Game Math Exercises

Frank Luna
www.gameinstitute.com

The following exercises will help prepare you for the exams. In general, exam questions will be like the following exercise questions, but with the numbers changed. Therefore, if you can do the exercises well, then that is a good indicator that you will do well on the exams. Answers to most problems are provided, but not solutions. If you need a solution, or perhaps an example, then consult with your instructor. Note that you should be doing the exercises from this exercise set and not the textbook's exercises, except when this exercise set specifically asks you to do a problem from the main textbook. You will not be tested on material outside this exercise set.

Game Math Chapter 1 Exercises

Chapter Remarks: *The main skill from this chapter that I want you to walk away with is being comfortable with the terminology of sets, and how the basic operations work; always have the Venn diagrams in mind to organize your thoughts when reasoning about sets. At this point, also be familiar with the basic ideas of functions, which map elements from one set to another; remember that to be a function, an element in the domain cannot map to two different elements in the range, although two different elements in the domain can map to the same range element. Also know the condition for invertibility, and understand why this condition is necessary. Most of the topics covered in this chapter are mainly groundwork for later chapters. The following exercises should test what I expect you to be able to do after studying this chapter.*

1. True or false. If false, give a counterexample. Assume A and B are sets.

   a. $\{b, a, c, c, f, d, d, d, e\} = \{a, b, c, d, e, f\}$.
   b. If $A = \{a, b, c, d, e, f\}$, then $c \notin A$.
   c. $|A \cup B| = |A| + |B|$.
   d. If $A = \{a, b, c, d, e, f\}$ and $B = \{a, b, c\}$, then $A \subseteq B$.
   e. $\mathbb{Z} \subseteq \mathbb{R}$.
   f. If $A \subseteq B$ or $B \subseteq A$, then $A = B$.
   g. $\varnothing \subseteq \mathbb{R} \subseteq \mathbb{R}$.
   h. $\mathbb{Z} = \mathbb{Z}^{+} \cup \mathbb{Z}^{-} \cup \{0\}$.
   i. $\{0\} = \mathbb{Z}^{+} \cap \mathbb{Z}^{-} \cap \{0\}$.
   j. $A - B = B - A$

k. Suppose the universal set of discourse is $\mathbb{Z}$. If $A = \{0\}$, then $A^C = \mathbb{Z}^+ \cup \mathbb{Z}^-$.

2. **Definition:** An integer n is odd if and only if we can write $n = 2k + 1$, for some integer k. Define the set of all odd integers.

3. **Definition:** An integer n is even if and only if we can write $n = 2k$, for some integer k. Define the set of all even integers.

4. Prove the following simple facts about odd and even integers:
   a. The sum of two odd integers is even.
   b. The sum of two even integers is even.
   c. The sum of an even integer and an odd integer is odd.
   d. The product of two odd integers is odd.
   e. The product of two even integers is even.
   f. The product of even integer and an odd integer is even.

Example: We prove (a): Let $a = 2m + 1$ and $b = 2n + 1$, where $m, n \in \mathbb{Z}$. Then

$$
\begin{aligned}
a + b &= (2m+1) + (2n+1) && \text{Definition of } a \text{ and } b \text{ odd integers} \\
&= 2m + 2n + 2 && \text{Algebra} \\
&= 2(m + n + 1) && \text{More algebra} \\
&= 2k && \text{Set } k = m + n + 1, \text{ for } k \in \mathbb{Z}
\end{aligned}
$$

The number k is obviously an integer, so we have shown that the sum of two odd integers has the form of an even integer $a + b = 2k$ (i.e., it is an even integer). Thus we are done.

Remarks: Note that in this proof we kept things very general. We did not assume specific values for a and b; if we did, then our proof would not be valid for all integers—it would only be valid for those specific integers chosen. In the proof, we pick arbitrary odd integers a and b, and then just use the fact that they are odd integers to move to the next step. The rest is just algebra until we show the resulting form is that of an even integer. The key is, because we have shown it is true for arbitrary odd integers a and b, it is true *for all* odd integers.

5. Let $U = \{a, b, c, d, e, f, g, h, i, j\}$ be the universal set of discourse, and let $A = \{a, c, f, h\}$ and $B = \{c, h, j\}$. Find the following:

   a. $A \cup B$
   b. $A \cap B$
   c. $A - B$
   d. $B - A$
   e. U^C

f. $|A|$
g. A^C
h. B^C
i. $(B^C)^C$
j. $(A \cup B) - (A \cap B)$
k. $(A - B) \cap (B - A) \cap (A \cap B)$

6. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set of discourse, and let $A = \{1, 2, 9, 10\}$, $B = \{1, 2, 6, 8\}$, and $C = \{1, 3, 5, 7, 9\}$. Find the following:

   a. $A \cup B \cup C$
   b. $A \cap B \cap C$
   c. $U - A$
   d. C^C
   e. $|(A \cup B) - C|$
   f. $A \cap (B \cup C)$
   g. $(A \cap B) \cup (A \cap C)$
   h. $A \cup (B \cap C)$
   i. $(A \cup B) \cap (A \cup C)$
   j. $(A \cap B)^C$
   k. $A^C \cup B^C$

7. Let C be the set which includes all the elements in A or B, but not both. Express C in terms of the sets A and B. (Hint: A Venn diagram helps organize your thoughts.)

8. Let A and B be sets. Express $A \cup B$ as the union of three disjoint sets. (Hint: A Venn diagram helps organize your thoughts.)

9. Do all the exercises in the textbook for this chapter. For #5, just draw a Venn diagram to illustrate the fact.

Answers to Selected Problems

1a. True.
1b. False, $c \in A$.
1c. False, suppose $A = \{a, b, c\}$ and $B = \{c, d\}$. Then $|A \cup B| = 4$, but $|A| = 3$ and $|B| = 2$. Hence, $|A \cup B| \neq |A| + |B|$.

1d. False, $e \in A$ but $e \notin B$.
1e. True.
1f. False, $A = B$ if and only if $A \subseteq B$ *and* $B \subseteq A$.
1g. True.
1h. True.
1i. False, $\mathbb{Z}^+ \cap \mathbb{Z}^- \cap \{0\} = \varnothing$
1j. False, suppose $A = \{a, b, c\}$ and $B = \{c, d\}$. Then $A - B = \{a, b\}$ but $B - A = \{d\}$.
1k. True.

