Game Math Chapter 2 Solutions

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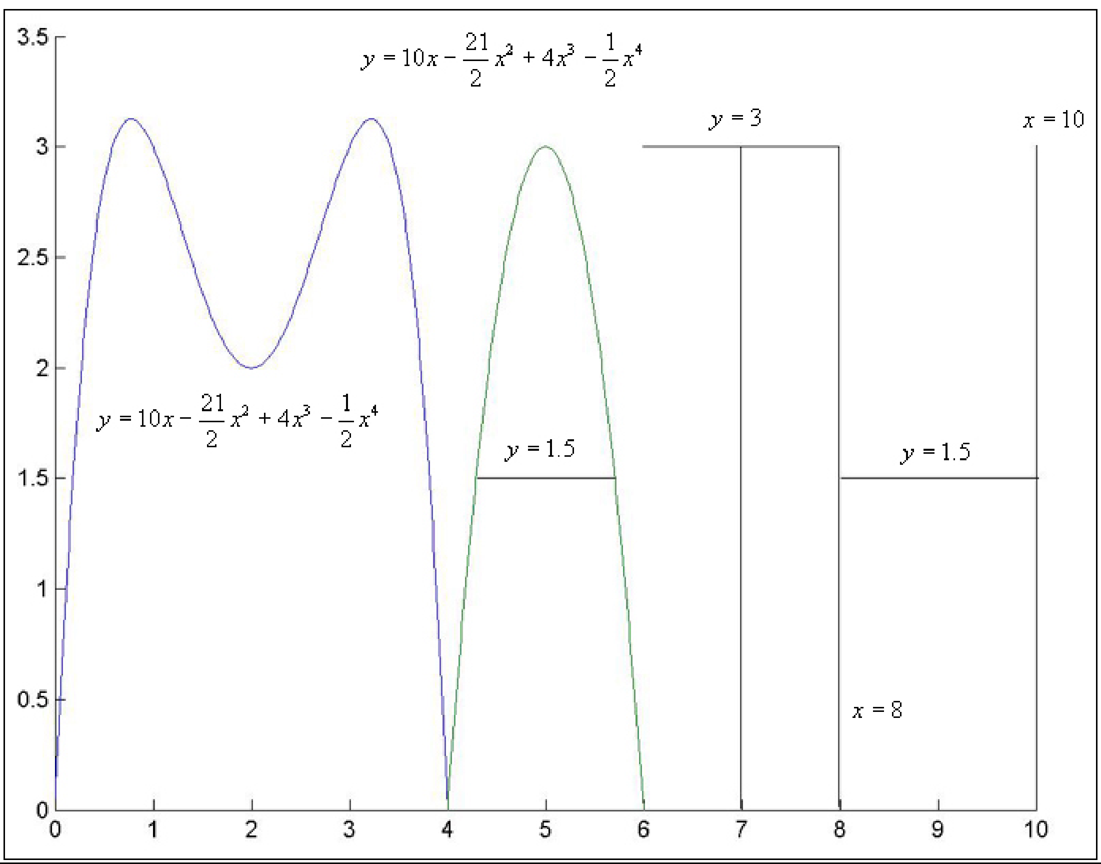
1. Plot the following graphs together:

$$\left\{(x, y) \mid 0 \le x \le 4,\ y = 10x - \frac{21}{2}x^2 + 4x^3 - \frac{1}{2}x^4\right\}$$
$$\left\{(x, y) \mid 4 \le x \le 6,\ y = -72 + 30x - 3x^2\right\}$$
$$\left\{(x, y) \mid 4.29289 \le x \le 5.70711,\ y = 1.5\right\}$$
$$\left\{(x, y) \mid 6 \le x \le 8,\ y = 3\right\}$$
$$\left\{(x, y) \mid 0 \le y \le 3,\ x = 7\right\}$$
$$\left\{(x, y) \mid 0 \le y \le 3,\ x = 8\right\}$$
$$\left\{(x, y) \mid 8 \le x \le 10,\ y = 1.5\right\}$$
$$\left\{(x, y) \mid 0 \le y \le 3,\ x = 10\right\}$$



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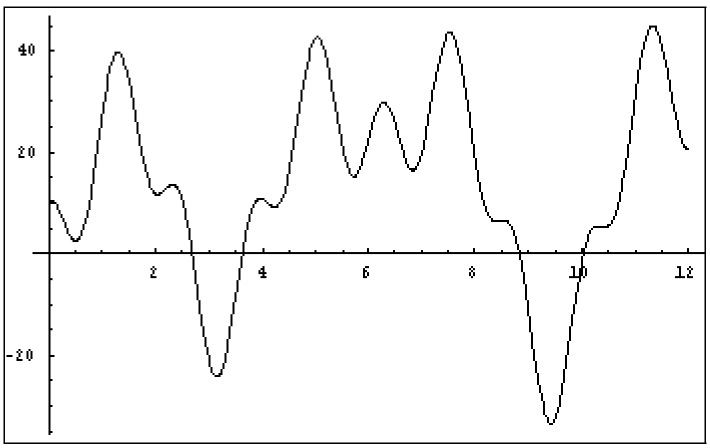
2. The following questions all refer to the graph of a certain function *f*, as shown.

   a. Approximately what is $f(5)$?
   b. Approximately what is $f(0)$?
   c. Is the function *f* invertible? Why or why not?
   d. If the function represented the temperature of an alien home world (the horizontal axis representing months, and the vertical axis representing degrees Celsius), about how many months would you say it takes for the alien planet to orbit its sun?



2a. 40

2b. 10

2c. No. Recall that for a function to be invertible it must be *one-to-one* and *onto*. A function is one-to-one if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. This function is not one-to-one since, from the graph, we can readily see that there exists a case where $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Thus the function is not invertible.

