Rotation about an Arbitrary Axis

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1 Eye Position

The standard way to compute a rotation about an arbitrary axis through the origin is to concatenate five rotation matrices. This note shows how vector algebra makes it easy to rotate about an arbitrary axis in a single step.

Figure 1 shows a point $\mathbf{P}$ which we want to rotate an angle θ about an axis that passes through $\mathbf{B}$ with a direction defined by unit vector $\mathbf{n}$. So, given the angle θ, the unit vector $\mathbf{n}$, and Cartesian coordinates for the points $\mathbf{P}$, $\mathbf{B}$, we want to find Cartesian coordinates for the point $\mathbf{P}'$.

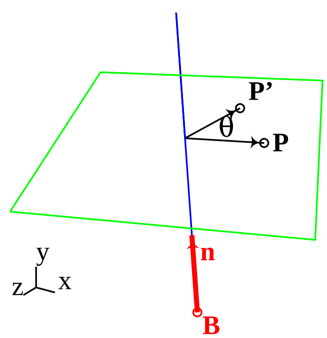

Figure 1: Rotation about an Arbitrary Axis

The key insight needed is shown in Figure 2. Let $\mathbf{u}$ and $\mathbf{v}$ be any two three-dimensional

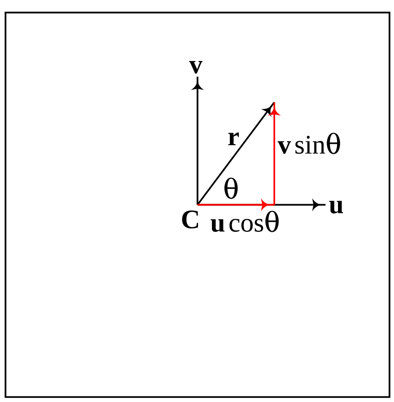

Figure 2: Key Insight

vectors that satisfy $u \cdot v = 0$ (that is, they are perpendicular) and $|\mathbf{u}| = |\mathbf{v}| \neq 0$ (that is, they are they same length but not necessarily unit vectors). We want to find a vector $\mathbf{r}$ that is obtained by rotating $\mathbf{u}$ an angle θ in the plane defined by $\mathbf{u}$ and $\mathbf{v}$. As suggested in Figure 2,

$$\mathbf{r} = \mathbf{u}\cos\theta + \mathbf{v}\sin\theta. \tag{1}$$

With that insight, it is easy to compute a rotation about an arbitrary axis. Referring

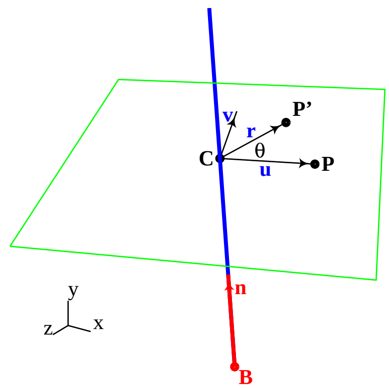

Figure 3: Rotation about an Arbitrary Axis

to Figure 3, we compute

$$\mathbf{C} = \mathbf{B} + [(\mathbf{P} - \mathbf{B}) \cdot \mathbf{n}]\mathbf{n}. \tag{2}$$

$$\mathbf{u} = \mathbf{P} - \mathbf{C} \tag{3}$$

$$\mathbf{v} = \mathbf{n} \times \mathbf{u} \tag{4}$$

Then, $\mathbf{r}$ is computed using equation (1), and

$$\mathbf{P}' = \mathbf{C} + \mathbf{r}. \tag{5}$$

It is possible to take these simple vector equations and to create from them a single 4×4 transformation matrix for rotation about an arbitrary axis. Let $\mathbf{P} = (x, y, z)$, $\mathbf{P}' = (x', y', z')$, $\mathbf{B} = (B_x, B_y, B_z)$, and $\mathbf{n} = (n_x.n_y, n_z)$. We seek a 4×4 matrix M such that

$$M \left\{ \begin{array}{c} x \\ y \\ z \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} x' \\ y' \\ z' \\ 1 \end{array} \right\}$$

$$(C_x, C_y, C_z) = (B_x, B_y, B_z) + [xn_x + yn_y + zn_z - \mathbf{B} \cdot \mathbf{n}](n_x, n_y, n_z) \tag{6}$$

