Game Math Chapter 4 Solutions

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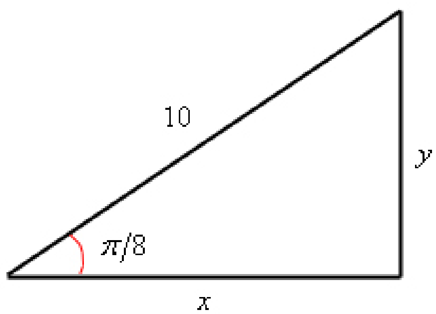
1. Given the following right triangle information, find x and y.



From definition, recall that $\sin(\theta) = y/r$ and $\cos(\theta) = x/r$, where y is the length of the opposite edge, x is the length of the adjacent edge, and r is the length of the hypotenuse. Using these definition with our given data we can solve for x and y like so:

$$\sin\left(\frac{\pi}{8}\right) = \frac{y}{10} \Leftrightarrow y = 10\sin\left(\frac{\pi}{8}\right) \approx 3.82683$$
$$\cos\left(\frac{\pi}{8}\right) = \frac{x}{10} \Leftrightarrow x = 10\cos\left(\frac{\pi}{8}\right) \approx 9.23879$$

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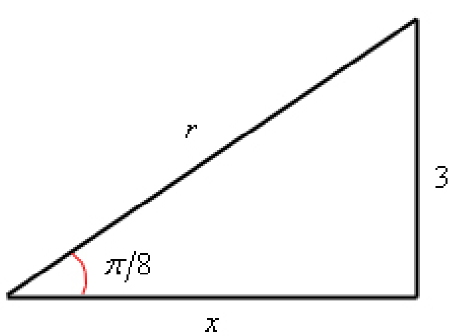
2. Given the following right triangle information, find x and r.



From definition, recall that $\sin(\theta) = y/r$ and $\cos(\theta) = x/r$, where y is the length of the opposite edge, x is the length of the adjacent edge, and r is the length of the hypotenuse. Using these definition with our given data we can solve for x and r like so:

$$\sin\left(\frac{\pi}{8}\right) = \frac{3}{r} \Leftrightarrow r = \frac{3}{\sin(\pi/8)} \approx 7.83937$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{x}{7.83937} \Leftrightarrow x = 7.83937\cos\left(\frac{\pi}{8}\right) \approx 7.24263$$

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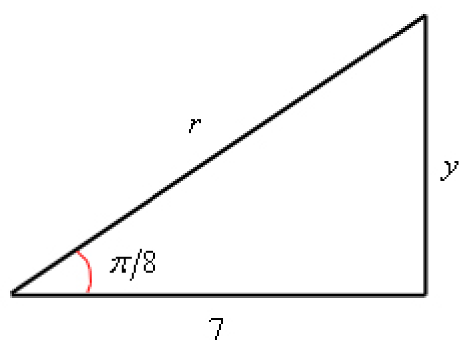
3. Given the following right triangle information, find y and r.



From definition, recall that $\sin(\theta) = y/r$ and $\cos(\theta) = x/r$, where y is the length of the opposite edge, x is the length of the adjacent edge, and r is the length of the hypotenuse. Using these definition with our given data we can solve for y and r like so:

$$\cos\left(\frac{\pi}{8}\right) = \frac{7}{r} \Leftrightarrow r = \frac{7}{\cos(\pi/8)} \approx 7.57674$$

$$\sin\left(\frac{\pi}{8}\right) = \frac{y}{7.57674} \Leftrightarrow y = 7.57674\sin\left(\frac{\pi}{8}\right) \approx 2.89949$$

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4. Suppose you have n functions, labeled $f_1(r, t), f_2(r, t), \ldots, f_n(r, t)$, and that the ith function models the height of a water ripple at time t at a distance r away from the impact point (x_i, y_i). Assuming all of the ripples occur in a single pond, write down an expression describing the height of an arbitrary point (x, y) at time t. (The distance between two points (a, b) and (c, d) is $\sqrt{(a-c)^2 + (b-d)^2}$.)

The key ideas here are that there are *n* ripples in the pond; the *i*th ripple originates from a point (x_i, y_i); and the *i*th function f_i models the height of the *i*th water ripple as a function of time and distance. So for example, take the first ripple f_1, with just this one ripple, the height of an arbitrary point (x, y) in the pond is given by

$$f_1\left(\sqrt{(x_1 - x)^2 + (y_1 - y)^2},\, t\right).$$

Notice the height of the point (x, y) is a function of the distance of the point from the origin of the ripple. This makes sense intuitively because we expect the ripple to be larger near the source and then diminish in magnitude with distance.

