ENGINEERING

MATHEMATICS

4TH EDITION

INSTRUCTOR'S MANUAL

WORKED SOLUTIONS TO THE ASSIGNMENTS

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INTRODUCTION

In **'ENGINEERING MATHEMATICS 4TH EDITION'** there are 61 chapters; each chapter contains numerous fully worked problems and further problems with answers. In addition, there are **16 Assignments** at regular intervals within the text. These Assignments do not have answers given since it is envisaged that lecturers could set the Assignments for students to attempt as part of there course structure. The **worked solutions to the Assignments** are contained in this instructor's manual and with each is a full suggested marking scheme. As a photocopiable resource the **main formulae** are also included.

CONTENTS

ASSIGNMENT 1 (Page 33)

This assignment covers the material contained in Chapters 1 to 4.

Problem 1. Simplify (a) $2\frac{2}{3} \div 3\frac{1}{3}$ (b) $\dfrac{1}{\left(\frac{4}{7} \times 2\frac{1}{4}\right)} \div \left(\frac{1}{3} + \frac{1}{5}\right) + 2\frac{7}{24}$

Marks

(a) $2\frac{2}{3} \div 3\frac{1}{3} = \frac{8}{3} \div \frac{10}{3} = \frac{8}{3} \times \frac{3}{10} = \frac{8}{10} = \mathbf{\frac{4}{5}}$ 4

(b) $\dfrac{1}{\left(\frac{4}{7} \times 2\frac{1}{4}\right)} \div \left(\frac{1}{3} + \frac{1}{5}\right) + 2\frac{7}{24} = \dfrac{1}{\left(\frac{4}{7} \times \frac{9}{4}\right)} \div \left(\frac{5+3}{15}\right) + 2\frac{7}{24}$ 2

$= \dfrac{1}{\frac{9}{7}} \div \frac{8}{15} + 2\frac{7}{24}$

$= \frac{7}{9} \times \frac{15}{8} + 2\frac{7}{24}$ 1

$= \frac{35}{24} + 2\frac{7}{24} = 1\frac{11}{24} + 2\frac{7}{24}$

$= 3\frac{18}{24} = \mathbf{3\frac{3}{4}}$ 2

total : 9

Problem 2. A piece of steel, 1.69 m long, is cut into three pieces in the ratio 2 to 5 to 6. Determine, in centimetres, the lengths of the three pieces.

Marks

Number of parts = 2 + 5 + 6 = 13

Length of one part = $\frac{1.69}{13}$ m = $\frac{169}{13}$ cm = 13 cm 1

Hence 2 parts ≡ 2 × 13 = 26

5 parts ≡ 5 × 13 = 65

6 parts ≡ 6 × 13 = 78

i.e. **2 : 5 : 6 :: 26 cm : 65 cm : 78 cm** 3

total : 4

Problem 3. Evaluate $\frac{576.29}{19.3}$ (a) correct to 4 significant figures

(b) correct to 1 decimal place

Marks

$\frac{576.29}{19.3}$ = 29.859585... by calculator

Hence (a) $\frac{576.29}{19.3}$ = **29.86**, correct to 4 significant figures | 1

(b) $\frac{576.29}{19.3}$ = **29.9**, correct to 1 decimal place | 1

total : 2

Problem 4. Determine, correct to 1 decimal place, 57% of 17.64 g.

Marks

57% of 17.64 g = $\frac{57}{100} \times 17.64$ g = 10.1 g, correct to 1 decimal place | 2

total : 2

Problem 5. Express 54.7 mm as a percentage of 1.15 m, correct to 3 significant figures.

Marks

54.7 mm as a percentage of 1.15 m is:

$\frac{54.7}{1150} \times 100\%$ = **4.76%**, correct to 3 significant figures | 3

total : 3

Problem 6. Evaluate the following: (a) $\frac{2^3 \times 2 \times 2^2}{2^4}$ (b) $\frac{\left(2^3 \times 16\right)^2}{(8 \times 2)^3}$ (c) $\left(\frac{1}{4^2}\right)^{-1}$

(d) $(27)^{-\frac{1}{3}}$ (e) $\frac{\left(\frac{3}{2}\right)^{-2} - \frac{2}{9}}{\left(\frac{2}{3}\right)^2}$

2

Marks

(a) $\frac{2^3 \times 2 \times 2^2}{2^4} = 2^{3+1+2-4} = 2^2 =$ **4** | 2

(b) $\frac{(2^3 \times 16)^2}{(8 \times 2)^3} = \frac{(2^3 \times 2^4)^2}{(2^3 \times 2)^3} = \frac{(2^7)^2}{(2^4)^3} = \frac{2^{14}}{2^{12}} = 2^{14-12} = 2^2 = \mathbf{4}$ 3

(c) $\left(\frac{1}{4^2}\right)^{-1} = (4^2)^{+1} = 4^2 = \mathbf{16}$ 3

(d) $(27)^{-\frac{1}{3}} = \frac{1}{27^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{27}} = \mathbf{\frac{1}{3}}$ 3

(e) $\frac{\left(\frac{3}{2}\right)^{-2} - \frac{2}{9}}{\left(\frac{2}{3}\right)^2} = \frac{\left(\frac{2}{3}\right)^2 - \frac{2}{9}}{\left(\frac{2}{3}\right)^2} = \frac{\frac{4}{9} - \frac{2}{9}}{\frac{4}{9}} = \frac{\frac{2}{9}}{\frac{4}{9}} = \frac{2}{9} \times \frac{9}{4} = \mathbf{\frac{1}{2}}$ 3

total : 14

Problem 7. Express the following in standard form: (a) 1623 (b) 0.076 (c) $145\frac{2}{5}$

Marks

(a) 1623 = $\mathbf{1.623 \times 10^3}$ 1

(b) 0.076 = $\mathbf{7.6 \times 10^{-2}}$ 1

(c) $145\frac{2}{5}$ = 145.4 = $\mathbf{1.454 \times 10^2}$ 1

total : 3

Problem 8. Determine the value of the following, giving the answer in standard form:

(a) $5.9 \times 10^2 + 7.31 \times 10^2$ (b) $2.75 \times 10^{-2} - 2.65 \times 10^{-3}$

Marks

(a) $5.9 \times 10^2 + 7.31 \times 10^2$ = 590 + 731 = 1321 = $\mathbf{1.321 \times 10^3}$ 2

(b) $2.75 \times 10^{-2} - 2.65 \times 10^{-3}$ = 0.0275 - 0.00265

= 0.02485 = $\mathbf{2.485 \times 10^{-2}}$ 2

total : 4

Problem 9. Convert the following binary numbers to decimal form:

(a) 1101 (b) 101101.0101

Marks

(a) $1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$= 8 + 4 + 0 + 1 = \mathbf{13_{10}}$ 2

(b) $101101.0101_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1$

$+ 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4}$

$= 32 + 0 + 8 + 4 + 0 + 1 + 0 + \frac{1}{4} + 0 + \frac{1}{16}$

$= \mathbf{45.3125_{10}}$ 3

total : 5

Problem 10. Convert the following decimal numbers to binary form:

(a) 27 (b) 44.1875

Marks

(a)

		Remainder
2	27	
2	13	1
2	6	1
2	3	0
2	1	1
	0	1

Hence $\mathbf{27_{10} = 11011_2}$ 2

(b)

		Remainder
2	44	
2	22	0
2	11	0
2	5	1
2	2	1
2	1	0
	0	1

Hence $44_{10} = 101100_2$ 2

$0.1875 \times 2 = 0.375$

$0.375 \times 2 = 0.75$

$0.75 \times 2 = 1.50$

$0.50 \times 2 = 1.00$

Hence $\mathbf{44.1875_{10} = 101100.0011_2}$ 2

total : 6

Problem 11. Convert the following decimal numbers to binary, via octal:

(a) 479 (b) 185.2890625

Marks

(a)

8 | 479 Remainder

8 | 59 7

8 | 7 3

0 7

7 3 7

From Table 3.1, page 19, $\mathbf{737_8 = 111\ 011\ 111_2}$ 2

(b)

8 | 185 Remainder

8 | 23 1

8 | 2 7

0 2

2 7 1

From Table 3.1, page 19, $271_8 = 010\ 111\ 001_2$ 2

$0.2890625 \times 8 = 2.3125$

$0.3125 \times 8 = 2.5$

$0.5 \times 8 = 4.0$

i.e. $0.2890625_{10} = .2\ 2\ 4_8 = .010\ 010\ 100_2$ from Table 3.1, page 19

Hence $\mathbf{185.2890625_{10} = 10111001.0100101_2}$ 2

total : 6

Problem 12. Convert (a) $5F_{16}$ into its decimal equivalent

(b) 132_{10} into its hexadecimal equivalent

(c) 110101011_2 into its hexadecimal equivalent

Marks

(a) $5F_{16} = 5 \times 16^1 + F \times 16^0$

$= 5 \times 16^1 + 15 \times 16^0 = 80 + 15 = \mathbf{95_{10}}$ 2

(b)

		Remainder
16	132	
16	8	4
	0	8

i.e. $\mathbf{132_{10} = 84_{16}}$ 2

(c) Grouping bits in 4's from the right gives:

$110101011_2 = 0001\ 1010\ 1011$

and assigning hexadecimal symbols to each group gives: 1 A B

Hence $\mathbf{110101011_2 = 1AB_{16}}$ 2

total : 6

Problem 13. Evaluate the following, each correct to 4 significant figure:

(a) 61.22^2 (b) $\frac{1}{0.0419}$ (c) $\sqrt{0.0527}$

Marks

(a) $61.22^2 = \mathbf{3748}$, correct to 4 significant figures 1

(b) $\frac{1}{0.0419} = \mathbf{23.87}$, correct to 4 significant figures 1

(c) $\sqrt{0.0527} = \mathbf{0.2296}$, correct to 4 significant figures 1

total : 3

Problem 14. Evaluate the following, each correct to 2 decimal places:

(a) $\left(\frac{36.2^2 \times 0.561}{27.8 \times 12.83}\right)^3$ (b) $\sqrt{\left(\frac{14.69^2}{\sqrt{17.42} \times 37.98}\right)}$

Marks

(a) $\left(\frac{36.2^2 \times 0.561}{27.8 \times 12.83}\right)^3 = \mathbf{8.76}$, correct to 2 decimal places 3

(b) $\sqrt{\left(\frac{14.69^2}{\sqrt{17.42} \times 37.98}\right)}$ = **1.17**, correct to 2 decimal places 4

total : 7

Problem 15. If 1.6 km = 1 mile, determine the speed of 45 miles/hour in kilometres per hour.

Marks

45 miles/hour = 45 × 1.6 km/h = **72 km/h** 3

total : 3

Problem 16. Evaluate B, correct to 3 significant figures, when W = 7.20, v = 10.0 and g = 9.81, given that B = $\frac{Wv^2}{2g}$

Marks

B = $\frac{Wv^2}{2g} = \frac{(7.20)(10.0)^2}{2(9.81)}$ = **36.7**, correct to 3 significant figures 3

total : 3

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 2 (Page 64)

This assignment covers the material contained in Chapters 5 to 8.

Problem 1. Evaluate $3xy^2z^3 - 2yz$ when $x = \frac{4}{3}$, $y = 2$ and $z = \frac{1}{2}$

	Marks
$3xy^2z^3 - 2yz = 3\left(\frac{4}{3}\right)(2)^2\left(\frac{1}{2}\right)^3 - 2(2)\left(\frac{1}{2}\right)$	
$= 2 - 2 = \mathbf{0}$	3

total : 3

Problem 2. Simplify the following: (a) $\frac{8a^2b\sqrt{c^3}}{(2a)^2\sqrt{b}\sqrt{c}}$ (b) $3x + 4 \div 2x + 5 \times 2 - 4x$

	Marks
(a) $\frac{8a^2b\sqrt{c^3}}{(2a)^2\sqrt{b}\sqrt{c}} = \frac{8a^2bc^{3/2}}{4a^2b^{1/2}c^{1/2}} = \mathbf{2b^{1/2}c}$ or $\mathbf{2\sqrt{b}\,c}$	3
(b) $3x + 4 \div 2x + 5 \times 2 - 4x = 3x + \frac{4}{2x} + 5 \times 2 - 4x$	
$= 3x + \frac{2}{x} + 10 - 4x$	
$= \mathbf{-x + \frac{2}{x} + 10}$ or $\mathbf{\frac{2}{x} - x + 10}$	3

total : 6

Problem 3. Remove the brackets in the following expressions and simplify:

(a) $(2x - y)^2$ (b) $4ab - [3\{2(4a - b) + b(2 - a)\}]$

	Marks
(a) $(2x - y)^2 = (2x - y)(2x - y) = 4x^2 - 2xy - 2xy + y^2$	1
$= \mathbf{4x^2 - 4xy + y^2}$	1
(b) $4ab - [3\{2(4a - b) + b(2 - a)\}] = 4ab - [3\{8a - 2b + 2b - ab\}]$	1
$= 4ab - [3\{8a - ab\}]$	
$= 4ab - [24a - 3ab]$	1
$= 4ab - 24a + 3ab$	

8

	Marks
$= \mathbf{7ab - 24a}$ or $\mathbf{a(7b - 24)}$	1

total : 5

Problem 4. Factorise $3x^2y + 9xy^2 + 6xy^3$

Marks

$3x^2y + 9xy^2 + 6xy^3 = \mathbf{3xy(x + 3y + 2y^2)}$ 3

total : 3

Problem 5. If x is inversely proportional to y and x = 12 when y = 0.4, determine (a) the value of x when y is 3, and (b) the value of y when x = 2

Marks

$x \propto \frac{1}{y}$ i.e. $x = \frac{k}{y}$

x = 12 when y = 0.4, hence $12 = \frac{k}{0.4}$ from which, k = (12)(0.4) = 4.8

(a) When y = 3, $x = \frac{k}{y} = \frac{4.8}{3} = \mathbf{1.6}$ 2

(b) When x = 2, $2 = \frac{4.8}{y}$ and $y = \frac{4.8}{2} = \mathbf{2.4}$ 2

total : 4

Problem 6. Factorise $x^3 + 4x^2 + x - 6$ using the factor theorem . Hence solve the equation $x^3 + 4x^2 + x - 6 = 0$

Marks

Let $f(x) = x^3 + 4x^2 + x - 6$

then f(1) = 1 + 4 + 1 - 6 = 0, hence (x - 1) is a factor

f(2) = 8 + 16 + 2 - 6 = 20

f(-1) = - 1 + 4 - 1 - 6 = - 4

f(-2) = - 8 + 16 - 2 - 6 = 0, hence (x + 2) is a factor

f(-3) = - 27 + 36 - 3 - 6 = 0, hence (x + 3) is a factor

Thus $\mathbf{x^3 + 4x^2 + x - 6 = (x - 1)(x + 2)(x + 3)}$ 3

If $x^3 + 4x^2 + x - 6 = 0$ then $(x - 1)(x + 2)(x + 3) = 0$

from which, **x = 1, -2 or -3** 3

total : 6

Problem 7. Use the remainder theorem to find the remainder when $2x^3 + x^2 - 7x - 6$ is divided by (a) $(x - 2)$ (b) $(x + 1)$

Hence factorise the cubic expression.

Marks

(a) When $2x^3 + x^2 - 7x - 6$ is divided by $(x - 2)$, the remainder is given by $ap^3 + bp^2 + cp + d$, where a = 2, b = 1, c = -7, d = -6 and p = 2,

i.e. **the remainder is**: $2(2)^3 + 1(2)^2 - 7(2) - 6$

$= 16 + 4 - 14 - 6 = \mathbf{0}$ 2

hence $(x - 2)$ is a factor of $2x^3 + x^2 - 7x - 6$

(b) When $2x^3 + x^2 - 7x - 6$ is divided by $(x + 1)$, the **remainder is** given

by: $2(-1)^3 + 1(-1)^2 - 7(-1) - 6 = -2 + 1 + 7 - 6 = \mathbf{0}$ 2

hence $(x + 1)$ is a factor of $2x^3 + x^2 - 7x - 6$

Either by dividing $2x^3 + x^2 - 7x - 6$ by $(x - 2)(x + 1)$ or by using the factor or remainder theorems the third factor is found to be $(2x + 3)$

Hence $\mathbf{2x^3 + x^2 - 7x - 6 = (x - 2)(x + 1)(2x + 3)}$ 3

total : 7

Problem 8. Simplify $\dfrac{6x^2 + 7x - 5}{2x - 1}$ by dividing out

Marks

$$\begin{array}{r|l} & 3x + 5 \\ 2x - 1 & 6x^2 + 7x - 5 \\ & 6x^2 - 3x \\ & \quad 10x - 5 \\ & \quad 10x - 5 \\ & \quad \;\; . \quad . \end{array}$$

Hence $\dfrac{6x^2 + 7x - 5}{2x - 1} = \mathbf{3x + 5}$ 5

total : 5

Problem 9. Resolve the following into partial fractions:

(a) $\frac{x-11}{x^2-x-2}$ (b) $\frac{3-x}{(x^2+3)(x+3)}$ (c) $\frac{x^3-6x+9}{x^2+x-2}$

Marks

(a)Let $\frac{x-11}{x^2-x-2} \equiv \frac{x-11}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$

Hence $x - 11 = A(x + 1) + B(x - 2)$

Let x = 2: $-9 = 3A$ hence **A = -3**

Let x = -1: $-12 = -3B$ hence **B = 4**

Hence $\mathbf{\frac{x-11}{x^2-x-2} = \frac{4}{(x+1)} - \frac{3}{(x-2)}}$ 6

(b)Let $\frac{3-x}{(x^2+3)(x+3)} \equiv \frac{Ax+B}{(x^2+3)} + \frac{C}{(x+3)} = \frac{(Ax+B)(x+3)+C(x^2+3)}{(x^2+3)(x+3)}$

Hence $3 - x = (Ax + B)(x + 3) + C(x^2 + 3)$

Let x = -3: $6 = 0 + 12C$ hence $\mathbf{C = \frac{1}{2}}$

x^2 coefficients: $0 = A + C$ hence $\mathbf{A = -\frac{1}{2}}$

x coefficients: $-1 = 3A + B$ hence $-1 = -\frac{3}{2} + B$ and $\mathbf{B = \frac{1}{2}}$

Hence $\mathbf{\frac{3-x}{(x^2+3)(x+3)}} = \frac{-\frac{1}{2}x+\frac{1}{2}}{(x^2+3)} + \frac{\frac{1}{2}}{(x+3)}$ or $\mathbf{\frac{1-x}{2(x^2+3)} + \frac{1}{2(x+3)}}$ 8

(c)Dividing out gives:

$$\begin{array}{r|l} & x - 1 \\ x^2 + x - 2 & x^3 \qquad - 6x + 9 \\ & \underline{x^3 + x^2 - 2x} \\ & \quad - x^2 - 4x + 9 \\ & \quad \underline{- x^2 - x + 2} \\ & \qquad -3x + 7 \end{array}$$

Hence $\frac{x^3-6x+9}{x^2+x-2} \equiv x - 1 + \frac{-3x+7}{x^2+x-2}$ 5

Let $\frac{-3x+7}{x^2+x-2} \equiv \frac{A}{x+2} + \frac{B}{x-1} \equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

from which, $-3x + 7 = A(x - 1) + B(x + 2)$

Let $x = -2$: $13 = -3A$ hence $\mathbf{A = -\frac{13}{3}}$

Let $x = 1$: $4 = 3B$ hence $\mathbf{B = \frac{4}{3}}$

Hence $\frac{-3x + 7}{x^2 + x - 2} = \frac{-13/3}{x + 2} + \frac{4/3}{x - 1}$

and $\mathbf{\frac{x^3 - 6x + 9}{x^2 + x - 2} = x - 1 - \frac{13}{3(x + 2)} + \frac{4}{3(x - 1)}}$ 5

total : 24

Problem 10. Solve the following equations:

(a) $3t - 2 = 5t - 4$ (b) $4(k - 1) - 2(3k + 2) + 14 = 0$

(c) $\frac{a}{2} - \frac{2a}{5} = 1$ (d) $\sqrt{\left(\frac{s + 1}{s - 1}\right)} = 2$

Marks

(a) $3t - 2 = 5t - 4$ from which, $4 - 2 = 5t - 3t$

i.e. $2 = 2t$ and $\mathbf{t = 1}$ 2

(b) If $4(k - 1) - 2(3k + 2) + 14 = 0$

then $4k - 4 - 6k - 4 + 14 = 0$

and $-4 - 4 + 14 = 6k - 4k$

i.e. $6 = 2k$ and $\mathbf{k} = \frac{6}{2} = \mathbf{3}$ 3

(c) $\frac{a}{2} - \frac{2a}{5} = 1$ hence $10\left(\frac{a}{2}\right) - 10\left(\frac{2a}{5}\right) = 10(1)$

i.e. $5a - 4a = 10$ and $\mathbf{a = 10}$ 4

(d) $\sqrt{\left(\frac{s + 1}{s - 1}\right)} = 2$ hence $\frac{s + 1}{s - 1} = (2)^2 = 4$ 1

and $s + 1 = 4(s - 1)$ 1

i.e. $s + 1 = 4s - 4$ 1

hence $4 + 1 = 4s - s = 3s$

and $5 = 3s$

from which, $\mathbf{s = \frac{5}{3}}$ or $\mathbf{1\frac{2}{3}}$ 1

total : 13

Problem 11. A rectangular football pitch has its length equal to twice its width and a perimeter of 360 m. Find its length and width.

	Marks
Since length l is twice the width w, then l = 2w	
Perimeter of pitch = 2l + 2w = 2(2w) + 2w = 360	1
Hence 6w = 360 and w = 60 m	
If w = 60 m, l = 2w = 120 m	
Hence **length = 120 m** and **width = 60 m**	3

total : 4

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 3 (Page 94)

This assignment covers the material contained in Chapters 9 to 12.

Problem 1. Solve the following pairs of simultaneous equations:

(a) $7x - 3y = 23$
$2x + 4y = -8$

(b) $3a - 8 + \frac{b}{8} = 0$

$b + \frac{a}{2} = \frac{21}{4}$

	Marks
(a) $7x - 3y = 23$ (1)	
$2x + 4y = -8$ (2)	
$4 \times$ (1) gives: $28x - 12y = 92$ (3)	
$3 \times$ (2) gives: $6x + 12y = -24$ (4)	
(3) + (4) gives: $34x = 68$	
from which, $\mathbf{x} = \frac{68}{34} = \mathbf{2}$	3
When x = 2 in equation (1): $14 - 3y = 23$	
from which, $-3y = 23 - 14 = 9$	
i.e. $\mathbf{y} = \frac{9}{-3} = \mathbf{-3}$	2
(b) $3a - 8 + \frac{b}{8} = 0$	
$b + \frac{a}{2} = \frac{21}{4}$	
i.e. $3a + \frac{b}{8} = 8$ (1)	
$\frac{a}{2} + b = \frac{21}{4}$ (2)	
$8 \times$ (1) gives: $24a + b = 64$ (3)	
$4 \times$ (2) gives: $2a + 4b = 21$ (4)	
$4 \times$ (3) gives: $96a + 4b = 256$ (5)	

(5) - (4) gives: $94a = 235$	
from which, $\mathbf{a} = \frac{235}{94} = \mathbf{2.5}$	4

When a = 2.5 in equation (1): $7.5 + \frac{b}{8} = 8$

and $\frac{b}{8} = 8 - 7.5 = 0.5$

from which, **b** = 8(0.5) = **4** 3

total : 12

Problem 2. In an engineering process two variables x and y are related by the equation $y = ax + \frac{b}{x}$ where a and b are constants. Evaluate a and b if y = 15 when x = 1 and y = 13 when x = 3

			Marks
When y = 15 and x = 1,	$15 = a + b$	(1)	1
When y = 13 and x = 3,	$13 = 3a + \frac{b}{3}$	(2)	1
3 × (2) gives;	$39 = 9a + b$	(3)	
(3) - (1) gives:	$24 = 8a$ from which, **a = 3**		1
When a = 3 in equation (1):	$15 = 3 + b$ from which, **b = 12**		1

total : 4

Problem 3. Transpose the following equations:

(a) $y = mx + c$ for m

(b) $x = \frac{2(y - z)}{t}$ for z

(c) $\frac{1}{R_T} = \frac{1}{R_A} + \frac{1}{R_B}$ for R_A

(d) $x^2 - y^2 = 3ab$ for y

(e) $K = \frac{p - q}{1 + pq}$ for q

Marks

(a) Since $y = mx + c$ then $y - c = mx$ and $\mathbf{m = \frac{y - c}{x}}$ 3

(b) $x = \frac{2(y - z)}{t}$ hence $xt = 2(y - z)$

from which, $\frac{xt}{2} = y - z$ and $\mathbf{z = y - \frac{xt}{2}}$ or $\mathbf{\frac{2y - xt}{2}}$ 4

(c) $\frac{1}{R_T} = \frac{1}{R_A} + \frac{1}{R_B}$ hence $\frac{1}{R_A} = \frac{1}{R_T} - \frac{1}{R_B} = \frac{R_B - R_T}{R_T R_B}$

and $\mathbf{R_A = \frac{R_T R_B}{R_B - R_T}}$ 4

(d) $x^2 - y^2 = 3ab$ from which, $x^2 - 3ab = y^2$

and $\mathbf{y = \sqrt{x^2 - 3ab}}$ 4

(e) $K = \frac{p - q}{1 + pq}$ from which, $K(1 + pq) = p - q$

thus $K + Kpq = p - q$

and $Kpq + q = p - K$

Then $q(Kp + 1) = p - K$

and $\mathbf{q = \frac{p - K}{Kp + 1}}$ 5

total : 20

Problem 4. The passage of sound waves through walls is governed by the equation:

$v = \sqrt{\frac{K + \frac{4}{3}G}{\rho}}$. Make the shear modulus G the subject of the formula.

Marks

$v = \sqrt{\frac{K + \frac{4}{3}G}{\rho}}$ from which, $v^2 = \frac{K + \frac{4}{3}G}{\rho}$

$v^2\rho = K + \frac{4}{3}G$

$v^2\rho - K = \frac{4}{3}G$

and $\mathbf{G = \frac{3}{4}(v^2\rho - K)}$ 4

total : 4

Problem 5. Solve the following equations by factorisation:

(a) $x^2 - 9 = 0$ (b) $2x^2 - 5x - 3 = 0$

Marks

(a) Since $x^2 - 9 = 0$ then $(x + 3)(x - 3) = 0$ and $x = \pm$ **3** 2

(b) $2x^2 - 5x - 3 = 0$ i.e. $(2x + 1)(x - 3) = 0$ 2

Hence $2x + 1 = 0$ and $x - 3 = 0$

i.e. $\mathbf{x = -\frac{1}{2}}$ **and** $\mathbf{x = 3}$ 2

total : 6

Problem 6. Determine the quadratic equation in x whose roots are 1 and -3

Marks

The quadratic equation whose roots are 1 and -3 is given by:

$(x - 1)(x + 3) = 0$ 2

i.e. $x^2 + 3x - x - 3 = 0$

i.e. $\mathbf{x^2 + 2x - 3 = 0}$ 2

total : 4

Problem 7. Solve the equation $4x^2 - 9x + 3 = 0$ correct to 3 decimal places

Marks

If $4x^2 - 9x + 3 = 0$ then $x = \frac{- -9 \pm \sqrt{[(-9)^2 - 4(4)(3)]}}{2(4)}$ 2

$= \frac{9 \pm \sqrt{81 - 48}}{8} = \frac{9 \pm \sqrt{33}}{8} = \frac{9 \pm 5.7446}{8}$

i.e. $x = \frac{9 + 5.7446}{8}$ and $x = \frac{9 - 5.7446}{8}$

Hence $\mathbf{x = 1.843}$ and $\mathbf{x = 0.407}$ 3

total : 5

Problem 8. The current i flowing through an electronic device is given by:

$i = 0.005\, v^2 + 0.014\, v$

where v is the voltage. Calculate the values of v when $i = 3 \times 10^{-3}$.

