum_{k,s} \alpha_{jk} S \alpha'_{jk} \alpha'_{js} S \theta_{1s} = \Sigma \alpha_{jk} S \theta_{1k},$$

we have  $\theta_{1k} = \alpha'_{1k}$  and therefore

$$f_{j1} = \Sigma \alpha_{jk} S \alpha'_{1k}.$$

and finally

$$(18) \quad f_{ij} = f_{i1}f_{1j} = \sum_{k,s} \alpha_{ik}S\alpha'_{1k}\alpha_{1s}S\alpha'_{js} = \Sigma\alpha_{ik}S\alpha'_{jk}.$$

If we now set  $\alpha_{ik} = Pe_{(i-1)r+k}$ , then by §1.09  $P$  is non-singular and  $\alpha'_{ik} = (P')^{-1}e_{(i-1)r+k}$ ; hence, if

$$(19) \quad h_{ij} = \sum_{k=1}^r e_{(i-1)r+k, (j-1)r+k},$$

we have

$$(20) \quad f_{ij} = Ph_{ij}P^{-1}.$$

Also if

$$(21) \quad k_{ij} = \sum_{s=0}^{m-1} e_{sr+i, sr+j},$$

then

$$(22) \quad k_{ij}h_{p+1, q+1} = e_{pr+i, qr+j} = n_{p+1, q+1}k_{ij}$$

so that the set  $(e_{ij})$  of all matrices of order  $n$  may be regarded as the direct product of the sets  $(h_{ij})$  and  $(k_{ij})$ . Finally, since any matrix can be expressed in the form  $\Sigma b_{ij}h_{ij}$ , where the  $b_{ij}$  depend on the basis  $(k_{ij})$ , it follows that an arbitrary matrix can also be expressed in the form

$$P\Sigma b_{ij}h_{ij}P^{-1} = \Sigma Pb_{ij}P^{-1}f_{ij};$$

$Pb_{ij}P^{-1}$  depends on the basis  $(Pk_{ij}P^{-1})$  and hence, if we set

$$g_{ij} = Pk_{ij}P^{-1} \quad (i, j = 1, 2, \dots, r)$$

the  $g$ 's form a basis of  $\mathfrak{A}$  which satisfies (14).

**7.05 Functions of commutative matrices.** Let  $x$  and  $y$  be commutative matrices whose distinct roots are  $\lambda_1, \lambda_2, \dots$  and  $\mu_1, \mu_2, \dots$  respectively and let  $R_i$  be the principal idempotent unit of  $x$  corresponding to  $\lambda_i$  and similarly  $S_j$  the principal idempotent unit of  $y$  corresponding to  $\mu_j$ . Since  $R_i$  and  $S_j$  are scalar polynomials in  $x$  and  $y$ , they are commutative. If we set

$$T_{ij} = R_iS_j,$$

those  $T_{ij}$  which are not 0 are linearly independent; for if  $\Sigma \xi_{ij}T_{ij} = 0$ , then

$$0 = R_p\Sigma \xi_{ij}T_{ij}S_q = \xi_{pq}T_{pq},$$

since  $R_pR_i = \delta_{pi}R_p$ ,  $S_jS_q = \delta_{jq}S_q$ , so that either  $\xi_{pq} = 0$  or  $T_{pq} = 0$ .

From the definition of  $T_{ij}$  it follows that  $T_{ij}T_{pq} = 0$  when  $i \neq p$  or  $j \neq q$ , and  $T_{ij}^2 = T_{ij}$ ,  $\sum T_{ij} = 1$ ; hence

$$x = \sum [\lambda_i + (x - \lambda_i)]T_{ij}, \quad y = \sum [\mu_j + (y - \mu_j)]T_{ij}$$

where  $(x - \lambda_i)T_{ij}$  and  $(y - \mu_j)T_{ij}$  are nilpotent. If  $\psi(\lambda, \mu)$  is any scalar polynomial then

$$\psi(\lambda\mu) = \psi(\lambda_i, \mu_j) + \sum_{r,s} \psi_{r,s}^{i,j} (\lambda - \lambda_i)^r (\mu - \mu_j)^s$$

where  $\psi_{r,s}^{i,j}$  are scalars, we have therefore

$$\begin{aligned} \psi(x, y) &= \sum_{i,j} \left[ \psi(\lambda_i, \mu_j)T_{ij} + \sum_{r,s} \psi_{r,s}^{i,j} (x - \lambda_i)^r (y - \mu_j)^s T_{ij} \right] \\ &= \sum_{i,j} \psi(\lambda_i, \mu_j)T_{ij} + \sum_{i,j} \sum_{r,s} \psi_{r,s}^{i,j} T_{ij}^{r,s} \end{aligned}$$

where

$$T_{ij}^{r,s} = (x - \lambda_i)^r (y - \mu_j)^s T_{ij}$$

and  $r$  runs from 1 to  $\rho_i - 1$ , where  $\rho_i$  is the smallest integer for which  $(x - \lambda_i)^{\rho_i} R_i = 0$ , and  $s$  has a similar range with respect to  $y$ . The matrices  $T_{ij}^{r,s}$  are commutative and each is nilpotent; and hence any linear combination of them is also nilpotent.

Let

$$z = \sum \psi(\lambda_i, \mu_j)T_{ij}, \quad w = \sum \sum \psi_{r,s}^{i,j} T_{ij}^{r,s};$$

then  $w$ , being the sum of commutative nilpotent matrices, is nilpotent. If we take in  $z$  only terms for which  $T_{ij} \neq 0$ , we see immediately that the roots of  $z$  are the corresponding coefficients  $\psi(\lambda_i, \mu_j)$ ; and the reduced characteristic function of  $z$  is found as in §2.12. We have therefore the following theorem which is due to Frobenius.

**THEOREM 4.** *If  $R_i, S_j$  ( $i = 1, 2, \dots; j = 1, 2, \dots$ ) are the principal idempotent units of the commutative matrices  $x, y$  and  $T_{ij} = R_i S_j$ ; and if  $\lambda_i, \mu_j$  are the corresponding roots of  $x$  and  $y$ , respectively; then the roots of any scalar function  $\psi(x, y)$  of  $x$  and  $y$  are  $\psi(\lambda_i, \mu_j)$  where  $i$  and  $j$  take only those values for which  $T_{ij} \neq 0$ .*

This theorem extends immediately to any number of commutative matrices.

**7.06 Sylvester's identities.** It was shown in §2.08 that, if the roots of  $x$  are all distinct, the only matrices commutative with it are scalar polynomials in  $x$ ; and in doing so certain identities, due to Sylvester, were derived. We shall now consider these identities in more detail.

We have already seen that in

$$f(\lambda) = |\lambda - x| = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n$$

the coefficient  $a_r$  of  $\lambda^{n-r}$  is  $(-1)^r$  times the sum of the principal minors of  $x$  of order  $r$ ; these coefficients are therefore homogeneous polynomials of degree  $r$  in the coordinates of  $x$ . We shall now denote  $(-1)^r a_r$  by  $\left[ \begin{smallmatrix} x \\ r \end{smallmatrix} \right]$ . If  $x$  is replaced by  $\lambda x + \mu y$ , then  $\left[ \begin{smallmatrix} x \\ r \end{smallmatrix} \right]$  can be expressed as a homogeneous polynomial in  $\lambda, \mu$  of degree  $r$ , and we shall write

$$(23) \quad \left[ \begin{smallmatrix} \lambda x + \mu y \\ r \end{smallmatrix} \right] = \sum_{s=0}^r \left\{ \begin{smallmatrix} x & y \\ s & r-s \end{smallmatrix} \right\} \lambda^s \mu^{r-s}.$$

We shall further set, as in §2.08,

$$(24) \quad (\lambda x + \mu y)^r = \sum_{s=0}^r \left\{ \begin{smallmatrix} x & y \\ s & r-s \end{smallmatrix} \right\} \lambda^s \mu^{r-s}$$

where  $\left\{ \begin{smallmatrix} x & y \\ s & t \end{smallmatrix} \right\}$  is obtained by multiplying  $s$   $x$ 's and  $t$   $y$ 's together in every possible way and adding the terms so obtained.

In this notation the characteristic equation of  $\lambda x + \mu y$  is

$$(25) \quad 0 = \sum_{r=0}^n (-1)^r \left[ \begin{smallmatrix} \lambda x + \mu y \\ r \end{smallmatrix} \right] (\lambda x + \mu y)^{n-r} \\ = \sum_{r,s,t} \left[ \begin{smallmatrix} x & y \\ s & r-s \end{smallmatrix} \right] \left\{ \begin{smallmatrix} x & y \\ t-s & n+s-r-t \end{smallmatrix} \right\} \lambda^t \mu^{n-t},$$

where in the second summation  $\left[ \begin{smallmatrix} x & y \\ p & q \end{smallmatrix} \right]$  or  $\left\{ \begin{smallmatrix} x & y \\ p & q \end{smallmatrix} \right\}$  is to be replaced by 0 if either  $p$  or  $q$  is negative and  $\left[ \begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix} \right] = 1$ . Since  $\lambda$  is an independent variable, the coefficients of its various powers in (25) are identically 0, and therefore

$$(26) \quad \sum_{r,s=0}^n (-1)^r \left[ \begin{smallmatrix} x & y \\ s & r-s \end{smallmatrix} \right] \left\{ \begin{smallmatrix} x & y \\ t-s & n+s-r-t \end{smallmatrix} \right\} = 0 \quad (t = 0, 1, \dots, n)$$

a series of identical relations connecting two arbitrary matrices.

These identities can be generalized immediately. If  $x_1, x_2, \dots, x_m$  are any matrices and  $\lambda_1, \lambda_2, \dots$ , scalar variables, we may write

$$(27) \quad \left[ \begin{smallmatrix} \sum \lambda_i x_i \\ r \end{smallmatrix} \right] = \sum \left[ \begin{smallmatrix} x_1 & x_2 & \dots & x_m \\ r_1 & r_2 & \dots & r_m \end{smallmatrix} \right] \lambda_1^{r_1} \lambda_2^{r_2} \dots \lambda_m^{r_m} \\ (\sum \lambda_i x_i)^r = \sum \left\{ \begin{smallmatrix} x_1 & x_2 & \dots & x_m \\ r_1 & r_2 & \dots & r_m \end{smallmatrix} \right\} \lambda_1^{r_1} \lambda_2^{r_2} \dots \lambda_m^{r_m} \quad (\sum r_i = r)$$

and by the same reasoning as before we have

$$(28) \quad \sum_r \sum_{r_1, \dots, r_m} (-1)^r \left[ \begin{smallmatrix} x_1 & x_2 & \dots & x_m \\ r_1 & r_2 & \dots & r_m \end{smallmatrix} \right] \left\{ \begin{smallmatrix} x_1 & x_2 & \dots & x_m \\ s_1 - r_1 & s_2 - r_2 & \dots & s_m - r_m \end{smallmatrix} \right\} = 0 \\ \left( \sum_1^m r_i = r \right)$$

where  $s_1, s_2, \dots, s_m$  is any partition of  $n$ , zero parts included, and as before a bracket symbol is 0 when any exponent is negative.

Since  $\begin{bmatrix} \Sigma \lambda_i x_i \\ r \end{bmatrix}$  is the sum of the principal minors of  $\Sigma \lambda_i x_i$  of order  $r$ , we see that  $\begin{bmatrix} x_1 & x_2 & \dots & x_m \\ r_1 & r_2 & \dots & r_m \end{bmatrix}$  ( $\Sigma r_i = r$ ) is formed as follows. Take any principal minor of  $x_1$  of order  $r$  and the corresponding minors of  $x_2, x_3, \dots, x_m$  and replace  $r_2$  of its columns by the corresponding columns of  $x_2$ , then replace  $r_3$  of the remaining columns by the corresponding ones of  $x_3$ , and so on; do this in every possible way for each of the minors of order  $r$  of  $x_1$  and add all the terms so obtained.

There is a great variety of relations connecting the scalar functions defined above, a few of which we note here for convenience.

$$(i) \quad \begin{bmatrix} 1 \\ r \end{bmatrix} = \frac{n!}{r!(n-r)!}, \quad \begin{bmatrix} x & 1 \\ r & s \end{bmatrix} = \frac{(n-r)!}{s!(n-r-s)!} \begin{bmatrix} x \\ r \end{bmatrix},$$

$$\begin{bmatrix} x \\ n \end{bmatrix} = |x|, \quad \begin{bmatrix} x & x \\ r & s \end{bmatrix} = \frac{(r+s)!}{r!s!} \begin{bmatrix} x \\ r+s \end{bmatrix}.$$

(ii) The value of  $\begin{bmatrix} x_1 & x_2 & \dots & x_m \\ r_1 & r_2 & \dots & r_m \end{bmatrix}$  is unchanged by a cyclic permutation of the  $x$ 's.

$$(iii) \quad \begin{bmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_m \end{bmatrix} = \frac{n!}{\Pi(r_i!) (n - \Sigma r_i)!}$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_m & 1 \\ r_1 & r_2 & \dots & r_m & s \end{bmatrix} = \frac{(n - \Sigma r_i)!}{s!(n - s - \Sigma r_i)!} \begin{bmatrix} x_1 & x_2 & \dots & x_m \\ r_1 & r_2 & \dots & r_m \end{bmatrix}$$

$$\begin{bmatrix} x & x & \dots & x & y_1 & \dots & y_p \\ r_1 & r_2 & \dots & r_m & s_1 & \dots & s_p \end{bmatrix} = \frac{(\Sigma r_i)!}{\Pi(r_i!)} \begin{bmatrix} x & y_1 & \dots & y_p \\ \Sigma r_i & s_1 & \dots & s_p \end{bmatrix}$$

$$(iv) \quad \begin{bmatrix} x \\ r \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ 1 & 1 & \dots & 1 \end{bmatrix} = \Sigma \begin{bmatrix} xy_{i_1} & xy_{i_2} & \dots & xy_{i_r} & y_{i_{r+1}} & \dots & y_{i_n} \\ 1 & 1 & \dots & 1 & 1 & \dots & 1 \end{bmatrix}$$

where the summation extends over the  $n!/r!(n-r)!$  ways of choosing  $r$  integers out of  $1, 2, \dots, n$ , the order being immaterial.

**7.07 Similar matrices.** In addition to the identities discussed in the preceding section Sylvester gave another type, a modification of which we shall now discuss. If  $x, y, a$  are arbitrary matrices, we have

$$(29) \quad x^{r+1}a - ay^{r+1} = x(x^r a + x^{r-1}ay + x^{r-2}ay^2 + \dots + ay^r) \\ - (x^r a + x^{r-1}ay + x^{r-2}ay^2 + \dots + ay^r)y$$

or say

$$x^{r+1}a - ay^{r+1} = x(x, a, y)_r - (x, a, y)_r y$$

where

$$(30) \quad (x, a, y)_r = \sum_{i=0}^r x^{r-i} ay^i.$$

Suppose now that  $x$  and  $y$  satisfy the same equation  $f(\lambda) = 0$  where

$$f(\lambda) = \lambda^m + a_1\lambda^{m-1} + \dots + a_{m-1}\lambda + a_m$$

$x$  and  $y$  being commutative with each  $a_i$  and  $a$  commutative with every  $a_i$ . Let

$$(31) \quad u = \sum_0^{m-1} a_i(x, a, y)_{m-i-1};$$

then

$$(32) \quad 0 = f(x)a - af(y) = \sum a_i(x^{m-i}a - ay^{m-i}) = xu - uy.$$

If  $|u| \neq 0$ , it follows that  $y = u^{-1}xu$ , that is,  $x$  and  $y$  are similar.

It can be shown that  $a$  can be chosen so that  $|u| \neq 0$  provided  $x$  and  $y$  have the same invariant factors and  $f(\lambda)$  is the reduced characteristic function.

## CHAPTER VIII

### FUNCTIONS OF MATRICES

**8.01 Matrix polynomials.** The form of a polynomial in a matrix has already been discussed in §2.11 but we repeat the principal formulas here for convenience. If  $x$  is a matrix whose reduced characteristic function is

$$(1) \quad \varphi(\lambda) = \prod_{i=1}^r (\lambda - \lambda_i)^{\nu_i}, \quad \sum \nu_i = \nu,$$

and

$$m_i(\lambda) = \varphi(\lambda) / (\lambda - \lambda_i)^{\nu_i}, \quad M_i(\lambda)m_i(\lambda) + (\lambda - \lambda_i)^{\nu_i}N_i(\lambda) \equiv 1,$$

$$(2) \quad \varphi_i(\lambda) = M_i(\lambda)m_i(\lambda),$$

$$(3) \quad \varphi_i = \varphi_i(x), \quad h_i = (x - \lambda_i)\varphi_i,$$

and if  $g(\lambda)$  is a scalar polynomial in  $\lambda$ , then

$$(4) \quad g(x) = \sum_{i=1}^r \left[ g(\lambda_i)\varphi_i + g'(\lambda_i)h_i + \cdots + \frac{g^{(\nu_i-1)}(\lambda_i)h_i^{\nu_i-1}}{(\nu_i-1)!} \right].$$

This formula can still be interpreted when the coefficients of  $g(\lambda)$  are matrices, but in this case the notation  $g(x)$  is ambiguous. Let  $g(\lambda) = a_0 + a_1\lambda + \cdots + a_m\lambda^m$ ; then

$$a_0 + a_1x + \cdots + a_mx^m \quad \text{and} \quad a_0 + xa_1 + \cdots + x^ma_m$$

are called, respectively, the *dextro-* and *laevo-lateral* polynomials corresponding to  $g(\lambda)$ . It is clear that (4) holds for a dextro-lateral polynomial and will give the corresponding laevo-lateral polynomial if  $g(\lambda_i)\varphi_i$ ,  $g'(\lambda_i)h_i$ , etc., are replaced by  $\varphi_i g(\lambda_i)$ ,  $h_i g'(\lambda_i)$ , etc.

**8.02 Infinite series.** If  $a_0, a_1, \cdots$  are matrices and  $\lambda$  a scalar variable the coordinates of the matrix

$$(5) \quad g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots$$

are scalar infinite series in  $\lambda$ ; and if each of these series converges for mod  $\lambda$  less than  $\rho$ , we say that the series (5) converges. When this condition is satisfied, the series

$$(6) \quad g(x) = a_0 + a_1x + a_2x^2 + \cdots$$

converges for any matrix  $x$  for which the absolute value of the root of greatest absolute value is less than  $\rho$ . For if  $g_m$  is the sum of the first  $m$  terms of (6), then by (4)  $g_m = \sum_i g_{mi}$  where

$$g_{mi} = g_m(\lambda_i)\varphi_i + g'_m(\lambda_i)h_i + \dots + \frac{g_m^{(\nu_i-1)}(\lambda_i)h_i^{\nu_i-1}}{(\nu_i-1)!}.$$

The matrices  $\varphi_i, h_i$  are independent of  $m$  and, since the absolute value of each  $\lambda_i$  is less than  $\rho$ ,  $g_m(\lambda_i), g'_m(\lambda_i), \dots, g_m^{(\nu_i-1)}(\lambda_i)$  converge to  $g(\lambda_i), g'(\lambda_i), \dots, g^{(\nu_i-1)}(\lambda_i)$  when  $m$  approaches infinity.

As an illustration of such a series we may define  $\exp x$  and  $\log(1+x)$  by

$$(7) \quad \exp x = \epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \dots$$

$$(8) \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

The first of these converges for every matrix  $x$ , the second for matrices all of whose roots are less than 1 in absolute value.

The usual rules for adding series and for multiplying series whose coefficients are commutative with  $x$  and with each other hold for matrix series. For instance we can show by the ordinary proof that, if  $xy = yx$ , then  $\epsilon^x + \epsilon^y = \epsilon^{x+y}$  but this will not usually be the case if  $xy \neq yx$ .