Now, our pond does not have one ripple, but *n* ripples. So how to handle *n* ripples? Well you can think of a ripple as a wave, and there is a physics principle called the *superposition of waves*, which states that when you have multiple waves, you can sum the wave functions to yield a net wave function. So, in order to get the height of an arbitrary point with *n* ripples (waves), we sum the heights given from all *n* ripples to obtain a net height:

$$f(x, y, t) = f_1\left(\sqrt{(x_1 - x)^2 + (y_1 - y)^2},\, t\right) + f_1\left(\sqrt{(x_2 - x)^2 + (y_2 - y)^2},\, t\right) + \\ \ldots + f_n\left(\sqrt{(x_n - x)^2 + (y_n - y)^2},\, t\right)$$

Observe that since the ripples originate from different point sources, the distance from (x, y) to each ripple source point is different.

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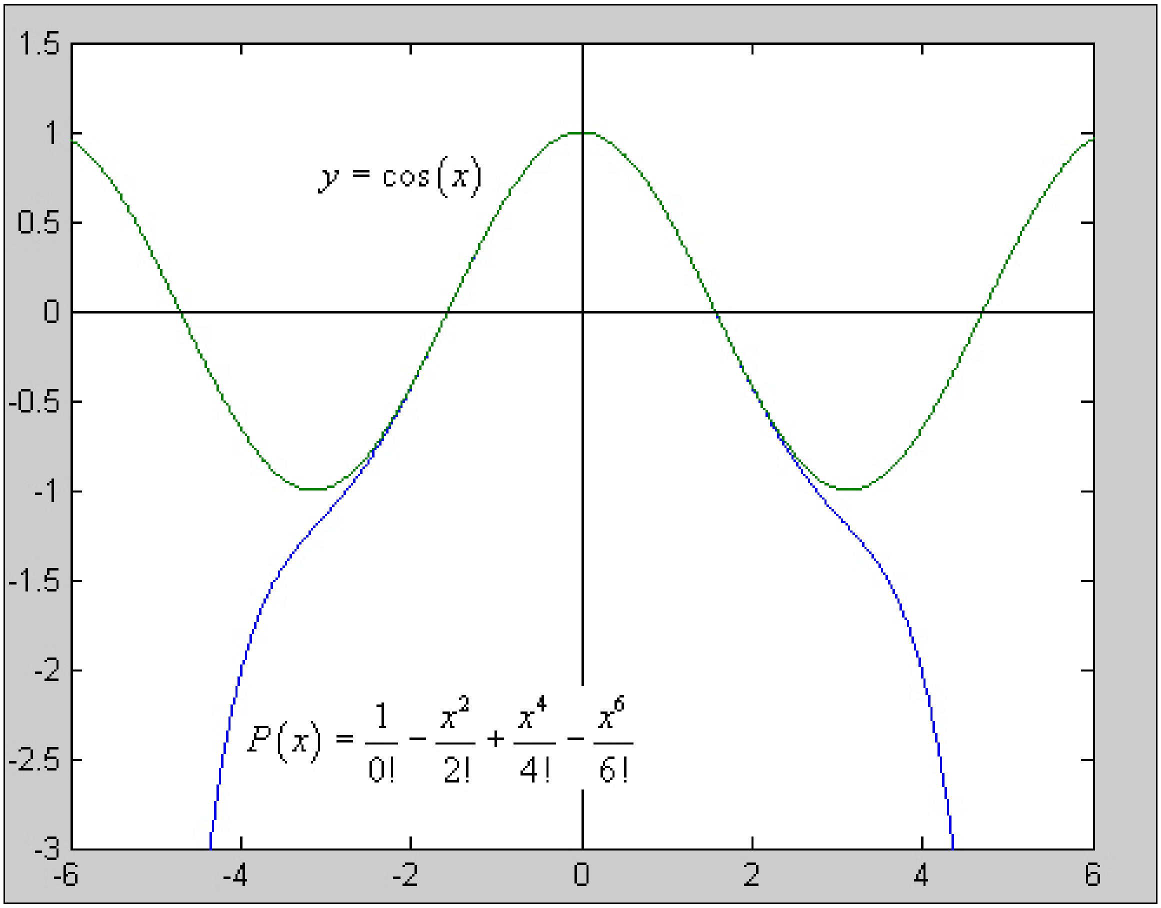
5. Polynomials can be used to represent can be used to approximate trigonometric functions (or even represent them perfectly, for polynomials of infinite degree). Determine which of the trig functions is approximated by the following polynomial:

$$f(x) = 1 - \frac{x^2}{1\times 2} + \frac{x^4}{1\times 2\times 3\times 4} - \frac{x^6}{1\times 2\times 3\times 4\times 5\times 6}$$

Graphing the function yields the following:



From the graph, we see that the function looks a lot like the cosine function for the interval [-2, 2].