Marks

$i = 0.005\, v^2 + 0.014\, v$ and when $i = 3 \times 10^{-3}$ then

$3 \times 10^{-3} = 0.005\, v^2 + 0.014\, v$

i.e. $0.005\, v^2 + 0.014\, v - 3 \times 10^{-3} = 0$

and $5\, v^2 + 14\, v - 3 = 0$

from which, $v = \frac{-14 \pm \sqrt{[14^2 - 4(5)(-3)]}}{2(5)}$ 2

$= \frac{-14 \pm \sqrt{256}}{10} = \frac{-14+16}{10}$ and $\frac{-14-16}{10}$

i.e. $v = \frac{2}{10}$ and $\frac{-30}{10}$

Hence **voltage v = $\mathbf{\frac{1}{5}}$** (or 0.2) and **-3** 3

total : 5

Problem 9. Evaluate $\log_{16} 8$

Marks

Let $x = \log_{16} 8$ then $16^x = 8$ from the definition of a logarithm

i.e. $2^{4x} = 2^3$

from which, $4x = 3$ and $x = \frac{3}{4}$

Hence $\mathbf{\log_{16} 8 = \frac{3}{4}}$ 3

total : 3

Problem 10. Solve (a) $\log_3 x = -2$ (b) $\log 2x^2 + \log x = \log 32 - \log x$

Marks

(a) If $\log_3 x = -2$ then $\mathbf{x} = 3^{-2} = \frac{1}{3^2} = \mathbf{\frac{1}{9}}$ 2

(b) If $\log 2x^2 + \log x = \log 32 - \log x$

then $\log 2x^3 = \log \frac{32}{x}$ from the laws of logarithms

i.e. $2x^3 = \frac{32}{x}$ from which, $x^4 = \frac{32}{2} = 16$

and $\mathbf{x} = \sqrt[4]{16} = \pm \mathbf{2}$ 4

total : 6

Problem 11. Solve the following equations, each correct to 3 significant figures:

(a) $2^x = 5.5$ (b) $3^{2t-1} = 7^{t+2}$ (c) $3e^{2x} = 4.2$

Marks

(a) Since $2^x = 5.5$ then $\lg 2^x = \lg 5.5$

from which, $x \lg 2 = \lg 5.5$

and $$\mathbf{x} = \frac{\lg 5.5}{\lg 2} = \mathbf{2.46}$$ 3

(b) Since $3^{2t-1} = 7^{t+2}$ then $\lg 3^{2t-1} = \lg 7^{t+2}$

and $(2t - 1)\lg 3 = (t + 2)\lg 7$

$2t \lg 3 - \lg 3 = t \lg 7 + 2 \lg 7$

$2t \lg 3 - t \lg 7 = 2 \lg 7 + \lg 3$

$0.9542\, t - 0.8451\, t = 2.1673$

$0.1091\, t = 2.1673$

hence $$\mathbf{t} = \frac{2.1673}{0.1091} = \mathbf{19.9}$$ 5

(c) Since $3e^{2x} = 4.2$ then $e^{2x} = \frac{4.2}{3} = 1.4$

and $\ln e^{2x} = \ln 1.4$

i.e. $2x = \ln 1.4$

and $$\mathbf{x} = \frac{\ln 1.4}{2} = \mathbf{0.168}$$ 3

total : 11

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 4 (PAGE 126)

This assignment covers the material contained in chapters 13 to 16.

Problem 1. Evaluate the following, each correct to 4 significant figures:

(a) $e^{-0.683}$ (b) $\dfrac{5(e^{-2.73} - 1)}{e^{1.68}}$

	Marks
(a) $e^{-0.683} = \mathbf{0.5051}$	1
(b) $\dfrac{5(e^{-2.73} - 1)}{e^{1.68}} = \mathbf{-0.8711}$	2

total : 3

Problem 2. Expand xe^{3x} to six terms

Marks

$$xe^{3x} = x\left\{1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \frac{(3x)^5}{5!} + ..\right\}$$

$$= x\left\{1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5 + ..\right\}$$

$$= \mathbf{x + 3x^2 + \frac{9}{2}x^3 + \frac{9}{2}x^4 + \frac{27}{8}x^5 + \frac{81}{40}x^6 + ..}$$ 5

total : 5

Problem 3. Plot a graph of $y = \frac{1}{2}e^{-1.2x}$ over the range x = -2 to x = +1 and hence determine, correct to 1 decimal place, (a) the value of y when x = -0.75, and (b) the value of x when y = 4.0

Marks

A table of values is drawn up as shown below

x	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0
$y = \frac{1}{2}e^{-1.2x}$	5.51	3.02	1.66	0.91	0.5	0.27	0.15

2

A graph of $y = \frac{1}{2}e^{-1.2x}$ is shown in Figure 1

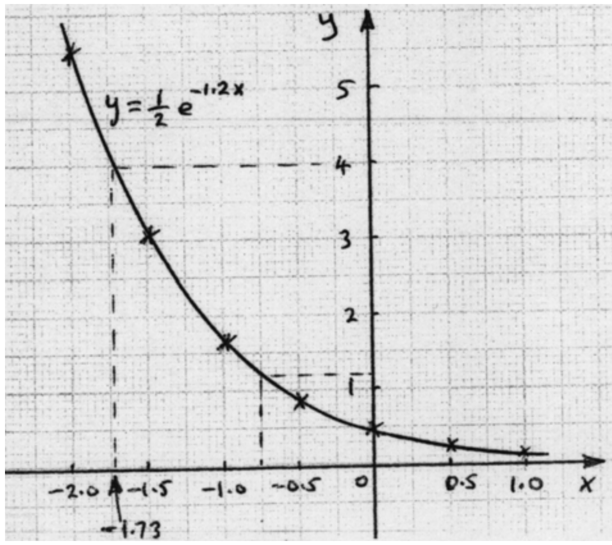


Figure 1 2

(a) When x = -0.75, **y = 1.2** 1

(b) When y = 4.0, **x = -1.7** 1

total : 6

Problem 4. Evaluate the following, each correct to 3 decimal places:

(a) ln 0.0753 (b) $\frac{\ln 3.68 - \ln 2.91}{4.63}$

Marks

(a) ln 0.0753 = **-2.586** 1

(b) $\frac{\ln 3.68 - \ln 2.91}{4.63}$ = **0.051** 1

total : 2

Problem 5. Two quantities x and y are related by the equation $y = ae^{-kx}$, where a and k are constants. Determine, correct to 1 decimal place, the value of y when a = 2.114, k = -3.20 and x = 1.429

Marks

$y = ae^{-kx} = 2.114\ e^{-(-3.20)(1.429)}$ = **204.7** 3

total : 3

Problem 6. Determine the 20th term of the series 15.6, 15, 14.4, 13.8, ...

Marks

The 20th term is given by: a + (n - 1)d i.e. $15.6 + (20 - 1)(-0.6)$

$= 15.6 - 19(0.6) = 15.6 - 11.4 =$ **4.2** 3

total : 3

Problem 7. The sum of 13 terms of an arithmetic progression is 286 and the common difference is 3. Determine the first term of the series.

Marks

$S_n = \frac{n}{2}[2a + (n - 1)d]$

i.e. $286 = \frac{13}{2}[2a + (13 - 1)3]$ 1

$286 = \frac{13}{2}[2a + 36]$

$\frac{286 \times 2}{13} = 2a + 36$ i.e. $44 - 36 = 2a$

from which, first term **a** $= \frac{44 - 36}{2} =$ **4** 3

total : 4

Problem 8. Determine the 11^{th} term of the series 1.5, 3, 6, 12, ..

Marks

The 11^{th} term is given by : ar^{n-1} where a = 1.5 and common ratio r = 2

i.e. 11^{th} term $= (1.5)(2)^{11-1} =$ **1536** 2

total : 2

Problem 9. A machine is to have seven speeds ranging from 25 rev/min to 500 rev/min. If the speeds form a geometric progression, determine their value, each correct to the nearest whole number.

Marks

The G.P. of n terms is given by: $a, ar, ar^2, .. ar^{n-1}$

The first term a = 25 rev/min

The seventh term is given by ar^{7-1} which is 500 rev/min

i.e. $ar^6 = 500$ from which, $r^6 = \frac{500}{a} = \frac{500}{25} = 20$

thus the common ratio $r = \sqrt[6]{20} = 1.64755$ 2

The first term is 25 rev/min

The second term $ar = (25)(1.64755) = 41.19$

The third term $ar^2 = (25)(1.64755)^2 = 67.86$

The fourth term $ar^3 = (25)(1.64755)^3 = 111.80$

The fifth term $ar^4 = (25)(1.64755)^4 = 184.20$

The sixth term $ar^5 = (25)(1.64755)^5 = 303.48$

Hence, correct to the nearest whole number the speeds of the machine

are: **25, 41, 68, 112, 184, 303 and 500 rev/min** 6

total : 8

Problem 10. Use the binomial series to expand $(2a - 3b)^6$

Marks

$$(2a - 3b)^6 = (2a)^6 + 6(2a)^5(-3b) + \frac{6(5)}{2!}(2a)^4(-3b)^2 + \frac{6(5)(4)}{3!}(2a)^3(-3b)^3$$

$$+ \frac{6(5)(4)(3)}{4!}(2a)^2(-3b)^4 + \frac{6(5)(4)(3)(2)}{5!}(2a)(-3b)^5 + (-3b)^6$$ 4

$$= \mathbf{64a^6 - 576a^5b + 2160a^4b^2 - 4320a^3b^3 + 4860a^2b^4}$$

$$\mathbf{- 2916ab^5 + 729b^6}$$ 3

total : 7

Problem 11. Expand the following in ascending powers of t as far as the term in t^3

(a) $\frac{1}{1+t}$ (b) $\frac{1}{\sqrt{(1-3t)}}$.

For each case, state the limits for which the expansion is valid

Marks

(a) $$\frac{1}{1+t} = (1+t)^{-1} = 1 + (-1)t + \frac{(-1)(-2)}{2!}t^2 + \frac{(-1)(-2)(-3)}{3!}t^3 + ..$$

$$= \mathbf{1 - t + t^2 - t^3 + ..}$$ 3

The expansion is valid when $|\mathbf{t}| < \mathbf{1}$ or $\mathbf{-1 < t < 1}$ 2

(b) $\dfrac{1}{\sqrt{(1-3t)}} = (1-3t)^{-\frac{1}{2}} = 1 + \left(-\dfrac{1}{2}\right)(-3t) + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-3t)^2$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-3t)^3 + ..$$

$$= \mathbf{1 + \frac{3}{2}t + \frac{27}{8}t^2 + \frac{135}{16}t^3 + ..}$$ 3

The expansion is valid when $|\mathbf{3t}| < \mathbf{1}$

i.e. $|\mathbf{t}| < \dfrac{\mathbf{1}}{\mathbf{3}}$ or $\mathbf{-\dfrac{1}{3} < x < \dfrac{1}{3}}$ 2

total : 10

Problem 12. The modulus of rigidity G is given by G = $\dfrac{R^4\theta}{L}$ where R is the radius, θ the angle of twist and L the length. Find the approximate percentage error in G when R is measured 1.5% too large, θ is measure 3% too small and L is measured 1% too small

Marks

The new values of R, θ and L are (1 + 0.015)R, (1 - 0.03)θ and (1 - 0.01)L

New modulus of rigidity = $\dfrac{[(1+0.015)R]^4[(1-0.03)\theta]}{[(1-0.01)L]}$ 3

$$= [(1 + 0.015)R]^4 [(1 - 0.03)\theta][(1 - 0.01)L]^{-1}$$

$$= (1 + 0.015)^4 R^4 (1 - 0.03)\theta(1 - 0.01)^{-1} L^{-1}$$

$$= (1 + 0.015)^4 (1 - 0.03)(1 - 0.01)^{-1} R^4 \theta L^{-1}$$

$$\approx [1 + 4(0.015)][1 - 0.03][1 - (-1)(0.01)] \frac{R^4\theta}{L}$$

neglecting products of small terms

$$\approx [1 + 0.06 - 0.03 + 0.01]G$$

$$= (1 + 0.04)G$$

i.e. **G is increased by 4%** 3

total : 6

Problem 13. The solution to a differential equation associated with the path taken by a projectile for which the resistance to motion is proportional to the velocity is given by: $y = 2.5(e^{x} - e^{-x}) + x - 25$

Use Newton's method to determine the value of x, correct to 2 decimal places, for which the value of y is zero.

Marks

If y = 0, $2.5e^{x} - 2.5e^{-x} + x - 25 = 0$

Let $f(x) = 2.5e^{x} - 2.5e^{-x} + x - 25$

$f(0) = 2.5 - 2.5 + 0 - 25 = -25$

$f(1) = -18.12$

$f(2) = -4.866$

$f(3) = 28.09$

Hence a root lies between x = 2 and x = 3. Let $r_1 = 2.2$ — 4

$f'(x) = 2.5e^{x} + 2.5e^{-x} + 1$

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)} = 2.2 - \frac{2.5e^{2.2} - 2.5e^{-2.2} + 2.2 - 25}{2.5e^{2.2} + 2.5e^{-2.2} + 1}$$

$$= 2.2 - \frac{-0.514474146}{23.83954165} = 2.222$$

$$r_3 = 2.222 - \frac{f(2.222)}{f'(2.222)} = 2.222 - \frac{0.01542961}{24.33539015} = \mathbf{2.221}$$

Hence **x = 2.22**, correct to 2 decimal places — 7

total : 11

TOTAL ASSIGNMENT MARKS: 70

ASSIGNMENT 5 (PAGE 168)

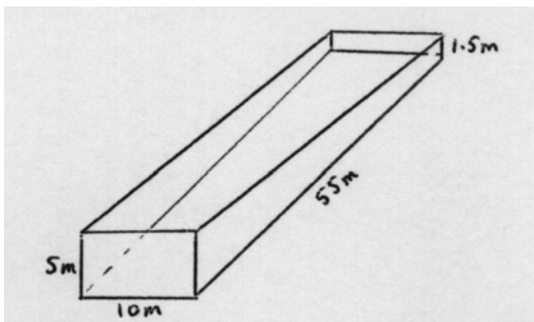
This assignment covers the material contained in chapters 17 to 20.

Problem 1. A swimming pool is 55 m long and 10 m wide. The perpendicular depth at the deep end is 5 m and at the shallow end is 1.5 m, the slope from one end to the other being uniform. The inside of the pool needs two coats of a protective paint before it is filled with water. Determine how many litres of paint will be needed if 1 litre covers 10 m^2

Marks

A sketch of the pool is shown in Figure 2.

Figure 2



Area to be painted = $(55 \times 10)+(10 \times 5)+(10 \times 1.5)+ 2\left[\frac{1}{2}(5 + 1.5)(55)\right]$

$= 550 + 50 + 15 + 357.5 = 972.5 \text{ m}^2$ 4

For two coats of paint, area to be covered = $2 \times 972.5 = 1945 \text{ m}^2$ 1

If 1 litre covers 10 m^2 then $\frac{1945}{10}$ = **194.5 litres will be needed** 2

total : 7

Problem 2. A steel template is of the shape shown in Fig. A10.1, the circular area being removed. Determine the area of the template, in square centimetres, correct to 1 decimal place.

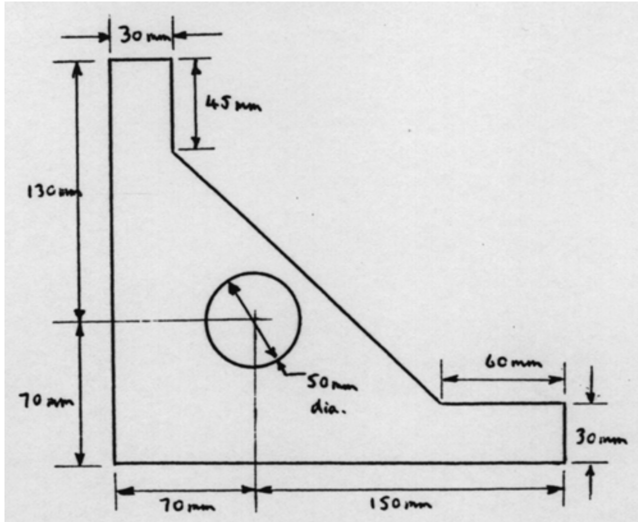


Figure A5.1

Marks

Figure A5.1 is re-drawn as shown in Figure 3

Area of Figure 3 = $(30 \times 200) + (190 \times 30) + \frac{1}{2}(130)(125) - \pi\frac{50^2}{4}$ 4

since the circle is removed

$= 6000 + 5700 + 8125 - 1963.5 = 17861.5\ mm^2$

= **178.6 cm^2**, correct to 1 decimal place 3

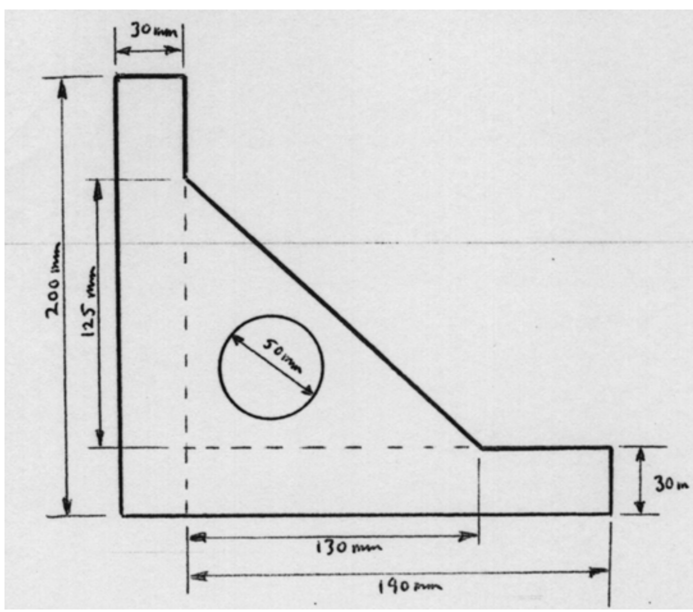


Figure 3

total : 7

Problem 3. The area of a plot of land on a map is 400 mm^2. If the scale of the map is 1 to 50000, determine the true area of the land in hectares. (1 hectare = 10^4 m^2)

Marks

True area = $(400)(50000)^2$ mm^2 = $\frac{(400)(50000)^2}{10^6}$ m^2 = 10^6 m^2

= $\frac{10^6}{10^4}$ hectare = 100 ha | 3

total : 3

Problem 4. Determine the shaded area in Figure A5.2, correct to the nearest square centimetre.

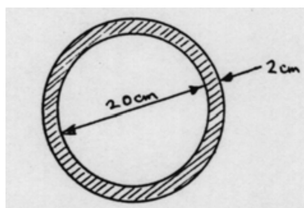


Figure A5.2

Marks

Shaded area in Figure A5.2 = $\frac{\pi(22)^2}{4} - \frac{\pi(20)^2}{4}$ = $\frac{\pi}{4}(22^2 - 20^2)$ = **66 cm^2** | 3

total : 3

Problem 5. Determine the diameter of a circle whose circumference is 178.4 cm.

Marks

Circumference c = πd, from which, diameter d = $\frac{c}{\pi}$ = $\frac{178.4}{\pi}$ = **56.79 cm** | 2

total : 2

Problem 6. Convert (a) 125°47′ to radians

(b) 1.724 radians to degrees and minutes

Marks

(a) 125°47′ = $125\frac{47}{60}$° = $(125\frac{47}{60} \times \frac{\pi}{180})$radians = **2.195 rad** | 1

(b) 1.724 rad = $1.724 \times \frac{180}{\pi}$ = 98.7779° = **98°47′** | 1

total : 2

Problem 7. Calculate the length of metal strip needed to make the clip shown in Figure A5.3

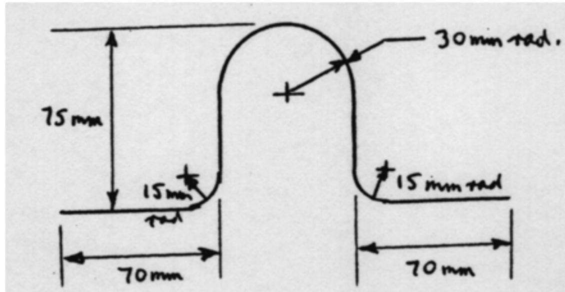


Figure A5.3

Marks

Length of metal strip $= (70 - 15) + \frac{1}{4}[2\pi(15)] + (75 - 15 - 30)$

$+ \frac{1}{2}[2\pi(30)] + (75 - 15 - 30) + \frac{1}{4}[2\pi(15)]$

$+ (70 - 15)$ 4

$= 55 + 23.56 + 30 + 94.25 + 30 + 23.56 + 55$

$=$ **311.4 mm** 2

total : 6

Problem 8. A lorry has wheels of radius 50 cm. Calculate the number of complete revolutions a wheel makes (correct to the nearest revolution) when travelling 3 miles. (Assume 1 mile = 1.6 km).

Marks

3 miles $= 3 \times 1.6$ km $= 4.8$ km $= 4800$ m 1

Circumference of wheel $= 2\pi r = 2\pi(0.5) = \pi$ m 1

Number of revolutions of wheel in 4800 m $= \frac{4800}{\pi} = 1527.9$

= **1528 revolutions**, correct to the nearest revolution 3

total : 5

Problem 9. The equation of a circle is: $x^2 + y^2 + 12x - 4y + 4 = 0$. Determine (a) the diameter of the circle, and (b) the co-ordinates of the centre of the circle.

Marks

The equation of a circle, centre (a, b), radius r is given by:

$$(x - a)^2 + (y - b)^2 = r^2$$

i.e. of the general form $x^2 + y^2 + 2ex + 2fy + c = 0$

Hence when the equation is $x^2 + y^2 + 12x - 4y + 4 = 0$

then $a = -\frac{2e}{2} = -\frac{12}{2} = -6$, $b = -\frac{2f}{2} = -\frac{-4}{2} = +2$

and $r = \sqrt{a^2 + b^2 - c} = \sqrt{(-6)^2 + (2)^2 - 4} = \sqrt{36} = 6$

Hence (a) the diameter of the circle is 12 2

(b) the centre of the circle is at (-6, 2) 3

total : 5

Problem 10. Determine the volume (in cubic metres) and the total surface area (in square metres) of a solid metal cone of base radius 0.5 m and perpendicular height 1.20 m. Give answers correct to 2 decimal places.

Marks

The cone is shown in Figure 4.

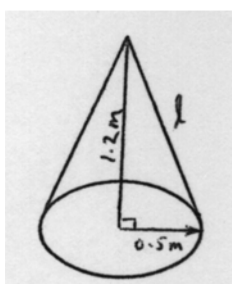


Figure 4

Volume of cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(0.5)^2(1.20) = \mathbf{0.31\ m^3}$ 2

Total surface area of cone = $\pi rl + \pi r^2$ where $l = \sqrt{0.5^2 + 1.2^2} = 1.3$ m 1

$= \pi(0.5)(1.3) + \pi(0.5)^2$

$= 0.65\pi + 0.25\pi$

$= 0.90\pi$

$= \mathbf{2.83\ m^2}$ 2

total : 5

Problem 11. Calculate the total surface area of a 10 cm by 15 cm rectangular pyramid of height 20 cm.

Marks

The pyramid is shown in Figure 5.

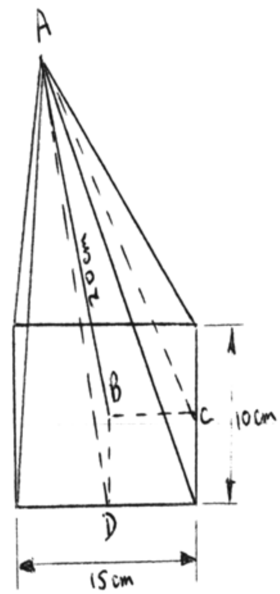


Figure 5

In Figure 5, AC = $\sqrt{20^2 + 7.5^2}$ = 21.36 cm 1

and AD = $\sqrt{20^2 + 5^2}$ = 20.62 cm 1

Hence total surface area = $2\left[\frac{1}{2}(10)(21.36)\right] + 2\left[\frac{1}{2}(15)(20.62)\right] + (15)(10)$

= 213.6 + 309.3 + 150 = **672.9 cm^2** 3

total : 5

Problem 12. A water container is of the form of a central cylindrical part 3.0 m long and diameter 1.0 m, with a hemispherical section surmounted at each end as shown in Figure A5.4. Determine the maximum capacity of the container, correct to the nearest litre. (1 litre = 1000 cm^3)

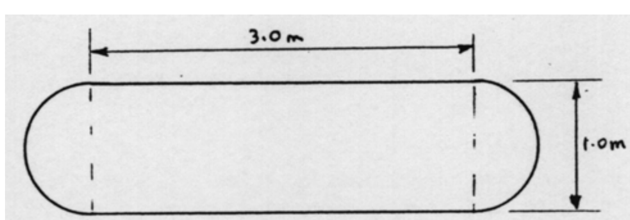


Figure A5.4

Marks

Volume of container = $\pi(0.5)^2(3.0) + \frac{4}{3}\pi(0.5)^3$

$= 2.3562 + 0.5236 = 2.8798 \text{ m}^3 = 2.8798 \times 10^6 \text{ cm}^3$ 3

Capacity, in litres = $\frac{2.8798 \times 10^6}{1000}$ = **2880 l**, correct to the nearest litre 2

total : 5

Problem 13. Find the total surface area of a bucket consisting of an inverted frustum of a cone, of slant height 35.0 cm and end diameters 60.0 cm and 40.0 cm.

Marks

The bucket is shown in Figure 6.

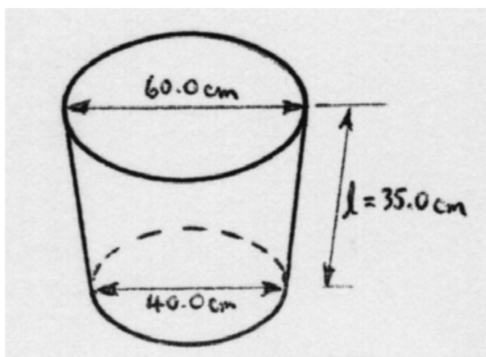


Figure 6

Surface area of bucket = $\pi l(R + r) + \pi r^2$

$= \pi(35)(30 + 20) + \pi(20)^2$

= $\mathbf{2150\pi \text{ cm}^2}$ or $\mathbf{6754 \text{ cm}^2}$ 4

total : 4

Problem 14. A boat has a mass of 20000 kg. A model of the boat is made to a scale of 1 to 80. If the model is made of the same material as the boat, determine the mass of the model (in grams).

Marks

Mass of model = $\left(\frac{1}{80}\right)^3(20000)$ kg = 0.039 kg = **39 g** 3

total : 3

Problem 15. Plot a graph of $y = 3x^2 + 5$ from x = 1 to x = 4. Estimate, correct to 2 decimal places, using 6 intervals, the area enclosed by the curve, the ordinates x = 1 and x = 4, and the x-axis by (a) the trapezoidal rule, (b) the mid-ordinate rule, and (c) Simpson's rule.

