Game Math Chapter 3 Solutions

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1. Calculate the following:

a. $(2x^2 - 4x^3) + (-2x^2 - 2x^3 + 9x)$
b. $(x + x^2 + x^3) \times (-2 + 3x + 5x^5)$
c. $(x^3 + 2x - 5) \div (x + 1)$

a. $(2x^2 - 4x^3) + (-2x^2 - 2x^3 + 9x)$
$= 2x^2 - 4x^3 - 2x^2 - 2x^3 + 9x$
$= -6x^3 + 9x$

b. $(x + x^2 + x^3) \times (-2 + 3x + 5x^5)$
$= -2x + 3x^2 + 5x^6 - 2x^2 + 3x^3 + 5x^7 - 2x^3 + 3x^4 + 5x^8$
$= 5x^8 + 5x^7 + 5x^6 + 3x^4 + x^3 + x^2 - 2x$

c. $(x^3 + 2x - 5) \div (x + 1)$

$$
\begin{array}{r|l}
 & x^2 - x + 3 \\
x+1 & x^3 + 0x^2 + 2x - 5 \\
 & -x^3 - x^2 \\
 & \quad -x^2 + 2x - 5 \\
 & \quad +x^2 + x \\
 & \qquad 3x - 5 \\
 & \qquad -3x - 3 \\
 & \qquad\quad -8
\end{array}
$$

$$(x^3 + 2x - 5) \div (x + 1) = x^2 - x + 3 + \frac{-8}{x + 1}$$

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2. Approximate the square root function with a quadratic polynomial over the interval [0, 10].

To do this we'll take three known points on the square root function over the interval [0, 10]: $(0, 0), (5, \sqrt{5}), (10, \sqrt{10})$. Given these three data points we can fit a parabola through them. This parabola will be our quadratic polynomial approximation. Note that it is important to select "good" points from the square root function that will give us a good approximation. So we pick the start and end points so that the quadratic polynomial agrees with the initial and end behavior; we also pick the middle point for our third point so that our approximation agrees with the square root function in the middle.

So we want to find the three coefficients of the quadratic polynomial $P(x) = ax^2 + bx + c$. Our given data tells us three points that our parabola must pass through, so we have the following conditions:

(1) $P(0) = 0 = a \cdot 0^2 + b \cdot 0 + c$

(2) $P(5) = \sqrt{5} = a \cdot 5^2 + b \cdot 5 + 0$

(3) $P(10) = \sqrt{10} = a \cdot 10^2 + b \cdot 10 + 0$

Condition (1) tells us directly that $c = 0$. Conditions (2) and (3) for a system of two linear equations, of which there are a myriad of ways to solve:

(4) $25a + 5b = \sqrt{5}$

(5) $100a + 10b = \sqrt{10}$.

Multiplying equation (4) by –2 gives:

(6) $-50a - 10b = -2\sqrt{5}$.

Adding equations (5) and (6) yields:

$$50a = \sqrt{10} - 2\sqrt{5}.$$

So,

(7) $a = \dfrac{\sqrt{10} - 2\sqrt{5}}{50}$.

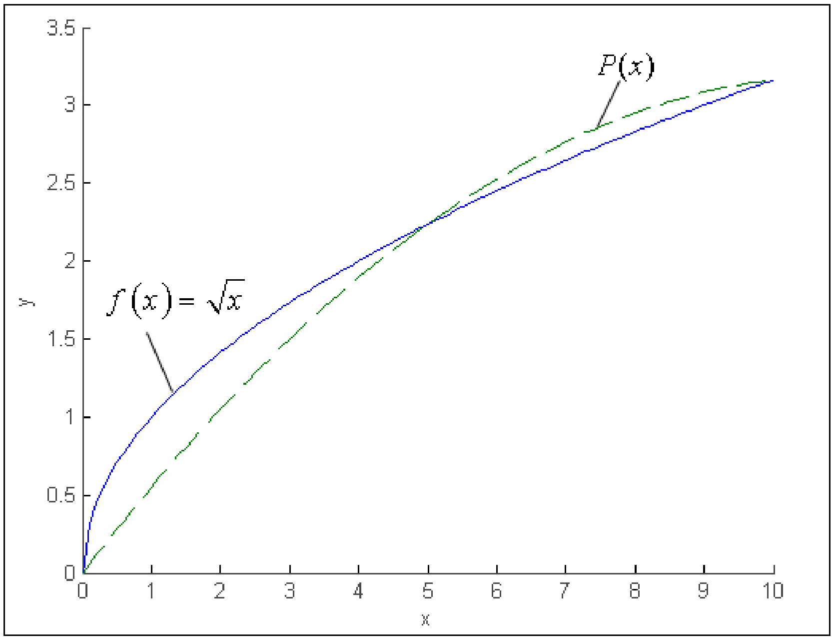
Substituting (7) into (4) and solving for b we get:

$$25\left(\frac{\sqrt{10} - 2\sqrt{5}}{50}\right) + 5b = \sqrt{5} \Rightarrow b = \frac{\sqrt{5}}{5} + \left(\frac{2\sqrt{5} - \sqrt{10}}{10}\right).$$

Thus our quadratic polynomial that fits the given data points is:

$$P(x) = \left(\frac{\sqrt{10} - 2\sqrt{5}}{50}\right)x^2 + \left(\frac{\sqrt{5}}{5} + \frac{2\sqrt{5} - \sqrt{10}}{10}\right)x$$

If we graph this polynomial onto of $f(x) = \sqrt{x}$ we get the following:



So you can visually see how accurate our approximation is. Note that the polynomial agrees exactly at the given data points $(0, 0), (5, \sqrt{5}), (10, \sqrt{10})$.

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3. Find the general form of a monomial that passes through the point (*x*, *y*).

A monomial is just a constant polynomial; that is, $P(x) = c$. And the graph of a monomial is a horizontal line at $y = c$. So the monomial that passes through the point (*x*, *y*) is $P(x) = y$.

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4. Find the general form of a binomial that passes through the points (x_0, y_0) and (x_1, y_1).

A binomial is of the form $P(x) = ax + c$, which is a line. So the problem wants us to find the line that passes through the two given points. You equation of a line is given by $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a fixed point on the line; *m* is the slope; and (x, y) is any point on the line. The slope of the line that passes through the given two points is:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Plugging that into our line equation yields:

$$y - y_1 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_1) \Leftrightarrow y = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)x - \left(\frac{y_1 - y_0}{x_1 - x_0}\right)x_1 + y_1 \Leftrightarrow y = mx + b,$$

where $b = -\left(\frac{y_1 - y_0}{x_1 - x_0}\right)x_1 + y_1$.

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5. In order to map the texture onto the polygon, as you display each pixel of the polygon you must find out which location in the bitmap corresponds to that pixel. Describe how you might use binomials to solve this problem.

Use linear interpolation (see pg., 16) to interpolate across the texture map.

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6. Suppose the angle between a surface and a light source is denoted by *x*, measured in degrees, from 0 to 180. Develop a quadratic function to model the intensity of the surface from 0 to 1 (assuming that at *x* = 90 degrees, the surface is at its brightest, and at *x* = 0 = 180 degrees, the surface is at its dimmest).

This problem is similar to problem #2. We are given the data points $(0, 0), (\pi/2, 1), (\pi, 0)$, and we want to fit a quadratic polynomial through them. So from our data we have the following conditions:

(8) $P(0) = 0 = a \cdot 0^2 + b \cdot 0 + c$

(9) $P(\pi/2) = 1 = a \cdot (\pi/2)^2 + b \cdot \pi/2 + 0$

(10) $P(\pi) = 0 = a \cdot \pi^2 + b \cdot \pi + 0$

Condition (8) tells us that $c = 0$. Equations (9) and (10) give a system of two equations and two unknowns. Solving (10) for a yields:

$$a \cdot \pi^2 + b \cdot \pi = 0 \Leftrightarrow a \cdot \pi^2 = -b \cdot \pi \Leftrightarrow a = -b/\pi \,.$$

Substituting a into equation (9) and solving for b gives:

$$a \cdot \left(\frac{\pi}{2}\right)^2 + b \cdot \frac{\pi}{2} = 1 \Leftrightarrow \frac{-b}{\pi} \cdot \frac{\pi^2}{4} + \frac{2b\pi}{4} = 1 \Leftrightarrow \frac{-b\pi}{4} + \frac{2b\pi}{4} = 1 \Leftrightarrow \frac{b\pi}{4} = 1 \Leftrightarrow b = \frac{4}{\pi} \,.$$