Students of calculus will recognize the function as the first four terms of the Maclaurin polynomial for the cosine function. (Students who have not studied Calculus can skip to the next exercise solution.) The *nth* Maclaurin polynomial for *f* is defined as follows:

$$P(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k = \frac{f(0)}{0!} + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n$$

For cosine we have the following derivatives:

$f(0) = \cos(0) = 1$
$f^{(1)}(0) = -\sin(0) = 0$
$f^{(2)}(0) = -\cos(0) = -1$
$f^{(3)}(0) = \sin(0) = 0$
$f^{(4)}(0) = \cos(0) = 1$
$f^{(5)}(0) = -\sin(0) = 0$
$f^{(6)}(0) = -\cos(0) = -1$

Plugging these values into the Maclaurin polynomial gives:

$$P(x) = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!},$$

which was the function given in the exercise.

Incidentally, we noticed that the given function only approximately agreed with the cosine function in the interval [-2, 2]. By adding more terms to the Maclaurin polynomial, say 15, you get a wider interval of agreement.

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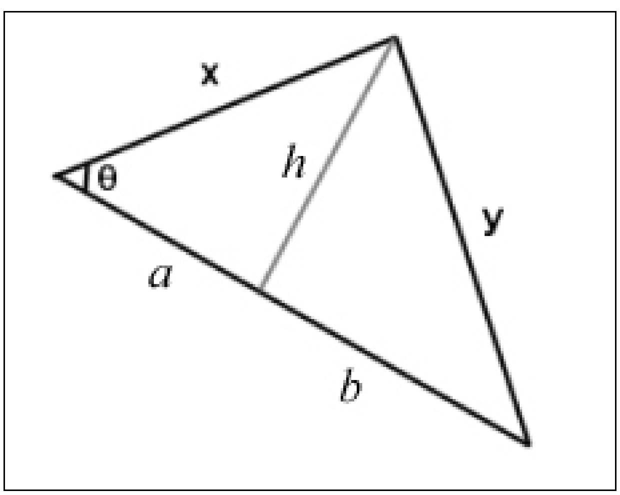
6. Given Pythagorean's theorem and what you know of trigonometry, determine the area of the triangle shown in the figure (the area of a triangle is used to compute its mass in some physics simulations).



From definition we have $\sin(\theta) = h/x \Leftrightarrow h = x\sin(\theta)$. Also by definition, we have $\cos(\theta) = a/x \Leftrightarrow a = x\cos(\theta)$. From the theorem of Pythagoras, we have

$b^2 + h^2 = y^2 \Leftrightarrow b = \pm\sqrt{y^2 - h^2}$. Since we are talking about lengths, a negative length does not make sense, and so we can abandon the negative solution and say that $b = \sqrt{y^2 - h^2}$. The area of a triangle is given by the formula $A = \frac{1}{2}wh$, where w is the length of the base, and h is the triangle height. In our figure, $w = a + b$. Thus, the area of our triangle is:

$$A = \frac{1}{2}(a+b)h = \frac{1}{2}\left(x\cos(\theta) + \sqrt{y^2 - h^2}\right)x\sin(\theta).$$