Marks

A table of values is shown below and a graph plotted as shown in Figure 7.

x	1	1.5	2	2.5	3	3.5	4
$y = 3x^2 + 5$	8	11.75	17	23.75	32	41.75	53

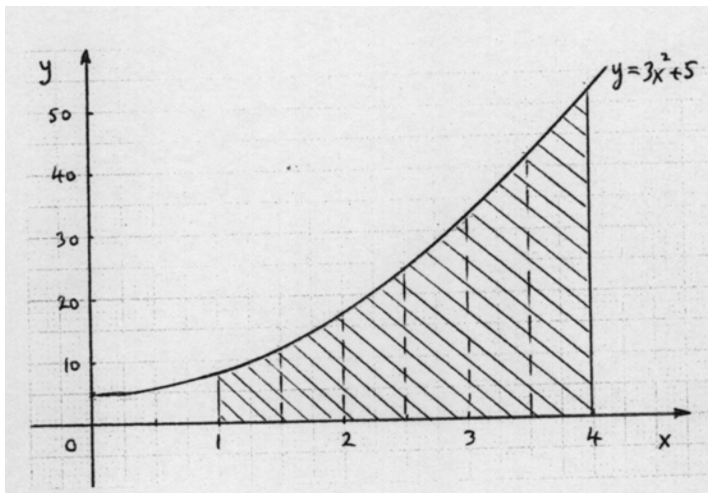


Figure 7

(a) Since six intervals are used ordinates lie at 1, 1.5, 2, 2.5, ..

By the **trapezoidal rule**,

$$\text{shaded area} \approx (0.5)\left\{\frac{1}{2}(8 + 53) + 11.75 + 17 + 23.75 + 32 + 41.75\right\}$$

= **78.38 square units** 4

(b) With the mid-ordinate rule, ordinates occur at 1.25, 1.75, 2.25, 2.75, 3.25 and 3.75

x	1.25	1.75	2.25	2.75	3.25	3.75
$y = 3x^2 + 5$	9.6875	14.1875	20.1875	27.6875	36.6875	47.1875

By the **mid-ordinate rule**,

$$\text{shaded area} \approx (0.5)(9.6875 + 14.1875 + 20.1875 + 27.6875 + 36.6875 + 47.1875)$$

= **77.81 square units** 4

(c) By **Simpson's rule**,

$$\text{shaded area} \approx \frac{1}{3}(0.5)\{(8 + 53) + 4(11.75 + 23.75 + 41.75) + 2(17 + 32)\}$$

$$= \frac{1}{3}(0.5)(61 + 309 + 98) = \textbf{78 square units}$$ 4

total : 12

Problem 16. A vehicle starts from rest and its velocity is measured every second for 6 seconds, with the following results:

Time t (s)	0	1	2	3	4	5	6
Velocity v (m/s)	0	1.2	2.4	3.7	5.2	6.0	9.2

Using Simpson's rule, calculate (a) the distance travelled in 6 s (i.e. the area under the v/t graph) and (b) the average speed over this period.

Marks

(a) Distance travelled

$$= \frac{1}{3}(1)[(0 + 9.2) + 4(1.2 + 3.7 + 6.0) + 2(2.4 + 5.2)]$$

$$= \frac{1}{3}[9.2 + 43.6 + 15.2] = \textbf{22.67 m}$$ 4

(b) Average speed $= \frac{22.67\ \text{m}}{6\ \text{s}} = \textbf{3.78 m/s}$ 2

total : 6

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 6 (PAGE 198)

This assignment covers the material contained in chapters 21 to 23.

Problem 1. Figure A6.l shows a plan view of a kite design. Calculate the lengths of the dimensions shown as a and b.

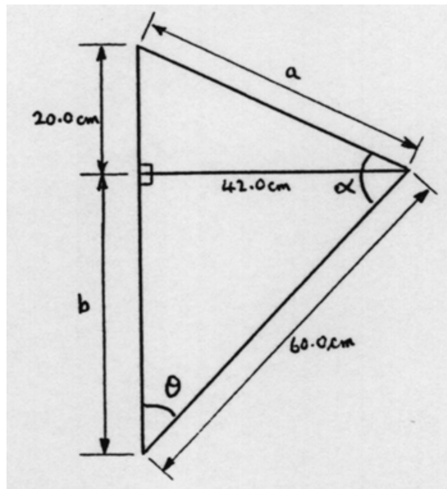


Figure A6.1

Marks

a = $\sqrt{42.0^2 + 20.0^2}$ = **46.52 cm** 2

b = $\sqrt{60.0^2 - 42.0^2}$ = **42.85 cm** 2

total : 4

Problem 2. In Figure A6.1 of Problem 1, evaluate (a) angle θ (b) angle α

Marks

(a) $\sin\theta = \frac{42.0}{60.0}$ from which, $\theta = \sin^{-1}\left(\frac{42.0}{60.0}\right)$ = **44.43°** 2

(b) In Figure 8, $\beta = \tan^{-1}\left(\frac{42.0}{20.0}\right)$ = 64.54°

∠ABC = 180° - 90° - 64.54° = 25.46°

and ∠CBD = 180° - 90° - 44.43° = 45.57°

Hence α = 25.46° + 45.57° = **71.03°** 3

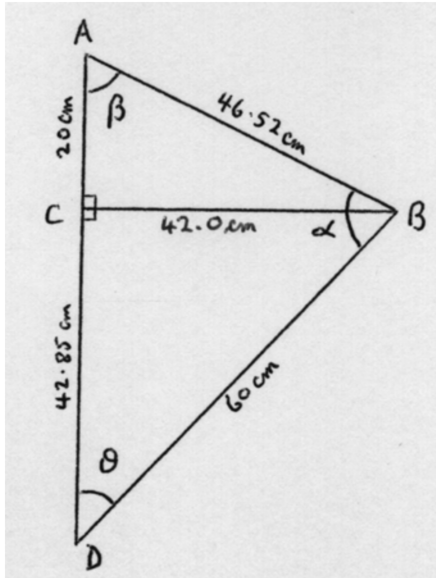


Figure 8

total : 5

Problem 3. Determine the area of the plan view of a kite shown in Figure A6.l of Problem 1.

Marks

Area of plan view = $\frac{1}{2}(42.0)(20.0) + \frac{1}{2}(42.0)(42.85)$ from Figure 8

= 420 + 899.85 = **1320 cm²** 4

total : 4

Problem 4. If the angle of elevation of the top of a 25 m perpendicular building from point A is measured as 27°, determine the distance to the building. Calculate also the angle of elevation at a point B, 20 m closer to the building than point A.

Marks

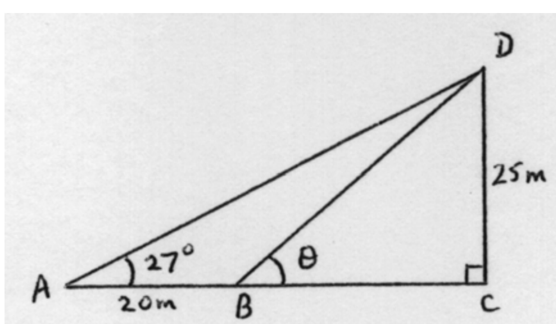


Figure 9

In Figure 9, $\tan 27° = \frac{25}{AC}$,

hence **distance to building**, $AC = \frac{25}{\tan 27°}$ = **49.07 m** 2

In Figure 9, BC = 49.07 - 20 = 29.07 m

Hence angle of elevation at B, $\theta = \tan^{-1}\left(\frac{25}{29.07}\right)$ = **40.70°** 3

total : 5

Problem 5. Evaluate, each correct to 4 significant figures:

(a) sin 231.78° (b) cos 151°16′ (c) $\tan\frac{3\pi}{8}$

Marks

(a) sin 231.78° = **-0.7856** 1

(b) cos 151°16′ = $\cos 151\frac{16}{60}°$ = **-0.8769** 1

(c) $\tan\frac{3\pi}{8}$ = **2.414** 1

total : 3

Problem 6. Sketch the following curves labelling relevant points:

(a) y = 4 cos(θ + 45°) (b) y = 5 sin(2t - 60°)

Marks

(a) y = 4 cos(θ + 45°) is sketched in Figure 10.

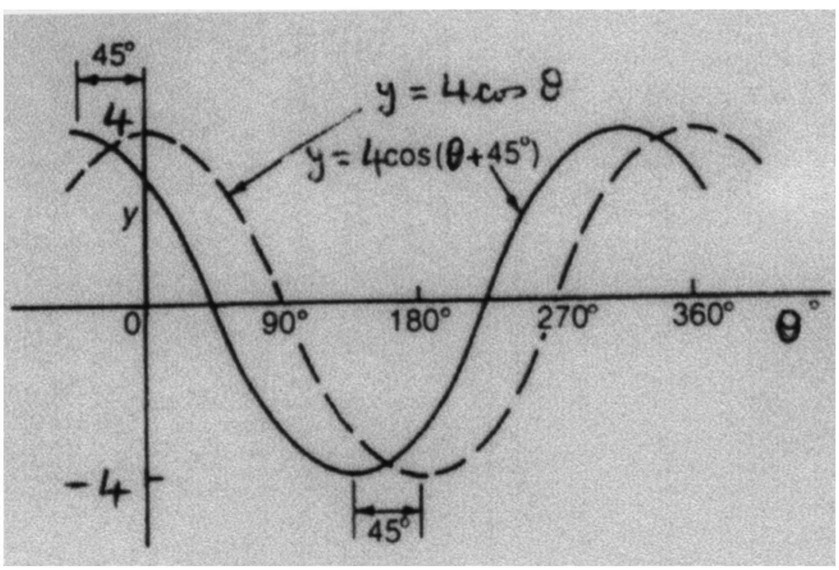


Figure 10 3

(b) y = 5 sin(2t - 60°) is sketched in Figure 11.

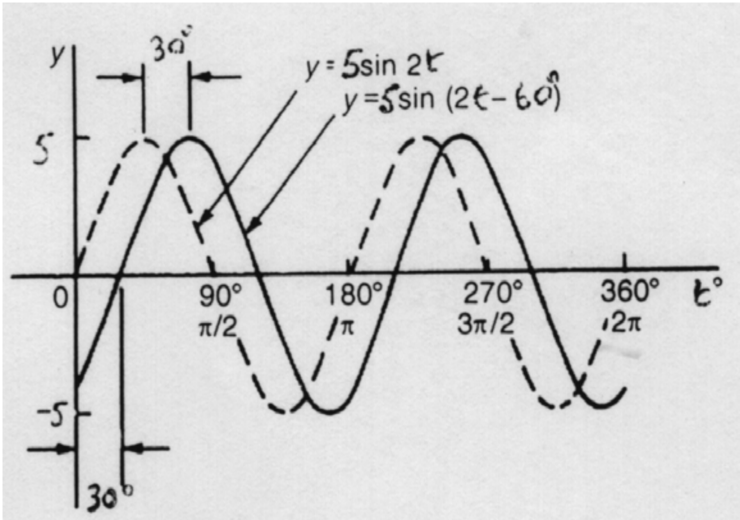


Figure 11

3

total : 6

Problem 7. Solve the following equations in the range 0° to 360°

(a) $\sin^{-1}(-0.4161) = x$ (b) $\cot^{-1}(2.4198) = \theta$

Marks

(a) If $\sin^{-1}(-0.4161) = x$ then from Figure 12, $\alpha = 24.59°$

and **x = 204.59° and 335.41°** 3

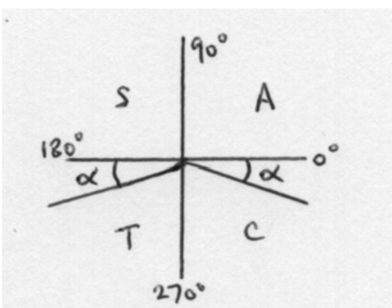


Figure 12

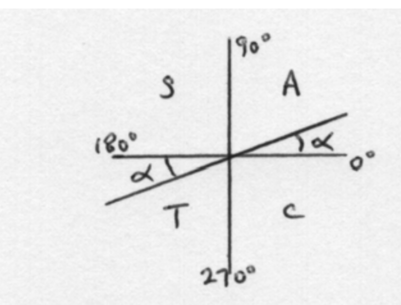


Figure 13

(b) If $\cot^{-1}(2.4198) = \theta$, then $\tan^{-1}\left(\frac{1}{2.4198}\right) = \theta$, and from Figure 13,

$\alpha = 22.45°$ and **θ = 22.45° and 202.45°** 3

total : 6

Problem 8. The current in an alternating current circuit at any time t seconds is given by: $i = 120 \sin(100\pi t + 0.274)$ amperes. Determine:

(a) the amplitude, periodic time, frequency and phase angle (with reference to $120 \sin 100\pi t$)

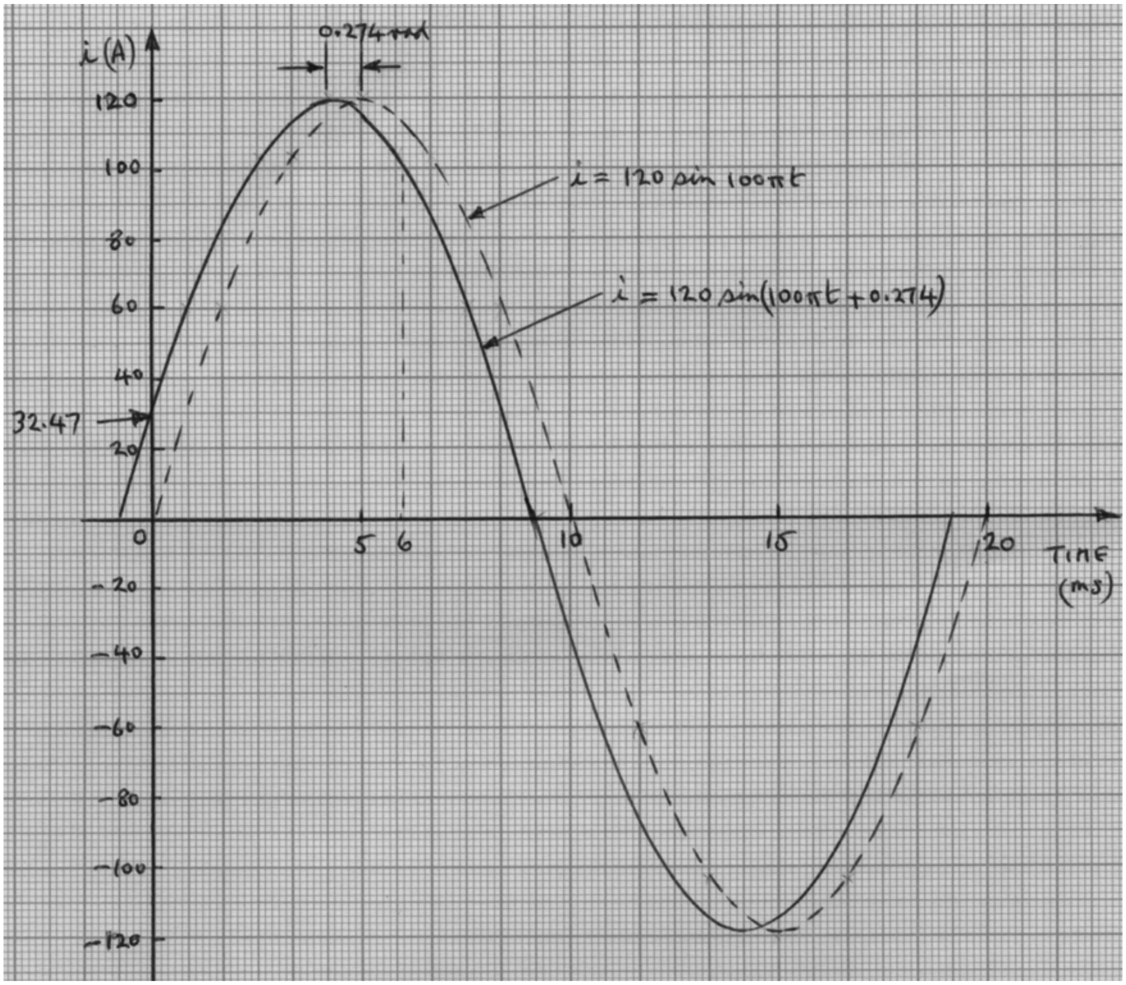
(b) the value of current when t = 0

(c) the value of current when t = 6 ms

(d) the time when the current first reaches 80 A

Sketch one cycle of the oscillation.

	Marks
(a) **Amplitude = 120 A**	1
$\omega = 100\pi$, hence **periodic time T** $= \dfrac{2\pi}{100\pi} = \dfrac{1}{50}$ s $=$ **20 ms**	1
Frequency f = 50 Hz	1
Phase angle = 0.274 rad = 15.70° leading	1
(b) When t = 0, **i** $= 120 \sin 0.274 =$ **32.47 A**	2
(c) When t = 6 ms, **i** $= 120 \sin(100\pi \times 6 \times 10^{-3} + 0.274)$	
$= 120 \sin 2.1589556 =$ **99.84 A**	3
(d) When i = 80 A, $80 = 120 \sin(100\pi t + 0.274)$	
from which, $\dfrac{80}{120} = \sin(100\pi t + 0.274)$	
and $\sin^{-1}\dfrac{80}{120} = 100\pi t + 0.274$	
i.e. $0.72972766 = 100\pi t + 0.274$	
Hence, $0.72972766 - 0.274 = 100\pi t$	
and **time t** $= \dfrac{0.72972766 - 0.274}{100\pi} =$ **1.451 ms**	4
One cycle of the current waveform is shown in Figure 14.	



4

Figure 14 **total : 17**

Problem 9. Change the following Cartesian co-ordinates into polar co-ordinates, correct to 2 decimal places, in both degrees and in radians:

(a) (-2.3, 5.4) (b) (7.6, -9.2)

Marks

(a) From Figure 15, $r = \sqrt{2.3^2 + 5.4^2} = 5.87$ and $\alpha = \tan^{-1}\frac{5.4}{2.3} = 66.93°$

Hence $\theta = 180° - 66.93° = 113.07°$

Thus (-2.3, 5.4) = **(5.87, 113.07°)** or **(5.87, 1.97 rad)** 3

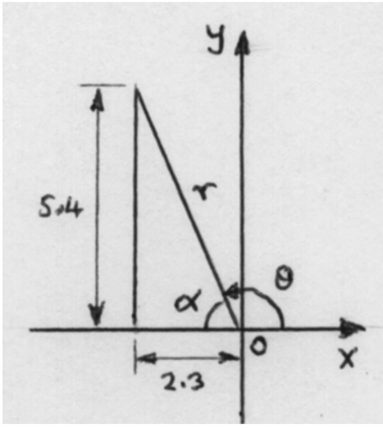


Figure 15

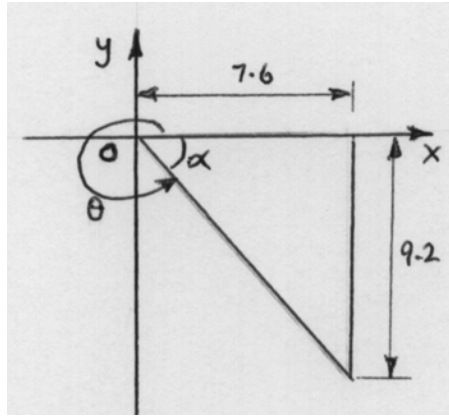


Figure 16

(b) From Figure 16, r = $\sqrt{7.6^2 + 9.2^2}$ = 11.93 and $\alpha = \tan^{-1}\frac{9.2}{7.6}$ = 50.44°

Hence θ = 360° - 50.44° = 309.56°

Thus (7.6, -9.2) = **(11.93, 309.56°)** or **(11.93, 5.40 rad)** 3

total : 6

Problem 10. Change the following polar co-ordinates into Cartesian co-ordinates, correct to 3 decimal places:

(a) (6.5, 132°) (b) (3, 3 rad)

Marks

(a) (6.5, 132°) = (6.5 cos 132°, 6.5 sin 132°) = **(-4.349, 4.830)** 2

(b) (3, 3 rad) = (3 cos 3, 3 sin 3) = **(-2.970, 0.423)** 2

total : 4

TOTAL ASSIGNMENT MARKS: 60

ASSIGNMENT 7 (PAGE 224)

This assignment covers the material contained in chapters 24 to 26.

Problem 1. A triangular plot of land ABC is shown in Figure A7.1. Solve the triangle and determine its area.

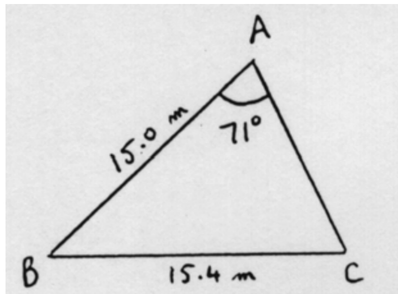


Figure A7.1

Marks

Using the sine rule: $\dfrac{15.4}{\sin 71^\circ} = \dfrac{15.0}{\sin C}$

from which, $\sin C = \dfrac{15.0 \sin 71^\circ}{15.4} = 0.9210$

and $\angle \mathbf{C} = \sin^{-1} 0.9210 = \mathbf{67.07^\circ}$ 3

Hence $\angle \mathbf{B} = 180^\circ - 71^\circ - 67.07^\circ = \mathbf{41.93^\circ}$ 1

By the cosine rule: $b^2 = a^2 + c^2 - 2ac \cos B$

$= 15.4^2 + 15.0^2 - 2(15.4)(15.0)\cos 41.93^\circ$

$= 118.45$

and **b = 10.88 m** 3

Area of triangle ABC $= \dfrac{1}{2} ac \sin B$

$= \dfrac{1}{2}(15.4)(15.0)\sin 41.93^\circ$

$= \mathbf{77.18\ m^2}$ 3

total : 10

Problem 2. Figure A7.2 shows a roof truss PQR with rafter PQ = 3 m. Calculate the length of (a) the roof rise PP' (b) rafter PR, and (c) the roof span QR. Find also (d) the cross-sectional area of the roof truss.

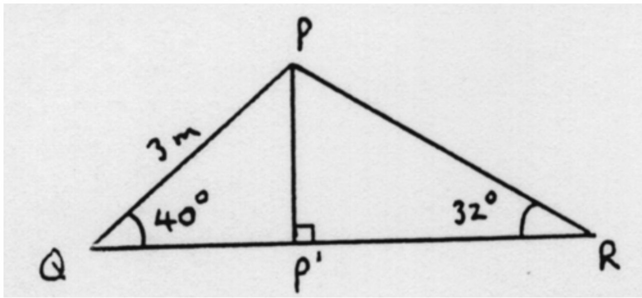


Figure A7.2

Marks

(a) In triangle PQP′, $\sin 40° = \dfrac{PP'}{3}$

from which, **PP′** = 3 sin 40° = **1.928 m** — 3

(b) From triangle PRP′, $\sin 32° = \dfrac{1.928}{PR}$

from which, **PR** = $\dfrac{1.928}{\sin 32°}$ = **3.638 m** — 3

(c) ∠QPR = 180° - 40° - 32° = 108°

Using the sine rule: $\dfrac{QR}{\sin 108°} = \dfrac{3}{\sin 32°}$

from which, **QR** = $\dfrac{3 \sin 108°}{\sin 32°}$ = **5.384 m** — 3

(d) Cross-sectional area of roof truss = $\dfrac{1}{2}$(QR)(PP′)

= $\dfrac{1}{2}$(5.384)(1.928

= **5.190 m²** — 2

total : 11

Problem 3. Prove the following identities:

(a) $\sqrt{\left[\dfrac{1 - \cos^2\theta}{\cos^2\theta}\right]} = \tan\theta$ (b) $\cos\left(\dfrac{3\pi}{2} + \phi\right) = \sin\phi$

Marks

(a) L.H.S. = $\sqrt{\left[\frac{1-\cos^2\theta}{\cos^2\theta}\right]} = \sqrt{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}$ since $\cos^2\theta + \sin^2\theta = 1$

$= \frac{\sin\theta}{\cos\theta} = \tan\theta$ = R.H.S. 3

(b) L.H.S. = $\cos\left(\frac{3\pi}{2} + \phi\right) = \cos\frac{3\pi}{2}\cos\phi - \sin\frac{3\pi}{2}\sin\phi$ from compound angles

$= 0 - (-1)\sin\phi = \sin\phi$ = R.H.S. 3

total : 6

Problem 4. Solve the following trigonometric equations in the range $0° \le x \le 360°$:

(a) $4\cos x + 1 = 0$ (b) $3.25 \operatorname{cosec} x = 5.25$ (c) $5\sin^2 x + 3\sin x = 4$

Marks

(a) Since $4\cos x + 1 = 0$ then $\cos x = -\frac{1}{4}$ and $x = \cos^{-1}(-0.25)$

i.e. **x = 104.48°** (or **104° 29′**) and **255.52°** (or **255° 31′**) 4

(b) Since $3.25 \operatorname{cosec} x = 5.25$ then $\operatorname{cosec} x = \frac{5.25}{3.25}$ and $\sin x = \frac{3.25}{5.25}$

i.e. **x** = $\sin^{-1}\left(\frac{3.25}{5.25}\right)$ = **38.25°** (or **38° 15′**) and **141.75°** (or **141° 45′**) 4

(c) Since $5\sin^2 x + 3\sin x = 4$ then $5\sin^2 x + 3\sin x - 4 = 0$

and $\sin x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-4)}}{2(5)} = \frac{-3 \pm \sqrt{89}}{10} = 0.6434$ or -1.2434

Ignoring the latter, $\sin x = 0.6434$ and **x** = $\sin^{-1} 0.6434$

= **40.05°** **or** **139.95°** 5

total : 13

Problem 5. Solve the equation $5\sin(\theta - \pi/6) = 8\cos\theta$ for values $0 \le \theta \le 2\pi$

Marks

$5\sin(\theta - \pi/6) = 8\cos\theta$

i.e. $5[\sin\theta\cos\pi/6 - \cos\theta\sin\pi/6] = 8\cos\theta$ 2

Thus $4.33 \sin\theta - 2.5 \cos\theta = 8 \cos\theta$

and $4.33 \sin\theta = 10.5 \cos\theta$ 2

Hence $\frac{\sin\theta}{\cos\theta} = \frac{10.5}{4.33} = 2.42494226$

i.e. $\tan\theta = 2.42494226$

and $\theta = \tan^{-1}(2.42494226)$

i.e. $\boldsymbol{\theta = 67.59°}$ **and** $\boldsymbol{247.59°}$ 4

total : 8

Problem 6. Express 5.3 cos t - 7.2 sin t in the form R sin(t + α). Hence solve the equation 5.3 cos t - 7.2 sin t = 4.5 in the range $0 \le t \le 2\pi$.

Marks

Let $5.3 \cos t - 7.2 \sin t = R \sin(t + \alpha)$

$= R[\sin t \cos\alpha + \cos t \sin\alpha]$

$= (R \cos\alpha)\sin t + (R \sin\alpha)\cos t$

Hence $5.3 = R \sin\alpha$ i.e. $\sin\alpha = \frac{5.3}{R}$

and $-7.2 = R \cos\alpha$ i.e. $\cos\alpha = \frac{-7.2}{R}$

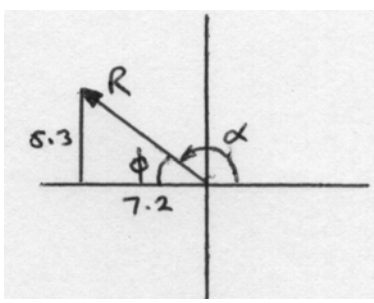


Figure 17

There is only one quadrant where sine is positive **and** cosine is negative, i.e. the second, as shown in Figure 17. 3

$R = \sqrt{(5.3^2 + 7.2^2)} = 8.94$ and $\phi = \tan^{-1}\frac{5.3}{7.2} = 0.6346$ rad

hence $\alpha = \pi - 0.6346 = 2.507$ rad

Thus $5.3 \cos t - 7.2 \sin t = \mathbf{8.94 \sin(t + 2.507)}$ 3

If $5.3 \cos t - 7.2 \sin t = 4.5$ then $8.94 \sin(t + 2.507) = 4.5$

and $\sin(t + 2.507) = \dfrac{4.5}{8.94} = 0.50336$

$$t + 2.507 = \sin^{-1} 0.50336 = 0.5275 \text{ rad or } 2.6141 \text{ rad}$$

and $\mathbf{t} = 0.5275 - 2.507 = -1.97952$

$$\equiv -1.97952 + 2\pi = \mathbf{4.304\ s}$$

or $\mathbf{t} = 2.6141 - 2.507 = \mathbf{0.107\ s}$ 6

total : 12

TOTAL ASSIGNMENT MARKS: 60

ASSIGNMENT 8 (PAGE 279)

This assignment covers the material contained in chapters 27 to 31.

