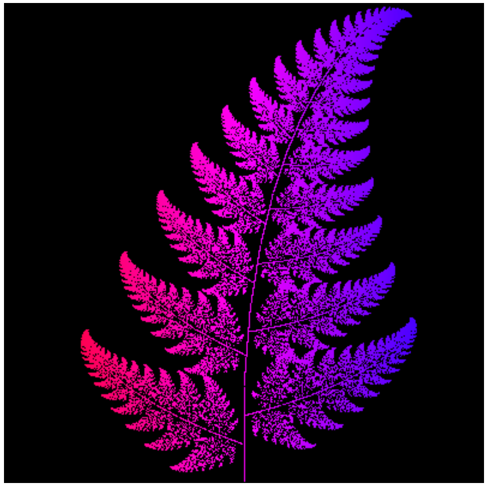
Transformations

Outline

- Assignment 0 Recap
- Intro to Transformations
- Classes of Transformations
- Representing Transformations
- Combining Transformations
- Change of Orthonormal Basis

Cool Results from Assignment 0



emmav

(Courtesy of Emily Vincent.
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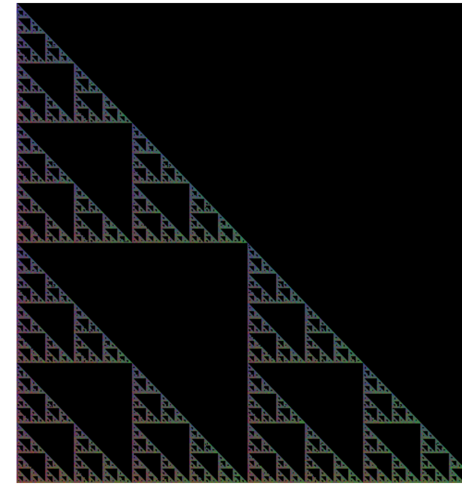

psoto

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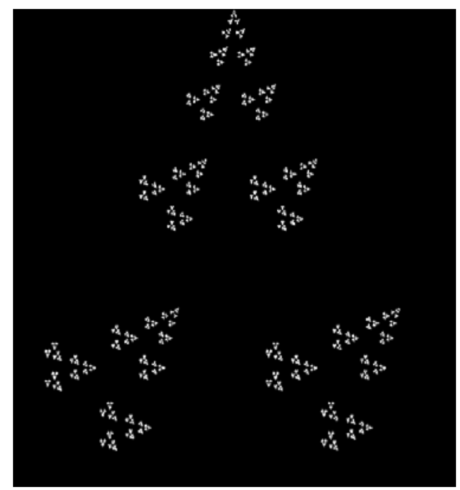

scyudits

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Notes on Assignments

- Make sure you turn in a *linux* or *windows* executable (so we can test your program)
- Collaboration Policy
  - Share ideas, not code
- Tell us how much time you spent on each assignment

Quick Review of Last Week

- Ray representation
- Generating rays from eyepoint / camera
  - orthographic camera
  - perspective camera
- Find intersection point & surface normal
- Primitives:
  - spheres, planes, polygons, triangles, boxes

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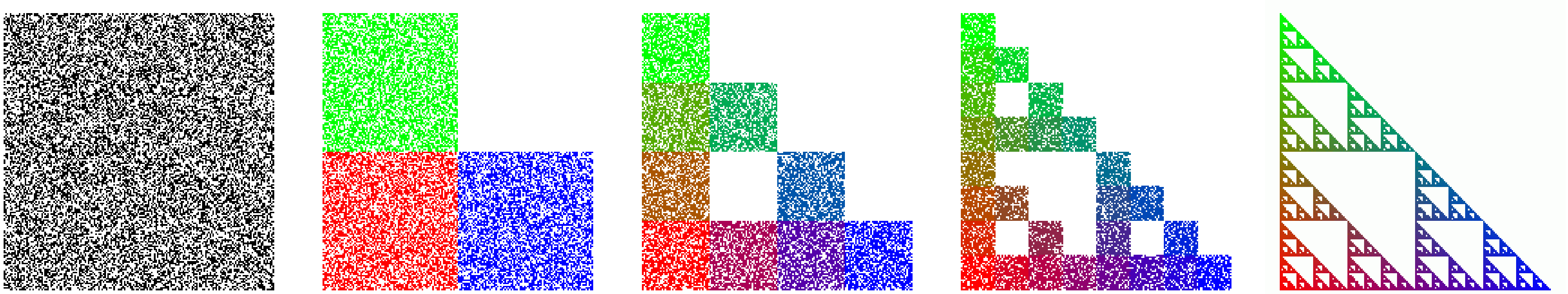
What is a Transformation?

- Maps points (x, y) in one coordinate system to points (x', y') in another coordinate system

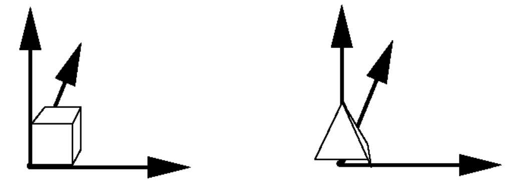
$$x' = ax + by + c$$
$$y' = dx + ey + f$$

- For example, IFS:

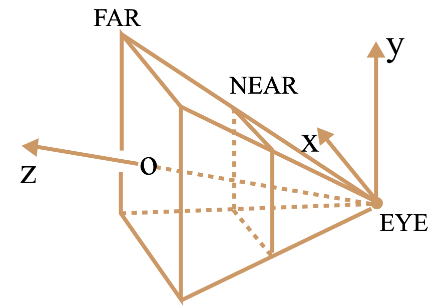


Common Coordinate Systems

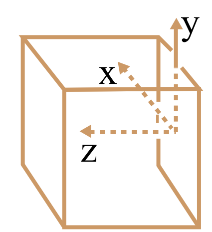
- Object space
  - local to each object



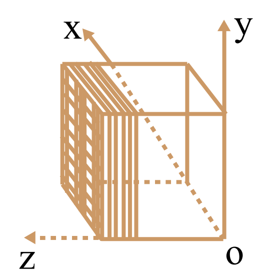
- World space
  - common to all objects



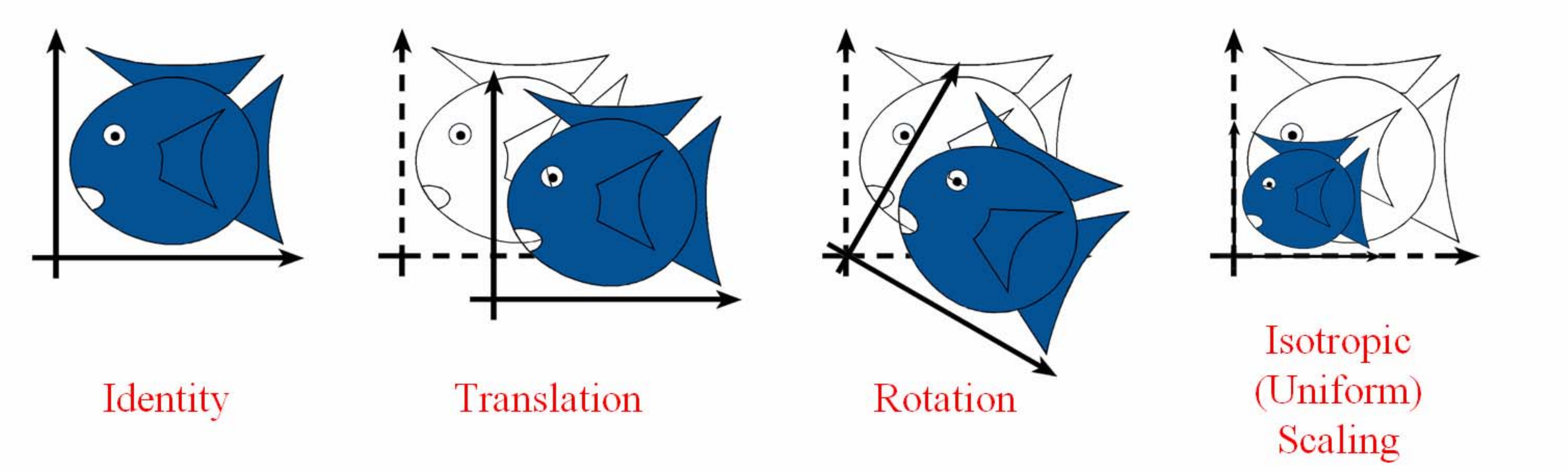
- Eye space / Camera space
  - derived from view frustum



- Screen space
  - indexed according to hardware attributes



Simple Transformations



- Can be combined
- Are these operations invertible?