**8.03 The canonical form of a function.** In the case of multiform functions (4) does not always give the most general determination of the function which is only obtained by taking into account the partial as well as the principal elements of the variable  $x$ . As in §3.06 suppose that  $x$  has the canonical form

$$(9) \quad x = \begin{vmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & a_r \end{vmatrix}$$

where  $a_i$  is a block of terms

$$(10) \quad \begin{matrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{matrix} \quad (r_i \text{ rows and columns}).$$

It is convenient to let  $a_i$  stand also for the matrix derived from (9) by replacing every  $a_j$  by 0 except when  $j = i$ . We can then write

$$(11) \quad x = \Sigma a_i, \quad a_i a_j = 0 \quad (i \neq j)$$

and we may set

$$(12) \quad a_i = \lambda_i e_i + z_i$$

where (cf. §3.07)  $e_i^2 = e_i$ ,  $z_i$  is a nilpotent matrix of index  $r_i$ , and

$$\sum_i e_i = 1, \quad e_i z_i = z_i = z_i e_i, \quad e_i e_j = 0, \quad e_i z_j = 0 = z_j e_i \quad (i \neq j).$$

The part of  $z_i$  which is not 0 is given by the oblique line of 1's in (10);  $z^2$  is obtained by moving all the 1's one place to the right except the last which disappears, and in general  $z^{m-1}$  has a line of 1's starting in the  $m$ th column of (10) and running parallel to the main diagonal till it meets the boundary of the block.

It is now easy to see the form of a scalar polynomial  $g(x)$  or of a convergent power series with scalar coefficients; for

$$(13) \quad g(x) = \Sigma g(a_i) e_i = \Sigma \left[ g(\lambda_i) e_i + g'(\lambda_i) z_i + \dots + \frac{g^{(r_i-1)}(\lambda_i) z_i^{r_i-1}}{(r_i-1)!} \right]$$

and the block of terms in  $g(x)$  which corresponds to  $a_i$  in (10) is, omitting the subscripts for clearness,

$$(14) \quad \begin{array}{ccccccc} g(\lambda) & g'(\lambda) & \frac{g''(\lambda)}{2!} & \frac{g'''(\lambda)}{3!} & \dots & \frac{g^{(r-1)}(\lambda)}{(r-1)!} & \\ & & & & & & \\ & g(\lambda) & g'(\lambda) & \frac{g''(\lambda)}{2!} & \dots & \frac{g^{(r-2)}(\lambda)}{(r-2)!} & \\ & & & & & & \\ & & g(\lambda) & g'(\lambda) & \dots & \frac{g^{(r-3)}(\lambda)}{(r-3)!} & \\ & & & & \dots & & \\ & & & & & & \\ & & & & & & \\ & & & & & & g'(\lambda) \\ & & & & & & g(\lambda) \end{array}$$

where all the terms to the left of the main diagonal are 0, the coordinates in the first row are as indicated, and all those on a line parallel to the main diagonal are the same as the one where this line meets the first row.

If the characteristic function is the same as the reduced function, no two blocks of terms in (9) correspond to the same root and  $e_i$ ,  $z_i$  are the principal idempotent and nilpotent elements of  $x$  corresponding to  $\lambda_i$  and (13) is the same as (4). This is not the case when the same root occurs in more than one

of the blocks (10) and, when this is so, the  $a_i$  are not necessarily uniquely determined. For instance let

$$(15) \quad x = \left\| \begin{array}{ccc} \gamma & 0 & 0 \\ 0 & \gamma & 1 \\ 0 & 0 & \gamma \end{array} \right\| = \gamma + e_{23}.$$

Here we have

$$\begin{aligned} a_1 &= \gamma e_{11}, & a_2 &= \gamma(e_{22} + e_{33}) + e_{23}, \\ e_1 &= e_{11}, & e_2 &= e_{22} + e_{33}, & z_1 &= 0, & z_2 &= e_{23}. \end{aligned}$$

But if

$$\begin{aligned} f_{11} &= e_{11} - e_{13} & f_{12} &= e_{12} & f_{13} &= e_{13} \\ f_{21} &= e_{21} - e_{23} & f_{22} &= e_{22} & f_{23} &= e_{23} \\ f_{31} &= e_{31} - e_{13} + e_{11} - e_{33}, & f_{32} &= e_{32} + e_{12}, & f_{33} &= e_{33} + e_{13}, \end{aligned}$$

the  $f$ 's form a set of matrix units and

$$x = \gamma f_{11} + \gamma(f_{22} + f_{33}) + f_{23}$$

so that we might have chosen  $f_1 = f_{11}$ ,  $f_2 = f_{22} + f_{33} = e_{22} + e_{33} + e_{13}$  as idempotent elements in place of  $e_1$  and  $e_2$ .

It should be carefully noted that  $f_1, f_2$  are not commutative with  $e_1, e_2$  and in consequence different determinations of a multiform function may not be commutative with each other. For instance, if  $x$  is the matrix given in (15) with  $\gamma \neq 0$ , and  $\gamma^\dagger$  is a particular determination of the square root of  $\gamma$ , we have already seen in §2.13 that determinations of  $x^\dagger$  are given by

$$\begin{aligned} u_1 &= \gamma^\dagger e_{11} - \gamma^\dagger(e_{22} + e_{33}) - e_{23}/2\gamma^\dagger \\ u_2 &= \gamma^\dagger f_{11} - \gamma^\dagger(f_{22} + f_{33}) - f_{23}/2\gamma^\dagger \\ &= \gamma^\dagger(e_{11} - e_{13}) - \gamma^\dagger(e_{22} + e_{33} + e_{13}) - e_{23}/2\gamma^\dagger \\ &= u_1 - 2\gamma^\dagger e_{13}, \end{aligned}$$

and these two values of  $x^\dagger$  are not commutative.

**8.04 Roots of 0 and 1.** The reduced equation of a nilpotent matrix of index  $m$  is  $x^m = 0$  and this matrix can therefore be defined as a primitive  $m$ th root of 0; the index  $m$  cannot be greater than  $n$  and it exceeds 1 unless  $x = 0$ . The canonical form of  $x$  must contain at least one block of order  $r_1 = m$ , similar to (10) but with  $\lambda_i = 0$ , and a number of like blocks of orders, say,  $r_i$  ( $i = 2, 3, \dots$ ) where  $r_i \leq r_1$  and  $\sum_1 r_i = n$ . This gives rise to a series of distinct types in number equal to the number of partitions of  $n - m$  into parts no one of which exceeds  $m$ , and  $x$  is a primitive  $m$ th root if, and only if, it is similar to one of these types.



there being here no question of convergence since any power series in a nilpotent matrix terminates. It follows from (21) and (22) that the indices of  $g_{ijk}$  and  $h_{ijk}$  are the same.

We shall now show that the matrices  $f_{ijk}$  and  $g_{ijk}$  form a set of partial idempotent and nilpotent elements of  $x$  provided always that  $x$  is not singular. If this were not so, then  $f_{ijk}$  must be the sum of two or more partial idempotent elements; for the sake of brevity we shall assume that it is the sum of two since the proof proceeds in exactly the same way if more components are taken. Let  $f_{ijk} = d_1 + d_2$  where  $d_1$  and  $d_2$  are partial idempotent elements of  $x$  and let  $c_1, c_2$  be the corresponding nilpotent elements; then

$$g_{ijk} = c_1 + c_2, \quad c_1c_2 = 0 = c_2c_1.$$

Hence also  $h_{ijk} = b_1 + b_2, b_1b_2 = 0 = b_2b_1$  where  $b_\alpha$  is obtained by putting  $c_\alpha$  for  $g_{ijk}$  in (22); and this is impossible since we assumed that  $f_{ijk}$  and  $h_{ijk}$  were partial idempotent and nilpotent elements of  $y$ . We have therefore the following theorem.

**THEOREM 1.** *If  $x$  is a non-singular matrix, any determination of  $y = x^{1/m}$  can be obtained by expressing  $x$  in terms of partial idempotent and nilpotent elements, say  $x = \Sigma(\lambda_i f_i + g_i)$  and putting*

$$y = \Sigma(\lambda_i f_i + g_i)^{1/m} = \Sigma \lambda_i^{1/m} (f_i + m^{-1} \lambda_i^{-1} g_i + \frac{1}{2} m^{-1} (m^{-1} - 1) \lambda_i^{-2} g_i^2 + \dots).$$

*Here the binomial series terminates and  $\lambda_i^{1/m}$  is a determination of the  $m$ th root of  $\lambda_i$  which may be different for different terms of the summation if this root occurs with more than one partial element.*

There is thus a two-sided multiplicity of  $m$ th roots of  $x$ ; the  $\lambda_i^{1/m}$  have  $m$  possible determinations in each term and also there is in certain cases an infinity of ways of choosing the set of partial elements. Since the canonical form is independent of the actual choice of the set of partial elements out of the possible sets, any choice of such a set can be derived from any other such set by transforming it by a matrix  $u$ ; and since  $x$  itself is the same no matter what set of partial elements is chosen, we have  $uxu^{-1} = x$ , that is,  $u$  is commutative with  $x$ . It follows from the development given in §§7.01, 7.02 and 7.04 that a matrix  $u$  which is commutative with every partial idempotent element is a polynomial in  $x$ .

8.06 We must now consider the case in which  $x$  is singular and in doing so it is sufficient to discuss  $m$ th roots of a nilpotent matrix; for the principal idempotent element of  $x$  which corresponds to a root  $\mu$  is the sum of those principal idempotent elements of  $y$  which correspond to those roots whose  $m$ th power is  $\mu$ , so that the principal idempotent element corresponding to the root 0 is the same for both  $x$  and  $y$ . Let the elementary divisors of  $y$  be  $\lambda^{m_1}, \lambda^{m_2}, \dots, \lambda^{m_p}$ ; then

$$y = y_1 + y_2 + \dots + y_p$$

where  $y_i$  is a nilpotent matrix of index  $m_i$ , and we may suppose the fundamental basis so chosen that the significant part of  $y_i$  is

$$(23) \quad \begin{matrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{matrix} \quad (m_i \text{ rows and columns}).$$

To simplify the notation we shall consider for the moment only one part  $y_i$  and replace it by  $y$  and  $m_i$  by  $n$  so that  $y^n = 0$  and

$$y = \sum_1^{n-1} e_i S e_{i+1}.$$

If we now form the  $m$ th power of  $y$ , then  $y^m = 0$  if  $m \geq n$  and if  $m < n$

$$y^m = \sum_1^{n-m} e_i S e_{i+m}.$$

If we define  $r$  and  $k$  by

$$(24) \quad (r - 1)m + k = n \leq rm \quad (k > 0)$$

then  $r \geq 2$  and

$$(25) \quad y^m e_i = 0, y^m e_{i+m} = e_i, y^m e_{i+2m} = e_{i+m}, \dots, y^m e_{i+(r-1)m} = e_{i+(r-2)m} \quad (i = 1, 2, \dots, k)$$

giving  $k$  chains of order  $r$  of invariant vectors, and similarly for  $i = k + 1, k + 2, \dots, m$ , we have  $m - k$  chains whose order is  $r - 1$  since for these values of  $i$  the last equation in (25) is missing. If we set  $u$  and  $v$  for blocks of terms like (23) only with  $r$  and  $r - 1$  rows and columns, respectively, then we can find a non-singular matrix  $P$  which permutes the rows and columns in  $y^m$  so that

$$(26) \quad P^{-1}y^mP = \left\| \begin{array}{cccc} u & & & \\ & u & & \\ & & \ddots & \\ & & & u \\ & & & & v \\ & & & & & \ddots \\ & & & & & & v \end{array} \right\| \quad (k \text{ } u\text{'s and } m - k \text{ } v\text{'s}).$$

We are now in a position to consider the solution of  $y^m = x$  where  $x$  is a nilpotent matrix of index  $r$ . In the elementary divisors of  $x$  suppose  $p_1$  expo-

nents equal  $r$ ,  $p_2$  equal  $r - 1$ , and in general  $p_j$  equal  $r - j + 1$  ( $j = 1, 2, \dots, r$ ); here the  $p$ 's are integers equal to or greater than 0 such that

$$\sum_j (r - j + 1)p_j = n, \quad p_1 \neq 0.$$

The maximum possible exponent for any elementary divisor of  $y$  is  $rm$ ; let  $q_{ji}$  ( $i = 1, 2, \dots, m; j = 1, 2, \dots, r$ ) be the number of exponents which equal

$$(r - j + 1)m - i + 1 = (r - j)m + (m - i + 1).$$

Forming  $y^m$  and using (24) and the results of (25) we then see that

$$(27) \quad q_{j-1,2} + 2q_{j-1,3} + \dots + (m-1)q_{j-1,m} + mq_{j1} + (m-1)q_{j2} \\ + \dots + q_{jm} = p_j \quad (j = 1, 2, \dots, r; q_{0i} = 0) \\ \sum_{i,j} [(r-j+1)m - i + 1] q_{ji} = n.$$

These relations form a set of Diophantine equations for the  $q$ 's. When a set of  $q$ 's have been found, we can find the matrix  $P$  (cf. (26)) for each part of  $y^m$  and then set  $y = \Sigma R^{-1}y_i R$  where  $R$  has the form  $\Sigma P_i Q_i$ ,  $Q_i$  being commutative with  $P_i^{-1}y_i^m P_i$  and so chosen that  $R$  is not singular.

**8.07 The exponential and logarithmic functions.** The function  $\exp y = \epsilon^y$  has already been defined in §8.02 by the series

$$\epsilon^y = 1 + \sum_1^{\infty} y^m/m!$$

or in §8.03 in terms of the partial units of  $y$ . Let the distinct roots of  $y$  be  $\mu_1, \mu_2, \dots, \mu_s$  and let a choice of the partial idempotent and nilpotent elements corresponding to  $\mu_i$  be  $f_{ij}, h_{ij}$  ( $j = 1, 2, \dots, k_i$ ) so that

$$(28) \quad f_i = \sum_{j=1}^{k_i} f_{ij}, \quad h_i = \sum_{j=1}^{k_i} h_{ij} \quad (i = 1, 2, \dots, s)$$

are the principal idempotent and nilpotent elements of  $y$ . If we set  $x = \epsilon^y$ , we have

$$(29) \quad x = \epsilon^y = \sum_i \sum_j \left[ \epsilon^{\mu_i f_{ij}} + h_{ij} + \frac{h_{ij}^2}{2!} + \dots + \frac{h_{ij}^{\nu_{ij}-1}}{(\nu_{ij}-1)!} \right] \\ = \sum_i \sum_j (\epsilon^{\mu_i f_{ij}} + g_{ij})$$

where  $\nu_{ij}$  is the index of  $h_{ij}$  and

$$(30) \quad g_{ij} = h_{ij} + \frac{h_{ij}^2}{2} + \dots + h_{ij}^{\nu_{ij}-1}/(\nu_{ij}-1)!.$$

The index of  $g_{ij}$  is clearly  $\nu_{ij}$ . Solving (30) for  $h_{ij}$ , we have the formal solution  $h_{ij} = \log(1 + g_{ij})$  and on using this or inverting the power series in (30) we get

$$(31) \quad h_{ij} = g_{ij} - \frac{1}{2}g_{ij}^2 + \cdots + (-1)^{\nu_{ij}}g_{ij}^{\nu_{ij}-1}/(\nu_{ij} - 1).$$

As in §8.05 it follows that  $f_{ij}, g_{ij}$  form a set of partial elements for  $x$  and, when  $x$  is given so that  $y = \log x$ , the method there used gives the following theorem.

**THEOREM 2.** *If  $x$  is a non-singular matrix whose distinct roots are  $\lambda_1, \lambda_2, \dots, \lambda_r$ , and if  $\log \lambda_1, \log \lambda_2, \dots, \log \lambda_r$  are particular determinations of the logarithms of these roots, then the general determination of  $\log x$  is found as follows. Take any set of partial elements of  $x$ , say  $f_{ij}, g_{ij}$  ( $i = 1, 2, \dots, r; j = 1, 2, \dots, k_i$ ) where  $f_{ij}, g_{ij}$  correspond to  $\lambda_i$  and the index of  $g_{ij}$  is  $\nu_{ij}$ , let  $h_{ij}$  be the nilpotent matrix defined by (31), and let  $k_{ij}$  be any integers, then*

$$(32) \quad \log x = \sum_i \sum_j [(\log \lambda_i + k_{ij}\pi\sqrt{-1})f_{ij} + h_{ij}].$$

The discussion of the relation between different determinations of  $\log x$  is practically the same as for  $x^{1/m}$  and need not be repeated.

If  $f_i$  and  $h_i$  are defined by (28), a particular determination of  $\log x$  is given by

$$(33) \quad \text{Log } x = \sum_i [(\log \lambda_i + k_i\pi\sqrt{-1})f_i + h_i].$$

This form of  $\log x$  has the same principal elements as  $x$  provided  $\log \lambda_i + k_i \neq \log \lambda_j + k_j$  for any  $i \neq j$ , and even when this condition is not satisfied, it is convenient to refer to (33) as a *principal* determination of  $\log x$ . This determination is the one given by the series (cf. §8.02 (8))

$$(34) \quad \log x = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \cdots$$

provided each  $k_i$  is 0 and the principal determination of  $\log \lambda_i$  is used. The series converges only when the roots of  $x - 1$  are all less than 1 in absolute value.

**8.08 The canonical form of a matrix in a given field.** If the coefficients of a matrix are restricted to lie in a given field of rationality, the canonical form used in the preceding sections requires some modification. The definition of the invariant factors is rational as are also the theorems regarding similar matrices which were derived from them in Chapter 3; and hence if  $X$  and  $x$  are rational matrices which have the same invariant factors there exists a rational matrix  $P$  for which  $P^{-1}xP = X$ . The definition of elementary divisors requires only the natural alteration of substituting powers of irreducible polynomials for  $(\lambda - \lambda_i)^{\nu_{ij}}$ .

Let

$$\alpha(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_m$$

be a scalar polynomial in a field  $F$  which is irreducible in  $F$ ; then the matrix of order  $m$  defined by

$$(35) \quad x_a = \begin{matrix} & -a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\ & 1 & 0 & \cdots & 0 & 0 \\ & 0 & 1 & \cdots & 0 & 0 \\ & \cdot & \cdot & \cdots & \cdot & \cdot \\ & \cdot & \cdot & \cdots & \cdot & \cdot \\ & 0 & 0 & \cdots & 1 & 0 \end{matrix}$$

has  $\alpha(\lambda)$  as its characteristic function; since  $\alpha(\lambda)$  is irreducible in  $F$ , it follows immediately that  $x_a$  is an irreducible matrix in  $F$  and that  $\alpha(\lambda)$  is also the reduced characteristic function. It is easily seen that the invariant factors of  $\lambda - x$  are given by  $m - 1$  units followed by  $\alpha(\lambda)$ .

Again, if we consider

$$(36) \quad x_a = \begin{matrix} x_a & 1_m & & & & \\ & x_a & 1_m & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1_m \\ & & & & & x_a \end{matrix} \quad (r \text{ rows and columns})$$

which is a matrix of order  $rm$ , we see as in §8.03 (14) that, if  $g(\lambda)$  is a scalar polynomial in  $\lambda$ ,

$$g(x_a) = \begin{matrix} g(x_a) & g'(x_a) & \cdots & g^{(r-1)}(x_a)/(r-1)! \\ & g(x_a) & \cdots & g^{(r-2)}(x_a)/(r-2)! \\ & & \cdots & \cdot \\ & & \cdot & \cdot \\ & & & g'(x_a) \\ & & & & g(x_a) \end{matrix}$$

It follows that, if  $g(x_a) = 0$ , we must have  $g^{(r-1)}(x_a) = 0$  and therefore  $\alpha(\lambda)$  is a factor of  $g^{(r-1)}(\lambda)$  so that  $[\alpha(\lambda)]^r$  is a factor of  $g(\lambda)$ . But if we put  $g(\lambda) = [\alpha(\lambda)]^r$  the first  $(r - 1)$  derivatives of  $g(\lambda)$  have  $\alpha(\lambda)$  as a factor and so vanish when  $\lambda$  is replaced by  $x_a$ ; hence  $g(x_a) = 0$ . It follows that the reduced characteristic function of  $x_a$  is  $[\alpha(\lambda)]^r$  and, since the degree of this polynomial equals the order  $rm$  of  $x_a$ , it is also the characteristic function so that the invariant factors of  $x_a$  are given by 1 repeated  $rm - 1$  times followed by  $[\alpha(\lambda)]^r$ . The argument used in §3.06 then gives the following theorem.