2. $S = \{x : x = 2k + 1, k \in \mathbb{Z}\}$
3. $S = \{x : x = 2k, k \in \mathbb{Z}\}$

5a. $A \cup B = \{a, c, f, h, j\}$
5b. $A \cap B = \{c, h\}$
5c. $A - B = \{a, f\}$
5d $B - A = \{j\}$
5e. $U^C = \varnothing$
5f. $|A| = 4$
5g. $A^C = \{b, d, e, g, i, j\}$
5h. $B^C = \{a, b, d, e, f, g, i\}$
5i. $\left(B^C\right)^C = B$
5j. $(A \cup B) - (A \cap B) = \{a, f, j\}$
5k. $(A - B) \cap (B - A) \cap (A \cap B) = \varnothing$

6a. $A \cup B \cup C = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$
6b. $A \cap B \cap C = \{1\}$
6c. $U - A = \{3, 4, 5, 6, 7, 8\}$
6d. $C^C = \{2, 4, 6, 8, 10\}$
6e. $|(A \cup B) - C| = 4$
6f. $A \cap (B \cup C) = \{1, 2, 9\}$
6g. $(A \cap B) \cup (A \cap C) = \{1, 2, 9\}$
6h. $A \cup (B \cap C) = \{1, 2, 9, 10\}$
6i. $(A \cup B) \cap (A \cup C) = \{1, 2, 9, 10\}$
6j. $(A \cap B)^C = \{3, 4, 5, 6, 7, 8, 9, 10\}$

6k. $A^C \cup B^C = \{3, 4, 5, 6, 7, 8, 9, 10\}$

7. $C = (A \cup B) - (A \cap B)$. Draw a Venn diagram to see this!

8. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$. Draw a Venn diagram to see this!

Game Math Chapter 2 Exercises

Chapter Remarks: *The key objectives in this chapter are for you to become familiar with common functions, and how to plot and interpret their graphs. In addition, you should become familiar with algebraically solving some simple equations involving the exponential and logarithmic functions. Finally, you are asked to develop functions that fit certain constraints to model certain situations mathematically. The following exercises should test what is expected of you after studying this chapter.*

1. True or false. If false, give a counterexample. Assume $x \in \mathbb{R}$.

   a. There exists an x such that $|x| \leq 0$.
   b. There exists an x such that $f(x) = 2x - 1 = 0$.
   c. For all x, $e^x > 0$.
   d. $\ln(0) = 0$.
   e. $\ln(e^1) = 1$.
   f. $e^0 = 0$.
   g. $\log_b(b) = b$.
   h. The domain of $f(x) = \log_b(x)$ is for $x > 0$.
   i. The range of $f(x) = \log_b(x)$ is all real numbers.
   j. The range of $f(x) = e^x$ is all real numbers.
   k. The range of $f(x) = \sqrt{x}$ is all real numbers for $x \geq 0$.
   l. The range of $f(x) = |x|$ is all real numbers.
   m. The range of $f(x) = x$ is all real numbers.

2. Evaluate each of the following functions at $x = 0$, $x = -1$, $x = 1$, $x = 2$, and $x = -3$; that is, find $f(0)$, $f(1)$, $f(-1)$, $f(2)$, and $f(-3)$. In addition, plot each function for $-4 \leq x \leq 4$.

   a. $f(x) = 3x - 2$.
   b. $f(x) = x$.
   c. $f(x) = -x^2$.
   d. $f(x) = \sqrt{x}$
   e. $f(x) = |x|$
   f. $f(x) = e^x$
   g. $f(x) = \ln(x)$

h. $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ 3 & \text{if } x > 0 \end{cases}$

i. $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 3 & \text{if } x > 0 \end{cases}$

3. Plot the following functions for $-4 \le x \le 4$.

   a. $f(x) = 2x + 3$
   b. $f(x) = -3x + 1$
   c. $f(x) = 2$
   d. $f(x) = x$
   e. $f(x) = 2x - 4$
   f. $f(x) = x^2$
   g. $f(x) = -x^2 + 3$
   h. $f(x) = (x-1)^2$
   i. $f(x) = (x+2)^2 - 3$
   j. $f(x) = \sqrt{x}$
   k. $f(x) = \sqrt{x} + 2$
   l. $f(x) = \sqrt{x-1} - 2$
   m. $f(x) = |x|$
   n. $f(x) = |x-2| - 4$
   o. $f(x) = \ln(x)$
   p. $f(x) = \ln(x+3)$
   q. $f(x) = \ln(x+3) - 4$
   r. $f(x) = e^x$
   s. $f(x) = e^{-x}$

4. Based on the graphs you did in the previous exercise, how does $f(x) = (x-1)^2$, $f(x) = -x^2 + 3$ and $f(x) = (x+2)^2 - 3$ compare to $f(x) = x^2$?

5. Solve, in simplest form, each of the following for *x*:

   a. $4x - 3 = 1$

b. $-2x^2 + 3x = -5$
c. $\ln(e \cdot x) = 5$
d. $\ln(4x^2) - \ln(2x) = 1$
e. $2^x = 1024$
f. $\log_2(256) = x$
g. $x = \log_7(23)$
h. $2e^x - 4xe^x = 0$
i. $\ln(8x) - 2\ln(x^2) = \ln(4)$
j. $\log_5(5^x) = 7$

6. Prove log properties 2 and 3 in Table 2.1

7. Do problems 3, 4, and 5 from the textbook:

3. Create an exponential function that models the population of rabbits on an island. Suppose the population begins at 10, and that after 12 months, there are 1000 rabbits. How many months until the rabbit population exceeds one million? (Hint: Try using $f(x) = ae^{bx}$, where *a* and *b* are the unknown constants; you'll need the natural log function to solve for *b*.)

4. Suppose the rabbit population in the preceding problem can't grow exponentially because of predators and limited resources. Try modeling the population using a log function, making sure the new function satisfies the original constraints. (Hint: If you run into difficulty solving for the constants, try graphing the log function to see what could be causing the problem.)