Problem 1. Determine the gradient and intercept on the y-axis for the following equations: (a) $y = -5x + 2$ (b) $3x + 2y + 1 = 0$

Marks

(a) $y = -5x + 2$ hence **gradient = -5** and **intercept = 2** — 2

(b) Rearranging $3x + 2y + 1 = 0$

gives: $2y = -3x - 1$ and $y = -\frac{3}{2}x - \frac{1}{2}$ — 1

hence **gradient =** $\mathbf{-\frac{3}{2}}$ and **intercept =** $\mathbf{-\frac{1}{2}}$ — 2

total : 5

Problem 2. The equation of a line is $2y = 4x + 7$. A table of corresponding values is produced and is as shown below. Complete the table and plot a graph of y against x. Determine the gradient of the graph.

x	-3	-2	-1	0	1	2	3
y	-2.5					7.5	

Marks

Since $2y = 4x + 7$ then $y = 2x + \frac{7}{2}$

The table of values is shown below

x	-3	-2	-1	0	1	2	3
y	-2.5	**-0.5**	**1.5**	**3.5**	**5.5**	7.5	**9.5**

2

A graph of y against x is shown plotted in Figure 18.

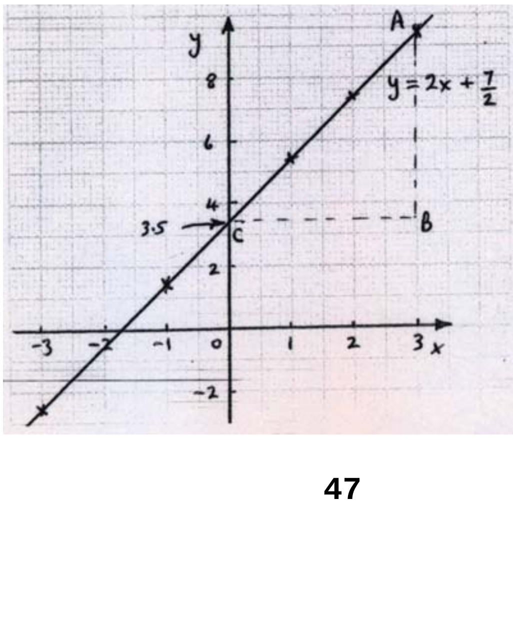


Figure 18 — 3

Gradient = $\frac{AB}{BC} = \frac{9.5 - 3.5}{3 - 0} = \frac{6.0}{3}$ = **2** 1

total : 6

Problem 3. Plot the graphs y = 3x + 2 and $\frac{y}{2} + x = 6$ on the same axes and determine the co-ordinates of their point of intersection.

Marks

A table of values is shown for y = 3x + 2 (just three values since it is a straight line graph)

x	0	2	4
y = 3x + 2	2	8	14

2

$\frac{y}{2} + x = 6$, hence $\frac{y}{2} = 6 - x$ and y = 12 - 2x or y = -2x + 12

A table of values is shown below

x	0	2	4
y = -2x + 12	12	8	4

2

Graphs of y = 3x + 2 and y = -2x + 12 are shown in Figure 19

They intersect at **(2, 8)** 1

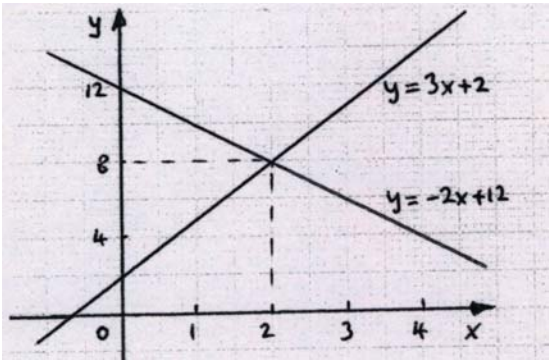


Figure 19 2

total : 7

Problem 4. The velocity v of a body over varying time intervals t was measured as follows:

t s	2	5	7	10	14	17
v m/s	15.5	17.3	18.5	20.3	22.7	24.5

Plot a graph with velocity vertical and time horizontal. Determine from the graph (a) the gradient, (b) the vertical axis intercept, (c) the equation of the graph, (d) the velocity after 12.5s, and (e) the time when the velocity is 18 m/s

Marks

A graph of v against t is shown in Figure 20.

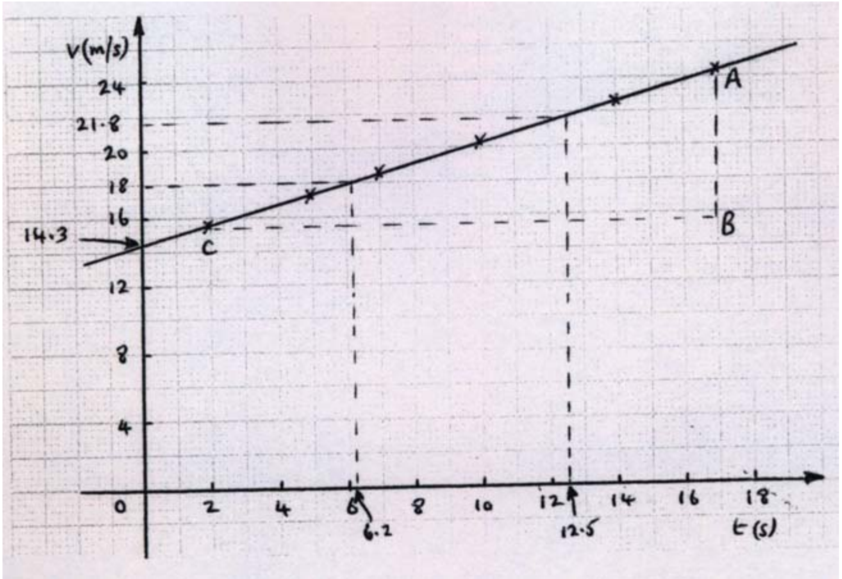


Figure 20 3

(a) Gradient = $\frac{AB}{BC} = \frac{24.5 - 15.5}{17 - 2} = \frac{9}{15} = \mathbf{0.6}$ 2

(b) Vertical axis intercept = **14.3 m/s** 1

(c) The equation of the graph is: **v = 0.6t + 14.3** 1

(d) When t = 12.5 s, velocity v = **21.8 m/s** 1

(e) When v = 18 m/s, time t = **6.2 s** 1

total : 9

Problem 5. The following experimental values of x and y are believed to be related by the law $y = ax^2 + b$, where a and b are constants. By plotting a suitable graph verify this law and find the approximate values of a and b.

x	2.5	4.2	6.0	8.4	9.8	11.4
y	15.4	32.5	60.2	111.8	150.1	200.9

Marks

Since $y = ax^2 + b$ then y is plotted vertically against x^2 to give a straight line form with gradient a and vertical axis intercept b 1

A table is produced as shown below:

x^2	6.25	17.64	36.0	70.56	96.04	129.96
y	15.4	32.5	60.2	111.8	150.1	200.9

2

A graph is plotted as shown in Figure 21.

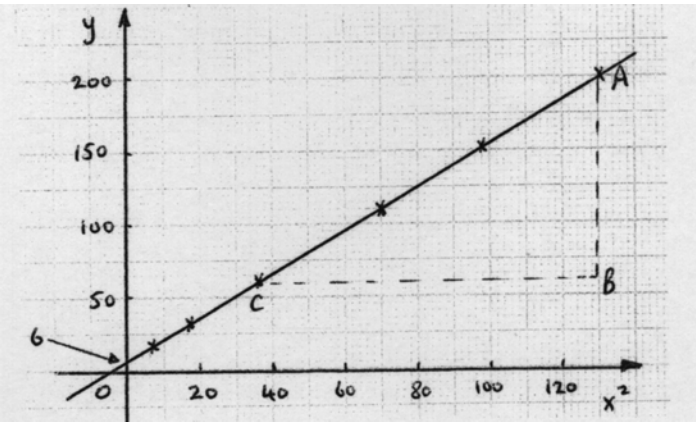


Figure 21 3

Gradient **a** = $\frac{AB}{BC} = \frac{200.9 - 60.2}{129.96 - 36.0}$ = **1.5** 2

and y-axis intercept **b = 6** 1

total : 9

Problem 6. Determine the law of the form $y = ae^{kx}$ which relates the following values:

y	0.0306	0.285	0.841	5.21	173.2	1181
x	-4.0	5.3	9.8	17.4	32.0	40.0

Marks

$y = ae^{kx}$ hence $\ln y = \ln(ae^{kx})$

i.e. $\ln y = \ln a + \ln e^{kx}$

i.e. $\ln y = kx + \ln a$

Hence ln y is plotted vertically and x is plotted horizontally to produce a straight line graph of gradient k and vertical-axis intercept ln a 2

A table of values is shown below:

ln y	-3.49	-1.26	-0.17	1.65	5.15	7.07
x	-4.0	5.3	9.8	17.4	32.0	40.0

1

A graph of ln y/x is shown in Figure 22.

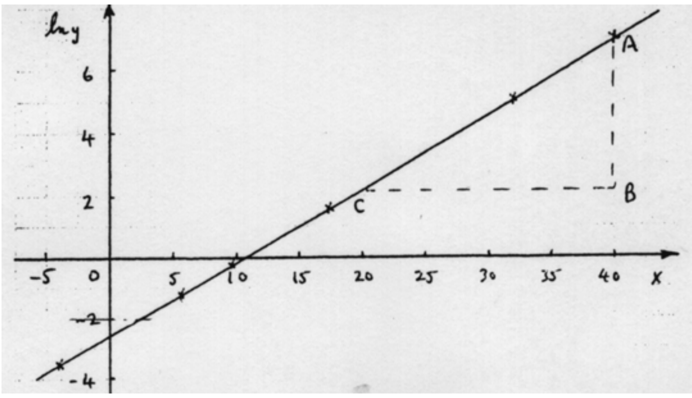


Figure 22 3

Gradient **k** = $\frac{AB}{BC} = \frac{7 - 2.2}{40 - 20} = \frac{4.8}{20}$ = **0.24** 1

Vertical axis intercept, ln a = -2.55

from which, **a** = $e^{-2.55}$ = **0.08**, correct to 1 significant figure 1

Hence the law of the graph is: $\mathbf{y = 0.08\ e^{0.24x}}$ 1

total : 9

Problem 7. State the minimum number of cycles on logarithmic graph paper needed to plot a set of values ranging from 0.073 to 490.

Marks

5 cycles are needed – from 0.01 to 0.1, 0.1 to 1, 1 to 10, 10 to 100 and 100 to 1000 2

total : 2

Problem 8. Plot a graph of $y = 2x^2$ from x = -3 to x = +3 and hence solve the equations: (a) $2x^2 - 8 = 0$ (b) $2x^2 - 4x - 6 = 0$

Marks

A graph of $y = 2x^2$ is shown in Figure 23. 2

(a) When $2x^2 - 8 = 0$ then $2x^2 = 8$

The points of intersection of $y = 2x^2$ and y = 8 occur when

x = -2 and x = 2, which is the solution of $2x^2 - 8 = 0$ 3

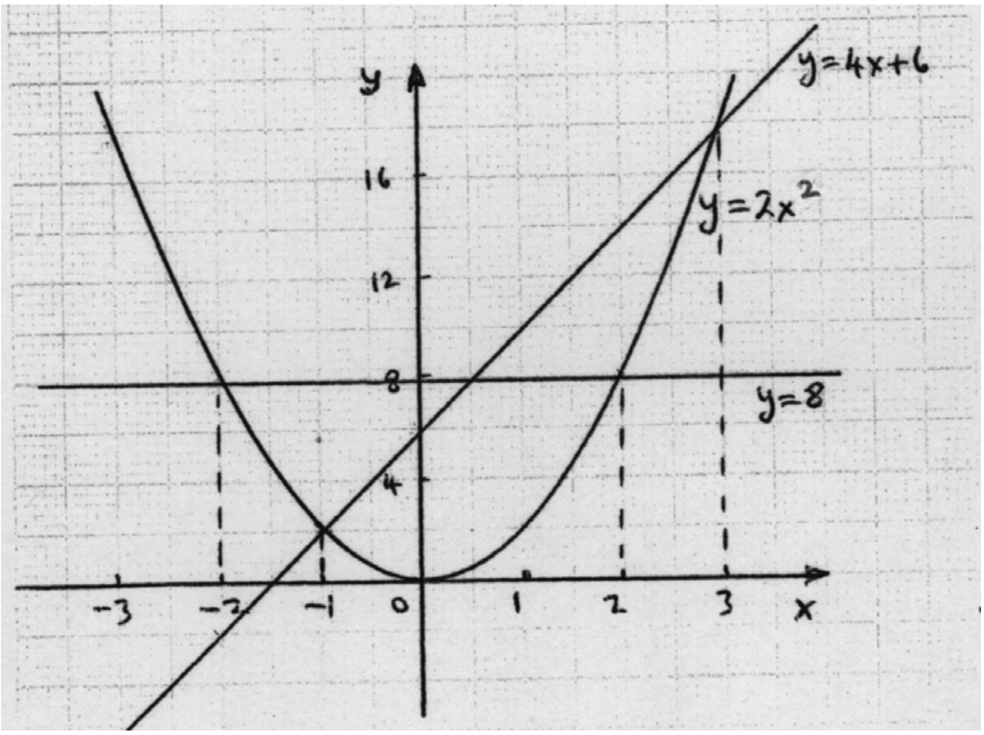


Figure 23

(b) When $2x^2 - 4x - 6 = 0$ then $2x^2 = 4x + 6$

The points of intersection of $y = 2x^2$ and $y = 4x + 6$ occur when

x = -1 and x = 3, which is the solution of $2x^2 - 4x - 6 = 0$ 4

total : 9

Problem 9. Plot the graph of $y = x^3 + 4x^2 + x - 6$ for values of x between $x = -4$ and $x = +2$. Hence determine the roots of the equation $x^3 + 4x^2 + x - 6 = 0$

Marks

A table of values is drawn up as shown below

x	-4	-3	-2	-1	0	1	2	
$y = x^3 + 4x^2 + x - 6$	-10	0	0	-4	-6	0	20	2

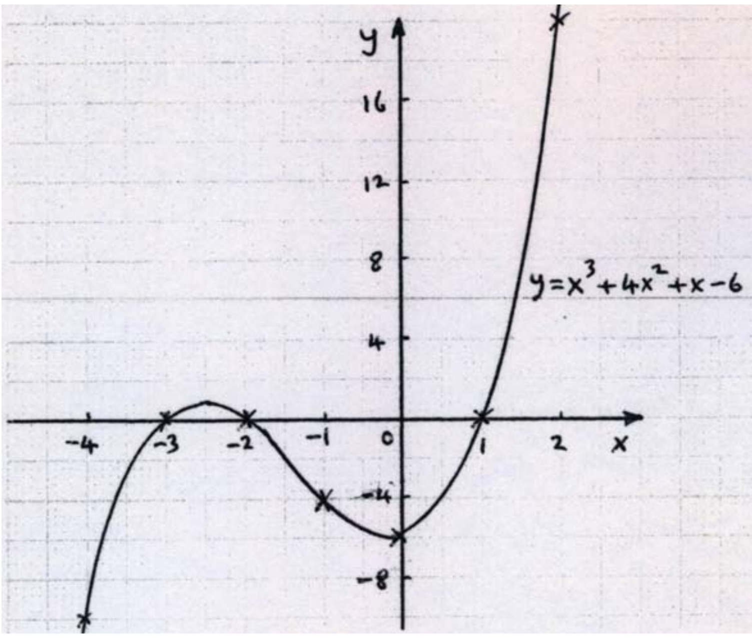


Figure 24 3

A graph of $y = x^3 + 4x^2 + x - 6$ is shown in Figure 24.

From the graph, the roots of $x^3 + 4x^2 + x - 6 = 0$ are seen to be at

x = -3, -2 and 1 2

total : 7

Problem 10. Sketch the following graphs, showing the relevant points:

(a) $y = (x - 2)^2$ (b) $y = 3 - \cos 2x$ (c) $f(x) = \begin{cases} -1 & -\pi \le x \le -\frac{\pi}{2} \\ x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le x \le \pi \end{cases}$

Marks

(a) A graph of $y = (x - 2)^2$ is shown in Figure 25.

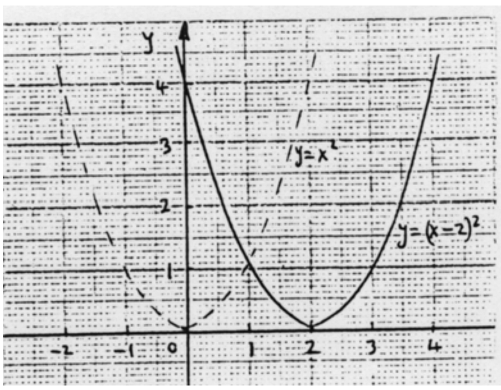


Figure 25 3

(b) A graph of $y = 3 - \cos 2x$ is shown in Figure 26.

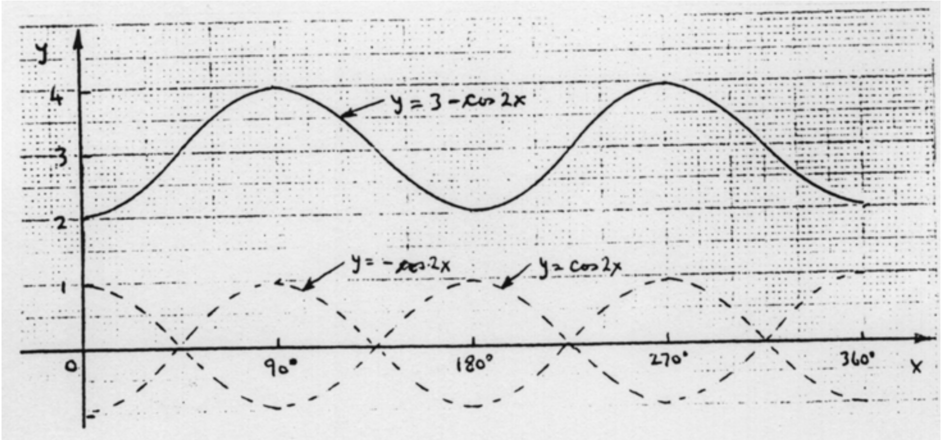


Figure 26 3

(c) A graph of $f(x) = \begin{cases} -1 & -\pi \le x \le -\frac{\pi}{2} \\ x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le x \le \pi \end{cases}$ is shown in Figure 27.

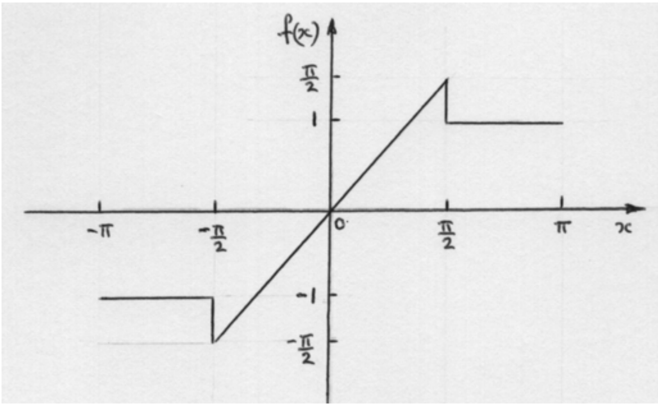


Figure 27 4

total : 10

Problem 11. Determine the inverse of f(x) = 3x + 1

Marks

Let f(x) = y then y = 3x + 1

Transposing for x gives: $x = \dfrac{y-1}{3}$

and interchanging x and y gives: $y = \dfrac{x-1}{3}$

Hence the inverse of f(x) = 3x + 1 is $\mathbf{f^{-1}(x) = \dfrac{x-1}{3}}$ 3

total : 3

Problem 12. Evaluate, correct to 3 decimal places:

$$2 \arctan 1.64 + \text{arcsec}\, 2.43 - 3\, \text{arccosec}\, 3.85$$

Marks

2 arctan 1.64 + arcsec 2.43 - 3 arccosec 3.85

$$= 2 \arctan 1.64 + \arccos\left(\frac{1}{2.43}\right) - 3 \arcsin\left(\frac{1}{3.85}\right)$$

$$= 2(1.02323409\ ..) + 1.14667223\ .. - 3(0.26275322\ ..)$$

= **2.405** 4

total : 4

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 9 (PAGE 306)

This assignment covers the material contained in chapters 32 to 35.

Problem 1. Four coplanar forces act at a point A as shown in Figure A9.1. Determine the value and direction of the resultant force by (a) drawing (b) by calculation.

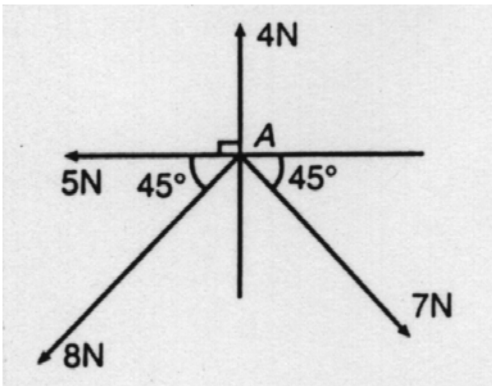


Figure A9.1

Marks

(a)From Figure 28, by drawing, resultant R = 8.7 N and $\theta = 230°$

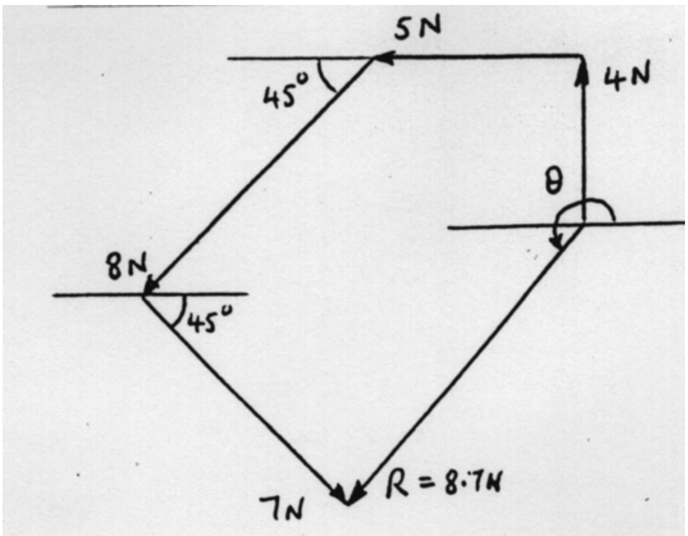


Figure 28 5

(b)By calculation:

Total horizontal component,

$H = 4\cos 90° + 5\cos 180° + 8\cos 225° + 7\cos 315° = -5.7071$

Total vertical component,

$V = 4\sin 90° + 5\sin 180° + 8\sin 225° + 7\sin 315° = -6.6066$

Hence, resultant $R = \sqrt{(-5.7071)^2 + (-6.6066)^2} = 8.73$ N and $\theta = \tan^{-1}\frac{V}{H} = \tan^{-1}\frac{-6.6066}{-5.7071} = 229.18°$ | 6

since both H and V are negative

total : 11

Problem 2. The instantaneous values of two alternating voltages are given by:

$$v_1 = 150\sin\left(\omega t + \frac{\pi}{3}\right) \text{ volts} \quad \text{and} \quad v_2 = 90\sin\left(\omega t - \frac{\pi}{6}\right) \text{ volts}$$

Plot the two voltages on the same axes to scales of 1 cm = 50 volts and 1 cm = $\frac{\pi}{6}$ rad. Obtain a sinusoidal expression for the resultant $v_1 + v_2$ in the form $R\sin(\omega t + \alpha)$: (a) by adding ordinates at intervals, and (b) by calculation

Marks

(a) From Figure 29, by adding ordinates at intervals, the waveform of $v_1 + v_2$ is seen to have a maximum value of 175 V and is leading by 30° or $\frac{\pi}{6}$ rad, i.e. 0.52 rad.

Hence **$v_1 + v_2 = 175\sin(\omega t + 0.52)$ volts** | 7

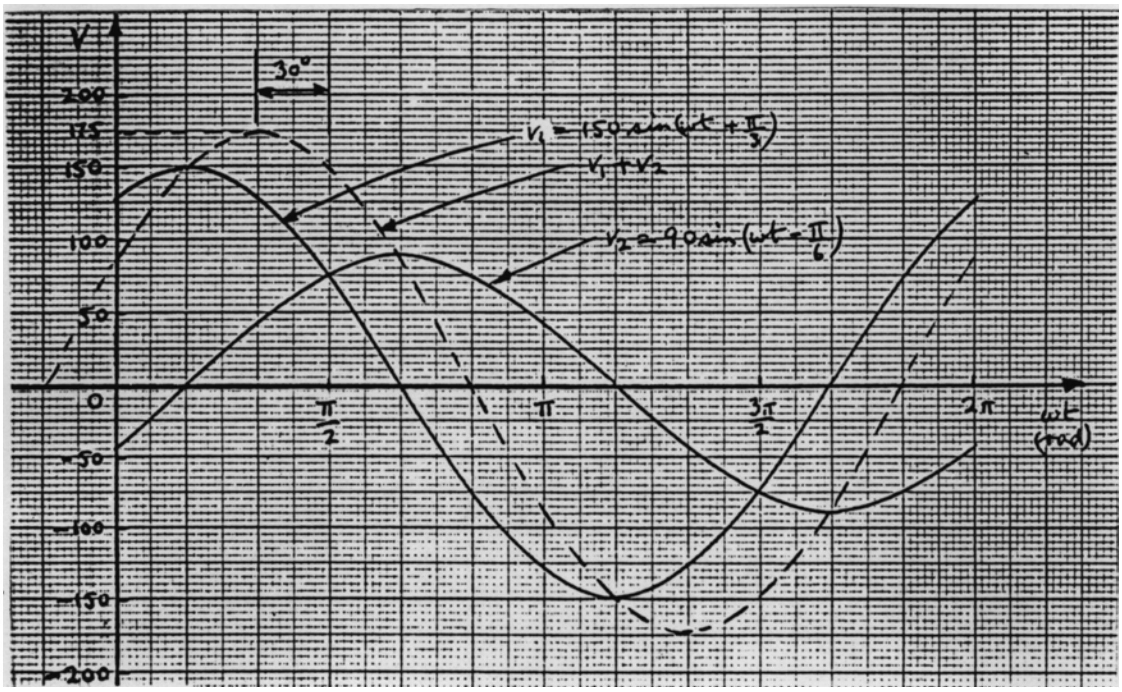


Figure 29

(b) By calculation:

At time t = 0, the phasors v_1 and v_2 are shown in Figure 30.