**THEOREM 3.** *Let  $\alpha_1(\lambda), \alpha_2(\lambda), \dots, \alpha_k(\lambda)$  be polynomials, not necessarily distinct, which are rational and irreducible in a field  $F$  and whose degrees are  $m_1, m_2, \dots, m_k$  respectively; and let  $r_1, r_2, \dots, r_k$  be any positive integers such that*

$\sum_1^k r_i m_i = n$ ; then, if  $x_{\alpha_i}$  is the matrix of order  $r_i m_i$  formed from  $\alpha_i(\lambda)$  in the same way as  $x_{\alpha}$  in (36) is formed from  $\alpha(\lambda)$ , the matrix of order  $n$  defined by

$$(37) \quad x = \begin{matrix} & x_{\alpha_1} & & & \\ & & x_{\alpha_2} & & \\ & & & \ddots & \\ & & & & x_{\alpha_k} \end{matrix}$$

has  $\alpha_1(\lambda), \alpha_2(\lambda), \dots, \alpha_k(\lambda)$  as its elementary divisors in  $F$ .

If  $X$  is any matrix with the same elementary divisors as  $x$ , it follows from §3.04 Theorem 5 that we can find a rational nonsingular matrix  $P$  such that  $PXP^{-1} = x$ . We may therefore take (37) as a canonical form for a matrix in the given field  $F$ .

**8.09 The absolute value of a matrix.** The absolute value of a matrix  $a = || a_{pq} ||$  is most conveniently defined as

$$(38) \quad |a| = \left( \sum_{p,q=1}^n a_{pq} \bar{a}_{pq} \right)^{\frac{1}{2}}$$

where the heavy bars are used to distinguish between the absolute value and the determinant  $|a|$ . It must be carefully noted that the absolute value of a scalar matrix  $\lambda$  is not the same as the ordinary absolute value or modulus of  $\lambda$ , the relation between them being

$$(39) \quad |\lambda| = n^{\frac{1}{2}} \text{mod } \lambda.$$

It follows immediately from (37) that

$$(40) \quad \text{mod} (|a| - |b|) \leq |a + b| \leq |a| + |b|;$$

and from

$$\sum_{p,r} a_{pr} \bar{a}_{pr} \sum_{q,s} b_{sq} \bar{b}_{sq} = \sum_{p,q} \left[ \frac{1}{2} \sum_{r,s} (a_{pr} \bar{b}_{sq} - a_{ps} \bar{b}_{rq}) (\bar{a}_{pr} b_{sq} - \bar{a}_{ps} b_{rq}) + \sum_r a_{pr} b_{rq} \sum_s \bar{a}_{ps} \bar{b}_{sq} \right],$$

we have

$$(41) \quad |ab| \leq |a| |b|.$$

Since the trace of  $a\bar{a}'$  is  $\sum a_{pq} \bar{a}_{pq}$ , the absolute value of  $a$  might also have been defined by

$$(42) \quad |a|^2 = \text{tr } a\bar{a}' = \text{tr } \bar{a}'a.$$

From this we see immediately that, if  $a$  is unitary, that is,  $a\bar{a}' = 1$ , then

$$(43) \quad |a| = n^{\frac{1}{2}}, \quad |ab| = |b|$$

where  $b$  is any matrix.

No matter what matrix  $a$  is,  $a\bar{a}'$  is a positive hermitian matrix, semi-definite or definite according as  $a$  is or is not singular; the roots  $g_1, g_2, \dots, g_n$  of  $a\bar{a}'$  are therefore real and not negative. If we set

$$p_r = \Sigma g_1 g_2 \cdots g_r, \quad p_r \equiv p_n = (\text{mod } |a|)^2, \quad s \equiv p_1 = |a|^2 = \Sigma g_i,$$

then

$$r^r p_r \leq \Sigma (g_1 + g_2 + \cdots + g_r)^r \leq \binom{n-1}{r-1} s^r, \quad p_r \geq \binom{n}{r} p^{\binom{n-1}{r-1} / \binom{n}{r}} = \binom{n}{r} p^{r/n},$$

whence<sup>1</sup>

$$(44) \quad \binom{n}{r} p^{r/n} \leq p_r \leq r^{-r} \binom{n-1}{r-1} s^r.$$

If  $C^r(a)$  is the  $r$ th supplementary compound of  $a$  (cf. §5.03), and  $\alpha = p^{\frac{1}{n}}$  is put for  $\text{mod } |a|$ , then  $p_r = |C^r(a)|^2$  and we may write for (44)

$$\binom{n}{r} \alpha^{2r/n} \leq |C^r(a)|^2 \leq r^{-r} \binom{n-1}{r-1} |a|^{2r}$$

and, since  $(a')^{-1} = C^{n-1}(a)/|a|$ , we have

$$(45) \quad n^{\frac{1}{2}} \alpha^{-1/n} \leq |\alpha^{-1}| \leq (n-1)^{-\frac{n-2}{2}} |a|^{n-1/\alpha}$$

provided  $|a| \neq 0$ . This inequality enables us to deal with expressions involving negative powers of  $a$ .

Since  $a^{-1} = \exp(-\log a)$ , we also have

$$|a^{-1}b| = |(1 - \log a + \frac{1}{2}(\log a)^2 - \cdots)b| \leq (1 + \frac{1}{2}|\log a|^2 + \cdots)|b|$$

and therefore

$$(46) \quad |a^{-1}b| \leq \epsilon^{|\log a|} |b|.$$

Putting  $b = 1$  we also have

$$(47) \quad |a^{-1}| \leq n^{\frac{1}{2}} - 1 + \epsilon^{|\log a|} \quad \text{and} \quad |a^{-1}| \leq n^{\frac{1}{2}} \epsilon^{|\log a|}.$$

It is also sometimes convenient to note as a consequence of (41) with  $b = a^{-1}$  that

$$(48) \quad |a^{-1}| \geq n^{\frac{1}{2}} |a|^{-1}.$$

<sup>1</sup> If  $r = n$ , (44) gives Hadamard's expression for the maximum value of  $\text{mod } |a|$ .

**8.10 Infinite products.** As an illustration of the use of the preceding section we shall now investigate briefly the convergence of an infinite product. Let

$$(49) \quad \begin{aligned} P_m &= (1 + a_1)(1 + a_2) \cdots (1 + a_m) \\ Q_m &= (1 + |a_1|)(1 + |a_2|) \cdots (1 + |a_m|), \end{aligned}$$

then, if  $c$  is an arbitrary matrix,

$$(50) \quad |P_m - 1| \leq Q_m - 1 < \epsilon^{\Sigma |a_i|} - 1,$$

$$(51) \quad |P_m c| \leq \epsilon^{\Sigma |a_i|} |c|,$$

$$(52) \quad |P_m - P_k| \leq Q_m - Q_k \leq \sum_k^m |a_i| \epsilon^{\sum_1^m |a_i|}.$$

For on expanding  $P_m$  we have

$$P_m = 1 + \sum_p a_p + \sum_{p < q} a_p a_q + \sum_{p < q < r} a_p a_q a_r + \cdots$$

therefore

$$|P_m - 1| \leq \sum_p |a_p| + \sum_{p < q} |a_p| |a_q| + \cdots = Q_m - 1$$

$$|P_m c| \leq (1 + \sum_p |a_p| + \sum_{p < q} |a_p| |a_q| + \cdots) |c| = Q_m |c| < \epsilon^{\Sigma |a_i|} |c|.$$

The proof of (52) follows in the same manner.

Hence  $P_m$  converges when  $Q_m$  does, for which it is sufficient that  $\Sigma a_i$  is absolutely convergent in the sense that  $\Sigma |a_i|$  converges.

**8.11 The absolute value of a tensor.** If  $w$  is a tensor of grade  $r$ , we define the absolute value of  $w$  by<sup>2</sup>

$$(53) \quad \text{mod } w = (r! S \bar{w} w)^{\frac{1}{r}}.$$

We shall for the most part consider only vectors of grade 1 as the extension to tensors of higher grade is usually immediate.

If  $x$  and  $y$  are any vectors, we derive from (53)

$$(54) \quad \text{mod } (x + y) \leq \text{mod } x + \text{mod } y, \quad \text{mod } Sxy \leq \text{mod } x \text{ mod } y.$$

If  $A$  is a matrix,

$$(\text{mod } Ax)^2 = S \bar{A} \bar{x} A x = S \bar{x} \bar{A}' A x.$$

By §6.02

$$\bar{A}' A = \Sigma g_i \bar{a}_i S a_i$$

<sup>2</sup> The  $r!$  enters here only because of the numerical factor introduced in defining  $Suw$  (cf. §5.16).

where the  $g$ 's are real and not negative, and  $S\bar{a}_i a_i = \delta_{ij}$  so that  $\text{mod } a_i = 1$ ; hence

$$\begin{aligned} S\bar{x}\bar{A}'Ax &= \Sigma g_i S\bar{x}\bar{a}_i S a_i x = \Sigma g_i (\text{mod } S a_i x)^2 \\ &\leq \Sigma g_i (\text{mod } a_i \text{ mod } x)^2 = (\Sigma g_i) (\text{mod } x)^2 \\ &= |A|^2 (\text{mod } x)^2, \end{aligned}$$

or

$$(55) \quad \text{mod } Ax \leq |A| \text{ mod } x.$$

From (54) we then have

$$(56) \quad \text{mod } SyAx \leq |A| \text{ mod } x \text{ mod } y.$$

**8.12 Matric functions of a scalar variable.** If the coordinates of a matrix  $a(t) = || a_{pq}(t) ||$  are functions of a scalar variable  $t$ , the matrix itself is called a matric function of  $t$ . The derivative, when it exists, is defined as

$$(57) \quad \frac{da}{dt} = \lim_{h \rightarrow 0} \frac{a(t+h) - a(t)}{h} = \left\| \frac{da_{pq}}{dt} \right\|$$

$h$  being a scalar. The fundamental rules of differentiation are

$$\frac{d(a+b)}{dt} = \frac{da}{dt} + \frac{db}{dt}, \quad \frac{d(ab)}{dt} = \frac{da}{dt} b + a \frac{db}{dt}, \quad \frac{da'}{dt} = \left( \frac{da}{dt} \right)'$$

to which we may add, when  $|a| \neq 0$ ,

$$(58) \quad \frac{da^{-1}}{dt} = -a^{-1} \frac{da}{dt} a^{-1}.$$

Other examples are

$$\frac{da^2}{dt} = \frac{da}{dt} a + a \frac{da}{dt}, \quad \frac{da^3}{dt} = \frac{da}{dt} a^2 + a \frac{da}{dt} a + a^2 \frac{da}{dt}$$

and in general, if  $m$  is any positive integer,

$$(59) \quad \frac{da^m}{dt} = \left\{ \begin{matrix} a & da/dt \\ m-1 & 1 \end{matrix} \right\}.$$

Under the usual conditions each of the coordinates of  $a(t)$  is expansible as a Taylor series and this is therefore also true of  $a(t)$ . If  $f(t)$  is a scalar function,  $f(a)$  may or may not have a meaning. For instance, if  $f(t)$  can be expanded in a power series which converges for  $\text{mod } t < \alpha$ , then the same power series<sup>3</sup> in

<sup>3</sup> If  $g(t) = \Sigma u_n t^n$ ,  $u_n$  scalar, the series intended here is  $\Sigma u_n a^n$ . Other definitions are possible, e.g., if we set

$$G(a) = \sum_n u_n \sum_i c_{i_0}^{(n)} a c_{i_1}^{(n)} a c_{i_2}^{(n)} \cdots c_{i_{n-1}}^{(n)} a c_{i_n}^{(n)}$$

where  $\sum c_{i_0}^{(n)} c_{i_1}^{(n)} \cdots c_{i_n}^{(n)} = 1$ , we still have  $G(t) = g(t)$ .

$a$  converges when  $|a| < \alpha$ ; but if  $f(t)$  is defined by a Fourier series which is not differentiable,  $f(a)$  will not have a meaning when the elementary divisors of  $a$  are not simple, as is seen immediately on referring to the form of §8.03 (14). If  $f(a)$  and  $f'(a)$  have a meaning and if  $da/dt$  is commutative with  $a$ , we have  $df(a)/dt = f'(a)da/dt$ . For instance, if  $x$  is a constant matrix and  $a = t - x$ , then

$$(60) \quad \frac{d \log (t - x)}{dt} = \frac{1}{t - x}$$

as is also easily proved directly.

The integral of  $a(t)$  is defined as follows. If  $C$  is a regular contour in the  $t$ -plane, we shall set

$$(61) \quad \int_C a(t) dt = \left\| \int_C a_{pq}(t) dt \right\|$$

or if  $t_1, t_2, \dots$  is a series of points on  $C$  and  $t'_i$  a point on the arc  $(t_i, t_{i+1})$ , and if the number of points is increased indefinitely in such a way that  $\text{mod}(t_{i+1} - t_i)$  approaches 0 for every interval, then

$$(62) \quad \int_C a(t) dt = \lim \Sigma a(t'_i)(t_{i+1} - t_i).$$

The conditions for the existence of this limit are exactly the same as in the scalar theory.

If  $M$  is the least upper bound of  $|a|$  on  $C$  and  $L$  is the length of  $C$ , it follows in the usual manner that

$$(63) \quad \left| \int_C a(t) dt \right| \leq \int |a(t)| \text{mod } dt \leq ML.$$

As an illustration of these definitions we shall now employ contour integration to prove some of our earlier results. If  $x$  is an arbitrary constant matrix and  $C$  a circle with center  $t = 0$  and radius greater than  $|x|$ , then all the roots of  $x$  lie inside  $C$  and on  $C$  the series

$$\frac{1}{t - x} = \frac{1}{t} + \frac{x}{t^2} + \frac{x^2}{t^3} + \dots$$

is uniformly convergent. Hence

$$(64) \quad \frac{1}{2\pi i} \int_C \frac{dt}{t - x} = \frac{1}{2\pi i} \sum_0^{\infty} x^n \int_C \frac{dt}{t^{n+1}} = 1$$

a result which may also be derived from the definition of  $\log(t - x)$  in §8.07 and

$$\int_C \frac{dt}{t - x} = [\log(t - x)]_C.$$

We then have

$$\frac{1}{2\pi i} \int_C \frac{t dt}{t - x} = \frac{1}{2\pi i} \int_C \left(1 + \frac{x}{t - x}\right) dt = x$$

and in general, if  $g(t)$  is a scalar function of  $t$  which is analytic in a region enclosing  $C$ ,

$$(65) \quad g(x) = \frac{1}{2\pi i} \int_C \frac{g(t)}{t - x} dt.$$

Suppose now that  $|t - x| = \theta(t)\varphi(t)$ ,  $\text{adj}(t - x) = \theta(t)a(t)$  where  $\theta(t)$  is the highest common factor of  $|t - x|$  and the coordinates of  $\text{adj}(t - x)$ . We then have

$$(66) \quad g(x) = \frac{1}{2\pi i} \int_C \frac{g(t)}{\varphi(t)} a(t) dt$$

and under the given conditions this vanishes if, and only if,  $g(t)/\varphi(t)$  has no singularities inside  $C$ , that is, if  $\varphi$  is a factor of  $g$ . We have therefore the theorem of §2.05 that  $\varphi(t)$  is the reduced characteristic function of  $x$  and that  $g(x) = 0$  only when  $\varphi(t)$  is a factor of  $g(t)$ .

Since  $a(t) \equiv \varphi(t)/(t - x)$  is a polynomial in  $x$  with scalar coefficients and with degree 1 less than the degree of  $\varphi(t)$ , say

$$a(t) = \alpha_1 x^{m-1} + \alpha_2 x^{m-2} + \dots + \alpha_m,$$

equation (66) shows that  $g(x)$  can be expressed as a polynomial in  $x$ , namely,

$$(67) \quad g(x) = \frac{1}{2\pi i} \sum x^{m-s} \int_C \frac{g(t)\alpha_s(t)}{\varphi(t)} dt.$$

We may also note that (66) leads to the interpolation formula §8.01 (4) if the integral is expanded in terms of the residues at the zeros of  $\varphi(t)$ .

All of these results can be extended to unilateral series in  $x$  with matrix coefficients if care is taken to use  $g(t)(t - x)^{-1}$  or  $(t - x)^{-1}g(t)$  according as dextro- or laevo-lateral series are desired.

**8.13 Functions of a variable vector.** Before considering functions of a variable matrix, we shall consider briefly those of a variable vector; for more extended and systematic treatments the reader is referred to treatises on vector and tensor analysis.

The differential of a function of a variable in any non-commutative algebra

was defined by Hamilton as follows. Let  $f(x)$  be a function of a variable  $x$ ,  $dx$  a variable independent of  $x$  and  $t$  a scalar variable; then

$$(68) \quad df(x) = \lim_{t \rightarrow 0} \frac{f(x + tdx) - f(x)}{t}.$$

We shall assume tacitly hereafter that this limit exists for all the functions we shall consider.

An immediate consequence of (68) is that  $df(x)$  is linear and homogeneous in  $dx$ . Hence, if  $x = \Sigma \xi_i e_i$ ,  $dx = \Sigma d\xi_i e_i$ , then

$$df(x) = \lim_{t \rightarrow 0} \frac{f(\Sigma(\xi_i + td\xi_i)e_i) - f(\Sigma \xi_i e_i)}{t} = \Sigma \frac{\partial f}{\partial \xi_i} d\xi_i.$$

This leads to Hamilton's differential operator

$$(69) \quad \nabla = \Sigma e_i \frac{\partial}{\partial \xi_i}$$

in terms of which we may write (68) in the form

$$(70) \quad df(x) = (Sdx\nabla)f(x).$$

In using this operator it is frequently convenient to place it after its operand and, when this is done, some artifice is necessary to indicate the connection between them. This is done by attaching the same subscript to both; the method of doing this will be clear from the following examples in which  $a = \Sigma \alpha_i e_i$ ,  $b = \Sigma \beta_i e_i$  are vectors and  $A = || a_{ij} ||$  is a matrix.

$$\begin{aligned} \nabla a &= \Sigma \frac{\partial \alpha_j}{\partial \xi_i} e_i e_j, & a_\alpha \nabla_\alpha &= \Sigma \frac{\partial \alpha_i}{\partial \xi_j} e_i e_j, \\ \nabla Sab &= \nabla_\alpha S a_\alpha b + \nabla_\alpha S a b_\alpha = \sum_i \left( \sum_j \left( \frac{\partial \alpha_j}{\partial \xi_i} \beta_j + \frac{\partial \beta_j}{\partial \xi_i} \alpha_j \right) \right) e_i, \\ S \nabla_\alpha \nabla_\beta S a_\alpha b_\beta &= \sum_{i,j} \frac{\partial \alpha_i}{\partial \xi_j} \frac{\partial \beta_i}{\partial \xi_j}, \\ A_\alpha \nabla_\alpha &= \sum_i \left( \sum_j \frac{\partial a_{ij}}{\partial \xi_j} \right) e_i, & \nabla_\alpha A_\alpha &= \Sigma \frac{\partial a_{jk}}{\partial \xi_i} e_i e_j S e_k, \\ S a_\alpha A \nabla_\alpha &= \sum_{i,j} a_{ij} \frac{\partial \alpha_i}{\partial \xi_j}, & S a A_\alpha \nabla_\alpha &= \sum_{i,j} \alpha_i \frac{\partial a_{ij}}{\partial \xi_j}. \end{aligned}$$

We can now consider the effect of a change of variable from  $x$  to<sup>4</sup>  $\bar{x}$ . Let  $\bar{x} = \Sigma \bar{\xi}_i e_i$ ,  $\nabla = \Sigma e_i \partial / \partial \bar{\xi}_i$ ; then

$$(71) \quad d\bar{x} = \bar{x}_\alpha S \nabla_\alpha dx = J dx$$

<sup>4</sup> Here  $\bar{x}$  denotes a new variable and not the conjugate imaginary. Instead of considering a change of variable, we may regard  $\bar{x}$  as a vector function of  $x$ .

where

$$(72) \quad J = \bar{x}_\alpha S \nabla_\alpha = \|\partial \bar{\xi}_i / \partial \xi_j\|$$

is the Jacobian matrix of the transformation. Similarly

$$dx = x_\alpha S \nabla_\alpha d\bar{x} = J d\bar{x}.$$

Hence

$$(73) \quad J\bar{J} = 1.$$

Again, since  $Sdx\nabla = Sd\bar{x}\bar{\nabla} = SdxJ'\bar{\nabla}$ , hence

$$(74) \quad \nabla = J'\bar{\nabla}, \quad \bar{\nabla} = (J')^{-1}\nabla = \bar{J}\nabla.$$

From (70) and (72) we see that the differentials of  $J$  and  $J'$  are given by

$$(75) \quad \begin{aligned} dJ &= Sdx\nabla_\alpha \cdot J_\alpha, & \text{and} & \quad dJ = d\bar{x}_\alpha S \nabla_\alpha = J_\alpha dx S \nabla_\alpha \\ dJ' &= Sdx\nabla_\alpha \cdot J'_\alpha, & dJ' &= \nabla_\alpha S J_\alpha dx = \nabla_\alpha S dx J'_\alpha. \end{aligned}$$

This leads to the notion of contravariant and covariant vectors. If  $u$  is a vector function of  $x$  and  $\bar{u}$  the corresponding<sup>5</sup> function after the change of variable,  $u$  is called contravariant if

$$(76) \quad \bar{u} = Ju,$$

and covariant when

$$(77) \quad \bar{u} = \bar{J}'u, \quad u = J'\bar{u}.$$

If  $d_1, d_2$  denote two independent variations so that  $d_1(d_2x) = d_2(d_1x)$ , then

$$(78) \quad \begin{aligned} d_2 d_1 \bar{x} &= d_2 (J d_1 x) = d_2 J d_1 x + J d_1 d_2 x \\ &= J_\alpha d_2 x S \nabla_\alpha d_1 x + J d_1 d_2 x. \end{aligned}$$

Hence second differentials are neither contra- nor co-variant.