  Yes, except scale = 0

Transformations are used:

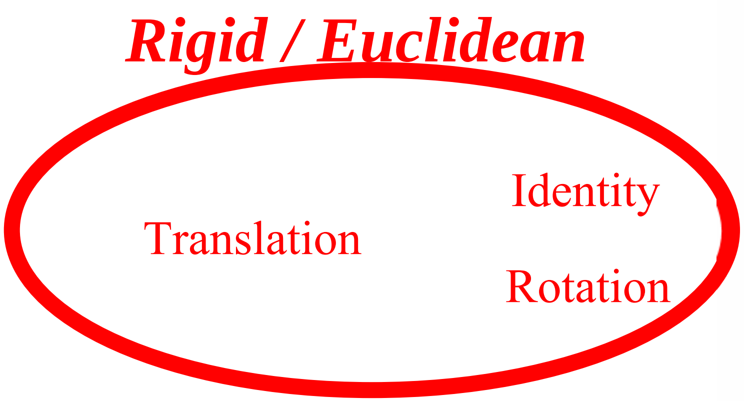
- Position objects in a scene (modeling)
- Change the shape of objects
- Create multiple copies of objects
- Projection for virtual cameras
- Animations

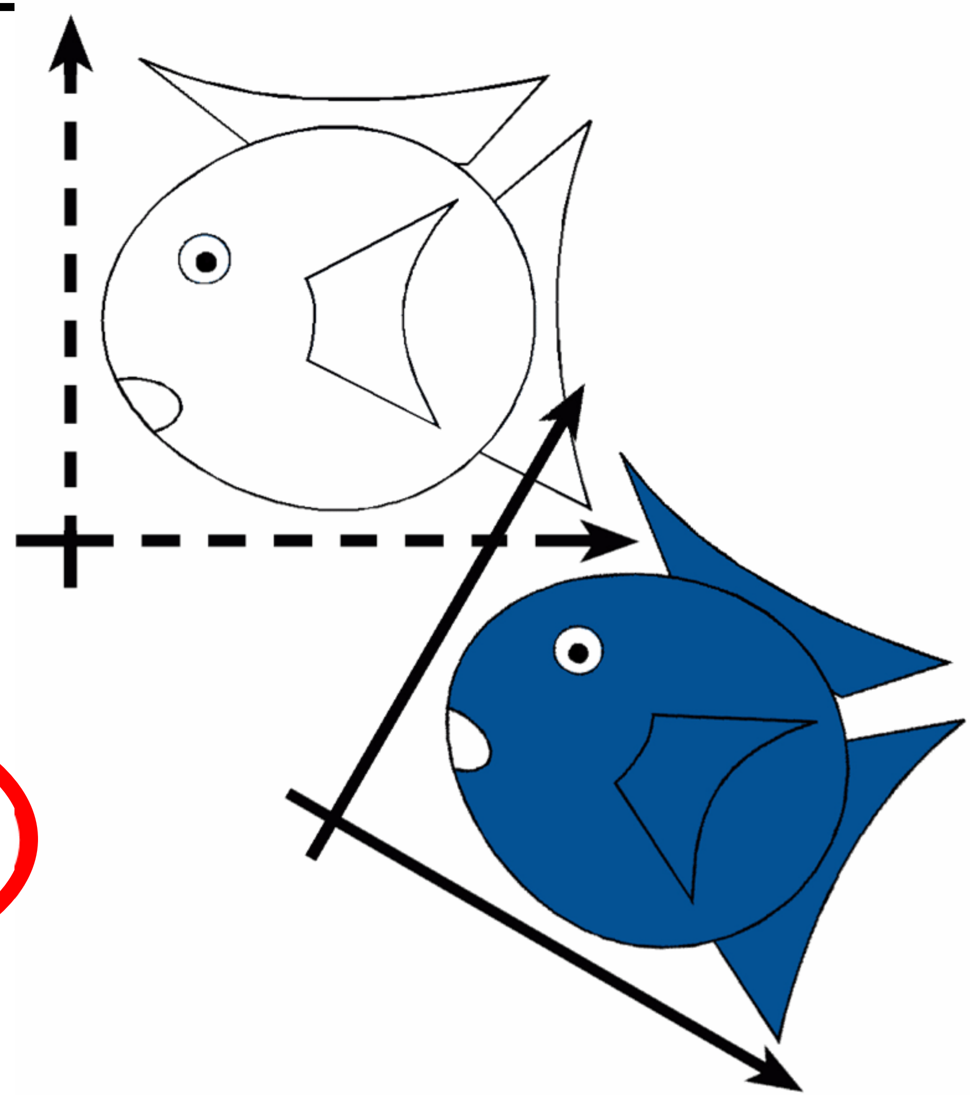
Outline

- Assignment 0 Recap
- Intro to Transformations
- Classes of Transformations
    - Rigid Body / Euclidean Transforms
    - Similitudes / Similarity Transforms
    - Linear
    - Affine
    - Projective
- Representing Transformations
- Combining Transformations
- Change of Orthonormal Basis

Rigid-Body / Euclidean Transforms

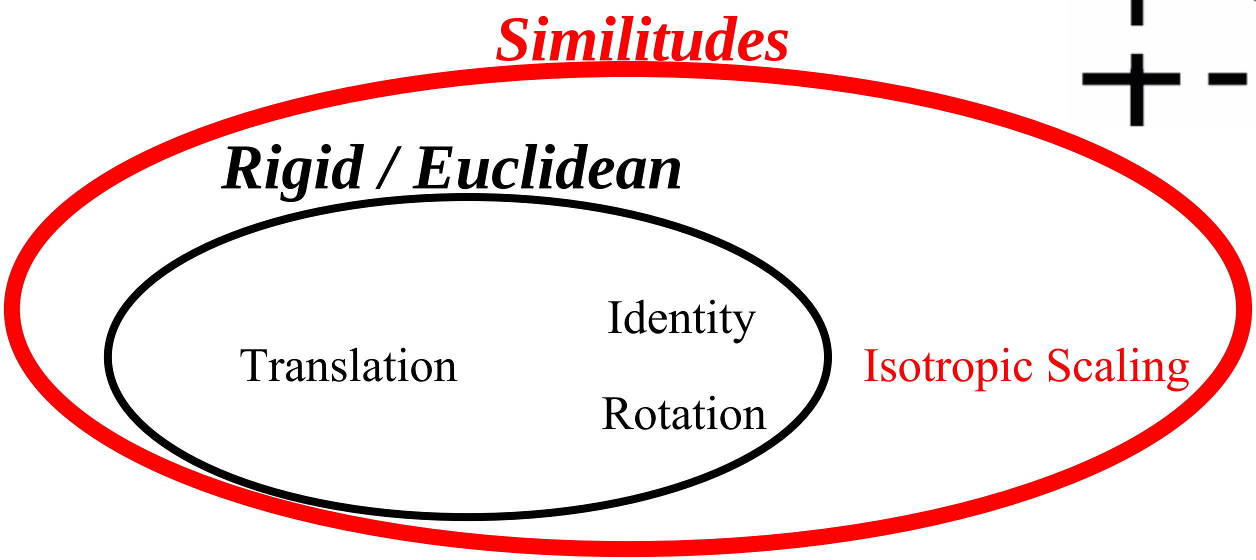
- Preserves distances
- Preserves angles