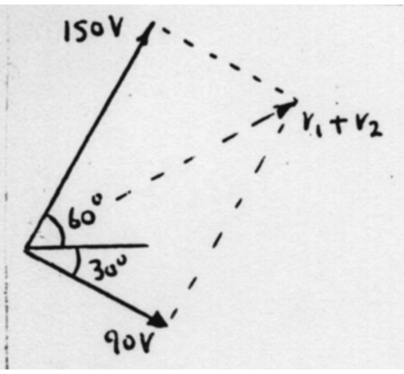


Figure 30

Total horizontal component H = 150 cos 60° + 90 cos -30° = 152.942

Total vertical component V = 150 sin 60° + 90 sin -30° = 84.90

Resultant, $v_1 + v_2 = \sqrt{(152.942)^2 + (84.90)^2}$ = 174.93 volts

Direction of $v_1 + v_2 = \tan^{-1}\dfrac{84.90}{152.942}$ = 29.04° or 0.507 rad

Hence **$v_1 + v_2$ = 174.93 sin(ωt + 0.507) volts** 6

total : 13

Problem 3. Solve the quadratic equation $x^2 - 2x + 5 = 0$ and show the roots on an Argand diagram.

Marks

$$\mathbf{x} = \frac{-\,-2 \pm \sqrt{[(-2)^2 - 4(1)(5)]}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm j4}{2} = \mathbf{1 \pm j2}$$ 6

The two roots are shown on the Argand diagram in Figure 31.

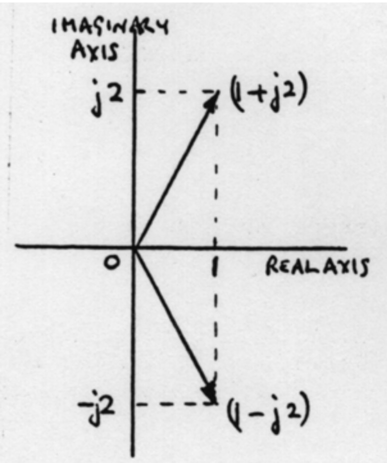


Figure 31 2

total : 8

Problem 4. If $Z_1 = 2 + j5$, $Z_2 = 1 - j3$ and $Z_3 = 4 - j$ determine, in both Cartesian and polar forms, the value of $\dfrac{Z_1 Z_2}{Z_1 + Z_2} + Z_3$, correct to 2 decimal places.

Marks

$$\frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(2 + j5)(1 - j3)}{(2 + j5) + (1 - j3)} = \frac{2 - j6 + j5 - j^2 15}{3 + j2} = \frac{17 - j}{3 + j2} = \frac{17 - j}{3 + j2} \times \frac{3 - j2}{3 - j2}$$

$$= \frac{51 - j34 - j3 + j^2 2}{3^2 + 2^2} = \frac{49 - j37}{13} = 3.77 - j2.85$$ 4

Hence $\dfrac{Z_1 Z_2}{Z_1 + Z_2} + Z_3 = 3.77 - j2.85 + 4 - j =$ **$7.77 - j3.85$** 2

or **$8.67\angle -26.36°$** 2

total : 8

Problem 5. Determine in both polar and rectangular forms:

(a) $[3.2 - j4.8]^5$ (b) $\sqrt{(-1 - j3)}$

Marks

(a) $[3.2 - j4.8]^5 = (5.769\angle -56.31°)^5 = 5.769^5 \angle 5 \times -56.31°$

$= 6390\angle -281.55° =$ **$6390\angle 78.45°$** $=$ **$1279 + j6261$** 5

(b) $\sqrt{(-1 - j3)} = [\sqrt{10}\angle -108.435°]^{\frac{1}{2}} = \sqrt{10}^{\frac{1}{2}} \angle \frac{1}{2} \times -108.435°$

$=$ **$1.778\angle -54.22°$** and **$1.778\angle 125.78°$** $=$ **$\pm(1.04 - j1.44)$** 5

total : 10

TOTAL ASSIGNMENT MARKS: 50

ASSIGNMENT 10 (PAGE 339)

This assignment covers the material contained in chapters 36 to 39.

Problem 1. A company produces five products in the following proportions:

Product A 24 Product B 16 Product C 15 Product D 11 Product E 6

Present these data visually by drawing (a) a vertical bar chart (b) a percentage component bar chart (c) a pie diagram.

Marks

(a) A vertical bar chart is shown in Figure 32.

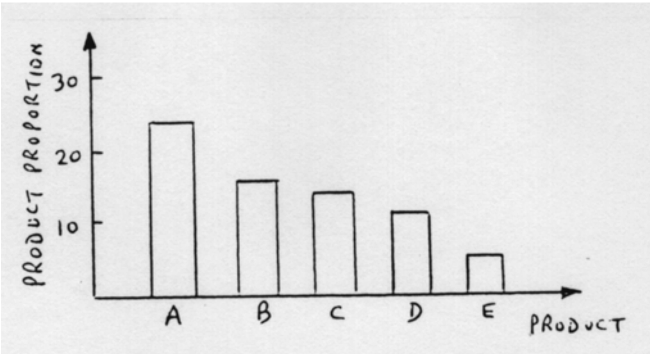


Figure 32 3

(b)For the percentage bar chart, 24 + 16 + 15 + 11 + 6 = 72

hence $A \equiv \frac{24}{72} \times 100\% = 33.3\%$, $B \equiv \frac{16}{72} \times 100\% = 22.2\%$,

$C \equiv \frac{15}{72} \times 100\% = 20.8\%$, $D \equiv \frac{11}{72} \times 100\% = 15.3\%$

and $E \equiv \frac{6}{72} \times 100\% = 8.3\%$ 3

A percentage component bar chart is shown in Figure 33.

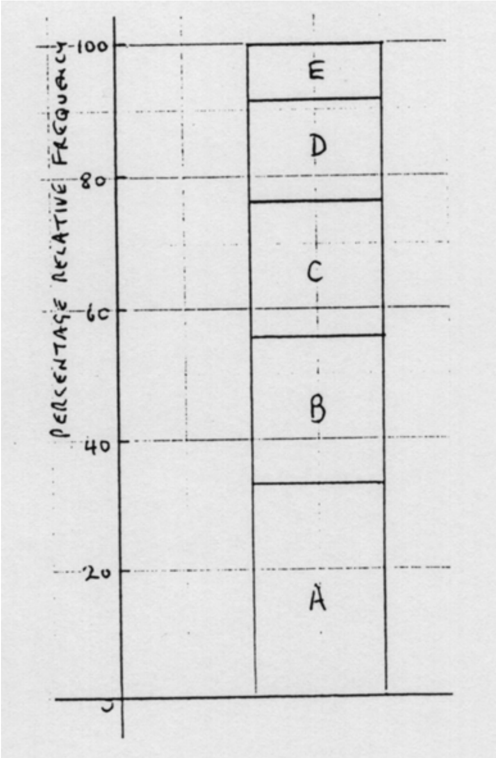


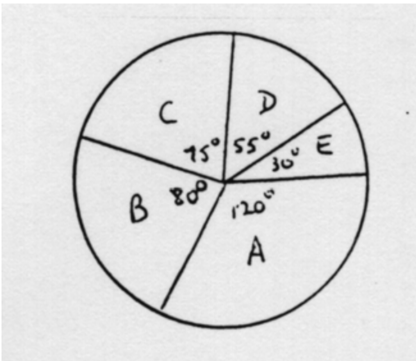
Figure 33 2

(c)Total number of products = 24 + 16 + 15 + 11 + 6 = 72

Hence A $\equiv \frac{24}{72} \times 360^\circ = 120^\circ$, B $\equiv \frac{16}{72} \times 360^\circ = 80^\circ$, C $\equiv \frac{15}{72} \times 360^\circ = 75^\circ$

D $\equiv \frac{11}{72} \times 360^\circ = 55^\circ$, E $\equiv \frac{6}{72} \times 360^\circ = 30^\circ$ 3

A pie diagram is shown in Figure 34.



2

Figure 34 **total : 13**

Problem 2. The following lists the diameters of 40 components produced by a machine, each measured correct to the nearest hundredth of a centimetre:

1.39	1.36	1.38	1.31	1.33	1.40	1.28	1.40	1.24	1.28	1.42	1.34
1.43	1.35	1.36	1.36	1.35	1.45	1.29	1.39	1.38	1.38	1.35	1.42
1.30	1.26	1.37	1.33	1.37	1.34	1.34	1.32	1.33	1.30	1.38	1.41
1.35	1.38	1.27	1.37								

(a) Using 8 classes form a frequency distribution and a cumulative frequency distribution.

(b)For the above data draw a histogram, a frequency polygon and an ogive

Marks

(a)Range = 1.24 to 1.47 i.e. 0.23

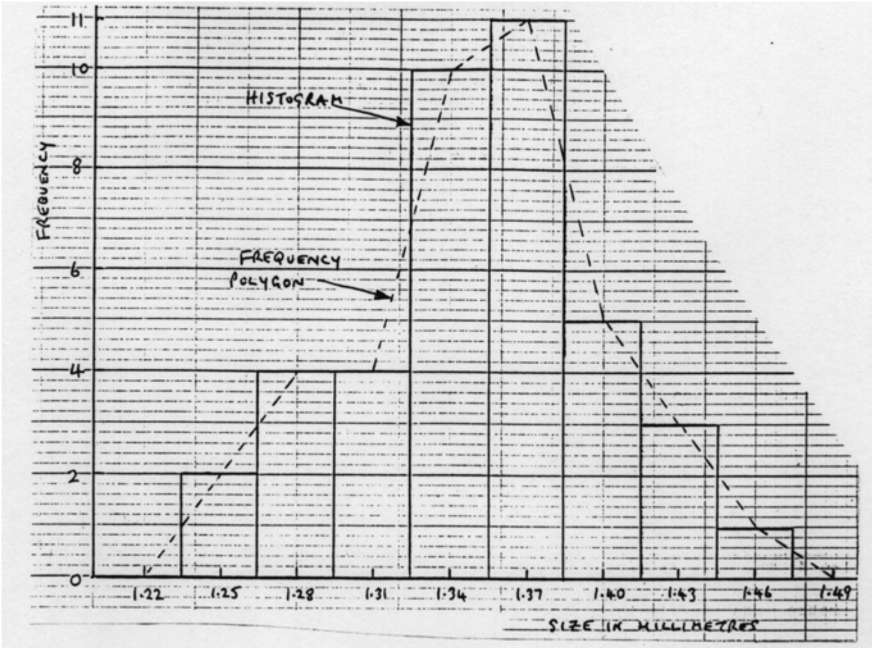
Hence let classes be 1.24 - 1.26, 1.27 - 1.29, 1.30 - 1.32, ..

A frequency distribution and cumulative frequency distribution is shown in the following table:

Class	Tally	Frequency	Cumulative frequency
1.24 - 1.26	11	2	2
1.27 - 1.29	1111	4	6
1.30 - 1.32	1111	4	10
1.33 - 1.35	~~1111~~ ~~1111~~	10	20
1.36 - 1.38	~~1111~~ ~~1111~~ 1	11	31
1.39 - 1.41	~~1111~~	5	36
1.42 - 1.44	111	3	39
1.45 - 1.47	1	1	40

8

(b)A histogram and frequency polygon are shown in Figure 35



histogram	6
frequency polygon	2

Figure 35

An ogive is shown in Figure 36.

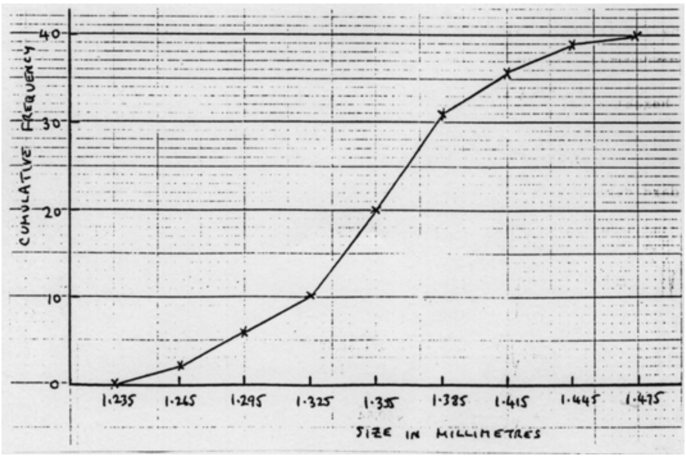


Figure 36 5

total : 21

Problem 3. Determine for the 10 measurements of lengths shown below:
(a) the arithmetic mean, (b) the median, (c) the mode, and (d) the standard deviation.

28 m, 20 m, 32 m, 44 m, 28 m, 30 m, 30 m, 26 m, 28 m and 34 m

Marks

(a)**Arithmetic mean** = $\dfrac{28+20+32+44+28+30+30+26+28+34}{10}$ = **30 m** 2

(b)Ranking gives: 20 26 28 28 28 30 30 32 34 34

Median = $\dfrac{28+30}{2}$ = **29 m** 2 (c)The **mode is 28 m** 1

(d)**Standard deviation** $\sigma = \sqrt{\left[\frac{(28-30)^2 + (20-30)^2 + \ldots + (34-30)^2}{10}\right]}$

$= \sqrt{\left(\frac{344}{10}\right)} = \mathbf{5.865}$ 4

total : 9

Problem 4. The heights of 100 people are measured correct to the nearest centimetre with the following results:

150 - 157 cm	5	158 - 165 cm	18	166 - 173 cm	42
174 - 181 cm	27	182 - 189 cm	8		

Determine for the data (a) the mean height, and (b) the standard deviation.

Marks

(a)**Mean height** $= \frac{153.5 \times 5 + 161.5 \times 18 + 169.5 \times 42 + 177.5 \times 27 + 185.5 \times 8}{100}$

$= \frac{17070}{100} = \mathbf{170.7}$ 3

(b)Standard deviation

$$\sigma = \sqrt{\left\{\frac{\sum f(x-\bar{x})^2}{\sum f}\right\}} = \sqrt{\left\{\frac{5(153.5-170.7)^2 + 18(161.5-170.7)^2 + 42(169.5-170.7)^2 + 27(177.5-170.7)^2 + 8(185.5-170.7)^2}{100}\right\}}$$

$$= \sqrt{\left\{\frac{1479.2 + 1523.52 + 60.48 + 1248.48 + 1752.32}{100}\right\}}$$

$$= \sqrt{\left\{\frac{6064}{100}\right\}} = \sqrt{60.64} = \mathbf{7.787\ cm}$$ 7

total : 10

Problem 5. Determine the probabilities of:

(a) drawing a white ball from a bag containing 6 black and 14 white balls

(b) winning a prize in a raffle by buying 6 tickets when a total of 480 tickets are sold

(c) selecting at random a female from a group of 12 boys and 28 girls

(d) winning a prize in a raffle by buying 8 tickets when there are 5 prizes and a total of 800 tickets are sold.

	Marks
(a) $p = \frac{14}{6+14} = \mathbf{\frac{14}{20}}$ or **0.70**	2
(b) $p = \frac{6}{480} = \mathbf{\frac{1}{80}}$ or **0.0125**	2
(c) $p = \frac{28}{12+28} = \frac{28}{40} = \mathbf{\frac{7}{10}}$ or **0.70**	2
(d) $p = 8 \times \frac{5}{800} = \mathbf{\frac{5}{100}}$ or **0.05**	2

total : 8

Problem 6. In a box containing 120 similar transistors 70 are satisfactory, 37 give too high a gain under normal operating conditions and the remainder give too low a gain.

Calculate the probability that when drawing two transistors in turn, at random, **with replacement,** of having (a) two satisfactory, (b) none with low gain, (c) one with high gain and one satisfactory, (d) one with low gain and none satisfactory.

Determine the probabilities in (a), (b) and (c) above if the transistors are drawn **without replacement.**

	Marks
With replacement	
(a) $p = \frac{70}{120} \times \frac{70}{120} = \frac{49}{144}$ or **0.3403**	2
(b) $p = \frac{70+37}{120} \times \frac{70+37}{120} = \left(\frac{107}{120}\right)^2 = \mathbf{0.7951}$	2
(c) $p = \frac{37}{120} \times \frac{70}{120} + \frac{70}{120} \times \frac{37}{120} = \mathbf{0.3597}$	2
(d) $p = \frac{13}{120} \times \frac{50}{120} + \frac{50}{120} \times \frac{13}{120} = \mathbf{0.0903}$	2
Without replacement	
(a) $p = \frac{70}{120} \times \frac{69}{119} = \mathbf{0.3382}$	2
(b) $p = \frac{107}{120} \times \frac{106}{119} = \mathbf{0.7943}$	2
(c) $p = \frac{37}{120} \times \frac{70}{119} + \frac{70}{120} \times \frac{37}{119} = \mathbf{0.3627}$	2

total : 14

Problem 7. A machine produces 15% defective components. In a sample of 5, drawn at random, calculate, using the binomial distribution, the probability that:

(a) there will be 4 defective items

(b) there will be not more than 3 defective items

(c) all the items will be non-defective

Marks

Let p = 0.15, q = 0.85 and n = 5

$$(q + p)^5 = q^5 + 5q^4 p + \frac{5 \times 4}{2!} q^3p^2 + \frac{5 \times 4 \times 3}{3!} q^2p^3 + \frac{5 \times 4 \times 3 \times 2}{4!} q\,p^4 + p^5$$

(a) The probability of 4 defective items = $\frac{5 \times 4 \times 3 \times 2}{4!} q\,p^4$

$= 5(0.85)(0.15)^4 = \mathbf{0.00215}$ — 4

(b) Not more than 3 defective items means the sum of the first 4 terms

$= (0.85)^5 + 5(0.85)^4(0.15) + 10(0.85)^3(0.15)^2 + 10(0.85)^2(0.15)^3$

$= 0.4437 + 0.3915 + 0.1382 + 0.0244 = \mathbf{0.9978}$ — 7

(c) The probability that all items will be non-defective is **0.4437** — 2

total : 13

Problem 8. 2% of the light bulbs produced by a company are defective. Determine, using the Poisson distribution, the probability that in a sample of 80 bulbs:

(a) 3 bulbs will be defective, (b) not more than 3 bulbs will be defective,

(c) at least 2 bulbs will be defective.

Marks

λ = 2% of 80 = 1.6

The probability of 0, 1, 2, .. defective items are given by

$$e^{-\lambda},\ \lambda e^{-\lambda},\ \frac{\lambda^2 e^{-\lambda}}{2!},\ ..$$

(a)The probability of 3 defective bulbs = $\frac{\lambda^3 e^{-\lambda}}{3!} = \frac{1.6^3 e^{-1.6}}{6} = \mathbf{0.1378}$ — 3

(b)The probability of not more than 3 defective bulbs is given by:

$$e^{-1.6} + 1.6e^{-1.6} + \frac{1.6^2 e^{-1.6}}{2!} + \frac{1.6^3 e^{-1.6}}{3!}$$

= 0.2019 + 0.3230 + 0.2584 + 0.1378 = **0.9211** 5

(c)The probability that at least two bulbs will be defective is given

by: $1 - (e^{-\lambda} + \lambda e^{-\lambda}) = 1 - (0.2019 + 0.3230) =$ **0.4751** 4

total : 12

TOTAL ASSIGNMENT MARKS: 100

ASSIGNMENT 11 (PAGE 368)

This assignment covers the material contained in chapters 40 to 43.

Problem 1. Some engineering components have a mean length of 20 mm and a standard deviation of 0.25 mm. Assume the data on the lengths of the components is normally distributed.

In a batch of 500 components, determine the number of components likely to:

(a) have a length of less than 19.95 mm, (b) be between 19.95 mm and 20.15 mm,

(c) be longer than 20.54 mm.

Marks

(a) $z = \frac{x - \bar{x}}{\sigma} = \frac{19.95 - 20}{0.25}$ = -0.2 standard deviations

From Table 40.1, page 339, when z = - 0.2 the partial area under the standardised curve is 0.0793(i.e. the shaded area in Figure 37 is 0.0793 of the total area)

The area to the left of the shaded area = 0.5 - 0.0793 = 0.4207

Thus for 500 components, 0.4207 × 500 are likely to have a length less than 19.95 mm, i.e. **210** 4

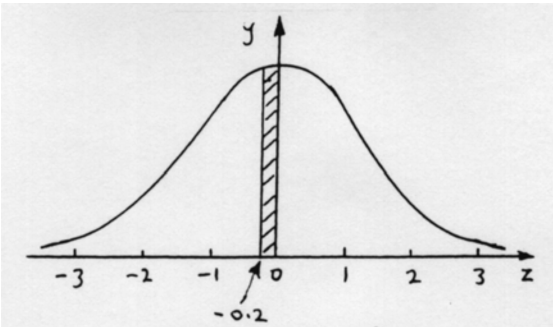


Figure 37

(b) When length is 20.15 mm, $z = \frac{20.15 - 20}{0.25}$ = 0.6 and from Table 42.1,

the area under the standardised curve is 0.2257

Hence the total partial area between z = - 0.2 and z = 0.6 is 0.0793 + 0.2257 = 0.3050 as shown shaded in Figure 38.

It is likely that 0.3050 × 500 components will lie between 19.95 mm and 20.15 mm, i.e. **153** 4

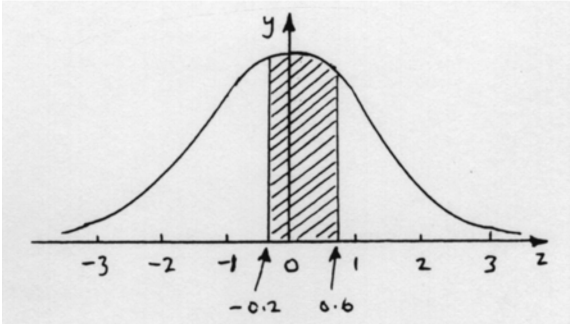


Figure 38

(c) When the length is 20.54 mm, $z = \frac{20.54 - 20}{0.25}$ = 2.16 and the partial area corresponding to this z-value is 0.4846

The area to the right of the shaded area shown in Figure 39 is 0.5 - 0.4846 i.e. 0.0154

Hence 0.0154 × 500 components are likely to be greater than 20.54 mm, i.e. **8** 4

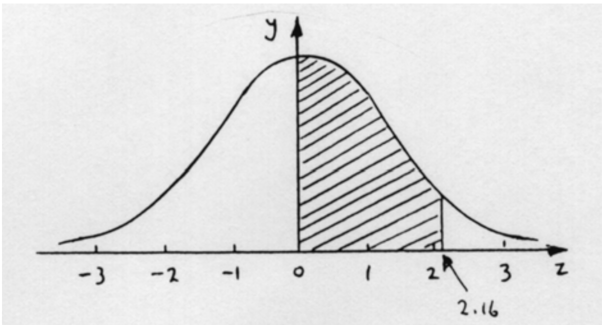


Figure 39

total : 12

Problem 2. In a factory, cans are packed with an average of 1.0 kg of a compound and the masses are normally distributed about the average value. The standard deviation of a sample of the contents of the cans is 12 g.
Determine the percentage of cans containing (a) less than 985 g, (b) more than 1030 g, (c) between 985 g and 1030 g.

Marks

(a) The z-value for 985 g is $\frac{985 - 1000}{12}$ = -1.25

From Table 40.1, page 339, the corresponding area under the standard-ised normal curve is 0.3944. Hence the area to the left of 1.25 standard deviations is 0.5 - 0.3944 = 0.1056, i.e. **10.56% of the cans contain less than 985 g** 4

(b) The z-value for 1030 g is $\frac{1030 - 1000}{12} = 2.5$

From Table 40.1, the area under the normal curve is 0.4938. hence the area to the right of 2.5 standard deviations is 0.5 - 0.4938 = 0.0062, i.e. **0.62% of the cans contain more than 1030 g.** 4

(c) The area under the normal curve corresponding to -1.25 to 2.5 standard deviations is 0.3944 + 0.4938 = 0.8882, i.e. **88.82% of the cans contain between 985 g and 1030 g.** 2

total : 10

Problem 3. The data given below gives the experimental values obtained for the torque output, X, from an electric motor and the current, Y, taken from the supply.

Torque X	0	1	2	3	4	5	6	7	8	9
Current Y	3	5	6	6	9	11	12	12	14	13

Determine the linear coefficient of correlation for this data.

Marks

Using a tabular approach:

X	Y	$x = X - \bar{x}$	$y = Y - \bar{y}$	xy	x^2	y^2
0	3	-4.5	-6.1	27.45	20.25	37.21
1	5	-3.5	-4.1	14.35	12.25	16.81
2	6	-2.5	-3.1	7.75	6.25	9.61
3	6	-1.5	-3.1	4.65	2.25	9.61
4	9	-0.5	-0.1	0.05	0.25	0.01
5	11	0.5	1.9	0.95	0.25	3.61
6	12	1.5	2.9	4.35	2.25	8.41
7	12	2.5	2.9	7.25	6.25	8.41
8	14	3.5	4.9	17.15	12.25	24.01
9	13	4.5	3.9	17.55	20.25	15.21
$\sum X = 45$ $\bar{x} = \frac{45}{10} = 4.5$	$\sum Y = 91$ $\bar{y} = \frac{91}{10} = 9.1$			$\sum xy =$ 101.5	$\sum x^2 =$ 82.5	$\sum y^2 =$ 132.9

12

Coefficient of correlation $r = \frac{\sum xy}{\sqrt{(\sum x^2)(\sum y^2)}} = \frac{101.5}{\sqrt{(82.5)(132.9)}} = \mathbf{0.969}$ 4

There is therefore good direct correlation between X and Y.

total : 16

Problem 4. Some results obtained from a tensile test on a steel specimen are shown below:

Tensile force (kN)	4.8	9.3	12.8	17.7	21.6	26.0
Extension (mm)	3.5	8.2	10.1	15.6	18.4	20.8

Assuming a linear relationship:

(a) determine the equation of the regression line of extension on force,

(b) determine the equation of the regression line of force on extension,

(c) estimate (i) the value of extension when the force is 16 kN, and

(ii) the value of force when the extension is 17 mm.

Marks

Force X	Extension Y	X^2	XY	Y^2
4.8	3.5	23.04	16.80	12.25
9.3	8.2	86.49	76.26	67.24
12.8	10.1	163.84	129.28	102.01
17.7	15.6	313.29	276.12	243.36
21.6	18.4	466.56	379.44	338.56
26.0	20.8	676.00	540.80	432.64
$\sum X =$ 92.2	$\sum Y =$ 76.6	$\sum X^2 =$ 1729.22	$\sum XY =$ 1418.70	$\sum Y^2 =$ 1196.06

9

(a) $\sum Y = a_0 N + a_1 \sum X$

$\sum XY = a_0 \sum X + a_1 \sum X^2$

Hence $76.6 = 6\,a_0 + 92.2\,a_1$ (1)

$1418.70 = 92.2\,a_0 + 1729.22\,a_1$ (2)

$92.2 \times (1)$ gives: $92.2(76.6) = 92.2(6)a_0 + 92.2(92.2)a_1$ (3)

6 × (2) gives: $6(1418.70) = 6(92.2)a_0 + 6(1729.22)a_1$ (4)

(3) - (4) gives: $-1449.68 = 0 - 1874.48\, a_1$

from which, $a_1 = \frac{1449.68}{1874.48} = \mathbf{0.773}$

Substituting in equation (1) gives: $76.6 = 6\, a_0 + 92.2(0.773)$

from which, $a_0 = \frac{76.6 - 92.2(0.773)}{6} = \mathbf{0.888}$

Hence **the regression line of extension on tensile force is:**

$Y = a_0 + a_1 X$ i.e. **$Y = 0.888 + 0.773\, X$** 5

(b) $\sum X = b_0 N + b_1 \sum Y$

$\sum XY = b_0 \sum Y + b_1 \sum Y^2$

Hence $92.2 = 6\, b_0 + 76.6\, b_1$ (1)

$1418.70 = 76.6\, b_0 + 1196.06\, b_1$ (2)

and $76.6(92.2) = 76.6(6)b_0 + 76.6(76.6)b_1$ (3)

$6(1418.70) = 6(76.6)b_0 + 6(1196.06)b_1$ (4)

(3) - (4) gives: $-1449.68 = -1308.80\, b_1$

from which, $b_1 = \frac{1449.68}{1308.80} = \mathbf{1.108}$

Substituting in (1) gives: $92.2 = 6\, b_0 + 76.6(1.108)$

from which, $b_0 = \frac{92.2 - 76.6(1.108)}{6} = \mathbf{1.221}$

Hence **the regression line of force on extension is:**

$X = b_0 + b_1 Y$ i.e. **$X = 1.221 + 1.108\, Y$** 5

(c)(i)Extension when the force is 16 kN is $Y = 0.888 + 0.773(16)$

= **13.26 mm** 1

(ii)Force when the extension is 17 mm is $X = 1.221 + 1.108(17)$

= **20.06 kN** 1

total : 21

Problem 5. 1200 metal bolts have a mean mass of 7.2 g and a standard deviation of 0.3 g. Determine the standard error of the means. Calculate also the probability that a sample of 60 bolts chosen at random, without replacement, will have a mass of (a) between 7.1 g and 7.25 g, and (b) more than 7.3 g.