If  $A$  is a matrix whose coordinates are functions of  $x$ , the bilinear differential form  $Sd_1 x A d_2 x$  when transformed becomes

$$Sd_1 \bar{x} \bar{A} d_2 \bar{x} = Sd_1 x J' \bar{A} J d_2 x$$

so that, if this form is invariant, that is,  $Sd_1 \bar{x} \bar{A} d_2 \bar{x} = Sd_1 x A d_2 x$ , we must have

$$(79) \quad A = J' \bar{A} J, \quad A' = J' \bar{A}' J, \quad A dx = J' \bar{A} d\bar{x}.$$

<sup>5</sup> As will be seen below, this does not necessarily mean merely the result of substituting  $\bar{x}$  for  $x$  in the coordinates of  $u$ .

Hence when  $\bar{A}$  is defined in this manner,  $A dx$  is a covariant vector. If by analogy with (78) we form a second differential of this vector and of  $A' dx$ , we get, using  $d\bar{x} = J dx$ ,

$$\begin{aligned}d_2(A d_1 x) &= d_2 J' \bar{A} d_1 \bar{x} + J' d_2(\bar{A} d_1 \bar{x}) \\d_1(A' d_2 x) &= d_1 J' \bar{A}' d_2 \bar{x} + J' d_1(\bar{A}' d_2 \bar{x}).\end{aligned}$$

From (75)  $d_2 J' = \nabla_\alpha S J_\alpha d_2 x$ ,  $d_1 J' = \nabla_\alpha S d_1 x J'_\alpha$ ; hence after a simple reduction  $d_2(A d_1 x) + d_1(A' d_2 x) = \nabla_\alpha S d_1 x (J' \bar{A}' J_\alpha + J'_\alpha \bar{A}' J) d_2 x + J' (d_2(\bar{A} d_1 \bar{x}) + d_1(\bar{A}' d_2 \bar{x})) = \nabla_\alpha S d_1 x (A'_\alpha - J' \bar{A}'_\alpha J) d_2 x + J' (d_2(\bar{A} d_1 \bar{x}) + d_1(\bar{A}' d_2 \bar{x}))$

which may be written

$$\begin{aligned}(80) \quad a &\equiv d_2(A d_1 x) + d_1(A' d_2 x) - \nabla_\alpha S d_1 x A'_\alpha d_2 x \\ &= J' (d_2(\bar{A} d_1 \bar{x}) + d_1(\bar{A}' d_2 \bar{x}) - \nabla_\alpha S d_1 \bar{x} \bar{A}'_\alpha d_2 \bar{x}) \\ &= J' \bar{a}\end{aligned}$$

so that  $a$  is a covariant vector. This vector may also be written

$$a = d_2 A d_1 x + d_1 A' d_2 x - \nabla_\alpha S d_1 x A'_\alpha d_2 x + (A + A') d_1 d_2 x.$$

Using a notation suggested by the Christoffel symbols we now write

$$\begin{aligned}(81) \quad [A; d_1 x, d_2 x] &= \frac{1}{2} (d_2 A d_1 x + d_1 A' d_2 x - \nabla_\alpha S d_1 x A'_\alpha d_2 x) \\ &= \sum_{i,j,k} \frac{1}{2} \left( \frac{\partial a_{kj}}{\partial \xi_i} + \frac{\partial a_{ki}}{\partial \xi_j} - \frac{\partial a_{ji}}{\partial \xi_k} \right) d_1 \xi_i d_2 \xi_j d_2 \xi_k\end{aligned}$$

$$\begin{aligned}(82) \quad \left\{ \begin{matrix} A \\ d_1 x, d_2 x \end{matrix} \right\} &= (A + A')^{-1} (d_2 A d_1 x + d_1 A' d_2 x - \nabla_\alpha S d_1 x A'_\alpha d_2 x) \\ &= 2(A + A')^{-1} [A; d_1 x, d_2 x]\end{aligned}$$

provided that  $|A + A'| \neq 0$ . If we now set

$$b = \left\{ \begin{matrix} A \\ d_1 x, d_2 x \end{matrix} \right\} + d_1 d_2 x$$

and use the relation  $(A + A')^{-1} J' = J^{-1} (\bar{A} + \bar{A}')^{-1}$ , we have from (80)

$$(83) \quad \bar{b} = J b$$

so that  $\bar{b}$  is contravariant.

If we set  $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = B + C$ , we get from (81) and (80)

$$\begin{aligned}(84) \quad [A; d_1 x, d_2 x] &= [B; d_1 x, d_2 x] + [C; d_1 x, d_2 x] \\ [B; d_1 x, d_2 x] + B d_1 d_2 x &\simeq J' ([\bar{B}; d_1 \bar{x}, d_2 \bar{x}] + \bar{B} d_1 d_2 \bar{x}) \\ [C; d_1 x, d_2 x] &= J' [\bar{C}; d_1 \bar{x}, d_2 \bar{x}].\end{aligned}$$

We shall require two transverses of the Christoffel matrices; these are defined by

$$Sb[A; a, c] = Sc[A; a, b]', \quad Sb \left\{ \begin{matrix} A \\ a, c \end{matrix} \right\} = Sc \left\{ \begin{matrix} A \\ a, b \end{matrix} \right\}'$$

whence

$$(85) \quad \begin{aligned} 2[A; a, b]' &= \nabla_\alpha S a A'_\alpha b + A_\alpha b S a \nabla_\alpha - A_\alpha a S b \nabla_\alpha \\ \left\{ \begin{matrix} A \\ a, b \end{matrix} \right\}' &= [A; a, (A + A')^{-1}b]'. \end{aligned}$$

To illustrate partial differentiation we shall consider functions which depend not only on  $x$  but also on a contravariant variable vector  $u = \Sigma \omega_i e_i$ . Since  $\bar{u} = Ju$  and  $J = \|\partial \bar{\xi}_i / \partial \xi_j\|$  is independent of  $u$ , we have

$$\begin{aligned} \frac{\partial}{\partial \omega_i} &= \sum \frac{\partial \bar{\omega}_j}{\partial \omega_i} \frac{\partial}{\partial \bar{\omega}_j} = \sum \frac{\partial \bar{\xi}_j}{\partial \xi_i} \frac{\partial}{\partial \bar{\omega}_j} \\ \frac{\partial}{\partial \xi_i} &= \sum \frac{\partial \bar{\xi}_j}{\partial \xi_i} \frac{\partial}{\partial \bar{\xi}_j} + \sum \frac{\partial \bar{\omega}_j}{\partial \xi_i} \frac{\partial}{\partial \bar{\omega}_j}. \end{aligned}$$

Hence, if  $\nabla' = \Sigma e_i \partial / \partial \omega_i$ ,  $\bar{\nabla}' = \Sigma e_i \partial / \partial \bar{\omega}_i$ , then

$$(86) \quad \nabla' = J' \bar{\nabla}', \quad \bar{\nabla} = J' \bar{\nabla} + d_u J' \cdot \bar{\nabla}'$$

where  $d_u J' \equiv Su \nabla_\alpha \cdot J'_\alpha$ . Here  $\nabla'$  is covariant but  $\bar{\nabla}$  is neither covariant nor contravariant, which means that formulae dependent on it will not usually be invariant in form under a change of variable. This difficulty is avoided as follows. If we combine (83) with (78) and replace  $d_1 x$ ,  $d_2 x$  by contravariant vectors  $a$ ,  $b$ , then

$$(87) \quad d_b J a = d_a J b = J_\alpha b S \nabla_\alpha a = J \left\{ \begin{matrix} A \\ a, b \end{matrix} \right\} - \left\{ \begin{matrix} \bar{A} \\ \bar{a}, \bar{b} \end{matrix} \right\}'.$$

Hence

$$(87') \quad d_a J' b = \left\{ \begin{matrix} A \\ a, J' b \end{matrix} \right\}' - J' \left\{ \begin{matrix} \bar{A} \\ \bar{a}, b \end{matrix} \right\}'$$

and therefore

$$\bar{\nabla} = J' \bar{\nabla}' + \left\{ \begin{matrix} A \\ u, \nabla' \end{matrix} \right\}' - J' \left\{ \begin{matrix} \bar{A} \\ \bar{u}, \bar{\nabla}' \end{matrix} \right\}'$$

whence

$$(88) \quad D \equiv \bar{\nabla} - \left\{ \begin{matrix} A \\ u, \nabla' \end{matrix} \right\}' = J' \left( \bar{\nabla}' - \left\{ \begin{matrix} \bar{A} \\ \bar{u}, \bar{\nabla}' \end{matrix} \right\}' \right) = J' \bar{D};$$

$D$  is therefore a covariant differential operator.

Let  $v$  be any contravariant vector and set  $d_u v = S u \nabla_\alpha \cdot v_\alpha$ ; then, if  $f$  is any function of  $x$  and  $u$ ,

$$(89) \quad d_u f \equiv (S v \nabla_\alpha + S d_u v \nabla'_\alpha) f_\alpha = (S v D_\alpha + S \delta_u v \nabla'_\alpha) f_\alpha$$

where  $\delta_u v$  is the contravariant vector defined by

$$(90) \quad \delta_u v = d_u v + \left\{ \begin{matrix} A \\ u, v \end{matrix} \right\} = V u.$$

The tensor corresponding to the matrix  $V$  is known as the covariant derivative of  $v$ .

**8.14 Functions of a variable matrix.** The general theory of analytic functions of a variable matrix  $x = || \xi_{ij} ||$  is co-extensive with that of  $n^2$  scalar variables and hence is so general as to be void of properties peculiar to matrices. This follows immediately from the obvious relation

$$\xi_{ij} = \sum_{p=1}^n e_{pi} x e_{jp}$$

which expresses the  $(i, j)$  coordinate as a linear function of  $x$ .

The differential operator<sup>6</sup> corresponding to  $x$  is

$$(91) \quad \Delta = \left\| \frac{\partial}{\partial \xi_{ij}} \right\|.$$

It is often convenient to have a special notation for the transverse  $\Delta'$  and when this is so we shall set

$$(92) \quad \partial = \left\| \frac{\partial}{\partial \xi_{ji}} \right\| = \Delta'.$$

These operators may stand after their operands and the same convention as was used for subscripts attached to  $\nabla$  will also be used for  $\Delta$  and  $\partial$  when necessary.

The fundamental property of  $\Delta$  is

$$(93) \quad df = \text{tr}(dx \Delta') f = \text{tr}(dx \partial) f$$

where  $f$  is any function of  $x$  and  $\text{tr}(A)$  stands for the trace of the matrix  $A$ . This result follows immediately from

$$\text{tr}(dx \Delta') \equiv \sum_i \sum_k d \xi_{ik} \frac{\partial}{\partial \xi_{ik}}.$$

<sup>6</sup> This operator first occurs in a paper by Taber (1890, (84)) who however did not make any systematic use of it. Macaulay in a tract published in 1893 (110) but written in 1887 used  $\Delta$  consistently in applying quaternions to physical problems; he used the notation  $\Omega$  for  $\Delta$ . Later Born (385) used the same operator to great effect in his theory of quantum matrices. Turnbull (436) uses  $\Omega$  for  $\Delta'$ .

**8.15 Differentiation formulae.** We collect here the principal formulae of differentiation; in each case the operand is  $x$  or  $x'$  and the dummy subscript is omitted except when the meaning is ambiguous without it. To simplify the expressions we set  $\alpha$  and  $\beta$  for the traces of the matrices  $a$  and  $b$ , and  $\xi_r$  for the trace of  $x^r$ .

$$(94) \quad \begin{aligned} \Delta axb &= a'b = a'xb'\Delta, & \Delta ax'b &= \alpha b = bx'a\Delta \\ \Delta' axb &= \alpha b = bxa\Delta', & \Delta' ax'b &= a'b = a'x'b'\Delta' \end{aligned}$$

$$(95) \quad \begin{aligned} \Delta x^r &= x^{r-1} + x'x^{r-2} + (x')^2x^{r-3} + \dots + (x')^{r-1} = \Delta'(x')^r \\ x^r\Delta &= x^{r-1} + x^{r-2}x' + x^{r-3}(x')^2 + \dots + (x')^{r-1} = (x')^r\Delta' \\ \Delta'x^r &= nx^{r-1} + \xi_1x^{r-2} + \xi_2x^{r-3} + \dots + \xi_r = x^r\Delta' \end{aligned}$$

$$(96) \quad \begin{aligned} \Delta \operatorname{tr}(axb) &= \Delta \operatorname{tr}(bax) = a'b' \\ \partial \operatorname{tr}(axb) &= ba \end{aligned}$$

$$(97) \quad \Delta \operatorname{tr}(x^r) \equiv \Delta \xi_r = r(x')^{r-1}, \quad \partial \operatorname{tr}(x^r) = rx^{r-1}.$$

$$(98) \quad \begin{aligned} \operatorname{tr}(\Delta)axb &= ab = \operatorname{tr}(\partial)axb \\ \operatorname{tr}(\Delta)x^r &= rx^{r-1} = \operatorname{tr}(\partial)x^r \end{aligned}$$

$$(99) \quad \begin{aligned} \partial \operatorname{tr} \begin{Bmatrix} x & a \\ r & s \end{Bmatrix} &= (r+s) \begin{Bmatrix} x & a \\ r-1 & s \end{Bmatrix} = \operatorname{tr}(\partial) \begin{Bmatrix} x & a \\ r & s \end{Bmatrix} \\ \Delta \operatorname{tr} \begin{Bmatrix} x & a \\ r & s \end{Bmatrix} &= (r+s) \begin{Bmatrix} x' & a' \\ r-1 & s \end{Bmatrix}. \end{aligned}$$

$$(100) \quad \begin{aligned} \Delta'_\alpha x_\alpha^{r+1} - x \Delta'_\alpha x_\alpha^r &= \operatorname{tr}(x^r) \equiv \xi_r \\ (\Delta_\alpha x_\alpha - x' \Delta_\alpha) x_\alpha^r &= (x')^r. \end{aligned}$$

The proofs of these formulae are all very similar and we shall consider here only the most important leaving the remainder to the reader. If  $a = ||a_{ij}||$ ,  $b = ||b_{ij}||$ , then

$$\Delta axb = \left\| \frac{\partial}{\partial \xi_{ip}} a_{pq} \xi_{qr} b_{ri} \right\| = \left\| \sum_p a_{pi} b_{pi} \right\| = a'b;$$

hence also

$$a'x'b'\Delta' = (\Delta bxa)' = (b'a)' = a'b.$$

The remaining parts of (94) follow in the same way. It follows also from (94) that

$$\begin{aligned} \Delta_\alpha x_\alpha^r &= \Delta_\alpha x_\alpha x_\alpha^{r-1} + \Delta_\alpha x_\alpha x_\alpha^{r-1} = x^{r-1} + \Delta_\alpha x x_\alpha x_\alpha^{r-2} + \Delta_\alpha x^2 x_\alpha^{r-2} \\ &= x^{r-1} + x'x^{r-2} + \Delta_\alpha x^2 x_\alpha x_\alpha^{r-3} + \Delta_\alpha x^3 x_\alpha^{r-3}, \end{aligned}$$

and so on; the remaining parts of (95) follow from (94) in the same way.

To prove (96) we notice first that  $\text{tr}(ab) = \text{tr}(ba)$  and hence in  $\text{tr}(a_1 a_2 \cdots a_r)$  the factors may be permuted cyclically. Then, if  $c = \|c_{ij}\|$ ,

$$\partial \text{tr}(cx) = \sum_{i,j} e_{ij} \frac{\partial}{\partial \xi_{ji}} \sum_{p,q} c_{pq} \xi_{qp} = \sum c_{ij} e_{ij} = c.$$

Formula (97) follows by repeated application of (96); thus

$$\begin{aligned} \Delta \text{tr}(x^r) &= \Delta_\alpha \text{tr}(x_\alpha x^{r-1}) + \Delta_\alpha \text{tr}(x_\alpha^{r-1} x) \\ &= (x')^{r-1} + \Delta_\alpha \text{tr}(x_\alpha x^{r-1}) + \Delta_\alpha \text{tr}(x_\alpha^{r-2} x^2) \\ &= (x')^{r-1} + (x')^{r-1} + \dots \\ &= r(x')^{r-1}. \end{aligned}$$

The remaining formulae are proved in the same way.

If  $\Sigma \alpha_r x^r$  is a scalar power series and  $f(x) = \Sigma \alpha_r x^r$ , then from (97) and (98)

$$\partial \text{tr}(f(x)) = \Sigma r \alpha_r x^{r-1} = f'(x) = \text{tr}(\partial f(x))$$

so that the operators  $\partial \text{tr}(\ )$  and  $\text{tr}(\partial)$  have the same effect on such functions.

Similarly, if  $F(x) = \Sigma \alpha_r \begin{Bmatrix} x & a \\ r & s_r \end{Bmatrix}$ , it follows from (99) that

$$(101) \quad \partial \text{tr}(F) = \text{tr}(\partial)F = \Sigma(r + s_r) \begin{Bmatrix} x & a \\ r - 1 & s_r \end{Bmatrix}.$$

8.16 As an illustration of the application of the formulae of the preceding section we shall give some parts of the theory of quantum matrices which are applicable to matrices of finite order.