5. Say you're building a space-based resource management game and you want the cost of research and development to increase exponentially. Suppose that having spent nothing on research, you want it to cost 10 credits to advance to the next level of development, but that after you have spent 10,000 credits, you want it to cost twice that much to advance to the next level. Develop a function to model this behavior.

Solutions for the above three problems are available for download in the class download section.

Answers to Selected Problems

1a. True.
1b. True.
1c. True.
1d. False, $\ln(0)$ is undefined.
1e. True.
1f. False, $e^0 = 1$.

1g. False, $\log_b(b) = 1$.
1h. True.
1i. True.
1j. False, $e^x > 0$.
1k. False, $f(x) = \sqrt{x} \geq 0$ for $x \geq 0$.
1l. False, $f(x) = |x| \geq 0$.
1m. True.

2. Use a graphing tool to check your work and plots (e.g., http://www.gnuplot.info/).

3. Use a graphing tool to check your plots (e.g., http://www.gnuplot.info/).

4. $(x-1)^2$ is like x^2 except it is shifted to the right by one unit. $-x^2+3$ is like x^2 except that it is reflected about the *x*-axis and shifted three units up. $(x+2)^2-3$ is like x^2 except that it is shifted to the left by two units and shifted three units down.

5a. $x = 1$
5b. $x = 5/2$, $x = -1$
5c. $x = e^4$
5d. $x = e/2$
5e. $x = 10$
5f. $x = 8$
5g. $x \approx 1.611325$
5h. $x = 1/2$
5i. $x = \sqrt[3]{2}$
5j. $x = 7$

Game Math Chapter 3 Exercises

Chapter Remarks: *The main objectives for this chapter are for you to be able to solve quadratic equations, graph polynomials, fit polynomials through data points, and to be able to use polynomials for interpolation and prediction purposes. The following exercises should test what is expected of you after studying this chapter.*

1. Solve the following quadratic equations for x:

   a. $x^2 + 3x + 1 = 0$
   b. $3x^2 + 4x - 2 = 0$
   c. $-x^2 + 3x - 1 = 0$

Example

For exercises 2-8, you will need to fit a polynomial through some specified data points. For example, to fit a second order polynomial through three data points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$*, you start with the general form of a second order polynomial,* $p(x) = ax^2 + bx + c$*. Then you use the given data points to algebraically solve for the constants a, b, and c:*

$$p(x_1) = ax_1^2 + bx_1 + c = y_1$$
$$p(x_2) = ax_2^2 + bx_2 + c = y_2$$
$$p(x_3) = ax_3^2 + bx_3 + c = y_3$$

Since the three data points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ *are known, the above gives three equations and three unknowns, namely a, b, and c. So you can algebraically solve the system of equations. (Solving systems of equations is taught in High School algebra courses.) If you are to fit a first order polynomial, then the general form is* $p(x) = ax + b$*, and you only need to be given two data points since there are only two unknown constants a and b. Likewise, if you are to fit a third order polynomial, the general form is* $p(x) = ax^3 + bx^2 + cx + d$*, and you will need four points since there are four unknown constants a, b, c, and d. Example: Find a second order polynomial through the points* $(-1, 0), (0, 2), (1, 0)$*. This yields the system of equations:*

(1) $p(-1) = a(-1)^2 + b(-1) + c = 0$

(2) $p(0) = a0^2 + b0^2 + c = 2$

(3) $p(1) = a1^2 + b1 + c = 0$

Equation (2) immediately implies $c = 2$*. Rewriting Equations (1) and (2) we have:*

(4) $a - b + 2 = 0$

(5) $a + b + 2 = 0$

Adding Equations (4) and (5) together gives:

(6) $2a = -4 \Rightarrow a = -2$

We now know a, and plug it into Equation (4) and solve for b:

(7) $-2 - b + 2 = 0 \Rightarrow b = 0$.

Thus, the second order polynomial that passes through the three data points is given by:

$$p(x) = ax^2 + bx + c$$
$$= -2x^2 + 2$$

We double-check that this polynomial does indeed pass through the three given points:

$$p(-1) = -2(-1)^2 + 2 = 0$$
$$p(0) = -2(0)^2 + 2 = 2$$
$$p(1) = -2(1)^2 + 2 = 0$$

It does, so we are done.

2. Find the first order polynomial (linear) through the following data points and plot your polynomial over $-5 \le x \le 5$. Be sure to verify that your polynomial does indeed pass through the given data points. For example, for 2a, if your polynomial is $p(x)$, then you should have $p(-3) = -2$ and $p(4) = 1$.

   a. $(-3, -2), (4, 1)$
   b. $(-1, -1), (1, 4)$

3. Find the second order polynomial (quadratic) through the following data points and plot your polynomial over $-5 \le x \le 5$. Be sure to verify that your polynomial does indeed pass through the given data points. For example, for 3a, if your polynomial is $p(x)$, then you should have $p(-4) = 0$, $p(0) = 5$ and $p(3) = 0$.

   a. $(-4, 0), (0, 5), (3, 0)$
   b. $(-5, 0), (0, -3), (5, 0)$

4. Find the third order polynomial through the following data points and plot your polynomial over $-5 \le x \le 5$. Be sure to verify that your polynomial does indeed pass through the given data points. For example, for 4a, if your polynomial is $p(x)$, then you should have $p(-4) = 0$, $p(-1) = 0$, $p(0) = 1$ and $p(5) = -1$.

   a. $(-4, 0), (-1, 0), (0, 1), (5, -1)$

5. Suppose water is being added to a container at an almost constant rate. At time t_0 the container has 7 liters of water. Three seconds later (i.e., at $t_0 + 3$), the container has 11 liters of water. Use linear interpolation to approximate the number of liters the container has at $t_0 + 1$.

6. A point on a hill as coordinates $(2, 5)$. Another point on the hill has coordinates $(10, 11)$. Use linear interpolation to estimate the point $(6, y)$, that is, find y.

7. A ball is thrown into the air and passes through the points $(0, 0), (4, 10), (8, 0)$. Use quadratic interpolation to estimate the position of the ball at $x = 3$.

8. Two seconds ago, a player's x-coordinate was 5. One second ago, it was 8. And it currently is 10. Predict the player's x-coordinate one second from now using a second order polynomial.

9. Do problems 1, 2, 3, and 4 from the textbook. Solutions for these four problems are available for download in the class download section.