	Marks
For the population: number of bolts, $N_p = 1200$	
standard deviation, $\sigma = 0.3$ g; mean $\mu = 7.2$ g	
For the sample: number in sample, N = 60	
Mean of sampling distribution of means, $\mu_{\bar{x}} = \mu = 7.2$ g	
Standard error of the means, $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}\sqrt{\left(\frac{N_p - N}{N_p - 1}\right)} = \frac{0.3}{\sqrt{60}}\sqrt{\left(\frac{1200 - 60}{1200 - 1}\right)}$	
= **0.03776 g**	3
(a) $z = \frac{x - \bar{x}}{\sigma_{\bar{x}}}$ When x = 7.1 g, $z = \frac{7.1 - 7.2}{0.03776}$ = -2.65 standard deviations	1
When x = 7.25 g, $z = \frac{7.25 - 7.2}{0.03776}$ = 1.32 standard deviations	1
From Table 40.1, page 339, the area corresponding these z-values are 0.4960 and 0.4066. Hence **the probability of the mean mass lying between 7.1 g and 7.25 g is** 0.4960 + 0.4066 = **0.9026**	2
(b) When x = 7.3 g, $z = \frac{7.3 - 7.2}{0.03776}$ = 2.65	1
From Table 40.1, the area corresponding to this z-value is 0.4960. The area lying to the right of this is 0.5 - 0.4960 = 0.0040 hence **the probability that a sample will have a mass of more than 7.3 g is 0.0040**	3
total : 11	

Problem 6. A sample of 10 measurements of the length of a component are made, and the mean of the sample is 3.650 cm. The standard deviation of the samples is 0.030 cm. Determine (a) the 99% confidence limits, and (b) the 90% confidence limits for an estimate of the actual length of the component.

Marks

For the sample: sample size, N = 10, mean, $\bar{x}$ = 3.650 cm

standard deviation, s = 0.030 cm

(a) The percentile value corresponding to a confidence coefficient value of $t_{0.99}$ and a degree of freedom value of ν = 10 - 1 = 9, is 2.82 from Table 43.2, page 363.

Estimated value of the mean of the population $= \bar{x} \pm \frac{t_c\, s}{\sqrt{(N-1)}}$

$$= 3.650 \pm \frac{(2.82)(0.030)}{\sqrt{(10-1)}}$$

$$= 3.650 \pm 0.0282$$

Thus **the 99% confidence limits are 3.622 cm to 3.678 cm.** 5

(b) For $t_{0.90}$, ν = 9, t_c = 1.38 from Table 43.2.

Estimated value of the 90% confidence limits $= \bar{x} \pm \frac{t_c\, s}{\sqrt{(N-1)}}$

$$= 3.650 \pm \frac{(1.38)(0.030)}{\sqrt{(10-1)}}$$

$$= 3.650 \pm 0.0138$$

Thus **the 90% confidence limits are 3.636 cm to 3.664 cm** 5

total : 10

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 12 (PAGE 406)

This assignment covers the material contained in chapters 44 to 46.

Problem 1. Differentiate the following with respect to the variable:

(a) $y = 5 + 2\sqrt{x^3} - \frac{1}{x^2}$ (b) $s = 4\,e^{2\theta}\sin 3\theta$ (c) $y = \frac{3\ln 5t}{\cos 2t}$

(d) $x = \frac{2}{\sqrt{(t^2 - 3t + 5)}}$

Marks

(a) $y = 5 + 2\sqrt{x^3} - \frac{1}{x^2} = 5 + 2x^{3/2} - x^{-2}$

$\frac{dy}{dx} = 0 + (2)\left(\frac{3}{2}x^{1/2}\right) - (-2x^{-3}) = \mathbf{3\sqrt{x} + \frac{2}{x^3}}$ 3

(b) $s = 4\,e^{2\theta}\sin 3\theta$ i.e. a product

$\frac{ds}{d\theta} = (4\,e^{2\theta})(3\cos 3\theta) + (\sin 3\theta)(8\,e^{2\theta}) = \mathbf{4\,e^{2\theta}(3\cos 3\theta + 2\sin 3\theta)}$ 4

(c) $y = \frac{3\ln 5t}{\cos 2t}$ i.e. a quotient

$\frac{dy}{dt} = \frac{(\cos 2t)\left(\frac{3}{t}\right) - (3\ln 5t)(-2\sin 2t)}{(\cos 2t)^2} = \mathbf{\frac{\frac{3}{t}\cos 2t + 6\ln 5t\sin 2t}{\cos^2 2t}}$ 4

(d) $x = \frac{2}{\sqrt{(t^2 - 3t + 5)}}$ Let $u = t^2 - 3t + 5$ then $\frac{du}{dt} = 2t - 3$

Hence $x = \frac{2}{\sqrt{u}} = 2u^{-1/2}$ and $\frac{dx}{du} = -u^{-3/2} = -\frac{1}{\sqrt{u^3}}$

$\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt} = \left(-\frac{1}{\sqrt{u^3}}\right)(2t - 3) = \mathbf{\frac{3 - 2t}{\sqrt{(t^2 - 3t + 5)^3}}}$ 4

total : 15

Problem 2. If $f(x) = 2.5x^2 - 6x + 2$ find the co-ordinates at the point at which the gradient is -1

Marks

$f(x) = 2.5x^2 - 6x + 2$

Gradient $= f'(x) = 5x - 6 = -1$ from which, $5x = 5$ and $x = 1$ 3

When $x = 1$, $f(x) = f(1) = 2.5(1)^2 - 6(1) + 2 = -1.5$

Hence **the gradient is -1 at the point (1, -1.5)** 2

total : 5

Problem 3. The displacement s cm of the end of a stiff spring at time t seconds is given by: $s = ae^{-kt}\sin 2\pi ft$. Determine the velocity and acceleration of the end of the spring after 2 seconds if a = 3, k = 0.75 and f = 20.

Marks

$s = ae^{-kt}\sin 2\pi ft$ i.e. a product

Velocity $= \frac{ds}{dt} = (ae^{-kt})(2\pi f\cos 2\pi ft) + (\sin 2\pi ft)(-k\,ae^{-kt})$ 2

When t = 2, a = 3, k = 0.75 and f = 20,

velocity $= (3e^{-(0.75)(2)})[2\pi \times 20\cos 2\pi(20)(2)]$

$- [\sin 2\pi(20)(2)](0.75)(3e^{-(0.75)(2)})$

$= 120\pi\,e^{-(0.75)(2)} - 0 =$ **84.12 cm/s** 2

Acceleration $= \frac{d^2s}{dt^2} = (ae^{-kt})[-(2\pi f)^2\sin 2\pi ft] + (2\pi f\cos 2\pi ft)(-k\,ae^{-kt})$

$+ (\sin 2\pi ft)(k^2 ae^{-kt}) + (-k\,ae^{-kt})(2\pi f\cos 2\pi ft$ 3

When t = 2, a = 3, k = 0.75 and f = 20,

acceleration $= 0 + (40\pi)(-2.25\,e^{-1.5}) + 0 + (-2.25\,e^{-1.5})(40\pi)$

$= -180\pi\,e^{-1.5} =$ **-126.2 cm/s²** 3

total : 10

Problem 4. Find the co-ordinates of the turning points on the curve

$y = 3x^3 + 6x^2 + 3x - 1$ and distinguish between them.

Marks

Since $y = 3x^3 + 6x^2 + 3x - 1$

then $\frac{dy}{dx} = 9x^2 + 12x + 3 = 0$ for a turning point

$= (3x + 3)(3x + 1) = 0$

from which, $x = -1$ or $x = -\frac{1}{3}$ 3

When x = -1, y = $3(-1)^3 + 6(-1)^2 + 3(-1) - 1$ = $-3 + 6 - 3 - 1 = -1$

When x = $-\frac{1}{3}$, y = $3\left(-\frac{1}{3}\right)^3 + 6\left(-\frac{1}{3}\right)^2 + 3\left(-\frac{1}{3}\right) - 1 = -\frac{1}{9} + \frac{2}{3} - 1 - 1 = -1\frac{4}{9}$

Hence turning points occur at (-1, -1) and $\left(-\frac{1}{3}, -1\frac{4}{9}\right)$ 4

$\frac{d^2y}{dx^2} = 18x + 12$

When x = -1, $\frac{d^2y}{dx^2}$ is negative, hence **(-1, -1) is a maximum point** 1

When x = $-\frac{1}{3}$, $\frac{d^2y}{dx^2}$ is positive, hence **$\left(-\frac{1}{3}, -1\frac{4}{9}\right)$ is a minimum point** 1

total : 9

Problem 5. The heat capacity C of a gas varies with absolute temperature θ as shown: $C = 26.50 + 7.20 \times 10^{-3}\theta - 1.20 \times 10^{-6}\theta^2$

Determine the maximum value of C and the temperature at which it occurs.

Marks

$\frac{dC}{d\theta} = 7.20 \times 10^{-3} - 2.40 \times 10^{-6}\theta = 0$ for a maximum or minimum value,

from which, $\theta = \frac{7.20 \times 10^{-3}}{2.40 \times 10^{-6}}$ = **3000** 3

$\frac{d^2C}{d\theta^2} = -2.40 \times 10^{-6}$ which is negative and hence θ = 3000 gives a maximum value

$\mathbf{C_{max}} = 26.50 + (7.20 \times 10^{-3})(3000) - (1.20 \times 10^{-6})(3000)^2$

$= 26.50 + 21.6 - 10.8 =$ **37.3** 4

Hence the maximum value of C is 37.3 which occurs at a temperature of 3000

total : 7

Problem 6. Determine for the curve $y = 2x^2 - 3x$ at the point (2, 2):

(a) the equation of the tangent (b) the equation of the normal

Marks

(a)Gradient m = $\frac{dy}{dx} = 4x - 3$

At the point (2, 2), x = 2 and m = 4(2) - 3 = 5   2

Hence equation of tangent is: $y - y_1 = m(x - x_1)$

i.e. $y - 2 = 5(x - 2)$

i.e. $y - 2 = 5x - 10$

or **$y = 5x - 8$**   2

(b) Equation of normal is: $y - y_1 = -\frac{1}{m}(x - x_1)$

i.e. $y - 2 = -\frac{1}{5}(x - 2)$

i.e. $y - 2 = -\frac{1}{5}x + \frac{2}{5}$

or $5y - 10 = - x + 2$

or **$5y + x = 12$**   3

total : 7

Problem 7. A rectangular block of metal with a square cross-section has a total surface area of 250 cm^2. Find the maximum volume of the block of metal.

Marks

The rectangular block is shown in Figure 40 having dimensions x by x by y

Surface area, $A = 2x^2 + 4xy = 250$ (1)

Volume, $V = x^2 y$

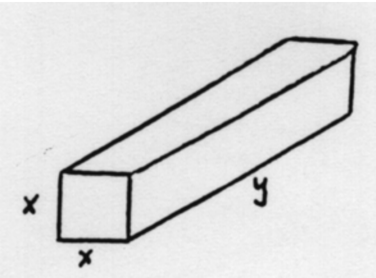


Figure 40

From equation (1), $4xy = 250 - 2x^2$ and $y = \frac{250 - 2x^2}{4x}$

Hence $V = x^2\left(\frac{250 - 2x^2}{4x}\right) = 62.5x - \frac{1}{2}x^3$ 3

$\frac{dV}{dx} = 62.5 - \frac{3}{2}x^2 = 0$ for a maximum or minimum value i.e. $62.5 = \frac{3}{2}x^2$ from which, $x = \sqrt{\frac{2(62.5)}{3}} = \pm 6.455$ cm

$\frac{d^2V}{dx^2} = -3x$ and when x = + 6.455, $\frac{d^2V}{dx^2}$ is negative, indicating a maximum value

Hence **maximum volume** = $62.5x - \frac{1}{2}x^3 = 62.5(6.455) - \frac{1}{2}(6.455)^3 =$ **269 cm^3** 4

total : 7

TOTAL ASSIGNMENT MARKS: 60

ASSIGNMENT 13 (PAGE 425)

This assignment covers the material contained in chapters 47 to 49.

Problem 1. Determine (a) $\int 3\sqrt{t^5}\,dt$ (b) $\int \frac{2}{\sqrt[3]{x^2}}\,dx$ (c) $\int (2+\theta)^2\,d\theta$

	Marks
(a) $\int 3\sqrt{t^5}\,dt = \int 3\,t^{\frac{5}{2}}\,dt = 3\frac{t^{\frac{7}{2}}}{\frac{7}{2}} + c = \mathbf{\frac{6}{7}\sqrt{t^7} + c}$	3
(b) $\int \frac{2}{\sqrt[3]{x^2}}\,dx = \int 2\,x^{-\frac{2}{3}}\,dx = 2\frac{x^{\frac{1}{3}}}{\frac{1}{3}} + c = \mathbf{6\sqrt[3]{x} + c}$	3
(c) $\int (2+\theta)^2\,d\theta = \int (4 + 4\theta + \theta^2)\,d\theta = 4\theta + \frac{4\theta^2}{2} + \frac{\theta^3}{3} + c$	
$= \mathbf{4\theta + 2\theta^2 + \frac{1}{3}\theta^3 + c}$	3

total : 9

Problem 2. Evaluate the following integrals, each correct to 4 significant figures:

(a) $\int_0^{\pi/3} 3\sin 2t\,dt$ (b) $\int_1^2 \left(\frac{2}{x^2} + \frac{1}{x} + \frac{3}{4}\right)dx$

	Marks
(a) $\int_0^{\pi/3} 3\sin 2t\,dt = -\frac{3}{2}\left[\cos 2t\right]_0^{\pi/3} = -\frac{3}{2}\left[\cos\frac{2\pi}{3} - \cos 0\right]$	
$= -\frac{3}{2}[-0.5 - 1] = \mathbf{2.250}$	5
(b) $\int_1^2 \left(\frac{2}{x^2} + \frac{1}{x} + \frac{3}{4}\right)dx = \int_1^2 \left(2x^{-2} + \frac{1}{x} + \frac{3}{4}\right)dx = \left[\frac{2x^{-1}}{-1} + \ln x + \frac{3}{4}x\right]_1^2$	
$= \left(-\frac{2}{2} + \ln 2 + \frac{6}{4}\right) - \left(-\frac{2}{1} + \ln 1 + \frac{3}{4}\right) = \mathbf{2.443}$	5

total : 10

Problem 3. Determine the following integrals:

(a) $\int 5(6t+5)^7\,dt$ (b) $\int \frac{3\ln x}{x}\,dx$ (c) $\int \frac{2}{\sqrt{(2\theta-1)}}\,d\theta$

Marks

(a) $\int 5(6t+5)^7\,dt$ Let $u = 6t + 5$ then $\frac{du}{dt} = 6$ and $dt = \frac{du}{6}$

Hence $\int 5(6t+5)^7\,dt = \int 5u^7 \frac{du}{6} = \frac{5}{6}\int u^7 du = \frac{5}{6}\frac{u^8}{8} + c$

$= \mathbf{\frac{5}{48}(6t+5)^8 + c}$ 3

(b) $\int \frac{3\ln x}{x}\,dx$ Let $u = \ln x$ then $\frac{du}{dx} = \frac{1}{x}$ and $dx = x\,du$

Hence $\int \frac{3\ln x}{x}\,dx = \int \frac{3u}{x}\,x\,du = \int 3u\,du = \frac{3u^2}{2} + c = \mathbf{\frac{3}{2}(\ln x)^2 + c}$ 3

(c) $\int \frac{2}{\sqrt{(2\theta-1)}}\,d\theta$ Let $u = 2\theta - 1$ then $\frac{du}{d\theta} = 2$ and $d\theta = \frac{du}{2}$

Hence $\int \frac{2}{\sqrt{(2\theta-1)}}\,d\theta = \int \frac{2}{\sqrt{u}}\frac{du}{2} = \int u^{-1/2}du = \frac{u^{1/2}}{\frac{1}{2}} + c = 2\sqrt{u} + c$

$= \mathbf{2\sqrt{(2\theta-1)} + c}$ 3

total : 9

Problem 4. Evaluate the following definite integrals:

(a) $\int_0^{\pi/2} 2\sin\left(2t+\frac{\pi}{3}\right)dt$ (b) $\int_0^1 3xe^{4x^2-3}\,dx$

Marks

(a) $\int_0^{\pi/2} 2\sin\left(2t+\frac{\pi}{3}\right)dt$ Let $u = 2t + \frac{\pi}{3}$ then $\frac{du}{dt} = 2$ and $dt = \frac{du}{2}$

Hence $\int 2\sin\left(2t+\frac{\pi}{3}\right)dt = \int 2\sin u\frac{du}{2} = \int \sin u\,du = -\cos u + c$

$= -\cos\left(2t+\frac{\pi}{3}\right) + c$

Thus $\int_0^{\pi/2} 2\sin\left(2t + \frac{\pi}{3}\right) dt = \left[-\cos\left(2t + \frac{\pi}{3}\right)\right]_0^{\pi/2} = -\left[\cos\left(\pi + \frac{\pi}{3}\right) - \cos\frac{\pi}{3}\right]$

$= -[-0.5 - 0.5] = \mathbf{1}$ 5

(b) $\int_0^1 3xe^{4x^2-3}\,dx$ Let $u = 4x^2 - 3$ then $\frac{du}{dx} = 8x$ and $dx = \frac{du}{8x}$

Hence $\int 3xe^{4x^2-3}\,dx = \int 3xe^u \frac{du}{8x} = \frac{3}{8}\int e^u\,du = \frac{3}{8}e^u + c = \frac{3}{8}e^{4x^2-3} + c$

Thus $\int_0^1 3xe^{4x^2-3}\,dx = \frac{3}{8}\left[e^{4x^2-3}\right]_0^1 = \frac{3}{8}[e^1 - e^{-3}] = \mathbf{1.001}$ 5

total : 10

Problem 5. Determine the following integrals:

(a) $\int \cos^3 x \sin^2 x\,dx$ (b) $\int \frac{2}{\sqrt{(9 - 4x^2)}}\,dx$

Marks

(a) $\int \cos^3 x \sin^2 x\,dx = \int \cos x \cos^2 x \sin^2 x\,dx = \int \cos x(1 - \sin^2 x)\sin^2 x\,dx$

$= \int (\cos x \sin^2 x - \cos x \sin^4 x)\,dx$

$= \mathbf{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c}$ 4

(b) $\int \frac{2}{\sqrt{(9 - 4x^2)}}\,dx = \int \frac{2}{\sqrt{\left[4\left(\frac{9}{4} - x^2\right)\right]}}\,dx = \int \frac{2}{2\sqrt{\left[\left(\frac{3}{2}\right)^2 - x^2\right]}}\,dx$

$= \int \frac{1}{\sqrt{\left[\left(\frac{3}{2}\right)^2 - x^2\right]}}\,dx = \arcsin\frac{x}{\frac{3}{2}} + c = \mathbf{\arcsin\frac{2x}{3} + c}$ 4

total : 8

Problem 6. Evaluate the following definite integrals, correct to 4 significant figures: (a) $\int_0^{\pi/2} 3\sin^2 t\,dt$ (b) $\int_0^{\pi/3} 3\cos 5\theta \sin 3\theta\,d\theta$ (c) $\int_0^2 \frac{5}{4 + x^2}\,dx$

Marks

(a) $\int_0^{\pi/2} 3\sin^2 t\,dt = \int_0^{\pi/2} 3\,\frac{1}{2}(1 - \cos 2t)\,dt = \frac{3}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2}$

$$= \frac{3}{2}\left[\left(\frac{\pi}{2} - \frac{\sin 2\frac{\pi}{2}}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$= \mathbf{\frac{3}{4}}\pi$ or **2.356** correct to 4 significant figures | 4

(b) $\int_0^{\pi/3} 3\cos 5\theta \sin 3\theta \, d\theta = \int_0^{\pi/3} \frac{3}{2}\left[\sin(5\theta + 3\theta) - \sin(5\theta - 3\theta)\right] d\theta$

$$= \frac{3}{2}\int_0^{\pi/3} (\sin 8\theta - \sin 2\theta)\, d\theta = \frac{3}{2}\left[-\frac{\cos 8\theta}{8} + \frac{\cos 2\theta}{2}\right]_0^{\pi/3}$$

$$= \frac{3}{2}\left[\left(-\frac{\cos\frac{8\pi}{3}}{8} + \frac{\cos\frac{2\pi}{3}}{2}\right) - \left(-\frac{\cos 0}{8} + \frac{\cos 0}{2}\right)\right]$$

$= \frac{3}{2}$ [(0.0625 - 0.25) - (-0.125 + 0.5)]

= **-0.8438** correct to 4 significant figures | 6

(c) $\int_0^2 \frac{5}{4 + x^2}\, dx = 5\int_0^2 \frac{1}{2^2 + x^2}\, dx = \left[\frac{5}{2}\arctan\frac{x}{2}\right]_0^2$

$= \frac{5}{2}$ [arctan 1 - arctan 0] = **1.963** correct to 4 significant figures | 4

total : 14

TOTAL ASSIGNMENT MARKS: 60

ASSIGNMENT 14 (PAGE 447)

This assignment covers the material contained in chapters 50 to 53.

Problem 1. Determine (a) $\int \frac{x - 11}{x^2 - x - 2}\,dx$ (b) $\int \frac{3 - x}{(x^2 + 3)(x + 3)}\,dx$

Marks

(a) Let $\frac{x - 11}{x^2 - x - 2} \equiv \frac{x - 11}{(x - 2)(x + 1)} = \frac{A}{(x - 2)} + \frac{B}{(x + 1)} = \frac{A(x + 1) + B(x - 2)}{(x - 2)(x + 1)}$

Hence $x - 11 = A(x + 1) + B(x - 2)$

Let x = 2: $-9 = 3A$ hence $\mathbf{A = -3}$

Let x = -1: $-12 = -3B$ hence $\mathbf{B = 4}$

Hence $\mathbf{\frac{x - 11}{x^2 - x - 2} = \frac{4}{(x + 1)} - \frac{3}{(x - 2)}}$ 5

$\int \frac{x - 11}{x^2 - x - 2}\,dx = \int \frac{4}{(x + 1)} - \frac{3}{(x - 2)}\,dx = \mathbf{4 \ln(x + 1) - 3 \ln(x - 2) + c}$

or $\mathbf{\ln\left\{\frac{(x + 1)^4}{(x - 2)^3}\right\} + c}$ 4

(b) Let $\frac{3 - x}{(x^2 + 3)(x + 3)} \equiv \frac{Ax + B}{(x^2 + 3)} + \frac{C}{(x + 3)} = \frac{(Ax + B)(x + 3) + C(x^2 + 3)}{(x^2 + 3)(x + 3)}$

Hence $3 - x = (Ax + B)(x + 3) + C(x^2 + 3)$

Let x = -3: $6 = 0 + 12C$ hence $\mathbf{C = \frac{1}{2}}$

x^2 coefficients: $0 = A + C$ hence $\mathbf{A = -\frac{1}{2}}$

x coefficients: $-1 = 3A + B$ hence $-1 = -\frac{3}{2} + B$ and $\mathbf{B = \frac{1}{2}}$

Hence $\int \frac{3 - x}{(x^2 + 3)(x + 3)}\,dx = \int \frac{-\frac{1}{2}x + \frac{1}{2}}{(x^2 + 3)} + \frac{\frac{1}{2}}{(x + 3)}\,dx$ 6

$$= \int \frac{-\frac{1}{2}x}{(x^2 + 3)} + \frac{\frac{1}{2}}{(x^2 + 3)} + \frac{\frac{1}{2}}{(x + 3)}\,dx$$

$$= -\frac{1}{4}\ln(x^2 + 3) + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}\right) + \frac{1}{2}\ln(x + 3) + c$$

$$= -\frac{1}{4}\ln(x^2+3) + \frac{1}{2\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + \frac{1}{2}\ln(x+3) + c$$ 6

total : 21

Problem 2. Evaluate $\int_1^2 \frac{3}{x^2(x+2)}\,dx$ correct to 4 significant figures.

