Game Math Chapter 5 Solutions

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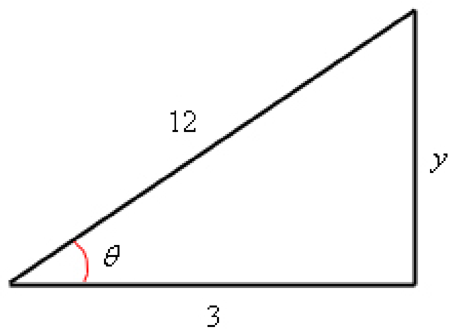
1. Given the following right triangle information, find y and θ.



We have, $\cos(\theta) = x/r = 3/12$. To solve for θ we take the inverse cosine of both sides of the equation giving,

$$\begin{aligned} \cos^{-1}\left(\cos(\theta)\right) &= \cos^{-1}(3/12) \\ \theta = \cos^{-1}(3/12) &\approx 75.522° \end{aligned}.$$

To solve for y we use the definition, $\sin\theta = y/r$, so $y = 12\sin(75.522°) \approx 11.618$.

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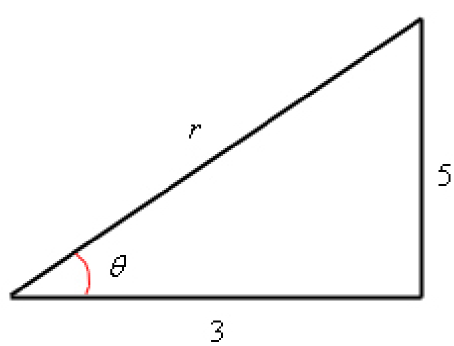
2. Given the following right triangle information, find r and θ.



To solve for θ we use the definition, $\tan(\theta) = y/x$, and then apply the inverse tangent. So, with our numbers we have,

$$\tan(\theta) = \frac{5}{3}$$
$$\tan^{-1}(\tan(\theta)) = \tan^{-1}\left(\frac{5}{3}\right)$$
$$\theta = \tan^{-1}\left(\frac{5}{3}\right) \approx 59.036°.$$

Then, to solve for *r*, we employ the theorem of Pythagoras, $a^2 + b^2 = c^2$, as follows,

$$3^2 + 5^2 = r^2$$
$$r = \sqrt{3^2 + 5^2} = \sqrt{34} \approx 5.8309.$$

Note you can also solve for *r* using the definition of either sine or cosine.

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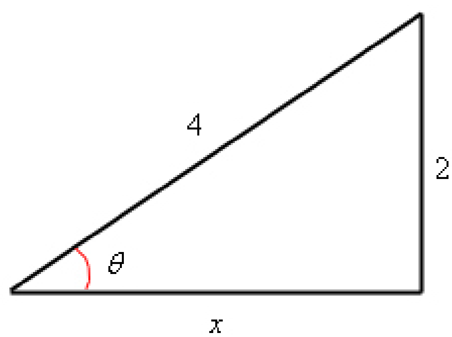
3. Given the following right triangle information, find *x* and θ.



We start with the definition of sine: $\sin(\theta) = y/r$. Substituting our numbers then gives, $\sin(\theta) = 2/4 = 1/2$. To solve for θ we take the inverse sine of both sides of the equation, which yields $\theta = \sin^{-1}(1/2) = 30°$, and recalling that $\sin^{-1}(\sin(\theta)) = \theta$, which follows from definition. To solve for *x* we recall that $\cos(\theta) = x/r$, so $x = r\cos(\theta) = 4\cos(30°) = 2\sqrt{3} = 3.4641$.

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4. Prove $1 + \cot^2\theta = \csc^2\theta$.

We know $\sin^2(\theta) + \cos^2(\theta) = 1$. Dividing both sides of the equation by $\sin^2(\theta)$ gives:

$$\frac{\sin^2(\theta)}{\sin^2(\theta)}+\frac{\cos^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)} \Leftrightarrow 1+\cot^2(\theta)=\csc^2(\theta).$$

QED.