Answers to Selected Problems

1a. $x = \dfrac{-3+\sqrt{5}}{2}$, $x = \dfrac{-3+\sqrt{5}}{2}$

1b. $x = \dfrac{-2+\sqrt{10}}{3}$, $x = \dfrac{-2-\sqrt{10}}{3}$

1c. $x = \dfrac{\sqrt{5}+3}{2}$, $x = \dfrac{3-\sqrt{5}}{2}$

2a. $p(x) = \dfrac{3}{7}x - \dfrac{5}{7}$

2b. $p(x) = \dfrac{5}{2}x + \dfrac{3}{2}$

3a. $p(x) = -\dfrac{5}{12}x^2 - \dfrac{5}{12}x + 5$

3b. $p(x) = \dfrac{6}{50}x^2 - 3$

4a. $p(x) = -\dfrac{29}{540}x^3 - \dfrac{1}{54}x^2 + \dfrac{559}{540}x + 1$

5. $\dfrac{25}{3}$

6. $y = 8$

7. $p(x) = -\frac{5}{8}x^2 + 5x$; $p(3) = 75/8$

8. $p(x) = -\frac{1}{2}x^2 + \frac{7}{2}x + 5$; $p(3) = 11$

Game Math Chapter 4 Exercises

Chapter Remarks: *After studying this chapter you need to be able to convert between radian and degree measure, convert between Cartesian and polar coordinates, apply similar triangles, and solve right triangle type problems. The following exercises should test these things.*

Converting between radian and degree measure. *The angle* π *in radian measure represents the same angle* $180°$ *does in degree measure.*
So the ratio $\pi/180° = 1$ *since they are the same thing, and of course,* $180°/\pi = 1$. *We can multiply any quantity by one without changing it; therefore, the previous two ratios can be used to convert from degrees to radians and vice versa. For example, we convert* $\pi/3$ *from radians to degrees as follows:*

$$\frac{\pi}{3} = \frac{\pi}{3} \cdot 1 = \frac{\pi}{3} \cdot \frac{180°}{\pi} = \frac{180°}{3} = 60°.$$

On the other hand, we convert $45°$ *from degrees to radians as follows:*

$$45° \cdot 1 = 45° \cdot \frac{\pi}{180°} = \frac{\pi}{4}.$$

1. Convert the following angles measured in radians to degrees:

   a. $\frac{\pi}{8}$
   b. 2.0
   c. $\frac{5\pi}{4}$
   d. $\frac{3\pi}{2}$

2. Convert the following angles measured in degrees to radians:

   a. $60°$
   b. $135°$
   c. $300°$
   d. $360°$

3. Suppose you have a right triangle with an angle θ. The side opposite to θ has length 2 and the side adjacent to θ has length 3. Find the length of the hypotenuse.

4. Suppose you have a right triangle with an angle θ. The side opposite to θ has length 4 and the hypotenuse has length 10. Find the length of the side adjacent to θ.

5. Suppose you have a right triangle with an angle θ. The side opposite to θ has length 2 and the side adjacent to θ has length 3. In addition, suppose that you have another right triangle also with the same angle θ, but for this triangle the side opposite to θ has length 5 and the side adjacent to θ has length x. Find x.

6. Suppose you have a right triangle with an angle θ. The side opposite to θ has length 4 and the side adjacent to θ has length 2. In addition, suppose that you have another right triangle also with the same angle θ, but for this triangle the side opposite to θ has length y and the side adjacent to θ has length 6. Find y.

7. Suppose you have a right triangle with an angle $\theta = 30°$. The side opposite to θ has length 4. Find the length of the hypotenuse and the length of the side adjacent to θ.

8. Suppose you have a right triangle with an angle $\theta = \pi/5$. The hypotenuse has length 12. Find the length of the side opposite to θ and the length of the side adjacent to θ.

9. Do problems 1, 2, and 3 from the textbook.

Answers to Selected Problems

1a. $22.5°$
1b. $114.59°$
1c. $225°$
1d. $270°$

2a. $\pi/3$
2b. $\frac{3\pi}{4}$
2c. $\frac{5\pi}{3}$
2d. 2π

3. $x = \sqrt{13}$

4. $x = 2\sqrt{21}$

5. $x = 7.5$

6. $y = 12$

7. Hypotenuse: $r = 8$, Adjacent: $x = 4\sqrt{3}$

8. Adjacent: $x = 9.71$, Opposite: $y = 7.05$

Game Math Chapter 5 Exercises

Chapter Remarks: *After studying this chapter you need to be able to apply the inverse trigonometric functions. These functions are typically used when you need to solve for an angle in an equation. You also need to be able to apply the trigonometric identities to simplify trigonometric expressions/equations. You also need to know how to apply the Law of Cosines and be familiar with the rotation equations.*

1. Suppose you have a right triangle with angle θ. The side opposite to θ has length 5 and the side adjacent to θ has length 8. Find θ.

2. Suppose you have a right triangle with angle θ. The side adjacent to θ has length 6 and the hypotenuse has length 12. Find θ.

3. Suppose you have a right triangle with angle θ. The side opposite to θ has length 5 and the hypotenuse has length 6. Find θ.

4. Suppose you have a triangle (not a right triangle) where one angle is $\theta = \pi/6$. The length of the side opposite to θ is c. The other two sides have length 5 and 8. Find c.

5. Suppose you have a triangle (not a right triangle) where one angle is θ. The length of the side opposite to θ is 5. The other two sides have length 4 and 6. Find θ.

6. Convert the following points in Cartesian coordinates (x, y) to polar coordinates (r, θ):
   a. $(2, 3)$
   b. $(-1, -1)$
   c. $(2, 0)$

7. Convert the following points in polar coordinates (r, θ) to Cartesian coordinates (x, y):
   a. $(2, 45°)$
   b. $(3, 3\pi/2)$
   c. $(1, 5\pi/4)$

8. Show $\sqrt{x^2 + y^2} = r$ by converting x and y to polar coordinates and applying the appropriate trig identities.

9. Show $\sec^2\theta - \left(\cos^2\theta + \sin^2\theta\right) = \tan^2\theta$ by applying the appropriate trig identities.