2d. Observe the two troughs that dip past –20 C°. These two troughs can be interpreted as corresponding to winter, thus the distance between these two troughs indicates a year (one complete orbit around the sun). It is approximately 6 months.

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3. Create an exponential function that models the population of rabbits on an island. Suppose the population begins at 10, and that after 12 months, there are 1000 rabbits.

How many months until the rabbit population exceeds one million? (Hint: Try using $f(x) = ae^{bx}$, where a and b are the unknown constants; you'll need the natural log function to solve for b.)

We want to fit a function of the form $f(x) = ae^{bx}$ through the points $(0, 10)$ and $(12, 1000)$.

Condition 1: $10 = ae^{0 \cdot b} \Leftrightarrow a = 10$
Condition 2: $1000 = 10e^{12b} \Leftrightarrow 100 = e^{12b} \Leftrightarrow \ln(100) = 12b \Leftrightarrow b = \ln(100)/12$

Thus the following gives us the exponential function we seek:

$$f(x) = 10\exp\left(x\frac{\ln(100)}{12}\right)^{1}$$

Now we want to know which x (month) gives us a population of $f(x) = 1000000$. To find this, we solve the following equation for x:

$$1000000 = 10\exp\left(x\frac{\ln(100)}{12}\right)$$

$$\Leftrightarrow$$

$$100000 = \exp\left(x\frac{\ln(100)}{12}\right)$$

$$\Leftrightarrow$$

$$\ln(100000) = x\frac{\ln(100)}{12}$$

$$\Leftrightarrow$$

$$x = \frac{12\ln(100000)}{\ln(100)} = 30$$

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4. Suppose the rabbit population in the preceding problem can't grow exponentially because of predators and limited resources. Try modeling the population using a log

[1] Note that *exp*(x) is another way to express e^x. We use this alternate notation because it gives us more room to write complex exponent expressions; using the exponent notation means we have to write very small.

function, making sure the new function satisfies the original constraints. (Hint: If you run into difficulty solving for the constants, try graphing the log function to see what could be causing the problem.)

If you graph a log function such as $f(x) = a\ln(bx)$ you will see that it approaches infinity as x goes to zero. So with this behavior we will have trouble fitting the curve to the data point $(0, 10)$. To get around this problem we will add an offset of 2 to the natural logarithm argument: $f(x) = a\ln(bx+2)$. Thus, as x approaches to 0, $f(x)$ approaches $a\ln(2)$, and we no longer have the problem of approaching infinity.

So we want to fit a function of the form $f(x) = a\ln(bx+2)$ through the points $(0, 10)$ and $(12, 1000)$.

Condition 1: $10 = a\ln(0b+2) \Leftrightarrow a = \dfrac{10}{\ln(2)} \approx 14.4269$

Condition 2:

$$1000 = \frac{10}{\ln(2)}\ln(12b+2)$$

$$\Leftrightarrow$$

$$100 = \frac{\ln(12b+2)}{\ln(2)}$$

$$\Leftrightarrow$$

$$\exp(100\ln(2)) = 12b+2$$

$$\Leftrightarrow$$

$$b = \frac{\exp(100\ln(2)) - 2}{12} \approx 1.05637\times10^{29}$$

Thus the following gives us the logarithmic function we seek:

$$f(x) = 14.4269\ln\left(1.05637\times10^{29}x+2\right)$$

We can test it to verify that it agrees at the points $(0, 10)$ and $(12, 1000)$:

$$f(0) = 14.4269\ln\left(1.05637\times10^{29}\cdot 0+2\right) \approx 10$$
$$f(12) = 14.4269\ln\left(1.05637\times10^{29}\cdot 12+2\right) \approx 1000$$

5. Say you're building a space-based resource management game and you want the cost of research and development to increase exponentially. Suppose that having spent nothing on research, you want it to cost 10 credits to advance to the next level of development, but that after you have spent 10,000 credits, you want it to cost twice that much to advance to the next level. Develop a function to model this behavior.

This problem is similar to problem 3. We want to fit a function of the form $f(x) = ae^{bx}$ through the points $(0, 10)$ and $(10000, 20000)$.

Condition 1: $10 = a\exp(0b) \Leftrightarrow a = 10$
Condition 2:

$$20000 = 10\exp(10000b)$$
$$\Leftrightarrow$$
$$2000 = \exp(10000b)$$
$$\Leftrightarrow$$
$$\ln(2000) = 10000b$$
$$\Leftrightarrow$$
$$b = \frac{\ln(2000)}{10000} \approx 0.00076$$

Thus the following gives us the exponential function we seek:

$$f(x) = 10\exp(0.00076x)$$

We can test it to verify that it agrees at the points $(0, 10)$ and $(10000, 20000)$:

$$f(0) = 10\exp(0.00076 \cdot 0) = 10$$
$$f(10000) = 10\exp(0.00076 \cdot 10000) \approx 20000$$

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6. Prove that you cannot force an exponential function to pass through just any three points. (Hint: Find three points that no exponential function can pass through.)

Intuitively, it is not hard to see this. Consider three points on an arbitrary line. If we superimpose an arbitrary exponential function we see that at most the curves can intersect at two points (see figure). Thus no exponential function can pass through three points on a line. Moreover, an exponential cannot even be made to fit two arbitrary points. Consider the points (u, v) and (w, v), where $u \neq w$.

Condition 1: $v = ae^{bu}$
Condition 2: $v = ae^{bw}$

So $ae^{bu} = ae^{bw} \Leftrightarrow e^{bu} = e^{bw} \Leftrightarrow bu = bw \Leftrightarrow u = w$, but that contradicts the fact that $u \neq w$. Thus an exponential cannot fit these two points, and if it can't fit these two points, it won't be able to fit three points if we added a third.