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5. Prove the following,

a. $\tan(\theta+\beta)=\dfrac{\tan\theta+\tan\beta}{1-\tan\theta\tan\beta}$ b. $\tan(\theta-\beta)=\dfrac{\tan\theta-\tan\beta}{1+\tan\theta\tan\beta}$

c. $\cot(\theta+\beta)=\dfrac{\cot\beta\cot\theta-1}{\cot\beta+\cot\theta}$ d. $\cot(\theta-\beta)=\dfrac{\cot\beta\cot\theta+1}{\cot\beta-\cot\theta}$.

5a.

$$\tan(\theta+\beta)=\frac{\sin(\theta+\beta)}{\cos(\theta+\phi)}=\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\cos\theta\cos\beta-\sin\theta\sin\beta}=\frac{\frac{1}{\cos\theta\cos\beta}}{\frac{1}{\cos\theta\cos\beta}}\cdot\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\cos\theta\cos\beta-\sin\theta\sin\beta}$$

$$=\frac{\frac{\sin\theta\cos\beta}{\cos\theta\cos\beta}+\frac{\cos\theta\sin\beta}{\cos\theta\cos\beta}}{\frac{\cos\theta\cos\beta}{\cos\theta\cos\beta}-\frac{\sin\theta\sin\beta}{\cos\theta\cos\beta}}=\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\beta}{\cos\beta}}{1-\tan\theta\tan\beta}=\frac{\tan\theta+\tan\beta}{1-\tan\theta\tan\beta}$$

QED.

The trick to this proof is using the addition formulas for sine and cosine, and also finding a clever expression of the number 1 to multiply by (recall we can always multiple by 1 without changing the expression), which allows us to do some simplifications. In this example, the expression for 1, which we multiplied by, was

$$\frac{\frac{1}{\cos\theta\cos\beta}}{\frac{1}{\cos\theta\cos\beta}}=1.$$

5b. $\tan(\theta-\beta)=\tan(\theta+(-\beta))=\dfrac{\tan\theta+\tan(-\beta)}{1-\tan\theta\tan(-\beta)}=\dfrac{\tan\theta-\tan\beta}{1+\tan\theta\tan\beta}$.

QED.

So the trick here is rewrite the subtraction as an addition with a negative angle so that we can use the result from 5a. We also use the fact $\tan(-\beta)=-\tan\beta$.

The next two proofs are essentially analogous to the above two.

5c.

$$\cot(\theta+\beta)=\frac{\cos(\theta+\beta)}{\sin(\theta+\beta)}=\frac{\cos\theta\cos\beta-\sin\theta\sin\beta}{\sin\theta\cos\beta+\cos\theta\sin\beta}=\frac{\frac{1}{\sin\theta\sin\beta}}{\frac{1}{\sin\theta\sin\beta}}\cdot\frac{\cos\theta\cos\beta-\sin\theta\sin\beta}{\sin\theta\cos\beta+\cos\theta\sin\beta}$$

$$=\frac{\frac{\cos\theta\cos\beta}{\sin\theta\sin\beta}-\frac{\sin\theta\sin\beta}{\sin\theta\sin\beta}}{\frac{\sin\theta\cos\beta}{\sin\theta\sin\beta}+\frac{\cos\theta\sin\beta}{\sin\theta\sin\beta}}=\frac{\cot\theta\cot\beta-1}{\cot\beta+\cot\theta}$$

QED.

5d.

$$\cot(\theta-\beta)=\cot(\theta+(-\beta))=\frac{\cot\theta\cot(-\beta)-1}{\cot(-\beta)+\cot\theta}=\frac{-\cot\theta\cot\beta-1}{-\cot\beta+\cot\theta}$$

$$=-1\cdot\frac{\cot\theta\cot\beta+1}{-\cot\beta+\cot\theta}=\frac{\cot\theta\cot\beta+1}{\cot\beta-\cot\theta}$$

QED.

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6. Derive formulae to rotate any point around the point (x, y), using the translation trick mentioned in this lesson's material.

Recall the rotation equations to rotate a point $(x,\ y)$ counterclockwise about the origin:

$$x'=x\cos\theta-y\sin\theta$$
$$y'=x\sin\theta+y\cos\theta$$

Here we wish to rotate a point (a, b), not about the origin, but about a pivot point (x, y)—see Figure 6a.