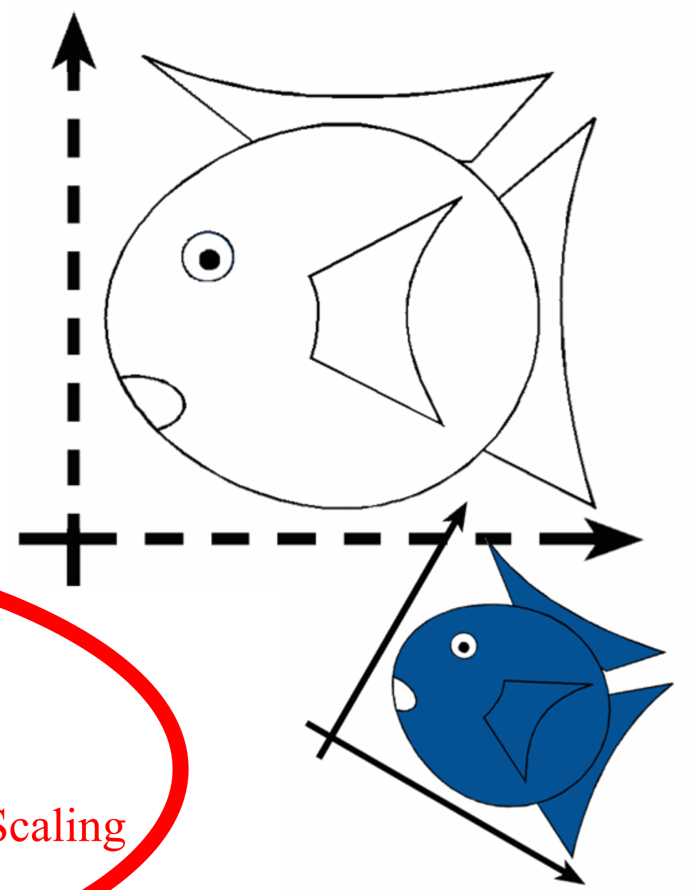




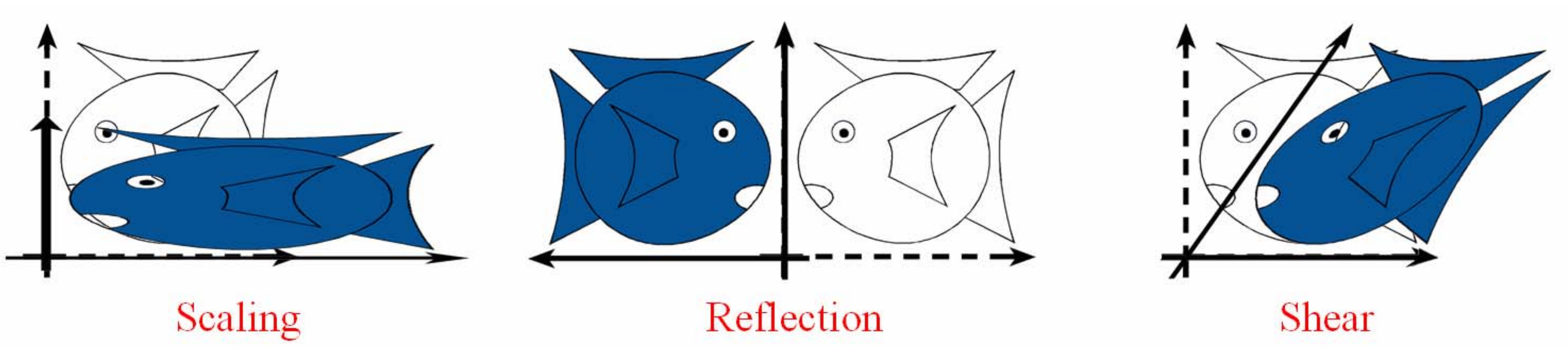
Similitudes / Similarity Transforms

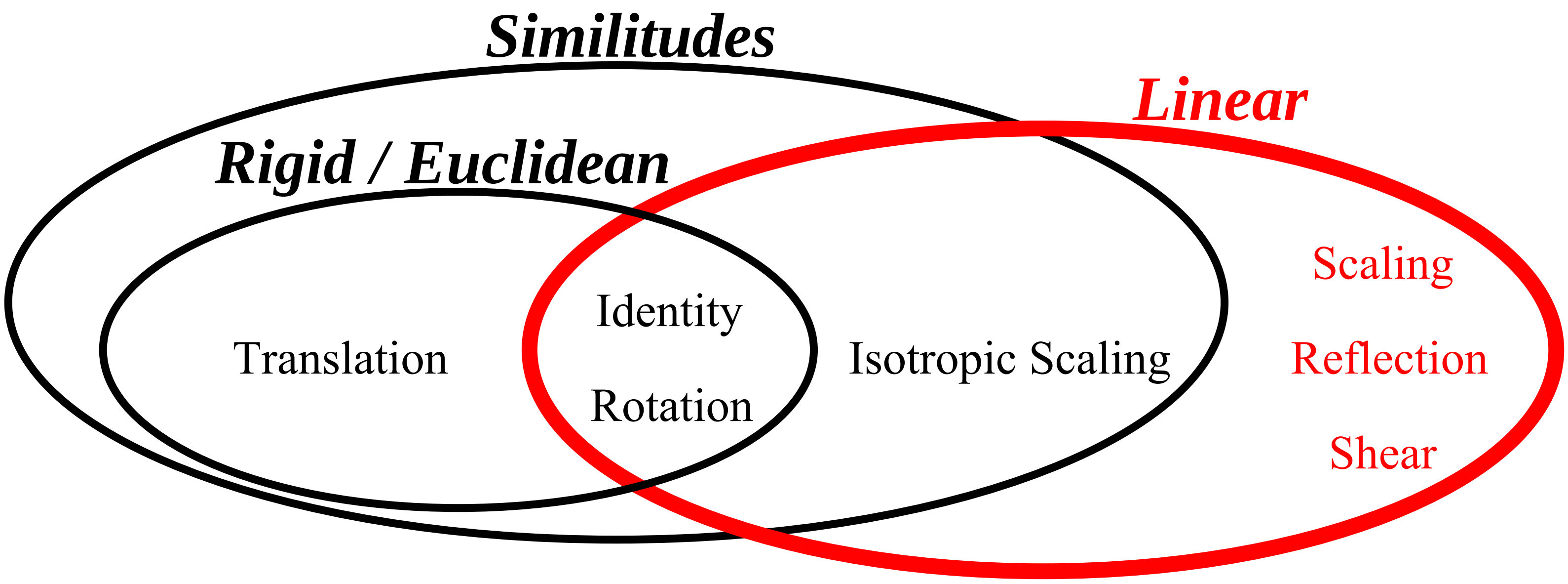
- Preserves angles