10. Show $\csc^2\theta - \cot^2\theta = \sec^2\theta - \tan^2\theta$ by applying the appropriate trig identities.

11. True or false.
    a. $\cos(-\theta) = \cos(\theta)$
    b. $\sin(-\theta) = -\sin(\theta)$
    c. $\cos\left(\frac{\pi}{2} + \theta\right) = \sin(\theta)$
    d. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$
    e. $\sin(2\theta) = \cos(\theta)$
    f. $1 = \cos(2\theta) + 2\sin^2\theta$

12. Verify that the trig identities:

    1) $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$;
    2) $\sin(2\theta) = 2\sin\theta\cos\theta$;
    3) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$;
    4) $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

    are true for $\alpha = 30°, \beta = 45°, \theta = 60°$.

13. Rotate the line defined by the two endpoints $(4,1),(4,-1)$ by an angle $\pi/4$. What are the coordinates of the rotated line?

14. Do all the problems from the textbook in this chapter; solutions for these problems are available in the download section of the class website.

Answers to Selected Problems

1. $32°$

2. $60°$

3. $56.44°$

4. 4.44

5. $55.77°$

6a. $\left(\sqrt{13}, 56.3°\right)$
6b. $\left(\sqrt{2}, 225°\right)$
6c. $(2, 0°)$

7a. $\left(\sqrt{2}, \sqrt{2}\right)$
7b. $(0, -3)$
7c. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

11a. T
11b. T
11c. F
11d. T
11e. F
11f. T

13. $(2.12, 3.54)$, $(3.54, 2.12)$

Game Math Chapter 6 Exercises

Chapter Remarks: *This chapter introduces some fundamental geometric entities like points, lines, ellipses, spheres, and planes. In addition, we explore some techniques for finding info about these geometric objects, such as where they intersect; such tools are useful for collision detection, for example. Geometric problems involving 3D geometry like spheres and planes will be given later, after vectors have been introduced.*

1. Find the distance between the following points:

   a. $(-3, 1, 2)$, $(2, 1, -1)$
   b. $(2, 0, 2)$, $(-1, -1, -1)$
   c. $(-1, 2, 3)$, $(3, -2, 0)$

2. The next four problems deal with 2D lines.

   a. Find the equation of the line that passes through the points $(-1, -2)$ and $(2, 4)$, and graph that line.

   b. Find the equation of the line that has slope –2 and intercepts the *y*-axis at $y = 1$, and graph that line.

   c. Find the intersection between the lines $y = \frac{1}{2}x - 3$ and $y = -x + 6$, and graph both lines to verify the point of intersection you found is correct.

   d. Find the intersection between the lines $y = x + 1$ and $y = 3$, and graph both lines to verify the point of intersection you found is correct.

3. Find the distance of the point $(2, 3)$ from each of the following lines:

   a. $y = \frac{1}{2}x - 3$
   b. $y = -x + 6$
   c. $y = x + 1$
   d. $y = 3$

4. Find the angle between the lines $y = \frac{1}{2}x - 3$ and $y = -x + 6$.
5. Find the angle between the lines $y = x + 1$ and $y = 3$.

6. Consider the 3D line defined by:

$$\begin{cases} x = 1 + 2t \\ y = 2 + 2t \\ z = 3 + 2t \end{cases}$$

Find the points on the line at $t = 0$, $t = 1$, $t = 2$, $t = 4$.

I want to show how the line-ellipse intersection formulas on page 143 were obtained. First note that the formulas assume the ellipse is at the origin (i.e., h = k = 0). We start with the equation of the line and ellipse:

$$y = mx + c$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We can rewrite the ellipse as:

$$b^2x^2 + a^2y^2 = a^2b^2$$

Now plugging in $y = mx + c$*, we obtain an equation solely in terms of x:*

$$b^2x^2 + a^2\left(mx + c\right)^2 = a^2b^2$$
$$b^2x^2 + a^2\left(m^2x^2 + 2mcx + c^2\right) = a^2b^2$$
$$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2$$
$$x^2\left(b^2 + a^2m^2\right) + x\left(2a^2mc\right) + \left(a^2c^2 - a^2b^2\right) = 0$$

As you can see, this is a quadratic equation since a, b, m and c are constants for a given line and ellipse. Applying the quadratic equation yields:

$$
\begin{aligned}
x &= \frac{-2a^2mc \pm \sqrt{4a^4m^2c^2 - 4\left(b^2 + a^2m^2\right)\left(a^2c^2 - a^2b^2\right)}}{2\left(b^2 + a^2m^2\right)} \\
&= \frac{-2a^2mc \pm \sqrt{4a^4m^2c^2 - 4\left(b^2a^2c^2 - b^4a^2 + a^4m^2c^2 - a^4m^2b^2\right)}}{2\left(b^2 + a^2m^2\right)} \\
&= \frac{-2a^2mc \pm \sqrt{4a^4m^2c^2 - 4b^2a^2c^2 + 4b^4a^2 - 4a^4m^2c^2 + 4a^4m^2b^2}}{2\left(b^2 + a^2m^2\right)} \\
&= \frac{-2a^2mc \pm \sqrt{-4b^2a^2c^2 + 4b^4a^2 + 4a^4m^2b^2}}{2\left(b^2 + a^2m^2\right)} \\
&= \frac{-a^2mc \pm ab\sqrt{-c^2 + b^2 + a^2m^2}}{b^2 + a^2m^2}
\end{aligned}
$$

We now know the two x-coordinates of the points of intersection. To find the two y-coordinates at the points of intersection, we just plug x into $y = mx + c$ *:*

$$
\begin{aligned}
y &= mx + c \\
&= m\left(\frac{-a^2mc \pm ab\sqrt{-c^2 + b^2 + a^2m^2}}{b^2 + a^2m^2}\right) + c \\
&= \frac{-a^2m^2c \pm abm\sqrt{-c^2 + b^2 + a^2m^2}}{b^2 + a^2m^2} + c \\
&= \frac{-a^2m^2c \pm abm\sqrt{-c^2 + b^2 + a^2m^2}}{b^2 + a^2m^2} + c\frac{b^2 + a^2m^2}{b^2 + a^2m^2} \\
&= \frac{-a^2m^2c \pm abm\sqrt{-c^2 + b^2 + a^2m^2}}{b^2 + a^2m^2} + \frac{cb^2 + ca^2m^2}{b^2 + a^2m^2} \\
&= \frac{cb^2 \pm abm\sqrt{-c^2 + b^2 + a^2m^2}}{b^2 + a^2m^2}
\end{aligned}
$$