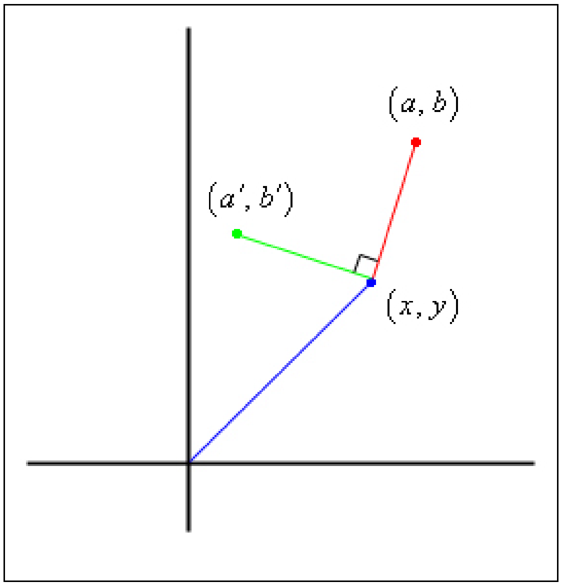

Figure 6a: We want to rotate a point (a, b) about a pivot point (x, y) into a new point (a', b').

Well, our rotation equations rotate about the origin of the coordinate system, so the first thing we must do is to pick a new frame of reference (coordinate system) where (x, y) is the origin, and then translate the geometry so that its coordinates are specified relative to this new coordinate system. This is easy enough, to transform points relative to (x, y) we offset their x coordinates by $-x$ and offset their y coordinates by $-y$. Thus, (a, b) relative to (x, y) is given by $(\tilde{a}, \tilde{b})$:

$$\tilde{a} = a - x$$
$$\tilde{b} = b - y$$

Our geometry relative to this new coordinate system looks like Figure 6b.

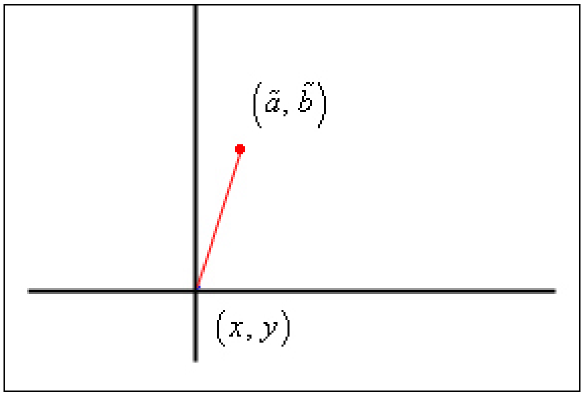

Figure 6b: We have transformed the geometry to be relative to the pivot point (x, y); in other words, we are using a different coordinate system. Don't let this fool you, there is nothing tricky going on, a coordinate system is just a frame of reference which is rather arbitrary except for the fact that particular systems are more convenient than others. This new system is more convenient than the other because the pivot point is the origin and so we can directly apply our rotation equations here.

Now, in this new coordinate system, the pivot is at the origin! Thus we can apply our rotation equations:

$$\tilde{a}' = \tilde{a}\cos\theta - \tilde{b}\sin\theta$$
$$\tilde{b}' = \tilde{a}\sin\theta + \tilde{b}\cos\theta$$

The result of this operation corresponds with Figure 6c.

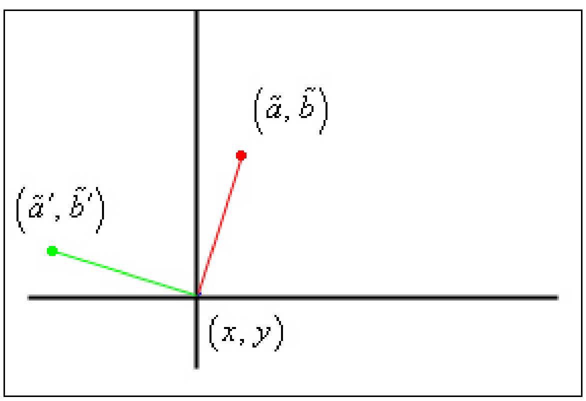

Figure 6c: Rotating in our new coordinate system.