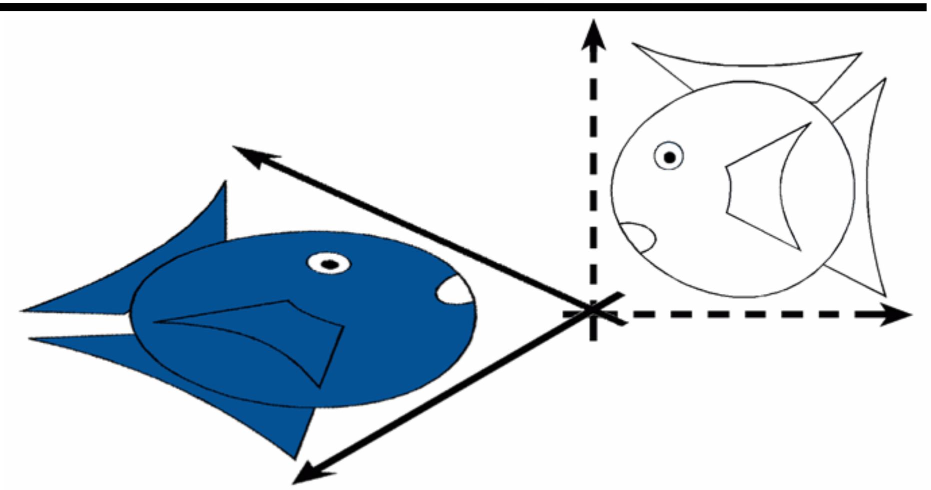
Linear Transformations

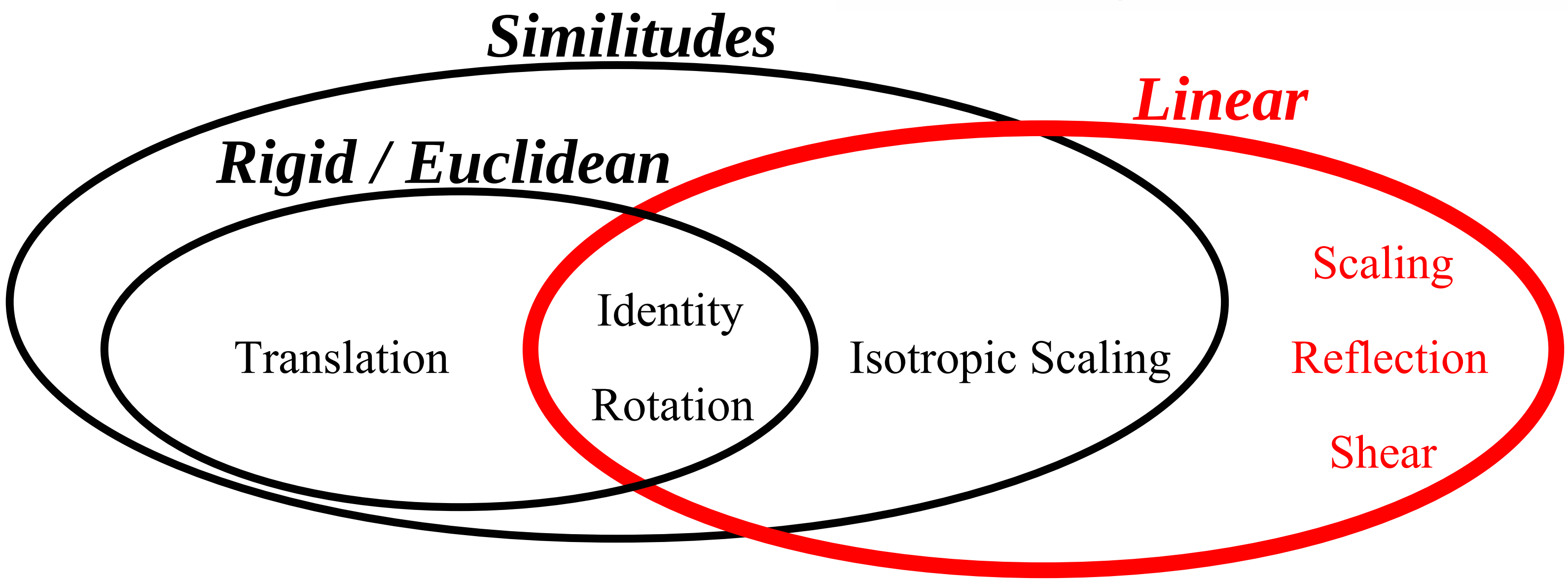




Linear Transformations

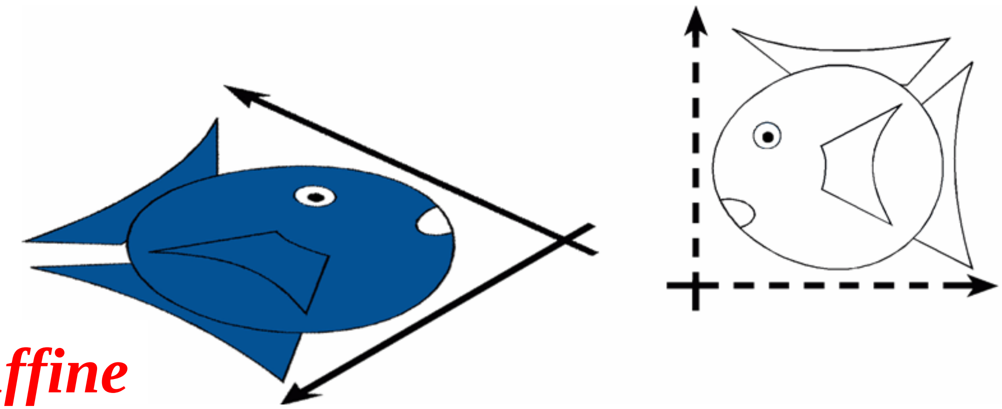
- $L(p + q) = L(p) + L(q)$
- $L(ap) = a\,L(p)$