$$\begin{aligned} C_x &= xn_x^2 + yn_xn_y + zn_xn_z + B_x - (\mathbf{B} \cdot \mathbf{n})n_x & (7) \\ C_y &= xn_xn_y + yn_y^2 + zn_yn_z + B_y - (\mathbf{B} \cdot \mathbf{n})n_y & (8) \\ C_z &= xn_xn_z + yn_yn_z + zn_z^2 + B_z - (\mathbf{B} \cdot \mathbf{n})n_z & (9) \end{aligned}$$

$$\mathbf{u} = (x, y, z) - (C_x, C_y, C_z) \tag{10}$$

$$
\begin{aligned}
u_x &= x(1 - n_x^2) - yn_xn_y - zn_xn_z + (\mathbf{B}\cdot\mathbf{n})n_x - B_x & (11)\\
u_y &= -xn_xn_y + y(1 - n_y^2) - zn_yn_z + (\mathbf{B}\cdot\mathbf{n})n_y - B_y & (12)\\
u_z &= -xn_xn_z - yn_yn_z + (1 - n_z^2) + (\mathbf{B}\cdot\mathbf{n})n_z - B_z & (13)
\end{aligned}
$$

$$
\begin{aligned}
v_x &= n_yu_z - n_zu_y & (14)\\
v_y &= n_zu_x - n_xu_z & (15)\\
v_z &= n_xu_y - n_yu_x & (16)
\end{aligned}
$$

$$
\begin{aligned}
r_x &= u_x\cos\theta + (n_yu_z - n_zu_y)\sin\theta & (17)\\
r_y &= u_y\cos\theta + (n_zu_x - n_xu_z)\sin\theta & (18)\\
r_y &= u_y\cos\theta + (n_xu_y - n_yu_x)\sin\theta & (19)
\end{aligned}
$$

$$(x', y', z') = (C_x + r_x, C_y + r_y, C_z + r_z) \tag{20}$$

$$
\begin{aligned}
x' \qquad &= xn_x^2 + yn_xn_y + zn_xn_z + B_x - (\mathbf{B}\cdot\mathbf{n})n_x + & (21)\\
&(x(1 - n_x^2) - yn_xn_y - zn_xn_z + (\mathbf{B}\cdot\mathbf{n})n_x - B_x)\cos\theta + & (22)\\
&n_y(-xn_xn_z - yn_yn_z + (1 - n_z^2) + (\mathbf{B}\cdot\mathbf{n})n_z - B_z)\sin\theta - & (23)\\
&n_z(-xn_xn_y + y(1 - n_y^2) - zn_yn_z + (\mathbf{B}\cdot\mathbf{n})n_y - B_y)\sin\theta & (24)
\end{aligned}
$$

$$
\begin{aligned}
x' &= x[n_x^2(1 - n_x^2)\cos\theta] + y[n_xn_y(1 - \cos\theta) - n_z\sin\theta]\\
&+ z[n_xn_z(1 - \cos\theta) + n_y\sin\theta] + (B_x - (\mathbf{B}\cdot\mathbf{n})n_x)(1 - \cos\theta) + n_zB_y - n_yB_z. \quad (25)
\end{aligned}
$$

Since $n_x^2 + n_y^2 + n_z^2 = 1$, $(1 - n_x^2) = n_y^2 + n_z^2$. In like manner we can come up with an expression for y' and z', and our matrix M is thus

$$
\begin{bmatrix}
n_x^2 + (n_y^2 + n_z^2)\cos\theta & n_xn_y(1 - \cos\theta) - n_z\sin\theta & n_xn_z(1 - \cos\theta) + n_y\sin\theta & T_1\\
n_xn_y(1 - \cos\theta) + n_z\sin\theta & n_y^2 + (n_x^2 + n_z^2)\cos\theta & n_yn_z(1 - \cos\theta) - n_x\sin\theta & T_2\\
n_xn_z(1 - \cos\theta) - n_y\sin\theta & n_yn_z(1 - \cos\theta) + n_x\sin\theta & n_z^2 + (n_x^2 + n_y^2)\cos\theta & T_3\\
0 & 0 & 0 & 1
\end{bmatrix} \tag{26}
$$

with

$$T_1 = (B_x - (\mathbf{B}\cdot\mathbf{n})n_x)(1 - \cos\theta) + n_zB_y - n_yB_z$$

$$T_2 = (B_y - (\mathbf{B}\cdot\mathbf{n})n_y)(1 - \cos\theta) + n_zB_x - n_xB_z$$

$$T_3 = (B_z - (\mathbf{B}\cdot\mathbf{n})n_z)(1 - \cos\theta) + n_xB_y - n_yB_x$$