We are not done, however, as we are still in the coordinate system where the pivot point is the origin, so what we want to do now is transform back to the original coordinate system. To do this we offset the *x* coordinates by +*x* and offset the *y* coordinates by +*y*:

$$a' = \left(\tilde{a}\cos\theta - \tilde{b}\sin\theta\right) + x$$
$$b' = \left(\tilde{a}\sin\theta + \tilde{b}\cos\theta\right) + y$$

Figure 6d shows where we are now, which is where we want to be. Making some substitutions allows us to express this in terms of the original variables:

$$a' = \left(\left(a - x\right)\cos\theta - \left(b - y\right)\sin\theta\right) + x$$
$$b' = \left(\left(a - x\right)\sin\theta + \left(b - y\right)\cos\theta\right) + y$$

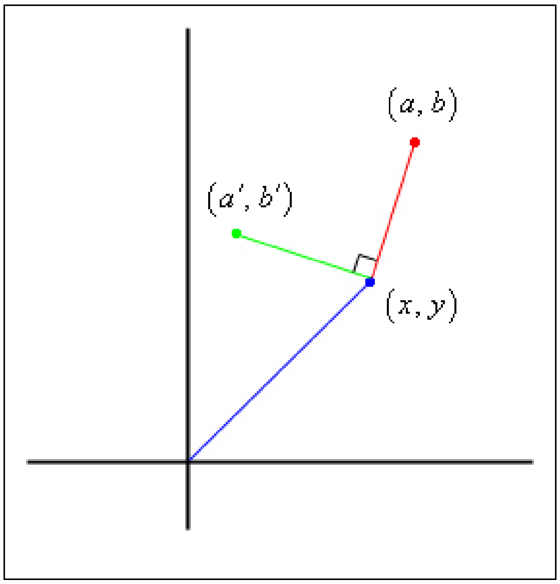

Figure 6d: Translating back to our original coordinate system.

In summary, we do the following steps:

1. Transform the geometry relative to a new coordinate system where the pivot point is the origin.

2. Apply the rotation equations.

3. Transform the geometry back to the original coordinate system.

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7. One problem in designing intelligent computer opponents in action games is determining if an enemy can see the player, which in turn determines if the enemy attacks. Suppose the position of the player is (x, y) and the position of an enemy is (s, t). Further, suppose the enemy's field of view is 2β, and that the angle the enemy's forward line of sight makes with the positive x-axis is θ. Derive equations to determine if the enemy can see the player. (You can also use this technique to optimize the performance of 3D games by only displaying the objects the player can see.)

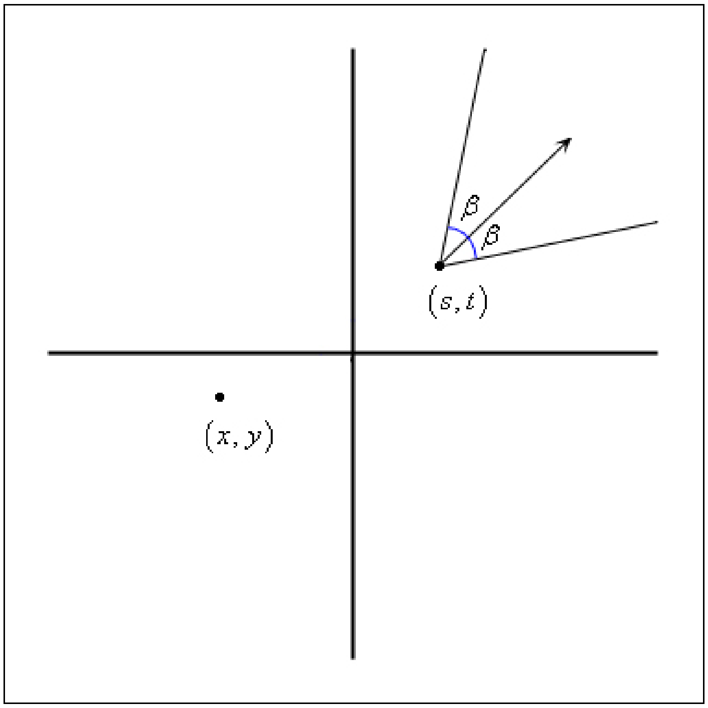

Figure 7a.

Figure 7a shows a general picture of our problem. However, the problem is easier to tackle if we move to the coordinate system relative to (s, t).