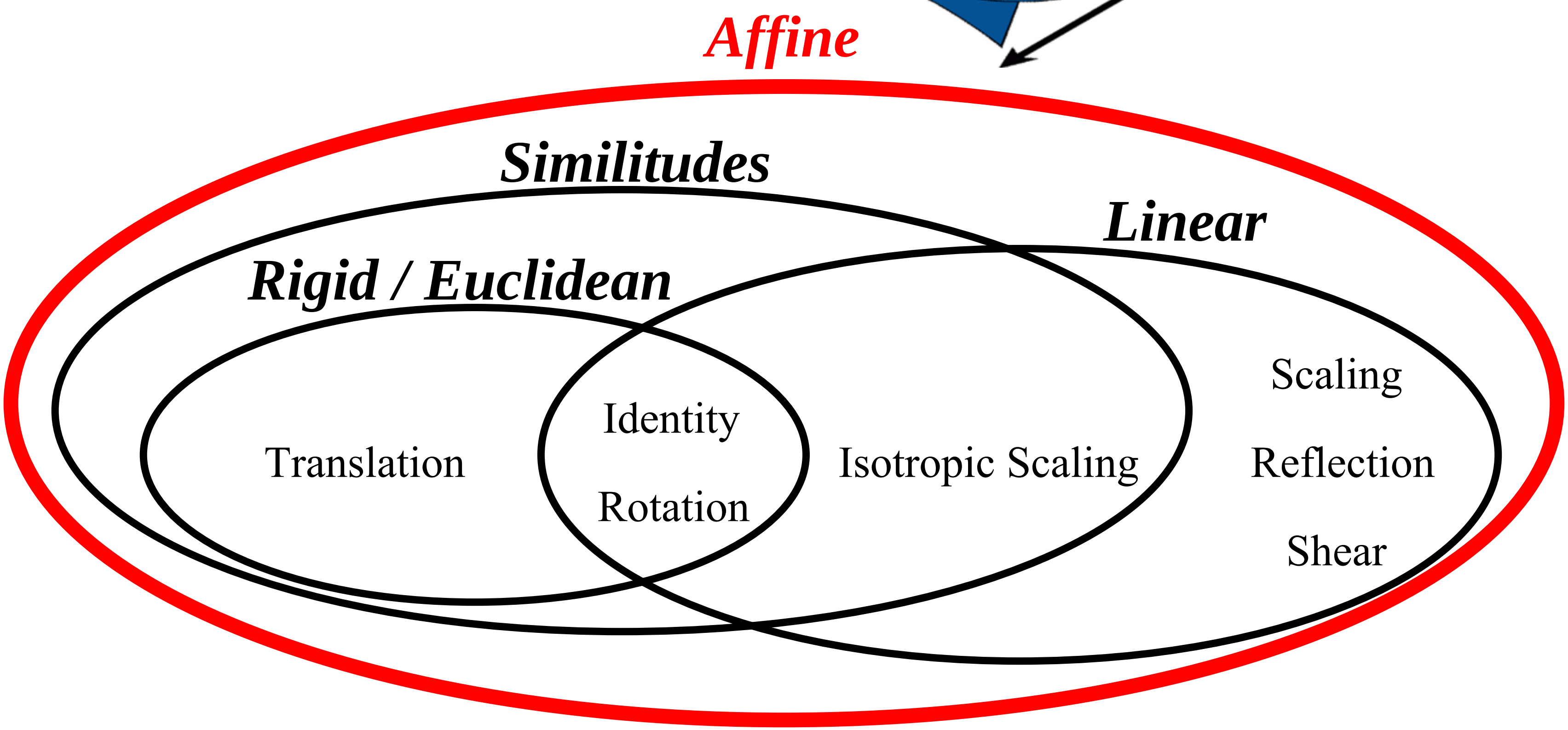




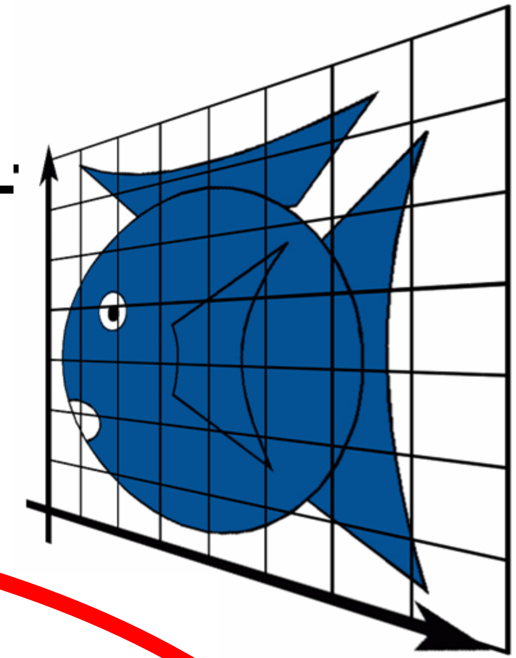
Affine Transformations

- preserves parallel lines

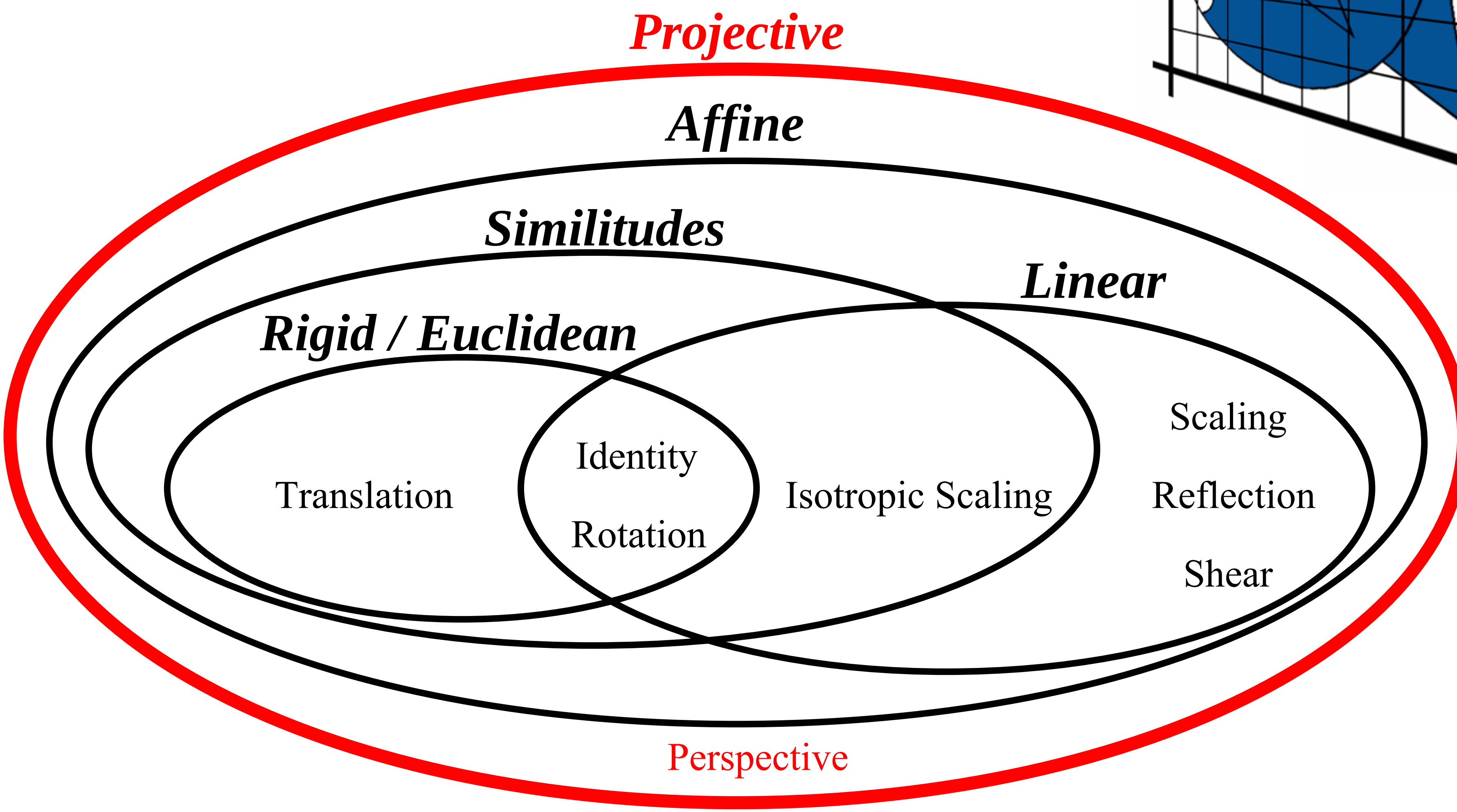




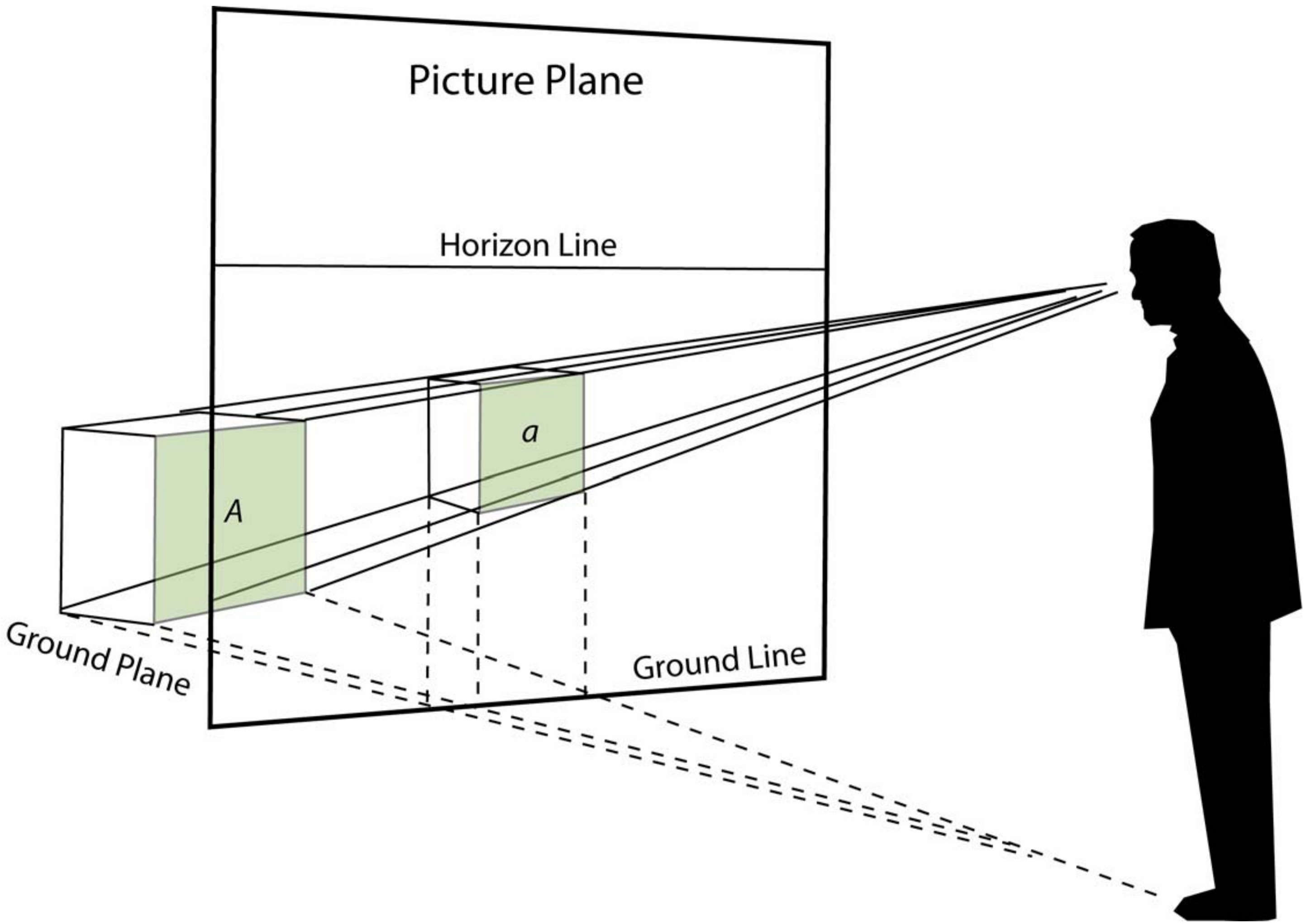
Projective Transformations



- preserves lines



Perspective Projection



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How are Transforms Represented?

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix}$$

$$p' = Mp + t$$

Homogeneous Coordinates

- Add an extra dimension
  - in 2D, we use 3 x 3 matrices
  - In 3D, we use 4 x 4 matrices
- Each point has an extra value, *w*