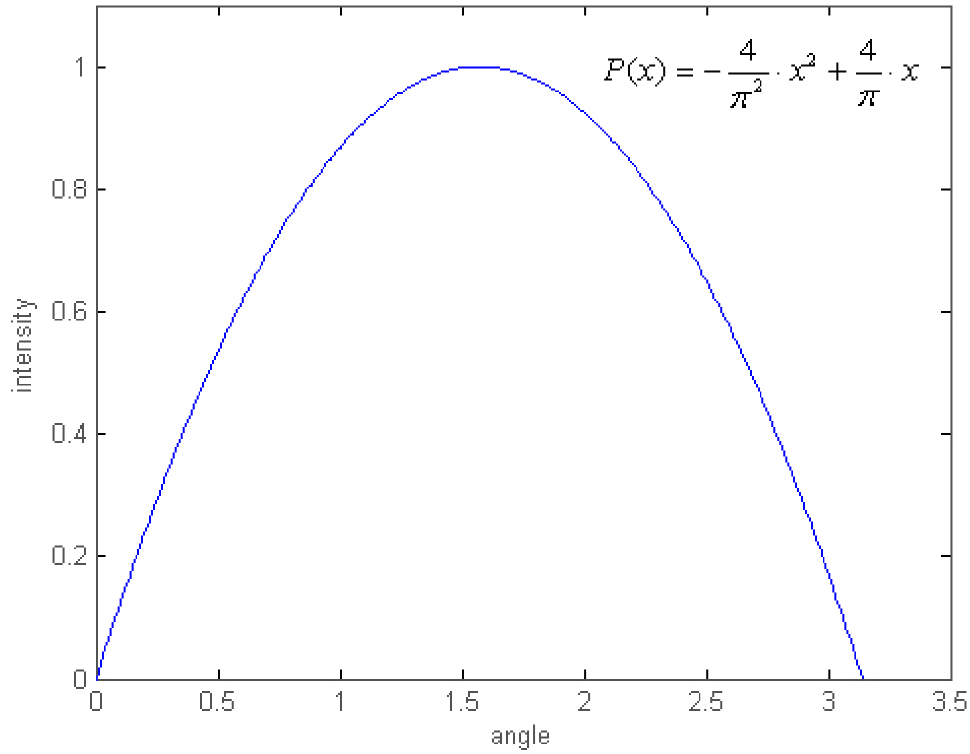
So our quadratic polynomial is,

$$P(x) = -\frac{4}{\pi^2} x^2 + \frac{4}{\pi} x \,.$$

To test our polynomial approximation, let us see if it agrees exactly with the given data points.

$$P(0) = -\frac{4}{\pi^2} 0^2 + \frac{4}{\pi} 0 = 0$$
$$P(\pi/2) = -\frac{4}{\pi^2} (\pi/2)^2 + \frac{4}{\pi} \pi/2 = -1 + 2 = 1$$
$$P(\pi) = -\frac{4}{\pi^2} \pi^2 + \frac{4}{\pi} \pi = -4 + 4 = 0$$

The graph of this polynomial is as follows:



Note that the graph of $P(x)$ agrees exactly at our condition points $(0, 0), (\pi/2, 1), (\pi, 0)$.

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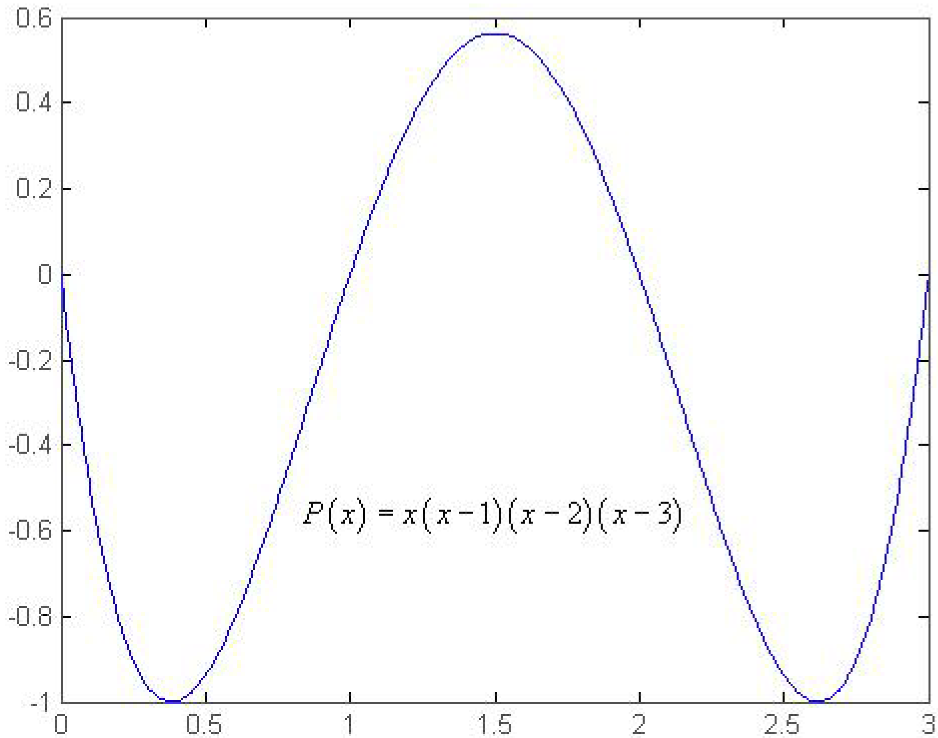
7. A product of n terms can only be zero if one of the terms is zero. Thus, the roots of the 4th degree polynomial $(x+2)(x+1)(x-1)(x-2)$ are at x = -2, -1, 1, and 2. Create a polynomial whose roots are at x = 0, 1, 2, and 3, and then scale the polynomial so that its maximum is 2.