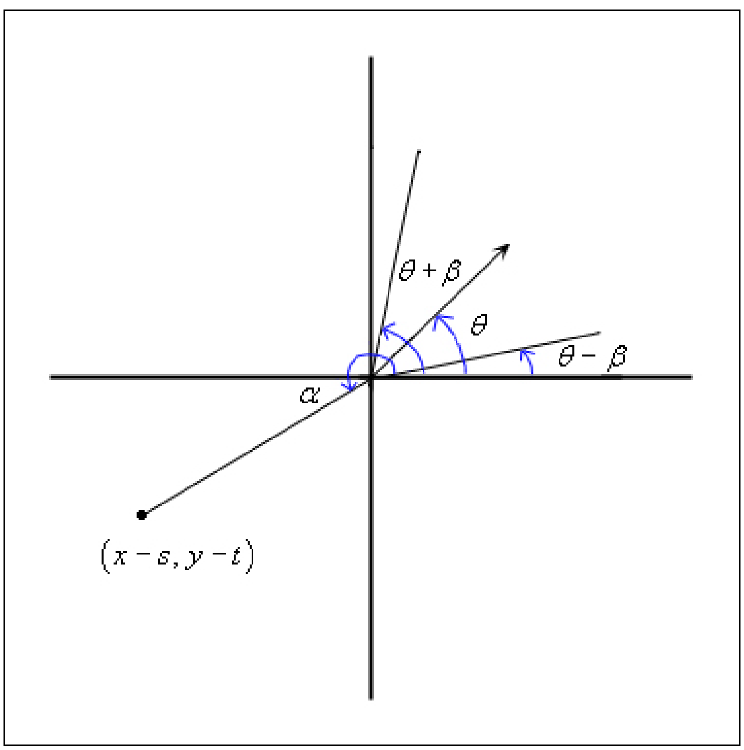

Figure 7b.

In this new coordinate system we can see that the field of view interval the enemy can see is $[\theta - \beta, \theta + \beta]$. Thus if the angle the player's position makes with the x-axis (call it α) falls in that interval, then the player is visible. More precisely,

IF $\theta - \beta \le \alpha \le \theta + \beta$**, THEN** *the player is visible.*

The angle α can be found in terms of $(x - s,\ y - t)$ as follows. First consider $\tan(\theta) = y/x$, where *y* and *x* are coordinates of some point in the 2D plane. Solving for theta we obtain $\theta = \tan^{-1}(y/x)$. But, recall from your studies of trigonometry that the inverse tangent function has some problems; in particular, its range is $[-90°, 90°]$, which means we cannot get angles outside quadrants 1 and 4. However, (x, y) can be in any quadrant. Clearly we have a problem, but making some observations easily solves it. Let us work with a concrete example to make this a bit easier.

Let $x = -4$ and $y = 4$. Clearly $(-4,\quad 4)$ lives in quadrant 2[1] and makes a 135° angle with the positive x-axis (sketch it out on paper). Because the point lies in quadrant 2, we know the inverse tangent will *not* return the correct angle since quadrant 2 is not in the inverse tangent's range. Are we stuck? Not yet. Let us calculate the inverse tangent just to see what happens: $\theta = \tan^{-1}(y/x) = \tan^{-1}(4/-4) = -45°$. Here we observe that if we add 180° to the inverse tangent result then we obtain the correct angle 135°; that is, -45° + 180° = 135°. In fact, if the angle θ falls in quadrant 2 or 3 (which we can determine by examining the signs of *x* and *y*), we will always be able to get the correct angle by adding 180°. In summary:

If θ is in quadrants 1 or 4 then $\theta = \tan^{-1}(y/x)$.
Else if θ is in quadrant 2 or 3 then $\theta = \tan^{-1}(y/x)$ + 180°.

Returning back to out original problem, based on the preceding discussion, we have,

$$\alpha = \begin{cases} \tan^{-1}(y - t/x - s),\ x - s,\ y - t > 0 \ or \ x - s > 0,\ y - t < 0 \\ \tan^{-1}(y - t/x - s) + 180°,\ x - s,\ y - t < 0 \ or \ x - s < 0,\ y - t > 0 \end{cases}$$

Note $x - s,\ y - t > 0$ implies quadrant 1; $x - s < 0,\ y - t > 0$ implies quadrant 2; $x - s,\ y - t < 0$ implies quadrant 3; $x - s > 0,\ y - t < 0$ implies quadrant 4.