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$p' = M p$$

Homogeneous Coordinates

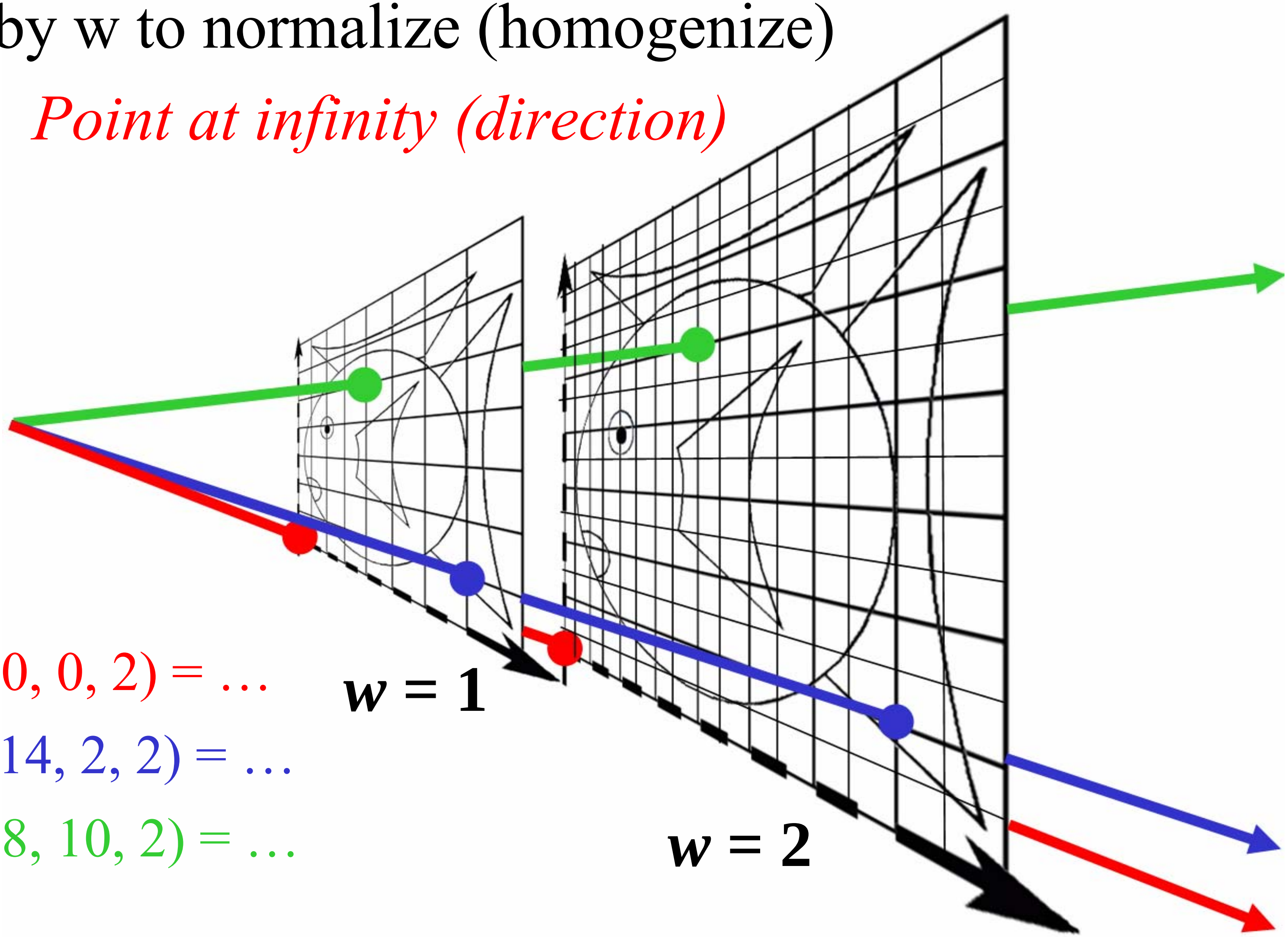
- Most of the time w = 1, and we can ignore it

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- If we multiply a homogeneous coordinate by an *affine matrix*, w is unchanged

Homogeneous Visualization

- Divide by w to normalize (homogenize)
- W = 0? *Point at infinity (direction)*



(0, 0, 1) = (0, 0, 2) = …

(7, 1, 1) = (14, 2, 2) = …

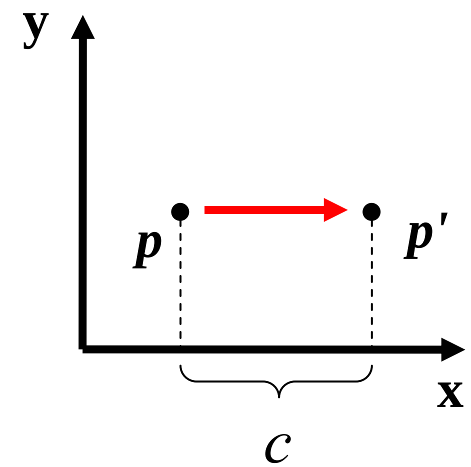
(4, 5, 1) = (8, 10, 2) = …

Translate (t_x, t_y, t_z)

- Why bother with the extra dimension?

  Because now translations can be encoded in the matrix!