7. Find the points where the given line intersects the given ellipse:

   a. $y = x$, $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$

   b. $y = -3$, $\frac{x^2}{3^2} + \frac{y^2}{3^2} = 1$

   c. $y = 2x + 9$, $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

8. Do problem 1 from the textbook.

Answers to Selected Problems

1a. $\sqrt{34} \approx 5.83$
1b. $\sqrt{19} \approx 4.36$
1c. $\sqrt{41} \approx 6.40$

2a. $y = 2x$
2b. $y = -2x + 1$
2c. $(6, 0)$
2d. $(2, 3)$

3a. $2\sqrt{5}$
3b. $1/\sqrt{2}$
3c. 0
3d. 0

4. $71.56°$

5. $45°$

6. $(1, 2, 3)$, $(3, 4, 5)$, $(5, 6, 7)$, $(9, 10, 11)$

7a. $\left(\frac{8\sqrt{20}}{20}, \frac{8\sqrt{20}}{20}\right), \left(-\frac{8\sqrt{20}}{20}, -\frac{8\sqrt{20}}{20}\right)$
7b. $(0, -3)$
7c. No intersection.

Game Math Chapter 7 Exercises

Chapter Remarks: *This chapter introduced vectors. You need to understand how they represent quantities that possess both magnitude and direction. You need to know how to graph them geometrically, and what the basic vector operations (e.g., addition, cross product) mean geometrically. Furthermore, you need to know how to perform the various vector operations computationally. Lastly, you need to be familiar with the vector representations of various geometric objects like lines, planes, and spheres. The following exercises should test what is expected of you after studying this chapter.*

1. Let $\vec{u} = (1, 3)$, $\vec{v} = (-2, 1)$, and $\vec{w} = (0, 2)$. Perform the indicated computation. Also draw the vectors relative to a coordinate system and perform the indicated operation geometrically. Does the geometry agree with your computations?

   a. $\vec{u} + \vec{v}$
   b. $\vec{u} - \vec{v}$
   c. $(\vec{u} + \vec{v}) - \vec{w}$
   d. $2\vec{u} + \frac{1}{2}\vec{v}$
   e. $2\vec{u} - 3\vec{v}$
   f. $(\vec{u} - 2\vec{v}) + \frac{1}{2}\vec{w}$

2. Let $\vec{u} = (1, 2, 3)$ and $\vec{v} = (-2, 0, 1)$.

   a. Compute $\vec{u} \cdot \vec{v}$.
   b. Find the length of $\vec{u}$ and $\vec{v}$.
   c. Find the angle between $\vec{u}$ and $\vec{v}$.
   d. Find $proj_{\vec{v}}(\vec{u})$.

3. Let $\vec{u} = (\sqrt{2}, \sqrt{2}, 0)$ and $\vec{v} = (\sqrt{2}, -\sqrt{2}, 0)$.

   a. Compute $\vec{u} \cdot \vec{v}$.
   b. Find the length of $\vec{u}$ and $\vec{v}$.
   c. Find the angle between $\vec{u}$ and $\vec{v}$.
   d. Find $proj_{\vec{v}}(\vec{u})$.

4. Let $\vec{u} = (1, 2, 3)$ and $\vec{v} = (-2, 0, 1)$.

   a. Compute $\vec{w} = \vec{u} \times \vec{v}$.
   b. Show $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0$.
   c. Compute $\vec{r} = \vec{v} \times \vec{u}$. How does $\vec{w}$ relate to $\vec{r}$?

5. Let $\vec{u} = \left(\sqrt{2}, \sqrt{2}, 0\right)$ and $\vec{v} = \left(\sqrt{2}, -\sqrt{2}, 0\right)$.

   a. Compute $\vec{w} = \vec{u} \times \vec{v}$.
   b. Show $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0$.
   c. Compute $\vec{r} = \vec{v} \times \vec{u}$. How does $\vec{w}$ relate to $\vec{r}$?

6. Consider the plane defined by the normal $\vec{n} = (1, 2, 3)$ and point $\vec{p}_0 = (-2, 3, 0)$. Find the distance between this plane and the points:

   a. $(0, 0, 0)$
   b. $(-1, 5, 3)$

I now want to derive formulas for line/plane intersections. Consider the plane $\vec{n} \cdot \vec{p} + d = 0$ and the line $\vec{r}(t) = \vec{q} + t\vec{v}$. We want to know where these two geometric objects intersect; that is, we wish to find the particular value of t such that $\vec{r}(t)$ satisfies the plane equation. To do this, we plug $\vec{r}(t)$ into the plane equation and solve for t:

$\vec{n} \cdot \vec{p} + d = 0$	*Start with plane equation*
$\vec{n} \cdot \vec{r}(t) + d = 0$	*Plug line into plane equation*
$\vec{n} \cdot (\vec{q} + t\vec{v}) + d = 0$	*Expand the line*
$\vec{n} \cdot \vec{q} + t(\vec{n} \cdot \vec{v}) + d = 0$	*Distribute and pull scalar out of dot product*
$t = \dfrac{-d - \vec{n} \cdot \vec{q}}{(\vec{n} \cdot \vec{v})}$	*Algebra manipulations*

The specific value $t = \dfrac{-d - \vec{n} \cdot \vec{q}}{(\vec{n} \cdot \vec{v})}$ is the parameter that yields the point of intersection. That is, the point of intersection is now given by:

$$\vec{r}\left(\frac{-d - \vec{n} \cdot \vec{q}}{(\vec{n} \cdot \vec{v})}\right) = \vec{q} + \left(\frac{-d - \vec{n} \cdot \vec{q}}{(\vec{n} \cdot \vec{v})}\right)\vec{v}$$

7. Consider the plane defined by the normal $\vec{n} = (1, 2, 3)$ and point $\vec{p}_0 = (-2, 3, 0)$. Also consider the line defined by $\vec{r}(t) = (1, 0, 0) + t\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Find the point where the line intersects the plane.