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[1] We know this by examining the signs of x and y: Since x is negative it has to be to the left of the y-axis, and since y is positive it must be above the x-axis. Therefore, the point lies in quadrant 2.

8. Suppose the player has a horizontal field of view of *h* radians, and a vertical field of view of *v* radians. Determine the angle of a cone that completely encompasses what the player sees. (This information can be used to quickly and easily determine which objects in a 3D world are visible to the player.)

So for the cone to completely encompass what the player sees, the cone must surround the screen. Moreover, we want the cone to be "tight fitting," which is to say, we want the cone to be just large enough to surround the screen. Now consider the circle produced when the cone intersects the plane the screen is on. Since the cone is tight fitting, the circle just barely surrounds the screen (Figure 8a). We note that the radius of the circle is *y*, and can be expressed in terms of *x*, *h*, and *v*, where *x* is the distance from the eye to the screen, *h* is the horizontal field of view angle, and *v* is the vertical field of view angle.

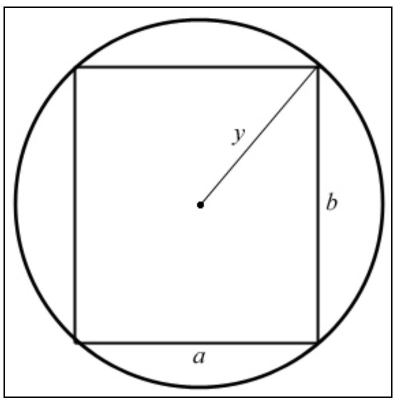

Figure 8a.

$$\tan\left(h/2\right) = \frac{a/2}{x} \Leftrightarrow a = 2x\tan\left(h/2\right)$$
$$\tan\left(v/2\right) = \frac{b/2}{x} \Leftrightarrow b = 2x\tan\left(v/2\right)$$

$$y = \sqrt{a^2 + b^2}$$

Now that we know the radius of this circle, we can solve for half the cone angle by writing,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

(see Figure 8b). Then the total cone angle is 2θ.

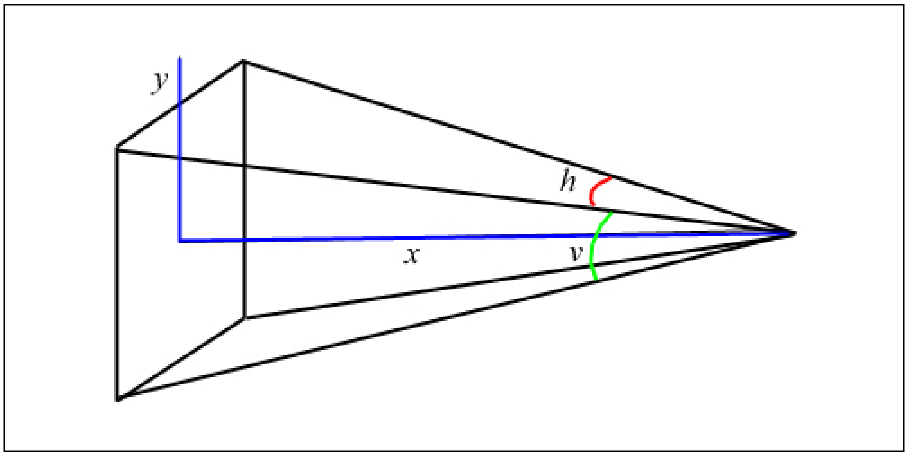

Figure 8b.

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9. Derive formulas for rotation around the *x*- and *y*-axes.

The idea is the same as rotation about the *z*-axis, except that we are working on different planes. So really all you need to do is switch the coordinates around so that it makes sense for the plane you are working on:

Rotation on *z*-axis (*xy*-plane):

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

Rotation on *y*-axis (*xz*-plane):

$$x' = x\cos\theta - z\sin\theta$$
$$z' = x\sin\theta + z\cos\theta$$

Rotation on *x*-axis (*zy*-plane):

$$y' = z\sin\theta + y\cos\theta$$
$$z' = z\cos\theta - y\sin\theta$$