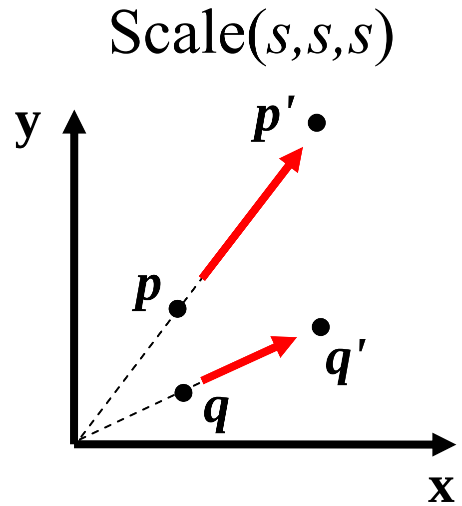
Translate($c,0,0$)



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \color{red}{t_x} \\ 0 & 1 & 0 & \color{red}{t_y} \\ 0 & 0 & 1 & \color{red}{t_z} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Scale (s_x, s_y, s_z)

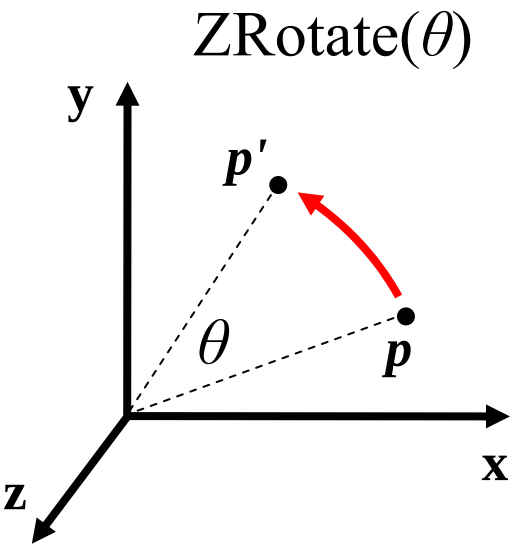
- Isotropic (uniform) scaling: $s_x = s_y = s_z$



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

- About z axis



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

- About x axis:

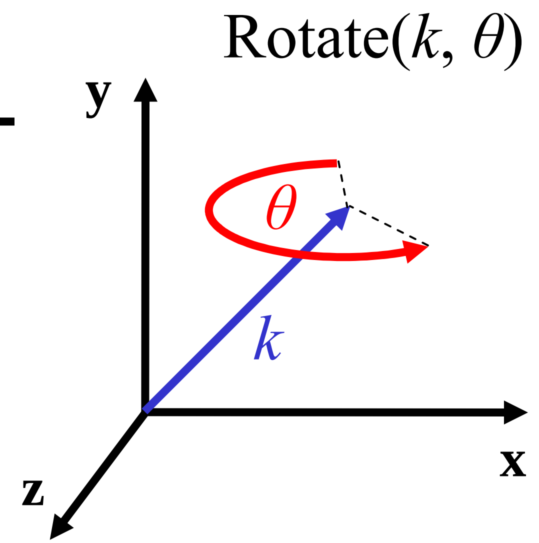
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- About y axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 1 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

- About (k_x, k_y, k_z), a unit vector on an arbitrary axis (Rodrigues Formula)



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} k_xk_x(1-c)+c & k_zk_x(1-c)-k_zs & k_xk_z(1-c)+k_ys & 0 \\ k_yk_x(1-c)+k_zs & k_zk_x(1-c)+c & k_yk_z(1-c)-k_xs & 0 \\ k_zk_x(1-c)-k_ys & k_zk_x(1-c)-k_xs & k_zk_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where $c = \cos\theta$ & $s = \sin\theta$

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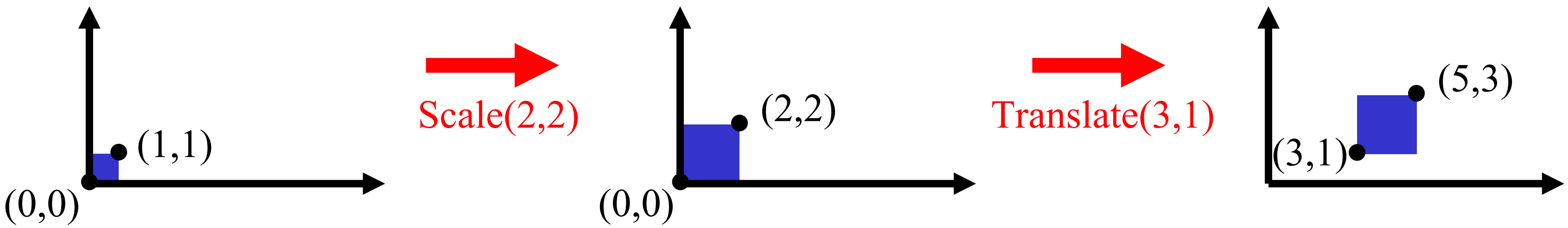
How are transforms combined?

Scale then Translate



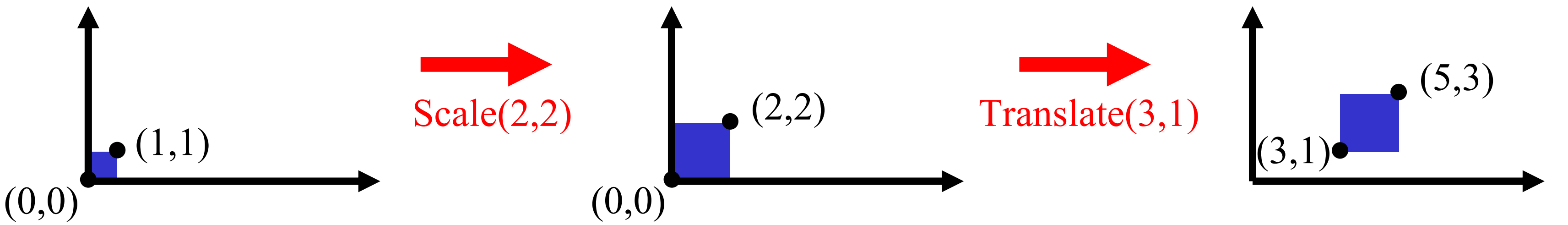
Use matrix multiplication: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

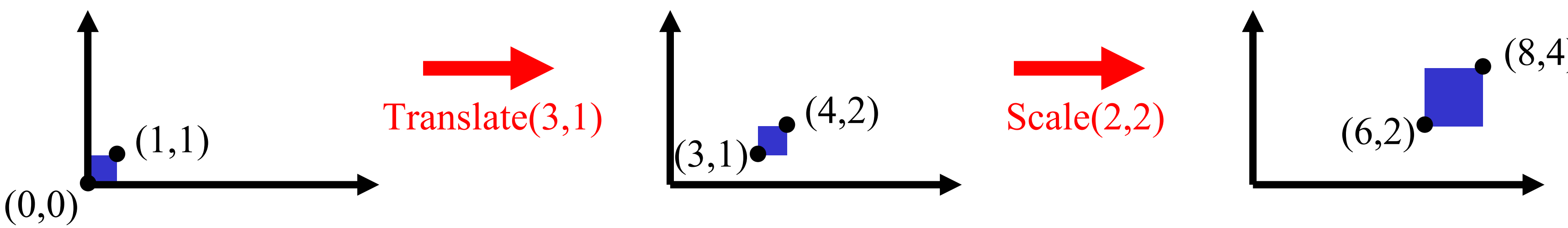
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$