8. Consider the plane defined by the normal $\vec{n} = (-1, 1, 0)$ and point $\vec{p}_0 = (0, 1, 0)$. Also consider the line defined by $\vec{r}(t) = (3, 0, 0) + t(0, 1, 0)$. Find the point where the line intersects the plane.

I now want to derive formulas for line/sphere intersections. Consider the sphere $\|\vec{p} - \vec{c}\| - r = 0$ and the line $\vec{r}(t) = \vec{q} + t\vec{v}$. We want to know where these two geometric objects intersect; that is, we wish to find the particular value of t such that $\vec{r}(t)$ satisfies the sphere equation. To do this, we plug $\vec{r}(t)$ into the sphere equation and solve for t:

$$
\begin{aligned}
&\|\vec{p} - \vec{c}\| - r = 0 \\
&\|\vec{r}(t) - \vec{c}\| - r = 0 \\
&\|\vec{q} + t\vec{v} - \vec{c}\| - r = 0 \\
&\|\vec{q} + t\vec{v} - \vec{c}\|^2 = r^2 \\
&(\vec{q} + t\vec{v} - \vec{c}) \cdot (\vec{q} + t\vec{v} - \vec{c}) = r^2 \\
&(\vec{q} + t\vec{v} - \vec{c}) \cdot \vec{q} + (\vec{q} + t\vec{v} - \vec{c}) \cdot t\vec{v} - (\vec{q} + t\vec{v} - \vec{c}) \cdot \vec{c} = r^2 \\
&\vec{q} \cdot \vec{q} + t\vec{v} \cdot \vec{q} - \vec{c} \cdot \vec{q} + t\vec{v} \cdot \vec{q} + t^2\vec{v} \cdot \vec{v} - t\vec{c} \cdot \vec{v} - \vec{c} \cdot \vec{q} - t\vec{c} \cdot \vec{v} + \vec{c} \cdot \vec{c} = r^2 \\
&t^2(\vec{v} \cdot \vec{v}) + (t\vec{v} \cdot \vec{q} + t\vec{v} \cdot \vec{q} - t\vec{c} \cdot \vec{v} - t\vec{c} \cdot \vec{v}) + (\vec{q} \cdot \vec{q} - \vec{c} \cdot \vec{q} - \vec{c} \cdot \vec{q} + \vec{c} \cdot \vec{c}) = r^2 \\
&t^2(\vec{v} \cdot \vec{v}) + 2t(\vec{v} \cdot \vec{q} - \vec{c} \cdot \vec{v}) + (\vec{q} \cdot \vec{q} - \vec{c} \cdot \vec{q} - \vec{c} \cdot \vec{q} + \vec{c} \cdot \vec{c}) = r^2 \\
&t^2(\vec{v} \cdot \vec{v}) + 2t\vec{v} \cdot (\vec{q} - \vec{c}) + (\vec{q} - \vec{c}) \cdot (\vec{q} - \vec{c}) = r^2
\end{aligned}
$$

For convenience, define $A = \vec{v} \cdot \vec{v}$, $B = 2\vec{v} \cdot (\vec{q} - \vec{c})$, $C = (\vec{q} - \vec{c}) \cdot (\vec{q} - \vec{c}) - r^2$. Then the equation has the form $At^2 + Bt + C = 0$, which is clearly quadratic. Applying the quadratic formula yields two solutions:

$$
t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.
$$

If $B^2 - 4AC < 0$, then there is no real solution, which means the line misses the sphere. If $B^2 - 4AC = 0$, then the two solutions are the same, which means the line is tangent to the sphere. If $B^2 - 4AC > 0$, then the line intersects the sphere twice.

9. Consider the sphere with center at the origin and radius 1. Also consider the line defined by $\vec{r}(t) = (0, 0, 0) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. Find the points where the line intersects the sphere.

10. Consider the sphere with center at the origin and radius 1. Also consider the line defined by $\vec{r}(t) = (0, 2, 0) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$. Find the points where the line intersects the sphere.

11. Consider the sphere with center at the origin and radius 1. Also consider the line defined by $\vec{r}(t) = (0, 1, 0) + t(1, 0, 0)$. Find the points where the line intersects the sphere.

12. Do problem 1 from the textbook.

Answers to Selected Problems

1a. $(-1, 4)$
1b. $(3, 2)$
1c. $(-1, 2)$
1d. $\left(1, \frac{13}{2}\right)$
1e. $(8, 3)$
1f. $(5, 2)$

2a. 1
2b. $\|\vec{u}\| = \sqrt{14}, \|\vec{v}\| = \sqrt{5}$
2c. 83.135°
2d. $\left(\frac{-2}{5}, 0, \frac{1}{5}\right)$

3a. 0
3b. $\|\vec{u}\| = 2, \|\vec{v}\| = 2$
3c. 90°
3d. $(0, 0, 0)$

4a. $(2, -7, 4)$
4c. $\vec{w} = -\vec{r}$

5a. $(0, 0, -4)$
5c. $\vec{w} = -\vec{r}$

6a. $4/\sqrt{14}$
6b. $14/\sqrt{14}$

7. $t = \sqrt{3}/2$, $\vec{p} = \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$

8. $t = 4$, $\vec{p} = (3, 4, 0)$

9. $t = \pm 1$, $\vec{p}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\vec{p}_2 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

10. No hit

11. $t = 0$, $\vec{p} = (0, 1, 0)$

Game Math Chapter 8 Exercises

Chapter Remarks: *For this chapter, you need to know how to perform the matrix operations, such as addition and multiplication. Matrix multiplication is particularly important. In addition, you need to know how to solve systems of equations with matrices. The following exercises should test what is expected of you after studying this chapter.*

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 3 \\ -1 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$. Perform the indicated operations.

   a. $A + B$
   b. $A - 2B$
   c. $-A + (B - C)$
   d. AB
   e. BA
   f. A^T
   g. $(AB)(B^{-1}A^{-1})$

2. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & 3 & 1 \end{bmatrix}$,

   $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} -1 & 2 & -3 & 0 \end{bmatrix}$, and $v = \begin{bmatrix} 2 & 0 & -2 & 1 \end{bmatrix}$. Perform the indicated multiplication.