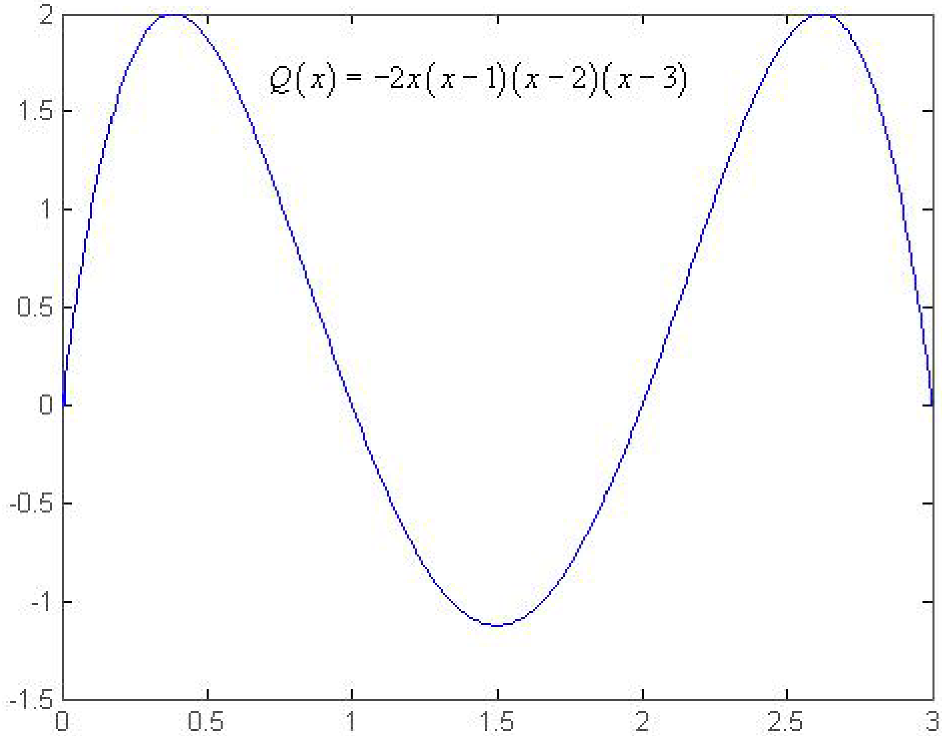
We start with the polynomial $P(x) = x(x-1)(x-2)(x-3)$, which has the desired roots. If we graph this polynomial we get the following.



We see that this polynomial has a minimum of –1. If we multiply the polynomial by 2 we will scale the curve so that it has a new minimum of –2. If we then multiply by –1, we will flip the curve about the x-axis so that the minimum of –2 becomes a maximum of 2. So, in total, we multiply the original polynomial $P(x)$ by –2 to produce a new polynomial $Q(x)$ with a maximum of two, and the same roots as $P(x)$.

$$Q(x) = -2P(x) = -2x(x-1)(x-2)(x-3)$$

Graphing $Q(x)$ gives us the following curve, which is the curve asked for.


$Q(x) = -2x(x-1)(x-2)(x-3)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3