Translate then Scale: $p' = S(T\,p) = ST\,p$



Non-commutative Composition

Scale then Translate: p' = T (S p) = TS p

$$TS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate then Scale: p' = S (T p) = ST p

$$ST = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

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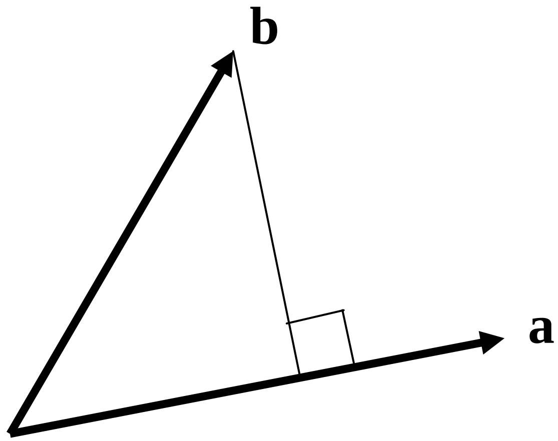
Review of Dot Product



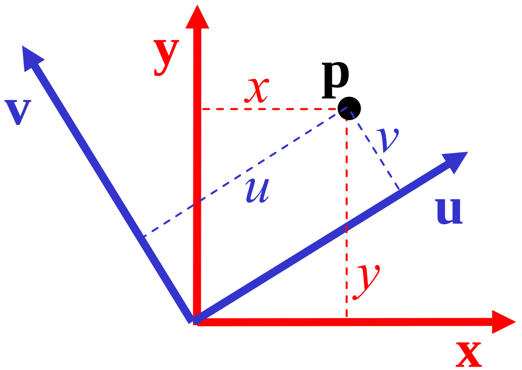
Change of Orthonormal Basis

- Given:

  coordinate frames

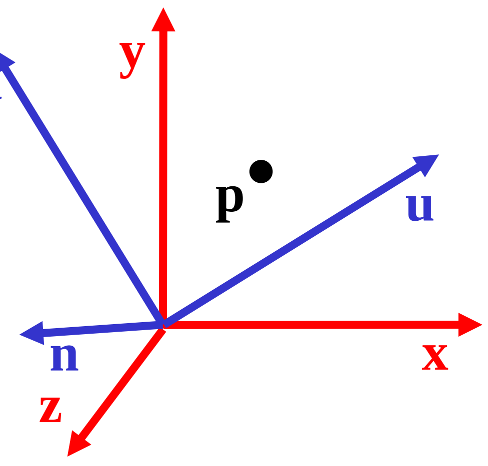
  xyz and **uvn**

  point $\mathbf{p} = (x,y,z)$

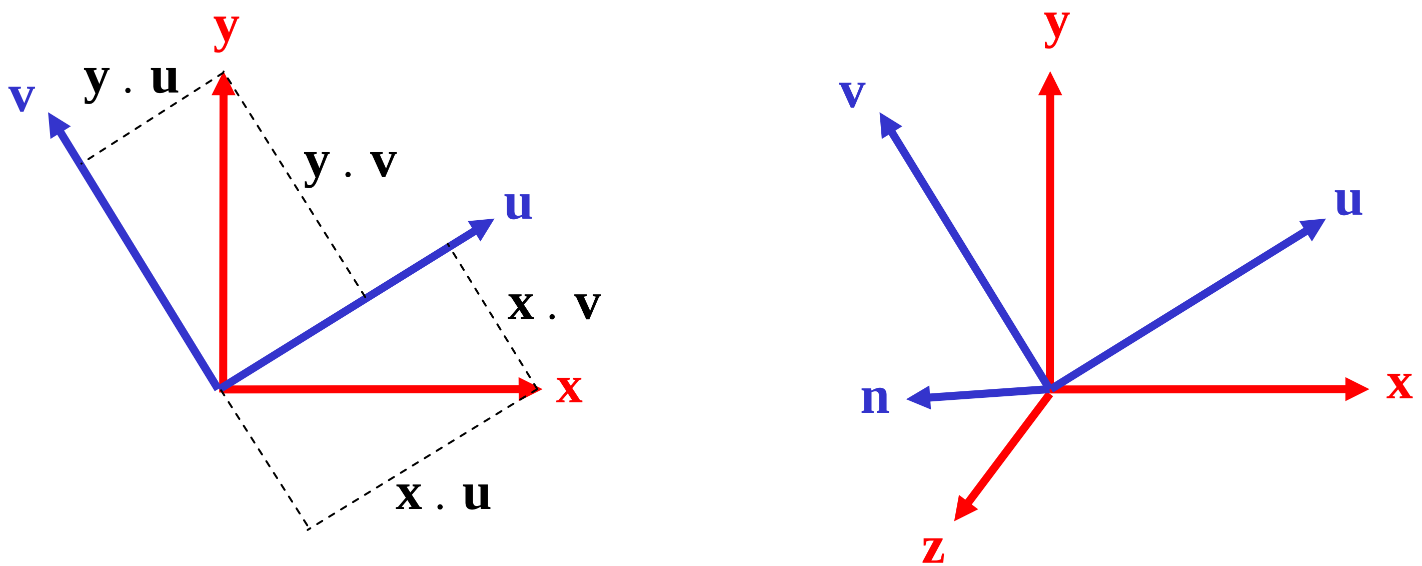


- Find:

  $\mathbf{p} = (u,v,n)$



Change of Orthonormal Basis



$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{z} = (\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}$$

Change of Orthonormal Basis

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{z} = (\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}$$

Substitute into equation for *p*:

$$\mathbf{p} = (x,y,z) = x\,\mathbf{x} + y\,\mathbf{y} + z\,\mathbf{z}$$

$$\begin{aligned}\mathbf{p} = \; & x\,[\,(\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}\,] + \\ & y\,[\,(\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}\,] + \\ & z\,[\,(\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}\,]\end{aligned}$$

Change of Orthonormal Basis

$$\mathbf{p} = x\,[\,(\mathbf{x}\,.\,\mathbf{u})\,\mathbf{u} + (\mathbf{x}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{x}\,.\,\mathbf{n})\,\mathbf{n}\,] +$$
$$y\,[\,(\mathbf{y}\,.\,\mathbf{u})\,\mathbf{u} + (\mathbf{y}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{y}\,.\,\mathbf{n})\,\mathbf{n}\,] +$$
$$z\,[\,(\mathbf{z}\,.\,\mathbf{u})\,\mathbf{u} + (\mathbf{z}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{z}\,.\,\mathbf{n})\,\mathbf{n}\,]$$

Rewrite:

$$\mathbf{p} = [\,x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u})\,]\,\mathbf{u} +$$
$$[\,x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v})\,]\,\mathbf{v} +$$
$$[\,x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n})\,]\,\mathbf{n}$$

Change of Orthonormal Basis

$$\mathbf{p} = [\; x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u}) \;]\,\mathbf{u} \;+$$
$$[\; x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v}) \;]\,\mathbf{v} \;+$$
$$[\; x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n}) \;]\,\mathbf{n}$$

$$\mathbf{p} = (u,v,n) = u\,\mathbf{u} + v\,\mathbf{v} + n\,\mathbf{n}$$

Expressed in **uvn** basis:

$$u = x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u})$$
$$v = x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v})$$
$$n = x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n})$$

Change of Orthonormal Basis

$$u = x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u})$$
$$v = x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v})$$
$$n = x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n})$$

In matrix form:

$$\begin{pmatrix} u \\ v \\ n \end{pmatrix} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where:

$u_x = \mathbf{x}\,.\,\mathbf{u}$

$u_y = \mathbf{y}\,.\,\mathbf{u}$

etc.

Change of Orthonormal Basis

$$\begin{bmatrix} u \\ v \\ n \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What's $\mathbf{M}^{-1}$, the inverse?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_n \\ y_u & y_v & y_n \\ z_u & z_v & z_n \end{bmatrix} \begin{bmatrix} u \\ v \\ n \end{bmatrix}$$

$$u_x = \mathbf{x} \,.\, \mathbf{u} = \mathbf{u} \,.\, \mathbf{x} = x_u$$

$$\mathbf{M}^{-1} = \mathbf{M}^T$$

Next Time:

Adding Transformations
to the Ray Caster
(Assignment 2)