   a. AB
   b. ABC
   c. CD
   d. uD
   e. vC
   f. $uABC$
   g. $vCBA$

3. Solve the system of equations using matrices:

$$3x_1 - 2x_2 = 5$$
$$-x_1 + 4x_2 = 2$$

4. Solve the system of equations using matrices:

$$3x_1 - 2x_2 + 4x_3 = 0$$
$$-x_1 + 4x_2 + 3x_3 = -1$$
$$x_1 + x_2 = 1$$

5. Solve the system of equations using matrices:

$$x_1 + 2x_2 - x_4 = 8$$
$$8x_1 + 3x_2 - 5x_3 + x_4 = 15$$
$$-2x_1 - 3x_2 + 4x_3 - x_4 = -4$$
$$-x_3 + x_4 = 0$$

6. Do problem 1 and 3 from the textbook.

Answers to Selected Problems

1a. $\begin{bmatrix} -1 & 5 \\ 2 & 7 \end{bmatrix}$

1b. $\begin{bmatrix} 5 & -4 \\ 5 & -2 \end{bmatrix}$

1c. $\begin{bmatrix} -4 & 2 \\ -6 & -1 \end{bmatrix}$

1d. $\begin{bmatrix} -4 & 9 \\ -10 & 21 \end{bmatrix}$

1e. $\begin{bmatrix} 7 & 8 \\ 8 & 10 \end{bmatrix}$

1f. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

1g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2a. $\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 2 & 0 & 0 \\ \frac{3\sqrt{2}}{2} & 0 & \frac{3\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2b. $\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 2 & 0 & 0 \\ \frac{3\sqrt{2}}{2} & 0 & \frac{3\sqrt{2}}{2} & 0 \\ -2 & -1 & 3 & 1 \end{bmatrix}$

2c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & 3 & 1 \end{bmatrix}$

2d. $\begin{bmatrix} -1 & 2 & -3 & 0 \end{bmatrix}$

2e. $\begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix}$

2f. $\begin{bmatrix} -5\sqrt{2} & 4 & -4\sqrt{2} & 0 \end{bmatrix}$

2g. $\begin{bmatrix} \frac{\sqrt{2}}{2} & -2 & \frac{3\sqrt{2}}{2} & 1 \end{bmatrix}$

3. $x_1 = \frac{12}{5}, x_2 = \frac{11}{10}$

4. $x_1 = \frac{26}{35}, x_2 = \frac{9}{35}, x_3 = -\frac{3}{7}$

5. $x_1 = \frac{49}{17}, x_2 = \frac{97}{17}, x_3 = \frac{107}{17}, x_4 = \frac{107}{17}$

Game Math Chapter 9 Exercises

Chapter Remarks: *This chapter deals with the matrix representation of transformations. In particular, you need to know how to use a transformation matrix to transform vertices. The following exercises should test what is expected of you after studying this chapter.*

1. Find the scaling matrix that scales 3 units on the *x*-axis, -1 units on the *y*-axis, and 2 units on the *z*-axis. Thus use this matrix to scale the vertex $[1,1,1,1]^T$.

2. Find the rotation matrix that rotates $30°$ on the *y*-axis. Thus use this matrix to rotate the vertex $[1,0,0,1]^T$.

3. Find the translation matrix that translates 3 units on the *x*-axis, -1 units on the *y*-axis, and 2 units on the *z*-axis. Thus use this matrix to translate the vertex $[1,1,1,1]^T$.

4. Find the transformation matrix that combines, in order, all three of the three previous transformations (from Exercises 1, 2, and 3). Then use this matrix to transform the vertex $[1,1,1,1]^T$.

5. Do problem 1, 2, 3, and 6 from the textbook.

Answers to Selected Problems

1. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{3\sqrt{3}}{2} & 0 & -1 & 3 \\ 0 & -1 & 0 & -1 \\ \frac{3}{2} & 0 & \sqrt{3} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{3\sqrt{3}}{2} + 2 \\ -2 \\ \sqrt{3} + \frac{7}{2} \\ 1 \end{bmatrix}$

Game Math Chapter 10 Exercises

Chapter Remarks: *The main skill from this chapter that I want you to walk away with is being comfortable with the terminology and operations of complex numbers and quaternions. You should be able to compute the basic operations by hand and apply the formulas. Quaternions are advance mathematical objects, so I do not expect you to have a deep understanding of them, but as said, you should be able to apply the formulas given in the textbook. The following exercises should test what is expected of you after studying this chapter.*

1. Perform the indicated complex number operation.

   a. $(3+2i)+(-1+i)$
   b. $(3+2i)-(-1+i)$
   c. $(3+2i)(-1+i)$
   d. $4(-1+i)$
   e. $(3+2i)/(-1+i)$
   f. $(3+2i)^*$
   g. $|(3+2i)|$

2. Perform the indicated complex operation.

   a. $(1-4i)+(2-3i)$
   b. $(1-4i)-(2-3i)$
   c. $(1-4i)(2-3i)$
   d. $-2(2-3i)$
   e. $(1-4i)/(2-3i)$
   f. $(1-4i)^*$
   g. $|(1-4i)|$

3. Let $p=1+2i+3j+4k$ and $q=2-i+j+-2k$. Perform the indicated quaternion operation.

   a. $p+q$
   b. $p-q$
   c. pq
   d. p^*

e. q^*
f. $p^* p$
g. $|p|$
h. $|q|$
i. p^{-1}
j. q^{-1}

4. Do problem 1 and 2 from the textbook.

Answers to Selected Problems

1a. $2+3i$
1b. $4+i$
1c. $-5+i$
1d. $-4+4i$
1e. $-\frac{1}{2}-\frac{5i}{2}$
1f. $3-2i$
1g. $\sqrt{13}$

2a. $3-7i$
2b. $-1-i$
2c. $-10-11i$
2d. $-4+6i$
2e. $\frac{14}{13}-\frac{5i}{13}$
2f. $1+4i$
2g. $\sqrt{17}$

3a. $3+i+4j+2k$
3b. $-1+3i+2j+6k$
3c. $9-7i+7j+11k$
3d. $1-2i-3j-4k$
3e. $2+i-j+2k$
3f. 30
3g. $\sqrt{30}$
3h. $\sqrt{10}$
3i. $\frac{1-2i-3j-4k}{30}$

3j. $\frac{2+i-j+2k}{10}$