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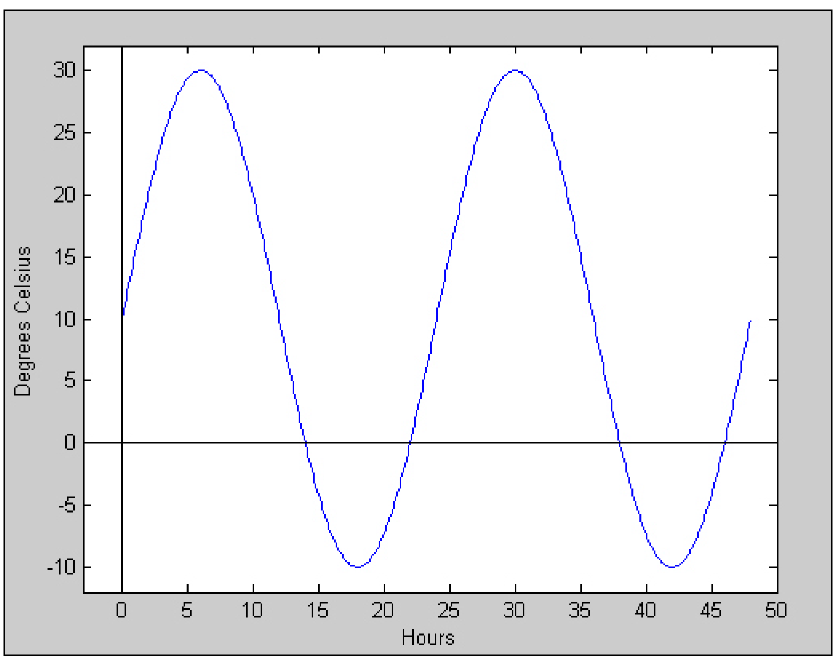
7. Suppose you are developing an RPG game and the speed at which a party can travel depends on the temperature (on the assumption that people travel slower in colder weather). Develop a function to model the temperature of the weather assuming trig-like behavior, subject to the following constraints: the time is measured in hours, the temperature, in degrees Celsius; the temperature reaches a low point and a high point exactly once every 24-hours; the minimum temperature is -10 C; the maximum temperature is 30 C.

We want a function that behaves like the following curve.



Notice that the curve meets the stated conditions: it repeats every 24 hours; it has a minimum of –10 C; it has a maximum of 30 C; and it has one maximum and one minimum every 24 hours. We can describe the above graph with a sine function. Recall that the since function has a range [-1, 1]. Therefore, if we multiply the since function by

20, it will have a range [-20, 20]. If we then offset the curve by adding 10, the range will become [10, 30]. Thus, so far we have $f(x) = 20\sin(\theta) + 10$. This, however, is not what we want. The sine function repeats every 2π, and we want it to repeat every 24 hours. In other words, one wavelength of 24 corresponds to an angle of 2π, thus we have the following ratio:

$$\frac{\theta}{2\pi} = \frac{x}{24} \Leftrightarrow \theta = \frac{2\pi}{24}x$$

Thus, out final function, which gives the graph shown in the above figure, is

$$f(x) = 20\sin\left(\frac{2\pi}{24}x\right) + 10.$$

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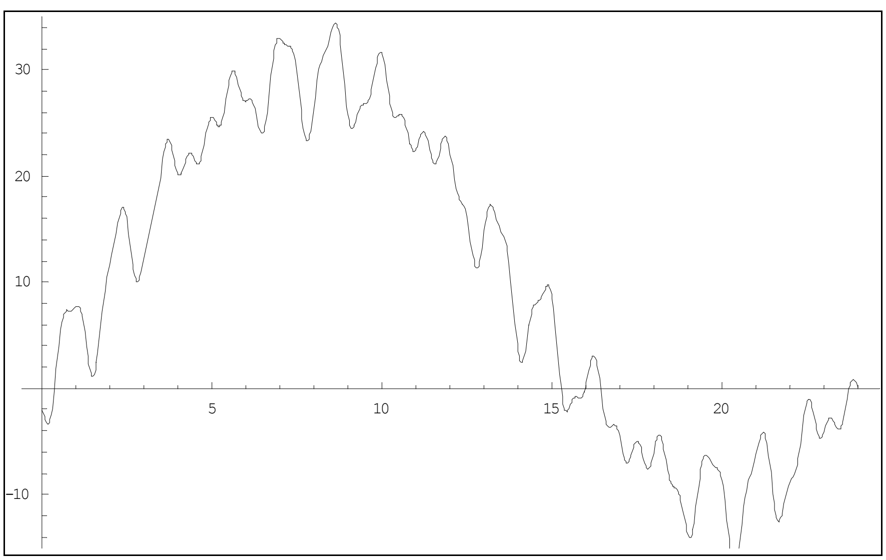
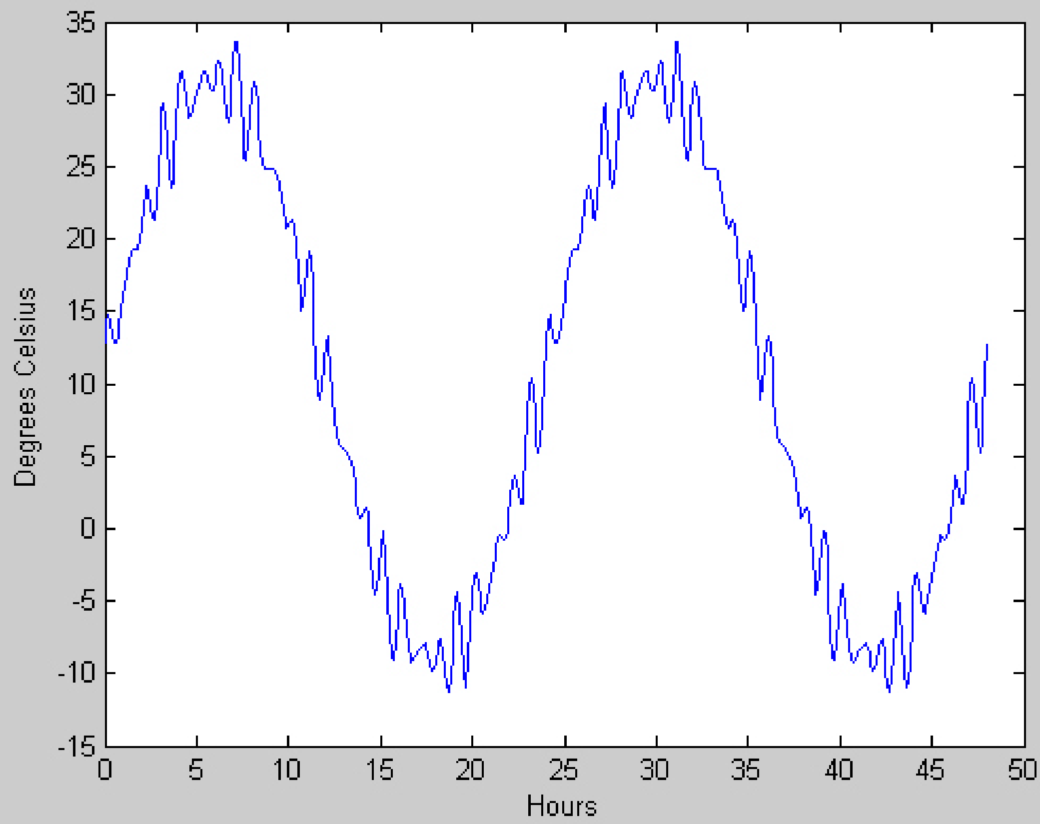
8. Improve upon the model you developed in problem 7 by incorporating small weather fluctuations throughout the day (caused by wind, changing locations or altitudes, etc.). The graph of your final function might look something like the graph shown in the figure. (Hint: You can obtain the graph shown in the figure by summing various sine or cosine functions.)



So the idea here is we have one function (the one found in Exercise 7), which gives the general temperature curve. We now want to add some chaotic fluctuations. As the hint says, we can do this by summing sine and cosine functions together. Because we want small chaotic fluctuations relative to the general temperature curve, the extra sine and cosine functions that we will sum, should have relatively smaller amplitudes, and should oscillate faster. By summing the following functions we get a fairly decent result.

$$f_1(x) = 20\sin\left(\frac{2\pi}{24}x\right) + 10$$
$$f_2(x) = 2\sin(2\pi x + 0.42)$$
$$f_3(x) = \sin(1.5\pi x)$$
$$f_4(x) = \cos(2.5\pi x + 0.1) + 1$$

$$\begin{aligned} f(x) &= f_1(x) + f_2(x) + f_3(x) + f_4(x) \\ &= 20\sin\left(\frac{2\pi}{24}x\right) + 10 + 2\sin(2\pi x + 0.42) + \sin(1.5\pi x) + \cos(2.5\pi x + 0.1) + 1 \end{aligned}$$



$$f(x) = 20\sin\left(\frac{2\pi}{24}x\right) + 10 + 2\sin(2\pi x + 0.42) + \sin(1.5\pi x) + \cos(2.5\pi x + 0.1) + 1$$

If you have a graphing utility, you might want to experiment and try to sum different sine and cosine functions to see the various results. A popular game related application of summing trig functions is to simulate water wave motion, which has a general pattern, along with chaotic variations.