Marks

Let $\frac{3}{x^2(x+2)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+2)} = \frac{Ax(x+2) + B(x+2) + Cx^2}{x^2(x+2)}$

Hence $3 = Ax(x+2) + B(x+2) + Cx^2$

Let $x = 0$: $3 = 0 + 2B + 0$ hence **$B = 3/2$**

Let $x = -2$: $3 = 0 + 0 + 4C$ hence **$C = 3/4$**

x^2 coefficients: $0 = A + C$ hence **$A = -3/4$** 6

Hence $\int_1^2 \frac{3}{x^2(x+2)}\,dx = \int \frac{-3/4}{x} + \frac{3/2}{x^2} + \frac{3/4}{(x+2)}\,dx$

$$= \left[-\frac{3}{4}\ln x - \frac{3}{2x} + \frac{3}{4}\ln(x+2)\right]_1^2$$

$$= \left(-\frac{3}{4}\ln 2 - \frac{3}{2(2)} + \frac{3}{4}\ln 4\right) - \left(-\frac{3}{4}\ln 1 - \frac{3}{2} + \frac{3}{4}\ln 3\right)$$

$= (-0.23014) - (-0.67604) =$ **0.4459** correct to 4 significant figures 6

total : 12

Problem 3. Determine: $\int \frac{dx}{2\sin x + \cos x}$

Marks

If $\tan\frac{x}{2}$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$

Thus $\int \frac{dx}{2\sin x + \cos x} = \int \frac{\frac{2\,dt}{1+t^2}}{2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{4t+1-t^2}{1+t^2}} = \int \frac{2\,dt}{1+4t-t^2}$ 2

$$= \int \frac{-2\,dt}{t^2-4t-1} = \int \frac{-2\,dt}{(t-2)^2-5} = \int \frac{2\,dt}{\left(\sqrt{5}\right)^2-(t-2)^2}$$

$$= 2\left[\frac{1}{2\sqrt{5}}\ \ln\left\{\frac{\sqrt{5}+(t-2)}{\sqrt{5}-(t-2)}\right\}\right] + c$$

i.e. $$\int \frac{dx}{2\sin x + \cos x} = \mathbf{\frac{1}{\sqrt{5}}\ \ln\left\{\frac{\sqrt{5}-2+\tan\frac{x}{2}}{\sqrt{5}+2-\tan\frac{x}{2}}\right\} + c}$$ 3

total : 5

Problem 4. Determine the following integrals:

(a) $\int 5xe^{2x}\ dx$ (b) $\int t^2 \sin 2t\ dt$

Marks

(a) $\int 5xe^{2x}\ dx$ Let $u = 5x$ then $\frac{du}{dx} = 5$ from which $du = 5\ dx$

and $dv = e^{2x}dx$ then $v = \int e^{2x}dx = \frac{1}{2}e^{2x}$

Hence $\int 5xe^{2x}\ dx = (5x)(\frac{1}{2}e^{2x}) - \int(\frac{1}{2}e^{2x})(5dx) = \frac{5}{2}xe^{2x} - \frac{5}{2}\int e^{2x}dx$

$$= \mathbf{\frac{5}{2}xe^{2x} - \frac{5}{4}e^{2x} + c}$$ 5

(b) $\int t^2 \sin 2t\ dt$ Let $u = t^2$ then $\frac{du}{dt} = 2t$ from which $du = 2t\ dt$

and $dv = \sin 2t\ dt$ then $v = \int \sin 2t\ dt = -\frac{1}{2}\cos 2t$

Hence $\int t^2 \sin 2t\ dt = (t^2)(-\frac{1}{2}\cos 2t) - \int(-\frac{1}{2}\cos 2t)(2t\ dt)$

$$= -\frac{1}{2}t^2\cos 2t + \left[\int t\cos 2t\ dt\right] \qquad (1)$$ 3

$\int t\cos 2t\ dt$ Let $u = t$ then $\frac{du}{dt} = 1$ from which $du = dt$

and $dv = \cos 2t\ dt$ then $v = \int \cos 2t\ dt = \frac{1}{2}\sin 2t$

Hence $\int t\cos 2t\ dt = (t)(\frac{1}{2}\sin 2t) - \int(\frac{1}{2}\sin 2t)dt$

$$= \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t$$

Substituting in equation (1) gives:

$$\int t^2 \sin 2t\,dt = -\frac{1}{2}t^2\cos 2t + \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t + c$$ 4

total : 12

Problem 5. Evaluate correct to 3 decimal places: $\int_1^4 \sqrt{x}\ln x\,dx$

Marks

$\int_1^4 \sqrt{x}\ln x\,dx$ Let $u = \ln x$ then $\frac{du}{dx} = \frac{1}{x}$ from which $du = \frac{dx}{x}$

and $dv = \sqrt{x}\,dx$ then $v = \int x^{1/2}dx = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$

Hence $\int \sqrt{x}\ln x\,dx = (\ln x)\left(\frac{2}{3}x^{3/2}\right) - \int\left(\frac{2}{3}x^{3/2}\right)\frac{du}{x}$

$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int x^{1/2}dx = \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\left(\frac{x^{3/2}}{3/2}\right) + c$

$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{4}{9}\sqrt{x^3} + c$ 5

Thus $\int_1^4 \sqrt{x}\ln x\,dx = \left[\frac{2}{3}\sqrt{x^3}\ln x - \frac{4}{9}\sqrt{x^3}\right]_1^4$

$= \left(\frac{2}{3}\sqrt{4^3}\ln 4 - \frac{4}{9}\sqrt{4^3}\right) - \left(\frac{2}{3}\sqrt{1^3}\ln 1 - \frac{4}{9}\sqrt{1^3}\right)$

$= \left(\frac{16}{3}\ln 4 - \frac{32}{9}\right) - \left(\frac{2}{3}\ln 1 - \frac{4}{9}\right)$

$= (3.83801) - (-0.44444) =$ **4.282** correct to 3 decimal places 5

total : 10

Problem 6. Evaluate $\int_1^3 \frac{5}{x^2}dx$ using (a) integration (b) the trapezoidal rule (c) the mid-ordinate rule (d) Simpson's rule. In each of the approximate methods use 8 intervals and give the answers correct to 3 decimal places.

Marks

(a) $\int_1^3 \frac{5}{x^2}dx = \int_1^3 5x^{-2} = \left[\frac{5x^{-1}}{-1}\right]_1^3 = -5\left[\frac{1}{x}\right]_1^3 = -5\left[\frac{1}{3} - 1\right]$

$=$ **3.333**, correct to 3 decimal places 4

(b) With the trapezoidal rule, width of interval = $\frac{3-1}{8}$ = 0.25, hence

the ordinates occur at 1, 1.25, 1.5, 1.75, ..

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
$\frac{5}{x^2}$	5	3.2	$2.\dot{2}$	1.6327	1.25	0.9877	0.8	0.6612	$0.\dot{5}$

Hence $\int_1^3 \frac{5}{x^2}\,dx \approx 0.25\left\{\frac{1}{2}(5 + 0.\dot{5}) + 3.2 + 2.\dot{2} + 1.6327 + 1.25 + 0.9877 + 0.8 + 0.6612\right\}$

$= 0.25(13.5316) = \mathbf{3.383}$ 4

(c) With the mid-ordinate rule, the mid-ordinates occur at 1.125, 1.375,

1.625, ..

x	1.125	1.375	1.625	1.875	2.125	2.375	2.625	2.875
$\frac{5}{x^2}$	3.9506	2.6446	1.8935	1.4222	1.1073	0.8864	0.7256	0.6049

Hence $\int_1^3 \frac{5}{x^2}\,dx \approx 0.25(3.9506 + 2.6446 + 1.8935 + 1.4222 + 1.1073$

$+ 0.8864 + 0.7256 + 0.6049)$

$= 0.25(13.2351) = \mathbf{3.309}$ 4

(d) Using Simpson's rule, using the table of values from part (a),

$\int_1^3 \frac{5}{x^2}\,dx \approx \frac{1}{3}(0.25)\{(5 + 0.\dot{5}) + 4(3.2 + 1.6327 + 0.9877 + 0.6612)$

$+ 2(2.\dot{2} + 1.25 + 0.8)\}$

$= \frac{1}{3}(0.25)\{5.\dot{5} + 25.9264 + 8.5444\} = \mathbf{3.336}$ 4

total : 16

Problem 7. An alternating current i has the following values at equal intervals of 5 ms

Time t(ms)	0	5	10	15	20	25	30
Current i(A)	0	4.8	9.1	12.7	8.8	3.5	0

Charge q, in coulombs, is given by $q = \int_0^{30\times 10^{-3}} i\,dt$

Use Simpson's rule to determine the approximate charge in the 30 ms period.

Marks

$$q = \int_0^{30\times10^{-3}} i\,dt \approx \frac{1}{3}(5 \times 10^{-3})\left\{ (0) + 4(4.8 + 12.7 + 3.5) + 2(9.1 + 8.8) \right\}$$

$$= \frac{1}{3}(5 \times 10^{-3})\left\{ (0) + 84 + 35.8 \right\} = 199.7 \times 10^{-3}$$

$$= \mathbf{0.1997\ C}$$ 4

total : 4

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 15 (PAGE 482)

This assignment covers the material contained in chapters 54 to 58.

Problem 1. The force F newtons acting on a body at a distance x metres from a fixed point is given by $F = 2x + 3x^2$. If work done $= \int_{x_1}^{x_2} F\,dx$, determine the work done when the body moves from the position when x = 1 m to that when x = 4 m.

Marks

Work done $= \int_{x_1}^{x_2} F\,dx = \int_1^4 2x + 3x^2\,dx = \left[x^2 + x^3 \right]_1^4 = (16 + 64) - (1 + 1)$

$= 80 - 2 = 78$ N 4

total : 4

Problem 2. Sketch and determine the area enclosed by the curve $y = 3\sin\frac{\theta}{2}$, the θ-axis and ordinates θ = 0 and $\theta = \frac{2\pi}{3}$.

Marks

A sketch of $y = 3\sin\frac{\theta}{2}$ is shown in Figure 41.

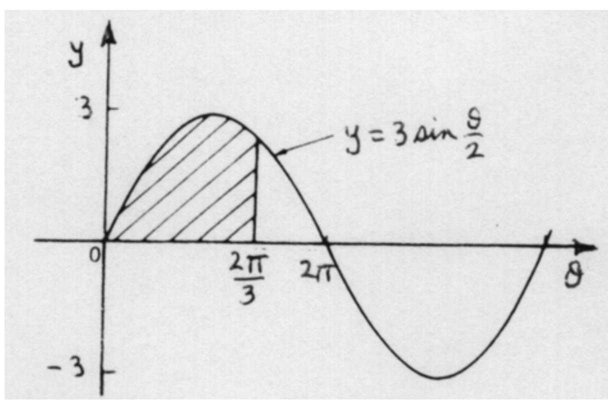


Figure 41 2

Shaded area $= \int_0^{2\pi/3} 3\sin\frac{\theta}{2}\,d\theta = \left[-\frac{3}{\frac{1}{2}}\cos\frac{\theta}{2} \right]_0^{2\pi/3} = -6\left[\cos\frac{2\pi/3}{2} - \cos 0\right]$

$= -6\left[\cos\frac{\pi}{2} - \cos 0\right]$ = **3 square units** 2

total : 4

Problem 3. Calculate the area between the curve $y = x^3 - x^2 - 6x$ and the x-axis.

Marks

$y = x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x - 3)(x + 2)$

When $y = 0$, $x = 0$, $x = 3$ or $x = -2$

When $x = 1$, $y = 1 - 1 - 6$ i.e. negative, hence the curve is as shown in Figure 42.

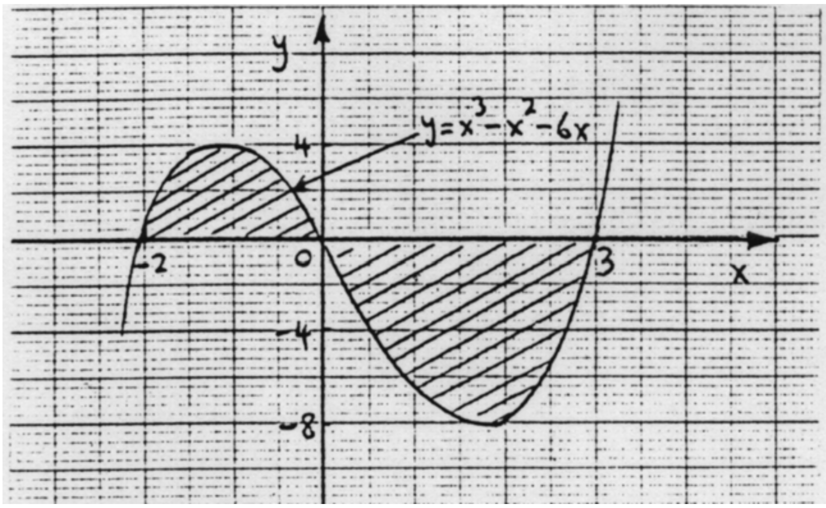


Figure 42 4

$$\text{Shaded area} = \int_{-2}^{0}(x^3 - x^2 - 6x)\,dx - \int_{0}^{3}(x^3 - x^2 - 6x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}\right]_{0}^{3}$$

$$= \left[(0) - \left(4 + \frac{8}{3} - 12\right)\right] - \left[\left(\frac{81}{4} - 9 - 27\right) - (0)\right]$$

$$= \left[5\frac{1}{3}\right] - \left[-15\frac{3}{4}\right] = \mathbf{21\frac{1}{12}} \quad \textbf{or} \quad \mathbf{21.08\dot{3}} \textbf{ square units}$$ 6

total : 10

Problem 4. A voltage $v = 25 \sin 50\pi t$ volts is applied across an electrical circuit. Determine, using integration, its mean and r.m.s. values over the range $t = 0$ to $t = 20$ ms, each correct to 4 significant figures.

Marks

$$\text{Mean value} = \frac{1}{20\times10^{-3}}\int_0^{20\times10^{-3}} 25\sin 50\pi t\,dt = (50(25)\left[-\frac{\cos 50\pi t}{50\pi}\right]_0^{20\times10^{-3}}$$

$$= -\frac{(50)(25)}{50\pi}\left[\cos 50\pi t\right]_0^{20\times10^{-3}} = -\frac{25}{\pi}[-1-1] = \frac{50}{\pi} = \mathbf{15.92\ volts}$$ 5

$$\text{Rms value} = \sqrt{\left\{\frac{1}{20\times10^{-3}}\int_0^{20\times10^{-3}}(25)^2\sin^2 50\pi t\,dt\right\}}$$

$$= \sqrt{\left\{50(25)^2\int_0^{20\times10^{-3}}\frac{1-\cos 2(50\pi t)}{2}\,dt\right\}}$$

$$= \sqrt{\left\{\frac{50(25)^2}{2}\left[t-\frac{\sin 100\pi t}{100\pi}\right]_0^{20\times10^{-3}}\right\}}$$

$$= \sqrt{\left\{\frac{50(25)^2}{2}\left[\left(20\times10^{-3}-\frac{\sin 100\pi(20\times10^{-3})}{100\pi}\right)-(0)\right]\right\}}$$

$$= \sqrt{\left\{\frac{50(25)^2}{2}(20\times10^{-3})\right\}} = \mathbf{17.68\ volts}$$ 7

total : 12

Problem 5. Sketch on the same axes the curves $x^2 = 2y$ and $y^2 = 16x$ and determine the co-ordinates of the points of intersection. Determine (a) the area enclosed by the curves, and (b) the volume of the solid produced if the area is rotated one revolution about the x-axis.

Marks

The curves are equal at the points of intersection. Thus, equating the

two y values gives: $\frac{x^2}{2} = 4\sqrt{x}$ or $\frac{x^4}{4} = 16x$

$x^4 = 64x$ and $x^4 - 64x = 0$

$x(x^3 - 64) = 0$ from which, $x = 0$ or $x = 4$

When $x = 0$, $y = 0$ and when $x = 4$, $y = 8$

Hence **(0, 0) and (4, 8) are the co-ordinates of the points of intersection.** 5

The curves are shown in Figure 43.

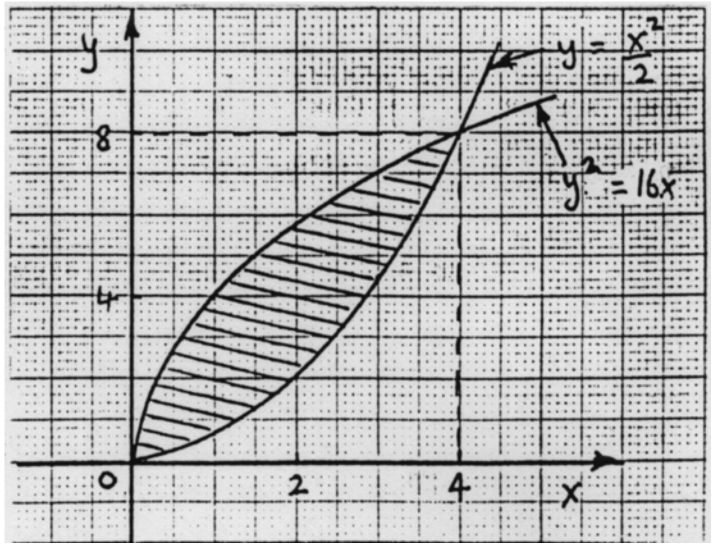


Figure 43

(a) Shaded area $= \int_0^4 \left(4\sqrt{x} - \frac{x^2}{2}\right) dx = \left[\frac{4x^{3/2}}{3/2} - \frac{x^3}{6}\right]_0^4 = \left(\frac{8}{3}\sqrt{4^3} - \frac{4^3}{6}\right) - (0)$

$= \mathbf{10\frac{2}{3}}$ **square units** | 4

(b) Volume $= \int_0^4 \pi y^2\, dx = \pi\int_0^4 \left(16x - \frac{x^4}{4}\right) dx = \pi\left[\frac{16x^2}{2} - \frac{x^5}{20}\right]_0^4$

$= \pi\left[\left(8(4)^2 - \frac{4^5}{20}\right) - (0)\right] = \pi[128 - 51.2] =$ **76.8π cubic units**

or **241.3 cubic units** | 4

total : 13

Problem 6. Calculate the position of the centroid of the sheet of metal formed by the x-axis and the part of the curve $y = 5x - x^2$ which lies above the x-axis.

Marks

$y = 5x - x^2 = x(5 - x)$ and when $y = 0$, i.e. the x-axis, $x = 0$ or $x = 5$. | 2

A sketch of $y = 5x - x^2$ is shown in Figure 44 where $\bar{x} = 2.5$ by symmetry. | 2

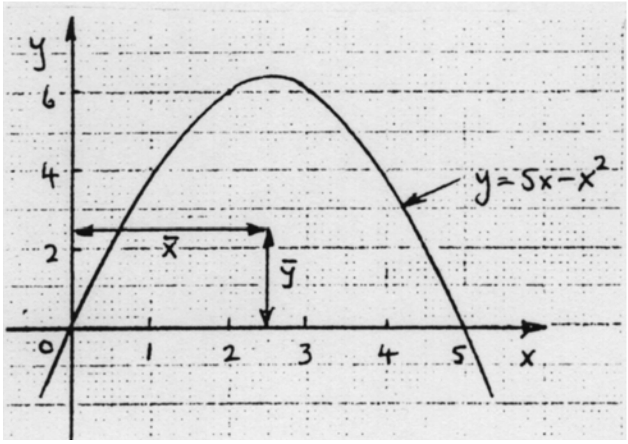


Figure 44

$$\bar{y} = \frac{\frac{1}{2}\int_0^5 y^2\,dx}{\int_0^5 y\,dx} = \frac{\frac{1}{2}\int_0^5 (5x - x^2)^2\,dx}{\int_0^5 5x - x^2\,dx} = \frac{\frac{1}{2}\int_0^5 (25x^2 - 10x^3 + x^4)\,dx}{\int_0^5 5x - x^2\,dx}$$

$$= \frac{\frac{1}{2}\left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5}{\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5} = \frac{\frac{1}{2}\left[\frac{25(5)^3}{3} - \frac{10(5)^4}{4} + \frac{(5)^5}{5}\right]}{\left[\frac{5(25)}{2} - \frac{125}{3}\right]} = \frac{52.08333}{20.8333} = 2.5$$

Hence **the co-ordinates of the centroid are at (2.5, 2.5)** 5

total : 9

Problem 7. A cylindrical pillar of diameter 500 mm has a groove cut around its circumference as shown in Figure A15.1. The section of the groove is a semicircle of diameter 40 mm. Given that the centroid of a semicircle from its base is $\frac{4r}{3\pi}$, use the theorem of Pappus to determine the volume of material removed, in cm^3, correct to 3 significant figures.

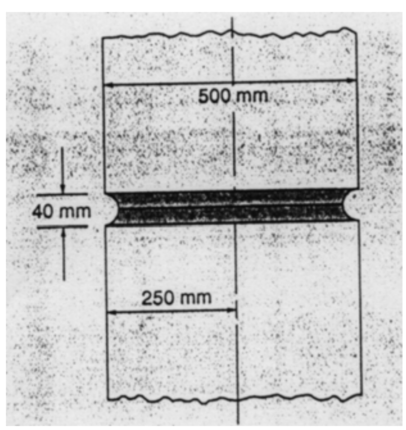


Figure A15.1

Marks

Distance of the centroid of the semicircle = $\frac{4r}{3\pi} = \frac{4(20)}{3\pi} = \frac{80}{3\pi}$ mm

Distance of the centroid from the centre of the pillar = $(250 - \frac{80}{3\pi})$mm

Distance moved by the centroid in one revolution = $2\pi(250 - \frac{80}{3\pi})$

$$= (500\pi - \frac{160}{3})\text{mm}$$ 4

From Pappus, volume = area × distance moved by centroid

$$= \left(\frac{1}{2}\pi 20^2\right)\left(500\pi - \frac{160}{3}\right) = 953450 \text{ mm}^3$$

i.e. volume of material removed = **953 cm^3** 4

total : 8

Problem 8. For each of the areas shown in Figure A15.2 determine the second moment of area and radius of gyration about axis XX.

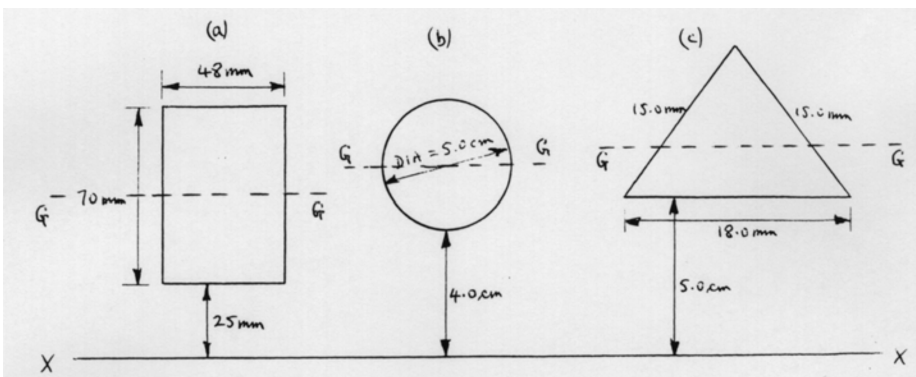


Figure A15.2

Marks

(b) In Figure A15.2(a), $I_{GG} = \frac{lb^3}{12} = \frac{(48)(70)^3}{12} = 1372000 \text{ mm}^4$

The second moment of area about axis XX,

$I_{XX} = I_{GG} + Ad^2 = 1372000 + (70 \times 48)(25 + 35)^2 =$ **13468000 mm^4** 3

$I_{XX} = Ak_{XX}^2$ from which, radius of gyration,

$$k_{XX} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{13468000}{70 \times 48}} = \textbf{63.31 mm}$$ 2

(b) In Figure A15.2(b), $I_{GG} = \frac{\pi r^4}{4} = \frac{\pi(2.50)^4}{4} = 9.765625\pi \text{ cm}^4$

$I_{XX} = I_{GG} + Ad^2 = 9.765625\pi + \pi(2.5)^2(4.0 + 2.50)^2$

$= 9.765625\pi + 264.0625\pi = \mathbf{860.26\ cm^4}$ 3

$I_{XX} = Ak_{XX}^2$ from which, radius of gyration,

$k_{XX} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{860.26}{\pi(2.50)^2}} = \mathbf{6.62\ cm}$ 2

(c) In Figure A15.2(c), perpendicular height of triangle = $\sqrt{15.0^2 - 9.0^2}$

$= 12.0$ cm

$I_{GG} = \frac{bh^3}{36} = \frac{(18)(12)^3}{36} = 864 \text{ cm}^4$

$I_{XX} = I_{GG} + Ad^2 = 864 + \left(\frac{1}{2}(18)(12)\right)\left(5.0 + \frac{12.0}{3}\right)^2 = 864 + 8748 = \mathbf{9612\ cm^4}$ 3

Radius of gyration, $k_{XX} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{9612}{\frac{1}{2}(18)(12)}} = \mathbf{9.43\ cm}$ 2

total : 15

Problem 9. A circular door is hinged so that it turns about a tangent. If its diameter is 1.0 m find its second moment of area and radius of gyration about the hinge.

Marks

From Table 58.1, page 474, second moment of area about a tangent

$= \frac{5\pi}{4} r^4 = \frac{5\pi}{4}\left(\frac{1}{2}\right)^4 = \mathbf{0.245\ m^3}$ 3

Radius of gyration $k = \frac{\sqrt{5}}{2} r = \frac{\sqrt{5}}{2}(0.50) = \mathbf{0.559\ m}$ 2

total : 5

TOTAL ASSIGNMENT MARKS: 80

ASSIGNMENT 16 (PAGE 521)

This assignment covers the material contained in chapters 59 to 61.

Problem 1. Use the laws and rules of Boolean algebra to simplify the following expressions: (a) $B.(A + \bar{B}) + A.\bar{B}$

(b) $\bar{A}.\bar{B}.\bar{C} + \bar{A}.B.\bar{C} + \bar{A}.B.C + \bar{A}.\bar{B}.C$

Marks

(a) $B.(A + \bar{B}) + A.\bar{B} = A.B + B.\bar{B} + A.\bar{B} = A.B + 0 + A.\bar{B} = A.B + A.\bar{B}$

$= A.(B + \bar{B}) = A.(1) = \mathbf{A}$ 4

(b) $\bar{A}.\bar{B}.\bar{C} + \bar{A}.\bar{B}.C = \bar{A}.\bar{B}.(\bar{C} + C) = \bar{A}.\bar{B}.(1) = \bar{A}.\bar{B}$

and $\bar{A}.B.\bar{C} + \bar{A}.B.C = \bar{A}.B.(\bar{C} + C) = \bar{A}.B.(1) = \bar{A}.B$

Thus $\bar{A}.\bar{B}.\bar{C} + \bar{A}.B.\bar{C} + \bar{A}.B.C + \bar{A}.\bar{B}.C = \bar{A}.\bar{B} + \bar{A}.B$

$= \bar{A}.(\bar{B} + B) = \bar{A}.(1) = \bar{\mathbf{A}}$ 5

total : 9

Problem 2. Simplify the Boolean expression $\overline{A.\bar{B} + A.B.\bar{C}}$ using de Morgan's laws

Marks

$\overline{A.\bar{B} + A.B.\bar{C}} = \overline{A.\bar{B}}.\overline{A.B.\bar{C}} = (\bar{A} + \bar{\bar{B}}).(\overline{A.B} + \bar{\bar{C}})$ 1

$= (\bar{A} + B).(\overline{A.B} + C) = (\bar{A} + B).(\bar{A} + \bar{B} + C)$ 1

$= \bar{A}.\bar{A} + \bar{A}.\bar{B} + \bar{A}.C + \bar{A}.B + B.\bar{B} + B.C = \bar{A} + \bar{A}.\bar{B} + \bar{A}.C + \bar{A}.B + 0 + B.C$ 1

$= \bar{A} + \bar{A}.(\bar{B} + C + B) + B.C = \bar{A} + \bar{A}(1 + C) + B.C$ 1

$= \bar{A} + \bar{A}(1) + B.C = \mathbf{\bar{A} + B.C}$ 1

total : 5

Problem 3. Use a Karnaugh map to simplify the Boolean expression:

$\bar{A}.\bar{B}.\bar{C} + \bar{A}.B.\bar{C} + \bar{A}.B.C + A.\bar{B}.C$

Marks

C \ AB	00	01	11	10
0	1	1		
1		1		1

3

The horizontal couple gives: $\bar{A}.\bar{B}.\bar{C} + \bar{A}.B.\bar{C} = \bar{A}.\bar{C}$

The vertical couple gives: $\bar{A}.B.\bar{C} + \bar{A}.B.C = \bar{A}.B$

The bottom right-hand corner square cannot be coupled and is $A.\bar{B}.C$

Hence $\bar{A}.\bar{B}.\bar{C} + \bar{A}.B.\bar{C} + \bar{A}.B.C + A.\bar{B}.C =$ $\mathbf{\bar{A}.\bar{C} + \bar{A}.B + A.\bar{B}.C}$ 3

total : 6

Problem 4. A clean room has two entrances, each having two doors, as shown in Fig. A16.1. A warning bell must sound if both doors A and B or doors C and D are open at the same time. Write down the Boolean expression depicting this occurrence, and devise a logic network to operate the bell using NAND-gates only.

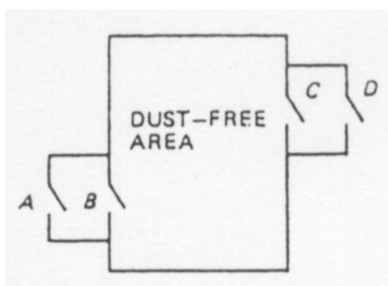
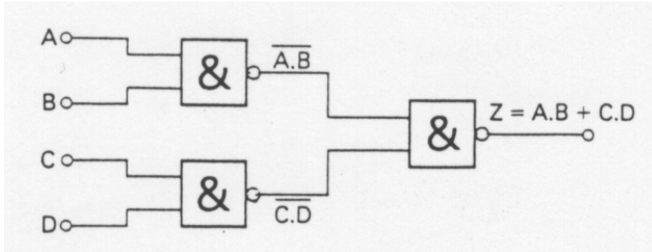


Figure A16.1

Marks

The Boolean expression which will ring the warning bell is **A.B + C.D** 2

A circuit using NAND-gates only is shown in Figure 45.