Let  $q_1, q_2, \dots, q_f; p_1, \dots, p_f$  be the coordinates of a dynamical system and  $\mathfrak{H}$  the Hamiltonian function; these coordinates satisfy the system of ordinary partial differential equations

$$(102) \quad \dot{q}_i = \frac{\partial \mathfrak{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathfrak{H}}{\partial q_i} \quad (i = 1, 2, \dots, f).$$

We may suppose that  $f = n^2$ , a perfect square; for, if  $(n - 1)^2 < f < n^2$ , we can introduce  $2(n^2 - f)$  additional coordinates  $q_{f+1}, p_{f+1}, \dots, q_n, p_n$  which do not occur in  $\mathfrak{H}$  so that these variables equated to constants are solutions of the extended system. When this is done, we can order the  $q$ 's and  $p$ 's in square arrays  $\|q_{ij}\|, \|p_{ij}\|$  in such a way that  $p_{ij}$  corresponds to  $q_{ji}$  for all  $i$  and  $j$ . Equation (102) then becomes

$$\dot{q}_{ij} = \frac{\partial \mathfrak{H}}{\partial p_{ji}}, \quad \dot{p}_{ij} = -\frac{\partial \mathfrak{H}}{\partial q_{ji}}$$

or, if the matrices  $\|q_{ij}\|$  and  $\|p_{ij}\|$  are denoted by  $q$  and  $p$  and the corresponding transverse differential operators by  $\partial_q$  and  $\partial_p$ ,

$$(103) \quad \dot{q} = \partial_p \mathfrak{H}, \quad \dot{p} = \partial_q \mathfrak{H}.$$

If we transform (103) by the substitution

$$q = \epsilon^{wt} Q \epsilon^{-wt}, \quad Q = \| Q_{ij} \|, \quad p = \epsilon^{wt} P \epsilon^{-wt}, \quad P = \| P_{ij} \|,$$

where  $w$  is a constant matrix, we get

$$\dot{q} = \epsilon^{wt} (\dot{Q} + wQ - Qw) \epsilon^{-wt}, \quad \dot{p} = \epsilon^{wt} (\dot{P} + wP - Pw) \epsilon^{-wt}.$$

Also, if  $t$  is kept constant,

$$\text{tr} (dq \partial_q) = \text{tr} (\epsilon^{wt} dQ \epsilon^{-wt} \partial_q) = \text{tr} (dQ \epsilon^{-wt} \partial_q \epsilon^{wt})$$

with a similar relation for  $p$ . Hence

$$\partial_q = \epsilon^{wt} \partial_Q \epsilon^{-wt}, \quad \partial_p = \epsilon^{wt} \partial_P \epsilon^{-wt}.$$

Using these results in (103) we get

$$(104) \quad (Q + wQ - Qw) = \partial_P \mathfrak{S}, \quad (\dot{P} + wP - Pw) = -\partial_Q \mathfrak{S},$$

$\mathfrak{S}$  being expressed in terms of  $P$  and  $Q$  and, if necessary, also  $t$ . Now from (96)

$$Qw - wQ = \partial_P \text{tr} [w(PQ - QP)] \quad - (Pw - wP) = \partial_Q \text{tr} [w(PQ - QP)]$$

and hence, if

$$(105) \quad \mathfrak{R} \equiv \mathfrak{S} + \text{tr} [w(PQ - QP)] = \mathfrak{S} + \text{tr} [w(pq - qp)],$$

we have in place of (103)

$$(106) \quad \dot{Q} = \partial_P \mathfrak{R}, \quad \dot{P} = -\partial_Q \mathfrak{R},$$

so that the transformation is canonical.

If  $\dot{Q} = 0 = \dot{P}$  in (104), then

$$wQ - Qw = \partial_P \mathfrak{S}, \quad wP - Pw = -\partial_Q \mathfrak{S}$$

or on restoring the exponential factor

$$(107) \quad wq - qw = \partial_P \mathfrak{S}, \quad wp - pw = -\partial_Q \mathfrak{S}.$$

When  $\mathfrak{S}$  is given, these are algebraic equations which can be solved for  $p$  and  $q$ ; the solution will of course generally contain arbitrary parameters.

Under the same assumptions (106) becomes

$$(108) \quad \partial_P \mathfrak{R} = 0 = \partial_Q \mathfrak{R},$$

and if  $P, Q$  are independent variables, the only solution is  $\mathfrak{R} = \text{constant}$  and the only solution for  $\mathfrak{S}$  in (107) then has the form

$$\mathfrak{S}^* = \text{tr} [w(pq - qp)]$$

apart from an additive constant. Equation (108) may then be written

$$(109) \quad \partial_P (\mathfrak{S} - \mathfrak{S}^*) = 0 = \partial_Q (\mathfrak{S} - \mathfrak{S}^*).$$

Now, if  $r_1, r_2, \dots, r_m$  are the parameters in the solution of (107) we have

$$\frac{\partial(\mathfrak{S} - \mathfrak{S}^*)}{\partial r_k} = \sum \frac{\partial p_{ij}}{\partial r_k} \frac{\partial(\mathfrak{S} - \mathfrak{S}^*)}{\partial p_{ij}} + \sum \frac{\partial q_{ij}}{\partial r_k} \frac{\partial(\mathfrak{S} - \mathfrak{S}^*)}{\partial q_{ij}}$$

which vanishes in virtue of (109). Hence, if  $\mathfrak{S}$  is expressed in terms of  $r_1, r_2, \dots, r_m$  by using the solutions of (107), it will differ by an additive constant at most from  $-\text{tr} [w(pq - qp)]$ .

## CHAPTER X

### LINEAR ASSOCIATIVE ALGEBRAS

**10.01 Fields and algebras.** A set of elements which are subject to the laws of ordinary rational algebra is called a *field*. We may make this idea more precise as follows. Let  $a, b, \dots$  be a set of entities,  $F$ , which are subject to two operations, addition and multiplication; this set is called a field if it satisfies the following postulates:<sup>1</sup>

- A1.  $a + b$  is a uniquely determined element of  $F$ .
- A2.  $a + b = b + a$ .
- A3.  $(a + b) + c = a + (b + c)$ .
- A4. There is a unique element  $0$  in  $F$  such that  $a + 0 = a$  for every element  $a$  in  $F$ .
- A5. For every element  $a$  in  $F$  there exists a unique element  $b$  in  $F$  such that  $a + b = 0$ .
- M1.  $ab$  is a unique element of  $F$ .
- M2.  $ab = ba$ .
- M3.  $ab \cdot c = a \cdot bc$ .
- M4. There is a unique element  $1$  in  $F$  such that  $a1 = a$  for every  $a$  in  $F$ .
- M5. For every element  $a \neq 0$  in  $F$  there exists a unique element  $b$  in  $F$  such that  $ab = 1$ .

AM.  $a(b + c) = ab + ac, (b + c)a = ba + ca$ .

R. If  $m$  is a whole number and  $ma$  denotes the element which results from adding together  $m$   $a$ 's, then  $ma \neq 0$  for any  $m > 0$  provided that  $a \neq 0$ .

If M2 is omitted the resulting set is said to be a *division algebra*. This does not imply that M2 does not hold, only that it is not presupposed; if it does hold, the algebra is said to be commutative. If M2, 4, 5 are all omitted, the corresponding set is called an *associative algebra*. If the algebra contains an identity, that is, an element satisfying the condition laid down in M4 for 1, this element is called the *principal unit* of the algebra. Postulate R is included merely as a matter of convenience; its effect is to exclude modular fields. In consequence of R every field which we shall consider contains<sup>2</sup> the field of rational numbers as a subset.

As an example of a field we may take the field of rational numbers extended by a cube root of unity,  $\omega = (-1 + \sqrt{-3})/2$ . Every number of this field can be put in the form

$$a = \alpha + \beta\omega = \alpha 1 + \beta\omega$$

<sup>1</sup> These postulates are not independent; they are formed so as to show the principal properties of the set. In place of M5 it is often convenient to take: M5' If  $a \neq 0, ax = 0$  implies  $x = 0$ .

<sup>2</sup> Strictly speaking, we should say that the field contains a subset simply isomorphic with the field  $R$  of rational numbers. This subset is then used in place of  $R$  in the same way as scalars are replaced by scalar matrices in §1.04.

where  $\alpha$  and  $\beta$  are rational numbers; the form of  $a$  is unique since  $\alpha + \beta\omega = \gamma + \delta\omega$  gives  $(\beta - \delta)\omega = \gamma - \alpha$  and, since  $\omega$  is not rational, this is impossible unless  $\beta - \delta = 0 = \gamma - \alpha$ . We say that  $1, \omega$  is a *basis* of  $F$  relative to the field  $R$  of rational numbers, and  $F$  is said to be a field of order 2 over  $R$ .

As an example of an associative algebra we may take the algebra of matrices with rational coordinates. Here any element  $a$  of the algebra can be put uniquely in the form  $a = \sum a_{ij}e_{ij}$ , where the  $a_{ij}$  are rational numbers; and  $e_{ij}(i, j = 1, 2, \dots, n)$  form a basis of the algebra, which is of order  $n^2$ . We also have an algebra if the coordinates  $a_{ij}$  are taken to be any elements of the field  $F = (1, \omega)$  described above. This algebra is one of order  $n^2$  over  $F$ . Instead of regarding it as an algebra over  $F$  we may clearly look on it as an algebra of order  $2n^2$  over  $R$  the basis being  $e_{ij}, \omega e_{ij}(i, j = 1, 2, \dots, n)$ .

**10.02 Algebras which have a finite basis.** Let  $A$  be a set of elements which form an associative algebra and  $G$  a subset which is also an algebra. We shall say that  $a_1, a_2, \dots, a_n$  form a basis of  $A$  relatively to  $G$  if (i) each  $a_i$  lies in  $A$ , (ii) if every element of  $A$  can be put uniquely in the form

$$(1) \quad a = \gamma_1 a_1 + \gamma_2 a_2 + \dots + \gamma_n a_n$$

where the  $\gamma$ 's belong to  $G$ . Though it is not altogether necessary to do so, we shall restrict ourselves to the case in which  $G$  is a field which contains the rational field, that is, we assume as a postulate:

BR. For every algebra  $A$  under consideration there exists a non-modular field  $F$  and a subset of elements  $a_1, a_2, \dots, a_n$  such that (i) every element of  $A$  can be put uniquely in the form

$$a = \sum_1^n \gamma_i a_i \quad (\gamma_i \text{ in } F)$$

and (ii) every element of this form belongs to  $A$ ; and further the elements of  $F$  are commutative with  $a_1, a_2, \dots, a_n$ .

Since the product of any two elements of  $A$  is also an element of  $A$  and can therefore be expressed in the form (1), we have

$$(2) \quad a_i a_j = \sum_k \gamma_{ijk} a_k \quad (i, j = 1, 2, \dots, n)$$

where  $\gamma_{ijk}$  are elements of  $F$ . Since the law of combination of the elements of  $F$  is supposed known, (2) defines the product of any two elements of  $A$ ; for

$$(3) \quad (\sum \alpha_i a_i) (\sum \beta_j a_j) = \sum \alpha_i \beta_j a_i a_j = \sum \alpha_i \beta_j \gamma_{ijk} a_k.$$

If the values of the  $\gamma$ 's are assigned arbitrarily in  $F$ , it is readily shown that the only postulate which is possibly violated is M3 which states that  $ab \cdot c = a \cdot bc$ ; and in order that this condition shall be satisfied it is necessary and sufficient

that  $a_i \cdot a_j a_k = a_i a_j \cdot a_k$  for all the elements of the basis. This gives immediately the 'associativity' condition

$$(4) \quad \sum_{\alpha} \gamma_{jka} \gamma_{ial} = \sum_{\alpha} \gamma_{ija} \gamma_{akl} \quad (i, j, k, l = 1, 2, \dots, n).$$

10.03 The matric representation of an algebra. If we set

$$(5) \quad A_i = \sum_{p, q=1}^n \gamma_{iqp} e_{pq} \quad (i = 1, 2, \dots, n),$$

the law of multiplication for matrices gives

$$A_i A_j = \sum \gamma_{iap} \gamma_{jqap} e_{pq}$$

and therefore from (4)

$$A_i A_j = \sum \gamma_{ija} \gamma_{aqp} e_{pq} = \sum \gamma_{ija} A_a.$$

Hence the set of matrices of the form  $\sum \alpha_i A_i$  is isomorphic with the given algebra in regard to both addition and multiplication. Further, if the algebra contains the identity, the isomorphism is simple; for, if there exist elements  $\alpha_i$  of the field such that  $\sum \alpha_i A_i = 0$ , it follows that

$$(\sum \alpha_i a_i) x = 0$$

for every element  $x$  of the algebra, and putting  $x = 1$  we get  $\sum \alpha_i a_i = 0$ .

If the algebra does not have a principal unit, all that is necessary is to replace (5) by

$$(6) \quad A_i = \sum_{p, q=1}^{n+1} \gamma_{iqp} e_{pq}$$

where  $\gamma_{i, j, n+1} = 0$  ( $i, j \leq n$ ) and  $\gamma_{n+1, i, j} = \delta_{ij} = \gamma_{i, n+1, j}$  for all  $i$  and  $j$ .

The importance of this representation is that it enables us to carry over the theory of the characteristic and reduced equations from the theory of matrices. The main theorem is as follows.

**THEOREM 1.** *The general element  $x = \sum \xi_i a_i$  satisfies an equation of the form*

$$(7) \quad \lambda^m + b_1 \lambda^{m-1} + \dots + b_m = 0$$

where  $b_p$  is a rational homogeneous polynomial in the  $\xi$ 's of degree  $p$ ; and if the variable coordinates  $\xi_p$  are given particular values in  $F$ , there exists a rational polynomial

$$(8) \quad \varphi(\lambda) = \lambda^\mu + \beta_1 \lambda^{\mu-1} + \dots + \beta_\mu$$

such that (i)  $\varphi(x) = 0$ , (ii) if  $\psi(\lambda)$  is any polynomial with coefficients in  $F$  such that  $\psi(x) = 0$ , then  $\varphi(\lambda)$  is a factor of  $\psi(\lambda)$ .

This theorem follows immediately from the theory of the reduced equation as given in §2.05 and from the fact that the equation which is satisfied by the general element must clearly be homogeneous in the coordinates of that element.

As in the theory of matrices,  $-b_1$  is called the *trace* of  $x$  and is written  $\text{tr}(x)$ . The trace is linear and homogeneous in the coordinates and hence  $\text{tr}(x + y) = \text{tr}(x) + \text{tr}(y)$ .

**10.04 The calculus of complexes.** If  $x_1, x_2, \dots, x_r$  are any elements of an algebra  $A$  in a field  $F$ , the set  $B$  of all elements of the form  $\sum \xi_i x_i$  ( $\xi_i$  in  $F$ ) is called a *complex*<sup>3</sup> or linear set. Any subset  $B$  of  $A$  which has the property that, when  $x, y$  are any two of its elements, then  $\xi x + \eta y$  is also an element of the set is a complex. This follows readily from the theory of linear dependence and the existence of a finite basis for  $A$ ; it is also easily shown that any subcomplex of  $A$  has a finite basis; the order of this basis is called the order of the complex.

We shall write  $B = (x_1, x_2, \dots, x_r)$ ; this does not imply that the  $x$ 's are necessarily linearly independent. If  $C = (y_1, y_2, \dots, y_s)$  is a second complex, the *sum* of  $B$  and  $C$  is defined by

$$B + C = (x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_s),$$

that is,  $B + C$  is the set of all elements of the form  $x + y$  where  $x$  lies in  $B$  and  $y$  in  $C$ . Similarly the *product* is defined by

$$BC = (x_i y_j; i = 1, 2, \dots, r; j = 1, 2, \dots, s).$$

The set of elements common to  $B$  and  $C$  forms a complex called the *intersection* of  $B$  and  $C$ ; it is denoted by  $B \cap C$ . If  $B$  and  $C$  have no<sup>4</sup> common element, we write  $B \cap C = 0$ . If every element of  $C$  lies in  $B$  but not every element of  $B$  in  $C$ , we shall write  $C < B$ ; in this case  $B + C = B$ . A complex of order 1 is defined by a single element, say  $x_1$ , and for most purposes it is convenient to denote the complex  $(x_1)$  simply by  $x_1$ ;  $x_1 < B$  then means that  $x_1$  is an element of  $B$ .

If a complex  $B$  is an algebra, the product of any two of its elements lies in  $B$  and hence  $B^2 \leq B$ ; conversely, if this condition is satisfied, the definition of the product  $BB \equiv B^2$  shows that  $B$  is an algebra.

We add a summary of the properties of the symbols introduced in this section.

$$\begin{aligned} B + C &= C + B, & (B + C) + D &= B + (C + D), & BC \cdot D &= B \cdot CD, \\ B \cap C &= C \cap B, & (B \cap C) \cap D &= B \cap (C \cap D), \\ B(C + D) &= BC + BD, & (C + D)B &= CB + DB, \\ B + (C \cap D) &\leq (B + C) \cap (B + D), & B(C \cap D) &\leq BC \cap BD. \end{aligned}$$

<sup>3</sup> The term 'complex,' which was introduced by Frobenius in the theory of groups, is more convenient than 'linear set' and no confusion is likely to arise between this meaning of the term and the one used in geometry.

<sup>4</sup> To avoid circumlocution we say the complexes have 'no element in common' in place of the more correct phrase 'no element in common except 0.'

If  $B \leq C$ , then  $B + C = C$ , and conversely.

If  $B < C$ , there exists  $D < C$  such that  $C = B + D, B \wedge D = 0$ .

If  $B = C + D$  and  $C \wedge D = 0$ , we shall say that  $B$  is congruent to  $C$  modulo  $D$ , or

$$B = C \pmod{D};$$

and if  $b, c, d$  are elements of  $B, C, D$ , respectively, such that  $b = c + d$ , then

$$b = c \pmod{D}, \quad c = b \pmod{D}.$$

**10.05 The direct sum and product.** If  $A = (a_1, a_2, \dots, a_\alpha)$  and  $B = (b_1, b_2, \dots, b_\beta)$  are associative algebras of orders  $\alpha, \beta$ , respectively, over the same field  $F$ , we can define a new algebra in terms of them as follows. Let  $C$  be the set of all pairs of elements  $(a, b)$  where  $a < A$  and  $b < B$  and two pairs  $(a, b), (a', b')$  are regarded as equal if, and only if,  $a = a', b = b'$ . If we define addition and multiplication by

$$(9) \quad \begin{aligned} (a, b) + (a', b') &= (a + a', b + b') \\ (a, b)(a', b') &= (aa', bb') \\ \xi(a, b) &= (\xi a, \xi b) \end{aligned} \quad (\xi \text{ in } F),$$

it is readily shown that the set  $C$  forms an associative algebra. This algebra is called the *direct sum* of  $A$  and  $B$  and is denoted by  $A \oplus B$ ; its order is  $\alpha + \beta$ .

The set  $\mathfrak{A}$  of all elements of the form  $(a, 0)$  forms an algebra which is simply isomorphic with  $A$ , and the set  $\mathfrak{B}$  of elements  $(0, b)$  forms an algebra which is simply isomorphic with  $B$ ; also

$$C = \mathfrak{A} + \mathfrak{B}, \quad \mathfrak{A}\mathfrak{B} = 0 = \mathfrak{B}\mathfrak{A}, \quad \mathfrak{A} \wedge \mathfrak{B} = 0.$$

In consequence of this it is generally convenient to say that  $C$  is the direct sum of  $\mathfrak{A}$  and  $\mathfrak{B}$ .

If we replace (9) by

$$(9') \quad \begin{aligned} \xi(a, b) &= (\xi a, b) = (a, \xi b) \\ (a, b)(a', b') &= (aa', bb'), \end{aligned} \quad (\xi \text{ in } F)$$

we get another type of algebra of order  $\alpha\beta$  which is called the *direct product* of  $A$  and  $B$  and is denoted by  $A \otimes B$  or by  $A \times B$  when there is no chance of confusion. If both  $A$  and  $B$  contain the identity, the set  $\mathfrak{A}$  of elements of the form  $(a, 1)$  forms an algebra simply isomorphic with  $A$  and the set  $\mathfrak{B}$  of elements  $(1, b)$  is an algebra simply isomorphic with  $B$ ; also<sup>5</sup>

$$C = \mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A}, \quad \mathfrak{A} \wedge \mathfrak{B} = (1, 1) = 1,$$

and the order of  $C$  is the product of the orders of  $\mathfrak{A}$  and  $\mathfrak{B}$ . As in the case of the direct sum it is convenient to say that  $A$  is the direct product of  $\mathfrak{A}$  and  $\mathfrak{B}$  and to indicate this by writing  $C = \mathfrak{A} \times \mathfrak{B}$ .

<sup>5</sup> Strictly speaking we should use different symbols here for the identity elements of the separate algebras.