Figure 45 

6

total : 8

Problem 5. Determine $\begin{pmatrix} -5 & 2 \\ 7 & -8 \end{pmatrix} \times \begin{pmatrix} 1 & 6 \\ -3 & -4 \end{pmatrix}$

Marks

$$\begin{pmatrix} -5 & 2 \\ 7 & -8 \end{pmatrix} \times \begin{pmatrix} 1 & 6 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} \mathbf{-11} & \mathbf{-38} \\ \mathbf{31} & \mathbf{74} \end{pmatrix}$$ 4

total : 4

Problem 6. Calculate $\begin{vmatrix} j3 & (1+j2) \\ (-1-j4) & -j2 \end{vmatrix}$

Marks

$$\begin{vmatrix} j3 & (1+j2) \\ (-1-j4) & -j2 \end{vmatrix} = -j^2 6 - (1+j2)(-1-j4)$$

$$= 6 - [-1 - j4 - j2 - j^2 8]$$

$$= 6 - [7 - j6] = \mathbf{-1 + j6}$$ 4

total : 4

Problem 7. Determine the inverse of $\begin{pmatrix} -5 & 2 \\ 7 & -8 \end{pmatrix}$

Marks

If $B = \begin{pmatrix} -5 & 2 \\ 7 & -8 \end{pmatrix}$ then $B^{-1} = \dfrac{1}{40-14}\begin{pmatrix} -8 & -2 \\ -7 & -5 \end{pmatrix} = \dfrac{\mathbf{1}}{\mathbf{26}}\begin{pmatrix} \mathbf{-8} & \mathbf{-2} \\ \mathbf{-7} & \mathbf{-5} \end{pmatrix}$

or $\begin{pmatrix} -\dfrac{\mathbf{4}}{\mathbf{13}} & -\dfrac{\mathbf{1}}{\mathbf{13}} \\ -\dfrac{\mathbf{7}}{\mathbf{26}} & -\dfrac{\mathbf{5}}{\mathbf{26}} \end{pmatrix}$ 4

total : 4

Problem 8. Determine $\begin{pmatrix} -1 & 3 & 0 \\ 4 & -9 & 2 \\ -5 & 7 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 3 \\ -5 & 1 & 0 \\ 4 & -6 & 2 \end{pmatrix}$

Marks

$$\begin{pmatrix} -1 & 3 & 0 \\ 4 & -9 & 2 \\ -5 & 7 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 3 \\ -5 & 1 & 0 \\ 4 & -6 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{-17} & \mathbf{4} & \mathbf{-3} \\ \mathbf{61} & \mathbf{-25} & \mathbf{16} \\ \mathbf{-41} & \mathbf{6} & \mathbf{-13} \end{pmatrix}$$ 9

total : 9

Problem 9. Calculate the determinate of $\begin{pmatrix} 2 & -1 & 3 \\ -5 & 1 & 0 \\ 4 & -6 & 2 \end{pmatrix}$

Marks

$$\begin{vmatrix} 2 & -1 & 3 \\ -5 & 1 & 0 \\ 4 & -6 & 2 \end{vmatrix} = 3\begin{vmatrix} -5 & 1 \\ 4 & -6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -5 & 1 \end{vmatrix}$$ using the third column

$= 3(30 - 4) + 2(2 - 5) = 3(26) + 2(-3) = 78 - 6 =$ **72** 5

tota l: 5

Problem 10. Using matrices to solve the following simultaneous equations:

$4x - 3y = 17$

$x + y + 1 = 0$

Marks

Since $4x - 3y = 17$

$x + y = -1$

then $\begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ -1 \end{pmatrix}$ 1

The inverse of $\begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$ is $\frac{1}{4 - -3}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$ 2

Hence $\frac{1}{7}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 17 \\ -1 \end{pmatrix}$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{7}\begin{pmatrix}14\\-21\end{pmatrix}$$

and $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix}$ i.e. **x = 2 and y = - 3** 3

total : 6

Problem 11. Use determinants to solve the following simultaneous equations:

4x + 9y + 2z = 21

-8x + 6y - 3z = 41

3x + y - 5z = -73

Marks

4x + 9y + 2z - 21 = 0

-8x + 6y - 3z - 41 = 0

3x + y - 5z + 73 = 0

Hence $\dfrac{x}{\begin{vmatrix}9 & 2 & -21\\6 & -3 & -41\\1 & -5 & 73\end{vmatrix}} = \dfrac{-y}{\begin{vmatrix}4 & 2 & -21\\-8 & -3 & -41\\3 & -5 & 73\end{vmatrix}} = \dfrac{z}{\begin{vmatrix}4 & 9 & -21\\-8 & 6 & -41\\3 & 1 & 73\end{vmatrix}} = \dfrac{-1}{\begin{vmatrix}4 & 9 & 2\\-8 & 6 & -3\\3 & 1 & -5\end{vmatrix}}$ 4

$$\frac{x}{9(-424) - 2(479) - 21(-27)} = \frac{-y}{4(-424) - 2(-461) - 21(49)} = \frac{z}{4(479) - 9(-461) - 21(-26)}$$

$$= \frac{-1}{4(-27) - 9(49) + 2(-26)}$$

i.e. $\dfrac{x}{-4207} = \dfrac{-y}{-1803} = \dfrac{z}{6611} = \dfrac{-1}{-601}$ 3

Hence **x** $= \dfrac{-4207}{601}$ = **-7** **y** $= \dfrac{1803}{601}$ = **3** and **z** $= \dfrac{6611}{601}$ = **11** 3

(or use Cramer's rule) **total : 10**

Problem 12. The simultaneous equations representing the currents flowing in an unbalanced, three-phase, star-connected, electrical network are as follows:

$2.4I_1 + 3.6I_2 + 4.8I_3 = 1.2$

$-3.9I_1 + 1.3I_2 - 6.5I_3 = 2.6$

$1.7I_1 + 11.9I_2 + 8.5I_3 = 0$

Using matrices, solve the equations for I_1, I_2 and I_3

Marks

$$\begin{pmatrix} 2.4 & 3.6 & 4.8 \\ -3.9 & 1.3 & -6.5 \\ 1.7 & 11.9 & 8.5 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 2.6 \\ 0 \end{pmatrix}$$

The inverse of the 3 by 3 matrix is

$$\frac{1}{2.4(88.4) - 3.6(-22.1) + 4.8(-48.62)}\begin{pmatrix} 88.4 & 26.52 & -29.64 \\ 22.1 & 12.24 & -3.12 \\ -48.62 & -22.44 & 17.16 \end{pmatrix}$$ 4

Hence $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \frac{1}{58.344}\begin{pmatrix} 88.4 & 26.52 & -29.64 \\ 22.1 & 12.24 & -3.12 \\ -48.62 & -22.44 & 17.16 \end{pmatrix}\begin{pmatrix} 1.2 \\ 2.6 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

i.e. **I_1 = 3, I_2 = 1 and I_3 = -2** 6

total : 10

TOTAL ASSIGNMENT MARKS: 80

LIST OF FORMUAE

Laws of indices: $a^m \times a^n = a^{m+n}$ $\qquad \dfrac{a^m}{a^n} = a^{m-n}$ $\qquad (a^m)^n = a^{mn}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $\qquad a^{-n} = \dfrac{1}{a^n}$ $\qquad a^0 = 1$

Factor theorem If $x = a$ is a root of the equation $f(x) = 0$, then $(x - a)$ is a factor of $f(x)$

Remainder theorem If $(ax^2 + bx + c)$ is divided by $(x - p)$, the remainder will be: $ap^2 + bp + c$

or if $(ax^3 + bx^2 + cx + d)$ is divided by $(x - p)$, the remainder will be: $ap^3 + bp^2 + cp + d$

Partial fractions

Provided that the numerator $f(x)$ is of less degree than the relevant denominator, the following identities are typical examples of the form of partial fractions used:

$$\frac{f(x)}{(x+a)(x+b)(x+c)} \equiv \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} \equiv \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{(ax^2+bx+c)(x+d)} \equiv \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Quadratic formula: If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Definition of a logarithm: If $y = a^x$ then $x = \log_a y$

Laws of logarithms: $\log(A \times B) = \log A + \log B$

$\log\left(\dfrac{A}{B}\right) = \log A - \log B$ $\qquad \log A^n = n \times \log A$

Exponential series: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ..$ (valid for all values of x)

Arithmetic progression:

If a = first term and d = common difference, then the arithmetic progression is: $a, a + d, a + 2d, ..$

The n'th term is : $a + (n - 1)d$ $\qquad$ Sum of n terms, $S_n = \dfrac{n}{2}[2a + (n - 1)d]$

Geometric progression:

If a = first term and r = common ratio, then the geometric progression

is: $a, ar, ar^2, ..$

The n'th term is: ar^{n-1} Sum of n terms, $S_n = \frac{a(1 - r^n)}{(1 - r)}$ or $\frac{a(r^n - 1)}{(r - 1)}$

If $-1 < r < 1$, $S_\infty = \frac{a}{(1 - r)}$

Permutation: ${}^nP_r = \frac{n!}{(n - r)!}$ **Combination:** ${}^nC_r = \frac{n!}{r!(n - r)!}$

Binomial series:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n - 1)}{2!}a^{n-2}b^2 + \frac{n(n - 1)(n - 2)}{3!}a^{n-3}b^3 + ..$$

$$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \frac{n(n - 1)(n - 2)}{3!}x^3 + ..$$

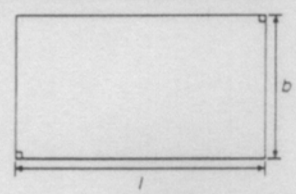
Newton-Raphson iterative method

If r_1 is the approximate value for a real root of the equation $f(x) = 0$, then a closer approximation to the root, r_2, is given by:

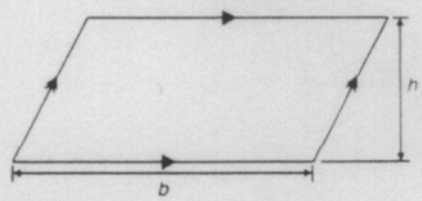
$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

Areas of plane figures:

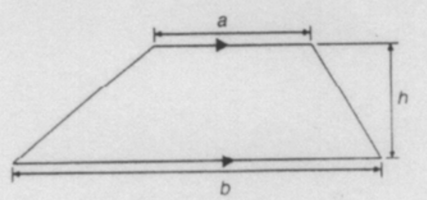
(i) **Rectangle** Area = $l \times b$



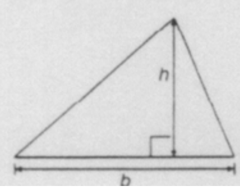
(ii) **Parallelogram** Area = $b \times h$



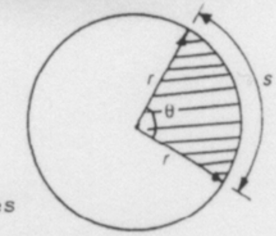
(iii) **Trapezium** Area = $\frac{1}{2}(a + b)h$



(iv) **Triangle** Area = $\frac{1}{2} \times b \times h$



(v) **Circle** Area = πr^2 Circumference = $2\pi r$

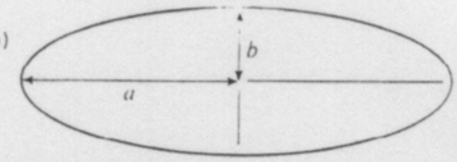


Radian measure: 2π radians = 360 degrees

For a sector of circle: arc length, $s = \frac{\theta°}{360}(2\pi r) = r\theta$ (θ in rad)

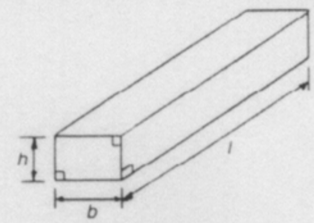
shaded area = $\frac{\theta°}{360}(\pi r^2) = \frac{1}{2}r^2\theta$ (θ in rad)

(vi) **Ellipse** Area = πab Perimeter $\approx \pi(a + b)$



Volumes and surface areas of regular solids:

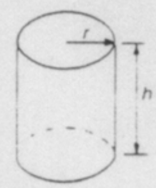
(i) **Rectangular prism (or cuboid)** Volume = $l \times b \times h$



Surface area = $2(bh + hl + lb)$

(ii) **Cylinder** Volume = $\pi r^2 h$

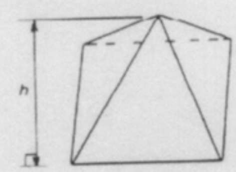
Total surface area = $2\pi rh + 2\pi r^2$

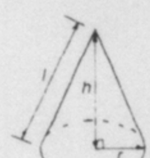


(iii) **Pyramid** If area of base = A and perpendicular height = h then:

Volume = $\frac{1}{3} \times A \times h$

Total surface area = sum of areas of triangles forming sides + area of base

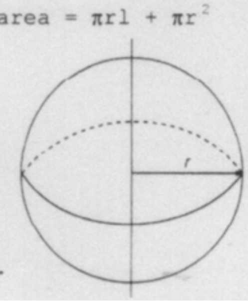




(iv) **Cone** Volume = $\frac{1}{3}\pi r^2 h$

Curved surface area = πrl Total surface area = $\pi rl + \pi r^2$

(v) **Sphere** Volume = $\frac{4}{3}\pi r^3$ Surface area = $4\pi r^2$



Areas of irregular figures/numerical integration:

Trapezoidal rule

$$\text{Area} \approx \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + \text{sum of remaining ordinates}\right]$$

Mid-ordinate rule

Area ≈ (width of interval)(sum of mid-ordinates)

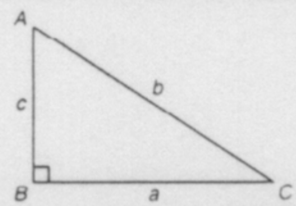
Simpson's rule

$$\text{Area} \approx \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

Mean or average value of a waveform

$$\text{mean value, } y = \frac{\text{area under curve}}{\text{length of base}} = \frac{\text{sum of mid - ordinates}}{\text{number of mid - ordinates}}$$

Theorem of Pythagoras: $b^2 = a^2 + c^2$

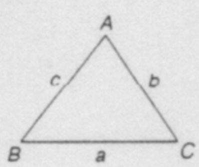


Trigonometry

Identities: $\sec\theta = \frac{1}{\cos\theta}$, $\text{cosec}\,\theta = \frac{1}{\sin\theta}$, $\cot\theta = \frac{1}{\tan\theta}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cos^2\theta + \sin^2\theta = 1$ $\quad 1 + \tan^2\theta = \sec^2\theta$ $\quad \cot^2\theta + 1 = \text{cosec}^2\theta$

Triangle formulae:



Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

Area of any triangle

(i) $\frac{1}{2} \times \text{base} \times \text{perpendicular height}$

(ii) $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$

(iii) $\sqrt{[s(s-a)(s-b)(s-c)]}$ where $s = \frac{a+b+c}{2}$

Compound angle formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

If $\mathbf{R \sin(\omega t + \alpha) = a \sin \omega t + b \cos \omega t}$,

then $a = R\cos\alpha$, $b = R\sin\alpha$, $R = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}\frac{b}{a}$

Double angles $\sin 2A = 2 \sin A \cos A$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]$$

Sums or differences of sines and cosines into products

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

For a **general sinusoidal function $y = A\sin(\omega t \pm \alpha)$**, then

A = amplitude ω = angular velocity = $2\pi f$ rad/s

$\frac{2\pi}{\omega}$ = periodic time T seconds $\frac{\omega}{2\pi}$ = frequency, f hertz

α = angle of lead or lag (compared with $y = A\sin\omega t$)

Cartesian and polar co-ordinates

If co-ordinate $(x, y) = (r, \theta)$ then $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

If co-ordinate $(r, \theta) = (x, y)$ then $x = r\cos\theta$ and $y = r\sin\theta$

Equations of functions

Equation of a straight line: $y = mx + c$

Equation of a parabola: $y = ax^2 + bx + c$

Circle, centre at origin, radius r: $x^2 + y^2 = r^2$

Circle, centre (a, b), radius r: $(x - a)^2 + (y - b)^2 = r^2$

Equation of an ellipse, centre at origin, semi-axes a and b: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of a hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Equation of a rectangular hyperbola: $xy = c^2$

Reduction of equations to linear form

If	then ... Vertical axis		Gradient	Horizontal axis		Intercept on vertical axis
$y = ax^n$	$\lg y$	=	n	$\lg x$	+	$\lg a$
$y = ab^x$	$\lg y$	=	$\lg b$	x	+	$\lg a$
$y = ae^{kx}$	$\ln y$	=	k	x	+	$\ln a$
$y = ax^n + bx^{n-1}$	$\frac{y}{x^{n-1}}$	=	a	x	+	b

Complex numbers

$z = a + jb = r(\cos\theta + j\sin\theta) = r\angle\theta = r\,e^{j\theta}$ where $j^2 = -1$

Modulus $r = |z| = \sqrt{(a^2 + b^2)}$ Argument $\theta = \arg z = \tan^{-1}\frac{b}{a}$

Addition: $(a + jb) + (c + jd) = (a + c) + j(b + d)$

Subtraction: $(a + jb) - (c + jd) = (a - c) + j(b - d)$

Complex equations: If $m + jn = p + jq$ then $m = p$ and $n = q$

Multiplication: $z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$

Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$

De Moivre's theorem: $[r\angle\theta]^n = r^n\angle n\theta = r^n(\cos n\theta + j\sin n\theta)$

Matrices and determinants

Matrices: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ then

$$A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \qquad A - B = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

$$A \times B = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then $A^{-1} = \dfrac{B^T}{|A|}$ where B^T = transpose of cofactors of matrix A

Determinants: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Standard derivatives

y or f(x)	$\frac{dy}{dx}$ or f′(x)
ax^n	anx^{n-1}
sin ax	a cos ax
cos ax	-a sin ax
tan ax	$a \sec^2 ax$
sec ax	a sec ax tan ax
cosec ax	-a cosec ax cot ax
cot ax	$-a \operatorname{cosec}^2 ax$
e^{ax}	ae^{ax}
ln ax	$\frac{1}{x}$

Product rule: When y = uv and u and v are functions of x then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule: When $y = \frac{u}{v}$ and u and v are functions of x then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Function of a function:

If u is a function of x then: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Maximum and minimum values: If $y = f(x)$ then $\frac{dy}{dx} = 0$ for stationary points.

Let a solution of $\frac{dy}{dx} = 0$ be $x = a$; if the value of $\frac{d^2y}{dx^2}$ when $x = a$ is:

positive, the point is a *minimum*, *negative*, the point is a *maximum*

Velocity and acceleration If distance $x = f(t)$, then

velocity $v = f'(t)$ or $\frac{dx}{dt}$ and acceleration $a = f''(t)$ or $\frac{d^2x}{dt^2}$

Tangents and normals

Equation of tangent to curve $y = f(x)$ at the point (x_1, y_1) is:

$$y - y_1 = m(x - x_1) \quad \text{where } m = \text{gradient of curve at } (x_1, y_1)$$

Equation of normal to curve $y = f(x)$ at the point (x_1, y_1) is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Standard integrals

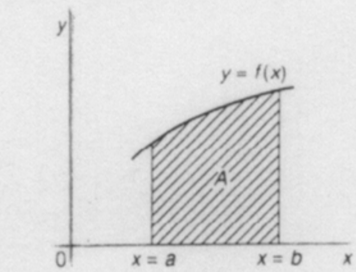
y	$\int y\,dx$
ax^n	$a\frac{x^{n+1}}{n+1} + c$ (except where $n = -1$)
$\cos ax$	$\frac{1}{a}\sin ax + c$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\sec^2 ax$	$\frac{1}{a}\tan ax + c$
$\text{cosec}^2\, ax$	$-\frac{1}{a}\cot ax + c$
$\text{cosec}\, ax \cot ax$	$-\frac{1}{a}\text{cosec}\, ax + c$
$\sec ax \tan ax$	$\frac{1}{a}\sec ax + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$
$\frac{1}{x}$	$\ln x + c$
$\tan ax$	$\frac{1}{a}\ln(\sec ax) + c$
$\cos^2 x$	$\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$
$\sin^2 x$	$\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$

$\tan^2 x$	$\tan x - x + c$
$\cot^2 x$	$-\cot x - x + c$
$\frac{1}{\sqrt{(a^2 - x^2)}}$	$\sin^{-1}\frac{x}{a} + c$
$\sqrt{(a^2 - x^2)}$	$\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(a^2 - x^2)} + c$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Integration by parts If u and v are both functions of x then:

$$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$

Area under a curve: $\text{area } A = \int_a^b y\,dx$



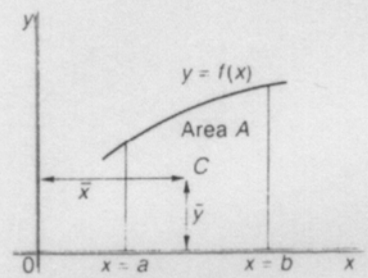
Mean value: $\text{mean value} = \frac{1}{b - a}\int_a^b y\,dx$

R.m.s. value: $\text{r.m.s. value} = \sqrt{\left\{\frac{1}{b - a}\int_a^b y^2\,dx\right\}}$

Volume of solid of revolution: $\text{volume} = \int_a^b \pi y^2\,dx$ about the x-axis

Centroids

$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$$



Theorem of Pappus When the curve y = f(x) is rotated one revolution about the x-axis between the limits x = a and x = b, the volume V generated is given by: $V = 2\pi A\bar{y}$

Summary of standard results of the second moments of areas of regular sections

Shape	Position of axis	Second moment of area, I	Radius of gyration, k
Rectangle	(1) Coinciding with b	$\frac{bl^3}{3}$	$\frac{l}{\sqrt{3}}$
length l	(2) Coinciding with l	$\frac{lb^3}{3}$	$\frac{b}{\sqrt{3}}$
breadth b	(3) Through centroid, parallel to b	$\frac{bl^3}{12}$	$\frac{l}{\sqrt{12}}$
	(4) Through centroid, parallel to l	$\frac{lb^3}{12}$	$\frac{b}{\sqrt{12}}$
Triangle	(1) Coinciding with b	$\frac{bh^3}{12}$	$\frac{h}{\sqrt{6}}$
Perpendic-ular height h	(2) Through centroid, parallel to base	$\frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
base b	(3) Through vertex, parallel to base	$\frac{bh^3}{4}$	$\frac{h}{\sqrt{2}}$
Circle radius r	(1) Through centre, perpendicular to plane (i.e. polar axis)	$\frac{\pi r^4}{2}$	$\frac{r}{\sqrt{2}}$
	(2) Coinciding with diameter	$\frac{\pi r^4}{4}$	$\frac{r}{2}$
	(3) About a tangent	$\frac{5\pi r^4}{4}$	$\frac{\sqrt{5}}{2} r$
Semicircle radius r	Coinciding with diameter	$\frac{\pi r^4}{8}$	$\frac{r}{2}$

Boolean algebra

Laws and rules of Boolean algebra

Commutative Laws:

$$A + B = B + A$$

$$A \cdot B = B \cdot A$$

Associative Laws:

$$A + B + C = (A + B) + C$$

$$A \cdot B \cdot C = (A \cdot B) \cdot C$$

Distributive Laws:

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$A + (B \cdot C) = (A + B) \cdot (A + C)$$

Sum rules:

$$A + \overline{A} = 1$$

$$A + 1 = 1$$

$$A + 0 = A$$

$$A + A = A$$

Product rules:

$$A \cdot \overline{A} = 0$$

$$A \cdot 0 = 0$$

$$A \cdot 1 = A$$

$$A \cdot A = A$$

Absorption rules:

$$A + A \cdot B = A$$

$$A \cdot (A + B) = A$$

$$A + \overline{A} \cdot B = A + B$$

De Morgan's Laws:

$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

Statistics

Mean, median, mode and standard deviation

If x = variate and f = frequency then:

$$\text{mean } \bar{x} = \frac{\sum fx}{\sum f}$$

The **median** is the middle term of a ranked set of data.
The **mode** is the most commonly occurring value in a set of data.

Standard deviation

$$\sigma = \sqrt{\left[\frac{\sum\{f(x-\bar{x})^2\}}{\sum f}\right]} \text{ for a population}$$

Binomial probability distribution

If n = number in sample, p = probability of the occurrence of an event and $q = 1 - p$, then the probability of 0, 1, 2, 3, ... occurrences is given by:

$$q^n, \quad nq^{n-1}p, \quad \frac{n(n-1)}{2!}q^{n-2}p^2,$$
$$\frac{n(n-1)(n-2)}{3!}q^{n-3}p^3, \ldots$$

(i.e. successive terms of the $(q+p)^n$ expansion).

Normal approximation to a binomial distribution:

Mean $= np$ Standard deviation $\sigma = \sqrt{(npq)}$

Poisson distribution

If λ is the expectation of the occurrence of an event then the probability of 0, 1, 2, 3, ... occurrences is given by:

$$e^{-\lambda}, \quad \lambda e^{-\lambda}, \quad \lambda^2\frac{e^{-\lambda}}{2!}, \quad \lambda^3\frac{e^{-\lambda}}{3!}, \ldots$$

Product-moment formula for the linear correlation coefficient

$$\text{Coefficient of correlation } r = \frac{\sum xy}{\sqrt{[(\sum x^2)(\sum y^2)]}}$$

where $x = X - \overline{X}$ and $y = Y - \overline{Y}$ and (X_1, Y_1), (X_2, Y_2), ... denote a random sample from a bivariate normal distribution and $\overline{X}$ and $\overline{Y}$ are the means of the X and Y values respectively.

Normal probability distribution

Partial areas under the standardized normal curve — see Table 40.1 on page 341.

Student's t distribution

Percentile values (t_p) for Student's t distribution with ν degrees of freedom — see Table 43.2 on page 365.

Symbols:

Population

number of members N_p, mean μ, standard deviation σ.

Sample

number of members N, mean $\bar{x}$, standard deviation s.

Sampling distributions

mean of sampling distribution of means $\mu_{\bar{x}}$ standard error of means $\sigma_{\bar{x}}$ standard error of the standard deviations σ_s.

Standard error of the means

Standard error of the means of a sample distribution, i.e. the standard deviation of the means of samples, is:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}\sqrt{\left(\frac{N_p - N}{N_p - 1}\right)}$$

for a finite population and/or for sampling without replacement, and

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

for an infinite population and/or for sampling with replacement.

The relationship between sample mean and population mean

$\mu_{\bar{x}} = \mu$ for all possible samples of size N drawn from a population of size N_p.

Estimating the mean of a population (σ known)

The confidence coefficient for a large sample size, ($N \geq 30$) is z_c where:

Confidence level %	Confidence coefficient z_c
99	2.58
98	2.33
96	2.05
95	1.96
90	1.645
80	1.28
50	0.6745

Table 1

The confidence limits of a population mean based on sample data are given by:

$$\bar{x} \pm \frac{z_c \sigma}{\sqrt{N}} \sqrt{\left(\frac{N_p - N}{N_p - 1}\right)}$$

for a finite population of size N_p, and by

$$\bar{x} \pm \frac{z_c \sigma}{\sqrt{N}} \text{ for an infinite population}$$

Estimating the mean of a population (σ unknown)

The confidence limits of a population mean based on sample data are given by: $\mu_{\bar{x}} \pm z_c \sigma_{\bar{x}}$.

Estimating the standard deviation of a population

The confidence limits of the standard deviation of a population based on sample data are given by: $s \pm z_c \sigma_s$.

Estimating the mean of a population based on a small sample size

The confidence coefficient for a small sample size ($N < 30$) is t_c which can be determined using Table 1 . The confidence limits of a population mean based on sample data is given by:

$$\bar{x} \pm \frac{t_c s}{\sqrt{(N - 1)}}$$