The following theorem gives an instance of the direct product which we shall require later.

**THEOREM 2.** *If an algebra  $A$ , which contains the identity, contains also the matrix algebra  $M$  ( $e_{ij}; i, j = 1, 2, \dots, n$ ), the identity being the same for  $A$  and  $M$ , then  $A$  can be expressed as the direct product of  $M$  and another algebra  $B$ .*

Let  $B$  be the set of elements of  $A$  which are commutative with every element of  $M$ ; these elements form an algebra since, if  $b_i e_{pq} = e_{pq} b_i$  ( $i = 1, 2, \dots$ ), then also

$$(b_i + b_j)e_{pq} = e_{pq}(b_i + b_j), \quad b_i b_j e_{pq} = e_{pq} b_i b_j.$$

Further  $B \cap M$  is the field  $F$ , since scalars are the only elements of  $M$  which are commutative with every element of  $M$ .

If  $x$  is any element of  $A$  and

$$x_{pq} = \sum_i e_{ip} x e_{qi},$$

then

$$x_{pq} e_{rs} = \sum_i e_{ip} x e_{qi} e_{rs} = e_{rp} x e_{qs} = e_{rs} \sum_i e_{ip} x e_{qi} = e_{rs} x_{pq}$$

so that  $x_{pq}$  belongs to  $B$ . Also

$$\sum_{p,q} x_{pq} e_{pq} = \sum_{p,q,i} e_{ip} x e_{qi} e_{pq} = \sum_{p,q} e_{pp} x e_{qq} = x.$$

so that  $A = BM$ , which proves the theorem.

**10.06 Invariant subalgebras.** If  $B$  is a subalgebra of  $A$  such that

$$(10) \quad AB \leq B, \quad BA \leq B,$$

then  $B$  is called an *invariant* subalgebra of  $A$ . If we set

$$A = B + C, \quad B \cap C = 0,$$

the product of any two elements  $c_i, c_j$  of  $C$  lies in  $A$  and hence

$$c_i c_j = c_{ij} + b_{ij}, \quad c_{ij} \in C, \quad b_{ij} \in B.$$

If we now introduce a new operation  $\times$  defined by

$$(11) \quad c_i \times c_j = c_{ij},$$

then the operations  $+$  and  $\times$ , when used to combine elements of  $C$ , satisfy all the postulates for an associative algebra. To prove this we need only consider the associativity postulate M3 since the proofs of the others are immediate. If  $c_1, c_2, c_3$  are any elements of  $C$ , then both  $c_1 \times (c_2 \times c_3)$  and  $(c_1 \times c_2) \times c_3$  differ by an element of  $B$  from  $c_1 c_2 c_3$ ; their difference is therefore an element of both  $B$  and  $C$  and hence is 0

The elements of  $C$  therefore form an associative algebra relatively to the

operations  $+$  and  $\times$ . When this algebra is considered abstractly, the operation  $\times$  may be called multiplication; the resulting algebra is called the *difference algebra* of  $A$  and  $B$  and is denoted by  $(A - B)$ .

The difference algebra may also be defined as follows. Let  $b_1, b_2, \dots, b_\beta$  be a basis of  $B$  and  $c_1, c_2, \dots, c_\gamma$  a basis of  $C$ , so that  $b_1, b_2, \dots, b_\beta, c_1, \dots, c_\gamma$  is a basis of  $A$ . Since  $A$  is an algebra, the product  $c_i c_j$  can be expressed in terms of this basis and we may therefore set

$$(12) \quad c_i c_j = \sum \gamma_{ijk} c_k + \sum \delta_{ijk} b_k.$$

The argument used above then shows that

$$(13) \quad d_i d_j = \sum \gamma_{ijk} d_k$$

defines an associative algebra when  $B$  is invariant.

It is readily seen that the form of the difference algebra is independent of the particular complex  $C$  which is used to supplement  $B$  in  $A$ . For if  $A = B + P$ ,  $B \cap P = 0$ , it follows that to an element  $p$  of  $P$  there corresponds an element  $c$  of  $C$  such that  $p - c \in B$ ; and we may therefore choose a basis for  $P$  for which

$$p_i = c_i + q_i \quad (q_i \in B; i = 1, 2, \dots, \gamma).$$

Equation (12) then gives

$$p_i p_j = \sum \gamma_{ijk} p_k + b_{ij}$$

where  $b_{ij} = q_i q_j + q_i c_j + c_i q_j + \sum \delta_{ijk} b_k - \sum \gamma_{ijk} b_k \in B$ ,

and the algebra derived from this in the same way as (13) is from (12) is abstractly the same as before.

If the algebra  $A$  does not contain the identity, it may happen that  $A^2 < A$ ,  $A^3 < A^2$ , and so on. Since the basis of  $A$  is finite, we must however have at some stage

$$A^m < A^{m-1}, \quad A^{m+1} = A^m;$$

the integer  $m$  is then called the *index* of  $A$ . The most interesting case is when  $A^m = 0$ ; the algebra is then said to be *nilpotent*.

When  $N_1$  and  $N_2$  are nilpotent subalgebras of  $A$  which are also invariant, then  $N_1 + N_2$  is a nilpotent invariant subalgebra of  $A$ . This is shown as follows. Let  $m_1, m_2$  be the indices of  $N_1$  and  $N_2$  respectively;  $N_3 = N_1 \cap N_2$  is nilpotent and, since  $N_3^{m_1} \leq N_1^{m_1} = 0$ , its index  $m_3$  is not greater than  $m_1$ . Now

$$\begin{aligned} (N_1 + N_2)^2 &= N_1^2 + N_2^2 + N_1 N_2 + N_2 N_1 \\ &\leq N_1^2 + N_2^2 + N_3 \leq N_1 + N_2 \end{aligned}$$

since it follows from the invariance of  $N_1$  and  $N_2$  that  $N_1 N_2$  and  $N_2 N_1$  are contained in both  $N_1$  and  $N_2$  and therefore in  $N_3$ . Similarly

$$(N_1 + N_2)^r \leq N_1^r + N_2^r + N_3$$

so that, if  $m$  is the greater of  $m_1$  and  $m_2$ ,

$$(N_1 + N_2)^m \leq N_1^m + N_2^m + N_3 = N_3$$

and hence  $N_1 + N_2$  is a nilpotent subalgebra. Further

$$\begin{aligned} A(N_1 + N_2) &= AN_1 + AN_2 \leq N_1 + N_2 \\ (N_1 + N_2)A &= N_1A + N_2A \leq N_1 + N_2 \end{aligned}$$

so that  $N_1 + N_2$  is invariant. It follows that the totality of all nilpotent invariant subalgebras is itself a nilpotent invariant subalgebra; this algebra is called the maximal nilpotent invariant subalgebra or *radical* of  $A$ .

An algebra  $A$  which is not nilpotent and which has no radical is said to be *semi-simple*; if in addition it has no invariant subalgebra, it is said to be *simple*.<sup>6</sup> We have then the following theorem whose proof we leave to the reader.

**THEOREM 3.** *If  $N$  is the radical of a non-nilpotent algebra  $A$ , then  $(A - N)$  is semi-simple.*

**10.07 Idempotent elements.** In the preceding section we defined a nilpotent algebra of index  $m$  as one for which  $A^m = 0$ ,  $A^{m-1} \neq 0$ . An immediate consequence of this definition is that every element of a nilpotent algebra is nilpotent; we shall now prove the converse by showing that, if  $A$  is not nilpotent, it contains an idempotent element.

**THEOREM 4.** *Every algebra which is not nilpotent contains an idempotent element.*

Let  $A = (a_1, a_2, \dots, a_\alpha)$  be an algebra of order  $\alpha$ . If  $aA = A$  for some element  $a$  in  $A$ , then  $ax = 0$  only when  $x = 0$ ; for  $aA = A$  implies that  $aa_1, aa_2, \dots, aa_\alpha$  is a basis, which means that there is no relation of the form

$$0 = \sum \xi_i aa_i = a \sum \xi_i a_i$$

except when every  $\xi_i = 0$ . Also, if  $aA = A$ , there must be an element  $e$  in  $A$  such that  $ae = a$ ; this gives  $ae^2 = ae$  or  $a(e^2 - e) = 0$  and hence  $e^2 = e$ .

The theorem is true of algebras of order 1; assume it true for algebras of order less than  $\alpha$ . If  $a_iA = A$  for some  $a_i$ , the theorem has just been shown to hold. If  $a_iA < A$  for every  $a_i$  in the basis of  $A$ , then, since  $(a_iA)^2 = a_iAa_iA \leq a_iA$ , either  $a_iA$  contains an idempotent element or, being of order less than  $\alpha$ , it is nilpotent. Now  $(Aa_iA)^r \leq A(a_iA)^r$  and therefore  $Aa_iA$  is also nilpotent; but

$$A \cdot Aa_iA \leq Aa_iA, \quad Aa_iA \cdot A \leq Aa_iA$$

so that  $Aa_iA$  is invariant and being nilpotent is contained in the radical  $N$  of  $A$ . Hence

$$A^3 = \sum_i Aa_iA \leq N$$

<sup>6</sup> Simple algebras are usually excluded from the class of semi-simple algebras; it seems more convenient however to include them.

The statement that  $A$  is not nilpotent is made in order to exclude the algebra of order 1 defined by a single element whose square is 0.

so that  $A^3$ , and therefore also  $A$ , is nilpotent, contrary to the hypothesis of the theorem. It follows that some  $a_i A$  is not nilpotent and being of lower order than  $A$  contains an idempotent element by assumption. The theorem is therefore proved.

The following lemma is an immediate consequence of Theorem 4.

LEMMA 1. *A non-nilpotent algebra cannot have a basis every element of which is nilpotent, nor a basis for which the trace of every element is 0.*

For, if every element of the basis is nilpotent, the trace of every element of the algebra is 0 whereas the trace of an idempotent element is not 0 since the only roots of its characteristic equation are 0 and 1.

If  $e$  is the only idempotent element in  $eAe$ , it is said to be *primitive*. An algebra which is not nilpotent contains at least one primitive idempotent element. For, if  $eAe$  contains an idempotent element  $e_1 \neq e$ , then  $e_1(e - e_1) = 0$  so that  $e_1 e A e e_1$  does not contain  $e - e_1$  and is therefore of lower order than  $eAe$ ; since the order of  $eAe$  is finite, a succession of such steps must lead to a primitive idempotent element.

THEOREM 4.5. *A simple algebra has a principal unit.*

If  $A$  is not nilpotent, it contains an idempotent element  $e$ . If  $a$  is any element of  $A$ , we may set  $a = a_1 + a_2$  where

$$a_1 = ea + ae - eae < eA + Ae, \quad a_2 = a - a_1, \quad ea_2 = 0 = a_2e.$$

We can therefore find a complex  $A_1$  such that

$$A = eA + Ae + A_1, \quad eA + Ae \cap A_1 = 0, \quad eA_1 = 0 = A_1e.$$

If  $A_1$  is not nilpotent, it contains an idempotent element  $e'$  and  $e + e'$  is also idempotent since  $ee' = 0 = e'e$ . We can therefore take  $e + e'$  in place of  $e$  so reducing the order of  $A_1$ , and after a finite number of such steps we arrive at a stage at which  $A_1$  contains no idempotent element and is therefore nilpotent; we shall now assume that  $e$  was chosen at the start so that  $A_1$  is nilpotent; we shall also assume that  $e$  is not an identity for  $A$  and there is no real loss of generality in assuming in addition that it is not a left-hand identity.

Let  $r$  be the index of  $A_1$ . If  $r > 1$  and  $x \neq 0$  is any element of  $A_1^{r-1}$  then  $xA_1 = 0 = A_1x$ ,  $ex = 0$ ; if  $r = 1$ , then  $A_1 = 0$  and since  $e$  is not a left-hand identity,  $e \leq eA < A$  so that there is an  $x \neq 0$  such that  $ex = 0$ ; we have therefore in both cases

$$xA_1 = 0 = A_1x, \quad ex = 0.$$

We now have  $Ax = eAx$ ,  $AxA = eAx$ ; hence  $Ax < A$ ,  $AxA < A$  and  $AxA$  is therefore an invariant subalgebra of  $A$ ; if  $AxA = 0$ , then  $Ax$  is invariant and not equal to  $A$ ; if  $Ax = 0$ , then  $xA$  is a proper invariant subalgebra unless it is 0 in which case  $X = \{x\}$  is a non-zero invariant subalgebra of  $A$ . In the case of a simple algebra it follows that  $e$  is an identity.

*Corollary.* An algebra without a principal unit is not semi-simple. For  $(Ax)^2 = AxAx = AxeAx = 0$  if  $A_1 \neq 0$ .

**10.08 Matric subalgebras.** Let  $A$  be an algebra which contains the identity and let  $e_1$  be a primitive idempotent element; then  $e_\alpha = 1 - e_1$  is also idempotent and, if  $e_i A e_j$  is denoted by  $A_{ij}$ , then

$$A = (e_1 + e_\alpha)A(e_1 + e_\alpha) = A_{11} + A_{1\alpha} + A_{\alpha 1} + A_{\alpha\alpha}.$$

Suppose in the first place that  $A_{\alpha 1}A_{1\alpha}$  is not nilpotent; there is then some  $a_{12} < A_{1\alpha}$  such that  $A_{\alpha 1}a_{12}$ , which is an algebra, is not nilpotent since otherwise  $A_{\alpha 1}A_{1\alpha}$  would have a basis of nilpotent elements, which is impossible by Lemma 1; hence some such  $A_{\alpha 1}a_{12}$  contains an idempotent element, say  $e_2 = a_{21}a_{12}$ . If  $e_2$  is not primitive in  $A$ , say  $e_2 = e' + e''$ ,  $e'e'' = 0 = e''e'$ , where  $e'$  is primitive in  $A$ , then  $a_{12}e' \neq 0$  since otherwise

$$e' = e_2e' = a_{21}a_{12}e' = 0;$$

also  $e' < A_{\alpha\alpha}$  since  $0 = e_1e_2 = e_1e' + e_1e''$  so that  $e_1e'' = -e_1e'$  and therefore

$$e_1e' = -e_1e''e' = 0$$

and similarly  $e'e_1 = 0$ ; we may therefore take  $a'_{12} = a_{12}e'$  and  $a'_{21} = e'a_{21}$  in place of  $a_{12}$  and  $a_{21}$ , which gives  $e'$  in place of  $e_2$ . We can therefore assume  $a_{12}$  so chosen that  $e_2$  is primitive in  $A$ ; also, since  $e_2a_{21}a_{12}e_2 = e_2^3 = e_2$ , then, replacing  $a_{21}$  by  $e_2a_{21}$ , if necessary, we may assume  $e_2a_{21} = a_{21}$  and similarly  $a_{12}e_2 = a_{12}$ .

The element  $a_{12}a_{21}$  is not 0 since

$$a_{21}a_{12}a_{21}a_{12} = a_{21}a_{12} \cdot a_{21}a_{12} = e_2^2 = e_2,$$

and it is idempotent since

$$(a_{12}a_{21})^2 = a_{12} \cdot a_{21}a_{12} \cdot a_{21} = a_{12}e_2a_{21} = a_{12}a_{21}.$$

But  $a_{12}a_{21} \leq A_{1\alpha}A_{\alpha 1} \leq A_{11}$  and, since  $e_1$  is primitive, it follows that  $a_{12}a_{21} = e_1$ . For the sake of symmetry we now put  $a_{11} = e_1$ ,  $a_{22} = e_2$ , and we then have a matric subalgebra of  $A$ , namely  $a_{11}, a_{12}, a_{21}, a_{22}$ .

Since  $A_{\alpha 1}(A_{1\alpha}A_{\alpha 1})^r A_{1\alpha} = (A_{\alpha 1}A_{1\alpha})^{r+1}$ , it follows that  $A_{1\alpha}A_{\alpha 1}$  and  $A_{\alpha 1}A_{1\alpha}$  are either both nilpotent or both not nilpotent. Suppose that both are nilpotent; then, since their product in either order is 0, their sum is nilpotent and, because  $(A_{1\alpha} + A_{\alpha 1})^2 = A_{1\alpha}A_{\alpha 1} + A_{\alpha 1}A_{1\alpha}$ , it follows that

$$N_1 = A_{1\alpha} + A_{\alpha 1} + A_{1\alpha}A_{\alpha 1} + A_{\alpha 1}A_{1\alpha}$$

is nilpotent. Now

$$\begin{aligned} AN_1 &= (A_{11} + A_{1\alpha} + A_{\alpha 1} + A_{\alpha\alpha})(A_{1\alpha} + A_{\alpha 1} + A_{1\alpha}A_{\alpha 1} + A_{\alpha 1}A_{1\alpha}) \\ &= A_{11}A_{1\alpha} + A_{11}A_{1\alpha}A_{\alpha 1} + A_{1\alpha}A_{\alpha 1} + A_{1\alpha}A_{\alpha 1}A_{1\alpha} + A_{\alpha 1}A_{1\alpha} \\ &\quad + A_{\alpha 1}A_{1\alpha}A_{\alpha 1} + A_{\alpha\alpha}A_{\alpha 1} + A_{\alpha\alpha}A_{\alpha 1}A_{1\alpha} \\ &\leq N_1 \end{aligned}$$

since  $A_{ij}A_{pq} = 0$  ( $p \neq j$ ),  $A_{ij}A_{jq} \leq A_{iq}$ . Similarly  $N_1A \leq N_1$ . Hence  $N_1$  lies in the radical of  $A$ .

Suppose that we have found a matric subalgebra  $a_{ij}$  ( $i, j = 1, 2, \dots, r - 1$ ) such that  $e_i = a_{ii}$  ( $i = 1, 2, \dots, r - 1$ ) are primitive idempotent elements of  $A$ ; let  $e_\alpha = 1 - \sum_1^{r-1} e_i$  and set  $A_{ij} = e_i A e_j$  as before. Suppose further that  $A_{\alpha i} A_{i \alpha}$  is not nilpotent for some  $i$ ; we may then take  $i = 1$  without loss of generality. By the argument used above there then exists a primitive idempotent element  $e_r = a_{rr} < A_{\alpha 1} A_{1 \alpha}$  and elements  $a_{r1} < A_{\alpha 1}$ ,  $a_{1r} < A_{1 \alpha}$  such that

$$\begin{aligned} a_{r1} a_{1r} &= a_{rr}, & a_{1r} a_{r1} &= a_{11}, \\ a_{rr} a_{r1} &= a_{r1}, & a_{1r} a_{rr} &= a_{1r}. \end{aligned}$$

If we set

$$a_{ir} = a_{i1} a_{1r}, \quad a_{ri} = a_{r1} a_{1i} \quad (i = 1, 2, \dots, r - 1),$$

then  $a_{ir} \neq 0$  since  $a_{1i} a_{ir} = a_{1r}$ , and  $a_{ij}$  ( $i, j = 1, 2, \dots, r$ ) form a matric algebra of higher order than before.

Again, if every  $A_{\alpha i} A_{i \alpha}$  is nilpotent, it follows as above that each  $A_{i \alpha} A_{\alpha i}$  is also nilpotent and hence

$$N_{r-1} = \sum_{i, j=1}^{r-1} (A_{i \alpha} + A_{\alpha i} + A_{i \alpha} A_{\alpha j} + A_{\alpha i} A_{i \alpha}),$$

having a nilpotent basis, is itself nilpotent; and it is readily seen as before that it is invariant and therefore belongs to the radical of  $A$ .

We can now treat  $A_{\alpha \alpha}$  in the same way as  $A$ , and by doing so we derive a set of matric algebras  $M_p(a_{ij}^p; i, j = 1, 2, \dots, r_p)$  with the identity elements

$$a_p = \sum_{i=1}^{r_p} a_{ii}^p,$$

such that  $\Sigma a_p = 1$ ; also

$$N' = \sum_{p \neq q} (a_p A a_q + a_p A a_q A a_p)$$

is contained in the radical  $N$  of  $A$ . We have therefore the following Lemma.

LEMMA 2. *If  $A$  is an algebra with an identity, there exists a set of matric subalgebras  $M_p = (a_{ij}^p; i, j = 1, 2, \dots, r_p)$  with the principal units*

$$a_p = \sum_{i=1}^{r_p} a_{ii}^p \quad (p = 1, 2, \dots, k)$$

such that  $a_p a_q = 0$  ( $p \neq q$ ) and  $\Sigma a_p = 1$ , and such that

$$N' = \sum_{p \neq q} (a_p A a_q + a_p A a_q A a_p)$$

lies in the radical  $N$  of  $A$ . Further each  $a_{ii}^p$  is a primitive idempotent element of  $A$ .

*Corollary.*  $B_k = a_k A a_k + N'$  is an invariant subalgebra of  $A$ . For

$$\begin{aligned} AB_k &= \sum a_i A (a_k A a_k + \sum_{p \neq q} a_p A a_q + a_p A a_q A a_p) \\ &= a_k A a_k + N' = B_k. \end{aligned}$$

10.09 We shall now consider the properties of the algebras  $a_p A a_p$  where  $a_p$  ( $p = 1, 2, \dots, k$ ) are the idempotent elements defined in Lemma 2.

LEMMA 3.  $a_p A a_p$  is the direct product of  $M_p$  and an algebra  $B_p$  in which the principal unit is the only idempotent element.

The first part of this lemma is merely a particular case of Theorem 2. That  $B_p$  contains only one idempotent element is seen as follows. If  $e$  is a primitive idempotent element of  $B_p$ , then  $a_{11}^p e$  and  $a_{11}^p (a_p - e)$  are distinct and, if not zero, are idempotent and lie in  $a_{11}^p A a_{11}^p$ ; but this algebra contains only one idempotent element since  $a_{11}^p$  is primitive; hence  $a_{11}^p (a_p - e) = 0$ , and therefore  $e = a_p$  is the only idempotent element in  $B_p$ .

LEMMA 4. If  $B$  is an algebra whose principal unit 1 is its only idempotent element, any element of  $B$  which is singular<sup>7</sup> is nilpotent; and the totality of such elements forms the radical of  $B$ .

The proof of the first statement is immediate; for, if  $a$  is singular, the algebra  $\{a\}$  generated by  $a$  does not contain the principal unit and, since  $B$  contains no other idempotent element,  $a$  is nilpotent by Theorem 4. To prove the second part, let  $x$  and  $y$  be nilpotent but  $z = x + y$  non-singular; then  $1 = z^{-1}x + z^{-1}y = x_1 + y_1$ . Here  $x_1$  and  $y_1$  are singular and therefore nilpotent. If  $m$  is the index of  $x_1$ , then

$$(1 - x_1)(1 + x_1 + x_1^2 + \dots + x_1^{m-1}) = 1$$

and this is impossible since  $y_1 = 1 - x_1$  is nilpotent. Hence  $z$  is also nilpotent and the totality of nilpotent elements forms an algebra; and this algebra is invariant since the product of any element of  $B$  into a nilpotent element is singular and therefore nilpotent. It follows that  $B$  is a division algebra whenever it has no radical, that is, when it is semi-simple.

10.10 **The classification of algebras.** We shall now prove the main theorem regarding the classification of algebras in a given field  $F$ .

THEOREM 5. (i) Any algebra which contains an identity can be expressed in the form

$$(14) \quad A = S + N$$

<sup>7</sup> An element of  $B$  is singular in  $B$  if it does not have an inverse relatively to the principal idempotent element of  $B$ .

where  $N$  is the radical of  $A$  and  $S$  is a semi-simple subalgebra;  $S$  is not necessarily unique but any two determinations of it are simply isomorphic.

(ii) A semi-simple algebra can be expressed uniquely as the direct sum of simple algebras.

(iii) A simple algebra can be expressed as the direct product of a division algebra  $D$  and a simple matrix algebra  $M$ ; these are not necessarily unique but, if  $D_1, M_1, D_2, M_2$  are any two determinations of  $D$  and  $M$ , then  $D_1 \simeq D_2, M_1 \simeq M_2$ .

We have seen in Lemma 2 that  $A = \sum a_p A a_p + N'$ , where  $N' \leq N$ , and also in Lemmas 3, 4 that  $a_p A a_p = M_p \times B_p$ , where  $M_p$  is a simple matrix algebra. The first part of the theorem therefore follows for  $A$  when it is proved for any algebra like  $B_p$  and when it is shown that the direct product of  $M_p$  by a division algebra is simple; for, if  $B_p = D_p + N_p$ , then  $D_p$  is a division algebra and

$$a_p A a_p = M_p \times D_p + M_p \times N_p, \quad M_p N_p \leq N.$$

If the field  $F$  is one in which every equation has a root, the field itself is clearly the only division algebra and hence  $M_p D_p = M_p$ ; in this case part (i) is already proved. Further, the theorem is trivial for algebras of order 1; we may, therefore, as a basis for a proof by induction assume it is true for algebras of order less than the order  $\alpha$  of  $A$ .

If the field  $F$  is extended to  $F(\xi)$  by the adjunction of an algebraic irrationality  $\xi$  of degree  $\rho + 1$ , we get in place of  $A$  an algebra  $A' = A(\xi)$  which has the same basis as  $A$  but which contains elements whose coordinates lie in  $F(\xi)$  but not necessarily in  $F$ ; all elements of  $A$  are also elements of  $A'$ . Regarding  $A'$  we have the following important lemma.

LEMMA 5. *If  $N$  is the radical of  $A$ , the radical of  $A' = A(\xi)$  is  $N' = N(\xi)$ .*

Let  $A = C + N, C \cap N = 0$ , and let the radical of  $A'$  be  $N''$ ; then clearly  $N'' \geq N'$ . If  $N'' > N'$ , there is an element of  $N''$  of the form

$$c'' = c_0 + c_1 \xi + \cdots + c_\rho \xi^\rho, \quad (c_i \in C, c_0 \neq 0).$$

Since  $c''$  is nilpotent,

$$0 = \text{tr}(c'') = \text{tr}(c_0) + \xi \text{tr}(c_1) + \cdots$$

and since  $\text{tr}(c_0), \text{tr}(c_1), \cdots$  are rational in  $F$ , each is separately 0. But, if  $a_1, a_2$  are arbitrary elements in  $A$ ,

$$a_1 c'' a_2 = a_1 c_0 a_2 + a_1 c_1 a_2 \xi + \cdots$$

lies in  $N''$  and, since each  $a_1 c_i a_2$  is rational in  $F$ , the trace of each is 0 as above. Hence the trace of every element in  $A c_0 A$  is 0 from which it follows by Lemma 1 that  $A c_0 A$  is nilpotent and being invariant and also rational it must lie in  $N$  (cf. §10.06). But  $A c_0 A$  contains  $c_0$  since  $A$  contains 1 whereas  $C \cap N = 0$ ; hence no elements of  $N''$  such as  $c''$  exist and the lemma is therefore true.

We may also note that, if  $B, C$  are complexes for which  $B \cap C = 0$ , and  $B', C'$  the corresponding complexes in  $A'$ , then also  $B' \cap C' = 0$ .

Suppose now that the identity is the only idempotent element of  $A$  and that the first part of the theorem is true for algebras of order less than  $\alpha$ . Let  $a \neq 1$  be an element of  $A$  corresponding to an element  $\bar{a}$  of  $(A - N)$  and let  $f(\lambda)$  be the reduced characteristic function of  $\bar{a}$ ;  $f(\lambda)$  is irreducible in  $F$  since  $(A - N)$  is a division algebra. Since  $f(\bar{a}) = 0$ , it follows that  $f(a) < N$  and hence, if  $r$  is the index of  $f(a)$ , the reduced characteristic function of  $a$  is  $[f(\lambda)]^r$ . If we adjoin to  $F$  a root  $\xi$  of  $f(\lambda)$ , this polynomial becomes reducible so that in  $A' = A(\xi)$  the difference algebra  $(A' - N')$  is no longer a division algebra though by Lemma 5 it is still semi-simple. If we now carry out in  $F(\xi)$  the reduction given in Lemma 2, say

$$A' = \sum e_p A' e_p + N^*,$$

either the algebras  $e_p A' e_p$  are all of lower order than  $\alpha$ , or, if  $A' = e_1 A' e_1$ , then it contains a matrix algebra  $M'$  of order  $n^2$  ( $n > 1$ ) and, if we set  $A' = M' B'$ , as previously,  $B'$  is of lower order than  $\alpha$ . In all cases, therefore, part (i) of the theorem follows for algebras in  $F(\xi)$  of order  $\alpha$  when it is true for algebras of order less than  $\alpha$ , and its truth in that case is assumed under the hypothesis of the induction.

We may now assume

$$\begin{aligned} A &= C + N, & C \frown N &= 0, \\ A' &= S' + N', & S' \frown N' &= 0, \end{aligned}$$

where  $S'$  is an algebra simply isomorphic with  $(A' - N')$ ;  $N'$  has a rational basis, namely that of  $N$  (cf. Lemma 5).

If  $c_1, c_2, \dots$  is a basis of  $C$  then, since  $A$  is contained in  $A'$  we have

$$c_i = s'_i + m'_i, \quad s'_i < S', \quad m'_i < N', \quad (i = 1, 2, \dots)$$

and, since  $C \frown N = 0$  implies  $C' \frown N' = 0$ , it follows that  $s'_1, s'_2, \dots$  form a basis of  $S'$ , that is, we may choose a basis for  $S'$  in which the elements have the form

$$c_i + n_{i0} + n_{i1}\xi + \dots \quad (c_i < C, n_{ij} < N)$$

where  $c_i, n_{i0}, \dots$  are rational in  $F$ . Moreover, since  $C$  is only determined modulo  $N$ , we may suppose it modified so that  $n_{i0}$  is absorbed in  $c_i$ ; we then have a basis for  $S'$

$$(15) \quad s'_i = c_i + n_{i1}\xi + \dots + n_{ip}\xi^p = c_i + n'_i.$$

When the basis is so chosen, the law of multiplication in  $S'$ , say

$$(16) \quad s'_i s'_j = \sum \sigma_{ijk} s'_k$$

has constants  $\sigma_{ijk}$  which are rational in  $F$ ; for  $s'_i = c_i \pmod{N'}$  and  $c_i$  is rational. If we now replace  $s'_k$  in (16) by its value from (15) and expand, we have

$$c_i c_j + c_i n'_j + n'_i c_j + n'_i n'_j = \sum \sigma_{ijk} c_k + \sum \sigma_{ijk} n'_k,$$

but  $n'_i n'_j < (N')^2$  and therefore

$$c_i c_j + c_i n'_j + n'_i c_j = \sum \sigma_{ijk} c_k + \sum \sigma_{ijk} n'_k \pmod{(N')^2},$$

a relation which is only possible if the coefficients of corresponding powers of  $\xi$  are also equivalent modulo  $(N')^2$  and in particular

$$c_i c_i = \sum \sigma_{ijk} c_k \pmod{(N')^2}.$$

Consequently the algebra  $A_1$  generated by  $c_i$  ( $i = 1, 2, \dots, \sigma$ ) contains no element of  $N$  which is not also in  $N^2$  and hence, except in the trivial case in which  $N = 0$ , the order of  $A_1$  is less than  $\alpha$ . By hypothesis we can therefore choose  $C$  rationally in such a way that  $c_i c_j = \sum \sigma_{ijk} c_k$ , that is, such that  $C$  is an algebra; part (i) of the theorem therefore follows by induction.

10.11 For the proof of part (ii) we require the following lemmas.

LEMMA 6. *If  $A$  contains the identity 1 and if  $B$  is an invariant subalgebra which has a principal unit  $e$ , then*

$$A = B \oplus (1 - e)A(1 - e).$$

Since  $e$  is the principal unit of  $B$ , which is invariant,  $eAe = B$ ; also  $eA(1 - e)$  and  $(1 - e)Ae$  are both 0 since  $Ae$  and  $eA$  lie in  $B$  and, if  $b$  is any element of  $B$ , then  $(1 - e)b = b - b = 0$ ,  $b(1 - e) = b - b = 0$ ; hence

$$(17) \quad A = eAe + (1 - e)A(1 - e), \quad eAe \wedge (1 - e)A(1 - e) = 0.$$

Further  $eAe \cdot (1 - e)A(1 - e) = 0 = (1 - e)A(1 - e) \cdot eAe$ , so that the sum in (17) is a direct sum.

LEMMA 7. *Every invariant subalgebra  $B$  of a semi-simple algebra  $A$  is semi-simple and therefore contains a principal unit.*

Suppose that  $B$  has a radical  $N$ ; then

$$AN \leq B, \quad (AN)^2 = ANA \cdot N \leq BN \leq N$$

so that  $AN$  is nilpotent. But, since  $A^2 = A$ , we have  $(ANA)^r = (AN)^r A$ ; hence  $ANA$  is a nilpotent invariant subalgebra of  $A$  which, since  $A$  contains an identity, is not 0 unless  $N$  is 0. But  $A$  has no radical; hence  $N = 0$  and  $B$  also has no radical.

In consequence of these lemmas a simple algebra is irreducible and a semi-simple algebra which is not also simple can be expressed as the direct sum of simple algebras. Let

$$A = B_1 \oplus B_2 \oplus \dots \oplus B_p = C_1 \oplus C_2 \oplus \dots \oplus C_q.$$

be two expressions of  $A$  as the direct sum of simple algebras and let the principal units of  $B_i$  and  $C_i$  be  $b_i$  and  $c_i$  respectively; then  $1 = \sum b_i = \sum c_i$ . We then have  $C_k \leq \sum b_i C_k b_i \leq C_k$  and therefore

$$C_k = \sum b_i C_k b_i = \sum b_i C_k b_i$$

since when  $i \neq j$  then  $b_i C_k b_j \leq B_i \wedge B_j = 0$  and  $b_i C_k b_i \cdot b_j C_k b_j = 0$ . If  $b_i C_k b_i \neq 0$ , it is an invariant subalgebra of  $C_k$  and, since the latter is simple, we have  $b_i C_k b_i = C_k$  for this value of  $i$  and all other  $b_j C_k b_j$  equal 0, and therefore  $C_k = b_i A b_i = B_i$ . The second part of the theorem is therefore proved.

10.12 We shall now prove part (iii) of Theorem 5 in two stages.

LEMMA 8. *If  $D$  is a division algebra and  $M$  the matrix algebra  $(e_{ij}; i, j = 1, 2, \dots, m)$ , and if  $D \times M = DM$ , then  $DM$  is simple.*

Let  $B$  be a proper invariant subalgebra of  $A = DM$ . If  $x$  is an element of  $B$ , then there exists an element  $y$  of  $A$  such that  $xy = 0$ , since otherwise we should have  $B \geq xA = A$ , in other words, every element of  $B$  is singular in  $A$  and hence  $B \wedge D = 0$ . But

$$x = \sum d_{ij} e_{ij}, \quad d_{ij} \in D$$

and  $d_{ij} = \sum_p e_{pi} x e_{jp}$  and is therefore contained in  $B$  as well as in  $D$ . Since  $B \wedge D = 0$ , every  $d_{ij} = 0$ , that is,  $x = 0$  so that  $B = 0$ . It follows immediately from Lemma 2 that a simple algebra always has the form  $D \times M$ , and also that  $D \simeq e_{ii} A e_{ii}$ .

LEMMA 9. *In a simple algebra all primitive idempotent elements are similar.*

Let  $e$  and  $a$  be primitive idempotent elements of a simple algebra  $A$ . We can then find a matrix algebra  $M = (e_{ij})$  for which  $e_{11} = e$  and such that  $A = D \times M$ , where  $D$  is a division algebra. If  $ea = 0 = ae$ , we can at the same time choose  $e_{22} = a$ ; and  $e_{22} = u e_{11} u^{-1}$  where

$$u = 1 - e_{11} - e_{22} + e_{12} + e_{21} = u^{-1},$$

so that the lemma is true in this case, and we may therefore assume that, say,  $ea \neq 0$ .

Suppose now that  $ea e \neq 0$ . Since  $A = D \times M$ , we can express  $a$  in the form  $\sum \alpha_{ij} e_{ij}$  ( $\alpha_{ij} \in D$ ), where  $\alpha_{11} \neq 0$  since  $ea e = \alpha_{11} e_{11}$ . We have then

$$(ea)^2 = (e_{11} a)^2 = (\alpha_{11} e_{11} + \alpha_{12} e_{12} + \dots)^2 = \alpha_{11} e_{11} a$$

and hence  $b = \alpha_{11}^{-1} ea$  is idempotent. We then have

$$eb = b, \quad be = e, \quad ba = b;$$

also  $ab = aba = abab$  and, since  $a$  is primitive, either  $aba = a$  or  $aba = 0$ ; but

$$eabe = eae \neq 0,$$

hence  $ab = a$ . We then have

$$\begin{aligned} b &= ueu^{-1}, & u &= 1 - b + e, & u^{-1} &= 1 + b - e, \\ b &= v^{-1}av, & v &= 1 - b + a, & v^{-1} &= 1 + b - a, \end{aligned}$$

and hence  $a$  and  $e$  are similar in this case also.

If  $ea e = 0$  but  $aea \neq 0$ , interchanging the rôles of  $e$  and  $a$  leads to results similar to those just obtained; we can therefore assume  $ea e = 0 = aea$ . If

$$u = 1 + e - ea + ae = 2(2 - e + ea - ae)^{-1},$$

then  $uau^{-1} = a - ae$ ; we can therefore assume  $ae = 0$ . If

$$v = (1 + e - 2ea) = 2(2 - e + 2ea)^{-1},$$

then  $vav^{-1} = a - ea$ ; we can therefore also assume  $ea = 0$ , which brings us back to the first case which we considered. The lemma is therefore proved.

Part (iii) of the theorem follows immediately. For, if  $e$  and  $a$  are primitive idempotent elements of  $M_1$  and  $M_2$  respectively, we can now find  $w$  such that  $a = wew^{-1}$ ; but  $D_1 \simeq eAe$  and  $D_2 \simeq aAa = weAew^{-1}$ , which is similar to  $eAe$  and therefore to  $D_1$ .

**10.13 Semi-invariant subalgebras.** If  $B$  is a subalgebra of  $A$  which is such that  $AB \leq B$  ( $BA \leq B$ ), it is called a right (left) semi-invariant subalgebra. We shall treat only the case in which  $A$  is semi-simple; it has then an identity and if we restrict ourselves, as we shall, to the case of right semi-invariant subalgebras, we may assume  $AB = B$ .

It is clear that, if  $A = A_1 \oplus A_2$ , then also  $B = B_1 \oplus B_2$ , where  $A_i B_i = B_i$ ,  $A_i B_j = 0$  ( $i \neq j$ ). It is sufficient then at first to consider only simple algebras, and in this case we have the added condition that  $ABA = A$ ; that is, we have simultaneously

$$(18) \quad AB = B, \quad ABA = A.$$

If we call  $B$  minimal when it contains no other semi-invariant subalgebra, we have

**LEMMA 10.** *A minimal right semi-invariant subalgebra of a simple algebra  $A$  has the form  $Au$ , where  $u$  is a primitive idempotent element of  $A$ . Conversely, if  $u^2 = u$  is primitive,  $Au$  is a minimal right semi-invariant subalgebra.*

Let  $AC = C$ ; if  $c_1 \neq 0$  is any element of  $C$ , and  $C_1 = Ac_1 \leq C$ , then  $AC_1 = C_1$ . Suppose  $C_1 < C$ ; then in the same way if  $c_2$  is any element of  $C_1$ , we have  $C_2 = Ac_2 \leq C_1$ . If  $C_2 < C_1$ , we may continue this process and after a finite number of steps we shall arrive at an algebra  $B \neq 0$  such that  $Ab = B$  for every element  $b$  of  $B$  which is not 0. Since  $A$  is simple,  $AbA = A$  and  $B^2 = B$ , so that  $B$  contains a primitive idempotent element  $u$  and  $Au = B$ . If  $u$  is not also primitive in  $A$ , let  $u = u_1 + u_2$ ,  $u_i u_j = 0$  ( $i \neq j$ ),  $u_i^2 = u_i \neq 0$ . Then  $u_1 u = u_1$  so that  $u_1$  is in  $B$ ; hence  $u$  must be primitive in  $A$ , if it is so in  $B$ .

Since  $B = Au$ , every  $x$  in  $B$  has the form  $au$  and hence  $xu = x$ . But also  $B = Ax$  and, from the manner in which  $B$  was chosen, either  $Bx = 0$  or  $Bx = B$ . If  $Bx = 0$ , then  $ux = 0$  and therefore

$$x^2 = xu \cdot xu = 0.$$

Also, if  $x$  is nilpotent, then  $x^2 = 0 = ux$ ; for  $uAu = uBu$  is simple since, by the proof of Lemma 4, it is a division algebra, and  $ux = uxu < uBu$ . If  $Bx = B$ , then there is a unique  $b$  such that  $bx = x$  and, since  $b$  is then idempotent, we have  $ux = x$ , that is,  $x$  lies in  $uAu$ . If  $B = Au$ , then  $AB = A^2u \leq B$  so that  $B$  is a right semi-invariant subalgebra of  $A$ . If  $C$  is minimal, then  $B = C$  as desired.

Conversely, let  $B = Au$ ,  $u$  primitive; then the only idempotent quantity of  $B$  has been shown above to be  $u$  and, if  $B$  were not primitive, we should have  $B > C = Av$ ,  $v$  primitive, which is impossible.

Suppose now that  $B$  is not minimal and let  $e_1, e_2, \dots, e_r$  be a complete set of primitive supplementary idempotent elements in  $B$ . Then  $B_r = Ae_1 + Ae_2 + \dots + Ae_r$  is semi-invariant in  $A$ . Let  $b$  be an element of  $B$  which is not in  $B_r$ ; since  $b \neq \Sigma be_i$ , we may replace  $b$  by  $b - \Sigma be_i$  and so assume every  $be_i = 0$  in which case clearly  $Ab \cap B_r = 0$ . But, if  $b \neq 0$ , then  $Ab$  contains an idempotent element  $e$  such that  $e_i e = 0$  ( $i = 1, 2, \dots, r$ ) and  $e_{r+1} = e - \Sigma e_i e$  is an idempotent element supplementary to the given complete set, which is impossible. We therefore have the following theorem.

**THEOREM 6.** *If  $A$  is simple and  $AB = B$  is a semi-invariant subalgebra, then*

$$B = Ae_1 + Ae_2 + \dots + Ae_r$$

where  $e_1, e_2, \dots, e_r$  is a complete supplementary set of primitive idempotent elements of  $B$ ; and these idempotent elements are also primitive in  $A$ .

We shall assume that  $A$  is semi-simple, say

$$(19) \quad A = S_1 \oplus \dots \oplus S_i,$$

when each  $S_i$  is simple and

$$(20) \quad S_i = D_i \times M_i.$$

As previously (cf. Lemma 2) we may set  $M_i = (e_{pq}^i)$ ,  $p, q = 1, 2, \dots, n_i$ , where  $e_{pp}^i$  form a set of supplementary primitive idempotent elements and  $\sum_{i, p} e_{pp}^i = 1$ .

If  $B$  is any invariant subalgebra, then  $B = \sum_{i, p} Be_{pp}^i$  and  $Be_{pp}^i$  is a right semi-invariant subalgebra; if  $B$  is minimal, we have already seen that it has the form  $Bu$  where  $u$  is a primitive idempotent element, and therefore we have  $B = Be_{pp}^i = S_i e_{pp}^i$  for some  $i$  and  $p$ . If set  $B_{ip} = S_i e_{pp}^i$ , then

$$B_{ip} e_{pq}^i = S_i e_{pp}^i e_{pq}^i = M_i D_i e_{pq}^i = S_i e_{iq}^i = B_{iq}.$$

We have therefore the following theorem.

**THEOREM 7.** *If  $A$  is semi-simple and is given by (19), and if  $e_{pp}^i$  form a complete*

set of supplementary primitive idempotent elements such that  $\sum_{p=1}^{n_i} e_{pp}^i = u_i$  is the identity of  $S_i$ , then every minimal right semi-invariant subalgebra has the form

$$(21) \quad B_{ip} = S_i e_{pp}^i.$$

Moreover, there is a number  $e_{pq}^i$  in  $S_i$  such that

$$(22) \quad B_{ip} e_{pq}^i = B_{iq}.$$

**10.14 The representation of a semi-simple algebra.** Let  $A$  be a linear associative algebra over  $F$  with the identity 1, and designate elements of  $A$  by  $a$ . A representation of  $A$  is a set,  $U(a)$ , of matrices of order  $n$  such that  $a \rightarrow U(a)$  is a correspondence between the elements of  $A$  and the matrices of the set in which the following conditions are satisfied

$$(23) \quad U(1) = 1_n, \quad U(a+b) = U(a) + U(b), \quad U(ab) = U(a)U(b), \\ U(\alpha a) = \alpha U(a)$$

for every  $a$  and  $b$  of  $A$  and every scalar  $\alpha$  in  $F$ .

We can now, as in chapter I, associate with the matrices  $U(a)$  a vector space  $R$  with a given fundamental basis, and a change of basis corresponds to replacing  $U(a)$  by  $PU(a)P^{-1}$ , an equivalent representation (cf. 1.08). A subspace  $R_1$  of  $R$  is invariant under  $A$  (cf. 5.16) if every matrix  $U(a)$  carries each vector of  $R_1$  into a vector of  $R_1$ . If  $R_1 \neq 0$ , we may set  $R = R_1 + R_2$  ( $R_1 \cap R_2 = 0$ ); and since we are only interested in the equivalence of representations, we may suppose the basis  $R$  so chosen that

$$(24) \quad U(a) = \begin{vmatrix} U_1(a) & U_3(a) \\ 0 & U_2(a) \end{vmatrix}.$$

The representation is said to be *reducible* in this case, and it is evident that both  $U_1(a)$  and  $U_2(a)$  give representations of  $A$ .

If  $R$  has no proper invariant subspace, then  $U(A)$  and  $R$  are said to be *irreducible*. It is now clear that we may write

$$R = R_1 + R_2 + \cdots + R_s$$

where  $R_t = R_1 + \cdots + R_t$  is the invariant subspace of least order which contains  $R_{t-1}$ , ( $R_0 = 0$ ), and in this case

$$(25) \quad U(a) = \begin{vmatrix} U_{11}(a) & U_{12}(a) & \cdots & U_{1s}(a) \\ 0 & U_{22}(a) & \cdots & U_{2s}(a) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & U_{ss}(a) \end{vmatrix}$$

and the representations  $U_1(a), \cdots, U_s(a)$  are irreducible. If in addition  $R_2, \cdots, R_t$  are themselves invariant for some  $t$ , then  $U_{ij}(a) = 0$  ( $i \neq j$ ;  $i, j = 1, 2, \cdots, t$ ), and we say that  $U(a)$  is *decomposable*.

A particular case of fundamental importance arises when we take  $R$  to be  $A$  itself, that is, if  $x$  is a variable element of  $A$ , then  $x' = ax$  corresponds to a linear transformation in the basis of  $R$  (or  $A$ ), say

$$x' = ax = U(a)(x),$$

and  $U(a)$  has the property given in (23) and so is a representation of  $A$ . It is obviously the representation of (6) and is one-to-one; it is called the *regular* representation.

The invariant subspaces of  $A$  are evidently its right semi-invariant subalgebras  $B$ . If  $e_1, e_2, \dots, e_t$  is a basis of  $B$  and

$$(26) \quad ae_i = \sum \alpha_{ij} e_j,$$

then the matrices  $U(a) = \|\alpha_{ij}\|$  give a representation of  $A$  on the subspace  $B$ . Suppose now that  $V(a)$  is a given representation,  $R$  the corresponding subspace, and  $B$  a right semi-invariant subalgebra of  $A$ . If  $y$  is any vector of  $R$ , then the set of vectors of the form  $V(b)y$  is an invariant subspace of  $R$ , since

$$(27) \quad V(a)V(b)y = V(ab)y = V(b_a)y, \quad b_a \in B.$$

From (27) it is seen immediately that the set  $B'$  of elements  $b'$  in  $B$  for which  $V(b')y = 0$  forms a right semi-invariant subalgebra of  $B$  and hence, if  $B$  is minimal, either  $B' = 0$  or  $B' = B$ . If  $B' = 0$ , then  $V(e_1)y, \dots, V(e_t)y$  is a basis of the set  $(V(b)y)$  and

$$V(a)V(e_i)y = V(ae_i)y = \sum \alpha_{ji} V(e_j)y.$$

But then the vectors of the form  $V(b)y$  give a representation of  $A$  equivalent to that determined by  $B$  in (26).

We shall now prove the following theorem.

**THEOREM 8.** *If the regular representation of an algebra is decomposable, then every representation is decomposable and its irreducible components are contained in the regular representation.*

Suppose that the regular representation of  $A$  is decomposable; then  $A = B_1 + B_2 + \dots + B_s$ , where the  $B_j$  are irreducible equivalent subspaces of  $A$ , that is, minimal semi-invariant subalgebras such that  $B_j \cap B_k = 0$  for  $j \neq k$ . Let  $y_1, y_2, \dots, y_n$  be a basis of the space  $R$  of a representation of  $A$ . Since  $A$  has an identity, we have

$$(28) \quad \begin{aligned} R &= AR = B_1R + B_2R + \dots + B_sR \\ &= B_1y_1 + B_2y_2 + \dots + B_2y_1 + \dots + B_sy_n. \end{aligned}$$

As we have seen above, if  $B_ky_j \neq 0$ , it is a subspace of  $R$  which gives a representation equivalent to that given by  $B_k$ ; it follows that either  $B_ky_j = 0$  or it is an invariant subspace of  $R$ .

The intersection of the invariant subspaces is also invariant so that either

$B_k y_j \cap B_p y_q = 0$  or  $B_k y_j = B_p y_q$ ; hence we may select from the spaces  $B_k y_q$  in (28) a set of independent irreducible invariant subspaces determining  $R$ . This proves Theorem 8.

Consider now a semi-simple algebra

$$A = S_1 \oplus \cdots \oplus S_r$$

where  $S_i$  is a simple algebra. We may write

$$1 = \sum u_{ij} \quad (i = 1, \dots, r; j = 1, \dots, n_i)$$

where the  $u_{ij}$  form a complete set of supplementary primitive idempotent elements of  $A$ . Then

$$A = \sum A u_{ij} = \sum B_{ij}$$

where  $B_{ij} = A u_{ij}$  is a minimal right invariant subalgebra of  $A$ . We have then decomposed  $A$  into irreducible invariant subspaces and have proved the first part of the following theorem.

**THEOREM 9.** *The regular representation of a semi-simple algebra is decomposable, and its reducible components are those obtained by the use of the  $B_{ij}$  as representation spaces. The representations given by any pair  $B_{ij}, B_{ik}$  are equivalent while  $B_{ij}, B_{ik}$  give inequivalent representations for  $j \neq k$ .*

For by Theorem 7 we have  $B_{ij} e_{jk}^i = B_{ik}$  so that the proof of Theorem 7 with  $y = e_{jk}^i$  shows that the representation by  $B_{ij}$  is equivalent to that by  $B_{ik}$ . In the representation by  $B_{ij}$  we have

$$e_i = \sum_{j=1}^{n_i} u_{ij} \rightarrow 1_{n_i}$$

where  $1_{n_i}$  is an identity matrix corresponding to the identity transformation on  $B_{ij}$  since  $e_i$  is the principal unit of  $B_{ij}$ . But in the representation by  $B_{ik}$ , we have  $e_i \rightarrow 0$ . Evidently these representations cannot be similar.

**10.15 Group algebras.** If  $\mathfrak{G} = (g_1 = 1, g_2, \dots, g_m)$  is a finite group, the group relation  $g_i g_j = g_{ij}$  is a particular case of the associative product defined in (2) and, when it is used in conjunction with addition, we get an associative algebra  $G$  of which  $(g_1, g_2, \dots, g_m)$  is a basis and  $g_1$  the identity.

The representation of  $\mathfrak{G}$  as a regular permutation group

$$h_i = \begin{pmatrix} g_1 & g_2 & \cdots & g_m \\ g_{i1} & g_{i2} & \cdots & g_{im} \end{pmatrix}$$

corresponds to the representation of  $G$  as a set of matrices, the matrix  $h_i$  being

$$h_i = \sum_{p=1}^m e_{ip} g_p \quad (g_i g_p = g_{ip}).$$

Since  $i_p = p$ , that is,  $g_i g_p = g_p$ , only when  $g_i$  is the identity, the matrix  $h_i$  has no coordinate in the main diagonal except for  $i = 1$  in which case  $h_1$  is the identity matrix; hence

$$(29) \quad \text{tr}(h_1) = m, \quad \text{tr}(h_i) = 0 \quad (i \neq 1).$$

It follows from this that  $G$  is semi-simple. For if  $u = \sum \eta_i h_i$  is the matrix corresponding to some element of the radical  $N$ , then  $\text{tr}(u) = 0$  since  $u$  is nilpotent. If  $u \neq 0$ , some coordinate, say  $\eta_p$ , is not 0 and in  $h_p^{-1}u$ , which also corresponds to some element of  $N$ , the coefficient of  $h_1$  is not 0; we may therefore assume  $\eta_1 \neq 0$  provided  $N \neq 0$ . But using (18) we get

$$0 = \text{tr}(u) = \sum \eta_i \text{tr}(h_i) = m\eta_1;$$

hence the assumption that  $u \neq 0$  leads to a contradiction and therefore  $N = 0$ , that is,  $G$  is semi-simple. This gives the following theorem.

**THEOREM 10.** *A group algebra is semi-simple. It is therefore the direct sum of simple algebras and, if the field of the coefficients is sufficiently extended, it is the direct sum of simple matrix algebras.*

The whole of the representation theory developed in the previous section can now be applied to groups.

## APPENDIX I

### NOTES<sup>1</sup>

#### CHAPTER I

The calculus of matrices was first used in 1853 by Hamilton (1, p. 559ff, 480ff) under the name of "Linear and vector functions." Cayley used the term *matrix* in 1854, but merely for a scheme of coefficients, and not in connection with a calculus. In 1858 (2) he developed the basic notions of the algebra of matrices without recognizing the relation of his work to that of Hamilton; in some cases (e.g., the theory of the characteristic equation) Cayley gave merely a verification, whereas Hamilton had already used methods in three and four dimensions which extend immediately to any number of dimensions. The algebra of matrices was rediscovered by Laguerre (9) in 1867, and by Frobenius (18) in 1878.

1.03 Matric units seem to have been first used by B. Peirce (17); see also Grassmann (5, §381).

1.10 For the history of the notion of rank and nullity see Muir, *Theory of Determinants*, London 1906-1930; the most important paper is by Frobenius (290).

#### CHAPTER II

2.01-03 The principle of substitution given in §2.01 was understood by most of the early writers, but was first clearly stated by Frobenius, who was also the first to use the division transformation freely (20, p. 203).

2.04 The remainder theorem is implicit in Hamilton's proof of the characteristic equation; see also Frobenius (280).

2.05-12 The characteristic equation was proved by general methods for  $n = 3, 4$  by Hamilton (1, p. 567; 8, p. 484ff; cf. also 4, 6). The first general statement was given by Cayley (2); the first general proof by Frobenius (18). See also the work of Frobenius cited below and 9, 10, 39, 41, 56, 59.

Hamilton, Cayley and other writers were aware that a matrix might satisfy an equation of lower degree than  $n$ , but the theory of the reduced equation seems to be due entirely to Frobenius (18, 140).

The theory of invariant vectors was foreshadowed by Hamilton, but the general case was first handled by Grassmann (5).

2.10 See Sylvester (42, 44) and Taber (96); see also 252.

2.13 The square root of a matrix was considered by Cayley (3, 12), Frobenius (139) and many others.

#### CHAPTER III

3.01 The idea of an elementary transformation seems to be due in the main to Grassmann (5).

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<sup>1</sup> In these Notes, numbers refer to the Bibliography unless otherwise indicated.

3.02-07 The theory of pairs of bilinear forms, which is equivalent to that of linear polynomials, was first given in satisfactory form by Weierstrass (see Muth, 175) although the importance of some of the invariants had been previously recognized by Sylvester. The theory in its matrix form is principally due to Frobenius (18, 20).

The theory of matrices with integral elements was first investigated by Smith (see Muth, 175) but was first given in satisfactory form by Frobenius (20). The form given in the text is essentially that of Kronecker (92).

3.04 The proof of Theorem 3 is a slight modification of that of Frobenius (20).

3.08 Invariant vectors were discussed by Hamilton (1, 8) and other writers on quaternions and vector analysis. The earliest satisfactory account seems to be that of Grassmann (5).

#### CHAPTER IV

The developments of this chapter are, in the main, a translation of Kronecker's work (see Muth, 175, p. 93ff). See also de Séguier (259).

#### CHAPTER V

5.03 From the point of view of matrix theory, the principal references are Schur (198), Rados (105, 106), Stephanos (185), and Hurwitz (117). See Loewy (284, p. 138) for additional references; also Muir, *Theory of Determinants*, London 1906-1930.

5.09 Non-commutative determinants were first considered by Cayley (Phil. Mag. 26 (1845), 141-145); see also Joly (195) and Sylvester (43).

5.10-11 See Loewy (284, p. 149); also 176, 178, 185, 198.

5.12 The principal references are Schur (198) and Weyl (440, chap. 5).

#### CHAPTER VI

For general references see Loewy (284, pp. 118-137), also Muth (175), Hilton (314, chap. 6, 8) and Muir, *Theory of Determinants*, London 1906-1930.

6.01 The method of proving that the roots are real is essentially that of Tait (10, chap. 5); see also 36, 60, 228, 399.

6.03 See Loewy (284, pp. 130-137), Baker (215) and Frobenius (292). See also 7, 18, 99, 113, 114, 115, 124, 135, 139, 210, 221, 273, 302, 307, 320, 371, 400, 414, 466, 476.

6.04 See Dickson (392).

6.05 See Loewy (284, pp. 128-135).

6.07 For references see Muth (175, p. 125) and Frobenius (139).

#### CHAPTER VII

7.10-02 See Cayley (2), Frobenius (18), Bucheim (59), Taber (98, 112), and Hilton (314, chap. 5); also 83, 86, 98, 137, 184, 197, 209, 223, 242, 250, 264, 301, 382.

7.03 See Frobenius (280).

7.05 See Frobenius (140); also 350.

7.06-07 See Sylvester (42, 44) and Taber (96); see also 252.

## CHAPTER VIII

8.01-03 See Sylvester (36), Bucheim (59, 69); also 134, 371.

8.02,07 See Hamilton (1, p. 545ff; 8, §316), Grassmann (5, §454), Laguerre (9). Many writers define the exponential and trigonometric functions and consider the question of convergence, e.g., 79, 80, 103, 389, 449; also in connection with differential equations, 13, 133, 258.

8.04-05 Roots of 0 and 1 have been considered by a large number of writers; see particularly the suite of papers by Sylvester in 1882-84; also 18, 67, 76, 107, 242, 255, 264, 277, 279, 381, 411, 430, 474, 539.

8.08 See 20, 94, 246, 256, 257, 274, 303, 338, 399.

8.09-11 The absolute value of a matrix was first considered by Peano (75) in a somewhat different form from that given here; see also 273, 348, 389, 472, 473, 494. For infinite products see 133, 324, 326, 389, 494.

8.12 In addition to the references already given above, see 10, 16, 18, 187, 418, 419, and also many writers on differential equations.

## CHAPTER IX

The problem of the automorphic transformation in matrices was first considered by Cayley (3, 7) who, following a method used by Hermite, gave the solution for symmetric and skew matrices; his solution was put in simpler form by Frobenius (18). Cayley failed to impose necessary conditions in the general case which was first solved by Voss (85, 108, 162, 163). The properties of the principal elements were given by Taber (125, 134; see also 127, 149, 156, 158, 231). Other references will be found in Loewy (284, pp. 130-137); see also 9, 19, 153, 154, 161, 167, 168, 169, 187, 229, 371.

## APPENDIX II

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